An Easton-like Theorem for all Cardinals in ZF

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Summary

We show that in Zermelo-Fraenkel Set Theory without the Axiom of Choice (ZF), a surjectively modified Continuum Function $\theta(\kappa)$ can take almost arbitrary values on all cardinals. This is in sharp contrast to the situation in ZFC, where on the one hand, Easton's Theorem states that the Continuum Function on the class of all *regular* cardinals is essentially undetermined, but on the other hand, various results show that the value of 2^κ for *singular* cardinals κ is strongly influenced by the behavior of the Continuum Function below.

Without the Axiom of Choice (AC), the powerset of a cardinal $\wp(\kappa)$ is generally not well-orderable, and there are different ways how "largeness" can now be expressed. The *θ-function* maps any cardinal κ to the least cardinal α for which there is no surjective function from $\wp(\kappa)$ onto α, thus measuring the *surjective size* of the powersets $\wp(\kappa)$.

Our first theorem answers a question of Saharon Shelah, who asked whether there are any bounds on the θ-function in the theory ZF + DC + AX_4. Here, the axiom AX_4 is the assertion that for every cardinal λ, the set $[\lambda]^{\aleph_0}$ (the collection of all countable subsets of λ) can be well-ordered. Together with the Axiom of Dependent Choice (DC), the theory ZF + DC + AX_4 provides a rich framework for combinatorial set theory in the ¬AC-context, in which set theory is "not so far from normal" (Shelah). Nevertheless, we prove that the answer to Shelah's question is *no*: Given any "reasonable" behavior of the θ-function on a set of uncountable cardinals, we construct a symmetric extension $N \vDash$ ZF + DC + AX_4 where this behavior is realized. More precisely: For sequences of uncountable cardinals $(\kappa_\eta \mid \eta < \gamma)$ and $(\alpha_\eta \mid \eta < \gamma)$ with certain natural properties in our ground model V, we construct a cardinal-preserving symmetric extension $N \supseteq V$ with $N \vDash$ ZF + DC + AX_4 such that $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all $\eta < \gamma$.

Our forcing notion is based on ideas from the paper *"Violating the Singular Cardinals Hypothesis without Large Cardinals"* (2012) by Moti Gitik and Peter Koepke. We modify and generalize their construction in order to treat not only the cardinal $\aleph_\omega$, but the θ-values of all cardinals $(\kappa_\eta \mid \eta < \gamma)$ simultaneously.
For every $\eta < \gamma$, we add α_η-many new κ_η-subsets to the ground model, which are linked in a certain fashion in order not to accidentally raise the θ-values of the cardinals below. Our eventual model N contains surjections $s : \wp(\kappa_\eta) \to \alpha$ for every $\eta < \gamma$ and $\alpha < \alpha_\eta$, but N does **not** contain a surjection $s : \wp(\kappa_\eta) \to \alpha_\eta$ for any $\eta < \gamma$. Moreover, an *Approximation Lemma* holds: Any set of ordinals located in N can be captured in a "mild" V-generic extension that preserves cardinals and the GCH. Thus, cardinals are N-V-absolute.

This great freedom provided to the Continuum Function in ZF + DC + AX_4 differs drastically from the limitations and restrictions prominent in ZFC.

Our second theorem deals with the question whether also any "reasonable" behavior of the θ-function on a **class** of infinite cardinals can be realized in ZF. (The construction explained above can not be straightforwardly generalized to a class-sized forcing notion

and is therefore only suitable for treating **set** many θ-values at the same time.)

Given a ground model V with a function $F : \mathrm{Card} \to \mathrm{Card}$ on the class of infinite cardinals such that F is weakly monotone and $F(\kappa) \geq \kappa^{++}$ holds for all κ, is there is a cardinal-preserving extension $N \supseteq V$ with $N \vDash \mathsf{ZF}$ such that $\theta^N(\kappa) = F(\kappa)$ for all $\kappa \in \mathrm{Card}$?

We introduce a new notion of class forcing $\mathbb{P}$, consisting of functions on trees with finitely many maximal points. The trees' levels are indexed by cardinals, and on any level κ, there are finitely many vertices (κ, i) with $i < F(\kappa)$. Below any vertex (κ, i), we will add a new κ-subset to the ground model. Since we do not allow splitting at limits for the trees, it follows that this forcing notion indeed adds $F(\kappa)$-many new κ-subsets for every κ. Our eventual model N is a symmetric extension by this class forcing $\mathbb{P}$. We prove that $N \vDash \mathsf{ZF}$, although $\mathbb{P}$ is not pretame and collapses all cardinals. Moreover, N can be approximated from within by rather "mild" V-generic extensions and hence, preserves all cardinals. Finally, we prove that $\theta^N(\kappa) = F(\kappa)$ holds for all κ.
Note that by finiteness of the trees however, it is not possible to retain DC in the symmetric extension N.

We conclude that any "reasonable" behavior of the θ-function can be realized in ZF – the only restrictions are the obvious ones. In other words: An analogue of Easton's theorem holds for all cardinals.

References

[1] Anne Fernengel and Peter Koepke, *An Easton-like Theorem for Zermelo-Fraenkel Set Theory with the Axiom of Dependent Choice*, 2018. `arXiv:1812.00960`.

[2] Anne Fernengel and Peter Koepke, *An Easton-like Theorem for Zermelo-Fraenkel Set Theory without Choice*, 2016. `arXiv:1607.00205`.

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Chapter 0

Introduction

0.1 The Continuum Function in ZFC

Investigations of the *Continuum Function* $\kappa \mapsto 2^\kappa$ lead back to the very beginnings of set theory. In 1878, at a time where the theory of transfinite ordinal numbers had not been developed yet, Georg Cantor postulated his first version of the **Continuum Hypothesis (CH)**:

There is no set the cardinality of which is strictly between the cardinality of the set $\mathbb{R}$ *of real numbers and the cardinality of the set* $\mathbb{N}$ *of natural numbers.*

A few years later, the theory of cardinals and their exponentiation lead Cantor to the final form:

$$2^{\aleph_0} = \aleph_1 \qquad (\mathsf{CH}) \ .$$

The question about the truth or falsehood of the Continuum Hypothesis became the first on David Hilbert's famous list of important open problems, presented at the International Congress of Mathematicians in 1900. Cantor was convinced that the Continuum Hypothesis should be true, and tried in vain to prove it for many years. It was not until 1963, 45 years after Cantor's death, that Paul Cohen gave a proof in [Coh63] and [Coh64] that $2^{\aleph_0}$ can be any cardinal κ of uncountable cofinality in ZFC. Before that, Kurt Gödel had shown in [Göd40] that CH holds in his constructible universe L. Thus, not only was the *Continuum Hypothesis* among the first statements that were shown to be independent of ZF, but the technique of *forcing* that Paul Cohen invented in his proofs has had a decisive impact on modern set theory as a powerful tool for establishing relative consistency and independence results.

However, there is no evidence that Georg Cantor or any of his contemporaries generalized the Continuum Hypothesis to arbitrary $\aleph_\alpha$ ([Moo11, p. 491]).

In 1904, a talk by Julius König at the International Congress of Mathematicians at Heidelberg attracted attention, where he gave a "proof" that CH is false. Shortly after, Felix Hausdorff discovered that the origin of the mistake was the theorem by Felix Bernstein

([Ber01]),

$$\aleph_\mu^{\aleph_\alpha} = 2^{\aleph_\alpha} \cdot \aleph_\mu,$$

which fails in the case that $\alpha = 0$ and $\mu = \omega$. This lead Hausdorff to the notion of *cofinality*, followed by extensive research on order types. In [Hau08], Hausdorff postulated the **Generalized Continuum Hypothesis (GCH)** in the following form:

If the ordinal α has a predecessor $\alpha - 1$ and $\aleph_\alpha$ is regular, then $\aleph_\alpha = \aleph_\alpha^{\aleph_{\alpha-1}} = 2^{\aleph_{\alpha-1}}$.

Replacing α by $\alpha + 1$ (taking in consideration that Hausdorff had shortly discovered that with the Axiom of Choice, it follows that any successor cardinal $\aleph_{\alpha+1}$ is regular), this yields:

$$\forall\, \alpha \in \mathrm{Ord} \;\; 2^{\aleph_\alpha} = \aleph_{\alpha+1} \quad (\mathsf{GCH})\ .$$

Hausdorff never took a clear position whether the GCH should be true or false, while methods for constructing independence results were still out of reach at that time.

In [Göd38], where Kurt Gödel introduced the class L of constructible sets, he proved the consistency of the GCH with ZFC. The first global result about possible behaviors of the Continuum Function $\kappa \mapsto 2^\kappa$ *contradicting* GCH was given by William B. Easton in [Eas70], seven years after Paul Cohen had invented the method of forcing. Easton's theorem states that any reasonable behavior of the 2^κ-function on the **regular** cardinals κ is consistent with ZFC. Indeed, the only constraints are *weak monotonicity* ($\kappa \leq \lambda \to 2^\kappa \leq 2^\lambda$) and *König's Theorem*, which implies $cf\left(2^\kappa\right) > \kappa$ for all κ.
Easton's Theorem reads as follows:

Theorem (William B. Easton)**.** *Let V be a ground model of ZFC + GCH with a class function F whose domain consists of regular cardinals and whose range consists of cardinals, such that for all κ, $\lambda \in \operatorname{dom} F$ the following properties hold:*

- $\kappa \leq \lambda \to F(\kappa) \leq F(\lambda)$,
- $\operatorname{cf} F(\kappa) > \kappa$.

Then there exists a generic extension $V[G] \vDash \mathsf{ZFC}$ by class forcing such that V and $V[G]$ have the same cardinals and cofinalities, and $V[G] \vDash 2^\kappa = F(\kappa)$ holds for all $\kappa \in \operatorname{dom} F$.

Easton's forcing construction takes "many" Cohen forcings, and combines them in a way that was henceforth known as the *Easton product.* This technique generalized results by Cohen and Solovay, which had only allowed for setting *finitely* many 2^κ-values simultaneously.

Summing up, the behavior of the Continuum Function on the class of regular cardinals follows the rules of "anything goes": There are no restrictions, except for the obvious ones.

For singular cardinals however, the situation is a lot more involved, since the value of 2^κ for singular κ is strongly influenced by the behavior of the Continuum Function below. In the model constructed by Easton, the 2^κ-values for singular κ are as small as possible, so the **Singular Cardinals Hypothesis (SCH)** holds, which is the following statement:

Whenever κ is a singular cardinal with $2^{\mathrm{cf}\,\kappa} < \kappa$, then $\kappa^{\mathrm{cf}\,\kappa} = \kappa^+$.

In particular:

Whenever κ is a singular cardinal with the property that $2^\lambda < \kappa$ holds for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$.

It turned out that the negation of the SCH is tightly linked with the existence of large cardinals. Among the first results in this direction was a theorem by Menachem Magidor ([Mag77b] and [Mag77a]) who proved that, assuming a huge cardinal, it is possible that GCH first fails at a singular strong limit cardinal. On the other hand, Ronald Jensen and Keith Devlin proved in [DJ75] that the negation of $0^\sharp$ implies SCH. Moti Gitik determined in [Git89] and [Git91] the consistency strength of $\neg$SCH being the existence of a measurable cardinal λ of Mitchell order $\sigma(\lambda) = \lambda^{++}$.

There are many more results about possible behaviors of the Continuum Function starting from large cardinals. For instance, a theorem of Carmi Merimovich shows that assuming some large enough cardinal, the theory $\mathsf{ZFC} + \forall\kappa\ (2^\kappa = \kappa^{+n})$ is consistent for each $n < \omega$ ([Mer07]).

On the other hand, *Silver's Theorem* ([Sil75]) imposes a restriction on possible values of the Continuum Function on singular cardinals, which came rather surprising at that time, since the general impression was that maybe Easton' theorem could be generalized to **all** cardinals.

Silver's Theorem reads as follows:

For any singular cardinal κ of uncountable cofinality such that $2^\lambda = \lambda^+$ holds for all $\lambda < \kappa$, it follows that $2^\kappa = \kappa^+$.

Moreover, the SCH holds if it holds for all singular cardinals of countable cofinality.
This result was extended by Fred Galvin and András Hajnal shortly after ([GH75]).

The probably most famous upper bound on the Continuum Function on singular cardinals is the following theorem by Saharon Shelah ([She94]):

If $2^{\aleph_n} < \aleph_\omega$ for all $n < \omega$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

In [GM96], William Mitchell and Moti Gitik prove that if there is no inner model with a strong cardinal, then even $2^{\aleph_\omega} < \aleph_{\omega_1}$.

This brief overview makes clear that there are significant constraints on possible behaviors of the Continuum Function in ZFC. In particular, a result like Easton's Theorem can not exist for all cardinals.
Today's research on the Continuum Function is concerned with firstly, finding restrictions on possible behaviors, and secondly, finding equiconsistency results of possible behaviors and large cardinals (cf. [Mer07, p. 2]).

All of these results essentially involve the Axiom of Choice.

In this thesis, we look at possible behaviors of the Continuum Function in $\mathsf{ZF} + \neg\mathsf{AC}$. Papers by Arthur Apter and Peter Koepke ([AK10]) and Moti Gitik and Peter Koepke ([GK12]), the latter of which this thesis is based on, show that an accordingly modified Continuum Function has a lot more freedom in ZF. We generalize their results towards an "Easton-like" theorem for regular **and** singular cardinals.

0.2 The Axiom of Choice

First formulated by Ernst Zermelo in [Zer04], the Axiom of Choice is still the most controversial mathematical axiom. It states:

For every family $\mathcal{S}$ of nonempty sets there exists a choice function*, i.e. a function f on $\mathcal{S}$ with the property that $f(X) \in X$ holds for every set X in $\mathcal{S}$.* (Axiom of Choice, AC)

The original purpose of Zermelo was to give a rigorous proof of the well-ordering theorem, but instead he started a debate about the tenability of this "new" axiom, providing the possibility of arbitrary choices without the slightest hint how the resulting function f could be defined.

Further criticism of the Axiom of Choice arose from the fact that it has some "unpleasant" ([Jec73, p. 2]) consequences that do not seem to agree with our basic intuition. The most famous one is probably the *Banach-Tarski* paradox ([BT24]), based on earlier work by Vitali ([Vit05]) and Hausdorff ([Hau14a]): Any solid sphere can be decomposed into finitely many subsets, which can be reassembled in a different way to obtain *two* solid spheres, each of which has the same size as the original one. This phenomenon seems to be a "paradox", since dividing a sphere into finitely many parts, moving them around and rotating them, should preserve the volume. The key point is that the subsets considered are *non-measurable sets* which do not have a volume in the ordinary sense. Their construction makes use of uncountably many choices.

On the other hand, the Axiom of Choice is indispensable for many important theorems of modern mathematics. For example, it is equivalent to *Zorn's Lemma*, *Tychonoff's Theorem* (the product of any family of compact topological spaces is compact), and the theorem that every vector space has a basis.

In [Göd38], Kurt Gödel proved the consistency of AC relative to ZF by constructing $L \vDash \mathsf{ZFC}$, starting off from a model of ZF. This paved the way for a broad acceptance of the Axiom of Choice (together with the matter of fact that many substantial theorems in mathematics do not get by without AC).

However, it was not until 1964 that a proof was given for the *independence* of AC from the axiom system ZF; for which Paul Cohen used his shortly invented technique of **forcing** (cf. [Coh63] and [Coh64]). He incorporated arguments by A. Fraenkel ([Fra22]), who had introduced *permutation models* more than 40 years ago, proving the independence of AC

from ZFA (an axiom system of set theory allowing the existence of *atoms*). We elaborate on this in Chapter 1.2, p. 13). Subsequent work by Mostowski ([Mos39]), Lindenbaum ([LM38]) and Specker ([Spe57]) lead to the formulation of *symmetric forcing*, which opened up the way for a huge variety of theorems and equiconsistency results. Starting off from a model of a theory ZFC + X, symmetric forcing leads to a model of a theory ZF + ¬AC + Y, thus showing: If ZFC + X is consistent, then so is ZF + ¬AC + Y.
Although there is no doubt that the Axiom of Choice should be generally accepted, it can still be a "worthwhile endeavor" ([She16, §0]) to construct models of a theory ZF + ¬AC + Y, which can give deep insight in the theory ZFC itself.

In [She14], Shelah reflects on the Axiom of Choice from a contemporary, pragmatic point of view. He starts with listing reasons why set theory without AC should be taken into account, although today's mathematicians are of course not impressed by the "paradox" of Banach-Tarski any more, and do not question the indispensability of the Axiom of Choice.
First, he reminds us that historically, essentially the lack of a "reasonable" theory without AC lead to its acceptance. Thus, the establishment of "nice" results in ZF (+ weaker forms of AC, for instance ZF + DC + AX_4, see below) on the other hand justifies considering ZF + ¬AC. Secondly, Shelah points out that a theory without AC bars the way to "mere existence theorem[s]" ([She14, p. 247]), but insists on "nicely definable" [She16, §0] solutions. Thus in a sense, existence theorems are "strengthen[ed]" [She16, §0] by weakening AC.

We now look at weak forms of AC, leading to rich theories.

The **Axiom of Dependent Choice (DC)** was introduced by Paul Bernays in 1942 ([Ber42]):

For every nonempty set X with a binary relation R such that for all $x \in X$ there is $y \in X$ with yRx, it follows that there is a sequence $(x_n \mid n < \omega)$ in X such that $x_{n+1} R x_n$ for all $n < \omega$. (Axiom of Dependent Choice, DC)

Over ZF, the axiom DC is equivalent to the Baire category theorem for complete metric spaces, and it is equivalent to the Löwenheim-Skolem theorem. Over ZF + DC, it is consistent that every set of reals is Lebesgue measurable ([Sol70]). (The construction of a non-measurable set requires uncountably many choices.)

The **Axiom of Countable Choice (CC or AC_ω)**, asserting that any countable collection of nonempty sets has a choice function, is strictly weaker than DC.

The Axiom of Dependent Choice can be generalized as follows, for κ a cardinal:

Let $\mathcal{S}$ be a nonempty set with a binary relation R on $\mathcal{S}$, such that for every $\alpha < \kappa$ and every function $f : \alpha \to \mathcal{S}$, there exists $y \in \mathcal{S}$ with $f\,R\,y$. Then there is $f : \kappa \to \mathcal{S}$ such that $f \restriction \alpha\, R\, f(\alpha)$ holds for all $\alpha < \kappa$. (DC_κ)

The Axiom of Choice implies that DC_κ holds for all κ – indeed, $\forall \kappa\ \mathsf{DC}_\kappa$ is equivalent to AC.

When dealing with real numbers, surprisingly often the Axiom of Dependent Choice is sufficient (instead of full AC), and the theory $\mathsf{ZF} + \mathsf{DC}$ provides an interesting framework for real analysis.
Concerning combinatorial set theory however, investigations under $\mathsf{ZF} + \mathsf{DC}$ seemed rather hopeless in the first place. A crucial step in the other direction was a paper by Saharon Shelah ([She97]) with the main result in $\mathsf{ZF} + \mathsf{DC}$ that whenever μ is a singular cardinal of uncountable cofinality such that $|H(\mu)| = \mu$, then μ^+ is regular and non-measurable. In the case that the power sets $\wp(\alpha)$ are well-orderable for all $\alpha < \aleph_{\omega_1}$ with $|\bigcup_{\alpha<\aleph_{\omega_1}} \wp(\alpha)| = \aleph_{\omega_1}$, it essentially follows that also $\wp(\aleph_{\omega_1})$ is well-orderable.
Subsequently (see [She10] and [She16]), Shelah showed that much of pcf theory is possible in $\mathsf{ZF} + \mathsf{DC}$, if an additional axiom is adopted:

$$\textit{For every cardinal } \lambda, \textit{ the set } [\lambda]^{\aleph_0} \textit{ can be well-ordered.} \qquad (\mathsf{AX}_4)$$

He calls AX_4 an "anti-thesis to considering $L[\mathbb{R}]$" ([She16, §0]) where roughly speaking, only $2^{\aleph_0}$ lacks a well-ordering.

Starting from a ground model $V \vDash \mathsf{ZFC}$, any symmetric extension by countably closed forcing yields a model of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ (see [She10, p.3 and p.15]). In [She16, 0.1], Shelah concludes that $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ is "a reasonable theory, for which much of combinatorial set theory can be generalized". For example, he proves a rather strong version of the pcf theorem, gives a representation of λ^κ for $\lambda >> \kappa$ (concluding that $[\lambda]^\kappa$ can be be "almost well-ordered" ([She14, p. 249])), and proves that certain covering numbers exist. Moreover, Shelah shows that in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ there is a proper class of regular successor cardinals. (There can still be singular successors, but "not too many", [She14, p. 249]). In [She14, p. 249], Shelah concludes that set theory in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ is "not so far from normal", which makes investigations in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ a worthwhile venture.

0.3 The Continuum Function in ZF

In Chapter 0.1, we saw that in ZFC, the Continuum Function on the class of all regular cardinals is essentially undetermined by Easton's Theorem, while for singular cardinals κ on the other hand, possible values of 2^κ are strongly influenced by the behavior of the Continuum Function below. In particular, an Easton-like theorem for regular and singular cardinals can not exist.

All results setting bounds on possible 2^κ-values for singular cardinals essentially involve the Axiom of Choice. Without AC, however, there is a lot more possible:

In [AK10], Arthur Apter and Peter Koepke examine the consistency strength of the negation of SCH in $\mathsf{ZF} + \neg\mathsf{AC}$. In this context, one has to distinguish between *injective* and *surjective* failures. An *injective failure of* SCH *at* κ is a model of $\mathsf{ZF} + \neg\mathsf{AC}$ with a singular cardinal κ such that GCH holds below κ, but there is an injective function $\iota : \lambda \hookrightarrow \wp(\kappa)$ for some $\lambda \geq \kappa^{++}$. A *surjective failure of* SCH *at* κ is a model of $\mathsf{ZF} + \neg\mathsf{AC}$ with a singular cardinal κ such that GCH holds below κ, but there is a surjective function $f : \wp(\kappa) \to \lambda$ for some cardinal $\lambda \geq \kappa^{++}$. On the one hand, Arthur Apter and Peter Koepke construct injective failures of the SCH at $\aleph_\omega$, $\aleph_{\omega_1}$ and $\aleph_{\omega_2}$ that would contradict the theorems by Shelah ([She94]) and Silver ([Sil75]) in the ZFC-context, but have fairly mild consistency strengths in $\mathsf{ZF} + \neg\mathsf{AC}$. For instance, they prove that the theory

$$\mathsf{ZFC} + \exists\kappa\left(\sigma(\kappa) = \kappa^{++} + \omega_2\right),$$

where $\sigma(\kappa)$ denotes the Mitchell order of the measurable cardinal, is equiconsistent with the theory

$$\mathsf{ZF} + \neg\mathsf{AC} + \text{“ }\mathsf{GCH}\text{ holds below } \aleph_{\omega_2}\text{” } +$$
$$+ \text{ “ there is an injective function } \iota : \aleph_{\omega_2+2} \to [\aleph_{\omega_2}]^{\omega_2} \text{ ”.}$$

On the other hand, regarding a surjective failure of the SCH, they prove that for every $\alpha \geq 2$, ZFC together with the existence of a measurable cardinal is equiconsistent with the theory

$$\mathsf{ZF} + \neg\mathsf{AC} + \text{“}\mathsf{GCH}\text{ holds below } \aleph_\omega \text{ ” } +$$
$$+ \text{ “ there is a surjective function } f : [\aleph_\omega]^\omega \to \aleph_{\omega+2} \text{ ”.}$$

It follows that also without the Axiom of Choice, injective failures of the SCH are inevitably linked to large cardinals. Regarding surjective failures however, one can not replace the surjective function $f : [\aleph_\omega]^\omega \to \aleph_{\omega+2}$ in their argument by a surjection $f : \wp(\aleph_\omega) \to \aleph_{\omega+2}$; so the following question remained:

Is it possible for $\lambda \geq \aleph_{\omega+2}$, to construct a model of $\mathsf{ZF} + \neg\mathsf{AC}$ where GCH holds below $\aleph_\omega$ and there is a surjection $f : \wp(\aleph_\omega) \to \lambda$ **without** any large cardinal assumptions?

This question was positively answered by Motik Gitik and Peter Koepke in [GK12], where a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with a cardinal $\lambda \geq \aleph_{\omega+2}$ is extended via symmetric forcing such that the extension $N = V(G)$ preserves all V-cardinals, the GCH holds in N below $\aleph_\omega$, and there is a surjective function $f : \wp(\aleph_\omega) \to \lambda$.

More generally, in the absence of the Axiom of Choice where $\wp(\kappa)$, the power set of a cardinal κ, is generally not well-ordered, the “size” of $\wp(\kappa)$ can be measured surjectively by the θ-function

$$\theta(\kappa) := \sup\{\alpha \in \mathrm{Ord} \mid \exists\, f : \wp(\kappa) \to \alpha \text{ surjective function}\},$$

generalizing the value $\theta := \theta(\omega)$ prominent in descriptive set theory. In the $\neg\mathsf{AC}$-context, the θ-function provides a surjective substitute for the Continuum Function $\kappa \mapsto 2^\kappa$. If $\theta(\kappa) = \mu$, there exists a surjective function $f : \wp(\kappa) \to \alpha$ for every $\alpha < \mu$, but there is no surjection function $f : \wp(\kappa) \to \mu$. Since also without the Axiom of Choice, there is always a surjection $f : \wp(\kappa) \to \kappa^+$, it follows that $\theta(\kappa) \geq \kappa^{++}$ for all cardinals κ.

One can show that in the model constructed in [GK12], it follows that indeed, $\theta(\aleph_\omega) = \lambda^+$. The question arises to what extent this result can be generalized: Is it possible to do a similar construction and replace $\aleph_\omega$ by a cardinal κ of uncountable cofinality? What happens if we want $\theta(\kappa)$ to be a limit cardinal? And is it possible to treat several cardinals κ at the same time and set their θ-values independently? Can we perhaps even modify the θ-function as we wish?
This leads us to our main question:

Is the θ-function essentially undetermined in ZF*?*

We will see that the answer is *yes*.

The construction introduced in [GK12] can roughly be described as follows: A ground model V is extended by a forcing notion $\mathbb{P}$ adding λ-many subsets of $\aleph_\omega$. These subsets are linked in a certain fashion to make sure that not too many $\aleph_n$-subsets are adjoined for $n < \omega$. The eventual model N is a symmetric submodel of the generic extension, generated by certain equivalence classes of these λ-many $\aleph_\omega$-subsets.

In Chapter 2, we modify and generalize this forcing notion. Given a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with a "reasonable" behavior of the θ-function on a set of (regular or singular) cardinals, our construction provides a cardinal-preserving symmetric extension where this behavior is realized.

One important modification is that we replace finiteness properties by the property of being countable, which gives a countably closed forcing notion $\mathbb{P}$. Together with a countably complete normal filter on our $\mathbb{P}$-automorphism group, it follows that the according symmetric extension N is a model of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ (cf. [Kar14, Lemma 1] and [She16, p. 3 + 15]).

Our first main theorem (see [FK18]) states:

Theorem. *Let V be a ground model of* $\mathsf{ZFC} + \mathsf{GCH}$ *with $\gamma \in \mathrm{Ord}$ and sequences of uncountable cardinals $(\kappa_\eta \mid \eta < \gamma)$ and $(\alpha_\eta \mid \eta < \gamma)$, such that $(\kappa_\eta \mid \eta < \gamma)$ is strictly increasing and closed, and the following properties hold:*

- $\forall\, \eta < \eta' < \gamma \quad \alpha_\eta \leq \alpha_{\eta'}$*, i.e. the sequence $(\alpha_\eta \mid \eta < \gamma)$ is increasing,*
- $\forall\, \eta < \gamma \quad \alpha_\eta \geq \kappa_\eta^{++}$*,*
- $\forall\, \eta < \gamma \quad \mathrm{cf}\ \alpha_\eta > \omega$*,*
- $\forall\, \eta < \gamma \quad (\alpha_\eta = \alpha^+ \rightarrow \mathrm{cf}\ \alpha > \omega)$*.*

Then there is a cardinal- and cofinality-preserving extension $N \supseteq V$ with $N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ such that that $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all $\eta < \gamma$.

Firstly, this result gives an answer to our main question for the theory $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$: *Yes*, the θ-function is essentially undetermined on any set of cardinals.
Secondly, the theorem above answers a question of Shelah: Firstly, in [She10, §0], he emphasizes that under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$, we "cannot say much" on possible cardinalities of $\wp(\kappa)$. In [She16, §0.2 1)], Shelah asks, referring to [GK12]: "*Can we bound* hrtg $\wp(\mu)$[1] $[= \theta(\mu)]$ *for μ singular?*" No, we can not.

[1] In [She16, §0.4 1)], Shelah defines "hrtg $(A) = \min\{\alpha \mid$ there is no function from A onto $\alpha\}$". Then *hrtg* $\wp(\kappa) = \min\{\alpha \in \mathrm{Ord} \mid \nexists f : \wp(\kappa) \to \alpha$ surjective function$\} = \sup\{\beta \in \mathrm{Ord} \mid \exists f : \wp(\kappa) \to \beta$ surjective function$\} = \theta(\kappa)$.
This does not coincide with the notion of the *Hartogs number* of a set X, which is usually defined as the least ordinal α such that there is no **injection** from α into X.

Chapter 3 deals with the question whether also any "reasonable" behavior of the θ-function on a **class** of (regular or singular) cardinals can be realized. Before that, in Chapter 2.7, we argue that a construction like in Chapter 2 can not be straightforwardly generalized to a class-sized forcing notion and is therefore only suitable for treating *set* many θ-values at the same time.

This gives rise to the following question: Given a ground model V with a "reasonable" function $F : \mathrm{Card} \to \mathrm{Card}$ on the class of all infinite cardinals, is there a cardinal-preserving extension $N \supseteq V$ where $\theta^N(\kappa) = F(\kappa)$ holds for all κ?

In Chapter 3.1, we introduce a new notion of forcing $\mathbb{P}$ whose elements p are functions on trees $(t, \leq_t)$ with finitely many maximal points. The trees' levels are indexed by cardinals, and on any level κ, there are finitely many vertices (κ, i) with $i < F(\kappa)$. For successor cardinals κ^+, the value $p(\kappa^+, i)$ is a partial 0-1-function on the interval $[\kappa, \kappa^+)$. Thus, for any condition p and $(\kappa, i) \in \operatorname{dom} p$, it follows that $\bigcup\{p(\nu^+, j) \mid (\nu^+, j) \leq_t (\kappa, i)\}$ is a partial function on κ with values in $\{0, 1\}$. Since we do not allow splitting at limits for the trees, it follows that this forcing indeed adds $F(\kappa)$-many new κ-subsets for every cardinal κ.

Our eventual model N is a symmetric extension by this class forcing $\mathbb{P}$. Although $\mathbb{P}$ is not pretame and collapses all cardinals, we will see that $N \vDash \mathsf{ZF}$. Moreover, cardinals are N-V-absolute, and $\theta^N(\kappa) = F(\kappa)$ holds for all κ.

In other words: In ZF, the θ-function can take almost arbitrary values on all cardinals. The only constraints are the obvious ones: weak monotonicity, and $\theta(\kappa) \geq \kappa^{++}$ for all κ.

This gives our second main theorem (see [FK16]):

Theorem. *Let V be a ground model of $\mathsf{ZFC} + \mathsf{GCH}$ with a function F on the class of infinite cardinals such that the following properties hold:*

- $\forall \kappa \; F(\kappa) \geq \kappa^{++}$
- $\forall \kappa, \lambda \; \big(\kappa \leq \lambda \to F(\kappa) \leq F(\lambda)\big)$.

Then there is a cardinal-preserving extension $N \supseteq V$ with $N \vDash \mathsf{ZF}$ such that $\theta^N(\kappa) = F(\kappa)$ holds for all κ.

This complements our results from Chapter 2 and gives another answer to our main question above: *Yes*, the Continuum Function is essentially undetermined in ZF – there is an Easton-like theorem for all cardinals.

This thesis is structured as follows: Following some preliminaries (see Chapter 0.4), Chapter 1 contains a comprehensive introduction to *Symmetric Forcing*. We start with the general forcing technique and then introduce symmetric forcing, largely following [Dim11], where the technical framework is given for symmetric forcing with partial orders (without using Boolean algebras). Chapter 1.2.3 complements the presentation from [Dim11] by including the case that one has to deal with automorphisms $\pi : D_\pi \to D_\pi$ defined not on the whole forcing notion $\mathbb{P}$, but only on a dense subset $D_\pi \subseteq \mathbb{P}$. Boolean algebras are avoided by working with equivalence classes of partial automorphisms $[\pi]$. We also set the framework for *Symmetric Class Forcing* without Boolean algebras, in the case that the class forcing $\mathbb{P}$ is considerably nice.

In Chapter 2, we first present the forcing notion introduced by Moti Gitik and Peter Koepke (cf. [GK12]). After that, we give a proof of our first main theorem: Any "reasonable" behavior of the θ-function on a set of uncountable cardinals (given by sequences in V, $(\kappa_\eta \mid \eta < \gamma)$, and the according θ-values $(\alpha_\eta \mid \eta < \gamma)$) can be realized in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$. We discuss what "reasonable" means in this context, and then introduce our countably closed forcing notion $\mathbb{P}$, based on the forcing notion constructed in [GK12]. The eventual model $N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ is a V-generic symmetric extension by $\mathbb{P}$. We show that N preserves all cardinals, and the θ-values are as desired: $\theta^N(\kappa_\eta) = \alpha_\eta$ for all $\eta < \gamma$.

In Chapter 3, we give a proof of our second main theorem: Given a ground model V with a function $F : \mathrm{Card} \to \mathrm{Card}$ on the class of infinite cardinals, respecting the rules of weak monotonicity and $F(\kappa) \geq \kappa^{++}$ for all $\kappa \in \mathrm{Card}$, we construct $N \supseteq V$ with $N \vDash \mathsf{ZF}$ such that cardinals are N-V-absolute and $\theta^N(\kappa) = F(\kappa)$ holds for all $\kappa \in \mathrm{Card}$. We first introduce our class-sized forcing notion $\mathbb{P}$, and then use our techniques from Chapter 1.4 to construct a V-generic symmetric extension, which will be our eventual model N. This yields an Easton-like theorem in ZF for all cardinals.

0.4 Preliminaries

The axiom system ZFC (*Zermelo-Fraenkel set theory with the Axiom of Choice*) is the most common foundation of mathematics. It consists of the following axioms:

Extensionality. If two sets X and Y have the same elements, then $X = Y$.

Foundation. Every nonempty set has an $\in$-minimal element.

Pairing. If X and Y are sets, then there exists a set $\{X, Y\}$ which contains exactly X and Y.

Union. If X is a set, there exists a set $Y = \bigcup X$ which is the union of all elements of X.

Infinity. There exists an infinite set.

Power Set. For every set X there exists a set $Y = \wp(X)$ which is the collection of all subsets of X.

Separation. If φ is a formula with its free variables among x, a, z; then for any sets a, z, also $\{x \in a \mid \varphi(x, a, z)\}$ is a set.

Replacement. If φ is a formula with its free variables among x, y, a, z; then for any sets a, z such that for all $x \in a$ there is exactly one y with $\varphi(x, y, a, z)$, also $\{y \mid \exists\, x \in a \ \varphi(x, y, a, z)\}$ is a set.

Axiom of Choice. Every family of nonempty sets has a choice function.

Separation and *Replacement* are *schemas* (they contain an axiom for every formula φ). The others are *single axioms*.

The Axiom System ZF (*Zermelo-Fraenkel Set Theory*) is ZFC without the *Axiom of Choice*. In context with class forcing, sometimes *Power Set* is excluded, which yields the theory ZFC^- (and ZF^-, respectively). Note that in ZF^-, not all commonly used choice principles are equivalent, so we have to insist on the strongest form:

Well-ordering. Every set can be well-ordered.

Then ZFC^- is the theory obtained from ZF^- by adding *Well-ordering*.

Also in context with class forcing, sometimes *Union* is weakened:

Weak Union. If X is a set, then there exists a set $Y \supseteq \bigcup X$ which is a superset of the union of all elements of X.

If *Separation* is available, then *Union* follows from *Weak Union*.

The following axiom schema implies *Replacement*, and is equivalent to *Replacement* if *Power Set* holds:

Collection. If a and z are sets and φ is a formula with its free variables among x, y, a, z, such that for every $x \in a$ there exists y with $\varphi(x, y, a, z)$, then there exists a set Y with the following property: For every $x \in a$ there is $y \in Y$ with $\varphi(x, y, a, z)$.

We denote by $\mathcal{L}_\in$ the language of set theory, i.e. first-order language with the binary predicate symbol "$\in$ ".

Throughout this thesis, we work in first-order set theory ZFC and forego introducing a second-order axiomatization like Gödel-Bernays set theory GB, which would allow for quantification over classes. In our setting, the classes are simply the definable classes in V, i.e. objects of the form $\{x \mid \varphi(x, x_0, \dots, x_{n-1})\}$, where $\varphi \in \mathcal{L}_\in$, with finitely many parameters $x_0, \dots, x_{n-1}$ from V. We will treat V-classes informally, but always take care that everything can be described in the language $\mathcal{L}_\in$.

Our notation is mostly standard and follows textbooks as [Jec06] or [Kun06].
We write *Ord* and *Card* for the class of ordinals and the class of infinite cardinals, respectively. The cofinality of an ordinal α is abbreviated *cf* α. We denote by *Reg* the class of all regular cardinals (all those $\kappa \in \mathrm{Card}$ with $\mathrm{cf}\,\kappa = \kappa$), and by *Sing* the class of all singular cardinals (all $\kappa \in \mathrm{Card}$ with $\mathrm{cf}\,\kappa < \kappa$).
A cardinal κ is *inaccessible* if it is uncountable, regular, and a *strong limit cardinal*, i.e. whenever $\lambda < \kappa$, then also $2^\lambda < \kappa$. An inaccessible cardinal is a type of large cardinal: It can not be reached from smaller cardinals by the common set-theoretic operations.

For a set X, we denote by $\wp(X) := \{Y \mid Y \subseteq X\}$ its *power set*, and by $TC(X)$ the *transitive closure of* X, i.e. the "$\subseteq$ "- smallest transitive set containing X. If X has cardinality $\geq \kappa$, then $[X]^\kappa$ denotes the collection of all $Y \in \wp(X)$ with $|Y| = \kappa$.

Given a function f, we denote by *dom* f its *domain*, and by *rg* f its *range*. We write $f : A \to B$ for $\mathrm{dom}\,f = A$ and $\mathrm{rg}\,f \subseteq B$. If the function f is injective, we write $f : A \hookrightarrow B$, if f is surjective, we sometimes write $f : A \twoheadrightarrow B$. For a set X, we denote by $id_X : X \to X$ the function that maps every $x \in X$ onto itself.

A *sequence* is a function whose domain is an ordinal α. We use the standard notation $(a_i \mid i < \alpha)$. A sequence of ordinals $(a_i \mid i < \alpha)$ is *normal* if it is *strictly increasing* (i.e. $i < j \to a_i < a_j$) and *closed* (i.e. for every limit ordinal $\beta < \alpha$, it follows that $a_\beta = \bigcup_{i<\beta} a_i$).

We assume familiarity with basic cardinal arithmetic as presented in [Jec06, Chapter 5].

For the sake of completeness, we state *König's Theorem*, which has decisive influence on behavior of the Continuum Function in ZFC:

Theorem 0.4.1 (König's Theorem, [Jec06, 5.10]). *Let $\mathcal{I}$ be a set and κ_i, $\lambda_i \in \mathrm{Card}$ for every $i \in \mathcal{I}$. Moreover, assume that $\kappa_i < \lambda_i$ holds for all $i \in \mathcal{I}$. Then*

$$\sum_{i \in \mathcal{I}} \kappa_i < \prod_{i \in \mathcal{I}} \lambda_i.$$

Corollary 0.4.2 ([Jec06, 5.12]). *Let κ be a cardinal. Then* $\mathrm{cf}\left(2^\kappa\right) > \kappa$.

The proof of König's Theorem essentially involves the Axiom of Choice.

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Chapter 1

Symmetric Forcing

In this chapter, we present symmetric forcing with partial orders following [Jec73, 5.1 + 5.2] and [Dim11, 1.1 + 1.2], and after that describe our construction of symmetric extensions for *partial automorphisms* $\pi : D_\pi \to D_\pi$, which are not defined on the whole forcing $\mathbb{P}$ but only on a dense subset $D_\pi \subseteq \mathbb{P}$. Thereby, we continue the presentation in [Dim11, 1.2] which contains a comprehensive approach to constructing symmetric extensions by partial orders without using Boolean Algebras. We extend the methods introduced there in order to incorporate *partial automorphisms* for set and class forcing.

We start this chapter with a short overview of the history of symmetric models.

In [Fra22], A. Fraenkel introduced the notion of a *permutation model* to provide a method for establishing independence results concerning the Axiom of Choice. His work was refined by Mostowski and Lindenbaum in [Mos39] and [LM38]. This approach starts from a ground model of ZFCA, which is a modified version of ZFC that allows *atoms*: An *atom* is not a set, and has no elements. Hence, *Extensionality* does not holds for the atoms.

A general theory of permutation models was developed by E. Specker in [Spe57]. The overall idea is that one can not use the axioms of ZFCA to distinguish between the atoms, which allows for constructing models in which the set A of atoms does not have a well-ordering.

Although this method can not be applied to models of ZFC, it gives some insight into the problem how the independence of the Axiom of Choice from the other axioms could be established.

Indeed, when Paul Cohen introduced the method of forcing in [Coh63] and [Coh64], set theorists noticed that certain sets derived from the generic filter behave similarly as the atoms in the theory ZFCA. The *symmetric extension* can roughly be constructed as follows: Automorphism of the partial order $\mathbb{P}$ can be extended to automorphisms of the name space, and the symmetric extension shall consist of the interpretations of all those names which are *hereditarily symmetric*, i.e. they hereditarily remain unchanged under "many" $\mathbb{P}$-automorphisms. In order to specify the phrase "many", one introduces an automorphism group A on the partial order, and a *normal filter* $\mathcal{F}$ on A. A $\mathbb{P}$-name $\dot{x}$ is *symmetric*, if the set $\{\pi \in A \mid \pi\dot{x} = \dot{x}\}$ is an element of $\mathcal{F}$, and recursively, a $\mathbb{P}$-name $\dot{x}$ is *hereditarily symmetric*, $\dot{x} \in HS$, if $\dot{x}$ is symmetric and $\dot{y} \in HS$ for all $\dot{y} \in \operatorname{dom}\dot{x}$. Then the *symmetric extension* $V(G) := \{\dot{x}^G \mid \dot{x} \in HS\}$ is a model of ZF.

This chapter is structured as follows:

In Chapter 1.1, we give the preliminaries for the method of *forcing*, in order to fix our notation and list some basic properties.
After that, in Chapter 1.2, we present the technique of *symmetric forcing with partial orders*. We follow the presentation given in [Dim11, 1.1 + 1.2], where the standard method of forcing with Boolean values from [Jec06] and [Jec73] is translated to partial orders.
In the case that the occuring automorphisms π are not defined on the whole forcing $\mathbb{P}$ but only on a dense subset $D_\pi \subseteq \mathbb{P}$ however, the standard approach is to turn back to Boolean-valued models, since any such $\pi : D_\pi \to D_\pi$ can be uniquely extended to an automorphism of the according complete Boolean algebra $B(\mathbb{P})$. The aim of Chapter 1.2.3 is to incorporate also this situation (which appears frequently in practical applications) into the technique of symmetric forcing with partial orders.
In Chapter 1.3, we give a brief introduction to *class forcing*. As an example, we discuss *Easton forcing*, a class-sized product forcing introduced by William Easton in [Eas70] in order to show that in ZFC, the Continuum Function on the class of all regular cardinals can take almost arbitrary values.
Finally, in Chapter 1.4, we set the framework for constructing symmetric extensions by class forcing in the case of *partial automorphisms* $\pi : D_\pi \to D_\pi$ on dense subsets $D_\pi \subseteq \mathbb{P}$. We restrict to the case that firstly, the class forcing $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ can be written as an increasing chain of set-sized complete subforcings (with certain additional properties) and secondly, any automorphism $\pi : D_\pi \to D_\pi$ defined on a dense class $D_\pi \subseteq \mathbb{P}$ is the canonical extension of some $\pi_\alpha : D_\pi \cap \mathbb{P}_\alpha \to D_\pi \cap \mathbb{P}_\alpha$, where $D_\pi \cap \mathbb{P}_\alpha$ is a dense subset of $\mathbb{P}_\alpha$. Symmetric class forcing will be used in Chapter 3.

1.1 Forcing: Notation, Basic Properties and Examples.

1.1.1 Forcing Preliminaries.

The method of *forcing* was invented by Paul Cohen in [Coh63] and [Coh64], where he proved the independence of the Continuum Hypothesis (CH) of ZFC. The idea is to extend a countable, transitive model of set theory V (the *ground model*) by a *generic filter* G, to obtain the *generic extension* $V[G]$ which is the smallest transitive ZFC-model with $G \in V[G]$ and $V \subseteq V[G]$. Forcing conditions in the ground model approximate G, and determine more and more properties of the generic extension $V[G]$.

Forcing is a very general and flexible method for producing a variety of models and establishing relative consistency results.

In this chapter, we give a short overview of the forcing technique and list some basic properties. A comprehensive introduction to forcing and generic extensions can be found in [Kun06, VII] or [Jec06, 14].

As an example, we will discuss several versions of *Cohen forcing*: Firstly, adding a *Cohen generic* is probably the simplest and most intuitive way of adding a new set to the ground model; and secondly, Cohen forcing is the starting point for modifying the Continuum Function $\kappa \mapsto 2^\kappa$.

We fix a *ground model* V, i.e. a countable, transitive model of ZFC. For a discussion of the metamathematical background, we refer to [Kun06, VII].

Definition 1.1.1. A *forcing* is a set $(\mathbb{P}, \leq, \mathbb{1})$ such that $(\mathbb{P}, \leq)$ is a *preorder* (i.e. the relation "$\leq$" is reflexive and transitive on $\mathbb{P}$) with greatest element $\mathbb{1}$.

The elements of $\mathbb{P}$ are the *conditions.* If $p, q \in \mathbb{P}$ with $q \leq p$, then q is *stronger* than p. Two forcing conditions $p, q \in \mathbb{P}$ are *compatible* (we write $p \| q$) if they have a *common extension* (i.e. there exists $r \in \mathbb{P}$ such that $r \leq q$ and $r \leq p$), and *incompatible* if they do not (we write $p \perp q$).

Most natural forcings are *antisymmetric*, i.e. for all $p, q \in \mathbb{P}$, we have $(q \leq p \wedge p \leq q) \rightarrow p = q$. Sometimes, antisymmetry is an additional requirement in Definition 1.1.1.

Note that, given a preorder $(\mathbb{P}, \leq)$ without a maximal element, one can easily construct a new one by adding a maximal element $\mathbb{1} \notin \mathbb{P}$ "on top", and work with the forcing notion $(\mathbb{P} \cup \mathbb{1}, \leq, \mathbb{1})$.

Definition 1.1.2. A forcing $\mathbb{P}$ is *separative* if for all $p_0, p_1 \in \mathbb{P}$ with $p_0 \nleq p_1$ there exists $p \leq p_0$ such that $p \perp p_1$.

Whenever a forcing notion $\mathbb{P}$ is *not* separative, it can be replaced by a separative partial order that yields the same generic extensions.

For the rest of this Chapter, let $(\mathbb{P}, \leq, \mathbb{1})$ denote a forcing.

We will always assume that a forcing notion $(\mathbb{P}, \leq, \mathbb{1})$ is separative (which is the case for most forcing notions that occur in practice).

Before we can construct generic extensions, we need the following notions:

Definition 1.1.3. A set $A \subseteq \mathbb{P}$ is an *antichain* if its elements are pairwise incompatible. A *maximal antichain* is an antichain $A \subseteq \mathbb{P}$ with the property that for all antichains $B \subseteq \mathbb{P}$ with $B \supseteq A$, it follows that $B = A$.

Definition 1.1.4. A set $D \subseteq \mathbb{P}$ is *dense* if for all $q \in \mathbb{P}$ there exists $q' \in D$ with $q' \leq q$. If in addition, D is *downwards closed* (i.e. $q \in D$ and $q' \leq q$ imply $q' \in D$), then D is called *open dense.* For a condition $p \in \mathbb{P}$, set $D \subseteq \mathbb{P}$ is *dense below* p if for every $q \leq p$ there exists $q' \in D$ with $q' \leq q$.

A set $D \subseteq \mathbb{P}$ is *predense* if for all $q \in \mathbb{P}$ there exists $q' \in D$ with $q' \parallel q$. For a condition $p \in \mathbb{P}$, a set $D \subseteq \mathbb{P}$ is *predense below* p if for every $q \leq p$ there exists $q' \in D$ with $q' \parallel q$.

Definition 1.1.5. A set $\varnothing \neq F \subseteq \mathbb{P}$ is a *filter on* $\mathbb{P}$ if the following holds:

(i) F is *upwards closed*: If $p \in F$ and $q \in \mathbb{P}$ with $q \geq p$, then also $q \in F$.

(ii) F is *directed*, i.e. for any p, $q \in F$, there exists $r \in F$ with $r \leq p$, q.

A filter $G \subseteq \mathbb{P}$ is *V-generic on* $\mathbb{P}$ if it intersects every dense set $D \subseteq \mathbb{P}$ with $D \in V$.

It is not difficult to verify that a filter $G \subseteq \mathbb{P}$ is V-generic on $\mathbb{P}$ if and only if it hits every maximal antichain / open dense set / predense set in V. If G is V-generic on $\mathbb{P}$ and $p \in G$, then $G \cap D \neq \varnothing$ for every set D that is dense below p.

In the case that $\mathbb{P}$ is *nonatomic* (or *nontrivial*), i.e. every $p \in \mathbb{P}$ has two incompatible extensions, an easy density argument yields that $\mathbb{P}$-generic filters never exist in the ground model V.

On the other hand, since we have assumed our ground model to be countable, one can enumerate the dense sets in V from the "outside" and find a filter G on $\mathbb{P}$ that hits every dense set in V. This settles the question of the existence of a V-generic filter (cf. [Kun06, 2.3] or [Jec06, 14.4]).

As an example, we look at *Cohen forcing* $\mathrm{Fn}(\omega, 2, \aleph_0)$, which adds an ω-subset to the ground model. It was introduced by Paul Cohen in[Coh63], who used a generalized version $\mathrm{Fn}(\omega_2 \times \omega, 2, \aleph_0)$ to construct a model in which $2^{\aleph_0} = \aleph_2$ holds, i.e. the Continuum Hypothesis fails. Further generalizations can be used to violate the *Generalized Continuum Hypothesis* GCH at any regular cardinal κ (see [Sol63]).

Example 1.1.6 (Cohen forcing). Let $\mathrm{Fn}(\omega, 2, \aleph_0) := \{p : \mathrm{dom}\, p \to 2 \mid \mathrm{dom}\, p \subseteq \omega\,, |\mathrm{dom}\, p| < \omega\}$ denote the set of all finite partial functions from ω into $\{0, 1\}$ ordered by reverse inclusion, i.e. $p \leq q$ iff $p \supseteq q$. For G a V-generic filter on $\mathrm{Fn}(\omega, 2, \aleph_0)$, it follows that any p, $q \in G$ are compatible; hence, $\bigcup G$ is a function. Moreover, for every $n < \omega$, the set $D_n := \{p \in \mathrm{Fn}(\omega, 2, \aleph_0) \mid n \in \mathrm{dom}\, p\}$ is dense; hence, $G \cap D_n \neq \varnothing$ and it follows that $n \in \mathrm{dom} \bigcup G$. Thus, $\bigcup G : \omega \to 2$, and since $\mathrm{Fn}(\omega, 2, \aleph_0)$ is nonatomic, we obtain $\bigcup G \notin V$. The function $\bigcup G : \omega \to 2$ is called a *Cohen real*.
In many cases, the generic filter G is confused with $\bigcup G$.

Cohen forcing is the starting point for changing the value of 2^κ and investigating possible behaviors of the Continuum Function.

We will now define the generic extension $V[G]$. Informally, $V[G]$ consists of all sets which can be constructed using G and finitely many elements of the ground model V. Every $x \in V[G]$ has a *name* $\dot{x} \in V$, which tells how x can be constructed from G.

Definition 1.1.7. The *class of* $\mathbb{P}$*-names for* V is defined as follows: Recursively, we define the set $\mathrm{Name}^V_\alpha(\mathbb{P})$ by setting $\mathrm{Name}^V_0(\mathbb{P}) := \varnothing$, $\mathrm{Name}^V_{\alpha+1}(\mathbb{P}) := \{\dot{x} \mid \dot{x} \subseteq \mathrm{Name}^V_\alpha(\mathbb{P}) \times \mathbb{P})\}$, and $\mathrm{Name}^V_\lambda(\mathbb{P}) := \bigcup_{\alpha<\lambda} \mathrm{Name}^V_\alpha(\mathbb{P})$ for λ a limit ordinal.
The *class of* $\mathbb{P}$*-names for* V is $\mathrm{Name}^V(\mathbb{P}) := \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{Name}^V_\alpha(\mathbb{P})$.

For $\dot{x} \in \mathrm{Name}^V(\mathbb{P})$, let $\mathsf{rk}_\mathbb{P} \dot{s} := \alpha$ if $\dot{x} \in \mathrm{Name}^V_{\alpha+1}(\mathbb{P}) \setminus \mathrm{Name}^V_\alpha(\mathbb{P})$. This is the $\mathbb{P}$*-rank of* $\dot{x}$. For any $\dot{x} \in \mathrm{Name}^V(\mathbb{P})$, it follows that $\mathsf{rk}_\mathbb{P}\dot{x} = \sup\{\mathsf{rk}_\mathbb{P}\dot{y} + 1 \mid \dot{y} \in \mathrm{dom}\, \dot{x}\}$, and $\mathrm{Name}^V_\alpha(\mathbb{P})$ is the collection of all $\dot{x} \in \mathrm{Name}^V(\mathbb{P})$ with $\mathsf{rk}_\mathbb{P}\dot{x} < \alpha$.

We introduce the following notation:

For a $\mathbb{P}$-name $\dot{x} \in \mathrm{Name}^V(\mathbb{P})$, let

$$d_0(\dot{x}) := \operatorname{dom}\dot{x}\,,\ d_{n+1}(\dot{x}) := \operatorname{dom} d_n(\dot{x}) \text{ for } n < \omega\,, \text{ and } \mathrm{Tdom}(\dot{x}) := \bigcup_{n<\omega} d_n(\dot{x}).$$

We say that a set $S \subseteq \mathrm{Name}^V(\mathbb{P})$ is *dom-transitive* if for every $\dot{x} \in S$ and $\dot{y} \in \operatorname{dom}\dot{x}$, it follows that also $\dot{y} \in S$. Then $\mathrm{Tdom}(\dot{x}) \cup \{\dot{x}\} \subseteq \mathrm{Name}^V(\mathbb{P})$ is the "$\subseteq$"- smallest *dom*-transitive set T with $\dot{x} \in T$.

Now, we are ready to define the *interpretation of* $\mathbb{P}$*-names* by a generic filter, along the well-founded relation $\dot{y}\, R\, \dot{x} :\leftrightarrow \dot{y} \in \operatorname{dom}\dot{x}$.

Definition 1.1.8. Let G be a V-generic filter on $\mathbb{P}$. If $\dot{x} \in \mathrm{Name}^V(\mathbb{P})$, then

$$\dot{x}^G := \{\dot{y}^G \mid \exists\, p \in G\ (\dot{y}, p) \in \dot{x}\}.$$

The *generic extension of* V *by* G is defined as follows:

$$V[G] := \{\dot{x}^G \mid \dot{x} \in \mathrm{Name}^V(\mathbb{P})\}.$$

Then $V[G]$ consists of all those x which are definable in $V[G]$ from G and finitely many elements of V.

The elements of the ground model a have canonical names $\check{a}$, defined recursively as follows:

$$\check{a} := \{(\check{b}, \mathbb{1}) \mid b \in a\}.$$

It follows inductively that $\check{a}^G = a$ holds for all $a \in V$. For ordinals α, the "$\check{\ }$" is usually omitted and we write α instead of $\check{\alpha}$.

The *canonical name for the generic filter* is

$$\dot{G} := \{(\check{p}, p) \mid p \in \mathbb{P}\}.$$

Then for any H a V-generic filter on $\mathbb{P}$, it follows that $\dot{G}^H = H$.

Theorem 1.1.9 ([Jec06, 14.5])**.** *Let G be a V-generic filter on $\mathbb{P}$. The generic extension $V[G]$ is the smallest transitive* ZFC*-model with the property that $V \cup \{G\} \subseteq V[G]$.*

It is not difficult to see that $\mathrm{Ord}^V = \mathrm{Ord}^{V[G]}$.

If $\dot{x}$, $\dot{y} \in \mathrm{Name}^V(\mathbb{P})$ with $(\dot{y}, p) \in \dot{x}$, then p *forces* that $\dot{y} \in \dot{x}$: For any G a V-generic filter on $\mathbb{P}$ with $p \in G$, it follows that $\dot{y}^G \in \dot{x}^G$. This concept is generalized in the following definition:

Definition 1.1.10. Let $\varphi(v_0, \ldots, v_{n-1}) \in \mathcal{L}_\in$ be a formula of set theory and $\dot{x}_0, \ldots, \dot{x}_{n-1} \in \mathrm{Name}(\mathbb{P})$. We say that p *forces* $\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$,

$$p \Vdash^V_{\mathbb{P}} \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}),$$

if $V[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ holds for every V-generic filter G on $\mathbb{P}$ with $p \in G$.

When V and $\mathbb{P}$ are clear from the context, we write just "$\Vdash$". Similarly, when the ground model V is clear, we write $\mathrm{Name}(\mathbb{P})$.

Of course, Definition 1.1.10 takes place in an outer model, not in V. However, one can show that for a formula of set theory $\varphi \equiv \varphi(v_0, \dots, v_{n-1})$ fixed, the forcing relation $p \Vdash^V_{\mathbb{P}} \varphi(\dot{x}_0, \dots, \dot{x}_{n-1})$ for $p \in \mathbb{P}$ and $\dot{x}_0, \dots, \dot{x}_{n-1} \in \mathrm{Name}(\mathbb{P})$ can be expressed in the ground model V. This *definability lemma* is an integral part of the theory of forcing and crucial for proving that the generic extension $V[G]$ is a model of ZFC. It is the first part of the *Forcing Theorem*, the second part of which (the *truth lemma*) states that every formula $\varphi \in \mathcal{L}_\in$ that holds true in the generic extension $V[G]$, is forced by some condition $p \in G$.

Theorem 1.1.11 (Forcing Theorem, [Kun06, VII 3.6])**.** *Let $\varphi(v_0, \dots, v_{n-1}) \in \mathcal{L}_\in$ denote a formula of set theory and $\mathbb{P}$ a forcing in V.*

- *The class $\{(p, \dot{x}_0, \dots, \dot{x}_{n-1}) \mid p \in \mathbb{P}, \dot{x}_0, \dots, \dot{x}_{n-1} \in \mathrm{Name}(\mathbb{P}), p \Vdash^V_{\mathbb{P}} \varphi(\dot{x}_0, \dots, \dot{x}_{n-1})\}$ is definable in V* (definability lemma).
- *If G is a V-generic filter on $\mathbb{P}$ and $\dot{x}_0, \dots, \dot{x}_{n-1} \in \mathrm{Name}(\mathbb{P})$ such that $V[G] \vDash \varphi(\dot{x}_0^G, \dots, \dot{x}_{n-1}^G)$, then there exists a condition $p \in G$ with $p \Vdash^V_{\mathbb{P}} \varphi(\dot{x}_0, \dots, \dot{x}_{n-1})$* (truth lemma).

The *definability lemma* implies that for every formula of set theory $\varphi(v_0, \dots, v_{n-1})$, there is another formula $\overline{\varphi}(y, z, v_0, \dots, v_{n-1})$ such that $\overline{\varphi}(p, \mathbb{P}, \dot{x}_0, \dots, \dot{x}_{n-1})$ holds true in V if and only if $\mathbb{P} \in V$ is a forcing, $p \in \mathbb{P}$, and $p \Vdash^V_{\mathbb{P}} \varphi(\dot{x}_0, \dots, \dot{x}_{n-1})$.

We quote the following list of important properties of forcing from [Jec06, 14.7]:

Proposition 1.1.12 (Properties of Forcing)**.** *Let φ, $\psi \in \mathcal{L}_\in$ denote formulas of set theory.*

(1) *If p, $q \in \mathbb{P}$ such that $p \Vdash \varphi$ and $q \leq p$, then also $q \Vdash \varphi$.*
(2) *There is no $p \in \mathbb{P}$ with both $p \Vdash \varphi$ and $p \Vdash \neg\varphi$.*
(3) *For every $p \in \mathbb{P}$, there is $q \leq p$ such that q* decides *φ, i.e. either $q \Vdash \varphi$, or $q \Vdash \neg\varphi$.*

For every $p \in \mathbb{P}$, the following holds:

(4) *$p \Vdash \neg\varphi$ iff there is no $q \leq p$ with $q \Vdash \neg\varphi$,*
(5) *$p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$,*
(6) *$p \Vdash \forall x\ \varphi$ iff $p \Vdash \varphi(\dot{x})$ for every $\dot{x} \in \mathrm{Name}(\mathbb{P})$,*
(7) *$p \Vdash \varphi \vee \psi$ iff for all $q \leq p$, there exists $r \leq q$ with $r \Vdash \varphi$ or $r \Vdash \psi$,*
(8) *$p \Vdash \exists x\varphi$ iff for all $q \leq p$, there exist $r \leq q$ and $\dot{x} \in \mathrm{Name}(\mathbb{P})$ with $r \Vdash \varphi(\dot{x})$.*

Finally, the Maximality Principle *states:*

(9) *If $p \in \mathbb{P}$ with $p \Vdash \exists x\ \varphi$, there exists $\dot{x} \in \mathrm{Name}(\mathbb{P})$ with $p \Vdash \varphi(\dot{x})$.*

The Maximality Principle is equivalent to the Axiom of Choice.

We introduce the following notation for names for ordered pairs:

Definition 1.1.13. For $\dot{x}$, $\dot{y} \in \mathrm{Name}(\mathbb{P})$, let $\mathrm{OR}^0_{\mathbb{P}}(\dot{x}, \dot{y}) := \{(\dot{x}, \mathbb{1})\}$, and $\mathrm{OR}^1_{\mathbb{P}}(\dot{x}, \dot{y}) := \{(\dot{x}, \mathbb{1}), (\dot{y}, \mathbb{1})\}$. Then

$$\mathrm{OR}_{\mathbb{P}}(\dot{x}, \dot{y}) := \{(\mathrm{OR}^0_{\mathbb{P}}(\dot{x}, \dot{y}), \mathbb{1}), (\mathrm{OR}^1_{\mathbb{P}}(\dot{x}, \dot{y}), \mathbb{1})\}$$

is the *canonical name for the ordered pair of* $\dot{x}$ *and* $\dot{y}$.

For G a V-generic filter on $\mathbb{P}$, it follows that the interpretation $\mathrm{OR}_{\mathbb{P}}(\dot{x},\dot{y})^G$ is the ordered pair $(\dot{x}^G,\dot{y}^G)$.
If the forcing $\mathbb{P}$ is clear from the context, we write just $\mathrm{OR}(\dot{x},\dot{y})$.

We continue with definitions and facts about *isomorphisms*, *embeddings* and *projections* of forcing notions.

Definition 1.1.14. Let $(\mathbb{P},\leq_{\mathbb{P}},\mathbb{1}_{\mathbb{P}})$, $(\mathbb{Q},\leq_{\mathbb{Q}},\mathbb{1}_{\mathbb{Q}})$ be forcing notions. A map $\mathsf{b}:\mathbb{P}\to\mathbb{Q}$ is an *isomorphism of forcings* if b is bijective, $\mathsf{b}(\mathbb{1}_{\mathbb{P}})=\mathbb{1}_{\mathbb{Q}}$, and for all p_0, $p_1\in\mathbb{P}$ it follows that $p_0\leq_{\mathbb{P}}p_1$ if and only if $\mathsf{b}(p_0)\leq_{\mathbb{Q}}\mathsf{b}(p_1)$. If additionally $\mathbb{P}=\mathbb{Q}$, then b is called an *automorphism of* $\mathbb{P}$.

Even if two forcing notions $\mathbb{P}$ and $\mathbb{Q}$ are not isomorphic, they can still produce the same generic extensions if there is a *dense embedding* between $\mathbb{P}$ and $\mathbb{Q}$:

Definition 1.1.15. Let $(\mathbb{P},\leq_{\mathbb{P}},\mathbb{1}_{\mathbb{P}})$, $(\mathbb{Q},\leq_{\mathbb{Q}},\mathbb{1}_{\mathbb{Q}})$ be forcing notions.

- A map $\varrho:\mathbb{P}\to\mathbb{Q}$ is an *embedding* if for all p_0, $p_1\in\mathbb{P}$, firstly, if $p_1\leq_{\mathbb{P}}p_0$, then $\varrho(p_1)\leq_{\mathbb{Q}}\varrho(p_0)$, and secondly, if $p_0\perp_{\mathbb{P}}p_1$, then $\varrho(p_0)\perp_{\mathbb{Q}}\varrho(p_1)$.
- A map $\varrho:\mathbb{P}\to\mathbb{Q}$ is a *complete embedding*, if ϱ is an embedding, and for every maximal antichain $A\subseteq\mathbb{P}$, it follows that the pointwise image $\varrho[A]=\{\varrho(p)\mid p\in A\}$ is a maximal antichain in $\mathbb{Q}$.
  If $\mathbb{P}\subseteq\mathbb{Q}$ and the inclusion $\iota:\mathbb{P}\hookrightarrow\mathbb{Q}$ with $\iota(p)=p$ for all $p\in\mathbb{P}$ is a complete embedding, then $\mathbb{P}$ is a *complete subforcing* of $\mathbb{Q}$. We write $\mathbb{P}\subseteq_c\mathbb{Q}$.
- A map $\varrho:\mathbb{P}\to\mathbb{Q}$ is a *dense embedding*, if ϱ is an embedding and the pointwise image $\varrho[\mathbb{P}]$ is dense in $\mathbb{Q}$.

Clearly, every dense embedding is complete. Whenever $\varrho:\mathbb{P}\to\mathbb{Q}$ is a complete embedding and H a V-generic filter on $\mathbb{Q}$, then $\varrho^{-1}[H]:=\{p\in\mathbb{P}\mid\varrho(p)\in H\}$ is a V-generic filter on $\mathbb{P}$ with $V[\varrho^{-1}[H]]\subseteq V[H]$. If $\varrho:\mathbb{P}\to\mathbb{Q}$ is a dense embedding, then $\mathbb{P}$ and $\mathbb{Q}$ produce the same generic extensions.

Definition 1.1.16. Let $(\mathbb{P},\leq_{\mathbb{P}},\mathbb{1}_{\mathbb{P}})$, $(\mathbb{Q},\leq_{\mathbb{Q}},\mathbb{1}_{\mathbb{Q}})$ be forcing notions. A map $\varrho:\mathbb{P}\to\mathbb{Q}$ is a *projection*, if the following hold:

(i) $\varrho(\mathbb{1}_{\mathbb{P}})=\mathbb{1}_{\mathbb{Q}}$
(ii) Whenever p_0, $p_1\in\mathbb{P}$ with $p_1\leq_{\mathbb{P}}p_0$, then $\varrho(p_1)\leq\varrho(p_0)$.
(iii) For all $p\in\mathbb{P}$ and $q\in\mathbb{Q}$ with $q\leq_{\mathbb{Q}}\varrho(p)$, there exists $p'\in\mathbb{P}$ with $p'\leq_{\mathbb{P}}p$ and $\varrho(p')\leq_{\mathbb{Q}}q$.

Whenever $\varrho:\mathbb{P}\to\mathbb{Q}$ is a projection and G a V-generic filter on $\mathbb{P}$, then the upwards closure $H:=\{q\in\mathbb{Q}\mid\exists\,p\in G\ \pi(p)\leq q\}$ is a V-generic filter on $\mathbb{Q}$.

1.1.2 Changing the Value of 2^κ.

In this chapter, we set the necessary preliminaries to construct a generic extension $V[G]$ where $2^{\aleph_0}=\aleph_2$ holds. Generalizing the forcing notion from Example 1.1.6, this was how Paul Cohen proved in [Coh63] and [Coh64] the consistency of $\neg\mathsf{CH}$.

After that, we discuss the generalization of Cohen forcing used by Robert Solovay in [Sol63] to obtain a generic extension with $2^\kappa = \lambda$, where κ is a regular cardinal, and λ is an arbitrary cardinal with $\operatorname{cf}\lambda > \kappa$.

Example 1.1.17 (Changing the value of $2^{\aleph_0}$). Let λ be an uncountable cardinal. Then

$$\operatorname{Fn}(\lambda \times \omega, 2, \aleph_0) := \{p : \operatorname{dom} p \to 2 \mid \operatorname{dom} p \subseteq \lambda \times \omega\,,\ |p| < \aleph_0\}$$

denotes the forcing notion consisting of all partial functions form $\lambda \times \omega$ into $\{0,1\}$ with finite domain, ordered by reverse inclusion. If G is a V-generic filter on $\operatorname{Fn}(\lambda \times \omega, 2, \aleph_0)$, then for every $\alpha < \lambda$, it follows that $G_\alpha : \omega \to 2$, $G_\alpha(n) := G(\alpha, n)$ for all $n < \omega$, is a total function on ω, thus adding a new ω-subset to the ground model. An easy density argument shows that $G_\alpha \neq G_\beta$ whenever $\alpha \neq \beta$. Hence, $\operatorname{Fn}(\lambda \times \omega, 2, \aleph_0)$ adds a λ-sequence of pairwise different function from ω into 2, and it follows that $(2^{\aleph_0})^{V[G]} \geq |\lambda|^{V[G]}$.

Setting $\lambda := \aleph_2^V$, this is not yet enough to make sure that $2^{\aleph_0} = \aleph_2$ holds true in $V[G]$, since it remains to prove that indeed, $\lambda = \aleph_2^{V[G]}$. This will follow from the fact that the partial order $\operatorname{Fn}(\lambda \times \omega, 2, \aleph_0)$ *preserves cardinals*, i.e. for any V-cardinal α, it follows that α is still a cardinal in $V[G]$.

Definition 1.1.18. A forcing $\mathbb{P}$ *preserves cardinals* if for every V-generic filter G on $\mathbb{P}$ and α an ordinal, it follows that α is a cardinal in V if and only if α is a cardinal in $V[G]$. A forcing $\mathbb{P}$ *preserves cofinalities* if for every V-generic filter G on $\mathbb{P}$ and γ a limit ordinal, it follows that $\operatorname{cf}^V(\gamma) = \operatorname{cf}^{V[G]}(\gamma)$.

Every forcing that preserves cofinalities, preserves cardinals, as well.

The following combinatorial property guarantees the preservation of cardinals and cofinalities:

Definition 1.1.19. A forcing $\mathbb{P}$ has the *countable chain condition (c.c.c.)* if every antichain in $\mathbb{P}$ is at most countable.

If $V[G]$ is a V-generic extension by some c.c.c.-forcing $\mathbb{P}$, then every function $f \in V[G]$, $f : A \to B$ with A, $B \in V$, can be approximated by a map $F \in V$, $F : A \to \wp(B)$ such that for all $a \in A$, it follows that $f(a) \in F(a)$ and $F(a)$ is at most countable.

This gives rise to the following lemma:

Lemma 1.1.20 ([Kun06, VII 5.10])**.** *Any c.c.c.-forcing $\mathbb{P}$ preserves cardinals and cofinalities.*

An easy application of the Δ*-system lemma* ([Kun06, II 1.6]) shows that the forcing notion $\operatorname{Fn}(\lambda \times \omega, 2, \aleph_0)$ from Example 1.1.17 has the c.c.c. Hence, by Lemma 1.1.20, it follows that $\operatorname{Fn}(\lambda \times \omega, 2, \aleph_0)$ preserves cardinals. In particular, $\aleph_2^V = (\aleph_2)^{V[G]}$. Hence, $V[G] \vDash 2^{\aleph_0} \geq \aleph_2$, and it follows that $V[G] \vDash \neg\mathsf{CH}$. This is probably the most famous consistency result in the theory of forcing:

Theorem 1.1.21 ([Coh63] , [Coh64])**.** *If* ZFC *is consistent, then also the theory* ZFC + ¬CH *is consistent.*

It was Robert Solovay who determined in [Sol63] the exact value of $2^{\aleph_0}$ in the generic extension from Example 1.1.17:

If λ is an infinite cardinal such that $\lambda^\omega = \lambda$ in V and G is a V-generic filter on $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$*, then $V[G] \vDash 2^{\aleph_0} = \lambda$.*

The proof uses *nice names* for subsets of ω:

Definition 1.1.22. Let α be an ordinal and $\mathbb{P}$ a notion of forcing. A *nice $\mathbb{P}$-name for a subset of α* is of the form

$$\dot{x} = \{(\beta, p) \mid \beta < \alpha, p \in A_\beta\},$$

where each $A_\beta \subseteq \mathbb{P}$ is an antichain. We denote by $Nice(\mathbb{P}, \alpha)$ the set of nice $\mathbb{P}$-names for subsets of α.

One can show that for every $X \subseteq \alpha$ in a $\mathbb{P}$-generic extension $V[G]$, there exists a nice name $\dot{X} \in \mathrm{Nice}(\mathbb{P}, \alpha)$ with $X = \dot{X}^G$.

The following lemma shows how nice names for subsets of κ can be used to put an upper bound on the value of 2^κ in the generic extension:

Lemma 1.1.23 ([Kun06, VII 5.13])**.** *Let κ and λ be cardinals, and $\mathbb{P}$ a forcing such that $|\mathbb{P}| \leq \lambda$ and $\mathbb{P}$ has the c.c.c.. Let G be a V-generic filter on $\mathbb{P}$. Then $V[G] \vDash 2^\kappa \leq (\lambda^\kappa)^V$.*

Proof. Since every antichain in $\mathbb{P}$ is countable, it follows that there are $\leq \lambda^{\aleph_0}$-many antichains in $\mathbb{P}$. Hence, there are only $(\lambda^{\aleph_0})^\kappa = \lambda^\kappa$-many nice $\mathbb{P}$-names for subsets of κ. Let $(\dot{X}_i \mid i < \lambda^\kappa)$ enumerate $\mathrm{Nice}(\mathbb{P}, \alpha)$ in V. For every $X \in \wp^{V[G]}(\kappa)$, there exists $\dot{X} \in \mathrm{Nice}(\mathbb{P}, \alpha)$ with $X = \dot{X}^G$. Hence, the map $F : (\lambda^\kappa)^V \to \wp^{V[G]}(\kappa)$, $F(i) = \dot{X}_i^G$ is surjective. $\square$

More general, for any generic extension $V[G]$ by a forcing notion $\mathbb{P}$, and κ a cardinal, it follows that $(2^\kappa)^{V[G]} \leq \left(2^{|\mathbb{P}| \cdot \kappa}\right)^V$, since $\mathrm{Nice}(\mathbb{P}, \kappa) \subseteq \wp(\kappa \times \mathbb{P})$, so there are at most $2^{|\mathbb{P}| \cdot \kappa}$-many nice $\mathbb{P}$-names for subsets of κ.

Applying Lemma 1.1.23 with $\kappa = \aleph_0$ to the forcing notion $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$, which has cardinality λ, it follows that $2^{\aleph_0} \leq \lambda^{\aleph_0}$ holds true in any $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$-generic extension.

If $V \vDash \mathsf{GCH}$, then $\lambda^{\aleph_0} = \lambda$ for all cardinals λ of uncountable cofinality; which gives the following result:

Theorem 1.1.24 ([Kun06, p. 209 + 210])**.** *Let V be a ground model of* $\mathsf{ZFC} + \mathsf{GCH}$ *with a cardinal λ such that $\mathrm{cf}^V(\lambda) > \omega$. Then there is a cardinal- and cofinality-preserving generic extension $V[G]$ with $V[G] \vDash 2^{\aleph_0} = \lambda$.*

König's Theorem (see 0.4.1) implies that $\mathrm{cf}(2^{\aleph_0}) > \aleph_0$ must hold in any model of ZFC. Thus, as Solovay wrote in [Sol65]: "$2^{\aleph_0}$ *can be anything it ought to be*".

This is a striking answer to more than eighty years of discussion after Cantor had advanced the *Continuum Hypothesis* in [Can78].

The question arises whether a similar forcing notion can also be used to modify the value of 2^κ for arbitrary cardinals.

If κ is regular, one can use the partial order

$$\mathrm{Fn}(\lambda \times \kappa, 2, \kappa) := \{p : \mathrm{dom}\, p \to 2\,,\ \mathrm{dom}\, p \subseteq \lambda \times \kappa\,,\ |\mathrm{dom}\, p| < \kappa\},$$

and proceed similarly as before:

Example 1.1.25 (Changing the value of 2^κ). Let κ, λ be cardinals. The forcing notion $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa) := \{p : \mathrm{dom}\, p \to 2\,,\ \mathrm{dom}\, p \subseteq \lambda \times \kappa\,,\ |\mathrm{dom}\, p| < \kappa\}$ consists of all partial functions p from $\lambda \times \kappa$ into $\{0, 1\}$ with $|\mathrm{dom}\, p| < \kappa$, ordered by reverse inclusion. Then $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ adds a λ-sequence of pairwise different functions from κ into 2. Hence, whenever G is a V-generic filter on $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$, it follows that $(2^\kappa)^{V[G]} \geq |\lambda|^{V[G]}$.

In order to establish $V[G] \vDash 2^\kappa = \lambda$, it will be necessary that κ is regular, $2^{<\kappa} = \kappa$, and $\lambda^\kappa = \lambda$ holds true in V.

Definition 1.1.26. Let κ be an infinite cardinal in V. A forcing notion $\mathbb{P}$ *preserves cardinals* $\geq \kappa$ (or $\leq \kappa$), if for every V-generic filter G on $\mathbb{P}$ and $\alpha \geq \kappa$ (respectively, $\alpha \leq \kappa$), it follows that α is a cardinal in V if and only if α is a cardinal in $V[G]$.
A forcing notion $\mathbb{P}$ *preserves cofinalites* $\geq \kappa$ (or $\leq \kappa$) if for every limit ordinal γ with $\mathrm{cf}^V(\gamma) \geq \kappa$ (respectively, $\mathrm{cf}^V(\gamma) \leq \kappa$), it follows that $\mathrm{cf}^V(\gamma) = \mathrm{cf}^{V[G]}(\gamma)$.

The following notion generalizes the c.c.c. :

Definition 1.1.27. Let κ be an uncountable cardinal. A forcing $\mathbb{P}$ has the *κ-chain condition (κ-c.c.)* if every antichain $A \subseteq \mathbb{P}$ has cardinality $< \kappa$.

By a theorem of Tarski, the least κ such that $\mathbb{P}$ satisfies the κ-c.c. is either finite, or regular and uncountable. This allows us to concentrate on the κ-c.c. for regular uncountable cardinals. Then every forcing with the κ-c.c. preserves cardinals and cofinalities $\geq \kappa$:

Proposition 1.1.28 ([Kun06, VII 6.9])**.** *Let $\mathbb{P}$ be a forcing and κ a regular uncountable cardinal such that $\mathbb{P}$ has the κ-chain condition. Then $\mathbb{P}$ preserves cofinalities and cardinals $\geq \kappa$.*

As for the c.c.c., the point is that whenever G is a V-generic filter on a κ-c.c. forcing $\mathbb{P}$ and $f : A \to B$ a function in the generic extension $V[G]$ with A, $B \in V$, then f can be approximated by a map $F \in V$, $F : A \to \mathcal{P}(B)$ such that $f(a) \in F(a)$ and $|F(a)| < \kappa$ for all $a \in A$.

In particular, any forcing $\mathbb{P}$ preserves all cofinalities and cardinals $\geq |\mathbb{P}|^+$.

Example 1.1.20 (Changing the value of 2^κ, *continued*). An application of the *Δ-system lemma* shows that $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ has the $(2^{<\kappa})^+$-c.c. Hence, $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ preserves cardinals and cofinalities $\geq (2^{<\kappa})^+$.

In the case that κ is regular in V, the preservation of cardinals and cofinalities up to κ will be guaranteed by a different combinatorial property of the partial order, the *$< \kappa$-closure*. Hence, in the case that $2^{<\kappa} = \kappa$ (for instance, if $V \vDash \mathsf{GCH}$), it follows that the forcing notion $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ preserves all cardinals.

In the case that there are V-cardinals α with $\kappa^+ \leq \alpha \leq 2^{<\kappa}$ however, it follows that in the generic extension $V[G]$, any such α is collapsed to κ. Thus, cardinals are **not** V-$V[G]$-absolute.

Definition 1.1.29. A forcing $\mathbb{P}$ is $\leq \kappa$*-closed* if for for any $\gamma \leq \kappa$ and $(p_i \mid i < \gamma)$ a descending sequence in $\mathbb{P}$ (i.e. $p_j \leq p_i$ for all $i \leq j < \gamma$) there exists a *lower bound*: There is $p \in \mathbb{P}$ with $p \leq p_i$ for all $i < \gamma$. A forcing $\mathbb{P}$ is $< \kappa$*-closed* if it is $\leq \gamma$-closed for all $\gamma < \kappa$.

Lemma 1.1.30 ([Kun06, VII 6.14]). *Let $\mathbb{P}$ be a $< \kappa$-closed forcing, G a V-generic filter on $\mathbb{P}$ and $f : \alpha \to V$ a function in $V[G]$, where $\alpha < \kappa$. Then $f \in V$.*

This immediately implies:

Corollary 1.1.31 ([Kun06, VII 6.15]). *If κ is a cardinal and $\mathbb{P}$ is $< \kappa$-closed, then $\mathbb{P}$ preserves cofinalities and cardinals $\leq \kappa$.*

Example 1.1.20 (Changing the value of 2^κ, ***continued*****).** If κ is a regular cardinal, then $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ is $< \kappa$-closed. Hence, the forcing $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ preserves cardinals $\geq \kappa$. If additionally $2^{<\kappa} = \kappa$, it follows that $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ preserves all cofinalities and cardinals.

Now, assume that κ is regular, $2^{<\kappa} = \kappa$ and $\lambda^\kappa = \lambda$ in V. Let G be a V-generic filter on $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$. Since $2^{<\kappa} = \kappa$, it follows that $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ has the κ^+-cc, and $|\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)| \leq \lambda^{<\kappa} = \lambda$. Hence, there are at most $\lambda^\kappa = \lambda$-many antichains, and $|\mathrm{Nice}\big(\mathrm{Fn}(\lambda \times \kappa, 2, \kappa), \kappa\big)| \leq \lambda^\kappa = \lambda$. As in the proof of Lemma 1.1.23, this implies $2^\kappa \leq \lambda$.

The following theorem is by Robert Solovay (see [Sol63]):

Theorem 1.1.32 ([Kun06, VII 6.17]). *Assume that κ is regular, $2^{<\kappa} = \kappa$ and $\lambda^\kappa = \kappa$ in V, and let G be a V-generic filter on $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$. Then $V[G] \vDash 2^\kappa = \lambda$.*

It follows that the forcing notion $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ can be used to violate GCH at any regular cardinal κ: We start with a model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with cardinals κ, λ such that κ is regular and $\mathrm{cf}(\lambda) > \kappa$. Then $2^{<\kappa} = \kappa$ and $\lambda^\kappa = \lambda$ holds; hence, for any V-generic extension $V[G]$ by $\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$, it follows that $V[G] \vDash 2^\kappa = \lambda$. For $\eta < \kappa$, it follows by $< \kappa$-closure of the forcing that $(2^\eta)^{V[G]} = (2^\eta)^V = (\eta^+)^V = (\eta^+)^{V[G}$.

By *König's Theorem* (see 0.4.1), it follows that always $\mathrm{cf}(2^\kappa) > \kappa$ must hold in any model of ZFC; hence, the requirement that $\mathrm{cf}(\lambda) > \kappa$ is not a restriction. Thus, for regular κ, the cardinality of the power set $\wp(\kappa)$ can take any possible value.

For singular cardinals κ, however, the forcing notion $\mathrm{Fn}(\kappa \times \lambda, 2, \kappa)$ is not $< \kappa$-closed and collapses the cardinal κ. Hence, $\mathrm{Fn}(\kappa \times \lambda, 2, \kappa)$ is not suitable for changing the value of 2^κ for singular κ. Indeed, investigating possible behaviors of the Continuum Function 2^κ for singular κ is a lot more involved, and there are restrictions beyond König's Theorem. For instance, *Silver's Theorem* implies that whenever κ is a singular cardinal of uncountable cofinality such that GCH holds below κ, then also $2^\kappa = \kappa^+$ follows. We elaborate on this in Chapter 0.1.

1.1.3 Products

The idea of taking products of forcing notions arose from the following question: Is it also possible to violate GCH at more than one regular cardinals at the same time? For example, can we use forcing to construct a model in which $2^{\aleph_1} = \aleph_3$ and $2^{\aleph_2} = \aleph_5$ holds? The answer is *yes*: Starting from a ground model $V \models \mathsf{ZFC} + \mathsf{GCH}$, one can first force with $\mathrm{Fn}(\aleph_5 \times \aleph_2, 2, \aleph_2)$ and obtain a model $V[G]$ where $2^{\aleph_2} = \aleph_5$, and $2^{\aleph_1} = \aleph_2$, $2^{\aleph_0} = \aleph_1$ holds. After that, one can force over $V[G]$ with $\mathrm{Fn}(\aleph_3 \times \aleph_1, 2, \aleph_1)$, and obtain a generic extension $V[G][H]$, where $2^{\aleph_2} = \aleph_5$, $2^{\aleph_1} = \aleph_3$, and $2^{\aleph_0} = \aleph_1$.
It is important that we proceed “backwards”, since we need for the second step that GCH holds below the relevant cardinal. Thus, this method only enables us to violate GCH at *finitely* many regular cardinals at the same time.

If we want to modify the powers of infinitely many regular cardinals simultaneously, we need *product forcing*:

Definition 1.1.33. Let $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$ be forcings. The *product forcing*

$$(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}) \times (\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}}) = (\mathbb{P} \times \mathbb{Q}, \leq_{\mathbb{P}\times\mathbb{Q}}, \mathbb{1}_{\mathbb{P}\times\mathbb{Q}})$$

is defined by setting $(p_1, q_1) \leq_{\mathbb{P}\times\mathbb{Q}} (p_0, q_0)$ if $p_1 \leq_{\mathbb{P}} p_0$ and $q_1 \leq_{\mathbb{Q}} q_0$, and $\mathbb{1}_{\mathbb{P}\times\mathbb{Q}} := (\mathbb{1}_{\mathbb{P}}, \mathbb{1}_{\mathbb{Q}})$.

Let G be a V-generic filter on $\mathbb{P} \times \mathbb{Q}$. It induces $G_0 := \{p \in \mathbb{P} \mid \exists\, q \in \mathbb{Q}\ (p,q) \in G\}$ and $G_1 := \{q \in \mathbb{Q} \mid \exists\, p \in \mathbb{P}\ (p,q) \in G\}$. It is not difficult to see that G_0 and G_1 are V-generic filters on $\mathbb{P}$ and $\mathbb{Q}$ respectively, and $G = G_0 \times G_1$. Hence, any V-generic filter G on a product $\mathbb{P} \times \mathbb{Q}$ has the form $G = G_0 \times G_1$, where G_0 is V-generic on $\mathbb{P}$, and G_1 is V-generic on $\mathbb{Q}$. The converse is not true: For genericity of the product $G_0 \times G_1$, it is additionally necessary that G_1 is generic over $V[G_0]$.

More precisely:

Lemma 1.1.34 (**Product Lemma**, [Kun06, VIII 1.4])**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be forcings, and $G_0 \subseteq \mathbb{P}$, $G_1 \subseteq \mathbb{Q}$. Then the following are equivalent:*

(i) *$G_0 \times G_1$ is a V-generic filter on $\mathbb{P} \times \mathbb{Q}$,*

(ii) *G_0 is a V-generic filter on $\mathbb{P}$ and G_1 is a $V[G_0]$-generic filter on $\mathbb{Q}$,*

(iii) *G_1 is a V-generic filter on $\mathbb{Q}$ and G_0 is a $V[G_1]$-generic filter on $\mathbb{P}$.*

If (i) - (iii) *hold, then $V[G_0 \times G_1] = V[G_0][G_1] = V[G_1][G_0]$.*

In many applications, one encounters the product of two forcings $\mathbb{P}$ and $\mathbb{Q}$ the first of which is not too large, and the second is sufficiently closed. Then the following lemma applies:

Lemma 1.1.35 ([Jec06, 15.19])**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions and κ a cardinal such that $\mathbb{P}$ satisfies the κ^+-chain condition and $\mathbb{Q}$ is $\leq \kappa$-closed. Let G be a V-generic filter on $\mathbb{P}$, H a V-generic filter on $\mathbb{Q}$, and $f : \kappa \to V$ a function in $V[G][H]$. Then $f \in V[G]$.*

We define products of infinitely many forcing notions:

Definition 1.1.36. Let $\mathcal{I}$ be an index set and $(\mathbb{Q}_i \mid i \in \mathcal{I})$ a collection of forcings where every $\mathbb{Q}_i$ is partially ordered by $\leq_i$ and has greatest element $\mathbb{1}_i$. The *product* (or *product with full support*)

$$\mathbb{P} = \prod_{i \in \mathcal{I}} \mathbb{Q}_i$$

consists of all $p : \mathcal{I} \to V$ with the property that $p(i) \in \mathbb{Q}_i$ for all $i \in \mathcal{I}$, with maximal element $\mathbb{1} := (\mathbb{1}_i \mid i \in \mathcal{I})$ and the partial order $\leq_{\mathbb{P}}$ defined by $q \leq p :\Leftrightarrow q(i) \leq_i p(i)$ for all $i \in \mathcal{I}$. For $p \in \mathbb{P}$, the *support* of p is $\operatorname{supp} p := \{i \in \mathcal{I} \mid p(i) \neq \mathbb{1}_i\}$.

If G is a V-generic filter on $\mathbb{P}$, then for every $i \in \mathcal{I}$, it follows that $G_i := \{p(i) \mid p \in G\}$, the *projection of G onto $\mathbb{Q}_i$*, is a V-generic filter on $\mathbb{Q}_i$.

For a collection of forcings $(\mathbb{Q}_i \mid i \in \mathcal{I})$ and κ a regular cardinal, the *κ-product* (or *product with $< \kappa$-support*) is

$$\prod_{i \in \mathcal{I}}^{<\kappa} \mathbb{Q}_i := \Big\{ p \in \prod_{i \in \mathcal{I}} \mathbb{Q}_i \;\Big|\; |\operatorname{supp} p| < \kappa \Big\},$$

with the ordering and maximal element as before. The $\aleph_1$-product is usually referred to as *product with countable support.*
Products with *finite support* appear frequently: For a collection of forcings $(\mathbb{Q}_i \mid i \in \mathcal{I})$, the *finite-support product* is

$$\prod_{i \in \mathcal{I}}^{\text{fin}} \mathbb{Q}_i := \Big\{ p \in \prod_{i \in \mathcal{I}} \mathbb{Q}_i \;\Big|\; |\operatorname{supp} p| < \aleph_0 \Big\},$$

with the ordering and maximal element as before.

Sometimes, the "product of $(\mathbb{Q}_i \mid i \in \mathcal{I})$" is defined like our *product with finite support.* In order to avoid misunderstandings, we always clarify what support we are using.

We conclude this chapter by introducing the forcing notion that William Easton used in [Eas70] to show that the Continuum Function on the class of all regular cardinals can take almost arbitrary values, as long as it meets the rules of monotonicity and König's Theorem.

Definition 1.1.37. For α an ordinal, $\mathcal{I} \subseteq \alpha$, and a collection of forcings $(\mathbb{Q}_i \mid i \in \mathcal{I})$, the *product with Easton support* is

$$\prod_{i \in \mathcal{I}}^{\text{Easton}} \mathbb{Q}_i \;:=\; \Big\{ p \in \prod_{i \in \mathcal{I}} \mathbb{Q}_i \;\Big|\; \forall \gamma \; (\gamma \text{ is inaccessible } \to |\operatorname{supp} p \cap \gamma| < \gamma) \Big\}.$$

In the case that $\mathcal{I}$ is a set of cardinals and GCH holds, Easton support is equivalent to requiring that $|\operatorname{dom} p \cap \gamma| < \gamma$ for all regular cardinals γ.

If we wish to change the value of 2^κ for "many" regular κ at the same time, we can use the product forcing

$$\mathbb{P}_F := \prod_{\kappa \in \operatorname{dom} F}^{\text{Easton}} \operatorname{Fn}(F(\kappa) \times \kappa, 2, \kappa),$$

where $F : \operatorname{dom} F \to \text{Card}$ is an *Easton function*, i.e.

(i) any $\kappa \in \operatorname{dom} F$ is a regular cardinal,

(ii) $\operatorname{cf} F(\kappa) > \kappa$ for all $\kappa \in \operatorname{dom} F$,

(iii) if κ, $\lambda \in \operatorname{dom} F$ with $\kappa < \lambda$, then $F(\kappa) \leq F(\lambda)$.

Starting from a model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$, one can show that in any V-generic extension by $\mathbb{P}_F$, it follows that $2^\kappa = F(\kappa)$ for all $\kappa \in \operatorname{dom} F$. In other words: Any "reasonable" behavior of the Continuum Function (i.e. meeting the rules of weak montonicity and König's Theorem) is consistent with ZFC.
In general, the domain of the Easton function F is a proper class, so we need *class forcing* to construct the appropriate generic extension. Therefore, further discussion of Easton forcing is deferred to Chapter 1.3.2.

1.2 Symmetric Forcing

In this chapter, we present the technique of constructing symmetric extensions by forcing with partial orders and symmetric names.

The idea of starting with a group of permutations A with a normal filter $\mathcal{F}$ on A, then considering *symmetric* objects (which are fixed by $\mathcal{F}$-many permutations), and taking all objects which are *hereditarily symmetric* (i.e. they are symmetric, and so are all elements in their transitive closure), already appeared in the construction of permutation models by Fraenkel ([Fra22]) and (in a precise version) Mostowski ([Mos39]), who proved the independence of the Axiom of Choice from ZFA (set theory with atoms).
The underlying idea – the axioms of ZFA do not distinguish between the atoms, which allows for constructing models in which the set of all atoms has no well-ordering – was adapted by Paul Cohen in [Coh63] and [Coh64], where he constructed a symmetric forcing extension where the reals cannot be well-ordered. As a consequence, it follows that $\neg\mathsf{AC}$ is consistent with ZF.

A general technique for constructing symmetric extension as submodels of Boolean-valued models was developed by Scott (unpublished) and reformulated by Jech ([Jec71]).
In practise, however, it is often more comfortable to work with automorphisms of partial orders. In [Dim11, 1.2], the method from [Jec71] is translated to forcing with partial orders; which allows for avoiding Boolean algebras, except in the case that one has to deal with automorphisms $\pi : D_\pi \to D_\pi$ that are not defined on the entire forcing $\mathbb{P}$, but only on a dense subset $D_\pi \subseteq \mathbb{P}$.

Our aim is to incorporate this situation (which appears frequently in practice) into the technique of constructing symmetric extensions by using automorphisms of partial orders. This will happen in Chapter 1.2.3. The overall idea is to call two isomorphisms $\pi : D_\pi \to D_\pi$ and $\sigma : D_\sigma \to D_\sigma$ *equivalent* (write $\pi \sim \sigma$), if they agree on the intersection $D_\pi \cap D_\sigma$. We then work with the equivalence classes $[\pi]$.

1.2.1 Constructing Symmetric Extensions

For this chapter, we fix a partial order $(\mathbb{P}, \leq, \mathbb{1})$.

Definition 1.2.1. An *automorphism of* $\mathbb{P}$ is a bijection $\pi : \mathbb{P} \to \mathbb{P}$ such that $\pi(\mathbb{1}) = \mathbb{1}$, and for any p, $q \in \mathbb{P}$, it follows that $p \leq q$ if and only if $\pi p \leq \pi q$. Let $Aut(\mathbb{P})$ denote the group of all $\mathbb{P}$-automorphisms, with the identity element $\mathrm{id}_{\mathbb{P}} : \mathbb{P} \to \mathbb{P}$, $p \mapsto p$ for all $p \in \mathbb{P}$.

Any $\pi \in \mathrm{Aut}(\mathbb{P})$ can be extended to an automorphism $\widetilde{\pi}$ of the name space $\mathrm{Name}(\mathbb{P})$ by the following recursive definition along the $\mathrm{Name}_\alpha(\mathbb{P})$-hierarchy:

$$\widetilde{\pi}(\dot{x}) := \{ (\widetilde{\pi}(\dot{y}), \pi p) \mid (\dot{y}, p) \in \dot{x} \}.$$

We confuse any $\pi \in \mathrm{Aut}(\mathbb{P})$ with its extension $\widetilde{\pi}$ (which does not lead to ambiguity). Inductively, it follows that $\mathsf{rk}_{\mathbb{P}}\, \pi\dot{x} = \mathsf{rk}_{\mathbb{P}}\, \dot{x}$ for any $\pi \in \mathrm{Aut}(\mathbb{P})$ and $\dot{x} \in \mathrm{Name}(\mathbb{P})$. For any canonical name $\check{a}$ for an element a of the ground model, it follows recursively that $\pi\check{a} = \check{a}$. Moreover, whenever $\dot{s}$, $\dot{t} \in \mathrm{Name}(\mathbb{P})$ and $\mathrm{OR}_{\mathbb{P}}(\dot{s}, \dot{t})$ denotes the canonical name for their ordered pair, it follows that

$$\pi\big(\mathrm{OR}_{\mathbb{P}}(\dot{s}, \dot{t})\big) = \mathrm{OR}_{\mathbb{P}}(\pi\dot{s}, \pi\dot{t}).$$

Lemma 1.2.2 (**Symmetry Lemma**, [Dim11, 1.14])**.** *Let π be a $\mathbb{P}$-automorphism, $\varphi(v_0, \ldots, v_{n-1})$ a formula of set theory and $\dot{x}_0, \ldots, \dot{x}_{n-1} \in \mathrm{Name}(\mathbb{P})$. Then $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ if and only if $\pi p \Vdash \varphi(\pi\dot{x}_0, \ldots, \pi\dot{x}_{n-1})$.*

The proof is by induction over the complexity of φ, using the properties of the forcing relation (cf. Propositon 1.1.12).

Definition 1.2.3. Let A be a group.

(i) A *filter* $\mathcal{F}$ *on* A is a collection of subgroups $B \subseteq A$ such that $\varnothing \neq \mathcal{F}$ and $\mathcal{F}$ is closed under supersets and finite intersections.

(ii) A filter $\mathcal{F}$ on A is *normal* if for every $B \in \mathcal{F}$ and $\pi \in A$ it follows that the conjugate $\pi^{-1}B\pi$ is contained in $\mathcal{F}$, as well.

(iii) A filter $\mathcal{F}$ on A is *countably complete* if for every $\{A_i \mid i < \omega\}$ a family of A-subgroups with the property that $A_i \in \mathcal{F}$ for all $i < \omega$, it follows that also in the intersection $\bigcap_{i<\omega} A_i$ is an element of $\mathcal{F}$.

(iv) For κ a regular cardinal, a filter $\mathcal{F}$ on A is *κ-complete* if for every $\{A_i \mid i < \kappa\}$ a family of A-subgroups such that $A_i \in \mathcal{F}$ for all $i < \kappa$, it follows that also the intersection $\bigcap_{i<\kappa} A_i$ is an element of $\mathcal{F}$.

For constructing symmetric forcing extensions, we will need a group A of $\mathbb{P}$-automorphisms and a normal filter $\mathcal{F}$ on A.

The following example demonstrates the main ideas:

Example 1.2.4 (**Cohen Forcing**)**.** As an example, we consider the forcing notion $\mathrm{Fn}(\omega \times \omega, 2, \aleph_0)$ introduced in Example 1.1.17. Paul Cohen used it to construct a symmetric extension $V(G) \supseteq V$ where the reals have no well-ordering. Hence, $V(G) \vDash \mathsf{ZF} + \neg\mathsf{AC}$.
Let G be a V-generic filter on $\mathrm{Fn}(\omega \times \omega, 2, \aleph_0)$. As before , we can extract for every $i < \omega$ the following real number (a subset of ω):

$$G_i := \{n < \omega \mid \exists p \in G\ p(i, n) = 1\}.$$

We wish to construct the symmetric extension $V(G)$ in such a way that the set $X := \{g_i \mid i \in \omega\}$ is an element of $V(G)$, but X has no well-ordering in $V(G)$.

We consider the following group A of $\mathrm{Fn}(\omega \times \omega, 2, \aleph_0)$-automorphisms: Let $\mathsf{b} : \omega \to \omega$ denote a bijection on ω with *finite support*, i.e. $\operatorname{supp} \mathsf{b} := \{i < \omega \mid \mathsf{b}(i) \neq i\}$ is finite. Then b induces an automorphism $\pi = \pi_{\mathsf{b}}$ of $\mathrm{Fn}(\omega \times \omega, 2, \aleph_0)$ as follows: For a condition $p \in \mathrm{Fn}(\omega \times \omega, 2, \aleph_0)$, let $\operatorname{dom}(\pi p) := \{(\pi(i), n) \mid (i,n) \in \operatorname{dom} p\}$, and $(\pi p)(\pi(i), n) := p(i,n)$ for all $(i,n) \in \operatorname{dom} p$. Then the generic ω-subsets are permuted according to F.
Let A denote the group of all automorphism $\pi_{\mathsf{b}} : \mathrm{Fn}(\omega \times \omega, 2, \aleph_0) \to \mathrm{Fn}(\omega \times \omega, 2, \aleph_0)$ which are induced by a bijection $\mathsf{b} : \omega \to \omega$ with finite support as described above.
For every $i < \omega$, let $\mathrm{Fix}(i) := \{\pi_{\mathsf{b}} \in A \mid \mathsf{b}(i) = i\} = \{\pi \in A \mid \forall\, p \in \mathrm{Fn}(\omega \times \omega, 2, \aleph_0)\ \forall\, n < \omega\ ((i,n) \in \operatorname{dom} p \to (\pi p)(i,n) = p(i,n))\}$. Then $\mathrm{Fix}(i)$ is a subgroup of A. Let $\mathcal{F}$ denote the filter generated by finite intersections of these $\mathrm{Fix}(i)$:

$$\mathcal{F} := \{B \subseteq A \text{ subgroup } \mid \exists\, k < \omega\ \exists\, i_0, \ldots, i_{k-1} < \omega \quad B \supseteq \mathrm{Fix}(i_0) \cap \cdots \cap \mathrm{Fix}(i_{k-1})\}.$$

It is not difficult to see that $\mathcal{F}$ is normal.

For the rest of this chapter, we fix an automorphism group $A \subseteq \mathrm{Aut}(\mathbb{P})$ and a normal filter $\mathcal{F}$ on A.

Intuitively, a $\mathbb{P}$-name $\dot{x}$ should be *symmetric* if it is fixed by "many" automorphisms of $\mathbb{P}$, and the *symmetric extension* $V(G)$ should consist of all evaluated $\mathbb{P}$-names which are symmetric and have only symmetric names in their transitive closure.

Definition 1.2.5. A $\mathbb{P}$-name $\dot{x}$ is *symmetric for* $\mathcal{F}$ if the stabilizer group

$$\mathrm{sym}^A(\dot{x}) := \{\pi \in A \mid \pi\dot{x} = \dot{x}\}$$

is an element of $\mathcal{F}$. Recursively, a name $\dot{x}$ is *hereditarily symmetric*, $\dot{x} \in HS^{\mathcal{F}}$, if $\dot{x}$ is symmetric, and $\dot{y}$ is hereditarily symmetric for all $\dot{y} \in \operatorname{dom} \dot{x}$.

When A and $\mathcal{F}$ are clear from the context, we write just $\mathrm{sym}(\dot{x})$ and HS.

Recursively, it follows that a canonical name $\check{a}$ for an element a of the ground model is always hereditarily symmetric, since $\pi\check{a} = \check{a}$ holds for all $\pi \in \mathrm{Aut}(\mathbb{P})$.

For every $\dot{x} \in \mathrm{Name}(\mathbb{P})$ and $\pi \in A$, it follows that

$$\mathrm{sym}^A(\pi\dot{x}) = \pi \cdot \mathrm{sym}^A(\dot{x}) \cdot \pi^{-1},$$

since for any $\sigma \in A$, we have $\sigma(\pi\dot{x}) = \pi\dot{x}$ if and only if $(\pi^{-1}\sigma\pi)(\dot{x}) = \dot{x}$. Hence, by normality of $\mathcal{F}$, it follows that whenever a name $\dot{x}$ is symmetric and $\pi \in A$, then $\pi\dot{x}$ is symmetric, too. Thus, whenever $\dot{x} \in HS$ and $\pi \in A$, then also $\pi\dot{x} \in HS$.

Definition 1.2.6. For a V-generic filter G on $\mathbb{P}$, the *symmetric extension by* $\mathcal{F}$ *and* G is defined as follows:

$$V(G)^{\mathcal{F}} := \{\dot{x}^G \mid \dot{x} \in HS\}.$$

Then $V(G)^{\mathcal{F}}$ is a transitive class with $V \subseteq V(G)^{\mathcal{F}} \subseteq V[G]$.

In most cases, we write just $V(G)$, or use the letter N for a symmetric extension.

The *symmetric forcing relation* $\Vdash_s$ can be defined informally as follows:

Definition 1.2.7. If $\varphi(v_0, \ldots, v_{n-1}) \in \mathcal{L}_\in$ is a formula of set theory and $\dot{x}_0, \ldots, \dot{x}_{n-1} \in HS$, we write

$$p \ (\Vdash_s)^V_{\mathbb{P},\mathcal{F}} \ \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$$

if $V(G)^{\mathcal{F}} \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ for every V-generic filter G on $\mathbb{P}$.

When V, $\mathbb{P}$ and $\mathcal{F}$ are clear from the context, we write just "$\Vdash_s$".

Note that the symmetric forcing relation $\Vdash_s$ can be defined in the ground model similar as the ordinary forcing relation $\Vdash$, but with the variables and quantifiers ranging over HS. It has most basic properties of $\Vdash$. In particular, the analogue of Proposition 1.1.12 holds:

Proposition 1.2.8 (**Properties of Symmetric Forcing**, [Dim11, 1.20])**.** *Let φ, ψ be formulas of set theory.*

(1) *If p, $q \in \mathbb{P}$ such that $p \Vdash_s \varphi$ and $q \leq p$, then also $q \Vdash_s \varphi$.*
(2) *There is no $p \in \mathbb{P}$ with both $p \Vdash_s \varphi$ and $p \Vdash_s \neg\varphi$.*
(3) *For every $p \in \mathbb{P}$, there is $q \leq p$ such that q* decides *φ, i.e. either $q \Vdash_s \varphi$ or $q \Vdash_s \neg\varphi$.*

For every $p \in \mathbb{P}$, the following holds:

(4) *$p \Vdash_s \neg\varphi$ if and only if no there is no $q \leq p$ with $q \Vdash_s \neg\varphi$,*
(5) *$p \Vdash_s \varphi \wedge \psi$ if and only if $p \Vdash_s \varphi$ and $p \Vdash_s \psi$,*
(6) *$p \Vdash_s \forall x\, \varphi$ if and only if $p \Vdash_s \varphi(\dot{x})$ for every $\dot{x} \in HS$*
(7) *$p \Vdash_s \varphi \vee \psi$ if and only if for all $q \leq p$, there exists $r \leq q$ with $r \Vdash_s \varphi$ or $r \Vdash_s \psi$,*
(8) *$p \Vdash_s \exists x\, \varphi$ if and only if for all $q \leq p$, there exists $r \leq q$ and a name $\dot{x} \in HS$ with $r \Vdash_s \varphi(\dot{x})$.*

Whenever $\dot{x}$, $\dot{y} \in HS$ and $p \in \mathbb{P}$, then $p \Vdash_s \dot{y} \in \dot{x}$ if and only if $p \Vdash \dot{y} \in \dot{x}$ with the ordinary forcing relation "$\Vdash$", and $p \Vdash_s \dot{x} \subseteq \dot{y}$ if and only if $p \Vdash \dot{x} \subseteq \dot{y}$.

Moreover, the *Symmetry Lemma* holds for $\Vdash_s$ (with the same proof by induction on the complexity of formulae as for the ordinary forcing relation "$\Vdash$", using Proposition 1.2.8); and the *Forcing Theorem* holds true, as well: The proof for the atomic cases $p \Vdash_s \dot{x} \in \dot{y}$ and $p \Vdash_s \dot{x} \subseteq \dot{y}$ is the same as for the ordinary forcing relation "$\Vdash$", and also the induction on the complexity of formulae can be carried out as for "$\Vdash$", with the modification that names are ranging over HS in the existential quantifier case).

Theorem 1.2.9 ([Dim11, 1.21])**.** *Let A be a group of $\mathbb{P}$-automorphisms with a normal filter $\mathcal{F}$ on A, and let G be a V-generic filter on $\mathbb{P}$. Then $V(G)^{\mathcal{F}}$ is a transitive model of* ZF *with $V \subseteq V(G)^{\mathcal{F}} \subseteq V[G]$.*

A detailed proof of the axioms can be found in [Dim11]. We will prove the analogue of Theorem 1.2.9 for our more general construction in the case that we do not have $\mathbb{P}$-automorphisms, but automorphisms $\pi : D_\pi \to D_\pi$ on dense subsets $D_\pi \subseteq \mathbb{P}$ (cf. Theorem

1.2.21).

Example 1.2.4 (**Cohen Forcing**, *continued*). In our example above, a name $\dot{x}$ for $\mathrm{Fn}(\omega \times \omega, 2, \aleph_0)$ is *symmetric* if there are finitely many $i_0, \dots, i_{k-1} < \omega$ with $\pi\dot{x} = \dot{x}$ for all π which are contained in the intersection $\mathrm{Fix}(i_0) \cap \dots \cap \mathrm{Fix}(i_{k-1})$. Let G be a V-generic filter on $\mathrm{Fn}(\omega \times \omega, 2, \aleph_0)$, and let $N := V(G)^{\mathcal{F}}$ denote the according symmetric extension. For $i < \omega$, the i-th generic ω-subset $G_i = \{n < \omega \mid \exists\, p \in G\ p(i,n) = 1\}$ has a canonical name

$$\dot{G}_i := \{(n,p) \mid p \in \mathbb{P}\,,\, p(i,n) = 1\},$$

with the property that $\pi\dot{G}_i = \dot{G}_i$ holds true for all $\pi \in \mathrm{Fix}(i)$. Indeed, for any $i < \omega$ and $\pi = \pi_{\mathsf{b}} \in A$ induced by a bijection $\mathsf{b} : \omega \to \omega$, it follows that $\pi_{\mathsf{b}}\dot{G}_i = \dot{G}_{\mathsf{b}(i)}$.
Hence, $\dot{G}_i \in HS$ and $G_i \in N$ for all $i < \omega$. Note that $G_i \neq G_j$ whenever $i \neq j$.

The set $X := \{G_i \mid i < \omega\}$ has the canonical name

$$\dot{X} := \{\, \mathrm{OR}(i, \dot{G}_i) \mid i < \omega\,\},$$

where $\mathrm{OR}(i, \dot{G}_i)$ denotes the canonical name for the ordered pair. Then $\dot{X}$ is stabilized by all $\pi \in A$, since for any $\pi = \pi_{\mathsf{b}} \in A$ induced by a bijection $\mathsf{b} : \omega \to \omega$, it follows that

$$\pi\dot{X} = \{\, \mathrm{OR}(i, \pi\dot{G}_i) \mid i < \omega\,\} = \{\, \mathrm{OR}(i, \dot{G}_{\mathsf{b}(i)}) \mid i < \omega\,\} = \dot{X}.$$

Hence, $X \in N$. We claim that the set X has no well-ordering in N.

Assume towards a contradiction that there was a injective function $f : \omega \to X$ in N. Let $\dot{f} \in HS$ with $\dot{f}^G = f$ such that $\pi\dot{f} = \dot{f}$ holds for all $\pi \in \mathrm{Fix}(i_0) \cap \dots \cap \mathrm{Fix}(i_{k-1})$, where $k < \omega$ and $i_0, \dots, i_{k-1} < \omega$. Take $G_i \in \mathrm{rg}\, f$ with $i \notin \{i_0, \dots, i_{k-1}\}$, and let $m < \omega$ with $f(m) = G_i$. Take a condition $p \in \mathbb{P}$ such that

$$p \Vdash_s \dot{f} : \omega \to \dot{X} \text{ is an injective function} \qquad (*)$$

and

$$p \Vdash_s \mathrm{OR}(m, \dot{G}_i) \in \dot{f}.$$

The idea is to consider an isomorphism π induced by a permutation $i \leftrightarrow j$ such that $\pi p \parallel p$ and $\pi\dot{f} = \dot{f}$, which will contradict the fact that p forces the functionality of $\dot{f}$.

Take $j < \omega$ such that $j \neq i$, $j \notin \{i_0, \dots, i_{k-1}\}$, and $(j,n) \notin \mathrm{dom}\, p$ for all $n < \omega$. Let $\pi = \pi_{\mathsf{b}} \in A$ be the map induced by the permutation $\mathsf{b} : \omega \to \omega$ with $\mathsf{b}(i) = j$, $\mathsf{b}(j) = i$, and $\mathsf{b}(j') = j'$ for all $j' \in \omega \setminus \{i, j\}$. Then

$$\overline{p} := p \cup \Big\{\, ((j,n), p(i,n)) \mid (i,n) \in \mathrm{dom}\, p \Big\}$$

is a common extension of p and πp; and from $\pi \in \mathrm{Fix}(i_0) \cap \dots \cap \mathrm{Fix}(i_{k-1})$ it follows that $\pi\dot{f} = \dot{f}$. Hence, $\pi p \Vdash_s \mathrm{OR}(m, \pi\dot{G}_i) \in \pi\dot{f}$ implies that $\pi p \Vdash_s \mathrm{OR}(m, \dot{G}_j) \in \dot{f}$. Altogether, $\overline{p} \Vdash_s \mathrm{OR}(m, \dot{G}_i) \in \dot{f}$ and $\overline{p} \Vdash_s \mathrm{OR}(m, \dot{G}_j) \in \dot{b}$, which contradicts $\overline{p} \leq p$ and $(*)$.

It follows that the set X cannot be well-ordered in N. Hence, the Axiom of Choice fails in the symmetric extension $V(G)$.

This construction was used by Paul Cohen to prove the consistency of $\mathsf{ZF} + \neg\mathsf{AC}$.

1.2.2 A Model for $\theta(\aleph_0) = \lambda^+$

In this chapter, we throw a first glimpse at the question about possible values of the Continuum Function in the absence of AC.
Since the power sets $\wp(\kappa)$ can not necessarily be well-ordered in ZF + ¬AC, the first question is how "largeness" of the power sets could now be expressed.

The **θ-function** is defined on the class of all cardinals by setting

$$\theta(\kappa) := \sup\{\alpha \in \mathrm{Ord} \mid \exists\, f : \wp(\kappa) \to \alpha \text{ surjective function}\}.$$

It generalizes the value $\Theta = \theta(\aleph_0)$ prominent in descriptive set theory and provides a surjective substitute for the Continuum Function in ZF.

Note that if the Axiom of Choice holds and $2^\kappa = \lambda$, then $\theta(\kappa) = \lambda^+$.

Let $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ be a ground model with an uncountable cardinal λ. The aim of this chapter is to use Cohen Forcing $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$ to construct a symmetric extension $V(G) \supseteq V$ with $\theta(\aleph_0) = \lambda^+$, i.e. there exists a surjection $f : \wp(\omega) \to \lambda$, but there is no surjective function $f : \wp(\omega) \to \lambda^+$.

Let A be the group consisting of all $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$-automorphisms π of the following form: There is a finite set $\mathrm{dom}\,\pi \subseteq \lambda \times \omega$ (the *domain of* π), and for every $(i,n) \in \mathrm{dom}\,\pi$, there is a map $\pi(i,n) : 2 \to 2$, such that for every condition $p \in \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$, the image πp is defined as follows:

- $\mathrm{dom}\,\pi p = \mathrm{dom}\,p$, with
- $(\pi p)(i,n) = \pi(i,n)(p(i,n))$ in the case that $(i,n) \in \mathrm{dom}\,p \cap \mathrm{dom}\,\pi$,
- $(\pi p)(i,n) = p(i,n)$ for all $(i,n) \in \mathrm{dom}\,p \setminus \mathrm{dom}\,\pi$.

In other words: For every (i,n) in the domain of π, the value of $p(i,n)$ is switched or not according to whether $\pi(i,n) : 2 \to 2$ is the identity or not.

Our normal filter $\mathcal{F}$ on A is defined as in Example 1.2.4: For $i < \lambda$, let $\mathrm{Fix}(i) := \{\pi \in A \mid \forall\, n < \omega\ ((i,n) \in \mathrm{dom}\,\pi \to \pi(i,n) = \mathrm{id})\}$. Then

$$\mathrm{Fix}(i) = \left\{ \pi \in A \;\middle|\; \forall\, p \in \mathrm{Fn}(\lambda \times \omega, 2, \omega)\ \forall n < \omega\ \big((i,n) \in \mathrm{dom}\,p \to (\pi p)(i,n) = p(i,n)\big) \right\}.$$

Let $\mathcal{F}$ be the normal filter generated by finite intersections of these $\mathrm{Fix}(i)$:

$$\mathcal{F} := \left\{ B \subseteq A \text{ subgroup } \mid \exists\, k < \omega\ \exists\, i_0, \dots, i_{k-1} < \omega \quad B \supseteq \mathrm{Fix}(i_0) \cap \dots \cap \mathrm{Fix}(i_{k-1}) \right\}.$$

We take a V-generic filter G on $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$, and denote by $N := V(G)^{\mathcal{F}}$ the symmetric extension by $\mathcal{F}$ and G. Since the forcing notion $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$ preserves cardinals, it follows that cardinals are also absolute between V and N.

For $i < \lambda$, let

$$G_i := \{(n, \epsilon) \mid n < \omega\,,\ \epsilon \in \{0,1\}\,,\ \exists\, p \in G\ p(i,n) = \epsilon\}.$$

Its canonical name

$$\dot{G}_i := \left\{ (a,p) \mid p \in \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)\ \wedge\ \exists\, n < \omega\ \exists\, \epsilon \in \{0,1\}\ \big(a = \mathrm{OR}(n,\epsilon)\ \wedge\ p(i,n) = \epsilon\big) \right\}$$

is stabilized by all $\pi \in A$ with $\pi \in \mathrm{Fix}(i)$. Thus, $G_i \in N$ for all $i < \lambda$.

However, the sequence $(G_i \mid i < \lambda)$ is *not* contained in N. In order to obtain a surjection $f : \wp(\aleph_0) \to \lambda$ in N, we use the following technique: Around each G_i, we define a "cloud" $\widetilde{G}_i \subseteq \wp^N(\aleph_0)$, with the property that $G_i \in \widetilde{G}_i$ for all $i < \lambda$, but $\widetilde{G}_i \cap \widetilde{G}_j = \varnothing$ whenever $i \neq j$.

For $i < \lambda$, let

$$\dot{\widetilde{G}}_i := \{(\pi \dot{G}_i, p) \mid p \in \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)\,, \pi \in A\}.$$

Then $\widetilde{G}_i := (\dot{\widetilde{G}}_i)^G = \{(\pi^{-1}G)_i \mid \pi \in A\} = \{(\pi G)_i \mid \pi \in A\}$.

For any i, $j < \lambda$ with $i \neq j$, it follows that $\widetilde{G}_i \cap \widetilde{G}_j = \varnothing$: If not, there would be π, $\sigma \in A$ with $(\pi G)_i = (\sigma G)_j$. Since $\mathrm{dom}\,\pi \cup \mathrm{dom}\,\sigma$ is finite, it follows that the set $D := \{p \in \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0) \mid \exists\, n < \omega \ : \ (i,n) \notin (\mathrm{dom}\,\pi \cup \mathrm{dom}\,\sigma) \wedge (j,n) \notin (\mathrm{dom}\,\pi \cup \mathrm{dom}\,\sigma) \wedge p(i,n) \neq p(j,n)\}$ is dense; so by genericity, we can take a condition $q \in G \cap D$. Then $(\pi q)_i(n) = (\pi q)(i,n) = q(i,n) \neq q(j,n) = (\sigma q)(j,n) = (\sigma q)_j(n)$; contradicting $(\pi G)_i = (\sigma G)_j$.

Hence, the sets $\widetilde{G}_i$ are pairwise disjoint.

Lemma 1.2.10. $V(G) \vDash \theta(\aleph_0) \geq \lambda^+$.

Proof. The sequence $(\widetilde{G}_i \mid i < \lambda)$ is an element of N, since its canonical name

$$\Big(\big(\mathrm{OR}(i, \dot{\widetilde{G}}_i), \mathbb{1} \big) \ \Big| \ i < \lambda \Big)$$

is stabilized by all $\pi \in A$. Thus, we can define in N a surjective function $f : \wp(\omega) \to \lambda$ as follows: For $X \in N$, $X \subseteq \omega$, let $f(X) = i$ if $X \in \widetilde{G}_i$, if such i exists, and $f(X) = 0$, else. Then f is well-defined, since the $\widetilde{G}_i$ are pairwise disjoint, and f is surjective, since $G_i \in N$ with $f(G_i) = i$ for all $i < \lambda$. □

It remains to prove that $\theta^N(\aleph_0) \leq \lambda^+$; i.e. there is no surjective function $f : \wp(\aleph_0) \to \lambda^+$.

An important property of symmetric extensions by forcing notions with a high degree of symmetricity is the *Approximation Lemma*: Sets of ordinals in $V(G)$ can be captured in fairly "mild" V-generic extensions.

For finitely many $i_0, \ldots, i_{k-1} < \lambda$, it follows that $G_{i_0} \times \cdots \times G_{i_{k-1}}$ is a V-generic filter on $\mathrm{Fn}(\omega, 2, \aleph_0)^k$, since for any dense set $D \subseteq \mathrm{Fn}(\omega, 2, \aleph_0)^k$, it follows that $\overline{D} := \{p \in \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0) \mid (p_{i_0}, \ldots, p_{i_{k-1}}) \in D\}$ is dense in $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$.

These finite products $G_{i_0} \times \cdots \times G_{i_{k-1}}$ will describe our approximation models:

Lemma 1.2.11 (Approximation Lemma)**.** *For every set of ordinals $X \subseteq \alpha$ with $X \in V(G)$ there are finitely many $i_0, \ldots, i_{k-1} < \lambda$ such that*

$$X \in V[G_{i_0} \times \cdots \times G_{i_{k-1}}].$$

Proof. Let $X = \dot{X}^G$ with $\dot{X} \in HS$ such that $\pi \dot{X} = \dot{X}$ holds for all $\pi \in \mathrm{Fix}(i_0) \cap \cdots \cap \mathrm{Fix}(i_{k-1})$, where $k < \omega$, and $i_0, \ldots, i_{k-1} < \lambda$. Let

$$X' := \{\beta < \alpha \mid \exists\, p \in \mathrm{Fn}(\lambda \times \omega, 2, \omega) \ : \ p \Vdash_s \beta \in \dot{X}\,,\, p_{i_0} \in G_{i_0}, \ldots, p_{i_{k-1}} \in G_{i_{k-1}}\}.$$

Clearly, $X' \in V[G_{i_0} \times \cdots \times G_{i_{k-1}}]$, so it remains to show that $X = X'$. The inclusion "$\subseteq$" is clear by the Forcing Theorem.

Regarding "$\supseteq$", assume towards a contradiction there was $\beta \in X' \setminus X$. By construction of X', take a condition p with $p \Vdash_s \beta \in \dot{X}$, with the property that $p_{i_0} \in G_{i_0}, \ldots, p_{i_{k-1}} \in G_{i_{k-1}}$. Since $\beta \notin X$, it follows that there must be $p' \in G$ with $p' \Vdash_s \beta \notin \dot{X}$.

We now wish to construct an automorphism $\pi \in A$ such that $\pi p \parallel p'$ and $\pi \in \mathrm{Fix}(i_0) \cap \cdots \cap \mathrm{Fix}(i_{k-1})$. Then $\pi\dot{X} = \dot{X}$; hence, from $p \Vdash_s \beta \in \dot{X}$ and $\pi p \Vdash_s \beta \in \pi\dot{X}$, it follows that $\pi p \Vdash_s \beta \in \dot{X}$. If $\overline{p} \leq \pi p$, p' denotes a common extension of πp and p', then $\overline{p} \Vdash_s \beta \in \dot{X}$ and $\overline{p} \Vdash_s \beta \notin \dot{X}$; contradiction.

It remains to construct π. Let $\operatorname{dom}\pi := \operatorname{dom}p \cup \operatorname{dom}p'$. For every $(i,n) \in \operatorname{dom}\pi$, we define $\pi(i,n) : 2 \to 2$ as follows: $\pi(i,n) \neq id$ in the case that $(i,n) \in \operatorname{dom}p \cap \operatorname{dom}p'$ with $p(i,n) \neq p'(i,n)$, and $\pi(i,n) = id$, else. Then $\pi p \parallel p'$ by construction; and since $p' \in G$ and $p_{i_0} \in G_{i_0}, \ldots, p_{i_{k-1}} \in G_{i_{k-1}}$, it follows that $p(i_l,n) = p'(i_l,n)$ for all $l < k$, $n < \omega$ whenever $(i_l,n) \in \operatorname{dom}p \cap \operatorname{dom}p'$. Thus, $\pi(i_l,n) = id$ for all $l < k$, $n < \omega$ with $(i_l,n) \in \operatorname{dom}\pi$; which implies $\pi \in \mathrm{Fix}(i_0) \cap \cdots \cap \mathrm{Fix}(i_{k-1})$.

It follows that the automorphism π has all the desired properties. Hence, $X = X'$ and $X \in V[G_{i_0} \times \cdots \times G_{i_{k-1}}]$. $\square$

Since any approximation model $V[G_{i_0} \times \cdots \times G_{i_{k-1}}]$ satisfies GCH and cardinals are absolute between V and $V[G]$, it follows that there is in $V[G]$ an injection $\wp(\aleph_0) \cap V[G_{i_0} \times \cdots \times G_{i_{k-1}}] \hookrightarrow \aleph_1$. There are λ-many tuples $(i_0, \ldots, i_{k-1}) \in [\lambda]^{<\omega}$; hence, there is an injection

$$\iota : \bigcup \left\{ \wp(\aleph_0) \cap V[G_{i_0} \times \cdots \times G_{i_{k-1}}] \;\middle|\; k < \omega\,,\, i_0, \ldots, i_{k-1} < \lambda \right\} \to \lambda$$

in $V[G]$.

Since

$$\wp^N(\aleph_0) \subseteq \bigcup \left\{ \wp(\aleph_0) \cap V[G_{i_0} \times \cdots \times G_{i_{k-1}}] \;\middle|\; k < \omega\,,\, i_0, \ldots, i_{k-1} < \lambda \right\}$$

by the *Approximation Lemma* 1.2.11, a surjective function $f : \wp(\aleph_0)^N \to \lambda^+$ in $N \subseteq V[G]$ would yield a bijection $\lambda \leftrightarrow \lambda^+$ in $V[G]$. This is a contradiction, since $\mathrm{Fn}(\lambda \times \omega, 2, \omega)$ preserves cardinals.

Thus, it follows that $\theta^N(\aleph_0) = \lambda^+$, i.e. there exists in N a surjection $f : \wp(\aleph_0) \to \lambda$, but there is not surjective function $f : \wp(\aleph_0) \to \lambda^+$.

This gives the following theorem:

Theorem 1.2.12. *Let V be a ground model of* $\mathsf{ZFC} + \mathsf{GCH}$ *with an uncountable cardinal λ. Then there exists a cardinal-preserving extension $N \supseteq V$ with $N \vDash$* ZF *such that cardinals are absolute between V and N and $\theta^N(\aleph_0) = \lambda^+$.*

In the case that $\mathrm{cf}\,\lambda = \omega$, for example $\lambda = \aleph_\omega$, this is in sharp contrast to the setting in ZFC, where König's Theorem requires $\mathrm{cf}\left(2^{\aleph_0}\right) > \omega$.

This example gives rise to the question whether for arbitrary cardinals κ, the values $\theta(\kappa)$ might be essentially undetermined in ZF. In [GK12], Motik Gitik and Peter Koepke construct a forcing notion giving rise to a symmetric extension $N = V(G)$ with $\theta(\aleph_\omega) = \lambda^+$, where $\lambda \geq \aleph_{\omega+1}$ is an arbitrary cardinal; while below $\aleph_\omega$, GCH is preserved. In Chapter 2.1, we give an overview of their construction.

1.2.3 Symmetric Forcing with Partial Automorphisms

For this chapter, let $(\mathbb{P}, \leq, \mathbb{1})$ be a separative forcing notion. In many applications, one encounters the situation that there are automorphisms π which can not be defined on the whole forcing notion $\mathbb{P}$, but only on a dense subset $D_\pi \subseteq \mathbb{P}$. We call such $\pi : D_\pi \to D_\pi$ a *partial automorphism*. The set A of partial automorphisms that should be considered, is usually not quite a group, but has a very similar structure:

- For any π, $\sigma \in A$ with $\pi : D_\pi \to D_\pi$, $\sigma : D_\sigma \to D_\sigma$ and $p \in D_\pi \cap D_\sigma$, the image $\sigma(p)$ is an element of $D_\pi \cap D_\sigma$ as well; and A contains a map $\nu : D_\nu \to D_\nu$ such that $D_\nu = D_\pi \cap D_\sigma$ and $\nu = \pi \circ \sigma$ on D_ν. (We call ν the *concatenation* $\pi \circ \sigma$.)
- For any $\pi \in A$, there is a map ν in A with $D_\nu = D_\pi$ such that $\pi \circ \nu = \nu \circ \pi = \mathrm{id}_{D_\nu} = \mathrm{id}_{D_\pi}$. (We call ν the *inverse* π^{-1}.)
- There is an *identity element* $\mathrm{id} \in A$, which is the identity map on its domain D_{id}, with $D_{id} \supseteq D_\pi$ for all $\pi \in A$.

This does not quite give a group structure: For instance, for any $\pi \in A$, the concatenation $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = \mathrm{id}_{D_\pi}$ is not the identity element id, which usually has a larger domain $D_{id} \supseteq D_\pi$.

In this setting, the standard approach would be using Boolean-valued models for the construction of the symmetric submodel N: Any automorphism $\pi : D_\pi \to D_\pi$ can be uniquely extended to an automorphism of the complete Boolean algebra $B(\mathbb{P})$, and thereby induces an automorphism of the Boolean valued model $V^{B(\mathbb{P})}$. Then one can consider the group consisting of these extended automorphisms, define a normal filter and construct the corresponding symmetric submodel as described in [Jec73, 5].

The aim of this chapter is to avoid Boolean valued models and find a way to incorporate this situation with *partial automorphisms* into symmetric forcing with partial orders.

Definition 1.2.13. A map π is a *partial* $\mathbb{P}$-*automorphism* if there is a dense set $D_\pi \subseteq \mathbb{P}$ such that $\pi : D_\pi \to D_\pi$, π is bijective, and for all p, $q \in D_\pi$ it follows that $q \leq p$ if and only if $\pi q \leq \pi p$.

Definition 1.2.14. Let $\mathcal{D}$ be a collection of dense subsets $D \subseteq \mathbb{P}$ which is closed under intersections (i.e. for any D, $D' \in \mathcal{D}$, it follows that the intersection $D \cap D'$ is contained in $\mathcal{D}$ as well) and has a *maximal element* $D_{\max}$ (i.e. $D_{\max} \supseteq D$ for all $D \in \mathcal{D}$). A set A is an *almost-group of partial* $\mathbb{P}$-*automorphisms for* $\mathcal{D}$ if the following hold:

- Every $\pi \in A$ is a partial $\mathbb{P}$-automorphism, $\pi : D_\pi \to D_\pi$, with $D_\pi \in \mathcal{D}$.
- For every $D \in \mathcal{D}$, the automorphisms $\{\pi \in A \mid D_\pi = D\}$ form a group, denoted by A_D.
- For every D, $D' \in \mathcal{D}$ with $D \subseteq D'$ and $\pi \in A_{D'}$, it follows that $\pi[D] = D$, and the restriction $\pi \restriction D$ is an element of A_D.

We will now see how any almost-group A of partial $\mathbb{P}$-automorphisms for $\mathcal{D}$ can be turned into a group, using the direct limit:

For any D, $D' \in \mathcal{D}$ with $D \subseteq D'$, there is a canonical homomorphism $\phi_{D'D} : A_{D'} \to A_D$, $\pi \mapsto \pi \restriction D$. This gives a directed system

$$(A_D, \phi_{D'D})_{D,D' \in \mathcal{D},\, D \subseteq D'}$$

and we can take the direct limit

$$\overline{A} := \varinjlim A_D = \bigsqcup A_D \big/ \sim$$

with the following equivalence relation "$\sim$": Whenever $\pi \in A_D$ and $\pi' \in A_{D'}$, then $\pi \sim \pi'$ if there exists $D'' \in A$, $D'' \subseteq D \cap D'$, such that π and π' agree on D''. Since $\mathcal{D}$ is closed under intersections and $\mathbb{P}$ is separative, this is the case if and only if π and π' agree on the intersection $D \cap D'$.

The explicit definition of $\overline{A}$ reads as follows:

Definition/Proposition 1.2.15. Let A be an almost-group of partial $\mathbb{P}$-automorphisms for $\mathcal{D}$. We define on A the following equivalence relation:

$$\pi \sim \pi' \;:\leftrightarrow \pi \restriction (D_\pi \cap D_{\pi'}) = \pi' \restriction (D_\pi \cap D_{\pi'}).$$

For $\pi \in A$, we denote by $[\pi]$ its equivalence class:

$$[\pi] := [\pi]_\sim := \{\sigma \in A \mid \sigma \sim \pi\} = \{\sigma \in A \mid \pi \restriction (D_\pi \cap D_\sigma) = \sigma \restriction (D_\pi \cap D_\sigma)\}.$$

Then $\overline{A} = \big\{ [\pi] \mid \pi \in A \big\}$ becomes a group as follows: For π, $\sigma \in A$, let $[\pi] \circ [\sigma] := [\nu]$, where $\nu \in A$ with $D_\nu = D_\pi \cap D_\sigma$ and $\nu(p) = \pi(\sigma(p))$ for all $p \in D_\pi \cap D_\sigma$.

We call $\overline{A}$ the *group of partial $\mathbb{P}$-automorphisms derived from A.*

Proof. First, we have to make sure that the operation "$\circ$" is well-defined. For any π, $\sigma \in A$ and $D := D_\pi \cap D_\sigma$, it follows by Definition 1.2.14 that $\pi \restriction D$ and $\sigma \restriction D$ are elements of A_D. Since A_D is a group, it follows that there exists a map $\nu \in A_D \subseteq A$ with $D_\nu = D = D_\pi \cap D_\sigma$ such that $\nu = (\pi \restriction D) \circ (\sigma \restriction D)$, i.e. $\nu(p) = \pi(\sigma(p))$ for all $p \in D$.

If $[\pi] = [\pi']$, $[\sigma] = [\sigma']$ and ν, ν' as above with $D_\nu = D_\pi \cap D_\sigma$, $\nu(p) = \pi(\sigma(p))$ for all $p \in D_\nu$, and $D_{\nu'} = D_{\pi'} \cap D_{\sigma'}$ with $\nu'(p) = \pi'(\sigma'(p))$ for all $p \in D_{\nu'}$, then for all $p \in D_\nu \cap D_{\nu'} = (D_\pi \cap D_\sigma) \cap (D_{\pi'} \cap D_{\sigma'})$, it follows that $\nu(p) = \pi(\sigma(p)) = \pi'(\sigma'(p)) = \nu'(p)$. Hence, $[\nu] = [\nu']$ and it follows that "$\circ$" is well-defined.

The identity element id is the identity element of the group $A_{D_{\max}}$, with $D_{\mathrm{id}} = D_{\max} \supseteq D_\pi$ for all $\pi \in A$. Then $[\pi] \circ [\mathrm{id}] = [\mathrm{id}] \circ [\pi] = [\pi]$ for all $\pi \in A$ follows.

Finally, for $\pi \in A$, let $[\pi]^{-1} := [\nu]$, where ν is the inverse element of π in A_{D_π}. Then $D_\nu = D_\pi$ and $\nu(\pi(p)) = \pi(\nu(p)) = p$ for all $p \in D_\pi = D_\nu$; hence, $[\pi] \circ [\nu] = [\nu] \circ [\pi] = [\mathrm{id}_{D_\pi}] = [\mathrm{id}_{D_{\max}}] = \mathrm{id}$.
Again, such $[\nu]$ is well-defined: Whenever $[\pi] = [\pi']$ and ν, ν' with $D_\nu = D_\pi$, $\nu = \pi^{-1}$ on D_π and $D_{\nu'} = D_{\pi'}$, $\nu' = (\pi')^{-1}$ on $D_{\pi'}$, then for any $p \in D_\nu \cap D_{\nu'} = D_\pi \cap D_{\pi'}$, it follows that $\nu(p) = \nu\big((\nu^{-1} \circ \pi^{-1})(p)\big) = \pi^{-1}(p) = (\pi')^{-1}(p) = \nu'(p)$. Thus, $[\nu] = [\nu']$.

$\square$

We now have to find a way to extend our partial automorphisms π to the name space $Name(\mathbb{P})$.

We fix a collection $\mathcal{D}$ of dense sets which is closed under intersections and has a maximal element $D_{\max}$, and an almost-group A of partial $\mathbb{P}$-automorphisms; and denote by $\overline{A}$ the group of partial automorphisms derived from A as in 1.2.15.

For $D \in \mathcal{D}$, we define a hierarchy $\overline{\mathrm{Name}_\alpha(\mathbb{P})}^D$ recursively:

- $\overline{\mathrm{Name}_0(\mathbb{P})}^D := \varnothing$
- $\overline{\mathrm{Name}_{\alpha+1}(\mathbb{P})}^D := \left\{ \dot{x} \in \mathrm{Name}(\mathbb{P}) \mid \dot{x} \subseteq \overline{\mathrm{Name}_\alpha(\mathbb{P})}^D \times D \right\}$, and
- $\overline{\mathrm{Name}_\lambda(\mathbb{P})}^D := \bigcup_{\alpha<\lambda} \overline{\mathrm{Name}_\alpha(\mathbb{P})}^D$ for λ a limit ordinal.

Let

$$\overline{\mathrm{Name}(\mathbb{P})}^D := \bigcup_{\alpha \in \mathrm{Ord}} \overline{\mathrm{Name}_\alpha(\mathbb{P})}^D.$$

In other words: $\overline{\mathrm{Name}(\mathbb{P})}^D$ is the collection of all $\mathbb{P}$-names $\dot{x}$ in which only conditions $p \in D$ occur.

Whenever $\pi \in A$, $\pi : D_\pi \to D_\pi$ and $\dot{x} \in \mathrm{Name}(\mathbb{P})$, the image $\pi\dot{x}$ can be defined as usual if $\dot{x} \in \overline{\mathrm{Name}(\mathbb{P})}^{D_\pi}$. In the case that $\dot{x}$ is a $\mathbb{P}$-name with $\dot{x} \notin \overline{\mathrm{Name}(\mathbb{P})}^{D_\pi}$ however, it is not clear how to apply π, so $\dot{x}$ has to be modified.

Given $D \in \mathcal{D}$, we define recursively for $\dot{x} \in \mathrm{Name}(\mathbb{P})$:

$$\overline{x}^D := \{(\overline{y}^D, p) \mid \dot{y} \in \mathrm{dom}\,\dot{x}\,,\, p \in D\,,\, p \Vdash \dot{y} \in \dot{x}\}.$$

Then $\overline{x}^D \in \overline{\mathrm{Name}(\mathbb{P})}^D$ with $\dot{x}^G = (\overline{x}^D)^G$ for any V-generic filter G on $\mathbb{P}$, and $\mathsf{rk}_{\mathbb{P}}\overline{x}^D = \mathsf{rk}_{\mathbb{P}}\dot{x}$.

Let now $D = D_\pi \in \mathcal{D}$, and $\pi : D_\pi \to D_\pi$ a partial $\mathbb{P}$-automorphism. Whenever G is a V-generic filter on $\mathbb{P}$, then $\pi^{-1}G := \{q \in \mathbb{P} \mid \exists\, r \in D \cap G\ q \geq \pi^{-1}r\}$ is a V-generic filter on $\mathbb{P}$ as well; and for any name $\dot{z} \in \overline{\mathrm{Name}(\mathbb{P})}^D$, it follows that $(\pi\dot{z})^G = \dot{z}^{\pi^{-1}G}$. In particular, $(\pi\overline{z}^{D_\pi})^G = \dot{z}^{\pi^1 G}$ for any $\dot{z} \in \mathrm{Name}(\mathbb{P})$.

Moreover, it is not difficult to verify that whenever D, $D' \in \mathcal{D}$ and $\dot{x} \in \mathrm{Name}(\mathbb{P})$, then $\overline{\overline{x}^D}^{D'} = \overline{x}^{D'}$; and whenever π, $\pi' \in A$, $\pi : D_\pi \to D_\pi$, $\pi' : D_{\pi'} \to D_{\pi'}$ and $\dot{x} \in \overline{\mathrm{Name}(\mathbb{P})}^{D_\pi}$, then

$$\pi\overline{x}^{D_{\pi'}} = \overline{\pi\dot{x}}^{D_{\pi'}}.$$

Later on, we will take a normal filter $\mathcal{F}$ on $\overline{A}$ and call a $\mathbb{P}$-name $\dot{x}$ *symmetric* if the collection of all $[\pi]$ with $\pi\overline{x}^{D_\pi} = \overline{x}^{D_\pi}$ is contained in $\mathcal{F}$.
We have to make sure that this definition does not depend on which representative of $[\pi]$ we choose:

Lemma 1.2.16. *Let π, $\pi' \in A$ with $\pi \sim \pi'$, i.e. $\pi \upharpoonright (D_\pi \cap D_{\pi'}) = \pi' \upharpoonright (D_\pi \cap D_{\pi'})$. Then for any $\dot{x} \in \mathrm{Name}(\mathbb{P})$, it follows that $\pi\overline{x}^{D_\pi} = \overline{x}^{D_\pi}$ if and only if $\pi'\overline{x}^{D_{\pi'}} = \overline{x}^{D_{\pi'}}$.*

We prove the following more general statement by induction over α:

Lemma 1.2.17. *Let π, $\pi' \in A$ with $\pi \sim \pi'$, i.e. $\pi \restriction (D_\pi \cap D_{\pi'}) = \pi' \restriction (D_\pi \cap D_{\pi'})$, and $\alpha \in \mathrm{Ord}$. Then for any $\dot{y}$, $\dot{z} \in \mathrm{Name}(\mathbb{P})$ with $\mathsf{rk}_{\mathbb{P}}\dot{y} = \mathsf{rk}_{\mathbb{P}}\dot{z} = \gamma$, it follows that $\pi\overline{y}^{D_\pi} = \overline{z}^{D_\pi}$ if and only if $\pi'\overline{y}^{D_{\pi'}} = \overline{z}^{D_{\pi'}}$.*

Proof. W.l.o.g. we can assume that $D_{\pi'} \subseteq D_\pi$; since the map $\sigma := \pi \restriction (D_\pi \cap D_{\pi'}) = \pi' \restriction (D_\pi \cap D_{\pi'})$ is contained in A as well, with $D_\sigma = D_\pi \cap D_{\pi'}$ and $\sigma \sim \pi$, $\sigma \sim \pi'$. Hence, if we now that $\pi\overline{y}^{D_\pi} = \overline{z}^{D_\pi} \Leftrightarrow \sigma\overline{y}^{D_\sigma} = \overline{z}^{D_\sigma}$ for all $\dot{y}$, $\dot{z} \in \mathrm{Name}(\mathbb{P})$, and $\sigma\overline{y}^{D_\sigma} = \overline{z}^{D_\sigma} \Leftrightarrow \pi'\overline{y}^{D_{\pi'}} = \overline{z}^{D_{\pi'}}$ for all $\dot{y}$, $\dot{z} \in \mathrm{Name}(\mathbb{P})$; then it follows that whenever $\dot{y}$, $\dot{z} \in \mathrm{Name}(\mathbb{P})$, then $\pi\overline{y}^{D_\pi} = \overline{z}^{D_\pi}$ if and only if $\pi'\overline{y}^{D_{\pi'}} = \overline{z}^{D_{\pi'}}$.

Thus, assume $D_{\pi'} \subseteq D_\pi$. We consider $\gamma \in \mathrm{Ord}$, and assume inductively that the statement holds true for all $\beta < \gamma$: Whenever $\dot{x}$, $\dot{u} \in \mathrm{Name}(\mathbb{P})$ with $\mathsf{rk}_{\mathbb{P}}\dot{x} = \mathsf{rk}_{\mathbb{P}}\dot{u} < \gamma$, then $\pi\overline{x}^{D_\pi} = \overline{u}^{D_\pi}$ if and only if $\pi'\overline{x}^{D_{\pi'}} = \overline{u}^{D_{\pi'}}$.

Let $\dot{y}$, $\dot{z} \in \mathrm{Name}(\mathbb{P})$ with $\mathsf{rk}_{\mathbb{P}}\dot{y} = \mathsf{rk}_{\mathbb{P}}\dot{z} = \gamma$.

"$\Rightarrow$": First, assume that $\pi\overline{y}^{D_\pi} = \overline{z}^{D_\pi}$. We only prove $\overline{z}^{D_{\pi'}} \subseteq \pi'\overline{y}^{D_{\pi'}}$; the other inclusion is similar.

Let $(\overline{x}^{D_{\pi'}}, \overline{p}) \in \overline{z}^{D_{\pi'}}$, i.e. $\dot{x} \in \operatorname{dom}\dot{z}$, $\overline{p} \in D_{\pi'}$, and $\overline{p} \Vdash \dot{x} \in \dot{z}$. Then also $\overline{p} \in D_\pi$ holds. Hence, $(\overline{x}^{D_\pi}, \overline{p}) \in \overline{z}^{D_\pi}$, and $\overline{z}^{D_\pi} = \pi\overline{y}^{D_\pi}$ by assumption; so there must be $\dot{u} \in \operatorname{dom}\dot{y}$ with $\overline{x}^{D_\pi} = \pi\overline{u}^{D_\pi}$. Setting $\overline{q} := \pi^{-1}\overline{p}$, it follows that $\overline{q} \Vdash \overline{u}^{D_\pi} \in \overline{y}^{D_\pi}$ and $\overline{q} \Vdash \dot{u} \in \dot{y}$.

Since $\overline{x}^{D_\pi} = \pi\overline{u}^{D_\pi}$ with $\overline{x}^{D_\pi} \in \operatorname{dom}\overline{z}^{D_\pi}$, it follows that $\mathsf{rk}_{\mathbb{P}}\dot{u} = \mathsf{rk}_{\mathbb{P}}\dot{x} < \gamma$. Thus, our inductive assumption implies that $\overline{x}^{D_{\pi'}} = \pi'\overline{u}^{D_{\pi'}}$. Hence, $(\overline{x}^{D_{\pi'}}, \overline{p}) = (\pi\overline{u}^{D_{\pi'}}, \pi\overline{q})$, which is contained in $\pi'\overline{y}^{D_{\pi'}}$, since $\dot{u} \in \operatorname{dom}\dot{y}$, $\overline{q} \in D_{\pi'}$ (since $\overline{p} \in D_{\pi'}$, $\overline{q} = \pi^{-1}\overline{p}$, and $\pi^{-1}[D_{\pi'}] = D_{\pi'}$), and $\overline{q} \Vdash \dot{u} \in \dot{y}$.

"$\Leftarrow$": Now, assume $\pi'\overline{y}^{D_{\pi'}} = \overline{z}^{D_{\pi'}}$. As before, we only prove the inclusion $\overline{z}^{D_\pi} \subseteq \pi\overline{y}^{D_\pi}$.

Consider $(\overline{x}^{D_\pi}, \overline{p}) \in \overline{z}^{D_\pi}$, i.e. $\dot{x} \in \operatorname{dom}\dot{z}$, $\overline{p} \in D_\pi$ and $\overline{p} \Vdash \dot{x} \in \dot{z}$. Let $\widetilde{p} \leq \overline{p}$ with $\widetilde{p} \in D_{\pi'}$. Then $(\overline{x}^{D_{\pi'}}, \widetilde{p}) \in \overline{z}^{D_{\pi'}} = \pi'\overline{y}^{D_{\pi'}}$, so there must be $\dot{u} \in \operatorname{dom}\dot{y}$ with $\overline{x}^{D_{\pi'}} = \pi'\overline{u}^{D_{\pi'}}$. By the inductive assumption, it follows that $\overline{x}^{D_\pi} = \pi\overline{u}^{D_\pi}$, since $\mathsf{rk}_{\mathbb{P}}\dot{u} = \mathsf{rk}_{\mathbb{P}}\dot{x} < \gamma$. Let $\overline{q} := \pi^{-1}\overline{p}$. We have to show that $(\pi\overline{u}^{D_\pi}, \pi\overline{q}) \in \pi\overline{y}^{D_\pi}$. Since $\dot{u} \in \operatorname{dom}\dot{y}$ and $\overline{q} \in D_\pi$, it suffices to verify that $\overline{q} \Vdash \dot{u} \in \dot{y}$. We prove that whenever $r \leq \overline{q}$, $r \in D_{\pi'}$, then $r \Vdash \dot{u} \in \dot{y}$. Consider such $r \leq \overline{q}$ with $r \in D_{\pi'}$. Then $\pi r \in D_{\pi'}$, and $\pi r \leq \overline{p}$ implies that $\pi r \Vdash \dot{x} \in \dot{z}$. Hence, $(\overline{x}^{D_{\pi'}}, \pi r) \in \overline{z}^{D_{\pi'}}$, and $\overline{z}^{D_{\pi'}} = \pi'\overline{y}^{D_{\pi'}}$ by assumption. Now, $(\pi'\overline{u}^{D_{\pi'}}, \pi' r) = (\overline{x}^{D_{\pi'}}, \pi r) \in \pi'\overline{y}^{D_{\pi'}}$ implies that $r \Vdash \overline{u}^{D_{\pi'}} \in \overline{y}^{D_{\pi'}}$; hence, $r \Vdash \dot{u} \in \dot{y}$ as desired.

□

Let $\mathcal{F}$ be a normal filter on $\overline{A}$ (cf. Definition 1.2.3), i.e. $\mathcal{F}$ is a nonempty collection of $\overline{A}$-subgroups, closed under supersets and finite intersections, such that for any subgroup $B \in \mathcal{F}$ and $\pi \in A$, the conjugate $[\pi]B[\pi]^{-1}$ is an element of $\mathcal{F}$, as well.

We use $\mathcal{F}$ to establish our notion of symmetry:

Definition 1.2.18. A $\mathbb{P}$-name $\dot{x}$ is *symmetric for $\mathcal{F}$* if the stabilizer group

$$\operatorname{sym}^{\overline{A}}(\dot{x}) \coloneqq \{[\pi] \in \overline{A} \mid \pi\overline{x}^{D_\pi} = \overline{x}^{D_\pi}\}$$

is an element of $\mathcal{F}$. Recursively, a name $\dot{x}$ is *hereditarily symmetric*, $\dot{x} \in HS^{\mathcal{F}}$, if $\dot{x}$ is symmetric, and $\dot{y}$ is herditarily symmetric for all $\dot{y} \in \operatorname{dom}\dot{x}$.

By Lemma 1.2.16, this is well-defined, since whenever $\pi \sim \pi'$ and $\dot{x} \in \operatorname{Name}(\mathbb{P})$, it follows that $\pi\overline{x}^{D_\pi} = \overline{x}^{D_\pi}$ if and only if $\pi'\overline{x}^{D_{\pi'}} = \overline{x}^{D_{\pi'}}$.

When $\overline{A}$ and $\mathcal{F}$ are clear from the context, we write just $\operatorname{sym}(\dot{x})$ and HS.

We will use the following properties: If $\dot{x} \in HS^{\mathcal{F}}$ and $\pi \in A$, then firstly, it is not difficult to verify that also $\overline{x}^{D_\pi} \in HS^{\mathcal{F}}$ holds; and secondly, $\pi\overline{x}^{D_\pi} \in HS^{\mathcal{F}}$. For the second claim, one can check that whenever $\sigma \in A$ with $\sigma\overline{x}^{D_\sigma} = \overline{x}^{D_\sigma}$, then

$$(\pi\sigma\pi^{-1})\,\overline{\pi\overline{x}^{D_\pi}}^{\,D_{\pi\sigma\pi^{-1}}} = \overline{\pi\overline{x}^{D_\pi}}^{\,D_{\pi\sigma\pi^{-1}}},$$

and then use the normality of $\mathcal{F}$.

For any element of the ground model $a \in V$, it follows that the canonical name $\check{a} \coloneqq \{(\check{b}, \mathbb{1}) \mid b \in a\}$ is hereditarily symmetric:

For $\pi \in A$,

$$\overline{\check{a}}^{D_\pi} = \{(\overline{\check{b}}^{D_\pi}, p) \mid \check{b} \in \operatorname{dom}\check{a}\,,\, p \in D_\pi\,,\, p \Vdash \check{b} \in \check{a}\} = \{(\overline{\check{b}}^{D_\pi}, p) \mid b \in \operatorname{dom} a\,,\, p \in D_\pi\},$$

and

$$\pi\overline{\check{a}}^{D_\pi} = \{(\pi\overline{\check{b}}, \pi p) \mid b \in \operatorname{dom} a\,,\, p \in D_\pi\} = \{(\pi\overline{\check{b}}, p) \mid b \in \operatorname{dom} a\,,\, p \in D_\pi\},$$

so one can show recursively that $\pi\overline{\check{a}}^{D_\pi} = \overline{\check{a}}^{D_\pi}$ holds for every $a \in V$ and $\pi \in A$.

Now, we are ready to define the symmetric extension:

Definition 1.2.19. Let G be a V-generic filter on $\mathbb{P}$. The *symmetric extension by $\mathcal{F}$ and G* is

$$V(G)^{\mathcal{F}} \coloneqq \{\dot{x}^G \mid \dot{x} \in HS^{\mathcal{F}}\}.$$

When the normal filter $\mathcal{F}$ is clear from the context, we write just HS and $V(G)$.

The *symmetric forcing relation with partial automorphisms* $(\Vdash_s)^V_{\mathbb{P},\mathcal{F}}$ can be defined as in Definition 1.2.7, and we write just "$\Vdash_s$" if the ground model V, the forcing $\mathbb{P}$, and the normal filter $\mathcal{F}$ on a group $\overline{A}$ of partial $\mathbb{P}$-automorphisms are clear from the context.

Whenever $\dot{x}$, $\dot{y} \in HS$ and $p \in \mathbb{P}$, then $p \Vdash_s \dot{y} \in \dot{x}$ if and only if $p \Vdash \dot{y} \in \dot{x}$ with the ordinary forcing relation "$\Vdash$", and $p \Vdash_s \dot{x} = \dot{y}$ if and only if $p \Vdash \dot{x} = \dot{y}$. In particular, for any $\dot{x} \in HS^{\mathcal{F}}$ and $D \in \mathcal{D}$, we have

$$\overline{x}^D = \{(\overline{y}^D, p) \mid \dot{y} \in \operatorname{dom}\dot{x}\,,\, p \in D\,,\, p \Vdash_s \dot{y} \in \dot{x}\}.$$

The symmetric forcing relation with partial automorphisms satisfies the same basic properties as the ordinary symmetric forcing relation (see Proposition 1.2.8 — one has to use that for every $\dot{x} \in HS^{\mathcal{F}}$ and $\pi \in A$, it follows that $\pi\overline{x}^{D_\pi} \in HS^{\mathcal{F}}$, as well).

Moreover, the *Symmetry Lemma* holds true:

Lemma 1.2.20 (Symmetry Lemma). *Let $\pi \in A$, $\varphi(v_0, \ldots, v_{n-1})$ a formula of set theory and $\dot{x}_0, \ldots, \dot{x}_{n-1} \in HS^{\mathcal{F}}$. For $p \in D_\pi$, it follows that $p \Vdash_s \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ if and only if $\pi p \Vdash_s \varphi(\pi \overline{x}_0^{D_\pi}, \ldots, \pi \overline{x}_{n-1}^{D_\pi})$.*

The *Forcing Theorem* holds true, as well (with the same proof as for the ordinary forcing relation "$\Vdash$", except that for the existential quantifier case in the induction on the complexity of formulae, one has to adopt Definition 1.2.18).
It remains to verify that symmetric forcing with partial automorphisms always yields a model of ZF.

Theorem 1.2.21. *Let $\mathbb{P}$ be a notion of forcing, let A be an almost-group of partial $\mathbb{P}$-automorphisms, $\overline{A}$ the group of partial automorphisms derived from A, and $\mathcal{F}$ a normal filter on $\overline{A}$. If G denotes a V-generic filter on $\mathbb{P}$, then $V(G) = V(G)^{\mathcal{F}}$ is a transitive model of ZF with $V \subseteq V(G) \subseteq V[G]$.*

Proof. The inclusions $V \subseteq V(G) \subseteq V[G]$ are clear, and the transitivity of $V(G)$ follows by heredity of $HS^{\mathcal{F}}$. Hence, the axioms of *Extensionality*, *Foundation* and *Infinity* hold in $V(G)$.

Pairing. Let x, $y \in V(G)$ and $\dot{x}$, $\dot{y} \in HS$ with $x = \dot{x}^G$, $y = \dot{y}^G$. We have to show that the set $\{x, y\}$ is an element of $V(G)$ as well, i.e. we have to find a name $\dot{z} \in HS$ with $\dot{z}^G = \{x, y\}$.
Let $\dot{z} := \{(\dot{x}, \mathbb{1}), (\dot{y}, \mathbb{1})\}$ and consider $\pi \in A$ with $\pi \overline{x}^{D_\pi} = \overline{x}^{D_\pi}$ and $\pi \overline{y}^{D_\pi} = \overline{y}^{D_\pi}$. Since

$$\overline{z}^{D_\pi} = \{(\overline{x}^{D_\pi}, p) \mid p \in D_\pi\} \cup \{(\overline{y}^{D_\pi}, p) \mid p \in D_\pi\},$$

it follows that

$$\begin{aligned} \pi \overline{z}^{D_\pi} &= \{(\pi \overline{x}^{D_\pi}, \pi p) \mid p \in D_\pi\} \cup \{(\pi \overline{y}^{D_\pi}, \pi p) \mid p \in D_\pi\} \\ &= \{(\overline{x}^{D_\pi}, \pi p) \mid \pi p \in D_\pi\} \cup \{(\overline{y}^{D_\pi}, \pi p) \mid \pi p \in D_\pi\} \\ &= \overline{z}^{D_\pi} \end{aligned}$$

as desired. Thus, $\operatorname{sym}^{\overline{A}}(\dot{z}) \supseteq \operatorname{sym}^{\overline{A}}(\dot{x}) \cap \operatorname{sym}^{\overline{A}}(\dot{y}) \in \mathcal{F}$, and it follows that $\dot{z}$ is symmetric. Since $\operatorname{dom} \dot{z} = \{\dot{x}, \dot{y}\} \subseteq HS$, this implies $\dot{z} \in HS$ as desired.

Union. Let $x \in V(G)$, $x = \dot{x}^G$ with $\dot{x} \in HS$. We have to show that $\bigcup x \in V(G)$, i.e. we have to find $\dot{u} \in HS$ with $\bigcup x = \dot{u}^G$. Let

$$\dot{u} := \{(\dot{z}, p) \mid (\exists\, \dot{y} \in \operatorname{dom} \dot{x}\ \dot{z} \in \operatorname{dom} \dot{y}) \wedge p \Vdash_s (\exists\, y \in \dot{x}\ \dot{z} \in y)\}.$$

It is not difficult to see that indeed, $\dot{u}^G = \bigcup x$. Let $\pi \in A$ with $\pi \overline{x}^{D_\pi} = \overline{x}^{D_\pi}$. We will show that also $\pi \overline{u}^{D_\pi} = \overline{u}^{D_\pi}$: Then $\operatorname{sym}^{\overline{A}}(\dot{u}) \supseteq \operatorname{sym}^{\overline{A}}(\dot{x})$; so $\dot{u}$ is symmetric, and $\operatorname{dom} \dot{u} \subseteq HS$ implies $\dot{u} \in HS$ as desired.
By definition,

$$\overline{u}^{D_\pi} := \{(\overline{z}^{D_\pi}, p) \mid p \in D_\pi\,,\ \dot{z} \in \operatorname{dom} \dot{u}\,,\ p \Vdash_s \dot{z} \in \dot{u}\}.$$

We claim that

$$\overline{u}^{D_\pi} = \{(\overline{z}^{D_\pi}, p) \mid (\exists\, \dot{y} \in \operatorname{dom} \dot{x}\ \dot{z} \in \operatorname{dom} \dot{y}) \wedge p \in D_\pi \wedge p \Vdash_s (\exists\, y \in \dot{x}\ \dot{z} \in y)\}.$$

Regarding "$\supseteq$", consider $(\overline{z}^{D_\pi}, p)$ such that $\dot z \in \operatorname{dom} \dot y$ for some $\dot y \in \operatorname{dom} \dot x$, $p \in D_\pi$, and $p \Vdash_s (\exists\, y \in \dot x\ \dot z \in y)$. Then $(\dot z, p) \in \dot u$; hence, $\dot z \in \operatorname{dom} \dot u$ and $p \Vdash_s \dot z \in \dot u$. This gives $(\overline{z}^{D_\pi}, p) \in \overline{u}^{D_\pi}$ as desired.

For the other inclusion "$\subseteq$", take $(\dot z, p) \in \overline{u}^{D_\pi}$, i.e. $\dot z \in \operatorname{dom} \dot u$, $p \in D_\pi$, and $p \Vdash_s \dot z \in \dot u$. Then by construction of $\operatorname{dom} \dot u$, there must be $\dot y \in \operatorname{dom} \dot x$ with $\dot x \in \operatorname{dom} \dot y$. It remains to show that $p \Vdash_s (\exists\, y \in \dot x\ \dot z \in y)$. Let H be a V-generic filter with $p \in H$. Then $\dot z^H \in \dot u^H$; so there must be $(\dot z_0, q) \in \dot u$ with $\dot z_0^H = \dot z^H$ and $q \in H$. Then $q \Vdash_s \exists y \in \dot x\ \dot z_0 \in y$. Hence, there must be $y \in \dot x^H$ with $\dot z^H = \dot z_0^H \in y$. This implies $p \Vdash_s (\exists\, y \in \dot x\ \dot z \in y)$ as desired.

Thus, we have shown that

$$\begin{aligned}
\overline{u}^{D_\pi} &= \{(\overline{z}^{D_\pi}, p) \mid (\exists\, \dot y \in \operatorname{dom} \dot x\ \dot z \in \operatorname{dom} \dot y) \land p \in D_\pi \land p \Vdash_s (\exists\, y \in \dot x\ \dot z \in \dot y)\} \\
&= \{(\overline{z}^{D_\pi}, p) \mid (\exists\, y \in \operatorname{dom} \overline{x}^{D_\pi}\ \overline{z}^{D_\pi} \in \operatorname{dom} y) \land p \in D_\pi \land p \Vdash_s (\exists\, y \in \overline{x}^{D_\pi}\ \overline{z}^{D_\pi} \in y)\}.
\end{aligned}$$

Hence, $\pi \overline{u}^{D_\pi} =$

$$\begin{aligned}
&= \{(\pi\overline{z}^{D_\pi}, \pi p) \mid (\exists\, y \in \operatorname{dom} \overline{x}^{D_\pi}\ \overline{z}^{D_\pi} \in \operatorname{dom} y) \land p \in D_\pi \land p \Vdash_s (\exists\, y \in \overline{x}^{D_\pi}\ \overline{z}^{D_\pi} \in y)\} \\
&= \{(\pi\overline{z}^{D_\pi}, \pi p) \mid (\exists\, y \in \operatorname{dom} \pi\overline{x}^{D_\pi}\ \pi\overline{z}^{D_\pi} \in \operatorname{dom} y) \land \pi p \in D_\pi \land \pi p \Vdash_s (\exists\, y \in \pi\overline{x}^{D_\pi}\ \pi\overline{z}^{D_\pi} \in y)\}.
\end{aligned}$$

Since $\pi\overline{x}^{D_\pi} = \overline{x}^{D_\pi}$ by assumption, this implies $\pi\overline{u}^{D_\pi} = \overline{u}^{D_\pi}$ as desired.

Separation. Let $\varphi(v, w)$ be a formula of set theory, $a \in V(G)$, and $z \in V(G)$ some parameter. We claim that

$$b := \{x \in a \mid V(G) \vDash \varphi(x, z)\}$$

is an element of $V(G)$, as well. Let $a = \dot a^G$, $z = \dot z^G$ with $\dot a$, $\dot z \in HS$. Let

$$\dot b := \{(\dot x, p) \mid \dot x \in \operatorname{dom} \dot a\,,\, p \in \mathbb{P}\,,\, p \Vdash_s (\dot x \in \dot a \land \varphi(\dot x, \dot z))\}.$$

Clearly, $\dot b^G = b$, and $\operatorname{dom} \dot b \subseteq HS$. It remains to make sure that the name $\dot b$ is symmetric. We show that for every $\pi \in A$ with $\pi\overline{x}^{D_\pi} = \overline{x}^{D_\pi}$ and $\pi\overline{z}^{D_\pi} = \overline{z}^{D_\pi}$, it follows that $\pi\overline{b}^{D_\pi} = \overline{b}^{D_\pi}$. Then $\operatorname{sym}^{\overline{A}}(\dot b) \supseteq \operatorname{sym}^{\overline{A}}(\dot x) \cap \operatorname{sym}^{\overline{A}}(\dot z)$ implies $\operatorname{sym}^{\overline{A}}(\dot b) \in \mathcal{F}$ as desired.

For $\pi \in A$,

$$\overline{b}^{D_\pi} = \{(\overline{x}^{D_\pi}, p) \mid \dot x \in \operatorname{dom} \dot b\,, p \in D_\pi\,, p \Vdash_s \dot x \in \dot b\}.$$

We now claim that for any $\dot x \in \operatorname{dom} \dot a$ and $p \in \mathbb{P}$, it follows that $p \Vdash_s \dot x \in \dot b$ if and only if $p \Vdash_s (\dot x \in \dot a \land \varphi(\dot x, \dot z))$. The implication "$\Leftarrow$" is clear, since for any $\dot x \in \operatorname{dom} \dot a$ and $p \in \mathbb{P}$ with $p \Vdash_s (\dot x \in \dot a \land \varphi(\dot x, \dot z))$, we have $(\dot x, p) \in \dot b$ by definition. Hence, $p \Vdash_s \dot x \in \dot b$. Regarding "$\Rightarrow$", consider $\dot x \in \operatorname{dom} \dot a$ and $p \in \mathbb{P}$ with $p \Vdash_s \dot x \in \dot b$. Let H be a V-generic filter on $\mathbb{P}$ with $p \in H$. We have to show that $\dot x^H \in \dot a^H$ and $V(H) \vDash \varphi(\dot x^H, \dot z^H)$.
From $\dot x^H \in \dot b^H$, it follows that there must be $\dot y \in \operatorname{dom} \dot a$ and $q \in H$ with $(\dot y, q) \in \dot b$ and $\dot x^H = \dot y^H$. Then $q \Vdash_s \dot y \in \dot a \land \varphi(\dot y, \dot z)$, hence, $\dot y^H \in \dot a^H$ and $V(H) \vDash \varphi(\dot y^H, \dot z^H)$. Since $\dot y^H = \dot x^H$, this finishes the proof.

Thus,

$$\begin{aligned}
\overline{b}^{D_\pi} &= \{(\overline{x}^{D_\pi}, p) \mid \dot{x} \in \operatorname{dom} \dot{b}\,,\, p \in D_\pi\,,\, p \Vdash_s (\dot{x} \in \dot{a} \wedge \varphi(\dot{x}, \dot{z}))\} \\
&= \{(\overline{x}^{D_\pi}, p) \mid \dot{x} \in \operatorname{dom} \dot{a}\,,\, p \in D_\pi\,,\, p \Vdash_s (\dot{x} \in \dot{a} \wedge \varphi(\dot{x}, \dot{z}))\} \\
&= \{(\overline{x}^{D_\pi}, p) \mid \overline{x}^{D_\pi} \in \operatorname{dom} \overline{a}^{D_\pi}\,,\, p \in D_\pi\,,\, p \Vdash_s (\overline{x}^{D_\pi} \in \overline{a}^{D_\pi} \wedge \varphi(\overline{x}^{D_\pi}, \overline{z}^{D_\pi}))\} \\
&= \{(x, p) \mid x \in \operatorname{dom} \overline{a}^{D_\pi}\,,\, p \in D_\pi\,,\, p \Vdash_s (x \in \overline{a}^{D_\pi} \wedge \varphi(x, \overline{z}^{D_\pi}))\}
\end{aligned}$$

Hence, by the *Symmetry Lemma* 1.2.20

$$\begin{aligned}
\pi\overline{b}^{D_\pi} &= \{(\pi x, \pi p) \mid x \in \operatorname{dom} \overline{a}^{D_\pi}\,,\, p \in D_\pi\,,\, p \Vdash_s (x \in \overline{a}^{D_\pi} \wedge \varphi(x, \overline{z}^{D_\pi}))\} \\
&= \{(\pi x, \pi p) \mid \pi x \in \operatorname{dom} \pi\overline{a}^{D_\pi}\,,\, p \in D_\pi\,,\, \pi p \Vdash_s (\pi x \in \pi\overline{a}^{D_\pi} \wedge \varphi(\pi x, \pi\overline{z}^{D_\pi}))\}.
\end{aligned}$$

Since $\pi\overline{a}^{D_\pi} = \overline{a}^{D_\pi}$, $\pi\overline{z}^{D_\pi} = \overline{z}^{D_\pi}$, and $p \in D_\pi$ if and only if $\pi p \in D_\pi$, this gives

$$\pi\overline{b}^{D_\pi} = \{(\pi x, \pi p) \mid \pi x \in \operatorname{dom} \overline{a}^{D_\pi}\,,\, \pi p \in D_\pi\,,\, \pi p \Vdash_s (\pi x \in \overline{a}^{D_\pi} \wedge \varphi(\pi x, \overline{z}^{D_\pi}))\} = \overline{b}^{D_\pi}$$

as desired.

Power Set. Consider $X \in N$, $X = \dot{X}^G$ with $\dot{X} \in HS$. We have to show that $\wp^N(X) \in N$. Let

$$\dot{B} := \{(\dot{Y}, p) \mid \dot{Y} \in HS\,,\, \dot{Y} \subseteq \operatorname{dom} \dot{X} \times \mathbb{P}\,,\, p \in \mathbb{P}\,,\, p \Vdash_s \dot{Y} \subseteq \dot{X}\}.$$

Then $\dot{B}^G = \wp^N(X)$, since for any $Y \in N$ with $Y \subseteq X$, there exists a name $\dot{Y} \in HS$, $\dot{Y}^G = Y$, such that $\dot{Y} \subseteq \operatorname{dom} \dot{X} \times \mathbb{P}$.

It remains to make sure that the name $\dot{B}$ is symmetric. Consider $\pi \in A$ with $\pi\overline{X}^{D_\pi} = \overline{X}$. Then

$$\overline{B}^{D_\pi} = \left\{ (\overline{Y}^{D_\pi}, \overline{p}) \mid \dot{Y} \in HS\,,\, \dot{Y} \subseteq \operatorname{dom} \dot{X} \times \mathbb{P}\,,\, \overline{p} \in D_\pi\,,\, \overline{p} \Vdash_s \dot{Y} \in \dot{B} \right\}.$$

It is not difficult to check that

$$\overline{B}^{D_\pi} = \left\{ (\overline{Y}^{D_\pi}, \overline{p}) \mid \dot{Y} \in HS\,,\, \dot{Y} \subseteq \operatorname{dom} \dot{X} \times \mathbb{P}\,,\, \overline{p} \in D_\pi\,,\, \overline{p} \Vdash_s \dot{Y} \subseteq \dot{X} \right\},$$

since for any $p \in D_\pi$ and $\dot{Y} \in HS$, $\dot{Y} \subseteq \operatorname{dom} \dot{X} \times \mathbb{P}$, it follows that $p \Vdash_s \dot{Y} \in \dot{B}$ if and only if $p \Vdash_s \dot{Y} \subseteq \dot{X}$. Hence,

$$\pi\overline{B}^{D_\pi} = \left\{ (\pi\overline{Y}^{D_\pi}, \pi\overline{p}) \mid \dot{Y} \in HS\,,\, \dot{Y} \subseteq \operatorname{dom} \dot{X} \times \mathbb{P}\,,\, \pi\overline{p} \in D_\pi\,,\, \pi\overline{p} \Vdash_s \pi\overline{Y}^{D_\pi} \subseteq \pi\overline{X}^{D_\pi} \right\}.$$

It remains to show that $\overline{B}^{D_\pi} = \pi\overline{B}^{D_\pi}$; then

$$\left\{ [\pi] \in \overline{A} \mid \pi\overline{B}^{D_\pi} = \overline{B}^{D_\pi} \right\} \supseteq \left\{ [\pi] \in \overline{A} \mid \pi\overline{X}^{D_\pi} = \overline{X}^{D_\pi} \right\} \in \mathcal{F}$$

as desired.

For the inclusion $\overline{B}^{D_\pi} \subseteq \pi\overline{B}^{D_\pi}$, consider $(\overline{Y}^{D_\pi}, \overline{p}) \in \overline{B}^{D_\pi}$ as above. It suffices to construct $\dot{Y}_0 \in HS$, $\dot{Y}_0 \subseteq \operatorname{dom} \dot{X} \times \mathbb{P}$ with $\pi\overline{Y_0}^{D_\pi} = \overline{Y}^{D_\pi}$. Then setting $\overline{p}_0 := \pi^{-1}\overline{p}$, it follows that $(\overline{Y}^{D_\pi}, \overline{p}) = (\pi\overline{Y_0}^{D_\pi}, \pi\overline{p}_0) \in \pi\overline{B}^{D_\pi}$, since $\overline{p} \Vdash_s \overline{Y}^{D_\pi} \subseteq \overline{X}^{D_\pi}$ and $\pi\overline{X}^{D_\pi} = \overline{X}^{D_\pi}$ gives

$\pi\overline{p}_0 \Vdash_s \pi\overline{Y_0}^{D_\pi} \subseteq \pi\overline{X}^{D_\pi}$.

Let
$$\dot{Y}_0 := \left\{ (\dot{z}, p) \mid \dot{z} \in \operatorname{dom} \dot{X}\,,\, p \in D_\pi\,,\, \pi\overline{z}^{D_\pi} \in \operatorname{dom} \overline{Y}^{D_\pi}\,,\, \pi p \Vdash_s \pi\overline{z}^{D_\pi} \in \overline{Y}^{D_\pi} \right\}.$$

Then
$$\pi\overline{Y_0}^{D_\pi} = \left\{ (\pi\overline{z}^{D_\pi}, \pi p) \middle| \, \dot{z} \in \operatorname{dom} \dot{X}\,,\, p \in D_\pi\,,\, \pi\overline{z}^{D_\pi} \in \operatorname{dom} \overline{Y}^{D_\pi}\,,\, p \Vdash_s \dot{z} \in \dot{Y}_0 \right\}.$$

We first show that whenever $\dot{z} \in \operatorname{dom} \dot{X}$, $p \in D_\pi$ and $\pi\overline{z}^{D_\pi} \in \operatorname{dom} \overline{Y}^{D_\pi}$ as above, then $p \Vdash_s \dot{z} \in \dot{Y}_0$ if and only if $\pi p \Vdash_s \pi\overline{z}^{D_\pi} \in \overline{Y}^{D_\pi}$.

"$\Leftarrow$": If $\pi p \Vdash_s \pi\overline{z}^{D_\pi} \in \overline{Y}^{D_\pi}$, it follows that $(\dot{z}, p) \in \dot{Y}_0$; hence, $p \Vdash_s \dot{z} \in \dot{Y}_0$ as desired.

"$\Rightarrow$": Now, assume $p \Vdash_s \dot{z} \in \dot{Y}_0$. Let H be a V-generic filter on $\mathbb{P}$ with $\pi p \in H$. We have to show that $(\pi\overline{z}^{D_\pi})^H \in (\overline{Y}^{D_\pi})^H$. Let $H' := \pi^{-1}H$. Then $(\pi\overline{z}^{D_\pi})^H = \dot{z}^{H'}$, and $p \in H'$. Hence, $\dot{z}^{H'} \in \dot{Y}_0^{H'}$ implies that there must be $(\dot{u}, r) \in \dot{Y}_0$ with $\dot{u}^{H'} = \dot{z}^{H'}$ and $r \in H'$. Then $r \in D_\pi$ and $\pi r \Vdash_s \pi\overline{u}^{D_\pi} \in \overline{Y}^{D_\pi}$ by construction of $\dot{Y}_0$. Since $\pi r \in H$, it follows that $(\pi\overline{u}^{D_\pi})^H \in (\overline{Y}^{D_\pi})^H$, with $(\pi\overline{u}^{D_\pi})^H = \dot{u}^{H'} = \dot{z}^{H'} = (\pi\overline{z}^{D_\pi})^H$ as desired.

Hence,
$$\pi\overline{Y_0}^{D_\pi} = \left\{ (\pi\overline{z}^{D_\pi}, \pi p) \middle| \, \dot{z} \in \operatorname{dom} \dot{X}\,,\, p \in D_\pi\,,\, \pi\overline{z}^{D_\pi} \in \operatorname{dom} \overline{Y}^{D_\pi}\,,\, \pi p \Vdash_s \pi\overline{z}^{D_\pi} \in \overline{Y}^{D_\pi} \right\}.$$

We have to make sure that $\pi\overline{Y_0}^{D_\pi} = \overline{Y}^{D_\pi}$. The inclusion $\pi\overline{Y_0}^{D_\pi} \subseteq \overline{Y}^{D_\pi}$ is clear. Regarding "$\supseteq$", consider $(\overline{u}^{D_\pi}, q) \in \overline{Y}^{D_\pi}$ with $\dot{u} \in \operatorname{dom} \dot{Y} \subseteq \operatorname{dom} \dot{X}$, and $q \in D_\pi$ such that $q \Vdash_s \dot{u} \in \dot{Y}$. From $\overline{u}^{D_\pi} \in \operatorname{dom} \overline{X}^{D_\pi} = \operatorname{dom} \pi\overline{X}^{D_\pi}$, it follows that there must be $\dot{v} \in \operatorname{dom} \dot{X}$ with $\overline{u}^{D_\pi} = \pi\overline{v}^{D_\pi}$. Let $r := \pi^{-1}q$. Then $(\overline{u}^{D_\pi}, q) = (\pi\overline{v}^{D_\pi}, \pi r) \in \pi\overline{Y_0}^{D_\pi}$, since $q \Vdash_s \dot{u} \in \dot{Y}$ implies that $\pi r \Vdash_s \pi\overline{v}^{D_\pi} \in \overline{Y}^{D_\pi}$ as desired.

Thus, we have constructed $\dot{Y}_0 \subseteq \operatorname{dom} \dot{X} \times \mathbb{P}$ with $\pi\overline{Y_0}^{D_\pi} = \overline{Y}^{D_\pi}$. It remains to make sure that $\dot{Y}_0 \in HS$. Firstly, $\dot{X} \in HS$ implies that $\operatorname{dom} \dot{Y}_0 \subseteq HS$. Secondly, for any $\sigma \in A$ with $\sigma\overline{Y}^{D_\sigma} = \overline{Y}^{D_\sigma}$, it follows for the concatenation $\nu := \pi^1\sigma\pi$ that

$$\nu\overline{Y_0}^{D_\nu} = \nu\overline{\overline{Y_0}^{D_\pi}}^{D_\nu} = \nu\,\overline{\pi^{-1}\overline{Y}^{D_\pi}}^{D_\nu},$$

and since $\sigma\overline{Y}^{D_\sigma} = \overline{Y}^{D_\sigma}$, one can easily check that

$$\nu\,\overline{\pi^{-1}\overline{Y}^{D_\pi}}^{D_\nu} = \overline{\pi^{-1}\overline{Y}^{D_\pi}}^{D_\nu},$$

and

$$\overline{\pi^{-1}\overline{Y}^{D_\pi}}^{D_\nu} = \overline{\overline{Y_0}^{D_\pi}}^{D_\nu} = \overline{Y_0}^{D_\nu}.$$

Since the name $\dot{Y}$ is symmetric, it follows by normality of $\mathcal{F}$ that $\dot{Y}_0$ is symmetric, as well. Hence, $\dot{Y}_0$ has all the desired properties; and it follows that $\overline{B}^{D_\pi} \subseteq \pi\overline{B}^{D_\pi}$.

The inclusion $\pi\overline{B}^{D_\pi} \subseteq \overline{B}^{D_\pi}$ is similar.

Replacement. Consider $a \in N$ such that $N \vDash \forall x \in a\ \exists y\ \varphi(x,y)$. We have to show that there is $b \in N$ with

$$N \vDash \forall x \in a\ \exists\, y \in b\ \varphi(x,y).$$

Let $a = \dot{a}^G$ with $\dot{a} \in HS$. We proceed like in the proof of *Replacement* in ordinary forcing extensions. For $\dot{x} \in \operatorname{dom} \dot{a}$ and $p \in \mathbb{P}$, let

$$\alpha(\dot{x},p) := \min\{\alpha \mid \exists\, \dot{w} \in \mathrm{Name}_\alpha(\mathbb{P}) \cap HS : p \Vdash_s (\varphi(\dot{x},\dot{w}) \wedge \dot{x} \in \dot{a})\}$$

if such α exists, and $\alpha(\dot{x},p) := 0$, else.
By *Replacement* in V, take $\beta \in \mathrm{Ord}$ with $\beta \geq \sup\{\alpha(\dot{x},p) \mid \dot{x} \in \operatorname{dom}\dot{a}\,,\ p \in \mathbb{P}\}$. Let

$$\dot{b} := \{(\dot{y},\mathbb{1}) \mid \dot{y} \in \mathrm{Name}_\beta(\mathbb{P}) \cap HS\},$$

and $b := \dot{b}^G$. Then for all $x \in a$, it follows that there exists $y \in b$ with $N \vDash \varphi(x,y)$. It remains to show that the name $\dot{b}$ is symmetric. Let $\pi \in A$. Then

$$\overline{b}^{D_\pi} = \{(\overline{y}^{D_\pi},q) \mid \dot{y} \in \mathrm{Name}_\beta(\mathbb{P}) \cap HS\ ,\ q \in D_\pi\},$$

and

$$\pi\overline{b}^{D_\pi} = \{(\pi\overline{y}^{D_\pi},\pi q) \mid \dot{y} \in \mathrm{Name}_\beta(\mathbb{P}) \cap HS\ ,\ q \in D_\pi\}.$$

We show that $\pi\overline{b}^{D_\pi} = \overline{b}^{D_\pi}$.

Since it is not possible to apply π to arbitrary $\mathbb{P}$-names $\dot{y}$ with $\dot{y} \notin \overline{\mathrm{Name}(\mathbb{P})}^{D_\pi}$, we construct an alternative $\widetilde{\pi}$ that is enough for our purposes here.
Recursively, we define for $\dot{y} \in \mathrm{Name}(\mathbb{P})$:

$$\widetilde{\pi}(\dot{y}) := \big\{\, (\widetilde{\pi}(\dot{z}),\pi\overline{q}) \mid \exists\,(\dot{z},q) \in \dot{y}\ ,\ \overline{q} \leq q\,,\ \overline{q} \in D_\pi \,\big\}.$$

Then $\mathsf{rk}_{\mathbb{P}}\,\dot{y} = \mathsf{rk}_{\mathbb{P}}\,\widetilde{\pi}(\dot{y})$.

Whenever H is a V-generic filter on $\mathbb{P}$, $H' := \pi^{-1}H$ and $\dot{y} \in \mathrm{Name}(\mathbb{P})$, it is not difficult to see that $(\widetilde{\pi}(\dot{y}))^H = \dot{y}^{H'}$, and

$$\pi\overline{y}^{D_\pi} = \overline{\widetilde{\pi}(\dot{y})}^{D_\pi}.$$

Moreover, one can show recursively that whenever $\dot{y} \in \mathrm{Name}(\mathbb{P})$ and $\sigma \in A$ with $\sigma\overline{y}^{D_\sigma} = \overline{y}^{D_\sigma}$, then

$$(\pi\sigma\pi^{-1})\,\overline{\widetilde{\pi}(\dot{y})}^{D_{\pi\sigma\pi^{-1}}} = \overline{\widetilde{\pi}(\dot{y})}^{D_{\pi\sigma\pi^{-1}}}.$$

Hence,

$$\Big\{\,[\tau] \in \overline{A}\ \Big|\ \tau\,\overline{\widetilde{\pi}(\dot{y})}^{D_\tau} = \overline{\widetilde{\pi}(\dot{y})}^{D_\tau}\,\Big\} \supseteq \Big\{\,[\pi][\sigma][\pi]^{-1}\ \Big|\ [\sigma] \in \overline{A}\,,\ \sigma\overline{y}^{D_\sigma} = \overline{y}^{D_\sigma}\,\Big\}.$$

In the case that $\dot{y}$ is symmetric, i.e. $\big\{\,[\sigma] \in \overline{A}\ \big|\ \sigma\overline{y}^{D_\sigma} = \overline{y}^{D_\sigma}\,\big\} \in \mathcal{F}$, it follows by normality that also $\Big\{\,[\tau] \in \overline{A}\ \Big|\ \tau\,\overline{\widetilde{\pi}(\dot{y})}^{D_\tau} = \overline{\widetilde{\pi}(\dot{y})}^{D_\tau}\,\Big\} \in \mathcal{F}$. Hence, $\widetilde{\pi}(\dot{y}) \in HS$ whenever $\dot{y} \in HS$.

Now, we can show that $\pi\overline{b}^{D_\pi} = \overline{b}^{D_\pi}$: For the inclusion "$\subseteq$", consider $(\pi\overline{y}^{D_\pi}, \pi q) \in \pi\overline{b}^{D_\pi}$ with $\dot{y} \in \mathrm{Name}_\beta(\mathbb{P}) \cap HS$, $q \in D_\pi$. Then also $\pi q \in D_\pi$, and $\pi\overline{y}^{D_\pi} = \overline{\widetilde{\pi}(\dot{y})}^{D_\pi}$, where $\widetilde{\pi}(\dot{y}) \in \mathrm{Name}_\beta(\mathbb{P}) \cap HS$; so $(\pi\overline{y}^{D_\pi}, \pi q) = (\overline{\widetilde{\pi}(\dot{y})}^{D_\pi}, \pi q) \in \overline{b}^{D_\pi}$ follows. The inclusion "$\supseteq$" is similar.

Hence, $\dot{b} \in HS$ as desired.

This finishes the proof of $V(G) \vDash \mathsf{ZF}$. □

The following proposition is an adaptation of [Kar14, Lemma 1] to symmetric forcing with partial automorphisms.

Proposition 1.2.22. *Let $\mathbb{P}$ be a countably closed forcing, A an almost-group of partial $\mathbb{P}$-automorphisms, $\overline{A}$ the group of partial $\mathbb{P}$-automorphisms derived from A, and $\mathcal{F}$ a countably complete filter on $\overline{A}$. Let G be a V-generic filter on $\mathbb{P}$. Then $V(G)^{\mathcal{F}} \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$.*

Proof. $V(G)^{\mathcal{F}} \vDash \mathsf{ZF}$ follows from Theorem 1.2.21. We now prove that for any set $X \in N$ and $f : \omega \to X$ a function in $V[G]$, it follows that $f \in N$. Then $N \vDash \mathsf{DC}$: Consider a nonempty set X in N with a binary relation R such that for all $x \in X$ there exists $y \in X$ with yRx. Then DC in $V[G]$ gives a sequence $(x_n \mid n < \omega)$ with the property that $x_{n+1} R x_n$ for all $n < \omega$; so $(x_n \mid n < \omega) \in N$ as desired.

Consider $X \in N$, $X = \dot{X}^G$ with $\dot{X} \in HS$. Let $f : \omega \to X$ denote a function in $V[G]$, $f = \dot{f}^G$ with $\dot{f} \in \mathrm{Name}^V(\mathbb{P})$. Take $\overline{p}_0 \in G$ with

$$\overline{p}_0 \Vdash^V_{\mathbb{P}} \dot{f} : \omega \to \dot{X}.$$

In particular, $\overline{p}_0$ forces the functionality of $\dot{f}$.

We claim that the following set is dense in $\mathbb{P}$ below $\overline{p}_0$:

$$D := \left\{ p \in \mathbb{P} \;\middle|\; \exists\, (\dot{x}_n \mid n < \omega) \;\; \forall\, n < \omega \; \left(\dot{x}_n \in \operatorname{dom} \dot{X} \wedge p \Vdash^V_{\mathbb{P}} \dot{f}(n) = \dot{x}_n\right) \right\}.$$

Let $p_0 \leq \overline{p}_0$. We work in $V[G]$ and construct sequences $(p_n \mid n < \omega)$ and $(\dot{x}_n \mid n < \omega)$ as follows: Assume inductively that $m < \omega$, and $(p_n \mid n \leq m)$, $(\dot{x}_n \mid n < m)$ are already constructed. Then pick $p_{m+1} \in \mathbb{P}$, $\dot{x}_m \in \operatorname{dom} \dot{X} \subseteq HS$ such that $p_{m+1} \leq p_m$ and $p_{m+1} \Vdash^V_{\mathbb{P}} \dot{f}(m) = \dot{x}_m$. It follows that $(p_n \mid n < \omega) \in V$, $(\dot{x}_n \mid n < \omega) \in V$, since $\mathbb{P}$ is countably closed. Hence, there exists $p \in \mathbb{P}$ with the property that $p \leq p_n$ for all $n < \omega$. Then p is an extension of p_0 in D; so D is dense in $\mathbb{P}$ below $\overline{p}_0$. Pick $p \in D \cap G$ and $(\dot{x}_n \mid n < \omega) \in V$ as in the Definition of D. We define a name for f as follows:

$$\dot{g} := \left\{ \left(\mathrm{OR}_{\mathbb{P}}(n, \dot{x}_n), \mathbb{1} \right) \;\middle|\; n < \omega \right\}.$$

Then $\dot{g}^G = f$ by definition of D and since $p \in G$. It remains to make sure that $\dot{g} \in HS$. Since $\dot{x}_n \in HS$ for all $n < \omega$, it suffices to show that

$$\mathrm{sym}^{\overline{A}}(\dot{g}) = \left\{ [\pi] \in \overline{A} \mid \pi\overline{g}^{D_\pi} = \overline{g}^{D_\pi} \right\} \in \mathcal{F}.$$

For any $n < \omega$ and $[\pi] \in \mathrm{sym}^{\overline{A}}(\dot{x}_n)$, it is not difficult to check that

$$\pi\,\overline{\mathrm{OR}_{\mathbb{P}}(n, \dot{x}_n)}^{D_\pi} = \overline{\mathrm{OR}_{\mathbb{P}}(n, \dot{x}_n)}^{D_\pi}.$$

Hence, whenever $[\pi] \in \bigcap_{n<\omega} \mathrm{sym}^{\overline{A}}(\dot{x}_n)$, it follows that $\pi\overline{g}^{D_\pi} = \overline{g}^{D_\pi}$. Now, $\dot{x}_n \in HS$ gives $\mathrm{sym}(\dot{x}_n) \in \mathcal{F}$ for every $n < \omega$; and since $\mathcal{F}$ is countably complete, it follows that $\bigcap_{n<\omega} \mathrm{sym}(\dot{x}_n) \in \mathcal{F}$. Hence,

$$\mathrm{sym}^{\overline{A}}(\dot{g}) \supseteq \bigcap_{n<\omega} \mathrm{sym}^{\overline{A}}(\dot{x}_n) \in \mathcal{F}.$$

is yields $f = \dot{g}^G \in N$, which finishes the proof of $N \vDash \mathsf{DC}$.

Regarding $N \vDash \mathsf{AX}_4$, note that $([\lambda]^{\aleph_0})^{V[G]} = ([\lambda]^{\aleph_0})^V$ by the countable closure of $\mathbb{P}$. The ZFC-model V contains a wellordering of $[\lambda]^{\aleph_0}$, i.e. a bijection $\mathsf{b} : [\lambda]^{\aleph_0} \to \alpha$ for some ordinal α. Then b is also a wellordering of $([\lambda]^{\aleph_0})^N = ([\lambda]^{\aleph_0})^V$ in N.

□

1.3 Class Forcing

In this chapter, we briefly review the basic properties of *class forcing*, i.e. we look at what happens if we drop the requirement on forcings that the partial order $(\mathbb{P}, \leq, \mathbb{1})$ is a set (cf. Definition 1.1.1):

Definition 1.3.1. A *class forcing* is a class $(\mathbb{P}, \leq, \mathbb{1})$ such that $(\mathbb{P}, \leq)$ is a preorder (the relation $\leq$ is transitive and reflexive on $\mathbb{P}$) with greatest element $\mathbb{1}$.

Class Forcing was first used by William B. Easton in [Eas70], who proved that in ZFC, the Continuum Function $\kappa \mapsto 2^\kappa$ can behave almost arbitrarily on the class of regular cardinals, as long it obeys the rules of weak monotonicity and König's Theorem. We will discuss *Easton forcing* in Chapter 1.3.2.

In contrast to set forcing, forcing with a proper class need not preserve the axioms of ZFC – for example, the partial order $\mathrm{Fn}(\omega, \mathrm{Ord}, \aleph_0) := \{p : \mathrm{dom}\, p \to \mathrm{Ord} \mid \mathrm{dom}\, p \subseteq \omega\,,\, |p| < \aleph_0\}$ adds a surjective function from ω into the ordinals, and thereby destroys the axiom of *Replacement*. Moreover, it is not difficult to write down a class-sized partial order that adds a proper class of Cohen reals and hence destroys *Power Set*.

We continue working in first-order set theory ZFC, where the classes of V are the definable ones, i.e. objects of the form $\{x \mid \varphi(x, x_0, \ldots, x_{n-1})\}$, where $\varphi \in \mathcal{L}_\in$ with finitely many parameters $x_0, \ldots, x_{n-1} \in V$. Thus, it is not possible to quantify over classes, which can be sidestepped by regarding statements of the form "*For every class forcing* $\mathbb{P}$ ..." as schemes. We will treat V-classes informally, but always take care that every statement can be described in the language $\mathcal{L}_\in$ (with additional predicates for the ground model V and the generic filter G where necessary).

1.3.1 The Forcing Theorem, Pretameness and Increasing Chains.

We refer to [Fri00, 2] and [BT97, p.5 - 12] for a detailed introduction to class forcing, and merely concentrate on some aspects important for us when constructing symmetric extensions by class-sized partial orders (cf. Chapter 1.4).

We start with introductory definitions and remarks regarding generic extensions by class forcing, before we turn to the *Forcing Theorem* (Definition 1.3.8). Unlike as with set forcing, the Forcing Theorem does not always hold for class forcing, but it can be traced back to the *definability lemma* for atomic formulae (see [Kra17]). We look at *pretameness* of class forcings, a necessary and sufficient condition for the generic extension to satisfy ZFC^-, and *tameness*, a necessary and sufficient condition for the generic extension to satisfy ZFC. In our applications, we will only consider fairly nice class forcings, namely those $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ that can be written as an increasing chain of set-sized subforcings with certain properties (Definition 1.3.15), which always satisfy the Forcing Theorem.

Later on, in Chapter 1.4, we will consider symmetric extensions by class forcing, where it can be the case that $V(G) \vDash \mathsf{ZF}$ although ZFC fails in $V[G]$.

Most of the definitions form Chapter 1.1.1 can be given verbatim, or by just replacing "set" by "class" where necessary.

Definition 1.3.2. Let $\mathbb{P}$ be a class forcing for V. A filter $G \subseteq \mathbb{P}$ is *V-generic on* $\mathbb{P}$ if for every $D \subseteq \mathbb{P}$ a dense class in V, it follows that $G \cap D = \varnothing$.

Since V is countable, there are only countably many dense classes of V. Thus, as in the case for set forcing, one can enumerate them from the "outside", and use a diagonalization argument to show:

Lemma 1.3.3. *Let $(\mathbb{P}, \leq, \mathbb{1})$ be a class forcing for V and $p \in \mathbb{P}$. Then there exists a V-generic filter G on $\mathbb{P}$ with $p \in G$.*

The class of all $\mathbb{P}$-names is defined recursively:

Definition 1.3.4. A $\mathbb{P}$-*name* is a set $\dot{x}$ such that every $y \in \dot{x}$ is of the form $y = (\dot{y}, p)$ with a $\mathbb{P}$-name $\dot{y}$ and $p \in \mathbb{P}$. We denote by $Name^V(\mathbb{P})$ the class of all $\mathbb{P}$-names for V.

The *rank function* on $\mathrm{Name}^V(\mathbb{P})$ is defined as usual:

$$\mathsf{rk}_{\mathbb{P}} \dot{x} := \sup\{\mathsf{rk}_{\mathbb{P}} \dot{y} + 1 \mid \dot{y} \in \mathrm{dom}\, \dot{x}\}.$$

For $\alpha \in \mathrm{Ord}$, we denote by $\mathrm{Name}^V_\alpha(\mathbb{P})$ the class of all $\dot{x} \in \mathrm{Name}^V(\mathbb{P})$ with $\mathsf{rk}_{\mathbb{P}} \dot{x} < \alpha$.

Definition 1.3.5. Let $(\mathbb{P}, \leq, \mathbb{1})$ be a class forcing for V, and G a V-generic filter on $\mathbb{P}$. We define recursively for $\dot{x} \in \mathrm{Name}^V(\mathbb{P})$:

$$\dot{x}^G := \{\dot{y}^G \mid \exists\, p \in G \ (\dot{y}, p) \in \dot{x}\}.$$

Then $V[G] := \{\dot{x}^G \mid \dot{x} \in \mathrm{Name}^V(\mathbb{P})\}$ is the *generic extension of V by G.*

As for set forcing, it follows that $V[G]$ is a transitive class with $V \subseteq V[G]$ and $\mathrm{Ord}^{V[G]} = \mathrm{Ord}^V$.

When the ground model V is clear from the context, we write just $\mathrm{Name}(\mathbb{P})$.

For $\varphi(v_0, \ldots, v_{n-1}) \in \mathcal{L}_\in$ a formula of set theory, $p \in \mathbb{P}$ and $\dot{x}_0, \ldots, \dot{x}_{n-1} \in \mathrm{Name}(\mathbb{P})$, the forcing relation $p \Vdash_{\mathbb{P}}^V \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ can be defined as for set forcing (cf. Definition 1.1.10).

We will work with the structure $\langle V[G], \in, V, G\rangle$, where we have predicate symbols for the ground model and the generic filter.

We proceed as in [Git80, 4] and extend out language of set theory $\mathcal{L}_\in$ by unary predicate symbols A and B, where $A(x)$ will assert that $x \in V$, and $B(x)$ will assert that x is in the generic filter G. We denote this extended language by $\mathcal{L}_\in^{A,B}$.

Definition 1.3.6. For $p \in \mathbb{P}$, we define:

- $p \Vdash_{\mathbb{P}}^V A(\dot{x})$ iff $\forall q \leq p \; \exists r \leq q \; \exists a \; (r \Vdash_{\mathbb{P}}^V \dot{x} = \check{a})$
- $p \Vdash_{\mathbb{P}}^V B(\dot{x})$ iff $\forall q \leq p \; \exists r \leq q \; \exists s \in \mathbb{P} \; : \; ((r \Vdash_{\mathbb{P}}^V \dot{x} = \check{s}) \wedge r \leq s)$.

Moreover:

- $V[G] \vDash A(x)$ iff $x \in V$
- $V[G] \vDash B(x)$ iff $x \in G$.

Informally, the forcing relation can be defined as usual:

Definition 1.3.7. For a formula $\varphi(v_0, \ldots, v_{n-1}) \in \mathcal{L}_\in^{A,B}$, a condition $p \in \mathbb{P}$ and $\dot{x}_0, \ldots, \dot{x}_{n-1} \in \mathrm{Name}(\mathbb{P})$, we write

$$p \Vdash_{\mathbb{P}}^V \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$$

if for any G a V-generic filter on $\mathbb{P}$ with $p \in G$, it follows that $\varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ holds in the structure $\langle V[G], \in, V, G\rangle$.

We will abuse notation and do *not* write A and B in our formulas, but keep in mind that inside the structure $V[G]$, formulas φ can talk about V and G. We write $\varphi(x_0, \ldots, x_{n-1}, V, G)$ where these predicates are important.

Behind the forcing symbol $\Vdash_{\mathbb{P}}^V$, we will write "$p \Vdash_{\mathbb{P}}^V \dot{x} \in \check{V}$" instead of "$p \Vdash_{\mathbb{P}}^V A(\dot{x})$" (which corresponds to introducing the class name $\check{V} := \{(\check{a}, \mathbb{1}) \mid a \in V\}$), and "$p \Vdash_{\mathbb{P}}^V \dot{x} \in \dot{G}$" for "$p \Vdash_{\mathbb{P}}^V B(\dot{x})$"(which corresponds to introducing the class name $\dot{G} := \{(\check{p}, p) \mid p \in \mathbb{P}\}$).
We write $p \Vdash_{\mathbb{P}}^V \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}, \check{V}, \dot{G})$ when we need to mention the predicates $\check{V}$ and $\dot{G}$ behind the forcing symbol.

The forcing relation for class forcing satisfies most of the basic properties as the ordinary forcing relation for set forcing (see Proposition 1.1.12, (1) - (8)), and the *Symmetry Lemma* holds true, as well.

It is not difficult to see that whenever $p \Vdash_{\mathbb{P}}^{V} \dot{x} \in \check{V}$ and $\pi : D_\pi \to D_\pi$ is a partial $\mathbb{P}$-automorphism with $p \in D_\pi$, then

$$\pi p \Vdash_{\mathbb{P}}^{V} \pi \overline{x}^{D_\pi} \in \check{V}$$

(which corresponds to the fact that $\check{V}$ regarded as a class name, satisfies $\pi \overline{\check{V}}^{D_\pi} = \overline{\check{V}}^{D_\pi}$ for all $\pi : D_\pi \to D_\pi$). Moreover, from $p \Vdash_{\mathbb{P}}^{V} \dot{x} \in \dot{G}$ and $\pi : D_\pi \to D_\pi$ a partial $\mathbb{P}$-automorphism with $p \in D_\pi$, it follows that

$$\pi p \Vdash_{\mathbb{P}}^{V} \pi \overline{x}^{D_\pi} \in \pi \dot{G},$$

where $\pi \dot{G}$ is the canonical name for $\pi^{-1} G$:

$$p \Vdash_{\mathbb{P}}^{V} \dot{y} \in \pi \dot{G} \ \text{ iff } \ \forall q \le p \ \exists r \le q \ \exists s \in \mathbb{P} \ : \ ((r \Vdash_{\mathbb{P}}^{V} \dot{y} = \check{s}) \wedge \pi^{-1} r \le s).$$

However, unlike as with set forcing, the *Forcing Theorem* does not always hold for class forcing.

Definition 1.3.8. Let $\varphi \equiv \varphi(v_0, \dots, v_{n-1})$ be an $\mathcal{L}_{\in}^{A,B}$-formula.

- We say that $\mathbb{P}$ *satisfies the definability lemma for φ over V* if

  $$\left\{ (p, \dot{x}_0, \dots, \dot{x}_{n-1}) \mid p \in \mathbb{P}, \dot{x}_0, \dots, \dot{x}_{n-1} \in \mathrm{Name}(\mathbb{P}), p \Vdash_{\mathbb{P}}^{V} \varphi(\dot{x}_0, \dots, \dot{x}_{n-1}) \right\}$$

  is definable in V.

- We say that $\mathbb{P}$ *satisfies the truth lemma for φ over V* if for all $\dot{x}_0, \dots, \dot{x}_{n-1} \in \mathrm{Name}(\mathbb{P})$ and G a V-generic filter on $\mathbb{P}$ with

  $$\langle V[G], \in, V, G \rangle \vDash \varphi(\dot{x}_0^G, \dots, \dot{x}_{n-1}^G).$$

  it follows that there exists $p \in G$ with

  $$p \Vdash_{\mathbb{P}}^{V} \varphi(\dot{x}_0, \dots, \dot{x}_{n-1}).$$

- We say that $\mathbb{P}$ *satisfies the Forcing Theorem for φ over V* if $\mathbb{P}$ satifies the definability lemma and the truth lemma for φ over V.

We say that $\mathbb{P}$ *satisfies the Forcing Theorem (over V)* if $\mathbb{P}$ satisfies the Forcing Theorem for all $\mathcal{L}_{\in}^{A,B}$-formulas φ (over V).

We remark that any generic extension $V[G]$ by class forcing satisfies all single axioms of ZFC (i.e. all axioms of ZFC except for possibly instances pf *Power Set*, *Separation* and *Replacement*, cf. Chapter 0.4); with *Union* replaced by *Weak Union* (see [Kra17, 1.2.9]): For any $x \in V[G]$, there exists a set $y \in V[G]$ with $\bigcup x \subseteq y$.

In set forcing, the axioms of *Separation* and *Replacement* can be established using the Forcing Theorem. For class forcing, however, we need a stronger property:

Definition 1.3.9 ([Fri00, p.33])**.** A class forcing $(\mathbb{P}, \le, \mathbb{1})$ is *pretame* if for every $p \in \mathbb{P}$ and $(D_i \mid i \in \mathcal{I})$ a definable sequence of dense classes, there exists $q \le p$ and a sequence $(d_i \mid i \in \mathcal{I}) \in V$ such that for all $i \in \mathcal{I}$ it follows that $d_i \subseteq D_i$ and d_i is predense below q.

It is not difficult to see that any ZFC^--preserving class forcing has to be pretame (see [Fri00, 2.17]).

On the other hand:

Proposition 1.3.10 ([Fri00, 2.19]). *Assume that the class forcing* $(\mathbb{P}, \leq, \mathbb{1})$ *is pretame and satisfies the Forcing Theorem. Then* $\mathbb{P}$ *preserves* ZFC^-.

Proposition 1.3.11 ([Fri00, 2.18]). *If the class forcing* $(\mathbb{P}, \leq, \mathbb{1})$ *is pretame, then it satisfies the Forcing Theorem.*

Thus, it follows that a class forcing is pretame if and only if it preserves ZFC^-.

For the preservation of *Power Set*, one needs a stronger notion: *tameness.*

Definition 1.3.12 ([Fri10, p.9]). A class forcing $(\mathbb{P}, \leq, \mathbb{1})$ is *tame* if $\mathbb{P}$ is pretame, and $\mathbb{1} \Vdash_{\mathbb{P}}^{V}$ *Power Set.*

For pretame forcings, tameness can be described by a combinatorial property of the partial order using *predense partitions* (see [Fri00, p. 36]).

A class forcing $(\mathbb{P}, \leq, \mathbb{1})$ is tame if and only if it preserves ZFC.

In Chapter 3, we will construct a symmetric extension by a class-sized partial order $(\mathbb{P}, \leq, \mathbb{1})$. Even if a class forcing $\mathbb{P}$ is not pretame, one can sometimes arrange that the according symmetric extension is nevertheless a model of ZF. It is crucial, however, that the Forcing Theorem holds.

By the following theorem, it suffices to check the *definability lemma* for the atomic formulae:

Theorem 1.3.13 ([Kra17, 2.1.5]). *If the class forcing* $(\mathbb{P}, \leq, \mathbb{1})$ *satisfies the definability lemma over* V *either for "*$v_0 \in v_1$*" or "*$v_0 = v_1$*", then* $\mathbb{P}$ *satisfies the Forcing Theorem over* V *for every* $\mathcal{L}_{\in}^{A,B}$*-formula* φ.

In the case that the Forcing Theorem holds, there is also a product lemma for class forcings:

Lemma 1.3.14 ([Fri00, 2.27]). *Suppose that* $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}})$ *and* $(\mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$ *are class forcings.*

(i) If G *is* $\mathbb{P}$*-generic over* V*, and* H *is* $\mathbb{Q}$*-generic over* $\langle V[G], \in, V, G\rangle$*, then* $G \times H$ *is a* $\mathbb{P} \times \mathbb{Q}$ *- generic filter over* V*.*
(ii) Let K *denote a* V*-generic filter on* $\mathbb{P} \times \mathbb{Q}$*. Then* K *is of the form* $K = G \times H$*, where* G *is a* $\mathbb{P}$*-generic filter over* V*. If in addition, the Forcing Theorem holds for* $\mathbb{P}$*, then* H *is a* $\mathbb{Q}$*-generic filter over* $\langle V[G], \in, V, G\rangle$*.*

Our class forcing that we construct in Chapter 3 will satisfy the property that it can be written as the union of a sequence of set-sized forcings, each of which is a complete subforcing of those beyond:

Definition 1.3.15. A class forcing $(\mathbb{P}, \leq, \mathbb{1})$ is an *increasing chain of set-sized complete subforcings* if there is a class $((\alpha, \mathbb{P}_\alpha) \mid \alpha \in \mathrm{Ord})$ such that $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$, and each $\mathbb{P}_\alpha = (\mathbb{P}_\alpha, \leq_\alpha, \mathbb{1}_\alpha) = (\mathbb{P}_\alpha, \leq \restriction \mathbb{P}_\alpha, \mathbb{1})$ is a set forcing with the property that for all α, $\beta \in \mathrm{Ord}$ with $\alpha < \beta$, it follows that $\mathbb{P}_\alpha$ is a complete subforcing of $\mathbb{P}_\beta$.

The following properties can be found in [Sho71, 12], [Zar73, 3], and in [Rei06] in a more modern fashion using Boolean Algebras:

Assume that $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ is an increasing chain of set-sized complete subforcings. Then any $\mathbb{P}_\alpha$ is a complete subforcing of $\mathbb{P}$. Let G denote a V-generic filter on $\mathbb{P}$. Then for every $\alpha \in \mathrm{Ord}$, it follows that $G_\alpha := G \cap \mathbb{P}_\alpha$ is a V-generic filter on $\mathbb{P}_\alpha$. We define a rank function $\Delta(\dot{x})$ recursively on $\mathrm{Name}(\mathbb{P})$ as follows: Let $\Delta(\dot{x})$ be the smallest α such that for all $(\dot{y}, p) \in \dot{x}$, it follows that $\Delta(\dot{y}) \leq \alpha$ and $p \in \mathbb{P}_\alpha$.
Whenever $\dot{x} \in \mathrm{Name}(\mathbb{P})$ with $\Delta(\dot{x}) \leq \alpha$, then $\dot{x} \in \mathrm{Name}(\mathbb{P}_\alpha)$ and $\dot{x}^G = \dot{x}^{G_\alpha}$. Hence, it follows that $V[G_\alpha] \subseteq V[G_\beta]$ whenever $\alpha < \beta$, and $V[G] = \bigcup_{\alpha \in \mathrm{Ord}} V[G_\alpha]$.

The following theorem is proved in [Sho71, 12] and [Zar73, 3]:

Theorem 1.3.16. *If the class forcing $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ is an increasing chain of set-sized complete subforcings, then $\mathbb{P}$ satisfies the Forcing Theorem for every $\mathcal{L}_\in^{A,B}$-formula φ.*

The basic idea of the proof is that for $\dot{x}$, $\dot{y} \in \mathrm{Name}(\mathbb{P})$ with $\Delta(\dot{x})$, $\Delta(\dot{y}) \leq \alpha$ and $p \in \mathbb{P}_\alpha$, the forcing relations $p \Vdash_{\mathbb{P}}^V \dot{x} \in \dot{y}$ and $p \Vdash_{\mathbb{P}}^V \dot{x} = \dot{y}$ for $\mathbb{P}$ can be defined via $p \Vdash_{\mathbb{P}_\alpha}^V \dot{x} \in \dot{y}$ and $p \Vdash_{\mathbb{P}_\alpha}^V \dot{x} = \dot{y}$, so the *definability lemma* for set forcing yields the *definability lemma* for the atomic formulas "$v_0 \in v_1$" and "$v_0 = v_1$". Then Theorem 1.3.13 can be applied.

Another useful property is that for $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ an increasing chain of set-sized complete subforcings as above, the interpretation function of names $\dot{x} \mapsto \dot{x}^G$ is definable in any $\mathbb{P}$-generic extension $V[G]$. This is not necessarily true for arbitrary class forcing, since the recursive definition of $(\cdot)^G$ makes use of *Replacement*, which might fail in $V[G]$.

Proposition 1.3.17 ([Git80]). *Assume that the class forcing $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ is an increasing chain of set-sized complete subforcings, and let G be a V-generic filter on $\mathbb{P}$. Then there is a formula $\tau(u, v)$ such that for any $\dot{x}$, $x \in V[G]$, we have $\langle V[G], \in, V, G \rangle \vDash \tau(\dot{x}, x)$ if and only if $\dot{x} \in \mathrm{Name}^V(\mathbb{P})$ with $x = \dot{x}^G$.*

Proof. First, we construct a function f in $\langle V[G], \in, V, G \rangle$ such that

$$\mathrm{dom}\, f = \{(\alpha, \dot{x}) \mid \alpha \in \mathrm{Ord}\,,\ \dot{x} \in \mathrm{Name}^V(\mathbb{P}_\alpha)\},$$

and for all $(\alpha, \dot{x}) \in \mathrm{dom}\, f$, it follows that $f(\alpha, \dot{x}) = \dot{x}^{G_\alpha}$ $(= \dot{x}^G)$. Note that we cannot apply the recursion theorem directly, since it makes use of *Replacement*.

However, we can still define in $\langle V[G], \in, V, G \rangle$:

$f(\alpha, \dot{x}) = x$ iff

(∗) $\alpha \in \mathrm{Ord}$, $\dot{x} \in \mathrm{Name}(\mathbb{P}_\alpha)$, and there exists $F \in V[G]$, $F : \mathrm{dom}\, F \to V[G]$ such that $\mathrm{dom}\, F \subseteq \mathrm{Name}^V(\mathbb{P}_\alpha)$, $\mathrm{dom}\, F$ is *dom-transitive* (see p. 17), $F(\dot{x}) = x$ and for every $\dot{y} \in \mathrm{dom}\, F$ it follows that

$$\forall\, z\ \big(z \in F(\dot{y}) \leftrightarrow \exists\, (\dot{z}, p) \in \dot{y}\ (p \in G_\alpha \wedge z = F(\dot{z}))\big).$$

Then as in the proof of the recursion theorem in ZFC, it follows that this definition indeed yields a function (one considers a counterexample of least rank and obtains a contradiction), and for all $\alpha \in \mathrm{Ord}$, the set $\{\dot{x} \mid (\alpha, \dot{x}) \in \operatorname{dom} f\}$ is dom-transitive. However, in order to show that $(\alpha, \dot{x}) \in \operatorname{dom} f$ for all $\alpha \in \mathrm{Ord}$, the original argument cannot be employed here, since it needs the axiom of *Replacement*. Instead, we use that the interpretation function $(\cdot)^{G_\alpha}$ can be defined inside the ZFC-model $V[G_\alpha]$.

Let $\dot{x} \in \mathrm{Name}(\mathbb{P}_\alpha)$, and assume recursively that for all $\dot{y} \in \operatorname{Tdom} \dot{x}$ (cf. p. 17), we have $\dot{y} \in \operatorname{dom} f$ with $f(\dot{y}) = \dot{y}^G$. Let F denote the function that maps any $\dot{y} \in \operatorname{Tdom} \dot{x} \cup \{\dot{x}\}$ to its interpretation $\dot{y}^{G_\alpha}$. Then $F \in V[G_\alpha] \subseteq V[G]$, $\operatorname{dom} F = \{\dot{x}\} \cup \operatorname{Tdom} \dot{x}$ is dom-transitive, and for any $\dot{y} \in \operatorname{dom} F$, we have $F(\dot{y}) = \dot{y}^{G_\alpha} = \{\dot{z}^{G_\alpha} \mid \exists\, (\dot{z}, p) \in \dot{y}\ \ p \in G_\alpha\}$. In other words,

$$\forall\, z\ \big(z \in F(\dot{y}) \leftrightarrow \exists (\dot{z}, p) \in \dot{y}\ (p \in G_\alpha \wedge z = F(\dot{z}))\big).$$

It follows that F satisfies all the requirements from $(*)$. Hence, $(\alpha, \dot{x}) \in \operatorname{dom} f$ with $f(\alpha, \dot{x}) = F(\dot{x}) = \dot{x}^{G_\alpha} = \dot{x}^G$ as desired.

It follows that $(*)$ defines in $\langle V[G], \in, V, G\rangle$ a function f on $\{(\alpha, \dot{x}) \mid \alpha \in \mathrm{Ord}\,, \dot{x} \in \mathrm{Name}^V(\mathbb{P}_\alpha)\}$ with $f(\alpha, \dot{x}) = \dot{x}^{G_\alpha} = \dot{x}^G$ for all $(\alpha, \dot{x}) \in \operatorname{dom} f$.

Hence, there is a formula $\tau_0(u, v, w)$ such that $\langle V[G], \in, V, G\rangle \vDash \tau_0(\alpha, \dot{x}, x)$ iff

$$\langle V[G], \in, V, G\rangle \vDash \alpha \in \mathrm{Ord} \wedge \dot{x} \in \mathrm{Name}^V(\mathbb{P}_\alpha) \wedge f(\alpha, \dot{x}) = x.$$

Moreover, there is a formula $\tau(v, w)$ with $\langle V[G], \in, V, G\rangle \vDash \tau(\dot{x}, x)$ iff

$$\langle V[G], \in, V, G\rangle \Vdash \big(\dot{x} \in \mathrm{Name}^V(\mathbb{P}) \wedge \forall\, \alpha \in \mathrm{Ord}\ (\dot{x} \in \mathrm{Name}(\mathbb{P}_\alpha) \to \tau_0(\alpha, \dot{x}, x))\big).$$

Since $\dot{x}^{G_\beta} = \dot{x}^{G_\alpha} = \dot{x}^G$ whenever $\dot{x} \in \mathrm{Name}(\mathbb{P}_\alpha)$ and $\beta \geq \alpha$, it follows that $\langle V[G], \in, V, G\rangle \vDash \tau(\dot{x}, x)$ iff $\dot{x} \in \mathrm{Name}^V(\mathbb{P})$ with $x = \dot{x}^G$. $\square$

We will now introduce *class products*. For $\mathcal{D}$ a class of ordinals and $(\mathbb{Q}_\beta \mid \beta \in \mathcal{D})$ a definable sequence of set forcings, a product

$$\mathbb{P} = \prod_{\beta \in \mathcal{D}}^{\mathcal{I}} \mathbb{Q}_\beta$$

is always an increasing chain of set-sized complete subforcings.

Definition 1.3.18 ([Rei06, 122])**.** Let $\mathcal{D}$ be a class of ordinals and $\mathcal{I}$ a *sub-ideal on* $\mathcal{D}$, i.e. $\mathcal{I}$ is a class consisting of sets of ordinals $X \subseteq \mathcal{D}$ such that $\varnothing \in \mathcal{I}$, $\{\beta\} \in \mathcal{I}$ for all $\beta \in \mathcal{D}$, $\mathcal{I}$ is closed under finite unions, and whenever $X \in \mathcal{I}$ and $\beta \in \mathrm{Ord}$, then also $X \cap \beta \in \mathcal{I}$ (cf. [Rei06, 105 + 113]). Let $\big(\, (\mathbb{Q}_\beta, \leq_\beta, \mathbb{1}_\beta) \mid \beta \in \mathcal{D}\,\big)$ be a class such that each $\mathbb{Q}_\beta$ is a set forcing. The *product of* $\big(\, (\mathbb{Q}_\beta, \leq_\beta, \mathbb{1}_\beta) \mid \beta \in \mathcal{D}\,\big)$ *with supports in* $\mathcal{I}$

$$\mathbb{P} := \prod_{\beta \in \mathcal{D}}^{\mathcal{I}} \mathbb{Q}_\beta$$

consists of all $p : \operatorname{dom} p \to V$ with the property that $\operatorname{dom} p \in \mathcal{I}$ and $p(\beta) \in \mathbb{Q}_\beta$ for all $\beta \in \operatorname{dom} p$, with maximal element $\mathbb{1} := \varnothing$, and the ordering $\leq_{\mathbb{P}}$ defined by setting $q \leq p$ iff

$\operatorname{dom} q \supseteq \operatorname{dom} p$ and $q(\beta) \leq_\beta p(\beta)$ for all $\beta \in \operatorname{dom} p$.

If G is a V-generic filter on $\mathbb{P}$, then for every $\beta \in \mathcal{D}$ it follows that $G_\beta := \{p(\beta) \mid p \in G\,,\, \beta \in \operatorname{dom} p\}$, the *projection of G onto $\mathbb{Q}_\beta$*, is a V-generic filter on $\mathbb{Q}_\beta$.

In applications, $\mathcal{D}$ is for example the class of all cardinals or the class of all regular cardinals.

Lemma 1.3.19 ([Rei06, 123])**.** *Let $(\mathbb{P}, \leq, \mathbb{1})$ be the product of $\big((\mathbb{Q}_\beta, \leq_\beta, \mathbb{1}_\beta) \mid \beta \in \mathcal{D} \big)$ with supports in $\mathcal{I}$ as in Definition 1.3.18. For $\alpha \in \mathrm{Ord}$, let $\mathbb{P}_\alpha := \{p \in \mathbb{P} \mid \operatorname{dom} p \subseteq \alpha\}$, with maximal element $\mathbb{1}_\alpha := \mathbb{1} = \varnothing$ and the ordering $\leq_\alpha$ inherited from $\mathbb{P}$. Then $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ is an increasing chain of set-sized complete subforcings.*

Proof. Let γ, $\delta \in \mathrm{Ord}$ with $\gamma \leq \delta$. We have to show that $\mathbb{P}_\gamma$ is a complete subforcing of $\mathbb{P}_\delta$. Consider p, $q \in \mathbb{P}_\gamma$. Clearly, $q \leq_\gamma p$ if and only if $q \leq_\delta p$; and $q \perp_\gamma p$ if and only if $q \perp_\delta p$, since whenever $r \in \mathbb{P}_\delta$ is a common extension of p and q, then $\overline{r} := r \restriction \gamma$ is a common extension of p and q in $\mathbb{P}_\gamma$.
Let now $A \subseteq \mathbb{P}_\gamma$ be a maximal antichain in $\mathbb{P}_\gamma$. Consider $p \in \mathbb{P}_\delta$, and take $r \in A$ with $r \parallel_\gamma p \restriction \gamma$. Let $q \in \mathbb{P}_\gamma$ with $q \leq_\gamma r$, $q \leq_\gamma p \restriction \gamma$. Then the condition $\overline{q} \in \mathbb{P}_\delta$, defined by setting $\overline{q}(\beta) := q(\beta)$ for $\beta < \gamma$, and $q(\beta) := p(\beta)$ for $\gamma \leq \beta < \delta$ is a common extension of r and p in $\mathbb{P}_\delta$. Hence, the antichain A is also maximal in $\mathbb{P}_\delta$; and we conclude that indeed, $\mathbb{P}_\gamma$ is a complete subforcing of $\mathbb{P}_\delta$.

□

Definition 1.3.20. Fix a definable sequence of set forcings $\big((\mathbb{Q}_\beta, \leq_\beta, \mathbb{1}_\beta) \mid \beta \in \mathcal{D} \big)$ as above.
Let κ be a regular cardinal. If $\mathcal{I}$ is the class of all sets $X \subseteq \mathcal{D}$ of cardinality $< \kappa$, we obtain the *κ-product* (or *product with $< \kappa$-support*)

$$\mathbb{P} = \prod_{\beta \in \mathcal{D}}^{<\kappa} \mathbb{Q}_\beta,$$

which is the collection of all $p : \operatorname{dom} p \to V$ with the property that $\operatorname{dom} p \subseteq \mathcal{D}$ with $|\operatorname{dom} p| < \kappa$, and $p(\beta) \in \mathbb{Q}_\beta$ for all $\beta \in \operatorname{dom} p$. The $\aleph_1$-product is usually referred to as *product with countable support.*

If $\mathcal{I}$ is the class of all finite subsets of $\mathcal{D}$, we obtain the *product with finite support*,

$$\mathbb{P} = \prod_{\beta \in \mathrm{Ord}}^{\mathrm{fin}} \mathbb{Q}_\beta,$$

which is the collection of all $p : \operatorname{dom} p \to V$ such that $\operatorname{dom} p$ is a finite subset of $\mathcal{D}$, and $p(\beta) \in \mathbb{Q}_\beta$ for all $\beta \in \operatorname{dom} p$.

Finally, if $\mathcal{I}$ is the class of all sets $X \subseteq \mathcal{D}$ with the property that for all inaccessible cardinals γ it follows that $|X \cap \gamma| < \gamma$, we obtain the *product with Easton support*

$$\mathbb{P} = \prod_{\beta \in \mathcal{D}}^{\mathrm{Easton}} \mathbb{Q}_\beta,$$

which is the class of all $p : \operatorname{dom} p \to V$ such that $\operatorname{dom} p \subseteq \mathcal{D}$ with $|\operatorname{dom} p \cap \gamma| < \gamma$ for all inaccessible γ, and $p(\beta) \in \mathbb{Q}_\beta$ for all $\beta \in \mathcal{D}$.

If $\mathcal{D}$ consists of cardinals and GCH holds, then Easton support is equivalent to requiring $|\operatorname{dom} p \cap \gamma| < \gamma$ for all regular cardinals γ.

1.3.2 Easton Forcing

In this chapter, we discuss *Easton Forcing* as an example of tame class forcing. Introduced by William Easton in [Eas70], it was used to prove that the Continuum Function on the class of all regular cardinals can take almost arbitrary values:

Theorem 1.3.21 (William B. Easton)**.** *Let V be a ground model of ZFC + GCH with a class function F whose domain consists of regular cardinals and whose range consists of cardinals, such that for all κ, $\lambda \in \operatorname{dom} F$ the following properties holds:*

- $\kappa \leq \lambda \to F(\kappa) \leq F(\lambda)$ *(weak monotonicity),*
- $\operatorname{cf} F(\kappa) > \kappa$ *(König's Theorem).*

Then there exists a generic extension $V[G]$ by class forcing such $V[G] \vDash \mathsf{ZFC}$, V and $V[G]$ have the same cardinals and cofinalities, and $V[G] \vDash 2^\kappa = F(\kappa)$ holds for all $\kappa \in \operatorname{dom} F$.

We again remark that a similar construction is not possible for singular cardinals.

Our proof of Easton's Theorem follows [Jec06, 15.18]. We start from a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with an *Easton function* $F : \operatorname{dom} F \to \mathrm{Card}$, which is a class function with the following properties:

(i) any $\kappa \in \operatorname{dom} F$ is a regular cardinal,
(ii) $\operatorname{cf} F(\kappa) > \kappa$ for all $\kappa \in \operatorname{dom} F$,
(iii) κ, $\lambda \in \operatorname{dom} F$ with $\kappa < \lambda \to F(\kappa) \leq F(\lambda)$.

The corresponding *Easton forcing for F* is the *Easton support product* of the Cohen forcings $\mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa)$:

Definition 1.3.22. For $\kappa \in \operatorname{dom} F$, we denote by $\mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa)$ the set of all function $q : \operatorname{dom} q \to 2$ with $\operatorname{dom} q \subseteq F(\kappa) \times \kappa$ and $|q| < \kappa$.

The *Easton forcing for F* is the product with Easton support

$$\mathbb{P}_F := \prod_{\kappa \in \operatorname{dom} F}^{\text{Easton}} \mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa),$$

which is the class of all $p : \operatorname{supp} p \to 2$ where $\operatorname{supp} p \subseteq \operatorname{dom} F$ a set such that for all regular cardinals γ, it follows that $|\operatorname{supp} p \cap \gamma| < \gamma$, and $p(\kappa) \in \mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa)$ for all $\kappa \in \operatorname{supp} p$.

For p, $q \in \mathbb{P}_F$, we set $q \leq p$ iff $\operatorname{supp} q \supseteq \operatorname{supp} p$ with $q(\kappa) \supseteq p(\kappa)$ for all $\kappa \in \operatorname{supp} p$; and $\mathbb{1}_F := \varnothing$.

Let G be a V-generic filter on $\mathbb{P}_F$. For every $\kappa \in \operatorname{dom} F$ and $i < F(\kappa)$, it induces

$$G_i^\kappa := \{\alpha < \kappa \mid \exists p \in G\ p(\kappa)(i, \alpha) = 1\}.$$

By genericity, it follows that $G_i^\kappa \neq G_j^\kappa$ whenever $i \neq j$; thus, the forcing $\mathbb{P}_F$ indeed adds $F(\kappa)$-many new κ-subsets for every $\kappa \in \operatorname{dom} F$. Hence, $V[G] \vDash 2^\kappa \geq F(\kappa)$. It remains to show that $V[G] \vDash \mathsf{ZFC}$, that cardinals and cofinalities are absolute between V and $V[G]$, and that $V[G] \vDash 2^\kappa \leq F(\kappa)$ for all $\kappa \in \operatorname{dom} F$.

For γ a regular cardinal and $p \in \mathbb{P}_F$, we consider the following decomposition:

$$p^{\leq\gamma} := p \upharpoonright (\gamma + 1) \quad , \quad p^{>\gamma} := p \upharpoonright (\mathrm{Ord} \setminus (\gamma + 1)).$$

Then $p = p^{\leq\gamma} \cup p^{>\gamma}$ for all $p \in \mathbb{P}_F$ and γ regular.

Let

$$\mathbb{P}_F^{\leq\gamma} := \{p^{\leq\gamma} \mid p \in \mathbb{P}_F\} \quad , \quad \mathbb{P}_F^{>\gamma} := \{p^{>\gamma} \mid p \in \mathbb{P}_F\}.$$

Then $\mathbb{P}_F$ is isomorphic to the product $\mathbb{P}_F^{\leq\gamma} \times \mathbb{P}_F^{>\gamma}$.

Lemma 1.3.23 ([Jec06, 15.18]). *For every regular cardinal γ, the forcing $\mathbb{P}_F^{>\gamma}$ is $\leq \gamma$-closed.*

Proof. Let $(p^i \mid i < \gamma)$ be a descending sequence in $\mathbb{P}_F^{>\gamma}$, i.e. $p^j \leq p^i$ whenever $i < j$. We define a condition p as follows: Let $\operatorname{supp} p := \bigcup_{i<\gamma} \operatorname{supp} p^i \subseteq \mathrm{Ord} \setminus (\gamma+1)$; and for $\kappa \in \operatorname{supp} p$, let $p(\kappa) := \bigcup_{i<\gamma} p^i(\kappa)$, with $p^i(\kappa) := \varnothing$ in the case that $\kappa \notin \operatorname{supp} p^i$.
By compatibility of the p^i, it follows that any $p(\kappa)$ is a function $p(\kappa) : \operatorname{dom} p(\kappa) \to 2$ with $\operatorname{dom} p(\kappa) \subseteq F(\kappa) \times \kappa$. Moreover, for any $\kappa \in \operatorname{supp} p$, we have $\kappa > \gamma$ and κ is regular; hence, $|\operatorname{supp} p(\kappa)| = |\bigcup_{i<\gamma} \operatorname{supp} p^i(\kappa)| < \kappa$, which implies $p(\kappa) \in \mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa)$. Finally, for any λ a regular cardinal with $\lambda > \gamma$, it follows that $|\operatorname{supp} p \cap \lambda| = |\bigcup_{i<\gamma} (\operatorname{supp} p^i \cap \lambda)| < \lambda$. Hence, $p \in \mathbb{P}_F^{>\gamma}$ is a common extension of $(p^i \mid i < \gamma)$. □

Moreover, since GCH holds, an application of the Δ-system lemma yields:

Lemma 1.3.24 ([Jec06, 15.17 + 15.18]). *For every regular cardinal γ, the set forcing $\mathbb{P}_F^{\leq\gamma}$ satisfies the γ^+-cc.*

By Lemma 1.3.19 it follows that $\mathbb{P}_F = \bigcup_{\gamma \in \mathrm{Reg}} \mathbb{P}_F^{\leq\gamma}$ is an increasing chain of set-sized complete subforcings. Hence, $\mathbb{P}_F$ satisfies the Forcing Theorem (see Theorem 1.3.16), and $V[G] = \bigcup_{\gamma \in \mathrm{Reg}} V[G^{\leq\gamma}]$.

Now, Lemma 1.3.14 yields for every $\gamma \in \mathrm{Reg}^V$ that $G \cong G^{\leq\gamma} \times G^{>\gamma}$, where $G^{\leq\gamma} := \{p^{\leq\gamma} \mid p \in G\}$ is a $\mathbb{P}_F^{\leq\gamma}$-generic filter over V, and $G^{>\gamma} := \{p^{>\gamma} \mid p \in G\}$ is a $\mathbb{P}_F^{>\gamma}$-generic filter over $\langle V[G^{\leq\gamma}], \in, V, G^{\leq\gamma}\rangle$.

Moreover,

Proposition 1.3.25 ([Fri00, 2.26]). *Easton forcing $\mathbb{P}_F$ is pretame.*

Hence, it follows that $\mathbb{P}_F$ preserves ZFC^-. Regarding the preservation of *Power Set*, it is not difficult to see that Lemma 1.1.35 remains true when the second factor $\mathbb{Q}$ is a class forcing. Thus,

Lemma 1.3.26 ([Jec06, p. 236]). *Let γ be a regular cardinal in V. Then every function $f : \gamma \to V$ in $V[G]$ is already contained in $V[G^{\leq\gamma}]$. In particular,*

$$\wp^{V[G]}(\gamma) = \wp^{V[G^{\leq\gamma}]}(\gamma).$$

Let now $X \in V[G]$. Since $V[G] \vDash \mathsf{ZFC}^-$, there must be a cardinal $\gamma \in \mathrm{Reg}^V$ with an injection $\iota : X \hookrightarrow \gamma$ in $V[G]$. Now, $\wp^{V[G]}(X) = \{\iota^{-1}[y] \mid y \in \wp^{V[G]}(\gamma)\}$, and $\wp^{V[G]}(\gamma) = \wp^{V[G^{\leq\gamma}]}(\gamma)$ is a set in $V[G^{\leq\gamma}] \subseteq V[G]$, since $V[G^{\leq\gamma}] \vDash \mathsf{ZFC}$. Hence, $\wp^{V[G]}(X) \in V[G]$; and we conclude that $V[G] \vDash$ *Power Set.*

From the factorization $\mathbb{P}_F \cong \mathbb{P}_F^{\leq\gamma} \times \mathbb{P}_F^{>\gamma}$, we also obtain the preservation of cardinals and cofinalities:

Lemma 1.3.27 ([Jec06, 15.18]). *Any $\kappa \in \mathrm{Reg}^V$ is still a a regular cardinal in $V[G]$.*

Proof. Assume towards a contradiction there was $\gamma < \kappa$, $\gamma \in \mathrm{Reg}^V$, with a cofinal function $f : \gamma \to \kappa$ in $V[G]$. Then by Lemma 1.3.26, it follows that $f \in V[G^{\leq\gamma}]$; so κ is not regular in $V[G^{\leq\gamma}]$. But this is not possible, since $\mathbb{P}_F^{\leq\gamma}$ satifies the γ^+-chain condition. □

Thus, all cardinals and cofinalities are preserved by $\mathbb{P}_F$.

Proposition 1.3.28 ([Jec06, 15.18]). *For any $\lambda \in \mathrm{dom}\, F$, it follows that $(2^\lambda)^{V[G]} = F(\lambda)$.*

Proof. We have already argued that $(2^\lambda)^{V[G]} \geq F(\lambda)$, since Easton forcing adds $F(\lambda)$-many new λ-subsets.

In order to show that $(2^\lambda)^{V[G]} \leq F(\lambda)$, first note that by Lemma 1.3.26,

$$(2^\lambda)^{V[G]} = |\wp^{V[G]}(\lambda)|^{V[G]} = |\wp^{V[G^{\leq\lambda}]}(\lambda)|^{V[G]} \leq |\wp^{V[G^{\leq\lambda}]}(\lambda)|^{V[G^{\leq\lambda}]} = (2^\lambda)^{V[G^{\leq\lambda}]}.$$

Now, $(2^\lambda)^{V[G^{\leq\lambda}]}$ can be computed as in Lemma 1.1.23; and now it is important that the function F meets the requirements from König's Theorem:

For any regular $\kappa \leq \lambda$ with $\kappa \in \mathrm{dom}\, F$, it follows that the forcing notion $\mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa)$ has cardinality $\leq F(\kappa) \leq F(\lambda)$, since $V \vDash \mathsf{GCH}$, and $\mathrm{cf}\, F(\kappa) > \kappa$. Thus, $|\mathbb{P}_F^{\leq\lambda}| \leq F(\lambda)^{<\lambda} = F(\lambda)$. Since $\mathbb{P}_F^{\leq\lambda}$ has the λ^+-c.c. by Lemma 1.3.24, it follows that there are $\leq F(\lambda)^\lambda = F(\lambda)$-many antichains in $\mathbb{P}_F^{\leq\lambda}$. Hence, $|\mathrm{Nice}(\mathbb{P}_F^{\leq\lambda}, \lambda)| = F(\lambda)^\lambda = F(\lambda)$, which implies $(2^\lambda)^{V[G^{\leq\lambda}]} \leq F(\lambda)$ as desired.

□

This finishes the proof of Easton's Theorem.

1.4 Symmetric Extensions by Class Forcing

In this chapter we extend our technique of symmetric forcing with partial automorphisms introduced in Chapter 1.2.3 to class-sized forcing notions $\mathbb{P}$. We proceed similarly as in Chapter 1.2.3, but for working with proper classes a measure of extra care is needed.

We confine ourselves to the case that the class forcing $\mathbb{P}$ has a *nice hierarchy*: We demand that $\mathbb{P} = \bigcup_\alpha \mathbb{P}_\alpha$ should be an increasing chain of set-sized complete subforcings with projections $\rho_\alpha : \mathbb{P} \to \mathbb{P}_\alpha$, with certain properties as listed below. Any partial automorphism for $\mathbb{P}$ that we consider, say $\pi : D_\pi \to D_\pi$, will be fairly "set-like": There will $\alpha \in \mathrm{Ord}$ such that π can be constructed from an automorphism $\pi_\alpha : D_\pi \cap \mathbb{P}_\alpha \to D_\pi \cap \mathbb{P}_\alpha$ as follows: Any $p \in D_\pi$ is first projected down to $\mathbb{P}_\alpha$ via ρ_α, then the map π_α is employed to $\rho_\alpha(p)$, and then $\pi_\alpha(\rho_\alpha(p)) \in \mathbb{P}_\alpha$ is "glued together" with the "upper part" of p that is not taken into account for $\rho_\alpha(p)$. In order to formalize this "upper part", we demand that there is a definable sequence $(\mathbb{P}_{[\alpha,\infty)} \mid \alpha \in \mathrm{Ord})$ of class-sized forcing notions with a definable sequence of projections $(\rho_{[\alpha,\infty)} : \mathbb{P} \to \mathbb{P}_{[\alpha,\infty)} \mid \alpha \in \mathrm{Ord})$; and for every $\alpha \in \mathrm{Ord}$, there is a canonical isomorphism from $\mathbb{P}$ into a dense subforcing of $\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$. Then every $p \in \mathbb{P}$ can be viewed as a pair $(\rho_\alpha(p), \rho_{[\alpha,\infty)}(p))$.

We demand that the maps ρ_α and $\rho_{[\alpha,\infty)}$ have several natural properties that one expects from "cutting off" and "gluing together", see Definition 1.4.2.

Examples 1.4.1. (1) In Chapter 3, we will apply this idea to a finite support product

$$\mathbb{P} = \prod_{\kappa \in \mathrm{Card}}^{\mathrm{fin}} \mathbb{Q}_\kappa$$

of Cohen-like forcing notions $\mathbb{Q}_\kappa$. Setting $\mathbb{P}_\alpha := \prod^{\mathrm{fin}}_{\kappa<\alpha} \mathbb{Q}_\kappa$ and $\mathbb{P}_{[\alpha,\infty)} := \prod^{\mathrm{fin}}_{\kappa\in[\alpha,\infty)} \mathbb{Q}_\kappa$, it follows that $\mathbb{P} = \bigcup_{\alpha\in\mathrm{Ord}} \mathbb{P}_\alpha$ is an increasing chain of set-sized complete forcing notions; and we have projections $\rho_\alpha : \mathbb{P} \to \mathbb{P}_\alpha$, $p \mapsto p \upharpoonright \alpha$, and $\rho_{[\alpha,\infty)} : \mathbb{P} \to \mathbb{P}_{[\alpha,\infty)}$, $p \mapsto p \upharpoonright (\mathrm{Card} \smallsetminus \alpha)$. Then for every $\alpha \in \mathrm{Ord}$, it follows that any $p \in \mathbb{P}$ can be viewed as a pair $(\rho_\alpha(p), \rho_{[\alpha,\infty)}(p))$, and $\mathbb{P} \cong \mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$.

(2) Also in Chapter 3, will also employ this construction to forcing with partial functions on finitary trees (i.e. in this case, trees with finitely many maximal points) the levels of which are indexed by cardinals. Then $\rho_\alpha(p)$ is the lower part of the tree up to level α (including level α itself), and $\rho_{[\alpha,\infty)}(p)$ is the upper part of the tree (level α and higher). In this case, $\mathbb{P}$ is not isomorphic to the product $\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$ but only to a dense subforcing, since conditions in $\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$ might have additional "roots" at level α.

Definition 1.4.2. A class forcing $\mathbb{P}$ *has a nice hierarchy* if the following hold:

a) $\mathbb{P} = \bigcup_{\alpha\in\mathrm{Ord}} \mathbb{P}_\alpha$ is an increasing chain of set-sized complete subforcings, each of which is *upwards closed in* $\mathbb{P}$, i.e. for any $p \in \mathbb{P}_\alpha$ and $q \in \mathbb{P}$ with $q \geq p$, it follows that $q \in \mathbb{P}_\alpha$, as well. There are projections $(\rho_\alpha \mid \alpha \in \mathrm{Ord})$, $\rho_\alpha : \mathbb{P} \to \mathbb{P}_\alpha$, such that $\{(\alpha, p, \rho_\alpha(p)) \mid \alpha \in \mathrm{Ord}, p \in \mathbb{P}\}$ is a class in V, and for all α, $\beta \in \mathrm{Ord}$, $\alpha \leq \beta$, the following properties hold:

(i) $\forall\, p, q \in \mathbb{P}\ (p \leq q \to \rho_\alpha(p) \leq \rho_\alpha(q))$,
(ii) $\forall\, p \in \mathbb{P}_\alpha\ \ \rho_\alpha(p) = p$ (in particular, $\rho_\alpha(\mathbb{1}) = \mathbb{1} = \mathbb{1}_\alpha$),
(iii) $\forall\, p \in \mathbb{P}\ \ \rho_\alpha(\rho_\beta(p)) = \rho_\alpha(p)$,
(iv) $\forall\, p \in \mathbb{P}, q \in \mathbb{P}_\alpha \big(q \leq_\alpha \rho_\alpha(p) \to \exists\, p' \leq p\ \rho_\alpha(p') \leq_\alpha q \big)$,

(v) $\forall p \in \mathbb{P}\ p \leq \rho_\alpha(p)$.

b) There is a definable sequence of class-sized forcing notions $(\mathbb{P}_{[\alpha,\infty)} \mid \alpha \in \mathrm{Ord})$ with projections $(\rho_{[\alpha,\infty)} \mid \alpha \in \mathrm{Ord})$, $\rho_{[\alpha,\infty)} : \mathbb{P} \to \mathbb{P}_{[\alpha,\infty)}$, i.e. $\{(\alpha, p) \mid \alpha \in \mathrm{Ord}\,, p \in \mathbb{P}_{[\alpha,\infty)}\}$ is a class in V and $\{(\alpha, p, \rho_{[\alpha,\infty)}(p)) \mid \alpha \in \mathrm{Ord}\,, p \in \mathbb{P}\}$ is a class in V.
For every $\alpha \in \mathrm{Ord}$, the map $b_\alpha : \mathbb{P} \to \mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$ defined by $b_\alpha(p) := (\rho_\alpha(p), \rho_{[\alpha,\infty)}(p))$, is an isomorphism from $\mathbb{P}$ into a dense subforcing of $\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$.

For notational convenience, we will often identify $p \in \mathbb{P}$ with its image $b_\alpha(p) = (\rho_\alpha(p), \rho_{[\alpha,\infty)}(p)) \in \mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$.

We define projections $\overline{\rho}_\alpha : \mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)} \to \mathbb{P}_\alpha$ and $\overline{\rho}_{[\alpha,\infty)} : \mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)} \to \mathbb{P}_{[\alpha,\infty)}$ for $\alpha \in \mathrm{Ord}$ by setting $\overline{\rho}_\alpha(p_\alpha, q_{[\alpha,\infty)}) := p_\alpha$ and $\overline{\rho}_{[\alpha,\infty)}(p_\alpha, q_{[\alpha,\infty)}) := q_{[\alpha,\infty)}$ for $(p_\alpha, q_{[\alpha,\infty)}) \in \mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$. Then for every $p \in \mathbb{P}$, it follows that $\overline{\rho}_\alpha(b_\alpha(p)) = \rho_\alpha(p)$, and $\overline{\rho}_{[\alpha,\infty)}(b_\alpha(p)) = \rho_{[\alpha,\infty)}(p)$. We will often mix up ρ_α with $\overline{\rho}_\alpha$, and $\rho_{[\alpha,\infty)}$ with $\overline{\rho}_{[\alpha,\infty)}$.

c) Regarding the projections $(\rho_{[\alpha,\infty)} \mid \alpha \in \mathrm{Ord})$, we require for all p, $q \in \mathbb{P}$ and α, $\beta \in \mathrm{Ord}$ with $\beta < \alpha$:

(i) $p \leq q \to \rho_{[\alpha,\infty)}(p) \leq \rho_{[\alpha,\infty)}(q)$,
(ii) $p \in \mathbb{P}_\beta \to \rho_{[\alpha,\infty)}(p) = \mathbb{1}_{[\alpha,\infty)}$ (in particular, $\rho_{[\alpha,\infty)}(\mathbb{1}) = \mathbb{1}_{[\alpha,\infty)}$),
(iii) for every $q_{[\alpha,\infty)} \in \mathbb{P}_{[\alpha,\infty)}$ with $q_{[\alpha,\infty)} \leq_{[\alpha,\infty)} \rho_{[\alpha,\infty)}(p)$, there exists $p' \leq p$ with $\rho_{[\alpha,\infty)}(p') \leq_{[\alpha,\infty)} q_{[\alpha,\infty)}$.

d) Regarding the interplay of the maps ρ_α and $\rho_{[\alpha,\infty)}$, we require for all $\alpha \leq \beta$:

Let $p_\alpha \in \mathbb{P}_\alpha$, and $q \in \mathbb{P}$ such that $(p_\alpha, \rho_{[\alpha,\infty)}(q)) \in \mathbb{P}$. Then

(i) $\rho_\beta\big(p_\alpha, \rho_{[\alpha,\infty)}(q)\big) = \big(p_\alpha, \rho_{[\alpha,\infty)}(\rho_\beta(q))\big)$
(ii) $\big(\rho_\beta(p_\alpha, \rho_{[\alpha,\infty)}(q)), \rho_{[\beta,\infty)}(q)\big) = \big(p_\alpha, \rho_{[\alpha,\infty)}(q)\big)$.

For $p \in \mathbb{P}$, we call $\Delta(p) := \min\{\alpha \in \mathrm{Ord} \mid p \in \mathbb{P}_\alpha\}$ the *height of* p. Whenever $p \leq q$, then by the upwards closure of the $\mathbb{P}_\alpha$, it follows that $\Delta(p) \geq \Delta(q)$.

For a name $\dot{x} \in \mathrm{Name}(\mathbb{P})$, we define recursively:

$$\Delta(\dot{x}) := \sup\{\Delta(\dot{y}) \mid \dot{y} \in \mathrm{dom}\,\dot{x}\} \cup \sup\{\Delta(p) \mid p \in \mathrm{rg}\,\dot{x}\}.$$

Then $\Delta(\dot{x})$ is the smallest α such that for all $(\dot{y}, p) \in \dot{x}$, it follows that $\Delta(\dot{y}) \leq \alpha$ and $p \in \mathbb{P}_\alpha$.

For the rest of this chapter, let $\mathbb{P}$ denote a separative class forcing with a nice hierarchy. By Theorem 1.3.16 it follows that the Forcing Theorem holds. In particular, the forcing relation is definable in V.

We will now describe what type of dense classes D and partial $\mathbb{P}$-automorphisms $\pi : D \to D$ we will consider.

Definition 1.4.3. • A dense class $D \subseteq \mathbb{P}$ *allows projections* if for any $p \in D$ and $\alpha \in \mathrm{Ord}$, it follows that $\rho_\alpha(p) \in D$, as well.

- A dense class $D \subseteq \mathbb{P}$ *can be described below* α if for every $p \in \mathbb{P}$, it follows that $p \in D$ if and only if $\rho_\alpha(p) \in D \cap \mathbb{P}_\alpha$.
- An automorphism $\pi : D \to D$ on a dense class $D \subseteq \mathbb{P}$ *can be described below* α if D can be described below α, and there exists an automorphism $\pi_\alpha : D \cap \mathbb{P}_\alpha \to D \cap \mathbb{P}_\alpha$ such that for every $p \in D$,
$$\pi(p) = \big(\pi_\alpha(\rho_\alpha(p)), \rho_{[\alpha,\infty)}(p)\big).$$
We write $\pi = \overline{\pi_\alpha}$.
- An automorphism $\pi : D \to D$ on a dense class $D \subset \mathbb{P}$ *is nicely level-preserving*, if D allows projections, and for all $\beta \in \mathrm{Ord}$ and $p \in D$, it follows that $\pi(\rho_\beta(p)) = \rho_\beta(\pi(p))$. In particular, $\Delta(\pi(p)) = \Delta(p)$ for all $p \in D$.

Whenever $D \subseteq \mathbb{P}$ is a dense class that allows projections, then for any $\alpha \in \mathrm{Ord}$, it follows that $D \cap \mathbb{P}_\alpha$ is dense in $\mathbb{P}_\alpha$.

Example 1.4.4. Let $\mathbb{P} = \prod^{\text{fin}}_{\kappa \in \mathrm{Ord}} \mathbb{Q}_\kappa$ with $\mathbb{P}_\alpha := \prod^{\text{fin}}_{\kappa < \alpha} \mathbb{Q}_\kappa$ as in Example 1.4.1 (1). For φ a formula of set theory, and S a class of parameters in V, the dense classes considered could be of the form

$$D_{\alpha,s} = \{p \in \mathbb{P} \mid \forall\, \kappa \in \operatorname{dom} p \cap \alpha \;\; \varphi(p(\kappa), s)\}$$

for $\alpha \in \mathrm{Ord}$ and a parameter $s \in S$. Then for any $p \in D_{\alpha,s}$ and $\overline{\alpha} \in \mathrm{Ord}$, it follows that also $\rho_{\overline{\alpha}}(p) = p \restriction \overline{\alpha} \in D_{\alpha,s}$. Hence, $D_{\alpha,s}$ allows projections. Moreover, for any $\overline{\alpha} \geq \alpha$, we have $p \in D_{\alpha,s}$ if and only if $\rho_{\overline{\alpha}}(p) \in D_{\alpha,s}$. Thus, $D_{\alpha,s}$ can be described below α.

The automorphisms π could be of the form $\pi : D_{\alpha,s} \to D_{\alpha,s}$ for some $D_{\alpha,s}$ as above with $\pi = (\pi(\kappa) \mid \kappa < \alpha)$ such that each $\pi(\kappa)$ is a partial automorphism on $\mathbb{Q}_\kappa$; and whenever $p \in D_{\alpha,s}$, then $\operatorname{dom}(\pi p) := \operatorname{dom} p$ with $\pi p(\kappa) = \pi(\kappa)\big(p(\kappa)\big)$ for all $\kappa < \alpha$, and $\pi p(\kappa) = p(\kappa)$ for all $\kappa \geq \alpha$. Then π can be described below α, and π is nicely level-preserving.

In this setting that $\pi : D \to D$ can be described below α with $\pi = \overline{\pi_\alpha}$, we will sometimes abuse notation, confuse $\pi = \overline{\pi_\alpha}$ with π_α, and treat π as a set.
Similarly, if a dense class $D \subseteq \mathbb{P}$ can be described below α, then membership to D can be reduced to membership to $D \cap \mathbb{P}_\alpha$, and we will again sometimes abuse notation, confuse D with $D \cap \mathbb{P}_\alpha$, and treat D as a set.

We continue with two lemmas about basic properties that follow from Definition 1.4.2 and 1.4.3:

Lemma 1.4.5. *Assume that $\pi : D \to D$ can be described below α with $\pi = \overline{\pi_\alpha}$, and assume that for all $p \in \mathbb{P}_\alpha$ and $\beta \in \mathrm{Ord}$, it follows that $\pi_\alpha(\rho_\beta(p)) = \rho_\beta(\pi_\alpha(p))$. Then π is nicely level-preserving.*

Proof. First, consider $\beta < \alpha$ and let $p \in D$. Then

$$\begin{aligned} \pi(\rho_\beta(p)) &= \big(\pi_\alpha(\rho_\alpha(\rho_\beta(p)))\,,\, \rho_{[\alpha,\infty)}(\rho_\beta(p))\big) \\ &= \big(\pi_\alpha(\rho_\beta(p))\,,\, \mathbb{1}_{[\alpha,\infty)}\big) \ \ \text{(by 1.4.2 a) + c))} \\ &= \pi_\alpha(\rho_\beta(p)). \end{aligned}$$

On the other hand,

$$\begin{aligned}\rho_\beta(\pi(p)) &= \rho_\beta\big(\pi_\alpha(\rho_\alpha(p))\,,\,\rho_{[\alpha,\infty)}(p)\big)\\ &= (\rho_\beta\circ\rho_\alpha)\big(\pi_\alpha(\rho_\alpha(p)),\rho_{[\alpha,\infty)}(p)\big)\ \text{(by 1.4.2 a))}\\ &= \rho_\beta\big(\pi_\alpha(\rho_\alpha(p))\big).\end{aligned}$$

Now, $\rho_\beta(\pi_\alpha(\rho_\alpha(p))) = \pi_\alpha(\rho_\beta(\rho_\alpha(p)))$ by our assumption on π_α, and $\pi_\alpha(\rho_\beta(\rho_\alpha(p))) = \pi_\alpha(\rho_\beta(p))$ by 1.4.2 a). This finishes the proof for $\beta < \alpha$.

In the case that $\beta \geq \alpha$, we have

$$\begin{aligned}\pi(\rho_\beta(p)) &= \big(\pi_\alpha(\rho_\alpha(\rho_\beta(p))),\rho_{[\alpha,\infty)}(\rho_\beta(p))\big)\\ &= \big(\pi_\alpha(\rho_\alpha(p)),\rho_{[\alpha,\infty)}(\rho_\beta(p))\big)\ \text{(by 1.4.2 a))}\\ &= \rho_\beta\big(\pi_\alpha(\rho_\alpha(p)),\rho_{[\alpha,\infty)}(p)\big)\ \text{(by 1.4.2 d))}\\ &= \rho_\beta(\pi(p)).\end{aligned}$$

□

Lemma 1.4.6. *If $\pi : D \to D$ can be described below α and $\alpha < \beta$, then π can also be described below β.*

Proof. Let $\pi = \overline{\pi_\alpha}$ as above with $\pi_\alpha : D \cap \mathbb{P}_\alpha \overset{\cong}{\to} D \cap \mathbb{P}_\alpha$. Then

$$\pi(p) = (\pi_\alpha(\rho_\alpha(p)),\rho_{[\alpha,\infty)}(p))$$

for all $p \in D$. We define a map π_β as follows: For $p \in D \cap \mathbb{P}_\beta$, set

$$\pi_\beta(p) := \rho_\beta\big(\pi_\alpha(\rho_\alpha(p)),\rho_{[\alpha,\infty)}(p)\big) = \rho_\beta(\pi(p)).$$

Then

$$\begin{aligned}\big(\pi_\beta(\rho_\beta(p)),\rho_{[\beta,\infty)}(p)\big) &= \Big(\rho_\beta\big(\pi_\alpha(\rho_\alpha(\rho_\beta(p))),\rho_{[\alpha,\infty)}(\rho_\beta(p))\big)\,,\,\rho_{[\beta,\infty)}(p)\Big)\\ &= \Big(\rho_\beta\big(\pi_\alpha(\rho_\alpha(p)),\rho_{[\alpha,\infty)}(\rho_\beta(p))\big)\,,\,\rho_{[\beta,\infty)}(p)\Big)\ \text{by 1.4.2 a)}\\ &= \Big(\rho_\beta\big(\pi_\alpha(\rho_\alpha(p)),\rho_{[\alpha,\infty)}(p)\big)\,,\,\rho_{[\beta,\infty)}(p)\Big)\ \text{by 1.4.2 d)}\\ &= \big(\pi_\alpha(\rho_\alpha(p)),\rho_{[\alpha,\infty)}(p)\big)\ \text{by 1.4.2 d)}\\ &= \pi(p)\end{aligned}$$

for all $p \in D$.

Whenever $p \in D \cap \mathbb{P}_\beta$, it follows that $\pi_\beta(p) = \pi(p)$; hence, $\pi_\beta : D \cap \mathbb{P}_\beta \to D \cap \mathbb{P}_\beta$ is indeed an automorphism.

□

Now, we adapt our Definition 1.2.14 of an *almost-group of partial $\mathbb{P}$-automorphisms* to class forcing. We try to avoid introducing $\mathcal{D}$ (which would be a collection of classes), and therefore assume that all dense classes D considered are given by the same formula $\varphi(x,y)$, with parameters ranging over a class S. In other words: Any dense class D we are considering is of the form $D = D_s = \{p \in \mathbb{P} \mid \varphi(p,s)\}$ for some $s \in S$.
If necessary, one could also allow finitely many formulas $\varphi_0(x,y),\dots,\varphi_{n-1}(x,y)$ with parameters in $S_0,\dots,S_{n-1}$, respectively.

Definition 1.4.7. Let $\varphi(x,y)$ be a formula of set theory and S a class in V. A class A is an *almost-group of partial $\mathbb{P}$-automorphisms for φ and S* if the following hold:

a) For every $s \in S$, the class $D_s := \{p \in \mathbb{P} \mid \varphi(p,s)\}$ is dense in $\mathbb{P}$, and there exists $\alpha \in \mathrm{Ord}$ such that D_s can be described below α. The smallest such α will be denoted by $\alpha(s)$.
b) For every s_0, $s_1 \in S$, there exists $s_2 \in S$ with $\alpha(s_2) \leq \max\{\alpha(s_0), \alpha(s_1)\}$ such that $D_{s_0} \cap D_{s_1} = D_{s_2}$, i.e. $\{p \in \mathbb{P} \mid \varphi(p,s_0) \wedge \varphi(p,s_1)\} = \{p \in \mathbb{P} \mid \varphi(p,s_2)\}$.
c) There exists $s_{\max} \in S$ with $D_s \subseteq D_{s_{\max}}$ for all $s \in S$.
d) Every $\pi \in A$ is a nicely level-preserving automorphism $\pi : D_s \xrightarrow{\cong} D_s$ for some $s \in S$, and there exists $\alpha \in \mathrm{Ord}$ such that π can be described below α, i.e. there exists an automorphism $\pi_\alpha : D_s \cap \mathbb{P}_\alpha \to D_s \cap \mathbb{P}_\alpha$ with $\pi = \overline{\pi_\alpha}$.
e) For every $s \in S$ and $\alpha \in \mathrm{Ord}$ with $\alpha \geq \alpha(s)$,
$$A_{(s,\alpha)} := \left\{\pi = \overline{\pi_\alpha} \in A \;\middle|\; \pi_\alpha : D_s \cap \mathbb{P}_\alpha \xrightarrow{\cong} D_s \cap \mathbb{P}_\alpha \right\}$$
is a group.
f) Whenever s, $s' \in S$ with $D_s \subseteq D_{s'}$, then for every $\pi \in A_{(s',\alpha')}$ with $\alpha' \geq \alpha(s')$, it follows that $\pi[D_s] = D_s$; and $\pi \restriction D_s \in A_{(s,\alpha)}$ for every $\alpha \geq \max\{\alpha', \alpha(s)\}$.

   (Note that for every $\pi \in A_{(s',\alpha')}$ with $\pi = \overline{\pi_{\alpha'}}$ for some $\pi_{\alpha'} : D_{s'} \cap \mathbb{P}_{\alpha'} \xrightarrow{\cong} D_{s'} \cap \mathbb{P}_{\alpha'}$ and $D_s \subseteq D_{s'}$ as above, it follows that $\pi \restriction D_s = \overline{\pi_{\alpha'} \restriction D_s}$. If $\alpha \geq \max\{\alpha', \alpha(s)\}$, it follows automatically that $\pi \restriction D_s$ can also be described below α: Setting $\pi_\alpha(p) := \rho_\alpha\big(\pi_{\alpha'}(\rho_{\alpha'}(p)), \rho_{[\alpha',\infty)}(p)\big)$, we obtain that $\pi_\alpha \restriction D_s : D_s \cap \mathbb{P}_\alpha \xrightarrow{\cong} D_s \cap \mathbb{P}_\alpha$ is an automorphism satisfying $\pi \restriction D_s = \overline{\pi_\alpha \restriction D_s}$.)

Let $\varphi(x,y)$ be a formula of set theory, S a class in V, and A an almost-group for φ and S. Then A can be can be turned into a group if we use a construction similar to the *direct limit*, but keep in mind that we are working with proper classes:

Whenever s, $s' \in S$ and α, $\alpha' \in \mathrm{Ord}$ with $\alpha \geq \alpha(s)$, $\alpha' \geq \alpha(s')$ such that $D_s \subseteq D_{s'}$ and $\alpha \geq \alpha'$, there is a canonical homomorphism $\phi_{(s',\alpha')(s,\alpha)} : A_{(s',\alpha')} \to A_{(s,\alpha)}$, $\pi \mapsto \pi \restriction D_s$. More precisely: A map $\pi \in A_{(s',\alpha)}$, $\pi = \overline{\pi_{\alpha'}}$ is mapped to $\overline{\pi_\alpha}$, where $\pi_\alpha : D_s \cap \mathbb{P}_\alpha \xrightarrow{\cong} D_s \cap \mathbb{P}_\alpha$ is defined by setting $\pi_\alpha(p) := \rho_\alpha\big(\pi_{\alpha'}(\rho_{\alpha'}(p)), \rho_{[\alpha',\infty)}(p)\big)$ for all $p \in D_s \cap \mathbb{P}_\alpha$.

This gives a directed system
$$(A_{(s,\alpha)}, \phi_{(s',\alpha')(s,\alpha)})$$
for $s, s' \in S$, $\alpha \geq \alpha(s)$, $\alpha' \geq \alpha(s')$; and $D_s \subseteq D_{s'}$, $\alpha \geq \alpha'$.

We cannot straightforwardly take the direct limit, since there is a proper class of indices. However, *Scott's Trick* can be applied as follows:

We consider the following equivalence relation "$\sim$" on $\bigcup\{A_{(s,\alpha)} \mid s \in S\,,\, \alpha \geq \alpha(s)\}$: Whenever $\pi \in A_{(s,\alpha)}$ and $\pi' \in A_{(s',\alpha')}$, let $\pi \sim \pi'$ iff there exists $s'' \in S$ with $D_{s''} \subseteq D_s \cap D_{s'}$ and $\alpha'' \geq \alpha, \alpha'$ such that $\phi_{(s,\alpha)(s'',\alpha'')}(\pi) = \phi_{(s',\alpha')(s'',\alpha'')}(\pi')$. It is not difficult to see that this is the case if and only if π and π' agree on $D_s \cap D_{s'}$ (since we assumed our forcing $\mathbb{P}$ to be separative).

Definition/Proposition 1.4.8. Let $\varphi(x, y)$ be a formula of set theory, S a class in V, and A an almost-group for φ and S. We define on A the following equivalence relation:

For π, π' with $\pi : D_s \to D_s$, $\pi' : D_{s'} \to D_{s'}$ let

$$\pi \sim \pi' \ :\Leftrightarrow \ \pi \upharpoonright (D_s \cap D_{s'}) = \pi' \upharpoonright (D_s \cap D_{s'}).$$

Consider $\pi \in A$. We denote by $\Delta_l(\pi)$ (the *lower height of* π) the least ordinal α such that there exists $\pi' \in A$ with $\pi' \sim \pi$ such that π' can be described below α.
We define

$$[\pi] := [\pi]_\sim := \left\{\sigma = \overline{\sigma_\alpha} \;\middle|\; \alpha = \Delta_l(\pi)\,,\, \sigma \sim \pi\,,\, \sigma_\alpha : \operatorname{dom}\sigma \cap \mathbb{P}_\alpha \to \operatorname{dom}\sigma \cap \mathbb{P}_\alpha\right\}.$$

Let

$$\overline{A} := \left\{\, [\pi] \mid \pi \in A \,\right\}.$$

Then $\overline{A}$ becomes a group as follows: Consider π, $\sigma \in A$ with $\pi \in A_{(s,\alpha)}$, $\sigma \in A_{(s',\alpha')}$. Then by 1.4.7 b), there is $s'' \in S$ with $D_{s''} = D_s \cap D_{s'}$, and $\alpha(s'') \leq \alpha'' := \max\{\alpha, \alpha'\}$. Let $[\pi] \circ [\sigma] := [\nu]$, where ν is the map in $A_{(s'',\alpha'')}$ satisfying $\nu(p) = \pi(\sigma(p))$ for all $p \in D_{s''}$.

We call $\overline{A}$ the *group of partial* $\mathbb{P}$*-automorphisms derived from* A.

Proof. We have to make sure that the operation "$\circ$" is well-defined: Let π, σ as above with $\pi \in A_{(s,\alpha)}$, $\sigma \in A_{(s',\alpha')}$, and $s'' \in S$ with $D_{s''} = D_s \cap D_{s'}$ and $\alpha(s'') \leq \alpha'' := \max\{\alpha, \alpha'\}$. Then $\pi \upharpoonright D_{s''} \in A_{(s'',\alpha'')}$ and $\sigma \upharpoonright D_{s''} \in A_{(s'',\alpha'')}$ by 1.4.7 f). Now, since $A_{(s'',\alpha'')}$ is a group, there exists $\nu \in A_{(s'',\alpha'')}$ with $\nu(p) = \pi(\sigma(p))$ for all $p \in D_{s''}$. Setting $\nu_{\alpha''} := \nu \upharpoonright \mathbb{P}_{\alpha''}$, it is not difficult to see that $\nu = \overline{\nu_{\alpha''}}$, and $\nu_{\alpha''}(p) = \pi(\sigma(p))$ for all $p \in D_{s''} \cap \mathbb{P}_{\alpha''}$.
The rest of the proof is as in Definition / Proposition 1.2.15 for set forcing. □

For the rest of this chapter, we fix a formula $\varphi(x, y)$, a class S, an almost-group A for φ and S, and $\overline{A}$, the group of partial automorphisms derived from A.

Note that whenever π, $\sigma \in A$ with $\pi \sim \sigma$, then $\Delta_l(\pi) = \Delta_l(\sigma)$. Thus, for any π, $\sigma \in A$, it follows that $\pi \sim \sigma$ if and only if $[\pi] = [\sigma]$.

We will now extend our automorphisms $\pi \in A$ to the name space. Let $s \in S$. Recursively, we say that $\dot{x} \in \mathrm{Name}(\mathbb{P})$ *is a* $\mathbb{P}$*-name for* D_s if for all $(\dot{y}, p) \in \dot{x}$, it follows that $\dot{y}$ is a $\mathbb{P}$-name for D_s, and $p \in D_s$. We denote by $\overline{\mathrm{Name}(\mathbb{P})}^{D_s}$ the class of all $\mathbb{P}$-names for D_s.

Whenever $\dot{x}$ is a $\mathbb{P}$-name and $\pi : D_s \to D_s$, then $\pi\dot{x}$ can be defined as usual in the case that $\dot{x} \in \overline{\mathrm{Name}(\mathbb{P})}^{D_s}$, and recursively, it follows that $\Delta(\pi\dot{x}) = \Delta(\dot{x})$ for all $\dot{x} \in \overline{\mathrm{Name}(\mathbb{P})}^{D_s}$. In the case that $\dot{x} \notin \overline{\mathrm{Name}(\mathbb{P})}^{D_s}$ however, we have to proceed similarly as in Chapter 1.2.3 to modify $\dot{x}$ to obtain a name $\overline{x}^{D_s} \in \overline{\mathrm{Name}(\mathbb{P})}^{D_s}$ with the property that $\dot{x}^G = (\overline{x}^{D_s})^G$ for any G a V-generic filter on $\mathbb{P}$.

In order to make sure that $\overline{x}^{D_s}$ is a set, we have to modify our definition from Chapter 1.2.3, and require that for all $(\overline{y}^{D_s}, p) \in \overline{x}^{D_s}$, it follows that $\Delta(p) \leq \Delta(\dot{x})$:

Given $s \in S$, we define recursively for $\dot{x} \in \mathrm{Name}(\mathbb{P})$:

$$\overline{x}^{D_s} := \left\{ (\overline{y}^{D_s}, p) \;\middle|\; \dot{y} \in \operatorname{dom}\dot{x}\,,\, p \in D_s\,,\, \Delta(p) \leq \Delta(\dot{x})\,,\, p \Vdash \dot{y} \in \dot{x} \right\}.$$

Then $\overline{x}^{D_s} \in \overline{\mathrm{Name}(\mathbb{P})}^{D_s}$, and inductively, it follows that $\mathsf{rk}_{\mathbb{P}}\overline{x}^{D_s} = \mathsf{rk}_{\mathbb{P}}\dot{x}$.

Lemma 1.4.9. *Let $s \in S$. Whenever G is a V-generic filter on $\mathbb{P}$ and $\dot{x} \in \mathrm{Name}(\mathbb{P})$, then $\dot{x}^G = (\overline{x}^{D_s})^G$.*

Proof. Assume recursively that $\dot{y}^G = (\overline{y}^{D_s})^G$ holds for all $\dot{y} \in \mathrm{Name}(\mathbb{P})$ with $\mathsf{rk}_{\mathbb{P}}\dot{y} < \mathsf{rk}_{\mathbb{P}}\dot{x}$. The inclusion "$\dot{x}^G \supseteq (\overline{x}^{D_s})^G$ " is clear. Regarding "$\subseteq$", consider $y \in \dot{x}^G$, and let $(\dot{y}, p) \in \dot{x}$ with $y = \dot{y}^G$ and $p \in G$. By density of D_s, there exists $q \leq p$, $q \in D_s$, with $q \in G$. Let $\alpha := \Delta(\dot{x})$. Then $\Delta(p) \leq \alpha$. Setting $\widetilde{q} := \rho_\alpha(q)$, it follows that $\widetilde{q} \in D_s$, as well, since D_s allows projections. Moreover, $\Delta(\widetilde{q}) \leq \alpha = \Delta(\dot{x})$, and since $\widetilde{q} = \rho_\alpha(q) \leq \rho_\alpha(p) = p$, it follows that $\widetilde{q} \Vdash \dot{y} \in \dot{x}$. Hence, $(\overline{y}^{D_s}, \widetilde{q}) \in \overline{x}^{D_s}$; and $q \in G$ with $\widetilde{q} = \rho_\alpha(q) \geq q$ (by 1.4.2 a)) implies that $\widetilde{q} \in G$, as well. Hence, $y = \dot{y}^G = (\overline{y}^{D_s})^G \in (\overline{x}^{D_s})^G$ as desired. □

Lemma 1.4.10. *Whenever s, $s' \in S$ and $\dot{x} \in \mathrm{Name}(\mathbb{P})$, then $\overline{\overline{x}^{D_s}}^{D_{s'}} = \overline{x}^{D_{s'}}$.*

Proof. We first show that $\Delta(\overline{x}^{D_s}) = \Delta(\dot{x})$ holds true for all $\dot{x} \in \mathrm{Name}(\mathbb{P})$ and $s \in S$. Take $\dot{x} \in \mathrm{Name}(\mathbb{P})$, and assume recursively that $\Delta(\overline{y}^{D_s}) = \Delta(\dot{y})$ holds for all $\dot{y} \in \mathrm{Name}(\mathbb{P})$ with $\mathsf{rk}_{\mathbb{P}}\dot{y} < \mathsf{rk}_{\mathbb{P}}\dot{x}$. By definition,

$$\Delta(\overline{x}^{D_s}) = \sup\{\Delta(\overline{y}^{D_s}) \mid \dot{y} \in \operatorname{dom}\dot{x}\} \cup \sup\{\Delta(p) \mid p \in \operatorname{rg}\big(\overline{x}^{D_s}\big)\}.$$

For any $\dot{y} \in \operatorname{dom}\dot{x}$, it follows by assumption that $\Delta(\overline{y}^{D_s}) = \Delta(\dot{y}) \leq \Delta(\dot{x})$. For every $p \in \operatorname{rg}\big(\overline{x}^{D_s}\big)$, we have $\Delta(p) \leq \Delta(\dot{x})$ by construction. Hence, $\Delta(\overline{x}^{D_s}) \leq \Delta(\dot{x})$.

Regarding the proof of "$\Delta(\overline{x}^{D_s}) \geq \Delta(\dot{x})$", it suffices to show that $\sup\{\Delta(p) \mid p \in \operatorname{rg}\big(\overline{x}^{D_s}\big)\} \geq \sup\{\Delta(p) \mid p \in \operatorname{rg}\dot{x}\}$. Consider $(\dot{y}, p) \in \dot{x}$ with $\Delta(p) =: \alpha$. Our aim is to find $(\overline{y}^{D_s}, \widetilde{q}) \in \overline{x}^{D_s}$ with $\Delta(\widetilde{q}) = \alpha$. Take $q \leq p$ with $q \in D_s$, and let $\widetilde{q} := \rho_\alpha(q)$. Then $\widetilde{q} \in D_s$ as well, since D_s allows projections, and $\Delta(\widetilde{q}) \leq \alpha = \Delta(p) \leq \Delta(\dot{x})$. Moreover, $\widetilde{q} = \rho_\alpha(q) \leq \rho_\alpha(p) = p$ implies $\Delta(\widetilde{q}) \geq \Delta(p) = \alpha$, and $\widetilde{q} \Vdash \dot{y} \in \dot{x}$. Hence, $(\overline{y}^{D_s}, \widetilde{q}) \in \overline{x}^{D_s}$ with $\Delta(\widetilde{q}) = \Delta(p) = \alpha$ as desired. We conclude that $\sup\{\Delta(p) \mid p \in \operatorname{rg}\big(\overline{x}^{D_s}\big)\} \geq \sup\{\Delta(p) \mid p \in \operatorname{rg}\dot{x}\}$; which finishes the proof of $\Delta(\overline{x}^{D_s}) = \Delta(\dot{x})$.

Now, consider $\dot{x} \in \mathrm{Name}(\mathbb{P})$, and assume recursively that $\overline{\overline{y}^{D_s}}^{D_{s'}} = \overline{y}^{D_{s'}}$ holds true for all $\dot{y} \in \mathrm{Name}(\mathbb{P})$ with $\mathsf{rk}_{\mathbb{P}}\dot{y} < \mathsf{rk}_{\mathbb{P}}\dot{x}$. Then

$$\begin{aligned}
\overline{\overline{x}^{D_s}}^{D_{s'}} &= \left\{ (\overline{\overline{y}^{D_s}}^{D_{s'}}, p) \;\middle|\; \dot{y} \in \operatorname{dom}\dot{x}\,,\, p \in D_{s'}\,,\, \Delta(p) \leq \Delta(\overline{x}^{D_s})\,,\, p \Vdash \overline{y}^{D_s} \in \overline{x}^{D_s} \right\} \\
&= \left\{ (\overline{y}^{D_{s'}}, p) \;\middle|\; \dot{y} \in \operatorname{dom}\dot{x}\,,\, p \in D_{s'}\,,\, \Delta(p) \leq \Delta(\dot{x})\,,\, p \Vdash \dot{y} \in \dot{x} \right\} \\
&= \overline{x}^{D_{s'}}.
\end{aligned}$$

□

Moreover, it is not difficult to verify that whenever π, $\pi' \in A$, $\pi : D_s \to D_s$, $\pi' : D_{s'} \to D_{s'}$ and $\dot{x} \in \overline{\mathrm{Name}(\mathbb{P})}^{D_s}$, then

$$\pi \overline{x}^{D_{s'}} = \overline{\pi \dot{x}}^{D_{s'}}.$$

In order to establish a notion of symmetry, we need the following analogue of Lemma 1.2.16:

Lemma 1.4.11. *Let π, $\pi' \in A$ with $\pi : D_s \to D_s$, $\pi' : D_{s'} \to D_{s'}$ such that $\pi \sim \pi'$, i.e. $\pi \upharpoonright (D_s \cap D_{s'}) = \pi' \upharpoonright (D_s \cap D_{s'})$. Then for $\dot{x} \in \mathrm{Name}(\mathbb{P})$, it follows that $\pi \overline{x}^{D_s} = \overline{x}^{D_s}$ if and only if $\pi' \overline{x}^{D_{s'}} = \overline{x}^{D_{s'}}$.*

We prove the following analogue of Lemma 1.2.17 by induction over α:

Lemma 1.4.12. *Let π, $\pi' \in A$ with $\pi : D_s \to D_s$, $\pi' : D_{s'} \to D_{s'}$ such that $\pi \sim \pi'$, i.e. $\pi \upharpoonright (D_s \cap D_{s'}) = \pi' \upharpoonright (D_s \cap D_{s'})$, and $\gamma \in \mathrm{Ord}$. Then for any $\dot{y}$, $\dot{z} \in \mathrm{Name}(\mathbb{P})$ with $\mathsf{rk}_{\mathbb{P}} \dot{y} = \mathsf{rk}_{\mathbb{P}} \dot{z} = \gamma$, it follows that $\pi \overline{y}^{D_s} = \overline{z}^{D_s}$ if and only if $\pi' \overline{y}^{D_{s'}} = \overline{z}^{D_{s'}}$.*

Proof. As in the proof of Lemma 1.2.17, we can assume w.l.o.g. that $D_{s'} \subseteq D_s$, since the map $\widetilde{\pi} := \pi \upharpoonright (D_s \cap D_{s'}) = \pi' \upharpoonright (D_s \cap D_{s'})$ is contained in A, as well by 1.4.7 f).

Consider $\gamma \in \mathrm{Ord}$, and assume the statement is true for all $\overline{\gamma} < \gamma$. Let $\dot{y}$, $\dot{z} \in \mathrm{Name}(\mathbb{P})$ with $\mathsf{rk}_{\mathbb{P}} \dot{y} = \mathsf{rk}_{\mathbb{P}} \dot{z} = \gamma$.

"$\Rightarrow$": If $\pi \overline{y}^{D_s} = \overline{z}^{D_s}$, then similarly as in the proof of Lemma 1.2.17, one can show that $\overline{z}^{D_{s'}} \subseteq \pi \overline{y}^{D_{s'}}$ holds. The inclusion "$\supseteq$" is similar.

"$\Leftarrow$": Now, assume $\pi' \overline{y}^{D_{s'}} = \overline{z}^{D_{s'}}$. Following the proof from Lemma 1.2.17, we show that $\overline{z}^{D_s} \subseteq \pi \overline{y}^{D_s}$. (The inclusion "$\supseteq$" is similar.) Let $\alpha := \Delta(\dot{z})$. Consider $(\overline{x}^{D_s}, \overline{p}) \in \overline{z}^{D_s}$, i.e. $\dot{x} \in \mathrm{dom}\, \dot{z}$, $\overline{p} \in D_s$, $\Delta(\overline{p}) \leq \alpha$, and $\overline{p} \Vdash \dot{x} \in \dot{z}$. We have to show that $(\overline{x}^{D_s}, \overline{p}) \in \pi \overline{y}^{D_s}$. Let $\overline{p}' \leq \overline{p}$ with $\overline{p}' \in D_{s'}$, and set $\widetilde{p} := \rho_\alpha(\overline{p}')$. Then $\widetilde{p} \in D_{s'}$ as well (since $D_{s'}$ allows projections), $\Delta(\widetilde{p}) \leq \alpha = \Delta(\dot{z})$, and from $\widetilde{p} = \rho_\alpha(\overline{p}') \leq \rho_\alpha(\overline{p}) = \overline{p}$, it follows that $\widetilde{p} \Vdash \dot{x} \in \dot{z}$. Hence, $(\overline{x}^{D_{s'}}, \widetilde{p}) \in \overline{z}^{D_{s'}} = \pi' \overline{y}^{D_{s'}}$; so there must be $\dot{u} \in \mathrm{dom}\, \dot{y}$ with $\overline{x}^{D_{s'}} = \pi' \overline{u}^{D_{s'}}$. By inductive assumption, it follows that $\overline{x}^{D_s} = \pi \overline{u}^{D_s}$, since $\mathsf{rk}_{\mathbb{P}} \dot{u} = \mathsf{rk}_{\mathbb{P}} \dot{x} < \gamma$. Let $\overline{q} := \pi^{-1} \overline{p}$. We have to show that $(\pi \overline{u}^{D_s}, \pi \overline{q}) \in \pi \overline{y}^{D_s}$. We know that $\dot{u} \in \mathrm{dom}\, \dot{y}$ and $\overline{q} \in D_s$, so it remains to show that $\Delta(\overline{q}) \leq \Delta(\dot{y})$ and $\overline{q} \Vdash \dot{u} \in \dot{y}$.

Regarding "$\Delta(\overline{q}) \leq \Delta(\dot{y})$", let $\widetilde{q} := (\pi')^{-1}(\widetilde{p})$. Then $(\pi' \overline{u}^{D_{s'}}, \pi'(\widetilde{q})) = (\overline{x}^{D_{s'}}, \widetilde{p}) \in \pi' \overline{y}^{D_{s'}}$ gives $\Delta(\widetilde{q}) \leq \Delta(\dot{y})$; hence, $\Delta(\widetilde{p}) \leq \Delta(\dot{y})$, and from $\widetilde{p} \leq \overline{p}$, we obtain $\Delta(\overline{p}) \leq \Delta(\dot{y})$. Hence, from $\overline{q} = \pi^{-1} \overline{p}$, it follows that $\Delta(\overline{q}) = \Delta(\overline{p}) \leq \Delta(\dot{y})$ as desired.

Regarding the proof of "$\overline{q} \Vdash \dot{u} \in \dot{y}$", it suffices to show that $r \Vdash \dot{u} \in \dot{y}$ for all $r \leq \overline{q}$ with $r \in D_{s'}$. Consider $r \leq \overline{q}$ with $r \in D_{s'}$, and let $\widetilde{r} := \rho_\alpha(r)$, where $\alpha := \Delta(\dot{z})$ as above. Since $\Delta(\overline{q}) = \Delta(\overline{p}) \leq \alpha$, it follows that $\widetilde{r} = \rho_\alpha(r) \leq \rho_\alpha(\overline{q}) = \overline{q}$; and $\widetilde{r} \in D_{s'}$, since $D_{s'}$ allows projections. Then $\pi \widetilde{r}$ is contained in $D_{s'}$, as well. Moreover, from $\overline{p} \Vdash \dot{x} \in \dot{z}$ and $\widetilde{r} \leq \overline{q} = \pi^{-1} \overline{p}$, it follows that $\pi \widetilde{r} \Vdash \dot{x} \in \dot{z}$. Finally, $\Delta(\pi \widetilde{r}) = \Delta(\widetilde{r}) \leq \alpha = \Delta(\dot{z})$, and $\dot{x} \in \mathrm{dom}\, \dot{z}$; so we obtain $(\overline{x}^{D_{s'}}, \pi \widetilde{r}) \in \overline{z}^{D_{s'}}$. Thus, $(\pi' \overline{u}^{D_{s'}}, \pi \widetilde{r}) \in \pi' \overline{y}^{D_{s'}}$; which implies $\widetilde{r} \Vdash \dot{u} \in \dot{y}$. Since $r \leq \widetilde{r} = \rho_\alpha(r)$, it follows that $r \Vdash \dot{u} \in \dot{y}$ as desired; which finishes the proof.

□

Now, we introduce our notion of symmetricity in the context of class forcing with partial automorphisms. We try to get around employing the notion of a *filter* (which would be a collection of proper classes), but instead notice that in most applications, there are only finitely many *types* of $\overline{A}$-subgroups to be considered, each of which is given by a formula $\psi_i(v,w)$: For $i < l$ (where l is a finite number), we have $\overline{A}_i(y) = \{ [\pi] \in \overline{A} \mid \psi_i(\pi,y) \}$ for some parameter y. We usually want only parameters with certain properties; so for any $i < l$, there will be another formula $\chi_i(w)$ such that only subgroups $\overline{A}_i(y)$ are considered where $\chi_i(y)$ holds. (In applications, the formula $\chi_i(y)$ could state, for example, that y is a finite set of ordinals with certain properties.)

In order to have the groups $\overline{A}_i(y)$ well-defined, one has to require that for all y with $\chi_i(y)$ and π, $\pi' \in A$ with $\pi \sim \pi'$, it follows that $\psi_i(\pi,y)$ if and only if $\psi_i(\pi',y)$.

Finally, we have to ask for a *normality property*, corresponding to the requirement that in the case of set forcing, one has to use a *normal filter*.

This results in the following definition (recall that we have fixed an almost-group $\overline{A}$ of partial automorphisms for $\mathbb{P}$):

Definition 1.4.13. A *finitely generated symmetric system $\mathcal{S}$ for $\overline{A}$* consists of a finite number $l < \omega$, l-many formulas $\psi_0(x,y),\dots,\psi_{l-1}(x,y)$, and l-many formulas $\chi_0(y),\dots,\chi_{l-1}(y)$ such that the following properties hold:

a) Let $i < l$. Whenever y with $\chi_i(y)$, and π, $\pi' \in A$ with $\pi \sim \pi'$, then $\psi_i(\pi,y)$ if and only if $\psi_i(\pi',y)$. In other words: The formulas $\psi_i(\pi,y)$ respect "$\sim$".

b) For all $i < l$ and y with $\chi_i(y)$, it follows that

$$\overline{A}_i(y) := \left\{ [\pi] \in \overline{A} \mid \psi_i(\pi,y) \right\}$$

is a subgroup of $\overline{A}$.

c) The following *normality property* holds: Let $[\pi] \in \overline{A}$, $i < l$, and y with $\chi_i(y)$. Then there is $n < \omega$ and finitely many $i_0,\dots,i_{n-1} < l$; moreover, finitely many parameters $y_0,\dots,y_{n-1}$ such that $\chi_{i_j}(y_j)$ holds for all $j < n$, and

$$[\pi]\overline{A}_i(y)[\pi]^{-1} \supseteq \overline{A}_{i_0}(y_0) \cap \dots \cap \overline{A}_{i_{l-1}}(y_{l-1}).$$

The following definition corresponds to saying that a subgroup $B \subseteq \overline{A}$ is contained in the filter generated by the $\overline{A}_i(y)$:

Definition 1.4.14. Let $\mathcal{S}$ be a finitely generated symmetric system for $\overline{A}$. A subgroup $B \subseteq \overline{A}$ *gives rise to symmetry with respect to $\mathcal{S}$* if there is $n < \omega$ and finitely many $i_0,\dots,i_{n-1} < l$; moreover, finitely many parameters $y_0,\dots,y_{n-1}$ such that $\chi_{i_j}(y_j)$ holds for all $j < n$, and

$$B \supseteq \overline{A}_{i_0}(y_0) \cap \dots \cap \overline{A}_{i_{n-1}}(y_{n-1}).$$

Remark 1.4.15. Since there are only finitely many formulas $\psi_0(v,w),\dots,\psi_{l-1}(v,w)$ and $\chi_0(w),\dots,\chi_{l-1}(w)$ involved, Definition 1.4.13 and Definition 1.4.14 could be rephrased to a formula of set theory.

Remark 1.4.16. One could also allow intersections of cardinality $< \kappa$ for κ a regular cardinal. Then Definition 1.4.13 c) and 1.4.14 above have to be modified as follows:

- Regarding *normality*, one has to require that for any $[\pi] \in \overline{A}$, $i < l$ and y with $\chi_i(y)$, there are $\kappa_0, \ldots, \kappa_{l-1} < \kappa$, and for any $\iota < l$ a sequence $(y_j^\iota \mid j < \kappa_\iota)$ with $\chi_\iota(y_j^\iota)$ for all $j < \kappa_\iota$ such that
$$[\pi]\overline{A}_i(y)[\pi]^{-1} \supseteq \bigcap_{\iota < l} \bigcap_{j < \kappa_\iota} \overline{A}_\iota(y_j^\iota).$$
- A subgroup $B \subseteq \overline{A}$ *gives rise to symmetry with respect to* $\mathcal{S}$ if there are $\kappa_0, \ldots, \kappa_{l-1} < \kappa$; and for any $\iota < l$ a sequence $(y_j^\iota \mid j < \kappa_\iota)$ with $\psi_\iota(y_j^\iota)$ for all $j < \kappa_\iota$ such that
$$B \supseteq \bigcap_{\iota < l} \bigcap_{j < \kappa_\iota} \overline{A}_\iota(y_j^\iota).$$

For our purposes in the context of Chapter 3, we will only need symmetric systems that are generated by finite intersections.

Fix a finitely generated symmetric system $\mathcal{S}$ for $\overline{A}$. Now, we can use $\mathcal{S}$ to establish our notion of symmetry:

Definition 1.4.17. A $\mathbb{P}$-name $\dot{x}$ is *symmetric for* $\mathcal{S}$ if the stabilizer group
$$\operatorname{sym}^{\overline{A}}(\dot{x}) := \left\{ [\pi] \in \overline{A}, \, \pi : D_s \to D_s \;\middle|\; \pi \overline{x}^{D_s} = \overline{x}^{D_s} \right\}$$
gives rise to symmetry with respect to $\mathcal{S}$. Recursively, a name $\dot{x}$ is *hereditarily symmetric*, $\dot{x} \in HS^{\mathcal{S}}$, if $\dot{x}$ is symmetric, and $\dot{y}$ is hereditarily symmetric for all $\dot{y} \in \operatorname{dom} \dot{x}$.

By Lemma 1.4.11, this is well-defined, since for any π, $\pi' \in A$, $\pi : D_s \to D_s$, $\pi' : D_{s'} \to D_{s'}$ with $\pi \sim \pi'$ and $\dot{x} \in \operatorname{Name}(\mathbb{P})$, it follows that $\pi \overline{x}^{D_s} = \overline{x}^{D_s}$ if and only if $\pi' \overline{x}^{D_{s'}} = \overline{x}^{D_{s'}}$.

When $\overline{A}$ and $\mathcal{S}$ are clear from the context, we write just $\operatorname{sym}(\dot{x})$ and HS.

Like in the case for set forcing, one can show that whenever $\dot{x} \in HS^{\mathcal{S}}$ and $\pi \in A$, $\pi : D_s \to D_s$, then firstly, also $\overline{x}^{D_s} \in HS^{\mathcal{S}}$ holds; and secondly, $\pi \overline{x}^{D_s} \in HS^{\mathcal{S}}$. For the second claim, one has to use the *normality property* from Definition 1.4.13 c).

Moreover, for any element a of the ground model, it is not difficult to see that the canonical name $\check{a} := \{(\check{b}, \mathbb{1}) \mid b \in a\}$ is hereditarily symmetric.

We are now ready to define the symmetric extension:

Definition 1.4.18. Let G be a V-generic filter on $\mathbb{P}$. The *symmetric extension by* $\mathcal{S}$ and G is
$$V(G)^{\mathcal{S}} := \{\dot{x}^G \mid \dot{x} \in HS^{\mathcal{S}}\}.$$
When the symmetric system is clear from the context, we write just $V(G)$.

For φ a formula of set theory, $p \in \mathbb{P}$, and $\dot{x}_0, \ldots, \dot{x}_{n-1} \in HS$, the symmetric forcing relation $p \, (\Vdash_s)^V_{\mathbb{P}, \mathcal{S}} \, \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ can be defined as for set forcing (cf. Definition 1.2.7).

We will work with the structure $\langle V(G)^{\mathcal{S}}, \in, V\rangle$, where we have a unary predicate symbol for the ground model.

Similarly as in Chapter 1.3.1, we extend our language of set theory $\mathcal{L}_\in$ by a unary predicat symbol A, where $A(x)$ will assert that $x \in V$, and denote this extended language by $\mathcal{L}_\in^A$.

Definition 1.4.19. For $p \in \mathbb{P}$, we define:

- $p\,(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}\, A(\dot{x})$ iff $\forall q \leq p\ \exists r \leq q\ \exists a\ (r \Vdash^V_{\mathbb{P}} \dot{x} = \check{a})$
- $V[G] \vDash A(x)$ iff $x \in V$

Again, we will abuse notation and do not mention the predicate A in our formulas. Instead, we keep in mind that inside the structure $V(G)^{\mathcal{S}}$, formulas can talk about the ground model V; and write $\varphi(x_0, \ldots, x_{n-1}, V)$ where necessary. Moreover, behind the forcing symbol $(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}$, we write "$p\,(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}\, \dot{x} \in \check{V}$ " instead of "$p\,(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}\, A(\dot{x})$"; and $p\,(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}\, \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}, \check{V})$ for a formula $\varphi \in \mathcal{L}_\in^A$ when we need to mention the predicate $\check{V}$.

Informally, the symmetric forcing relation for class forcing can be defined as usual:

Definition 1.4.20. For a formula $\varphi(v_0, \ldots, v_{n-1}) \in \mathcal{L}_\in^A$, a condition $p \in \mathbb{P}$, and $\dot{x}_0, \ldots, \dot{x}_{n-1} \in HS$, we write

$$p\,(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}\, \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$$

if for any G a V-generic filter on $\mathbb{P}$ with $p \in G$, it follows that $\varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ holds in the structure $\langle V(G)^{\mathcal{S}}, \in, V\rangle$.

Then $(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}$ satisfies the same basic properties as the ordinary symmetric forcing relation (see Proposition 1.2.8); and the *Symmetry Lemma* holds true, as well.

Whenever $\dot{x}$, $\dot{y} \in HS$ and $p \in \mathbb{P}$, then $p\,(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}\, \dot{y} \in \dot{x}$ if and only if $p \Vdash^V_{\mathbb{P}} \dot{y} \in \dot{x}$; moreover, $p\,(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}\, \dot{x} = \dot{y}$ if and only if $p \Vdash^V_{\mathbb{P}} \dot{x} = \dot{y}$; and $p\,(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}\, \dot{x} \in \check{V}$ if and only if $p \Vdash^V_{\mathbb{P}} \dot{x} \in \check{V}$. Hence, the *definability lemma* for $(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}$ holds for the atomic formulas "$v_0 \in v_1$" ,"$v_0 = v_1$" and "$v_0 \in \check{V}$". As in [Kra17, 2.1.5], this implies that the Forcing Theorem holds for the symmetric forcing relation $(\Vdash_s)^V_{\mathbb{P},\mathcal{S}}$.

In most cases, when the ground model V, the forcing notion $\mathbb{P}$, and the symmetric system $\mathcal{S}$ are clear from the context, we write just "$\Vdash_s$".

Since $p \Vdash_s \dot{y} \in \dot{x}$ if and only if $p \Vdash \dot{y} \in \dot{x}$ for all $\dot{x}$, $\dot{y} \in HS$ and $p \in \mathbb{P}$, it follows that for any $\dot{x} \in HS$ and a parameter $s \in S$, we have

$$\overline{x}^{D_s} = \{(\overline{y}^{D_s}, p) \mid \dot{y} \in \operatorname{dom}\dot{x}\,,\, p \in D_s\,,\, \Delta(p) \leq \Delta(\dot{x})\,,\, p \Vdash_s \dot{y} \in \dot{x}\}.$$

In general, symmetric extensions by class forcing do not preserve ZF. However, the following proposition remains true:

Proposition 1.4.21. *Let $\mathbb{P}$ be a notion of forcing, A a group of partial $\mathbb{P}$-automorphisms for φ and S, and $\overline{A}$ the group of partial automorphisms derived from A. Let $\mathcal{S}$ be a finitely generated symmetric system for $\overline{A}$. Then $V(G) = V(G)^{\mathcal{S}}$ is transitive with $V \subseteq V(G) \subseteq V[G]$, and $V(G)$ satisfies all single axioms of ZF^- (that is, all axioms of ZF^- except* Separation *and* Replacement*), with* Union *replaced by* Weak Union *(cf. Chapter 0.4).*

Proof. The axioms of *Extensionality*, *Foundation* and *Infinity* are clear.

Regarding *Pairing*, consider x, $y \in V(G)$, $x = \dot{x}^G$, $y = \dot{y}^G$ with $\dot{x}$, $\dot{y} \in HS$. We have to find a name $\dot{z} \in HS$ with $\dot{z}^G = \{x, y\}$.

Let $\dot{z} := \{(\dot{x}, \mathbb{1}), (\dot{y}, \mathbb{1})\}$ and consider $\pi \in A$, $\pi : D_s \to D_s$ with $\pi\overline{x}^{D_s} = \overline{x}^{D_s}$ and $\pi\overline{y}^{D_s} = \overline{y}^{D_s}$. Then

$$\overline{z}^{D_s} = \{(\overline{x}^{D_s}, p) \mid p \in D_s\,,\, \Delta(p) \leq \Delta(\dot{x})\} \cup \{(\overline{y}^{D_s}, p) \mid p \in D_s\,,\, \Delta(p) \leq \Delta(\dot{y})\},$$

and

$$\pi\overline{z}^{D_s} = \{(\pi\overline{x}^{D_s}, \pi p) \mid p \in D_s\,,\, \Delta(p) \leq \Delta(\dot{x})\} \cup \{(\pi\overline{y}^{D_s}, \pi p) \mid p \in D_s\,,\, \Delta(p) \leq \Delta(\dot{y})\}.$$

Since $p \in D_s \Leftrightarrow \pi p \in D_s$ for all $p \in \mathbb{P}$, and $\Delta(p) = \Delta(\pi p)$ for all $p \in D_s$, it follows that $\pi\overline{z}^{D_s} = \overline{z}^{D_s}$. Hence, $\mathrm{sym}^{\overline{A}}(\dot{z}) \supseteq \mathrm{sym}^{\overline{A}}(\dot{x}) \cap \mathrm{sym}^{\overline{A}}(\dot{y})$, so $\mathrm{sym}^{\overline{A}}(\dot{z})$ gives rise to symmetry with respect to $\mathcal{S}$. Since $\mathrm{dom}\,\dot{z} = \{\dot{x}, \dot{y}\} \subseteq HS$, it follows that $\dot{z} \in HS$ as desired.

For *Weak Union*, consider $x \in V(G)$, $x = \dot{x}^G$ with $\dot{x} \in HS$. We have to find $u \in V(G)$ with $u \supseteq \bigcup x$. Let

$$\dot{u} := \{(\dot{z}, \mathbb{1}) \mid \exists\, \dot{y} \in \mathrm{dom}\,\dot{x} \;\; \dot{z} \in \mathrm{dom}\,\dot{y}\}.$$

Then $\dot{u}^G \supseteq \bigcup x$. It remains to make sure that $\dot{u} \in HS$.

Consider $\pi \in A$, $\pi : D_s \to D_s$ with $\pi\overline{x}^{D_s} = \overline{x}^{D_s}$. Then

$$\begin{aligned} \overline{u}^{D_s} &= \{(\overline{z}^{D_s}, p) \mid p \in D_s\,,\, \Delta(p) \leq \Delta(\dot{z})\,,\, \exists\, \dot{y} \in \mathrm{dom}\,\dot{x} \;\; \dot{z} \in \mathrm{dom}\,\dot{y}\} \\ &= \{(z, p) \mid p \in D_s\,,\, \Delta(p) \leq \Delta(z)\,,\, \exists\, y \in \mathrm{dom}\,\overline{x}^{D_s} \;\; z \in \mathrm{dom}\,y\}, \end{aligned}$$

since by the proof of Lemma 1.4.10 it follows that $\Delta(\dot{z}) = \Delta(\overline{z}^{D_s})$ for all $\overline{z}^{D_s} \in \mathrm{dom}\,\overline{u}^{D_s}$.

We obtain

$$\begin{aligned} \pi\overline{u}^{D_s} &= \{(\pi z, \pi p) \mid p \in D_s\,,\, \Delta(p) \leq \Delta(z)\,,\, \exists\, y \in \mathrm{dom}\,\overline{x}^{D_s} \;\; z \in \mathrm{dom}\,y\} \\ &= \{(\pi z, \pi p) \mid \pi p \in D_s\,,\, \Delta(\pi p) \leq \Delta(z)\,,\, \exists\, y \in \mathrm{dom}\,\pi\overline{x}^{D_s} \;\; \pi z \in \mathrm{dom}\,y\}, \end{aligned}$$

which is equal to $\overline{u}^{D_s}$, since we assumed $\pi\overline{x}^{D_s} = \overline{x}^{D_s}$.

Hence, $\mathrm{sym}^{\overline{A}}(\dot{u}) \supseteq \mathrm{sym}^{\overline{A}}(\dot{x})$, and $\mathrm{dom}\,\dot{u} \subseteq HS$; which gives $\dot{u} \in HS$ as desired. □

The axioms of *Separation* and *Replacement* hold true if $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ and A satisfy certain homogeneity properties as in Proposition 3.3.2 and 3.3.3. The axiom of *Power Set* does not holds true in general.

However, even if the generic extension $V[G]$ by some class forcing $\mathbb{P}$ does not satisfy ZFC, there can still be an intermediate symmetric extension $V(G) = V(G)^{\mathcal{S}}$ with $V(G) \vDash \mathsf{ZF}$ (see Chapter 3).

Chapter 2

An Easton-like Theorem for Set-many Cardinals in ZF + DC

In this chapter, we show that in the theory $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$, the θ-function can take almost arbitrary values on any set of cardinals. This answers a question of Saharon Shelah from [She16, §0.2 1)]: "*Can we bound* hrtg $(\wp(\mu))$ [$= \theta(\mu)$] *for μ singular?*" No, we can not.

On the one hand, this generalization of Easton's Theorem to regular **and** singular cardinals in a theory with weak choice is in sharp contrast to the AC-situation, where Silver's Theorem and pcf theory put prominent bounds on the Continuum Function. On the other hand, the theory $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ is surprisingly rich and allows for much of combinatorics. For instance, a version of the pcf theorem holds (see [She16, §1]) and certain covering numbers exist (see [She16, §2 (D)]).

We continue with a few words about $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$. Starting off from $\mathsf{ZF} + \mathsf{DC}$, most of real analysis is possible. Investigations in combinatorial set theory under $\mathsf{ZF} + \mathsf{DC}$ seemed rather hopeless in the first place, until Saharon Shelah proved various interesting results under $\mathsf{ZF} + \mathsf{DC}$ in [She97], thus initiating further projects under weak choice.

In [She10], he suggested to adopt the following additional axiom:

(AX_4) *For every cardinal λ, the set $[\lambda]^{\aleph_0}$ can be well-ordered.*

Given a ground model $V \vDash \mathsf{ZFC}$, any symmetric extension by countably closed forcing yields a model of $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ (cf. [She10, p.3 + p. 15] and [Kar14, Lemma 1]).
In [She16, 0.1], Shelah concludes that $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ is a "*reasonable theory, for which much of combinatorial set theory can be generalized*".

We stress that we aim to work without any large cardinal assumptions. By [AK10], raising the surjective size of $[\kappa]^{\mathrm{cf}\,\kappa}$ requires a measurable cardinal. This again underlines how differently $[\kappa]^{\mathrm{cf}\,\kappa}$ and $\wp(\kappa)$ behave in the absence of the Axiom of Choice; so our setting does not allow for investigating $[\kappa]^{\mathrm{cf}\,\kappa}$.

The starting point of this thesis was the paper "*Violating the Singular Cardinals Hypothesis Without Large Cardinals*" by Motik Gitik and Peter Koepke ([GK12]). Starting off from a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$, they construct a cardinal-preserving symmetric extension $N \supseteq V$ with $N \vDash \mathsf{ZF}$ such that in N, the GCH holds below $\aleph_\omega$, but there is a surjective function $s : \wp(\aleph_\omega) \to \lambda$ for some arbitrarily high fixed cardinal λ in V.

Note that under AC, this theory has rather high consistency strength for $\lambda \geq \aleph_{\omega+2}$, and is inconsistent for $\lambda \geq \aleph_{\omega_4}$ by pcf theory ([She94]). Hence, without the Axiom of Choice, the (surjectively modified) Continuum Function $\theta(\kappa)$ apparently has a lot more freedom. This result gives rise to the thesis that in the theory ZF, the θ-function can take almost arbitrary values on **all** cardinals.

In Chapter 2.1, we present the construction from [GK12]. Many questions arise: Is it possible to generalize the theorem to cardinals of uncountable cofinality? Is it also possible to set the θ-values of several cardinals independently? The forcing notion introduced in [GK12] relies on certain finiteness properties, so DC does not hold in the symmetric extension. Is it possible to modify the general construction and obtain a forcing notion that is countably closed? On page 75 - 77 we discuss what generalizations would be interesting, and sketch basic ideas.

In Chapter 2.2 we state our theorem: Given a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with $\gamma \in \mathrm{Ord}$ and "reasonable" sequences $(\kappa_\eta \mid \eta < \gamma)$, $(\alpha_\eta \mid \eta < \gamma)$ of uncountable cardinals, there is a cardinal-preserving extension $N \supseteq V$ with $N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ such that $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all $\eta < \gamma$. Here, "reasonable" means that the following properties hold:

- $\forall\, \eta < \eta' \;\; \alpha_\eta \leq \alpha'_\eta$
- $\forall\, \eta \;\; \alpha_\eta \geq \kappa_\eta^{++}$
- $\forall\, \eta \;\; \mathrm{cf}\, \alpha_\eta > \omega$
- $\forall\, \eta \; (\alpha_\eta = \alpha^+ \to \mathrm{cf}\, \alpha > \omega)$.

It is not difficult to see that we cannot remove any of the first three properties. The fourth property

$$\forall\, \eta \; (\alpha_\eta = \alpha^+ \to \mathrm{cf}\, \alpha > \omega)$$

is necessary in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$, as well (cf. Chapter 2.2). The more general question whether there could be a model $N \vDash \mathsf{ZF} + \mathsf{DC}$ with cardinals κ, α such that $\theta^N(\kappa) = \alpha^+$ and $cf\, \alpha = \omega$, is adressed in Chapter 2.7, where we show that this is not possible under $N \vDash \neg 0^\sharp$.

In Chapter 2.3, we introduce our countably closed forcing notion $\mathbb{P}$, the basic ingredients of which are based on the forcing notion introduced in [GK12]. We treat limit cardinals and successor cardinals separately, in order to obtain better gaps between the limit cardinals for subsequent factoring arguments: Our forcing notion will be a product $\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_0$ is in charge of setting the θ-values of the limit cardinals κ_η, and $\mathbb{P}_1$ is in charge of setting the θ-values of the successor cardinals $\kappa_\eta = \overline{\kappa_\eta}^{\,+}$.

Roughly speaking, $\mathbb{P}_0$ adds α_η-many κ_η-subsets $(G_i^\eta \mid i < \alpha_\eta)$ to the ground model, which are linked in a certain fashion in order not to accidentally raise the θ-values of the cardinals below. The forcing notion $\mathbb{P}_1$ is a countable support product of Cohen-like forcing notions P^η.

We establish our notion of symmetry in Chapter 2.4. Applying the technique introduced in Chapter 1.2.3, we define a collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$, and an *almost-group* A *of partial* $\mathbb{P}$*-automorphisms for* $\mathcal{D}$ (cf. Definition 1.2.14) in Chapter 2.4.1. We will make sure that the forcing $\mathbb{P}_0$ is *homogeneous with respect to* A, i.e. for any dense set $D \in \mathcal{D}$ and conditions p, $p' \in \mathbb{P}_0$, there exist $\overline{p} \leq p$, $\overline{p}' \leq p'$ with $\overline{p}$, $\overline{p}' \in D$ (such that $\overline{p}$ and $\overline{p}'$ have the same "shape") and an automorphism $\pi \in A$, $\pi : D \to D$ with $\pi\overline{p} = \overline{p}'$. Secondly, we will need that for any pair of generic κ_η-subsets G_i^η and G_j^η with i, $j < \alpha_\eta$, there is an automorphism $\pi \in A$ interchanging them. We turn A into a group $\overline{A}$ by considering equivalence classes $[\pi]$ for $\pi \in A$, where $\pi \sim \sigma$ if and only if π and σ agree on the intersection of their domains.

In Chapter 2.4.2, we introduce our $\overline{A}$-subgroups that will generate a normal filter $\mathcal{F}$ on $\overline{A}$. Firstly, we will have subgroups $\text{Fix}(\eta, i)$ for $\eta < \gamma$, $i < \alpha_\eta$ in order to make sure that any generic κ_η-subset G_i^η is contained in the eventual symmetric extension N. Secondly, for $\lambda < \gamma$, $k < \alpha_\lambda$, we include subgroups H_k^λ such that for any automorphism $\pi \in H_k^\lambda$, there is an interval $[\alpha, \kappa_\lambda)$, with the property that π does not affect the generics G_i^λ for $i \leq k$ "too much" on this interval $[\alpha, \kappa_\lambda)$. This will eventually give rise to a surjective function $f : \wp(\kappa_\lambda) \to k$ in N.

Countable intersections of these subgroups $Fix(\eta, i)$ and H_k^λ generate a normal filter $\mathcal{F}$ on $\overline{A}$ (Lemma 2.4.4).

In Chapter 2.5, we take a V-generic filter G on $\mathbb{P}$, and define $N := V(G)^{\mathcal{F}}$ as the symmetric extension by G and $\mathcal{F}$. Then $N \vDash \mathsf{ZF}$, and $N \vDash \mathsf{DC} + \mathsf{AX}_4$, since the forcing notion $\mathbb{P}$ is countably closed and our normal filter $\mathcal{F}$ is countably complete (see [Kar14, Lemma 1] and [She10, p.3 + p. 15]).

Moreover, an *Approximation Lemma* holds (Lemma 2.5.2): Any set of ordinals located in N can be captured in a "mild" V-generic extension that preserves cardinals and the GCH. Hence, cardinals are V-N-absolute.

It remains to prove that indeed, $\theta^N(\kappa_\eta) = \alpha_\eta$ for all $\eta < \gamma$, which is the task of Chapter 2.6. The direction "$\theta^N(\kappa_\eta) \geq \alpha_\eta$" follows by construction of the groups H_k^λ (see Chapter 2.6.1). Regarding "$\theta^N(\kappa_\eta) \leq \alpha_\eta$" (see Chapter 2.6.2 + 2.6.3), we proceed as follows:

We assume towards a contradiction that there was a surjective function $f : \wp(\kappa_\eta) \to \alpha_\eta$ in N for some $\eta < \gamma$. We fix $\beta < \alpha_\eta$ "large enough", and define a restriction f^β that is obtained from f as follows: In the domain of f^β, we allow only κ_η-subsets contained in those intermediate generic extensions from the *Approximation Lemma* that add for any $\sigma < \gamma$ only those κ_σ-subsets G_i^σ that have indices $i \leq \beta$. We ask ourselves whether f^β is still surjective onto α_η.

First, we assume towards a contradiction that $f^\beta : \text{dom} f^\beta \to \alpha_\eta$ **is** a surjection. We prove that f^β is contained in a model $V[G^\beta \restriction (\eta + 1)]$, which is a V-generic extension by a forcing notion $\mathbb{P}^\beta \restriction (\eta + 1)$, obtained from $\mathbb{P}$ by essentially "cutting off" at height $\eta + 1$ and width β. On the one hand, we prove that $\mathbb{P}^\beta \restriction (\eta + 1)$ preserves all cardinals $\geq \alpha_\eta$; but on the other hand, we will see that $V[G^\beta \restriction (\eta + 1)]$ contains a set $\widetilde{\wp}(\kappa_\eta) \supseteq \text{dom} f^\beta$ with an injection $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow \beta$. Contradiction. Hence, it follows that $f^\beta : \text{dom} f^\beta \to \alpha_\eta$ must not be surjective.

However, considering $\alpha \in \operatorname{rg} f \setminus \operatorname{rg} f^{\beta}$, an isomorphism argument yields a contradiction, again.

We conclude that our assumption of a surjective map $f : \wp(\kappa_\eta) \to \alpha_\eta$ in N must be wrong, and $\theta^N(\kappa_\eta) \leq \alpha_\eta$ follows.

In Chapter 2.6.4 and 2.6.5, we prove that in the symmetric extension N, the θ-values $\theta^N(\lambda)$ of cardinals $\lambda \in (\kappa_\eta, \kappa_{\eta+1})$ or $\lambda \geq \sup\{\kappa_\eta \mid \eta < \gamma\}$ are the smallest possible. This allows us to assume w.l.og. for our construction that the sequence $(\alpha_\eta \mid \eta < \gamma)$ is strictly increasing.

We conclude with several remarks in Chapter 2.7.

The contents of Chapter 2 have appeared in [FK18].

2.1 The Basic Construction

This chapter is concerned with with the paper "*Violating the Singular Cardinals Hypothesis Without Large Cardinals*" by Moti Gitik and Peter Koepke ([GK12]), where they prove the following theorem:

Theorem 2.1.1 ([GK12, Theorem 1])**.** *Let V be a ground model of* ZFC *and* GCH*, and λ a cardinal in V. Then there exists a cardinal-preserving extension $N \supseteq V$ with $N \vDash$* ZF *such that $N \vDash$ "*GCH *holds below $\aleph_\omega$", and $\theta^N(\aleph_\omega) \geq \lambda^+$.*

This theory for $\lambda \geq \aleph_{\omega+2}$ would have large consistency strength under AC (see [Git91]), and in the case that $\lambda \geq \aleph_{\omega_4}$, even contradict Shelah's pcf theory. In other words: Theorem 2.1.1 provides a strong surjective violation of pcf theory in the absence of the Axiom of Choice.

In this chapter, we first present their construction, and then look at possible generalizations. We describe the main issues that we will be dealing with in this thesis and suggest upcoming difficulties.

The procedure in [GK12] can roughly be described as follows: The ground model V is extended by a forcing notion $\mathbb{P}$ which contains firstly, a "square forcing" P_* adding $\aleph_{n+1}$-many Cohen subsets of $\aleph_{n+1}$ for every $n < \omega$, and secondly, a component adding λ-many subsets of $\aleph_\omega$ that are linked with the "square forcing" P_*. The eventual model N is a choiceless submodel of the generic extension, generated by certain equivalence classes of these λ-many adjoined $\aleph_\omega$-subsets.

Let V be a ground model of ZFC + GCH, and λ a cardinal in V. We will now give a definition of $\mathbb{P}$. The first basic ingredient is the following forcing notion P', adding one Cohen subset to each interval $[\aleph_n, \aleph_{n+1})$:

Definition 2.1.2 ([GK12])**.** The forcing $(P', \supseteq, \varnothing)$ consists of all functions $p : \operatorname{dom} p \to 2$ for which there is a sequence $(\delta_n \mid n < \omega)$ with $\delta_n \in [\aleph_n, \aleph_{n+1})$ for all $n < \omega$, such that

$$\operatorname{dom} p = \bigcup_{n<\omega} [\aleph_n, \delta_n).$$

A product analysis shows:

Lemma 2.1.3 ([GK12, Lemma 1]). *The forcing P' preserves cardinals and the* GCH.

The forcing notion P_* is a two-dimensional version of P', adjoining $\aleph_{n+1}$-many Cohen subsets to every interval $[\aleph_n, \aleph_{n+1})$:

Definition 2.1.4 ([GK12]). Denote by $(P_*, \supseteq, \varnothing)$ the forcing notion consisting of all functions $p_* : \operatorname{dom} p_* \to 2$ such that $\operatorname{dom} p_*$ is of the following form: There is a sequence $(\delta_n \mid n < \omega)$ with $\delta_n \in [\aleph_n, \aleph_{n+1})$ for all $n < \omega$, such that

$$\operatorname{dom} p = \bigcup_{n<\omega} [\aleph_n, \delta_n)^2.$$

For $p_* \in P_*$ and $\xi < \aleph_\omega$, let $p_*(\xi) := \{ (\zeta, p_*(\xi, \zeta)) \mid (\xi, \zeta) \in \operatorname{dom} p_* \}$ denote the ξ-th section of p_*.

As in Lemma 2.1.3, it follows that also P_* preserves all cardinals and the GCH.

We are now ready to define the eventual forcing notion $\mathbb{P}$. Every condition $p \in \mathbb{P}$ is of the form $p = (p_*, (p_i, a_i)_{i<\lambda})$ where $p_* \in P_*$ and $p_i \in P'$ for all $i < \lambda$. The *linking ordinals* a_i determine how the i-th generic $\aleph_\omega$-subset G_i will be eventually linked with the P_*-generic filter G_*.

Definition 2.1.5 ([GK12, Definition 1]). Let $\mathbb{P}$ be the collection of all $p = (p_*, (p_i, a_i)_{i<\lambda})$ with the following properties:

- The *support* of p, $\operatorname{supp} p$, is a finite subset of λ, and $p_i = a_i = \varnothing$ whenever $i \notin \operatorname{supp} p$.
- There is a sequence $(\delta_n \mid n < \omega)$ with $\delta_n \in [\aleph_n, \aleph_{n+1})$ for all $n < \omega$, such that $p_* : \bigcup_{n<\omega} [\aleph_n, \delta_n)^2 \to 2$ and $p_i : \bigcup_{n<\omega} [\aleph_n, \delta_n) \to 2$ for all $i \in \operatorname{supp} p$.

  Let *dom* $p := \bigcup_{n<\omega} [\aleph_n, \delta_n)$.
- Whenever $i \in \operatorname{supp} p$, then a_i is a finite subset of $\aleph_\omega$ with $|a_i \cap [\aleph_n, \aleph_{n+1})| \leq 1$ for all $n < \omega$.

  If $i_0 \neq i_1$, then $a_{i_0} \cap a_{i_1} = \varnothing$. (We call this the *independence property*).

Concerning the partial ordering "$\leq$", any linking ordinal $\{\xi\} = a_i \cap [\aleph_n, \aleph_{n+1})$ settles that whenever $q \leq p$, then the extension $q_i \supseteq p_i$ within in the interval $[\aleph_n, \aleph_{n+1})$ is determined by $q_*(\xi)$:

For $p = (p_*, (p_i, a_i)_{i<\lambda})$, $q = (q_*, (q_i, b_i)_{i<\lambda}) \in \mathbb{P}$, we let $q \leq p$ if the following holds: $q_* \supseteq p_*$; $q_i \supseteq p_i$, $b_i \supseteq a_i$ for all $i \in \operatorname{supp} p$; and whenever $\zeta \in (\operatorname{dom} q_i \setminus \operatorname{dom} p_i) \cap [\aleph_n, \aleph_{n+1})$ with $a_i \cap [\aleph_n, \aleph_{n+1}) = \{\xi\}$, then $\xi \in \operatorname{dom} q$ with $q_i(\zeta) = q_*(\xi, \zeta)$ (we call this the *linking property*).

The *maximal element* of $\mathbb{P}$ is $\mathbb{1} := (\varnothing, (\varnothing, \varnothing)_{i<\lambda})$.

Let G be a V-generic filter on $\mathbb{P}$. It induces

$$G_* := \{q_* \in P_* \mid \exists\, p = (p_*, (p_i, a_i)_{i<\lambda}) \in G \quad q_* \subseteq p_*\},$$

and for $i < \lambda$:

$$G_i := \{q_i \in P' \mid \exists\, p = (p_*, (p_i, a_i)_{i<\lambda}) \in G \ \ q_i \subseteq p_i\}.$$

As usual, these filters G_*, G_i are identified with their union $\bigcup G_*$, $\bigcup G_i$.

Let now $\xi \in [\aleph_n, \aleph_{n+1})$. We denote by

$$G_*(\xi) := \Big\{ q : [\aleph_n, \delta_n) \to 2 \ \Big| \ \delta_n \in [\aleph_n, \aleph_{n+1})\,,\ \exists\, p = (p_*, (p_i, a_i)_{i<\lambda}) \in G \ : \\ \forall\, \zeta \in \operatorname{dom} q \ \ q(\zeta) = p_*(\xi, \zeta) \Big\}$$

the ξ-th section of G_*. Identifying $G_*(\xi)$ with $\bigcup G_*(\xi)$, it follows that $G_*(\xi) : [\aleph_n, \aleph_{n+1}) \to 2$.

For $i < \lambda$, let $g_i := \bigcup\{a_i \mid p = (p_*, (p_i, a_i)_{i<\lambda}) \in G\}$. Then $g_i \subseteq \aleph_\omega$ hits every interval $[\aleph_n, \aleph_{n+1})$ in exactly one point, and by the *independence property* it follows that $g_{i_0} \cap g_{i_1} = \varnothing$ whenever $i_0 \neq i_1$.

The *linking property* implies that any $G_i \upharpoonright [\aleph_n, \aleph_{n+1})$ is equal to some $G_*(\xi)$ on a final segment, with $\{\xi\} = g_i \cap [\aleph_n, \aleph_{n+1})$. Hence, by the *independence property* it follows that distinct G_{i_0} and G_{i_1} correspond to parallel disjoint "paths through the forcing P_*" ([GK12, p.6]).

Before defining the symmetric submodel model N, we need the following notions:

Definition 2.1.6 ([GK12, Chapter 2])**.** For a set D and functions $F : D \to 2$, $F' : D \to 2$, the *pointwise exclusive or* $F \oplus F' : D \to 2$ is defined as follows: $(F \oplus F')(x) = 0$ *if* $F(x) = F'(x)$ *and* $(F \oplus F')(x) = 1$ *if* $F(x) \neq F'(x)$.

For functions $F : \aleph_\omega \smallsetminus \aleph_0 \to 2$, $F' : \aleph_\omega \smallsetminus \aleph_0 \to 2$, we set $F \sim F'$ if there exists $n < \omega$ with

- $(F \oplus F') \upharpoonright \aleph_{n+1} \in V[G_*]$,
- $(F \oplus F') \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V$.

Then "$\sim$" is an equivalence relation on $2^{\aleph_\omega \smallsetminus \aleph_0}$.

For a function $F : \aleph_\omega \smallsetminus \aleph_0 \to 2$, we denote by

$$\widetilde{F} := \{F' : \aleph_\omega \smallsetminus \aleph_0 \to 2 \mid F' \sim F\}$$

its equivalence class by "$\sim$".

The eventual model N will be of the form $N = \mathsf{HOD}^{V[G]}(V \cup A)$, where A is a transitive set. We refer to [Jec06, p. 194 – 196] for a detailed introduction to OD and HOD, and merely use that $X \in N$ if and only if for every $Y \in \mathsf{TC}(\{X\})$ there is a formula φ and parameters $v \in V$, $a_0, \dots, a_{n-1} \in A$, such that

$$Y = \big\{ y \in V[G] \ \big| \ V[G] \vDash \varphi(y, v, A, a_0, \dots, a_{n-1}) \big\}.$$

Definition 2.1.7. We define in $V[G]$:

$$T_* := \{X \in V[G_*] \mid \exists n < \omega \ X \subseteq \aleph_n\}.$$

For $i < \lambda$, we have the equivalence class

$$\widetilde{G_i} := \{F : \aleph_\omega \setminus \aleph_0 \to 2 \mid F \sim G_i\},$$

and set

$$\vec{G} := (\widetilde{G_i} \mid i < \lambda).$$

Let

$$N := \mathsf{HOD}^{V[G]}\Big(V \cup \mathsf{TC}(\{T_*, \vec{G}\})\Big).$$

Then $N \vDash \mathsf{ZF}$.

For Theorem 2.1.1, it remains to show that cardinals are N-V-absolute, $N \vDash$ "GCH holds below $\aleph_\omega$", and there exists in N a surjective function $f : \wp(\aleph_\omega) \to \lambda$.

We only sketch the basic ideas here, and refer to [GK12, Chapter 4 + 5] for detailed proofs.

Lemma 2.1.8 ([GK12, Lemma 3]). *There exists in N a surjective function $f : \wp^N(\aleph_\omega^V) \to \lambda$.*

Proof. Let $i, j < \lambda$ with $i \neq j$, and assume towards a contradiction that $G_i \sim G_j$. Then there exists $n < \omega$ with $v := (G_i \oplus G_j) \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V$, which contradicts the density of the set

$$D := \{p = (p_*, (p_i, a_i)_{i<\lambda}) \in \mathbb{P} \mid \exists \xi \in [\aleph_{n+1}, \aleph_\omega) \ p_i(\xi) \oplus p_j(\xi) \neq v(\xi)\}.$$

Hence, $G_i \nsim G_j$ whenever $i \neq j$, and we can define in N a function $f : \wp(\aleph_\omega^V) \to \lambda$ as follows: Let $f(X) := i$ for $X \in \widetilde{G_i}$, if such i exists, and $f(X) = 0$, else. Note that the definition of f uses only the sequence $\vec{G}$, which is contained in N. Moreover, f is well-defined by what we have just shown, and f is surjective, since $G_i \in N$ for all $i < \lambda$. $\square$

The model N can be approximated by fairly "mild" V-generic extensions, which is crucial for keeping control over the surjective size of $\wp(\aleph_\omega^V)$.

Lemma 2.1.9 ([GK12, Lemma 5], *Approximation Lemma*). *Let $X \in N$ with $X \subseteq$ Ord. Then there are $n < \omega$ and finitely many $i_0, \ldots, i_{l-1} < \lambda$ with*

$$X \in V[G_* \upharpoonright (\aleph_{n+1}^V)^2 \times G_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V) \times \cdots \times G_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V)],$$

where $G_* \upharpoonright (\aleph_{n+1}^V)^2 \times G_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V) \times \cdots \times G_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V)$ is a V-generic filter over the forcing notion $P_* \upharpoonright (\aleph_{n+1}^V)^2 \times (P' \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V))^l$.

The proof uses an isomorphism argument relying on homogeneity properties of $\mathbb{P}$.

Now, a product analysis shows that cardinals are absolute between V and $V[G_* \upharpoonright (\aleph_{n+1}^V)^2 \times G_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V) \times \dots \times G_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V)]$ (cf. [GK12, Lemma 6]); hence, cardinals are absolute between V and N. In particular, $\aleph_\omega^V = \aleph_\omega^N$.

Altogether, $N \supseteq V$ is a cardinal-preserving extension with $N \models \mathsf{ZF}$ and $\theta^N(\aleph_\omega) \geq \lambda^+$.

Below $\aleph_\omega$, the situation in N is to large extend like the situation in V:

Lemma 2.1.10 ([GK12, Lemma 8])**.** *GCH holds in N below $\aleph_\omega$: For every $n < \omega$, there exists in N a bijection $\mathsf{b}_n : \wp(\aleph_n) \to \aleph_{n+1}$.*

Summing up 2.1.1 provides a strong "surjective violation of pcf theory without the use of large cardinals" ([GK12, p.2]).

The starting point of this thesis was to look at possible generalizations.

A) The first obvious question to ask is whether $\theta(\aleph_\omega) = \lambda^+$ (not only "$\geq$") holds in the constructed model N, in the case that $\lambda \geq \aleph_\omega^+$. Indeed, assuming that there was a surjective function $f : \wp(\aleph_\omega) \to \overline{\lambda}$ for some $\overline{\lambda} > \lambda$, one can apply an isomorphism argument and obtain a contradiction. Hence, $\theta^N(\aleph_\omega) = \lambda^+$.

B) Chapter 6 in [GK12] suggests that their construction can be straightforwardly generalized to any cardinal κ of countable cofinality. In the case that $cf\, \kappa > \omega$, several modifications yield the same result: Let $(\kappa_j \mid j < \operatorname{cf} \kappa)$ denote a normal cofinal sequence in κ. The intervals $[\kappa_j, \kappa_{j+1})$ for $j < \operatorname{cf} \kappa$ now take the role of the intervals $[\aleph_n, \aleph_{n+1})$ for $n < \omega$. Regarding the forcing notions P' and P_*, one has to require that the conditions are bounded below all regular limit cardinals κ_j. The linking ordinals a_i are bounded below κ (instsead of finite), and hit every interval $[\kappa_j, \kappa_{j+1})$ in at most one point. The *support* of the conditions p remains finite. Then a similar proof shows that the constructed ZF-model $N \supseteq V$ is a cardinal-preserving extension with $\theta^N(\kappa) = \lambda^+$, and $\theta^N(\alpha) = \alpha^{++}$ for all $\alpha < \kappa$.

C) In a setting without AC, where the power sets $\wp(\kappa)$ are not necessarily well-ordered, it can happen that $\theta(\kappa) = \mu$ is a limit cardinal: If $\theta(\kappa) = \mu$, there exist surjections $f : \wp(\kappa) \to \alpha$ for all $\alpha < \mu$, but there is no surjective function $f : \wp(\kappa) \to \mu$. This situation can not be realized by the same construction:

   If one tries to use equivalence classes $\widetilde{G_i}$ for $i < \mu$ as in Definition 2.1.7, then for any $\alpha < \mu$, the sequence $(\widetilde{G_i} \mid i < \alpha)$ would have to be contained in N, while the whole sequence of equivalence classes $\overrightarrow{G} = (\widetilde{G_i} \mid i < \lambda)$ must **not** be contained in N.

   Instead of taking some $\mathsf{HOD}(\mathsf{A})$-inner model of $V[G]$, we will construct our ZF-model as a symmetric extension $N = V(G)$, using the technique of (partial) automorphisms and symmetric names as described in Chapter 1.2. By choosing the normal filter $\mathcal{F}$ carefully, we will make sure that any sequence $(\widetilde{G_i} \mid i < \alpha)$ for $\alpha < \mu$ has a symmetric name. The whole sequence $(\widetilde{G_i} \mid i < \mu)$ will *not* have a symmetric name, and we will prove that indeed, there exists no surjective function $f : \wp(\kappa) \to \mu$ in N.

D) The key question is whether it is possible to treat several cardinals κ_η at the same time and set their θ-values independently. More precisely: Given "reasonable" sequences of cardinals $(\kappa_\eta \mid \eta < \gamma)$, $(\alpha_\eta \mid \eta < \gamma)$ in V, is it possible to construct a cardinal-preserving symmetric extension $N \supseteq V$ such that $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all $\eta < \gamma$?

Dealing with "many" cardinals κ_η at the same time requires essentially new ideas; in particular, when the sequence $(\kappa_\eta \mid \eta < \gamma)$ has limit points. For instance, it is not any more possible to work with initial segments of G_* (such as $G_* \upharpoonright \kappa_\eta^2$), since they would interfere with the generic $\kappa_{\overline{\eta}}$-subsets $G_i^{\overline{\eta}}$ for $\overline{\eta} < \eta$. For this reason, we adjust the approximation models, and establish that the symmetric extension N can be approximated by intermediate generic extensions of the form

$$V[G_{i_0}^{\eta_0} \times \cdots \times G_{i_{m-1}}^{\eta_{m-1}}],$$

where G_i^η denotes the i-the generic κ_η-subset adjoined by the generic filter G.

The overall construction can be described as follows (assuming w.l.o.g. that the sequence $(\kappa_\eta \mid \eta < \gamma)$ is *normal*, i.e. strictly increasing and closed): For every $\kappa_{\eta+1}$ a limit cardinal, we take a normal sequence $(\kappa_{\eta,j} \mid j < \operatorname{cf} \kappa_{\eta+1})$ cofinal in $\kappa_{\eta+1}$ with $\kappa_{\eta,0} = \kappa_\eta$. First, we will define our forcing notion $\mathbb{P}_0$, which is a generalization of the forcing $\mathbb{P}$ from Definition 2.1.5. The basic ingredient P^η (for $\eta < \gamma$) is defined as follows: $\mathbb{P}^\eta$ is the collection of all $p : \mathrm{dom} \to 2$ for which there is a sequence $(\delta_{\nu,j} \mid \nu < \eta\,,\, j < \operatorname{cf} \kappa_{\nu+1})$ with $\delta_{\nu,j} \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for all $\nu < \eta$, $j < \operatorname{cf} \kappa_{\nu+1}$ such that

$$\mathrm{dom}\, p = \bigcup_{\substack{\nu<\eta \\ j<\operatorname{cf} \kappa_{\nu+1}}} [\kappa_{\nu,j}, \delta_{\nu,j}),$$

and for any regular $\kappa_{\nu,j}$, the domain $\mathrm{dom}\, p \cap \kappa_{\nu,j}$ is bounded below $\kappa_{\nu,j}$. The forcing notion P_* is a "square version" of P^γ.

Any condition $p \in \mathbb{P}_0$ is of the form $p = (p_*, (p_i^\eta, a_i^\eta)_{\eta<\gamma\,,\, i<\alpha_\eta})$, where $p_* \in P_*$, $p_i^\eta \in P^\eta$ for all $\eta < \gamma$; and the pairwise disjoint linking ordinals $a_i^\eta \subseteq \kappa_\eta$ are all bounded below κ_η and hit every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$ in at most one point.

However, we will need better closure properties for certain product analyses. Therefore, we treat successor cardinals $\kappa_\eta = \overline{\kappa_\eta}^+$ in a separate forcing $\mathbb{P}_1$ (which will be a product of Cohen-like forcing notions) and set $\mathbb{P} := \mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_0$ is in charge of the limit cardinals and $\mathbb{P}_1$ is in charge of the successor cardinals. Then for any $\kappa_{\eta+1}$ a limit cardinal, we choose the normal cofinal sequence $(\kappa_{\eta,j} \mid j < \operatorname{cf} \kappa_{\eta+1})$ in such a way that $\kappa_{\eta,j+1} \geq \kappa_{\eta,j}^{++}$ holds for all $j < \operatorname{cf} \kappa_{\eta+1}$.

Our model $N = V(G)$ will be a symmetric extension by $\mathbb{P}$, and we will show that N preserves cardinals and $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all $\eta < \gamma$.

The only requirements on the sequences $(\kappa_\eta \mid \eta < \gamma)$ and $(\alpha_\eta \mid \eta < \gamma)$ are the obvious ones: weak monotonicity and $\alpha_\eta \geq \kappa_\eta^{++}$ for all $\eta < \gamma$.

E) Finally, we ask whether it is also possible to work with a countably closed forcing notion $\mathbb{P}$ and a countably complete filter $\mathcal{F}$ generating the symmetric extension. Then $N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$, providing a model where the Axiom of Choice fails but still, surprisingly much of set theory can be realized (see [She16, 0.1]).

In order to obtain a countably closed forcing notion, finiteness properties in the forcing construction have to be replaced by the property of being countable. Regarding the linking ordinals a_i^η however, requiring that any $a_i^\eta \subseteq \kappa_\eta$ is a bounded subset of κ_η is not any more possible, since in the case that $\operatorname{cf} \kappa_\eta = \omega$ this would conflict with the requirement of $\mathbb{P}$ being countably closed.

Instead, the linking ordinals $a_i^\eta \subseteq \kappa_\eta$ will now hit every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$ in *exactly* one point. This adjustment makes a substantial difference: For any V-generic filter $G = G_0 \times G_1$ on $\mathbb{P}$ and $\sigma < \gamma$, $m < \alpha_\sigma$ with κ_σ a limit cardinal, the collection of linking ordinals

$$g_m^\sigma := \bigcup \left\{ a_m^\sigma \;\middle|\; p = (p_*, (p_i^\eta, a_i^\eta)_{\eta<\gamma, i<\alpha_\eta}) \in G_0 \right\}$$

is now contained in the ground model V. (Indeed, for any condition $p \in G_0$, $p = (p_*, (p_i^\eta, a_i^\eta)_{\eta<\gamma, i<\alpha_\eta})$ with $(\sigma, m) \in \operatorname{supp} p_0$, it follows that $g_m^\sigma = a_m^\sigma$.) By countable support, this implies that also countable sequences $(g_{m_j}^{\sigma_j} \mid j < \omega)$ of linking ordinals are in the ground model. However, for $\sigma < \gamma$ and κ_σ a limit cardinal, the sequence $(g_i^\sigma \mid i < \alpha_\sigma)$ can **not** be contained in V nor in the symmetric extension N: By the *independence property*, the sequence $(g_i^\sigma \mid i < \alpha_\sigma)$ would blow up any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\sigma$ to size α_σ and thereby collapse cardinals.

The aim of this Chapter 2 is to modify and generalize the forcing notion from [GK12] according to A) - E) and prove that given a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with "reasonable" sequences of cardinals $(\kappa_\eta \mid \eta < \gamma)$ and $(\alpha_\eta \mid \eta < \gamma)$ — see Chapter 2.2 for a precise definition of the term "reasonable" —, one can construct a cardinal-preserving symmetric extension $N \supseteq V$ with $N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ such that $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all $\eta < \gamma$.

In other words: Every possible behavior of the θ-function in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ can be realized. This version of Easton's Theorem for "nice" ZF-models with little choice, including regular *and* singular cardinals, is in sharp contrast to the situation in ZFC.

2.2 The Theorem

We start from a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ and a *reasonable behavior of the θ-function*: There are sequences of uncountable cardinals $(\kappa_\eta \mid 0 < \eta < \gamma)$ and $(\alpha_\eta \mid 0 < \eta < \gamma)$ in V (where γ is an ordinal) for which we aim to construct a symmetric extension $N \supseteq V$ with $N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$, such that V and N have the same cardinals and cofinalities and $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all η.
(Later on, we will set $\kappa_0 := \aleph_0$, $\alpha_0 := \aleph_2$ for technical reasons – therefore, we talk about sequences $(\kappa_\eta \mid 0 < \eta < \gamma)$, $(\alpha_\eta \mid 0 < \eta < \gamma)$ here, excluding κ_0 and α_0.)

First, we want to discuss what properties the sequences $(\kappa_\eta \mid 0 < \eta < \gamma)$ and $(\alpha_\eta \mid 0 < \eta < \gamma)$ must have to allow for such construction.

W.l.o.g. we can assume that $(\kappa_\eta \mid 0 < \eta < \gamma)$ is strictly increasing and closed.

The following conditions must be satisfied:

- For $\eta < \eta'$, it follows from $\kappa_\eta < \kappa_{\eta'}$ that $\alpha_\eta \leq \alpha_{\eta'}$ must hold, i.e. the sequence $(\alpha_\eta \mid 0 < \eta < \gamma)$ must be increasing.

- For any cardinal κ, it is possible to construct a surjection $s : \wp(\kappa) \to \kappa^+$ without making use of the Axiom of Choice. Hence, $\alpha_\eta \geq \kappa_\eta^{++}$ must hold for all η.

- Since $N \vDash \mathsf{AC}_\omega$, it follows that $\operatorname{cf} \alpha_\eta > \omega$ for all η: Assume towards a contradiction there were cardinals κ, α with $\theta^N(\kappa) = \alpha$, but $\operatorname{cf}^N(\alpha) = \omega$. Let $\alpha = \bigcup_{i<\omega} \alpha_i$. By definition of $\theta^N(\kappa)$, it follows that for every $i < \omega$, there exists in N a surjection from $\wp(\kappa)$ onto α_i. Now, AC_ω allows us to pick in N a sequence $(s_i \mid i < \omega)$ such that each $s_i : \wp(\kappa) \to \alpha_i$ is a surjection. This yields a surjective function $\overline{s} : \wp(\kappa) \times \omega \to \alpha$, defined by setting $\overline{s}(X, i) := s_i(X)$ for each $(X, i) \in \wp(\kappa) \times \omega$; which can be easily turned into a surjection $s : \wp(\kappa) \to \alpha$. Contradiction, since $\theta^N(\kappa) = \alpha$.
  Hence, it follows that $\operatorname{cf} \alpha_\eta > \omega$ for all η.

- Finally, for every α_η a successor cardinal with $\alpha_\eta = \alpha^+$, we will need that $\operatorname{cf} \alpha > \omega$.
  In our setting here, it is not possible to drop this requirement: We start from a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with sequences $(\kappa_\eta \mid 0 < \eta < \gamma)$, $(\alpha_\eta \mid 0 < \eta < \gamma)$, and aim to construct $N \supseteq V$ with $N \vDash \mathsf{ZF} + \mathsf{DC}$ such that V and N have the same cardinals and cofinalities and $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all η. If there was some cardinal κ_η with $\theta^N(\kappa_\eta) = \alpha^+$, where $\operatorname{cf} \alpha = \omega$, one could construct in N a surjective function $\overline{s} : \wp(\kappa_\eta) \to \alpha^+$ as follows:
  Take a surjection $s : \wp(\kappa_\eta) \to \alpha$ in N. Firstly, the canonical bijection $\kappa_\eta \leftrightarrow \kappa_\eta \times \omega$ gives a surjection $s_0 : 2^{\kappa_\eta} \to (2^{\kappa_\eta})^\omega$. Secondly, the surjection $s : \wp(\kappa_\eta) \to \alpha$ yields in N a surjection $s_1 : (2^{\kappa_\eta})^\omega \to \alpha^\omega$, defined by setting $s_1(X_i \mid i < \omega) := (s(X_i) \mid i < \omega)$. Then s_1 is surjective, since for any $(\alpha_i \mid i < \omega) \in \alpha^\omega$ given, one can use AC_ω to obtain a sequence $(Y_i \mid i < \omega)$ with $Y_i \in s^{-1}[\{\alpha_i\}]$ for all $i < \omega$. Then $s_1(Y_i \mid i < \omega) = (\alpha_i \mid i < \omega)$. Thirdly, it follows from $\operatorname{cf} \alpha = \omega$ that there is a surjection $\overline{s_2} : \alpha^\omega \to \alpha^+$ in V. Then $\overline{s_2} \in N$, and since $(\alpha^\omega)^N \supseteq (\alpha^\omega)^V$ and $(\alpha^+)^N = (\alpha^+)^V$, this gives a surjection $s_2 : \alpha^\omega \to \alpha^+$ in N.
  Thus, it follows that $s_2 \circ s_1 \circ s_0 : 2^{\kappa_\eta} \to \alpha^+$ is a surjective function in N; contradicting that $\theta^N(\kappa_\eta) = \alpha^+$.

  Hence, in our setting where we want to extend a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ cardinal-preservingly and obtain $N \vDash \mathsf{ZF} + \mathsf{DC}$, it is not possible to have $\alpha_\eta = \alpha^+$ with $\operatorname{cf} \alpha = \omega$.

  The following question arises: More generally, without referring to a ground model V, could there be $N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ with cardinals κ, α, such that $\operatorname{cf}^N(\alpha) = \omega$ and $\theta^N(\kappa) = \alpha^+$? The answer is *no*: Let $s : \wp(\kappa) \to \alpha$ denote a surjective function in N. Then with DC, it follows as before that there is also a surjective function

$s_1 : (2^\kappa)^\omega \to \alpha^\omega$ in N; and we also have a surjective function $s_0 : 2^\kappa \to (2^\kappa)^\omega$. It remains to make sure that there is also a surjection $s_2 : \alpha^\omega \to \alpha^+$ in N.

We proceed as follows: First, we show that there is no surjective function $\chi : \alpha \to [\alpha]^\omega$, then we use AX_4. For any set $M \in [\alpha]^\omega$, we denote by $(M(i) \mid i < \omega)$ its increasing enumeration. We apply a diagonalization argument similar as in *König's Theorem*: Let $\alpha = \bigcup_{i<\omega} \alpha_i$, and assume towards a contradiction that there **was** a surjection $\chi : \bigcup_{i<\omega} \alpha_i \to [\alpha]^\omega$. For every $i < \omega$, the set $\chi[\alpha_i]$ consists of countable $M \subseteq \alpha$, and we let $A_i := \{M(i) \mid M \in \chi[\alpha_i]\}$. Then $A_i \subseteq \alpha$ with $|A_i| \leq \alpha_i < \alpha$. We take a strictly increasing sequence $(\beta_j \mid j < \omega)$ such that $\beta_j \in \alpha \smallsetminus A_j$ for every $j < \omega$, and let $\overline{M} := \{\beta_j \mid j < \omega\}$. By surjectivity of χ, there must be $\overline{\alpha} < \alpha$ with $\chi(\overline{\alpha}) = \overline{M}$. Take $i < \omega$ with $\overline{\alpha} < \alpha_i$. Then $\overline{M} \in \chi[\alpha_i]$; hence, $\beta_i = \overline{M}(i) \in A_i$. Contradiction.

Hence, it follows that there can not be a surjective function $\chi : \alpha \to [\alpha]^\omega$ in N. Since $[\alpha]^\omega$ is well-ordered by AX_4, it follows that there must be a surjection $\overline{s}_2 : [\alpha]^\omega \to \alpha^+$. Together with the canonical surjection $\alpha^\omega \twoheadrightarrow [\alpha]^\omega$ (mapping any function $f : \omega \to \alpha$ to its range $\operatorname{rg} f = \{f(n) \mid n < \omega\}$ if $\operatorname{rg} f$ is countable, and to an arbitrary set $X \in [\alpha]^\omega$, else) this yields a surjection $s_2 : \alpha^\omega \to \alpha^+$ as desired.

Thus, $s_2 \circ s_1 \circ s_0 : 2^\kappa \to \alpha^+$ is a surjective function in N; which gives the desired contradiction.

We conclude that all the requirements on the sequences $(\kappa_\eta \mid 0 < \eta < \gamma)$ and $(\alpha_\eta \mid 0 < \eta < \gamma)$ listed above are necessary for a model $N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$.

In addition, one could ask if there exists a model $N \vDash \mathsf{ZF} + \mathsf{DC}$ (without AX_4) with cardinals κ, α such that $\theta^N(\kappa) = \alpha^+$ and $\operatorname{cf}^N(\alpha) = \omega$. It is not difficult to see that this is not possible under $\neg 0^\sharp$ (cf. Chapter 2.7); and we conclude that without large cardinal assumptions, it is not possible to construct a model $N \vDash \mathsf{ZF} + \mathsf{DC}$ with κ, α such that $\theta^N(\kappa) = \alpha^+$ and $\operatorname{cf} \alpha = \omega$.

Our main theorem states that the properties listed above are the only restrictions on the θ-function for set-many uncountable cardinals in $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$:

Theorem. *Let V be a ground model of $\mathsf{ZFC} + \mathsf{GCH}$ with $\gamma \in \mathrm{Ord}$ and sequences of uncountable cardinals $(\kappa_\eta \mid 0 < \eta < \gamma)$ and $(\alpha_\eta \mid 0 < \eta < \gamma)$, such that $(\kappa_\eta \mid 0 < \eta < \gamma)$ is strictly increasing and closed and the following properties hold:*

- $\forall\, 0 < \eta < \eta' < \gamma \quad \alpha_\eta \leq \alpha_{\eta'}$, *i.e. the sequence $(\alpha_\eta \mid 0 < \eta < \gamma)$ is increasing,*
- $\forall\, 0 < \eta < \gamma \quad \alpha_\eta \geq \kappa_\eta^{++}$,
- $\forall\, 0 < \eta < \gamma \quad \operatorname{cf} \alpha_\eta > \omega$,
- $\forall\, 0 < \eta < \gamma \quad (\alpha_\eta = \alpha^+ \to \operatorname{cf} \alpha > \omega)$.

Then there is a cardinal- and cofinality-preserving extension $N \supseteq V$ with $N \vDash \mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$ such that that $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all $0 < \eta < \gamma$.

In our construction, we will make sure that for any cardinal λ in a "gap" $(\kappa_\eta, \kappa_{\eta+1})$, the value $\theta^N(\lambda)$ is the smallest possible, i.e. $\theta^N(\lambda) = \max\{\alpha_\eta, \lambda^{++}\}$. Moreover, setting $\kappa_\gamma := \bigcup\{\kappa_\eta \mid 0 < \eta < \gamma\}$, $\alpha_\gamma := \bigcup\{\alpha_\eta \mid 0 < \eta < \gamma\}$, we will also make sure that $\theta^N(\lambda)$ takes the smallest possible value for every $\lambda \geq \kappa_\gamma$: We will have $\theta^N(\lambda) = \max\{\alpha_\gamma^{++}, \lambda^{++}\}$ in the case that $\operatorname{cf}\alpha_\gamma = \omega$; $\theta^N(\lambda) = \max\{\alpha_\gamma^{+}, \lambda^{++}\}$ in the case that $\alpha_\gamma = \alpha^+$ for some cardinal α with $\operatorname{cf}\alpha = \omega$; and $\theta^N(\lambda) = \max\{\alpha_\gamma, \lambda^{++}\}$, else.

This allows us to assume w.l.o.g. that the sequence $(\alpha_\eta \mid 0 < \eta < \gamma)$ is *strictly* increasing: If not, one can start with the original sequences $(\kappa_\eta \mid 0 < \eta < \gamma)$ and $(\alpha_\eta \mid 0 < \eta < \gamma)$, and successively strike out all κ_η for which the value α_η is not larger than the values $\alpha_{\overline{\eta}}$ before. This procedure results in sequences $(\widetilde{\kappa}_\eta \mid 0 < \eta < \widetilde{\gamma}) := (\kappa_{s(\eta)} \mid 0 < \eta < \widetilde{\gamma})$ and $(\widetilde{\alpha}_\eta \mid 0 < \eta < \widetilde{\gamma}) := (\alpha_{s(\eta)} \mid 0 < \eta < \widetilde{\gamma})$ for some $\widetilde{\gamma} \leq \gamma$ and a strictly increasing function $s : \widetilde{\gamma} \to \gamma$, such that $\widetilde{\alpha}_{\widetilde{\gamma}} := \bigcup\{\widetilde{\alpha}_\eta \mid 0 < \eta < \widetilde{\gamma}\} = \bigcup\{\alpha_{s(\eta)} \mid 0 < \eta < \widetilde{\gamma}\} = \bigcup\{\alpha_\eta \mid 0 < \eta < \gamma\} = \alpha_\gamma$, and $(\widetilde{\alpha}_\eta \mid 0 < \eta < \widetilde{\gamma}) = (\alpha_{s(\eta)} \mid 0 < \eta < \widetilde{\gamma})$ is strictly increasing. If we then use the sequences $(\widetilde{\kappa}_\eta \mid 0 < \eta < \widetilde{\gamma})$, $(\widetilde{\alpha}_\eta \mid 0 < \eta < \widetilde{\gamma})$ for our construction and make sure that not only $\theta^N(\widetilde{\kappa}_\eta) = \widetilde{\alpha}_\eta$ holds for all $0 < \eta < \widetilde{\gamma}$, but additionally, $\theta^N(\lambda)$ takes the smallest possible value for all cardinals λ within the "gaps" $(\widetilde{\kappa}_\eta, \widetilde{\kappa}_{\eta+1})$, and for all cardinals $\lambda \geq \widetilde{\kappa}_{\widetilde{\gamma}} := \bigcup\{\widetilde{\kappa}_\eta \mid 0 < \eta < \widetilde{\gamma}\}$; then it follows, that for all κ_η in the original sequence $(\kappa_\eta \mid 0 < \eta < \gamma)$, the values $\theta^N(\kappa_\eta) = \alpha_\eta$ are as desired.

Hence, from now on, we assume w.l.o.g. that the sequence $(\alpha_\eta \mid 0 < \eta < \gamma)$ is **strictly increasing**.

2.3 The Forcing

In this chapter, we define our forcing notion $\mathbb{P}$.

We start from a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with sequences $(\kappa_\eta \mid 0 < \eta < \gamma)$, $(\alpha_\eta \mid 0 < \eta < \gamma)$ that have all the properties mentioned in Chapter 2.2.

We will have to treat limit cardinals and successor cardinals separately. Let $Lim := \{0 < \eta < \gamma \mid \kappa_\eta \text{ is a limit cardinal}\}$, and $Succ := \{0 < \eta < \gamma \mid \kappa_\eta \text{ is a successor cardinal}\}$. For $\eta \in \text{Succ}$, we denote by $\overline{\kappa_\eta}$ the cardinal predecessor of κ_η; i.e. $\kappa_\eta = \overline{\kappa_\eta}^{\,+}$. Our forcing will be a product $\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_0$ deals with the limit cardinals κ_η, and $\mathbb{P}_1$ is in charge of the successor cardinals.

The forcing $\mathbb{P}_0$ is a generalized version of the forcing notion in [GK12].
Roughly speaking, for every $\eta \in \text{Lim}$ we add α_η-many κ_η-subsets, which will be linked in a certain fashion, in order to make sure that not too many κ-subsets for cardinals $\kappa < \kappa_\eta$ make their way into the eventual model N.

For technical reasons, let $\kappa_0 := \aleph_0$, $\alpha_0 := \aleph_2$. For all η with $\eta + 1 \in \text{Lim}$, we take a sequence of cardinals $(\kappa_{\eta,j} \mid j < \operatorname{cf}\kappa_{\eta+1})$ cofinal in $\kappa_{\eta+1}$, such that $\kappa_{\eta,0} = \kappa_\eta$, the sequence $(\kappa_{\eta,j} \mid j < \operatorname{cf}\kappa_{\eta+1})$ is strictly increasing and closed, and any $\kappa_{\eta,j+1}$ is a successor cardinal with $\kappa_{\eta,j+1} \geq \kappa_{\eta,j}^{++}$ for all $j < \operatorname{cf}\kappa_{\eta+1}$.
These "gaps" between the cardinals $\kappa_{\eta,j}$ and $\kappa_{\eta,j+1}$ will be necessary for further factoring arguments.

For all $0 < \eta < \gamma$ for which $\eta + 1 \in \mathrm{Succ}$, i.e. $\kappa_{\eta+1}$ is a successor cardinal, we set $\kappa_{\eta,0} := \kappa_\eta$, and $\mathrm{cf}\,\kappa_{\eta+1} := 1$ for reasons of homogeneity.

Now, in the case that $\eta \in \mathrm{Lim}$, the forcing P^η will be defined like an Easton-support product of Cohen forcings for the intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$:

Definition 2.3.1. For $\eta \in \mathrm{Lim}$, we let the forcing notion $(P^\eta, \supseteq, \varnothing)$ consist of all functions $p : \mathrm{dom}\, p \to 2$ such that $\mathrm{dom}\, p$ is of the following form:

There is a sequence $(\delta_{\nu,j} \mid \nu < \eta\,,\, j < \mathrm{cf}\,\kappa_{\nu+1})$ with $\delta_{\nu,j} \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for all $\nu < \eta$, $j < \mathrm{cf}\,\kappa_{\nu+1}$ with

$$\mathrm{dom}\, p = \bigcup_{\substack{\nu<\eta \\ j<\mathrm{cf}\,\kappa_{\nu+1}}} [\kappa_{\nu,j}, \delta_{\nu,j}),$$

and for any regular $\kappa_{\nu,j}$, the domain $\mathrm{dom}\, p \cap \kappa_{\nu,j}$ is bounded below $\kappa_{\nu,j}$.

For a set $S \subseteq \kappa_\eta$, we let $P^\eta \upharpoonright S := \{p \in P^\eta \mid \mathrm{dom}\, p \subseteq S\} = \{p \upharpoonright S \mid p \in P^\eta\}$. Then for any $\kappa_{\nu,j} < \kappa_\eta$, the forcing P^η is isomorphic to the product $P^\eta \upharpoonright \kappa_{\nu,j} \times P^\eta \upharpoonright [\kappa_{\nu,j}, \kappa_\eta)$, where the first factor has cardinality $\leq \kappa_{\nu,j}^+$, and the second factor is $\leq \kappa_{\nu,j}$-closed.

This helps to establish:

Lemma 2.3.2. *For all $\eta \in \mathrm{Lim}$, the forcing P^η preserves cardinals and the* GCH.

Proof. Let G^η denote a V-generic filter on P^η. It suffices to show that for all cardinals α in V,

$$(2^\alpha)^{V[G^\eta]} \leq (\alpha^+)^V,$$

which implies that cardinals are V-$V[G^\eta]$-absolute: If not, there would be a V-cardinal α with a surjection $s : \beta \to \alpha$ in $V[G^\eta]$ for some $V[G^\eta]$-cardinal $\beta < \alpha$. Then there is also a surjection $\overline{s} : \beta \to (\beta^+)^V$ in $V[G^\eta]$, which gives a surjection $\overline{\overline{s}} : \beta \to \big(2^\beta\big)^{V[G^\eta]}$. Contradiction.

- In the case that $\alpha \geq \kappa_\eta^+$, it follows that $(2^\alpha)^{V[G^\eta]} \leq |\wp(\alpha \cdot |P^\eta|)|^V \leq (2^\alpha)^V = (\alpha^+)^V$ by the GCH in V.
- Now, assume $\alpha \in (\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for some $\kappa_{\nu,j} < \kappa_\eta$. Then the forcing P^η can be factored as $P^\eta \upharpoonright \kappa_{\nu,j} \times P^\eta \upharpoonright [\kappa_{\nu,j}, \kappa_\eta)$, where $P^\eta \upharpoonright \kappa_{\nu,j}$ has cardinality $\leq \kappa_{\nu,j}^+ \leq \alpha$, and $P^\eta \upharpoonright [\kappa_{\nu,j}, \kappa_\eta)$ is $\leq \alpha$-closed. Hence,
$$\big(2^\alpha\big)^{V[G^\eta]} \leq \big(2^\alpha\big)^{V[G^\eta \upharpoonright \kappa_{\nu,j}]} \leq \big|\wp\big(\alpha \cdot |P^\eta \upharpoonright \kappa_{\nu,j}|\big)\big|^V \leq \big(2^\alpha\big)^V = (\alpha^+)^V.$$
- If $\alpha = \kappa_{\nu,j}$ for some regular $\kappa_{\nu,j} < \kappa_\eta$, then $|P^\eta \upharpoonright \kappa_{\nu,j}| = \kappa_{\nu,j}$ and $P^\eta \upharpoonright [\kappa_{\nu,j}, \kappa_\eta)$ is $\leq \kappa_{\nu,j}$-closed; so the same argument applies.
  If $\alpha = \kappa_\eta$ is regular, then $(2^\alpha)^{V[G^\eta]} \leq (\alpha^+)^V$ follows from $|P^\eta| \leq \kappa_\eta$.

It remains to show that $(2^{\kappa_{\nu,j}})^{V[G^\eta]} = (\kappa_{\nu,j}^+)^V$ for all singular $\kappa_{\nu,j} < \kappa_\eta$, and $(2^{\kappa_\eta})^{V[G^\eta]} \leq (\kappa_\eta^+)^V$ in the case that κ_η itself is singular.

We only prove the first part (the argument for κ_η is similar).

- Assume the contrary and let $\kappa_{\nu,j}$ least with $\lambda := \operatorname{cf} \kappa_{\nu,j} < \kappa_{\nu,j}$ and $(2^{\kappa_{\nu,j}})^{V[G^\eta]} > (\kappa_{\nu,j}^+)^V$. Take $(\alpha_i \mid i < \lambda)$ cofinal in $\kappa_{\nu,j}$. By assumption and by what we have shown before, it follows that $(2^\alpha)^{V[G^\eta]} = (\alpha^+)^V$ for all $\alpha < \kappa_{\nu,j}$. Hence, $\mathrm{Card}^V \cap (\kappa_{\nu,j} + 1) = \mathrm{Card}^{V[G]} \cap (\kappa_{\nu,j} + 1)$, and $(2^{\alpha_i})^{V[G^\eta]} = (\alpha_i^+)^V$ for all $i < \lambda$. Thus,
$$2^{\kappa_{\nu,j}} \leq \prod_{i<\lambda} 2^{\alpha_i} \leq \kappa_{\nu,j}^\lambda \leq \kappa_{\nu,j}^{\kappa_{\nu,j}} = 2^{\kappa_{\nu,j}}$$
holds true in V and $V[G^\eta]$. Let $\lambda \in [\kappa_{\mu,m}, \kappa_{\mu,m+1})$ for some $\kappa_{\mu,m} < \kappa_{\nu,j}$. If $\lambda > \kappa_{\mu,m}$, then $|P^\eta \upharpoonright \kappa_{\mu,m}| \leq (\kappa_{\mu,m})^+ \leq \lambda$, and $P^\eta \upharpoonright [\kappa_{\mu,m}, \kappa_\eta)$ is $\leq \lambda$-closed. In the case that $\lambda = \kappa_{\mu,m}$, it follows by regularity of λ that $|P^\eta \upharpoonright \kappa_{\mu,m}| \leq \kappa_{\mu,m} = \lambda$, as well. In either case,
$$\left(2^{\kappa_{\nu,j}}\right)^{V[G^\eta]} = \left(\kappa_{\nu,j}^\lambda\right)^{V[G^\eta]} \leq \left(\kappa_{\nu,j}^\lambda\right)^{V[G^\eta \upharpoonright \kappa_{\mu,m}]} \leq \left(2^{\kappa_{\nu,j}}\right)^{V[G^\eta \upharpoonright \kappa_{\mu,m}]} \leq$$
$$\leq \left|\wp\left(\kappa_{\nu,j} \cdot |P^\eta \upharpoonright \kappa_{\mu,m}|\right)\right|^V \leq \left|\wp\left(\kappa_{\nu,j} \cdot \kappa_{\mu,m}^+\right)\right|^V = (\kappa_{\nu,j}^+)^V,$$
which gives the desired contradiction.

□

Corollary 2.3.3. *For every $\eta \in Lim$, the forcing P^η preserves cofinalites.*

Proof. We show that every regular V-cardinal λ is still regular in $V[G^\eta]$. If not, there would be in $V[G^\eta]$ a regular cardinal $\overline{\lambda} < \lambda$ with a cofinal function $f : \overline{\lambda} \to \lambda$. Let $\overline{\lambda} \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$. The forcing P^η is isomorphic to the product $P^\eta \upharpoonright \kappa_{\nu,j} \times P^\eta \upharpoonright [\kappa_{\nu,j}, \kappa_\eta)$, where the second factor is $\leq \overline{\lambda}$-closed. If $\overline{\lambda} > \kappa_{\nu,j}$, then the first factor has cardinality $\leq \kappa_{\nu,j}^+ \leq \overline{\lambda}$. In the case that $\overline{\lambda} = \kappa_{\nu,j}$, the first factor has cardinality $\leq \kappa_{\nu,j} = \overline{\lambda}$ by regularity of $\overline{\lambda}$. Hence, $f \in V[G \upharpoonright \kappa_{\nu,j}]$. However, since $|P^\eta \upharpoonright \kappa_{\nu,j}| < \lambda$, it follows that λ is still a regular cardinal in the generic extension $V[G^\eta \upharpoonright \kappa_{\nu,j}]$. Contradiction.
Thus, it follows that P^η preserves cofinalities as desired. □

Our eventual forcing notion $\mathbb{P}_0$ will contain α_σ-many copies of P^σ for every $\sigma \in \mathrm{Lim}$. They will be labelled P_i^σ, where $i < \alpha_\sigma$. All the P_i^σ for $\sigma \in \mathrm{Lim}$, $i < \alpha_\sigma$, will be linked with a forcing notion P_*, which is a two-dimensional version of P^γ, adding $\kappa_{\nu,j+1}$-many Cohen subsets to every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$:

Definition 2.3.4. We denote by $(P_*, \supseteq \varnothing)$ the forcing notion consisting of all functions $p_* : \operatorname{dom} p_* \to 2$ such that $\operatorname{dom} p_*$ is of the following form:

There is a sequence $(\delta_{\nu,j} \mid \nu < \gamma\,,\, j < \operatorname{cf} \kappa_{\nu+1})$ with $\delta_{\nu,j} \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for all $\nu < \gamma$, $j < \operatorname{cf} \kappa_{\nu+1}$ with
$$\operatorname{dom} p_* = \bigcup_{\substack{\nu<\gamma \\ j<\operatorname{cf} \kappa_{\nu+1}}} [\kappa_{\nu,j}, \delta_{\nu,j})^2,$$
and for any $\kappa_{\nu,j}$ a regular cardinal, it follows that $|\operatorname{dom} p_* \cap \kappa_{\nu,j}^2| < \kappa_{\nu,j}$, and in the case that κ_γ itself is regular, we require that $|\operatorname{dom} p_*| < \kappa_\gamma$.

For $p_* \in P_*$ and $\xi < \kappa_\gamma$, let $p_*(\xi) := \{ (\zeta, p_*(\xi,\zeta)) \mid (\xi,\zeta) \in \operatorname{dom} p_* \}$ denote the ξ-th section of p_*. If $a \subseteq \kappa_\gamma$ is a set that hits every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ in at most one point, we let

$$p_*(a) := \{ (\zeta, p_*(\xi,\zeta)) \mid \xi \in a, (\xi,\zeta) \in \operatorname{dom} p_* \}.$$

As in Lemma 2.3.2, it follows that P_* preserves cardinals and the GCH.

Now, we are ready to define our forcing notion $\mathbb{P}_0$. Every $p_0 \in \mathbb{P}_0$ is of the form

$$p_0 = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma \in \mathrm{Lim}\,, i<\alpha_\sigma})$$

with $p_* \in P_*$ and $p_i^\sigma \in P^\sigma$ for all (σ, i).
The *linking ordinals* a_i^σ will determine how the i-th generic κ_σ-subset G_i^σ, given by the projection of the generic filter G onto P_i^σ, will be eventually linked with the P_*-generic filter G_*.

Definition 2.3.5. Let $\mathbb{P}_0$ be the collection of all $p_0 = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma \in \mathrm{Lim}\,, i<\alpha_\sigma})$ such that:

- The *support* of p_0, $\operatorname{supp} p_0$, is countable with $p_i^\sigma = a_i^\sigma = \varnothing$ whenever $(\sigma, i) \notin \operatorname{supp} p_0$.
- We have $p_* \in P_*$, and $p_i^\sigma \in P^\sigma$ for all $(\sigma, i) \in \operatorname{supp} p_0$.
- The domains of the p_i^σ are coherent in the following sense:
  If $\operatorname{dom} p_* = \bigcup_{\nu<\gamma, j<\operatorname{cf}\kappa_{\nu+1}} [\kappa_{\nu,j}, \delta_{\nu,j})^2$, then for every $(\sigma, i) \in \operatorname{supp} p_0$, it follows that $\operatorname{dom} p_i^\sigma = \bigcup_{\nu<\sigma, j<\operatorname{cf}\kappa_{\nu+1}} [\kappa_{\nu,j}, \delta_{\nu,j})$.
  We set $\operatorname{dom} p_0 := \bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j})$.
- For all $(\sigma, i) \in \operatorname{supp} p_0$, we have $a_i^\sigma \subseteq \kappa_\sigma$ with $|a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})| = 1$ for all intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\sigma$.
  If $(\sigma_0, i_0) \neq (\sigma_1, i_1)$, then $a_{i_0}^{\sigma_0} \cap a_{i_1}^{\sigma_1} = \varnothing$. (We call this the *independence property*).

Concerning the partial ordering $\leq_0$, any linking ordinal $\{\xi\} = a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ settles that whenever $q_0 \leq p_0$, the extension $q_i^\sigma \supseteq p_i^\sigma$ within in the interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ is determined by $q_*(\xi)$:

For $p_0 = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i})$, $q_0 = (q_*, (q_i^\sigma, b_i^\sigma)_{\sigma,i}) \in \mathbb{P}_0$, let $q_0 \leq_0 p_0$ if the following holds: $q_* \supseteq p_*$; $q_i^\sigma \supseteq p_i^\sigma$, $b_i^\sigma \supseteq a_i^\sigma$ for all $(\sigma, i) \in \operatorname{supp} p_0$, and whenever $\zeta \in (\operatorname{dom} q_i^\sigma \setminus \operatorname{dom} p_i^\sigma) \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \{\xi\}$, then $\xi \in \operatorname{dom} q_0$ with $q_i^\sigma(\zeta) = q_*(\xi, \zeta)$ (we call this the *linking property*).
The *maximal element* of $\mathbb{P}_0$ is $\mathbb{1}_0 := (\varnothing, (\varnothing, \varnothing)_{\sigma<\gamma, i<\alpha_\sigma})$.

Let G_0 denote a V-generic filter on $\mathbb{P}_0$, and $g_i^\sigma := \bigcup \{ a_i^\sigma \mid p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}) \in G_0 \}$.
Note that by our strong *independence property*, every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ will be blown up to size $\sup\{\alpha_\sigma \mid \sigma \in \mathrm{Lim}\}$ in a $\mathbb{P}_0$-generic extension.
Hence, since we want our eventual symmetric submodel N preserve all V-cardinals, we will have so make sure that N "does not know" the sequence of linking ordinals $(g_i^\sigma \mid \sigma \in \mathrm{Lim}\,, i < \alpha_\sigma)$.

A major difference between our forcing and the basic construction in [GK12] is the following: The forcing conditions in [GK12, Definition 2] have *finite* linking ordinals a_i^σ; so

the according generics g_i^σ are **not** contained in the ground model V. With our definition however, it follows for any $p \in G_0$ with $(\sigma, i) \in \operatorname{supp} p$ that $g_i^\sigma = a_i^\sigma \in V$. By countable support, also countable sequences of linking ordinals $(g_{i_j}^{\sigma_j} \mid j < \omega)$ are contained in V; but for $\sigma \in \mathrm{Lim}$ not the whole sequence $(g_i^\sigma \mid i < \alpha_\sigma)$.
This modification helps to establish that any generic G_i^σ can be described using only G_* and sets from the ground model V (see below).

Next, we define our forcing notion $\mathbb{P}_1$, which will be in charge of the successor cardinals. For every $\sigma \in \mathrm{Succ}$ with $\kappa_\sigma = \overline{\kappa_\sigma}^+$, it follows that $\sigma =: \overline{\sigma} + 1$ must be a successor ordinal, since we have assumed in the beginning that the sequence $(\kappa_\sigma \mid 0 < \sigma < \gamma)$ is closed.

We denote by P^σ the Cohen forcing

$$P^\sigma := \{p : \operatorname{dom} p \to 2 \;\big|\; \operatorname{dom} p \subseteq [\overline{\kappa_\sigma}, \kappa_\sigma)\,,\, |\operatorname{dom} p| < \kappa_\sigma\},$$

and let

$$C^\sigma := \{p : \operatorname{dom} p \to 2 \mid \operatorname{dom} p = \operatorname{dom}_x p \times \operatorname{dom}_y p \subseteq \alpha_\sigma \times [\overline{\kappa_\sigma}, \kappa_\sigma)\,,\, |\operatorname{dom} p| < \kappa_\sigma\}.$$

Then both P^σ and C^σ are $< \kappa_\sigma$-closed, and if $2^{<\kappa_\sigma} = \kappa_\sigma$, i.e. $2^{\overline{\kappa_\sigma}} = \kappa_\sigma$, then they satisfy the κ_σ^+-chain condition and hence, preserve cardinals.
In particular, any forcing P^σ or C^σ preserves cardinals if we are working in our ground model V with $V \vDash \mathsf{GCH}$, or any V-generic extension by $\leq \kappa_\sigma$-closed forcing.

Definition 2.3.6. The forcing notion $(\mathbb{P}_1, \leq_1, \varnothing_1)$ consists of all $p_1 = (p^\sigma)_{\sigma \in \mathrm{Succ}}$ with countable support $\operatorname{supp} p_1 := \{\sigma \in \mathrm{Succ} \mid p^\sigma \neq \varnothing\}$, and $p^\sigma \in C^\sigma$ for all $\sigma \in \mathrm{Succ}$. For $p_1 = (p^\sigma)_{\sigma \in \mathrm{Succ}}$, $q_1 = (q^\sigma)_{\sigma \in \mathrm{Succ}} \in \mathbb{P}_1$, we let $q_1 \leq_1 p_1$ if $q^\sigma \supseteq p^\sigma$ for all $\sigma \in \mathrm{Succ}$; and $\mathbb{1}_1 := (\varnothing)_{\sigma \in \mathrm{Succ}}$ is the maximal element.

For $\sigma \in \mathrm{Succ}$ and $i < \alpha_\sigma$, we set $p_i^\sigma = \{(\zeta, p^\sigma(i, \zeta)) \mid (i, \zeta) \in \operatorname{dom} p^\sigma\}$.

Our main forcing will be the product $\mathbb{P} := \mathbb{P}_0 \times \mathbb{P}_1$ with maximal element $\mathbb{1} := (\mathbb{1}_0, \mathbb{1}_1)$ and order relation $\leq$. In order to simplify notation, we write conditions $p \in \mathbb{P}$ in the form $p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma \in \mathrm{Lim}, i < \alpha_\sigma}, (p^\sigma)_{\sigma \in \mathrm{Succ}})$.

It is not difficult to verify:

Proposition 2.3.7. *$\mathbb{P}$ is countably closed.*

This is important to make sure that DC holds in our eventual symmetric extension N.

For $0 < \eta \leq \gamma$ (with $\eta \in Lim$ or $\eta \in Succ$ or $\eta = \gamma$), we define a forcing $\overline{P}^\eta$ like P^η is defined in the case that $\eta \in Lim$:

Let $\overline{P}^\eta$ consist of all functions $p : \operatorname{dom} p \to 2$ such that there is a sequence $(\delta_{\nu,j} \mid \nu < \eta, j < \operatorname{cf} \kappa_{\nu+1})$ with $\delta_{\nu,j} \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for all $\kappa_{\nu,j} < \kappa_\eta$, and

$$\operatorname{dom} p = \bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j}),$$

such that $|p \upharpoonright \kappa_{\overline{\nu},\overline{\jmath}}| < \kappa_{\overline{\nu},\overline{\jmath}}$ whenever $\kappa_{\overline{\nu},\overline{\jmath}}$ is a regular cardinal, and $|p| < \kappa_\eta$ in the case that κ_η itself is regular.

For any $0 < \eta < \lambda$ with κ_λ a limit cardinal, it follows that $\overline{P}^\eta = P^\lambda \upharpoonright \kappa_\eta$.

Let now G be a V-generic filter on $\mathbb{P}$. It induces

$$G_* := \{q_* \in P_* \mid \exists\, p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}, (p^\sigma)_\sigma) \in G \,:\, q_* \subseteq p_*\},$$

and for $\lambda \in Lim$, $k < \alpha_\lambda$:

$$G_k^\lambda := \{q_k^\lambda \in P^\lambda \mid \exists\, p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}, (p^\sigma)_\sigma) \in G \,:\, q_k^\lambda \subseteq p_k^\lambda\}.$$

As usually, these filters G_*, G_k^λ are identified with their union $\bigcup G_*$, $\bigcup G_k^\lambda$. Then any G_k^λ can be regarded a subset of κ_λ.

Moreover, let

$$g_k^\lambda := \bigcup\{a_k^\lambda \mid p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}, (p^\sigma)_\sigma) \in G\}.$$

Then $g_k^\lambda = a_k^\lambda$ for any $p \in G$ with $(\lambda, k) \in \operatorname{supp} p_0$; and g_k^λ hits any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\lambda$ in exactly one point. By the *independence property*, it follows that $g_{k_0}^{\lambda_0} \cap g_{k_1}^{\lambda_1} = \emptyset$ whenever $(\lambda_0, k_0) \neq (\lambda_1, k_1)$.

For $\lambda \in Succ$, set

$$G^\lambda := \{p^\lambda \mid p = (p_*, (p_i^\sigma, a_i^\sigma)_{\eta,i}, (p^\sigma)_\sigma) \in G\},$$

and

$$G_k^\lambda := \{p_k^\lambda \mid p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}, (p^\sigma)_\sigma) \in G\}$$

for any $k < \alpha_\lambda$.

Again, we confuse these filters G^λ, G_k^λ with their union $\bigcup G^\lambda$, $\bigcup G_k^\lambda$.

Let now $\xi \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$. We denote by

$$G_*(\xi) := \Big\{\, q : [\kappa_{\nu,j}, \delta_{\nu,j}) \to 2 \;\Big|\; \delta_{\nu,j} \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \;\;,\;\; \exists\, p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}, (p^\sigma)_\sigma) \in G \;: \\ \forall\, \zeta \in \operatorname{dom} q \;\; q(\zeta) = p_*(\xi, \zeta) \,\Big\}$$

the ξ-th section of G_*.

If $a \subseteq \kappa_\gamma$ is a set that hits any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\gamma$ in at most one point, we denote by $G_*(a)$ the set of all $q \in \overline{P}^\gamma$ such that there is $p \in G$ with $q \subseteq p_*(a)$.
As before, we identify any $G_*(\xi)$ and $G_*(a)$ with their union $\bigcup G_*(\xi)$ and $\bigcup G_*(a)$, respectively. Then any $G_*(\xi)$ with $\xi \in [\kappa_{\nu j}, \kappa_{\nu,j+1})$ can be regarded as a function $G_*(\xi) : [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \to 2$, and any $G_*(a)$ becomes a function $G_*(a) : dom\, G_*(a) \to 2$, where $dom\, G_*(a) \subseteq \kappa_\gamma$ is the union of those intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $a \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \neq \emptyset$.

Now, the *linking property* implies that any $G_k^\lambda \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $\lambda \in \mathrm{Lim}$, $k < \alpha_\lambda$, is eventually equal to $G_*(\xi)$, where $\{\xi\} := a_k^\lambda \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$.
Indeed, the symmetric difference $G_k^\lambda \oplus G_*(g_k^\lambda)$ is always an element of the ground model V: Take a condition $p \in G$ with $(\lambda, k) \in \operatorname{supp} p_0$, such that for any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\lambda$ with $\operatorname{dom} p_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \neq \emptyset$ and $\{\xi\} := a_k^\lambda \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$, it follows that $\xi \in \operatorname{dom} p_0$. (This does not interfere with the condition that $\operatorname{dom} p_0$ has to be bounded below all regular $\kappa_{\overline{\nu},\overline{\jmath}}$, since we do not bother the intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $\operatorname{dom} p_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \emptyset$.)

- Firstly, $G_k^\lambda(\zeta) \oplus G_*(g_k^\lambda)(\zeta) = 0$ whenever $\zeta \notin \operatorname{dom} p_0$: Let $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$, $\zeta \notin \operatorname{dom} p$ with $\{\xi\} := g_k^\lambda \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = a_k^\lambda \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$. Take $q \in G$, $q \le p$ with $\zeta \in \operatorname{dom} q_0$. Then by the *linking property*, it follows that $\xi \in \operatorname{dom} q_0$ with $q_k^\lambda(\zeta) = q_*(\xi, \zeta)$. Hence, $G_k^\lambda(\zeta) = q_k^\lambda(\zeta) = q_*(\xi,\zeta) = G_*(g_k^\lambda)(\zeta)$, and $G_k^\lambda(\zeta) \oplus G_*(g_k^\lambda)(\zeta) = 0$.
- If $\zeta \in \operatorname{dom} p$ then $G_k^\lambda(\zeta) \oplus G_*(g_k^\lambda)(\zeta) = p_k^\lambda(\zeta) \oplus p_*(\xi,\zeta)$, where again, $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ and $\{\xi\} := g_k^\lambda \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = a_k^\lambda \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$. Here we use that for any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $\operatorname{dom} p \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \neq \emptyset$, it follows that $a_k^\lambda \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \operatorname{dom} p_0$.

Hence, $G_k^\lambda \oplus G_*(g_k^\lambda)$ can be calculated in V.

This will be employed to keep control over the surjective size of $\wp(\kappa_\lambda)$ in the eventual symmetric extension N.

Now, we consider countable products $\prod_{m<\omega} P^{\sigma_m}$ and $\prod_{m<\omega} \overline{P}^{\sigma_m}$:

Definition 2.3.8. Let $((\sigma_m, i_m) \mid m < \omega)$ be a sequence of pairwise distinct pairs with $0 < \sigma_m < \gamma$, $i < \alpha_{\sigma_m}$ for all $m < \omega$. We denote by $\prod_{m<\omega} P^{\sigma_m}$ the set of all $(p(m) \mid m < \omega)$ with $p(m) \in P^{\sigma_m}$ for all $m < \omega$ (with full support), and similarly, $\prod_{m<\omega} \overline{P}^{\sigma_m} := \{ (p(m) \mid m < \omega) \mid \forall\, m < \omega \;\; p(m) \in \overline{P}^{\sigma_m} \}$.

For any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\gamma$, it follows that $\prod_{m<\omega} P^{\sigma_m} \upharpoonright \kappa_{\nu,j}$ has cardinality $\le \kappa_{\nu,j}$ in the case that $\kappa_{\nu,j}$ is regular, and cardinality $\le \kappa_{\nu,j}^+$, else. Moreover, $\prod_{m<\omega} P^{\sigma_m} \upharpoonright [\kappa_{\nu,j}, \kappa_{\sigma_m})$ is $\le \kappa_{\nu,j}$-closed. Hence, as in Lemma 2.3.2 and Corollary 2.3.3, one can show that the product $\prod_{m<\omega} P^{\sigma_m}$ preserves cardinals, cofinalities and the GCH.
Similarly, $\prod_{m<\omega} \overline{P}^{\sigma_m}$ preserves cardinals, cofinalities and the GCH.

The next lemma implies that countable products $\prod_{m<\omega} G_*(g_{i_m}^{\sigma_m})$ are V-generic over $\prod_{m<\omega} \overline{P}^{\sigma_m}$:

Lemma 2.3.9. *Consider a sequence $(a_m \mid m < \omega)$ of pairwise disjoint sets such that for all $m < \omega$, the following holds: a_m is a subset of κ_{σ_m} for some $0 < \sigma_m < \gamma$, and for all $\kappa_{\nu,j} < \kappa_{\sigma_m}$, it follows that $|a_m \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})| = 1$, i.e. a_m hits every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\sigma_m}$ in exactly one point. Then $\prod_{m<\omega} G_*(a_m) := \{ (p(m) \mid m < \omega) \mid \forall\, m < \omega \;\; p(m) \in G_*(a_m) \}$ is a V-generic filter on $\prod_{m<\omega} \overline{P}^{\sigma_m}$.*

Proof. Let $D \subseteq \prod_{m<\omega} P^{\sigma_m}$ be an open dense set in V. We show that $D \cap \prod_{m<\omega} G_*(a_m) \neq \emptyset$. Let

$$\overline{D} := \{ p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}, (p^\sigma)_\sigma) \in \mathbb{P} \mid (p_*(a_m) \mid m < \omega) \in D \}.$$

It suffices to prove that $\overline{D}$ is dense in $\mathbb{P}$. Assume $p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}, (p^\sigma)_\sigma) \in \mathbb{P}$ given, and denote by $(q_m \mid m < \omega)$ an extension of $(p_*(a_m) \mid m < \omega)$ in D. We have to construct $\overline{p} \le p$ such that $\overline{p}_*(a_m) \supseteq q_m$ for all $m < \omega$.
Consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\gamma$. In the case that $(\operatorname{dom} p \cup \bigcup_{m<\omega} \operatorname{dom} q_m) \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \emptyset$, let $\delta_{\nu,j} := \kappa_{\nu,j}$. Otherwise, we pick $\delta_{\nu,j} \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ such that firstly,

$$\big(\operatorname{dom} p \cup \bigcup_{m<\omega} \operatorname{dom} q_m\big) \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq [\kappa_{\nu,j}, \delta_{\nu,j});$$

secondly, for all $m < \omega$, it follows that $a_m \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq [\kappa_{\nu,j}, \delta_{\nu,j})$; and thirdly, $a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq [\kappa_{\nu,j}, \delta_{\nu,j})$ for all $(\sigma, i) \in \operatorname{supp} p$. This is possible, since the sets

$a_m \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ and $a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ are singletons or empty, all the domains $\operatorname{dom} p \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ and $\operatorname{dom} q_m \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for $m < \omega$ are bounded below $\kappa_{\nu,j+1}$, and $\kappa_{\nu,j+1}$ is always a successor cardinal.
Let

$$\operatorname{dom} \overline{p}_0 := \bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j}).$$

Then $\operatorname{dom} \overline{p}_0$ is bounded below all regular $\kappa_{\overline{\nu},\overline{j}}$, since this holds true for $\operatorname{dom} p$ and $\bigcup_{m<\omega} \operatorname{dom} q_m$. We define $\overline{p}_*$ on $\bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j})^2$ as follows: Consider an interval $[\kappa_{\nu,j}, \delta_{\nu,j}) \neq \varnothing$ and $\xi, \zeta \in [\kappa_{\nu,j}, \delta_{\nu,j})$. For $(\xi, \zeta) \in \operatorname{dom} p \times \operatorname{dom} p$, let $\overline{p}_*(\xi, \zeta) := p_*(\xi, \zeta)$. If $\{\xi\} = a_m \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for some $m < \omega$ and $\zeta \in \operatorname{dom} q_m$, we set $\overline{p}_*(\xi, \zeta) := q_m(\zeta)$. This is not a contradiction towards $\overline{p}_* \upharpoonright [\kappa_{\nu,j}, \delta_{\nu,j})^2 \supseteq p_* \upharpoonright [\kappa_{\nu,j}, \delta_{\nu,j})^2$, since $q_m \supseteq p_*(a_m)$ for all $m < \omega$. Also, the a_m are pairwise disjoint, so for any $\xi \in [\kappa_{\nu,j}, \delta_{\nu,j})$, there is at most one m with $\xi \in a_m$. For all the remaining $(\xi, \zeta) \in \operatorname{dom} \overline{p}_*$, we can set $\overline{p}_*(\xi, \zeta) \in \{0, 1\}$ arbitrarily. This defines $\overline{p}_*$ on $\bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j})^2$.

For all $(\sigma, i) \in \operatorname{supp} \overline{p}_0 := \operatorname{supp} p_0$, we set $\overline{a}_i^\sigma := a_i^\sigma$, and define $\overline{p}_i^\sigma \supseteq p_i^\sigma$ on the corresponding domain $\bigcup_{\kappa_{\nu,j} < \kappa_\sigma} [\kappa_{\nu,j}, \delta_{\nu,j})$ according to the *linking property*: Whenever $\zeta \in (\operatorname{dom} \overline{p}_0 \setminus \operatorname{dom} p) \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ and $a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \{\xi\}$, then $\xi \in \operatorname{dom} \overline{p}_0$ follows, so we can set $\overline{p}_i^\sigma(\zeta) := \overline{p}_*(\xi, \zeta)$. For the $\zeta \in \operatorname{dom} \overline{p}_i^\sigma \setminus \operatorname{dom} p_i^\sigma$ remaining, we can define $\overline{p}_i^\sigma(\zeta)$ arbitrarily. This completes the construction of $\overline{p}_0$.

Let $\overline{p}_1 := p_1$. It is not difficult to check that $\overline{p} \leq p$ indeed is a condition in $\mathbb{P}$ with $\overline{p}_*(a_m) \supseteq q_m$ for all $m < \omega$. Hence, $(\overline{p}_*(a_m) \mid m < \omega) \in D$, and $\overline{p} \in \overline{D}$ as desired. □

In particular, for $((\sigma_m, i_m) \mid m < \omega)$ a sequence of pairwise distinct pairs as before with $\sigma_m \in \mathrm{Lim}$, $i_m < \alpha_{\sigma_m}$ for all $m < \omega$, it follows that $\prod_{m<\omega} G_*(g_{i_m}^{\sigma_m})$ is a V-generic filter over $\prod_{m<\omega} \overline{P}^{\sigma_m}$.

Similarly, one can show:

Lemma 2.3.10. *Let $((\sigma_m, i_m) \mid m < \omega)$ denote a sequence of pairwise distinct pairs with $0 < \sigma_m < \gamma$, $i_m < \alpha_{\sigma_m}$ for all $m < \omega$. Then $\prod_{m<\omega} G_{i_m}^{\sigma_m} := \big\{ (p(m) \mid m < \omega) \;\big|\; \forall\, m < \omega \;\; p(m) \in G_{i_m}^{\sigma_m} \big\}$ is a V-generic filter on $\prod_{m<\omega} P^{\sigma_m}$.*

2.4 Symmetric Names

2.4.1 Constructing A and $\overline{A}$

For defining our symmetric extension N, we first need an almost-group A of partial $\mathbb{P}$-automorphisms. We will have $A = A_0 \times A_1$, where A_0 is an almost-group of partial $\mathbb{P}_0$-automorphisms, and A_1 is an almost-group of partial $\mathbb{P}_1$-automorphisms.

We start with the construction of A_0.

Every $\pi_0 \in A_0$ will be an order-preserving bijection $\pi_0 : D_{\pi_0} \to D_{\pi_0}$ with $D_{\pi_0} \in \mathcal{D}_0$, where $\mathcal{D}_0$ is defined as follows:

Let $\mathcal{D}_0$ denote the collection of all sets $D \subseteq \mathbb{P}_0$ given by

- a countable *support* $\operatorname{supp} D \subseteq \{(\sigma, i) \mid \sigma \in \mathrm{Lim}, i < \alpha_\sigma\}$, and
- a *domain* $\operatorname{dom} D := \bigcup_{\nu<\gamma\,,\, j<\operatorname{cf}\kappa_{\nu+1}} [\kappa_{\nu,j}, \delta_{\nu,j})$ such that $\delta_{\nu,j} \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for all $\nu < \gamma$, $j < \operatorname{cf}\kappa_{\nu+1}$; and for all regular $\kappa_{\overline{\nu},\overline{\jmath}}$, it follows that *dom* $D \cap \kappa_{\overline{\nu},\overline{\jmath}}$ is bounded below $\kappa_{\overline{\nu},\overline{\jmath}}$,

such that D is the set of all $p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}) \in \mathbb{P}_0$ with

- $\operatorname{supp} p \supseteq \operatorname{supp} D$, $\operatorname{dom} p \supseteq \operatorname{dom} D$, and
- for all intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $\operatorname{dom} p \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \neq \emptyset$, it follows that

$$\bigcup_{(\sigma,i)\in\operatorname{supp} p} a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \;\subseteq\; \operatorname{dom} p.$$

In other words: D is the collection of all $p \in \mathbb{P}_0$ the domain and support of which cover a certain domain and support given by D; with the additional property that all the linking ordinals $\{\xi\} = a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ contained in any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ hit by $\operatorname{dom} p$, are already contained in $\operatorname{dom} p$.

It is not difficult to see that any $D \in \mathcal{D}_0$ is dense in $\mathbb{P}_0$. The sets $D \in \mathcal{D}_0$ are not open dense; but whenever p, $q \in \mathbb{P}_0$ with $p \in D$ and $q \leq p$ such that $\operatorname{supp} q = \operatorname{supp} p$, then by the *linking property*, it follows that also $q \in D$.

Whenever D, $D' \in \mathcal{D}_0$, then the intersection $D \cap D'$ is contained in $\mathcal{D}_0$ as well, with *supp* $(D \cap D') =$ *supp* $D \cup$ *supp* D', *dom* $(D \cap D') =$ *dom* $D \cup$ *dom* D'.
The collection $\mathcal{D}_0$ has a *maximal element* $(D_0)_{\max}$ with $\operatorname{supp}(D_0)_{\max} := \emptyset$, $\operatorname{dom}(D_0)_{\max} = \emptyset$. Clearly, $(D_0)_{\max} \supseteq D$ for all $D \in \mathcal{D}$.

Hence, it follows that $\mathcal{D}$ has all the properties required in Definition 1.2.14.

We now describe the two types of partial $\mathbb{P}_0$-automorphisms that will generate A_0:

Our first goal is that for any two conditions p, $q \in \mathbb{P}$ with the same "shape", i.e. $\operatorname{dom} p = \operatorname{dom} q$, $\operatorname{supp} p = \operatorname{supp} q$ and $\bigcup a_i^\sigma = \bigcup b_i^\sigma$, there is an automorphism $\pi_0 \in A_0$ with $\pi_0 p = q$. This homogeneity property will be achieved by giving the maps $\pi_0 \in A_0$ a lot of freedom regarding what can happen on $\operatorname{supp} \pi_0$ and $\operatorname{dom} \pi_0$.

For $\kappa_{\nu,j} < \kappa_\gamma$, let

$$\operatorname{supp} \pi_0(\nu, j) := \{(\sigma, i) \in \operatorname{supp} \pi_0 \mid \kappa_{\nu,j} < \kappa_\sigma\}.$$

Concerning the linking ordinals, we want that for any $p \in D_{\pi_0}$, $p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i})$ with $\pi p = p' = ((p')_*, ((p')_i^\sigma, (a')_i^\sigma)_{\sigma,i})$, the sets of linking ordinals for p and p' are the same, i.e. $\bigcup a_i^\sigma = \bigcup (a')_i^\sigma$. In other words, for any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$, the linking ordinals $\xi \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ will be exchanged between the coordinates $(\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)$, which is described by an isomorphism $F_{\pi_0}(\nu, j) : \operatorname{supp} \pi_0(\nu, j) \to \operatorname{supp} \pi_0(\nu, j)$.

Regarding the $(p')_i^\sigma$ for $(\sigma, i) \in \operatorname{supp} \pi_0$, there will be for every $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap \operatorname{dom} \pi_0$ a bijection $\pi_0(\zeta) : 2^{\operatorname{supp} \pi_0(\nu,j)} \to 2^{\operatorname{supp} \pi_0(\nu,j)}$ with

$$\big(\, (p')_i^\sigma(\zeta) \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j) \,\big) := \pi_0(\zeta)\big(\, p_i^\sigma(\zeta) \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j) \,\big).$$

Concerning p'_*, we will have a similar construction for the $p'_*(\xi,\zeta)$ in the case that $\zeta \in \operatorname{dom}\pi_0$ and ξ is a linking ordinal contained in $\bigcup a_i^\sigma$. Moreover, for all $(\xi,\zeta) \in \operatorname{dom}\pi_0 \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})^2$, we will have a bijection $\pi_*(\xi,\zeta): 2 \to 2$, and set $p'_*(\xi,\zeta) = \pi_*(\xi,\zeta)(p_*(\xi,\zeta))$ whenever ξ, $\zeta \in \operatorname{dom}\pi_0$ and $\xi \notin \bigcup a_i^\sigma$.

Our second goal is that for any interval $[\kappa_{\nu,j},\kappa_{\nu,j+1})$ and (σ,i), $(\lambda,k) \in \operatorname{supp}\pi_0(\nu,j)$, there is an isomorphism $\pi_0 \in A_0$ with $(\pi_0 G)_k^\lambda \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) = G_i^\sigma \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$. Thus, every $\pi_0 \in A_0$ will be equipped with bijections $G_{\pi_0}(\nu,j) : \operatorname{supp}\pi_0(\nu,j) \to \operatorname{supp}\pi_0(\nu,j)$ for every $\kappa_{\nu,j} < \kappa_\gamma$, such that the following holds: Whenever $p \in D_{\pi_0}$, $p' := \pi p$ and $\zeta \in \operatorname{dom} p \setminus \operatorname{dom}\pi_0$, $(\sigma,i) \in \operatorname{supp}\pi_0(\nu,j)$, then $(p')_i^\sigma(\zeta) = p_k^\lambda(\zeta)$ with $(\lambda,k) := G_{\pi_0}(\nu,j)(\sigma,i)$.
Whenever $\zeta \in \operatorname{dom}\pi_0$ and $(\sigma,i) \in \operatorname{supp}\pi(\nu,j)$, then the values $(p')_i^\sigma(\zeta)$ are described by the maps $\pi_0(\zeta)$ mentioned above, which allows for setting $(p')_i^\sigma(\zeta) := p_k^\lambda(\zeta)$ for any pair (σ,i), $(\lambda,k) \in \operatorname{supp}\pi_0(\nu,j)$.

Roughly speaking, A_0 will be generated by these two types of isomorphism. Regarding the construction of p'_*, some extra care is needed concerning the values $p'_*(\xi,\zeta)$ for $\zeta \notin \operatorname{dom}\pi_0$ and $\xi \in \bigcup a_i^\sigma$ a linking ordinal, since we have to make sure that the maps $\pi_0 \in A_0$ are order-preserving. Whenever $p,q \in D_{\pi_0}$ with $q \le p$, then also $q' \le p'$ must hold; in particular, whenever $\{\xi_i^\sigma\} := a_i^\sigma \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$ is a linking ordinal and $\zeta \in \operatorname{dom} q \setminus \operatorname{dom} p$ (in particular, $\zeta \notin \operatorname{dom}\pi_0$), then $\{\xi_i^\sigma\} = (a')_k^\lambda \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$ with $(\sigma,i) = F_{\pi_0}(\nu,j)(\lambda,k)$, and $q'_*(\xi_i^\sigma,\zeta) = (q')_k^\lambda(\zeta)$ by the linking property for $q' \le p'$. Moreover, $(q')_k^\lambda(\zeta) = q_l^\mu(\zeta)$ with $(\mu,l) = G_{\pi_0}(\nu,j)(\lambda,k)$, and $q_l^\mu(\zeta) = q_*(\xi_l^\mu,\zeta)$ with $\xi_l^\mu = a_l^\mu \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$ by the linking property for $q \le p$. Hence, $q'_*(\xi_i^\sigma,\zeta) = q_*(\xi_l^\mu,\zeta)$ must hold, where $(\mu,l) = G_{\pi_0}(\nu,j) \circ (F_{\pi_0}(\nu,j))^{-1}(\sigma,i)$.

This gives rise to the following definition:

Definition 2.4.1. Let A_0 consist of all automorphisms $\pi_0 : D_{\pi_0} \to D_{\pi_0}$ such that there are

- a countable set $\operatorname{supp}\pi_0 \subseteq \{(\sigma,i) \mid \sigma \in \operatorname{Lim}, i < \alpha_\sigma\}$
  (for $\kappa_{\nu,j} < \kappa_\gamma$, we set $\operatorname{supp}\pi_0(\nu,j) := \{(\sigma,i) \in \operatorname{supp}\pi_0 \mid \kappa_{\nu,j} < \kappa_\gamma\}$),
- a domain $\operatorname{dom}\pi_0 = \bigcup_{\nu<\gamma,\, j<\operatorname{cf}\kappa_{\nu+1}} [\kappa_{\nu,j},\delta_{\nu,j})$ such that $\delta_{\nu,j} \in [\kappa_{\nu,j},\kappa_{\nu,j+1})$ for all $\nu < \gamma$, $j < \operatorname{cf}\kappa_{\nu+1}$; and for all regular $\kappa_{\overline{\nu},\overline{j}}$, it follows that $\operatorname{dom}\pi_0 \cap \kappa_{\overline{\nu},\overline{j}}$ is bounded below $\kappa_{\overline{\nu},\overline{j}}$
  (for $\kappa_{\nu,j} < \kappa_\gamma$, we set $\operatorname{dom}\pi_0(\nu,j) := \operatorname{dom}\pi_0 \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$),

such that

$$D_{\pi_0} = \Big\{ p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma,i}) \in \mathbb{P}_0 \;\Big|\; \operatorname{supp} p \supseteq \operatorname{supp}\pi_0\,,\ \operatorname{dom} p \supseteq \operatorname{dom}\pi_0,\ \text{and}$$

$$\forall\, \kappa_{\nu,j} < \kappa_\gamma :\ \Big(\operatorname{dom} p \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) \neq \varnothing \Rightarrow \bigcup_{(\sigma,i)\in\operatorname{supp} p} a_i^\sigma \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) \subseteq \operatorname{dom} p\Big)\Big\};$$

moreover, there are

- for all $\nu < \gamma$, $j < \operatorname{cf}\kappa_{\nu+1}$, a bijection

$$F_{\pi_0}(\nu,j) : \operatorname{supp}\pi_0(\nu,j) \to \operatorname{supp}\pi_0(\nu,j)$$

(which will be in charge of permuting the linking ordinals as mentioned above),

and a bijection

$$G_{\pi_0}(\nu,j):\operatorname{supp}\pi_0(\nu,j)\to\operatorname{supp}\pi_0(\nu,j)$$

(which will be in charge of permuting the verticals p_i^σ outside $\operatorname{dom}\pi_0$ on the interval $[\kappa_{\nu,j},\kappa_{\nu,j+1})$),

- for all $\nu<\gamma$, $j<\operatorname{cf}\kappa_{\nu+1}$ and $\zeta\in[\kappa_{\nu,j},\kappa_{\nu,j+1})\cap\operatorname{dom}\pi_0$, a bijection

$$\pi_0(\zeta):2^{\operatorname{supp}\pi_0(\nu,j)}\to 2^{\operatorname{supp}\pi_0(\nu,j)}$$

  (which will be in charge of setting the values $(\pi p)_i^\sigma(\zeta)$ for $(\sigma,i)\in\operatorname{supp}\pi_0(\nu,j)$, $\zeta\in\operatorname{dom}\pi_0$),

- for all $\nu<\gamma$, $j<\operatorname{cf}\kappa_{\nu+1}$, $\zeta\in[\kappa_{\nu,j},\kappa_{\nu,j+1})\cap\operatorname{dom}\pi_0$, and

$$\big(\xi_i^\sigma\mid(\sigma,i)\in\operatorname{supp}\pi_0(\nu,j)\big)\in(\operatorname{dom}\pi_0(\nu,j))^{\operatorname{supp}\pi_0(\nu,j)}$$

  a sequence of pairwise distinct ordinals, a bijection

$$(\pi_0)_*(\zeta)\big(\xi_i^\sigma\mid(\sigma,i)\in\operatorname{supp}\pi_0(\nu,j)\big):2^{\operatorname{supp}\pi_0(\nu,j)}\to 2^{\operatorname{supp}\pi_0(\nu,j)}$$

  (which will be in charge of setting the values $(\pi p)_*(\xi_i^\sigma,\zeta)$ for $\{\xi_i^\sigma\}=a_i^\sigma\cap[\kappa_{\nu,j},\kappa_{\nu,j+1})$ a linking ordinal and $\zeta\in[\kappa_{\nu,j},\kappa_{\nu,j+1})\cap\operatorname{dom}\pi_0$),

- for all $\nu<\gamma$, $j<\operatorname{cf}\kappa_{\nu+1}$ and $(\xi,\zeta)\in[\kappa_{\nu,j},\kappa_{\nu,j+1})^2$, a bijection

$$(\pi_0)_*(\xi,\zeta):2\to 2$$

  such that $\pi_*(\xi,\zeta)$ is the identity whenever $(\xi,\zeta)\notin(\operatorname{dom}\pi_0)^2$
  (which will be in charge of the values $(\pi p)_*(\xi,\zeta)$ in the case that $\xi\notin\bigcup_{\sigma,i}a_i^\sigma$ is not a linking ordinal);

which defines for $p\in D_{\pi_0}$, $p=(p_*,(p_i^\sigma,a_i^\sigma)_{\sigma,i})$, the image $\pi p=:p'=(p'_*,((p')_i^\sigma,(a')_i^\sigma)_{\sigma,i})$ as follows:

We will have $\operatorname{supp}p'=\operatorname{supp}p$, $\operatorname{dom}p'=\operatorname{dom}p$. Moreover:

- Concerning the linking ordinals, for all $(\sigma,i)\in\operatorname{supp}p'=\operatorname{supp}p$ and $\kappa_{\nu,j}<\kappa_\sigma$:
  - $(a')_i^\sigma\cap[\kappa_{\nu,j},\kappa_{\nu,j+1})=a_i^\sigma\cap[\kappa_{\nu,j},\kappa_{\nu,j+1})$ for $(\sigma,i)\notin\operatorname{supp}\pi_0(\nu,j)$,
  - $(a')_i^\sigma\cap[\kappa_{\nu,j},\kappa_{\nu,j+1})=a_k^\lambda\cap[\kappa_{\nu,j},\kappa_{\nu,j+1})$ with $(\lambda,k)=F_{\pi_0}(\nu,j)(\sigma,i)$ in the case that $(\sigma,i)\in\operatorname{supp}\pi_0(\nu,j)$.

- Concerning the $(p')_i^\sigma$ with $(\sigma,i)\in\operatorname{supp}\pi_0$:
  - for $\zeta\in\operatorname{dom}\pi_0$,

$$\big((p')_i^\sigma(\zeta)\mid(\sigma,i)\in\operatorname{supp}\pi_0(\nu,j)\big)=\pi_0(\zeta)\big(p_i^\sigma(\zeta)\mid(\sigma,i)\in\operatorname{supp}\pi_0(\nu,j)\big),$$

– and in the case that $\zeta \notin \operatorname{dom} \pi_0$,

$$(p')_i^\sigma(\zeta) = p_k^\lambda(\zeta) \text{ with } (\lambda, k) = G_{\pi_0}(\nu, j)(\sigma, i).$$

- Whenever $(\sigma, i) \notin \operatorname{supp} \pi_0$, then $(p')_i^\sigma = p_i^\sigma$.
- We now turn to p'_*. Consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$. For any $(\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)$, let $\{\xi_i^\sigma\} := a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$. For $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap \operatorname{dom} \pi_0$, we will have

$$\big(p'_*(\xi_i^\sigma, \zeta) \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)\big) =$$

$$= (\pi_0)_*(\zeta)\big(\xi_i^\sigma \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)\big)\big(p_*(\xi_i^\sigma, \zeta) \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)\big).$$

In the case that $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap (\operatorname{dom} p \smallsetminus \operatorname{dom} \pi_0)$, we will have for $(\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)$:

$$p'_*(\xi_i^\sigma, \zeta) := p_*(\xi_k^\lambda, \zeta),$$

where $(\lambda, k) = G_{\pi_0}(\nu, j) \circ (F_{\pi_0}(\nu, j))^{-1}(\sigma, i)$.
Finally, if $(\xi, \zeta) \in (\operatorname{dom} p)^2$ with $\xi, \zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ such that $\xi \notin \bigcup_{\sigma,i} a_i^\sigma$, then

$$p'_*(\xi, \zeta) = (\pi_0)_*(\xi, \zeta)\big(p_*(\xi, \zeta)\big).$$

For any $\pi \in A_0$, it follows that $D_{\pi_0} \in \mathcal{D}_0$ with $\operatorname{supp} D_{\pi_0} := \operatorname{supp} \pi_0$ and $\operatorname{dom} D_{\pi_0} := \operatorname{dom} \pi_0$. Moreover, whenever p is a condition in D_{π_0}, then $p' := \pi_0 p \in \mathbb{P}_0$ is well-defined with $p' \in D_{\pi_0}$, since $\operatorname{supp} p' = \operatorname{supp} p$, $\operatorname{dom} p' = \operatorname{dom} p$, and $\bigcup_{\sigma,i} a_i^\sigma = \bigcup_{\sigma,i} (a')_i^\sigma$ by construction.
Here we use that π_0 is only defined on D_{π_0} and not on the entire forcing $\mathbb{P}_0$, since we have to make sure that in our construction of the $p'_*(\xi_i^\sigma, \zeta)$ for $\zeta \notin \operatorname{dom} \pi_0$, we do not run out of $\operatorname{dom} p$.

It is not difficult to see that for any p, $q \in D_{\pi_0}$ with $q \leq p$, also $q' \leq p'$ holds. The *linking property* follows readily from our definition of the $p'_*(\xi_i^\sigma, \zeta)$ for $\zeta \notin \operatorname{dom} \pi_0$.

Whenever $\pi_0 \in A_0$ and $D \in \mathcal{D}_0$ with $D \subseteq D_{\pi_0}$, it follows that the map $\overline{\pi}_0 := \pi_0 \upharpoonright D$ is contained in A_0, as well. Here we have to use that the maps π_0 do not disturb the conditions' domain or support, and merely *permute* the linking ordinals. In particular, whenever $p \in D$, it follows that the image $\pi_0 p$ is contained in D, as well.

For Definition 1.2.14, it remains to verify that for any $D \in \mathcal{D}_0$, the collection $(A_0)_D := \{\pi_1 \in A_0 \mid D_{\pi_0} = D\}$ is a group.

Firstly, whenever $\pi_0 \in A_0$, $\pi_0 : D_{\pi_0} \to D_{\pi_0}$, it is not difficult to write down a map $\nu_0 \in A_0$ with $D_{\nu_0} = D_{\pi_0}$ such that ν_0 is the inverse of π_0:

Let $\operatorname{supp} \nu_0 := \operatorname{supp} \pi_0$ and $\operatorname{dom} \nu_0 := \operatorname{dom} \pi_0$. For any $\kappa_{\nu,j} < \kappa_\gamma$, we set $F_{\nu_0}(\nu, j) := (F_{\pi_0}(\nu, j))^{-1}$, $G_{\nu_0}(\nu, j) := (G_{\pi_0}(\nu, j))^{-1}$; and for $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$, we let $\nu_0(\zeta) := (\pi_0(\zeta))^{-1}$. Regarding $(\nu_0)_*$ we use the following notation:

For sets $\mathcal{I}$, $\mathcal{J}$ with a bijection $b : \mathcal{I} \to \mathcal{J}$ and a sequence $(x_j \mid j \in \mathcal{J})$, we denote by $\overline{b}(x_j \mid j \in \mathcal{J})$ the induced sequence parametrized by $\mathcal{I}$:

$$\overline{b}(x_j \mid j \in \mathcal{J}) := (y_i \mid i \in \mathcal{I}) \text{ with } y_i := x_{b(i)} \text{ for all } i \in \mathcal{I}.$$

Whenever $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap \operatorname{dom} \pi_0$, and

$$\big(\xi_i^\sigma \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)\big) \in (\operatorname{dom} \pi_0(\nu, j))^{\operatorname{supp} \pi_0(\nu,j)}$$

is a sequence of pairwise distinct ordinals, we set $(\nu_0)_*\big(\xi_i^\sigma \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)\big) :=$

$$\overline{F_{\pi_0}(\nu,j)} \circ \Big[(\pi_0)_*\Big(\overline{F_{\pi_0}(\nu,j)}^{\,-1}\big(\xi_i^\sigma \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)\big)\Big)\Big]^{-1} \circ \overline{F_{\pi_0}(\nu,j)}^{\,-1},$$

which is a bijection on $2^{\operatorname{supp} \pi_0(\nu,j)}$.

For $(\xi, \zeta) \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2$, let $(\nu_0)_*(\xi, \zeta) := \big((\pi_0)_*(\xi, \zeta)\big)^{-1}$.

It is not difficult to verify that indeed, $\pi_0(\nu_0(p)) = \nu_0(\pi_0(p)) = p$ holds for all $p \in D_{\pi_0} = D_{\nu_0}$.

Secondly, for any $\pi_0 : D \to D$, $\sigma_0 : D \to D$ in A_0, one can write down a map $\tau_0 \in A_0$ explicitly with $D_{\tau_0} = D$ such that $\tau_0(p) = \pi_0(\sigma_0(p))$ holds for all $p \in D$.

Finally, $(A_0)_D$ contains the identity element $(\mathrm{id}_0)_D$ (the identity on D); and it follows that $(A_0)_D$ is indeed a group.

Now, all the properties form Definition 1.2.14 are satisfied. Hence, A_0 is an *almost-group of partial* $\mathbb{P}_0$*-automorphisms.*

We turn to $\mathbb{P}_1$ and define A_1, our collection of partial $\mathbb{P}_1$-isomorphisms. Every $\pi_1 \in A_1$ will be a bijection $\pi_1 : D_{\pi_1} \to D_{\pi_1}$ with a dense set $D_{\pi_1} \in \mathcal{D}_1$, where $\mathcal{D}_1$ is defined as follows:

Let $\mathcal{D}_1$ denote the collection of all $D \subseteq \mathbb{P}_1$ given by:

- a countable *support* $\operatorname{supp} D \subseteq Succ$, and
- for every $\sigma \in \operatorname{supp} D$, $\kappa_\sigma = \overline{\kappa_\sigma}^{\,+}$, a *domain* $\operatorname{dom} D(\sigma) = \operatorname{dom}_x D(\sigma) \times \operatorname{dom}_y D(\sigma) \subseteq \alpha_\sigma \times [\overline{\kappa_\sigma}, \kappa_\sigma)$ with $|\operatorname{dom} \pi_1(\sigma)| < \kappa_\sigma$,

such that

$$D = \{p \in \mathbb{P}_1 \mid \operatorname{supp} p \supseteq \operatorname{supp} D \ \wedge\ \forall\, \sigma \in \operatorname{supp} D\ \ \operatorname{dom} p^\sigma \supseteq \operatorname{dom} D(\sigma)\}.$$

Then every set $D \in \mathcal{D}_1$ is open dense; and whenever D, $D' \in \mathcal{D}_1$, then the intersection $D \cap D'$ is contained in $\mathcal{D}_1$ as well, with $\operatorname{supp}(D \cap D') = \operatorname{supp} D \cup \operatorname{supp} D'$, and $\operatorname{dom}_x(D \cap D')(\sigma) = \operatorname{dom}_x D(\sigma) \cup \operatorname{dom}_x D'(\sigma)$, $\operatorname{dom}_y(D \cap D')(\sigma) = \operatorname{dom}_y D(\sigma) \cup \operatorname{dom}_y D'(\sigma)$ for all $\sigma \in \operatorname{supp}(D \cap D')$.

Moreover, the collection $\mathcal{D}_1$ has a *maximal element* $(D_1)_{\max}$ with $\operatorname{supp}(D_1)_{\max} := \varnothing$. Then for all $D \in \mathcal{D}_1$, we have $\mathbb{P}_1 = (D_1)_{\max} \supseteq D$.

It follows that $\mathcal{D}$ has all the properties required in Definition 1.2.14.

We now describe the two types of partial $\mathbb{P}_1$-isomorphisms that will generate A_1:

As with A_0, our first goal is that for any $p, q \in \mathbb{P}_1$ which have the same "shape", i.e. $\operatorname{supp} p = \operatorname{supp} q$ and $\operatorname{dom} p^\sigma = \operatorname{dom} q^\sigma$ for every $\sigma \in \operatorname{supp} p$, there is an isomorphism $\pi_1 \in A_1$ with

$\pi_1 p = q$. These isomorphisms will be of the following form: For every $\sigma \in \operatorname{supp} \pi_1$, we will have a collection of $\pi(\sigma)(i,\zeta) : 2 \to 2$ for $(i,\zeta) \in \operatorname{dom} \pi_1(\sigma)$, such that for any $p \in D_{\pi_1}$, the map π_1 changes the value of $p^\sigma(i,\zeta)$ if and only if $\pi_1(\sigma)(i,\zeta) \neq id$. In other words, $(\pi_1 p)^\sigma(i,\zeta) = \pi_1(\sigma)(i,\zeta)\big(p^\sigma(i,\zeta)\big)$.
This allows for constructing an isomorphism π_1 with $\pi_1 p = q$ for any pair of conditions p, q that have the same supports and domains: One can simply set $\pi_1(\sigma)(i,\zeta) = id$ if $p^\sigma(i,\zeta) = q^\sigma(i,\zeta)$, and $\pi_1(\sigma)(i,\zeta) \neq id$ in the case that $p^\sigma(i,\zeta) \neq q^\sigma(i,\zeta)$.

Secondly, for every pair of generic κ_σ-subsets G^σ_i and $G^\sigma_{i'}$ for $\sigma \in \mathrm{Succ}$ and $i, i' < \alpha_\sigma$, we want an isomorphism $\pi \in A_1$ with $\pi G^\sigma_i = G^\sigma_{i'}$. Therefore, we include into A_1 all isomorphisms $\pi_1 = (\pi_1(\sigma) \mid \sigma \in \operatorname{supp} \pi_1)$ such that for every $\sigma \in \operatorname{supp} \pi_1$, there is a bijection $f_{\pi_1}(\sigma)$ on a countable set $\operatorname{supp} \pi_1(\sigma) \subseteq \alpha_\sigma$; and π_1 is defined as follows: Whenever $p \in D_{\pi_1}$, then $(\pi_1 p)^\sigma(i,\zeta) = p^\sigma\big(f_{\pi_1}(\sigma)(i)\,,\,\zeta\big)$ for all $(i,\zeta) \in \operatorname{dom} p^\sigma$. Then $\pi G^\sigma_i = G^\sigma_{f_{\pi_1}(\sigma)(i)}$.

Roughly speaking, A_1 will be generated by these two types of isomorphisms. In order to retain a group structure, the values $(\pi_1 p)^\sigma(i,\zeta)$ for $(i,\zeta) \in \operatorname{dom} \pi_1(\sigma)$ and $i \in \operatorname{supp} \pi_1(\sigma)$ have to be treated separately: For every $\zeta \in \operatorname{dom}_y \pi_1(\sigma)$, there will be a bijection $\pi_1(\zeta) : 2^{\operatorname{supp} \pi_1(\sigma)} \to 2^{\operatorname{supp} \pi_1(\sigma)}$ such that $\big((\pi_1 p)^\sigma(i,\zeta) \mid i \in \operatorname{supp} \pi_1(\sigma)\big) = \pi_1(\zeta)\big(p^\sigma(i,\zeta) \mid i \in \operatorname{supp} \pi_1(\sigma)\big)$.

This yields the following definition:

Definition 2.4.2. A_1 consists of all isomorphisms $\pi_1 : D_{\pi_1} \to D_{\pi_1}$, $\pi_1 = (\pi_1(\sigma) \mid \sigma \in \operatorname{supp} \pi_1)$ with countable support $\operatorname{supp} \pi_1 \subseteq \mathrm{Succ}$, such that for all $\sigma \in \operatorname{supp} \pi_1$, $\kappa_\sigma = \overline{\kappa_\sigma}^+$, there are

- a countable set $\operatorname{supp} \pi_1(\sigma) \subseteq \alpha_\sigma$ with a bijection $f_{\pi_1}(\sigma) : \operatorname{supp} \pi_1(\sigma) \to \operatorname{supp} \pi_1(\sigma)$,
- a domain $\operatorname{dom} \pi_1(\sigma) = \operatorname{dom}_x \pi_1(\sigma) \times \operatorname{dom}_y \pi_1(\sigma) \subseteq \alpha_\sigma \times [\overline{\kappa_\sigma}, \kappa_\sigma)$ with $|\operatorname{dom} \pi_1(\sigma)| < \kappa_\sigma$, such that $\operatorname{supp} \pi_1(\sigma) \subseteq \operatorname{dom}_x \pi_1(\sigma)$,
- for every $(i,\zeta) \in \alpha_\sigma \times [\overline{\kappa_\sigma}, \kappa_\sigma)$ a bijection $\pi_1(\sigma)(i,\zeta) : 2 \to 2$, with $\pi_1(\sigma)(i,\zeta) = \mathrm{id}$ whenever $(i,\zeta) \notin \operatorname{dom} \pi_1(\sigma)$, and
- for every $\zeta \in \operatorname{dom}_y \pi_1(\sigma)$ a bijection $\pi_1(\zeta) : 2^{\operatorname{supp} \pi_1(\sigma)} \to 2^{\operatorname{supp} \pi_1(\sigma)}$

with
$$D_{\pi_1} = \{p \in \mathbb{P}_1 \mid \operatorname{supp} p \supseteq \operatorname{supp} \pi_1 \wedge \forall\, \sigma \in \operatorname{supp} \pi_1 \;\; \operatorname{dom} p^\sigma \supseteq \operatorname{dom} \pi_1(\sigma)\};$$
and for every $p \in D_{\pi_1}$, the image $\pi_1 p$ is defined as follows:

We will have $\operatorname{supp}(\pi_1 p) = \operatorname{supp} p$ with $(\pi_1 p)^\sigma = p^\sigma$ whenever $\sigma \notin \operatorname{supp} \pi_1$. Moreover, for $\sigma \in \operatorname{supp} \pi_1$,

- for every $(i,\zeta) \in \operatorname{dom} p^\sigma$ with $i \notin \operatorname{supp} \pi_1(\sigma)$, we have $(\pi_1 p)^\sigma(i,\zeta) = \pi_1(\sigma)(i,\zeta)\big(p^\sigma(i,\zeta)\big)$,
- for every $i \in \operatorname{supp} \pi_1(\sigma)$ and $\zeta \in \operatorname{dom}_y p^\sigma \setminus \operatorname{dom}_y \pi_1(\sigma)$,
$$(\pi_1 p)^\sigma(i,\zeta) = p^\sigma\big(f_{\pi_1}(\sigma)(i), \zeta\big), \text{ and}$$

- for all $\zeta \in \operatorname{dom}_y \pi_1(\sigma)$,

$$\big((\pi_1 p)^\sigma(i,\zeta) \mid i \in \operatorname{supp}\pi_1(\sigma)\big) = \pi_1(\zeta)\big(p^\sigma(i,\zeta) \mid i \in \operatorname{supp}\pi_1(\sigma)\big).$$

In other words: Outside the domain $\operatorname{dom}\pi_1(\sigma)$, we have a swap of the horizontal lines $p^\sigma(i,\cdot)$ for $i \in \operatorname{supp}\pi_1(\sigma)$, according to $f_{\pi_1}(\sigma)$. If $\zeta \in \operatorname{dom}_y \pi_1(\sigma)$, then the values $(\pi_1 p)^\sigma(i,\zeta)$ for $i \in \operatorname{supp}\pi_1(\sigma)$ are given by the map $\pi_1(\zeta)$. Any remaining value $(\pi_1 p)^\sigma(i,\zeta)$ with $i \notin \operatorname{supp}\pi_1(\sigma)$ is equal to $p^\sigma(\zeta,i)$ or not, depending on whether $\pi_1(\sigma)(i,\zeta) : 2 \to 2$ is the identity or not.

We need the dense sets D_{π_1} in order to make sure that $\operatorname{dom}(\pi_1 p)^\sigma = \operatorname{dom} p^\sigma$. In particular, we do not want to run out of $\operatorname{dom} p^\sigma$ when permuting the $p^\sigma(i,\cdot)$ for $i \in \operatorname{supp}\pi_1(\sigma)$.

It is not difficult to see that any map $\pi_1 : D_{\pi_1} \to D_{\pi_1}$ as in Definition 2.4.2 is order-preserving.

Whenever $\pi_1 \in A_1$ and $D \in \mathcal{D}_1$ with $D \subseteq D_{\pi_1}$, then the map $\overline{\pi}_1 := \pi_1 \restriction D$ is contained in A_1, as well. Here we have to use that the maps π_1 do not disturb the conditions' support or domain. In particular, whenever $p \in D$, it follows that $\pi_1 p \in D$, as well.

For Definition 1.2.14, it remains to verify that for any $D \in \mathcal{D}_1$, the collection $(A_1)_D := \{\pi_1 \in A_1 \mid D_{\pi_1} = D\}$ is a group; which happens similarly as for A_0:

Firstly, for any $\pi_1 \in A_1$, $\pi_1 : D_{\pi_1} \to D_{\pi_1}$, one can write down a map $\nu_1 \in A_1$ with $D_{\nu_1} = D_{\pi_1}$ such that ν_1 is the inverse of π_1.
Secondly, whenever $\pi_1, \sigma_1 \in A_1$, $\pi_1 : D \to D$, $\sigma_1 : D \to D$, one can explicitly write down a map $\tau_1 \in A_1$ with $D_{\tau_1} = D$ such that $\tau_1(p) = \pi_1(\sigma_1(p))$ holds for all $p \in D$.
Finally, $(A_1)_D$ contains the identity element $(\mathrm{id}_1)_D$ (the identity on D), and it follows that $(A_1)_D$ is indeed a group.

Hence, all the properties from Definition 1.2.14 are satisfied, so A_1 is indeed an *almost-group of partial* $\mathbb{P}_1$*-automorphisms.*

Definition 2.4.3. Let $A := A_0 \times A_1$, i.e. any $\pi \in A$ is of the form $\pi = (\pi_0, \pi_1)$, where $\pi_0 \in A_0$, $\pi_0 : D_{\pi_0} \to D_{\pi_0}$ is a partial $\mathbb{P}_0$-automorphism, and $\pi_1 \in A_1$, $\pi_1 : D_{\pi_1} \to D_{\pi_1}$ is a partial $\mathbb{P}_1$-automorphism.

Let $\mathcal{D} := \mathcal{D}_0 \times \mathcal{D}_1$. By what we have just shown, it follows that A is an *almost-group of partial* $\mathbb{P}$*-automorphisms for* $\mathcal{D}$.

Let $\overline{A}$ denote the *group of partial* $\mathbb{P}$*-automorphisms derived from* A as in Definition 1.2.15:

For π, $\pi' \in A$, $\pi : D_\pi \to D_\pi$, $\pi' : D_{\pi'} \to D_{\pi'}$, we set

$$\pi \sim \pi' \;:\Leftrightarrow\; \pi \restriction (D_\pi \cap D_{\pi'}) = \pi' \restriction (D_\pi \cap D_{\pi'}),$$

and let $\overline{A} := \{[\pi] \mid \pi \in A\}$, with concatenation $\overline{A}$ given by concatenation in A (cf. 1.2.15).

2.4.2 Constructing $\mathcal{F}$

Now, we define a collection of $\overline{A}$-subgroups that will generate a normal filter $\mathcal{F}$ on $\overline{A}$, establishing our notion of symmetry.

We will introduce two different types of $\overline{A}$-subgroups.

Firstly, for any $0 < \eta < \gamma$, $i < \alpha_\eta$ (with $\eta \in \mathrm{Lim}$ or $\eta \in \mathrm{Succ}$), let

$$Fix(\eta, i) := \{ [\pi] \in \overline{A} \mid \forall p \in D_\pi \ (\pi p)^\eta_i = p^\eta_i \}.$$

Whenever $\pi \sim \pi'$, it follows that $(\pi p)^\eta_i = p^\eta_i$ for all $p \in D_\pi$ if and only if $(\pi' p)^\eta_i = p^\eta_i$ for all $p \in D_{\pi'}$. Hence, $Fix(\eta, i)$ is well-defined, and any $Fix(\eta, i)$ is a subgroup of $\overline{A}$.

By including $Fix(\eta, i)$ into our filter $\mathcal{F}$, we make sure that any canonical name $\dot{G}^\eta_i$ for the i-th generic κ_η-subset G^η_i is hereditarily symmetric, since $\pi \overline{G^\eta_i}^{D_\pi} = \overline{G^\eta_i}^{D_\pi}$ for all $\pi \in Fix(\eta, i)$. Hence, our eventual model N will contain any generic κ_η-subset G^η_i.

Now, we turn to the second type of $\overline{A}$-subgroup. For any $0 < \lambda < \gamma$ and $k < \alpha_\lambda$ (with $\lambda \in \mathrm{Lim}$ or $\lambda \in \mathrm{Succ}$), we need in N a surjection $s : \wp(\kappa_\lambda) \to k$ in order to make sure that $\theta^N(\kappa_\lambda) \geq \alpha_\lambda$. However, the sequence $(G^\lambda_i \mid i < \alpha_\lambda)$ must *not* be included into N, since $\theta^N(\kappa_\lambda) \leq \alpha_\lambda$, so N must not contain a surjection $s : \wp(\kappa_\lambda) \to \alpha_\lambda$.

The idea is that for any $0 < \lambda < \gamma$ and $k < \alpha_\lambda$, we define a "cloud" around each G^λ_i for $i \leq k$, denoted by $(\widetilde{G^\lambda_i})^{(k)}$, and make sure that the "sequence of clouds" $\big((\widetilde{G^\lambda_i})^{(k)} \mid i < k \big)$ makes its way into N.

When defining the according $\overline{A}$-subgroups, we have to treat limit cardinals and successor cardinals separately.

For $\lambda \in \mathrm{Lim}$, $k < \alpha_\lambda$, let

$$H^\lambda_k := \Big\{ [\pi] \in \overline{A} \ \Big| \ \exists\, \kappa_{\overline{\nu}, \overline{\jmath}} < \kappa_\lambda \ \ \forall\, \kappa_{\nu, j} \in [\kappa_{\overline{\nu}, \overline{\jmath}}, \kappa_\lambda) \ \ \forall\, i \leq k \ : \\ \Big((\lambda, i) \notin \operatorname{supp} \pi_0(\nu, j) \ \vee \ G_{\pi_0}(\nu, j)(\lambda, i) = (\lambda, i) \Big) \Big\}.$$

It is not difficult to verify that any H^λ_k is well-defined and indeed a subgroup of $\overline{A}$.

Roughly speaking, H^λ_k contains all $[\pi] \in \overline{A}$ such that above some $\kappa_{\overline{\nu}, \overline{\jmath}} < \kappa_\lambda$, there is no permutation of the vertical lines $P^\lambda_i \restriction [\kappa_{\overline{\nu}, \overline{\jmath}}, \kappa_\lambda)$ for $i \leq k$.
This implies that for any $i, j < k$ with $i \neq j$ and $[\pi] \in H^\lambda_k$, it is not possible that $\pi G^\lambda_i = G^\lambda_j$. Hence, for any $i < k$, we can define a "cloud" around G^λ_i as follows:

$$(\dot{\widetilde{G^\lambda_i}})^{(k)} := \{ (\pi \overline{G^\lambda_i}^{D_\pi}, \mathbb{1}) \mid [\pi] \in H^\lambda_k \}.$$

With $(\widetilde{G^\lambda_i})^{(k)} := \big((\dot{\widetilde{G^\lambda_i}})^{(k)} \big)^G$, it follows that $(\widetilde{G^\lambda_i})^{(k)}$ is the orbit of G^λ_i under H^λ_k; so two distinct orbits $(\widetilde{G^\lambda_i})^{(k)}$ and $(\widetilde{G^\lambda_j})^{(k)}$ for $i \neq j$ are disjoint. The sequence $\big((\widetilde{G^\lambda_i})^{(k)} \mid i < k \big)$, which has a canonical symmetric name stabilized by all π with $[\pi] \in H^\lambda_k$, gives a surjection $s : \wp(\kappa_\lambda) \to k$ in N (see Chapter 2.6.1).

Now, we consider the case that $\lambda \in \mathrm{Succ}$. For $k < \alpha_\lambda$, let

$$H_k^\lambda := \left\{ [\pi] \in \overline{A} \mid \forall\, i \leq k \ \left(i \notin \operatorname{supp} \pi_1(\lambda) \vee f_{\pi_1}(\lambda)(i) = i \right) \right\}.$$

Again, one can easily check that H_k^λ is well-defined and indeed an $\overline{A}$-subgroup.

Whenever $[\pi]$ is contained in H_k^λ, then π does not interchange any G_i^λ and G_j^λ for $i, j < k$ in the case that $i \neq j$. Thus, as for $\lambda \in \mathrm{Lim}$, we can define "clouds" $(\widetilde{G_i^\lambda})^{(k)}$ for $i \leq k$ and obtain a surjection $s : \wp(\kappa_\eta) \to k$ in N (see Chapter 2.6.1).

We are now ready to define our normal filter $\mathcal{F}$ on $\overline{A}$. Note that the $Fix(\eta, i)$ and H_k^λ are *not* normal $\overline{A}$-subgroups: For instance, if $[\pi] \in Fix(\eta, i)$ for some $\eta \in \mathrm{Lim}$, $i < \alpha_\eta$, and $\sigma \in A$ with $G_{\sigma_0}(\nu, j)(\eta, i) = (\eta, i')$ for all $\kappa_{\nu,j}, < \kappa_\eta$ such that $[\pi] \notin Fix(\eta, i')$, then in general, $[\sigma]^{-1}[\pi][\sigma]$ is not contained in $Fix(\eta, i)$.

However, it is not difficult to verify:

Lemma 2.4.4.

- *For all $\sigma \in A$, and $\eta \in \mathrm{Lim}$, $i < \alpha_\eta$,*

  $$[\sigma] Fix(\eta, i)[\sigma]^{-1} \supseteq Fix(\eta, i) \cap \bigcap \{ Fix(\eta_m, i_m) \mid m < \omega, (\eta_m, i_m) \in \operatorname{supp} \sigma_0 \}.$$

  In the case that $\sigma \in A$, and $\eta \in \mathrm{Succ}$, $i < \alpha_\eta$,

  $$[\sigma] Fix(\eta, i)[\sigma]^{-1} \supseteq Fix(\eta, i) \cap \bigcap \{ Fix(\eta, i_m) \mid m < \omega, i_m \in \operatorname{supp} \sigma_1(\eta) \}.$$

- *For $\sigma \in A$ and $\lambda \in \mathrm{Lim}$, $k < \alpha_\lambda$,*

  $$[\sigma] H_k^\lambda [\sigma]^{-1} \supseteq H_k^\lambda \cap \bigcap \{ Fix(\eta_m, i_m) \mid m < \omega, (\eta_m, i_m) \in \operatorname{supp} \sigma_0 \}.$$

  In the case that $\lambda \in \mathrm{Succ}$, $k < \alpha_\lambda$,

  $$[\sigma] H_k^\lambda [\sigma]^{-1} \supseteq H_k^\lambda \cap \bigcap \{ Fix(\lambda, i_m) \mid m < \omega, i_m \in \operatorname{supp} \sigma_1(\lambda) \}.$$

Hence, it follows that countable intersections of the $\overline{A}$-subgroups $Fix(\eta, i)$ and H_k^λ generate a normal filter on $\overline{A}$:

Definition 2.4.5. We define $\mathcal{F}$ as follows:

A subgroup $B \subseteq \overline{A}$ is contained in $\mathcal{F}$ if there are $((\eta_m, i_m) \mid m < \omega)$, $((\lambda_m, k_m) \mid m < \omega)$ with

$$B \supseteq \bigcap_{m<\omega} Fix(\eta_m, i_m) \cap \bigcap_{m<\omega} H_{k_m}^{\lambda_m}.$$

Then by Lemma 2.4.4, it follows that $\mathcal{F}$ is a normal, countably complete filter on $\overline{A}$.

Now, we can use $\mathcal{F}$ to establish our notion of symmetry. The following Definition corresponds to Definition 1.2.18:

Definition 2.4.6. A $\mathbb{P}$-name $\dot{x}$ is *symmetric* if

$$\operatorname{sym}^{\overline{A}}(\dot{x}) := \left\{ [\pi] \in \overline{A} \mid \pi \overline{x}^{D_\pi} = \overline{x}^{D_\pi} \right\} \in \mathcal{F}.$$

Recursively, a name $\dot{x}$ is hereditarily symmetric, $x \in HS$, if $\dot{x}$ is symmetric, and $\dot{y}$ is hereditarily symmetric for all $\dot{y} \in \operatorname{dom} \dot{x}$.

2.5 The Symmetric Submodel

Let G be a V-generic filter on $\mathbb{P}$. The *symmetric extension by $\mathcal{F}$ and G* is

$$N := V(G) := V(G)^{\mathcal{F}} = \{\dot{x}^G \mid \dot{x} \in HS\}.$$

As set out in Chapter 1.2.3, the symmetric forcing relation with partial automorphisms "$\Vdash_s$" can be defined as usual, and satisfies the same basic properties as the ordinary symmetric forcing relation. In particular, the *Symmetry Lemma* holds, and the *Forcing Theorem* holds true, as well.

Whenever $\dot{x}$, $\dot{y} \in HS$ and $p \in \mathbb{P}$, then $p \Vdash_s \dot{y} \in \dot{x}$ if and only if $p \Vdash \dot{y} \in \dot{x}$ (with the ordinary forcing relation "$\Vdash$") and $p \Vdash_s \dot{x} = \dot{y}$ if and only if $p \Vdash \dot{x} = \dot{y}$. In particular, for any $\dot{x} \in HS$ and $D \in \mathcal{D}$, we have

$$\overline{x}^D = \{(\overline{y}^D, p) \mid \dot{y} \in \operatorname{dom} \dot{x}\,,\, p \in D\,,\, p \Vdash_s \dot{y} \in \dot{x}\}.$$

By Theorem 1.2.21, it follows that $N = V(G)$ is a transitive model of ZF with $V \subseteq N \subseteq V[G]$.

Proposition 2.5.1. $N \vDash \mathsf{DC} + \mathsf{AX}_4$.

Proof. $N \vDash \mathsf{DC}$ follows readily, since firstly, $\mathbb{P}$ is countably closed (Proposition 2.3.7) and secondly, our normal filter $\mathcal{F}$ generating N is countably complete. We give a proof for the sake of completeness, using the basic ideas from [Kar14, Lemma 1].

It suffices to show that whenever $X \in N$ and $f : \omega \to X$ is a function in $V[G]$, then $f \in N$. This implies DC: Assume that there was in N a set X with a binary relation R such that for any $x \in X$ there exists $y \in X$ with $y\,R\,x$. Since DC holds in $V[G]$, we obtain a sequence $(x_n \mid n < \omega)$ in $V[G]$ such that $x_{n+1}\,R\,x_n$ for every $n < \omega$. Then $(x_n \mid n < \omega)$ is already contained in N, as desired.

Let $X \in N$, $X = \dot{X}^G$ with $\dot{X} \in HS$, and consider a function $f : \omega \to X$ in $V[G]$. Let $f = \dot{f}^G$, where $\dot{f} \in \operatorname{Name}^V(\mathbb{P})$. Take a condition $\overline{p}_0 \in G$ such that

$$\overline{p}_0 \Vdash \dot{f} : \omega \to \dot{X}.$$

We claim that the following set (in V) is dense in $\mathbb{P}$ below $\overline{p}_0$:

$$D := \left\{ p \in \mathbb{P} \;\middle|\; \exists\, (\dot{x}_n \mid n < \omega)\ \forall\, n < \omega\ \left(\dot{x}_n \in \operatorname{dom} \dot{X} \,\wedge\, p \Vdash \dot{f}(n) = \dot{x}_n\right) \right\}$$

Let $p_0 \in \mathbb{P}$, $p_0 \leq \overline{p}_0$. We use recursion in V to construct sequences $(p_n \mid n < \omega)$ and $(\dot{x}_n \mid n < \omega)$ such that $\dot{x}_n \in \operatorname{dom} \dot{X}$, $p_{n+1} \leq p_n$, and $p_{n+1} \Vdash \dot{f}(n) = \dot{x}_n$ for all $n < \omega$.
By countable closure of $\mathbb{P}$, we can take a condition $p \in \mathbb{P}$ such that $p \leq p_n$ for all $n < \omega$. Then $p \Vdash \dot{f}(n) = \dot{x}_n$ for all $n < \omega$. Thus, p is an extension of p_0 in D; and it follows that D is dense in $\mathbb{P}$ below $\overline{p}_0$.

Thus, we can take $q \in G \cap D$, $q \leq \overline{p}_0$, and $(\dot{x}_n \mid n < \omega)$ as in the definition of D with $\dot{x}_n \in \operatorname{dom} \dot{X}$ and $q \Vdash \dot{f}(n) = \dot{x}_n$ for all $n < \omega$. Let

$$\dot{g} := \left\{ \left(\operatorname{OR}_{\mathbb{P}}(n, \dot{x}_n), \mathbb{1} \right) \;\middle|\; n < \omega \right\}.$$

Then $\dot{g}^G = f$. It remains to make sure that $\dot{g} \in HS$. Since $\dot{x}_n \in \operatorname{dom} \dot{X} \subseteq HS$ for all $n < \omega$, it follows that $\operatorname{sym}^{\overline{A}}(\dot{x}_n) \in \mathcal{F}$ for every $n < \omega$. Thus, $\operatorname{dom} \dot{g} \subseteq HS$. Moreover, $\mathcal{F}$ is countably complete, which gives $\bigcap_{n<\omega} \operatorname{sym}^{\overline{A}}(\dot{x}_n) \in \mathcal{F}$. Since $\pi \overline{g}^{D_\pi} = \overline{g}^{D_\pi}$ for all $\pi \in \bigcap_{n<\omega} \operatorname{sym}^{\overline{A}}(\dot{x}_n)$, it follows that $\dot{g} \in HS$ as desired.

Regarding $N \vDash \mathsf{AX}_4$ (see [She10, p.3 and p.15]), we note that $([\lambda]^{\aleph_0})^{V[G]} = ([\lambda]^{\aleph_0})^V$, since $\mathbb{P}$ is countably closed. Hence, $([\lambda]^{\aleph_0})^N = ([\lambda]^{\aleph_0})^V$. Thus, the set $[\lambda]^{\aleph_0}$ can be well-ordered in N, using the according well-ordering of $[\lambda]^{\aleph_0}$ in V.

□

Next, we want to show that N preserves all V-cardinals; which will follow from the fact that any set of ordinals $X \subseteq \alpha$, $X \in N$, can be captured in a "mild" V-generic extension by a forcing notion as in Lemma 2.3.9 and Lemma 2.3.10.

This *Approximation Lemma* demonstrates how our symmetric extension N can be approximated from within by fairly nice V-generic extensions. Later on, this will be a crucial step in keeping control over the values $\theta^N(\kappa_\eta)$.

Lemma 2.5.2 (Approximation Lemma). *Consider $X \in N$, $X \subseteq \alpha$ with $X = \dot{X}^G$ such that $\pi \overline{X}^{D_\pi} = \overline{X}^{D_\pi}$ holds for $\pi \in A$ with $[\pi]$ contained in the intersection*

$$\bigcap_{m<\omega} Fix(\eta_m, i_m) \cap \bigcap_{m<\omega} Fix(\overline{\eta}_m, \bar{i}_m) \cap \bigcap_{m<\omega} H^{\lambda_m}_{k_m} \cap \bigcap_{m<\omega} H^{\overline{\lambda}_m}_{\overline{k}_m},$$

where $((\eta_m, i_m) \mid m < \omega)$, $((\overline{\eta}_m, \bar{i}_m) \mid m < \omega)$, $((\lambda_m, k_m) \mid m < \omega)$ and $((\overline{\lambda}_m, \overline{k}_m) \mid m < \omega)$ denote sequences with $\eta_m \in \mathrm{Lim}$, $i_m < \alpha_{\eta_m}$; $\overline{\eta}_m \in \mathrm{Succ}$, $\bar{i}_m < \alpha_{\overline{\eta}_m}$ for all $m < \omega$; and $\lambda_m \in \mathrm{Lim}$, $k_m < \alpha_{\lambda_m}$; $\overline{\lambda}_m \in \mathrm{Succ}$, $\overline{k}_m < \alpha_{\overline{\lambda}_m}$ for all $m < \omega$.
Then

$$X \in V\Big[\prod_{m<\omega} G^{\eta_m}_{i_m} \times \prod_{m<\omega} G^{\overline{\eta}_m}_{\bar{i}_m}\Big].$$

Proof. Let

$$X' := \Big\{ \beta < \alpha \;\Big|\; \exists\, p = (p_*, (p^\sigma_i, a^\sigma_i)_{\sigma,i}, (p^\sigma)_\sigma) \,:\, p \Vdash_s \beta \in \dot{X} \;,\; \forall m:\; (\eta_m, i_m) \in \operatorname{supp} p_0 \;,$$

$$\forall m \,:\, a^{\eta_m}_{i_m} = g^{\eta_m}_{i_m} \;,\; (p^{\eta_m}_{i_m})_{m<\omega} \in \prod_{m<\omega} G^{\eta_m}_{i_m} \;,\; (p^{\overline{\eta}_i}_{\bar{i}_m})_{m<\omega} \in \prod_{m<\omega} G^{\overline{\eta}_m}_{\bar{i}_m} \Big\}.$$

Then

$$X' \in V\Big[\prod_{m<\omega} G^{\eta_m}_{i_m} \times \prod_{m<\omega} G^{\overline{\eta}_m}_{\bar{i}_m}\Big],$$

since the sequence $(g^{\eta_m}_{i_m})_{m<\omega}$ is contained in V. It remains to show that $X = X'$. The inclusion $X \subseteq X'$ follows from the Forcing Theorem. Concerning "$\supseteq$", assume towards a contradiction there was $\beta \in X' \setminus X$. Take p as above with $(\eta_m, i_m) \in \operatorname{supp} p_0$ for all $m < \omega$, and

$$p \Vdash_s \beta \in \dot{X} \;,\; \forall m \,:\, a^{\eta_m}_{i_m} = g^{\eta_m}_{i_m} \;,\; (p^{\eta_m}_{i_m})_{m<\omega} \in \prod_{m<\omega} G^{\eta_m}_{i_m} \;,\; (p^{\overline{\eta}_i}_{\bar{i}_m})_{m<\omega} \in \prod_{m<\omega} G^{\overline{\eta}_m}_{\bar{i}_m}.$$

Since $\beta \notin X$, we can take $p' \in G$, $p' = \big(p'_*, \big((p')^\sigma_i, (a')^\sigma_i\big)_{\sigma,i}, \big((p')^\sigma\big)_\sigma\big)$ with $p' \Vdash_s \beta \notin \dot{X}$, such that $(\eta_m, i_m) \in \operatorname{supp} p'_0$ for all $m < \omega$.

First, we want to extend p and p' and obtain conditions $\overline{p} \leq p$, $\overline{p}' \leq p'$, $\overline{p} = (\overline{p}_*, (\overline{p}_i^\sigma, \overline{a}_i^\sigma)_{\sigma,i}, (\overline{p}^\sigma)_\sigma)$, $\overline{p}' = \big(\overline{p}'_*, \big((\overline{p}')_i^\sigma, (\overline{a}')_i^\sigma\big)_{\sigma,i}, \big((\overline{p}')^\sigma\big)_\sigma\big)$ such that the following holds:

- $\forall\, m < \omega \;\; \overline{p}_{i_m}^{\eta_m} = (\overline{p}')_{i_m}^{\eta_m} \;\; , \;\; \overline{a}_{i_m}^{\eta_m} = (\overline{a}')_{i_m}^{\eta_m}$
- $\forall\, m < \omega \;\; \overline{p}_{\overline{i}_m}^{\overline{\eta}_m} = (\overline{p}')_{\overline{i}_m}^{\overline{\eta}_m}$
- $\operatorname{dom} \overline{p}_0 = \operatorname{dom} \overline{p}'_0$
- $\operatorname{supp} \overline{p}_0 = \operatorname{supp} \overline{p}'_0$
- $\bigcup_{(\sigma,i) \in \operatorname{supp} \overline{p}_0} \overline{a}_i^\sigma = \bigcup_{(\sigma,i) \in \operatorname{supp} \overline{p}'_0} (\overline{a}')_i^\sigma$
- $\forall\, (\nu, j) \;:\; \operatorname{dom} \overline{p}_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \neq \emptyset \;\to\; \big(\bigcup_{\sigma,i} \overline{a}_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})\big) \subseteq \operatorname{dom} \overline{p}_0$,
  $\forall\, (\nu, j) \;:\; \operatorname{dom} \overline{p}'_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \neq \emptyset \;\to\; \big(\bigcup_{\sigma,i} (\overline{a}')_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})\big) \subseteq \operatorname{dom} \overline{p}'_0$
- $\operatorname{supp} \overline{p}_1 = \operatorname{supp} \overline{p}'_1$
- $\forall\, \sigma \in \operatorname{supp} \overline{p}_1 = \operatorname{supp} \overline{p}'_1 \;:\; \operatorname{dom} \overline{p}_1(\sigma) = \operatorname{dom} \overline{p}'_1(\sigma)$.

We will now describe how $\overline{p}_0$ and $\overline{p}'_0$ can be constructed. First, we need a set $\operatorname{supp}_0 := \operatorname{supp} \overline{p}_0 = \operatorname{supp} \overline{p}'_0$. Consider

$$s := \sup\{\kappa_\sigma \mid \sigma \in \mathrm{Lim}\,,\ \exists\, i < \alpha_\sigma \,:\, (\sigma, i) \in \operatorname{supp} p_0 \cup \operatorname{supp} p'_0\}.$$

Then by closure of the sequence $(\kappa_\sigma \mid 0 < \sigma < \gamma)$, it follows that $s = \kappa_{\overline{\gamma}}$ for some $\overline{\gamma} \leq \gamma$. If $\overline{\gamma} = \gamma$, then $cf\, \kappa_\gamma = \omega$ and we can take $((\sigma_k, l_k) \mid k < \omega)$ with $\sigma_k \in \mathrm{Lim}$, $l_k < \alpha_{\sigma_k}$ for all $k < \omega$ such that $(\kappa_{\sigma_k} \mid k < \omega)$ is cofinal in κ_γ, and $(\sigma_k, l_k) \notin \operatorname{supp} p_0 \cup \operatorname{supp} p'_0$ for all $k < \omega$. Let

$$\operatorname{supp}_0 := \operatorname{supp} \overline{p}_0 := \operatorname{supp} \overline{p}'_0 := \operatorname{supp} p_0 \,\cup\, \operatorname{supp} p'_0 \,\cup\, \{(\sigma_k, l_k) \mid k < \omega\}.$$

If $\overline{\gamma} < \gamma$, we can set $\sigma_k := \overline{\gamma} \in \mathrm{Lim}$ for all $k < \omega$ and take $(l_k \mid k < \omega)$ such that $l_k < \alpha_{\sigma_k}$ with $(\sigma_k, l_k) = (\overline{\gamma}, l_k) \notin \operatorname{supp} p_0 \,\cup\, \operatorname{supp} p'_0$ for all $k < \omega$. Let

$$\operatorname{supp}_0 := \operatorname{supp} \overline{p}_0 := \operatorname{supp} \overline{p}'_0 := \operatorname{supp} p_0 \,\cup\, \operatorname{supp} p'_0 \,\cup\, \{(\sigma_k, l_k) \mid k < \omega\}$$

as before.
The next step is to define the linking ordinals. Take a set $X \subseteq \kappa_{\overline{\gamma}}$ such that for all intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\overline{\gamma}}$, it follows that $|X \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})| = \aleph_0$; and $X \cap \big(\bigcup_{(\sigma,i) \in \operatorname{supp} p_0} a_i^\sigma \,\cup\, \bigcup_{(\sigma,i) \in \operatorname{supp} p'_0} (a')_i^\sigma\big) = \emptyset$. Let

$$\overline{X} := X \,\cup\, \bigcup_{\sigma,i} a_i^\sigma \,\cup\, \bigcup_{\sigma,i} (a')_i^\sigma.$$

Our aim is to construct $\overline{p}$ and $\overline{p}'$ such that $\bigcup_{\sigma,i} \overline{a}_i^\sigma = \bigcup_{\sigma,i} (\overline{a}')_i^\sigma = \overline{X}$.

Consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\overline{\gamma}}$. For every $(\sigma, i) \in \operatorname{supp} p_0$ with $\kappa_\sigma > \kappa_{\nu,j}$, we let $\overline{a}_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) := a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$.

Define

$$\{\xi_k(\nu, j) \mid k < \omega\} := \big(\overline{X} \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})\big) \setminus \bigcup_{\sigma,i} a_i^\sigma.$$

This set has cardinality $\aleph_0$ by construction of X.

Moreover, let

$$\left\{ \left(\overline{\sigma}_k(\nu,j), \bar{l}_k(\nu,j)\right) \mid k < \omega \right\} =: \{(\sigma,i) \in \operatorname{supp} \overline{p}_0 \setminus \operatorname{supp} p_0 \mid \kappa_\sigma > \kappa_{\nu,j}\}.$$

This set also has cardinality $\aleph_0$ by construction of $\operatorname{supp} \overline{p}_0 = \operatorname{supp}_0$.

Now, for any $k < \omega$, let

$$\overline{a}^{\overline{\sigma}_k(\nu,j)}_{\bar{l}_k(\nu,j)} \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) := \{\xi_k(\nu,j)\}.$$

Together with same construction for $\overline{p}'$, we obtain the linking ordinals $\overline{a}^\sigma_i$, $(\overline{a}')^\sigma_i$ for $(\sigma,i) \in \operatorname{supp}_0 = \operatorname{supp} \overline{p}_0 = \operatorname{supp}(\overline{p}')_0$ such that the *independence property* holds, and $\bigcup_{\sigma,i} \overline{a}^\sigma_i = \bigcup_{\sigma,i} (\overline{a}')^\sigma_i = X$.

Next, we construct $\operatorname{dom}_0 := \operatorname{dom} \overline{p}_0 = \operatorname{dom}(\overline{p}')_0 := \bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j})$ as follows: Consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\gamma$. In the case that $\operatorname{dom} p_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \operatorname{dom}(p')_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \emptyset$, let $\delta_{\nu,j} := \kappa_{\nu,j}$. Otherwise, take $\delta_{\nu,j} \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ such that $(\operatorname{dom} p_0 \cup \operatorname{dom} p_0' \cup \overline{X}) \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq [\kappa_{\nu,j}, \delta_{\nu,j})$. (This is possible since the set $\overline{X} \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ is countable, and any $\kappa_{\nu,j+1}$ is a successor cardinal.)
Let

$$\operatorname{dom}_0 := \operatorname{dom} \overline{p}_0 := \operatorname{dom} \overline{p}_0' := \bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j}).$$

This set is bounded below all regular $\kappa_{\overline{\nu},\overline{j}}$ by construction, since $\operatorname{dom} p_0$ and $\operatorname{dom} p_0'$ are bounded below all regular $\kappa_{\overline{\nu},\overline{j}}$.

Now, for $(\sigma,i) \in \operatorname{supp} \overline{p}_0$, let $\overline{p}^\sigma_i : \operatorname{dom} \overline{p}^\sigma_i \to 2$ on the corresponding domain with $\operatorname{dom} \overline{p}^\sigma_i = \operatorname{dom} \overline{p}_0 \cap \kappa_\sigma$, such that $\overline{p}^\sigma_i \supseteq p^\sigma_i$ for all $(\sigma,i) \in \operatorname{supp} p_0$, and in the case that $(\sigma,i) = (\eta_m, i_m)$ for some $m < \omega$, we additionally require that $\overline{p}^{\eta_m}_{i_m} \supseteq (p')^{\eta_m}_{i_m}$. This is possible, since $p' \in G$ and $p^{\eta_m}_{i_m} \in G^{\eta_m}_{i_m}$, so $p^{\eta_m}_{i_m}$ and $(p')^{\eta_m}_{i_m}$ are compatible.

We define $\overline{p}_*$ on the according domain $\bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j})^2$ such that $\overline{p}_* \supseteq p_*$, and the *linking property* holds for $\overline{p}_0 \le p_0$: Consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $\delta_{\nu,j} > \kappa_{\nu,j}$. For $\zeta \in (\operatorname{dom} \overline{p}_0 \setminus \operatorname{dom} p_0) \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ and $\{\xi\} := a^\sigma_i \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for some $(\sigma,i) \in \operatorname{supp} p_0$, it follows by construction that $\xi \in \operatorname{dom} \overline{p}_0$. Let $\overline{p}_*(\xi,\zeta) := \overline{p}^\sigma_i(\zeta)$. For all ξ, $\zeta \in \operatorname{dom} p_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$, we set $\overline{p}_*(\xi,\zeta) := p_*(\xi,\zeta)$; and $\overline{p}_*(\xi,\zeta) \in \{0,1\}$ arbitrary for the ξ, $\zeta \in \operatorname{dom} \overline{p}_0$ remaining.

Concerning $\overline{p}'$, we set $(\overline{p}')^{\eta_m}_{i_m} = \overline{p}^{\eta_m}_{i_m}$ for all $m < \omega$. Then $(\overline{p}')^{\eta_m}_{i_m} \supseteq (p')^{\eta_m}_{i_m}$ by construction. For the $(\sigma,i) \in \operatorname{supp}(\overline{p}')_0 = \operatorname{supp} \overline{p}_0$ remaining, we can set $(\overline{p}')^\sigma_i$ arbitrarily on the given domain such that $(\overline{p}')^\sigma_i \supseteq (p')^\sigma_i$.
Finally, we let $(\overline{p}')_* \supseteq (p')_*$ according to the *linking property* for $\overline{p}_0' \le p_0'$ (same construction as for $\overline{p}_*$).

It follows that $\overline{p}_0 \le p_0$ and $\overline{p}_0' \le p_0'$, and $\overline{p}_0$ and $\overline{p}_0'$ have all the required properties.

The construction of $\overline{p}_1 \le p_1$ and $\overline{p}_1' \le p_1'$ is similar.

Our aim is to write down an isomorphism $\pi \in A$ with the following properties:

- $\overline{p} \in D_\pi$ with $\pi\overline{p} = \overline{p}'$,
- $\pi \in \bigcap_{m<\omega} Fix(\eta_m, i_m) \cap \bigcap_{m<\omega} Fix(\overline{\eta}_m, \overline{i}_m) \cap \bigcap_{m<\omega} H^{\lambda_m}_{k_m} \cap \bigcap_{m<\omega} H^{\overline{\lambda}_m}_{\overline{k}_m}$ (then $\pi\overline{X}^{D_\pi} = \overline{X}^{D_\pi}$ follows).

From $\overline{p} \Vdash_s \beta \in \dot{X}$, we will then obtain $\pi\overline{p} \Vdash_s \beta \in \pi\overline{X}^{D_\pi}$; hence, $\overline{p}' \Vdash_s \beta \in \overline{X}^{D_\pi}$. This will be a contradiction towards $p' \Vdash_s \beta \notin \dot{X}$.

We start with π_0. Let $\operatorname{dom}\pi_0 := \operatorname{dom}\overline{p}_0 = \operatorname{dom}\overline{p}'_0$, and $\operatorname{supp}\pi_0 := \operatorname{supp}\overline{p}_0 = \operatorname{supp}\overline{p}'_0$.

- Consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$. We define $F_{\pi_0}(\nu, j) : \operatorname{supp}\pi_0(\nu, j) \to \operatorname{supp}\pi_0(\nu, j)$ as follows: Let $F_{\pi_0}(\nu, j)(\sigma, i) := (\lambda, k)$ in the case that $(\overline{a}')^\sigma_i \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \overline{a}^\lambda_k \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$. This is well-defined by the *independence property*, and since we have arranged $\bigcup_{\sigma,i} \overline{a}^\sigma_i = \bigcup_{\sigma,i} (\overline{a}')^\sigma_i$.
- For every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$, let $G_{\pi_0}(\nu, j)(\sigma, i) = (\sigma, i)$ for all $(\sigma, i) \in \operatorname{supp}\pi_0(\nu, j)$.

  (These maps $G_{\pi_0}(\nu, j)$ will be the only parameters of π_0 which are *not* determined by the requirement that $\pi_0\overline{p}_0 = \overline{p}'_0$. However, in order to make sure that $\pi \in \bigcap_{m<\omega} Fix(\eta_m, i_m) \cap \bigcap_{m<\omega} H^{\lambda_m}_{k_m}$, we firstly need $G_{\pi_0}(\nu, j)(\eta_m, i_m) = (\eta_m, i_m)$ for all $m < \omega$; and secondly, whenever $m < \omega$ and $i \le k_m$, we need that $G_{\pi_0}(\nu, j)(\lambda_m, i) = (\lambda_m, i)$ for all $\kappa_{\nu,j}$ above a certain $\kappa_{\overline{\nu},\overline{j}}$.)
- For $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap \operatorname{dom}\pi_0$, we define $\pi_0(\zeta) : 2^{\operatorname{supp}\pi_0(\nu,j)} \to 2^{\operatorname{supp}\pi_0(\nu,j)}$ as follows: For $\big(\epsilon_{(\sigma,i)} \mid (\sigma, i) \in \operatorname{supp}\pi_0(\nu, j)\big) \in 2^{\operatorname{supp}\pi_0(\nu,j)}$ given, let $\pi_0(\zeta)\big(\epsilon_{(\sigma,i)} \mid (\sigma, i) \in \operatorname{supp}\pi(\nu, j)\big) := \big(\widetilde{\epsilon}_{(\sigma,i)} \mid (\sigma, i) \in \operatorname{supp}\pi_0(\nu, j)\big)$ such that $\widetilde{\epsilon}_{(\sigma,i)} = \epsilon_{(\sigma,i)}$ whenever $\overline{p}^\sigma_i(\zeta) = (\overline{p}')^\sigma_i(\zeta)$, and $\widetilde{\epsilon}_{(\sigma,i)} \ne \epsilon_{(\sigma,i)}$ in the case that $\overline{p}^\sigma_i(\zeta) \ne (\overline{p}')^\sigma_i(\zeta)$.
- Let now $\zeta \in \operatorname{dom}\pi_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$, and

  $$\big(\xi^\sigma_i(\nu, j) \mid (\sigma, i) \in \operatorname{supp}\pi_0(\nu, j)\big) \in \operatorname{dom}\pi_0(\nu, j)^{\operatorname{supp}\pi_0(\nu,j)}.$$

  The map $\pi_*(\zeta)\big(\xi^\sigma_i(\nu, j) \mid (\sigma, i) \in \operatorname{supp}\pi_0(\nu, j)\big) : 2^{\operatorname{supp}\pi_0(\nu,j)} \to 2^{\operatorname{supp}\pi_0(\nu,j)}$ is defined as follows: A sequence $\big(\epsilon_{(\sigma,i)} \mid (\sigma, i) \in \operatorname{supp}\pi_0(\nu, j)\big)$ is mapped to $\big(\widetilde{\epsilon}_{(\sigma,i)} \mid (\sigma, i) \in \operatorname{supp}\pi_0(\nu, j)\big)$ with $\widetilde{\epsilon}_{(\sigma,i)} = \epsilon_{(\sigma,i)}$ if $\overline{p}_*(\xi^\sigma_i(\nu, j), \zeta) = \overline{p}'_*(\xi^\sigma_i(\nu, j), \zeta)$, and $\widetilde{\epsilon}_{(\sigma,i)} \ne \epsilon_{(\sigma,i)}$ in the case that $\overline{p}_*(\xi^\sigma_i(\nu, j), \zeta) \ne \overline{p}'_*(\xi^\sigma_i(\nu, j), \zeta)$.
- For $(\xi, \zeta) \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2$, the map $\pi_*(\xi, \zeta) : 2 \to 2$ is defined as follows: We let $\pi_*(\xi, \zeta) = id$ in the case that $(\xi, \zeta) \notin (\operatorname{dom}\pi_0(\nu, j))^2$. If $\xi, \zeta \in \operatorname{dom}\pi_0(\nu, j)$, let $\pi_*(\xi, \zeta) = id$ if $\overline{p}_*(\xi, \zeta) = \overline{p}'_*(\xi, \zeta)$, and $\pi_*(\xi, \zeta) \ne id$ in the case that $\overline{p}_*(\xi, \zeta) \ne \overline{p}'_*(\xi, \zeta)$.

This defines π_0. Directly by construction, it follows that $\pi_0\overline{p}_0 = \overline{p}'_0$: Let

$$\pi_0\overline{p}_0 =: ((\pi\overline{p})_*, ((\pi\overline{p})^\sigma_i, (\pi\overline{a})^\sigma_i)_{\sigma,i}, (\pi\overline{p}^\sigma)_\sigma).$$

Then for any $(\sigma, i) \in \operatorname{supp}(\pi_0\overline{p}_0) = \operatorname{supp}\overline{p}_0$ and $\kappa_{\nu,j} < \kappa_\sigma$, we have $(\pi\overline{a})^\sigma_i \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \overline{a}^\lambda_k \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$, where $(\lambda, k) = F_{\pi_0}(\nu, j)(\sigma, i)$; hence, $\overline{a}^\lambda_k \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = (\overline{a}')^\sigma_i \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ as desired.

For any $\zeta \in \operatorname{dom} \overline{p}_0$, it follows by definition of $\pi_0(\zeta)$ that $\big((\pi\overline{p})_i^\sigma(\zeta) \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)\big) = \big((\overline{p}')_i^\sigma(\zeta) \mid (\sigma, i) \in \operatorname{supp} \pi_0(\nu, j)\big)$, and similarly, $(\pi\overline{p})_*(\xi, \zeta) = \overline{p}'_*(\xi, \zeta)$ for all $(\xi, \zeta) \in \operatorname{dom}(\pi\overline{p})_* = \operatorname{dom} \overline{p}_*$.
Hence, $\pi_0\overline{p}_0 = \overline{p}'_0$.

It remains to verify that $\pi \in \bigcap_m Fix(\eta_m, i_m) \cap \bigcap_m H_{k_m}^{\lambda_m}$. Consider a condition $r \in D_{\pi_0}$ and let $r' := \pi_0 r$. Take an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\gamma$. Then for any $m < \omega$ with $(\eta_m, i_m) \in \operatorname{supp} \pi_0(\nu, j)$ and $\zeta \in \operatorname{dom} \pi_0(\nu, j)$, it follows that $(r')_{i_m}^{\eta_m}(\zeta) = r_{i_m}^{\eta_m}(\zeta)$ by construction of the map $\pi_0(\zeta)$, since we have arranged $\overline{p}_{i_m}^{\eta_m}(\zeta) = (\overline{p}')_{i_m}^{\eta_m}(\zeta)$. In the case that $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $\zeta \in \operatorname{dom} \overline{r}_0 \setminus \operatorname{dom} \pi_0$, it follows for $m < \omega$ that $(r')_{i_m}^{\eta_m}(\zeta) = r_k^\lambda(\zeta)$, where $(\lambda, k) = G_{\pi_0}(\nu, j)(\eta_m, i_m) = (\eta_m, i_m)$ as desired. Hence, $(r')_{i_m}^{\eta_m} = r_{i_m}^{\eta_m}$ for all $m < \omega$. Since $r \in D_{\pi_0}$ was arbitrary, it follows that $\pi_0 \in \bigcap_{m<\omega} Fix(\eta_m, i_m)$.

Similarly, $\pi_0 \in \bigcap_m H_{k_m}^{\lambda_m}$ follows from the fact that $G_{\pi_0}(\nu, j) = id$ for all intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\gamma$.

Now, we turn to the map π_1.

Let $\operatorname{supp} \pi_1 := \operatorname{supp} \overline{p}_1 = \operatorname{supp} \overline{p}'_1$, and $\operatorname{dom} \pi_1(\sigma) := \operatorname{dom} \overline{p}_1(\sigma) = \operatorname{dom} \overline{p}'_1(\sigma)$ for $\sigma \in \operatorname{supp} \pi_1$. We set $\operatorname{supp} \pi_1(\sigma) := \varnothing$ for all $\sigma \in \operatorname{supp} \pi_1$. Then we only have to define maps $\pi_1(\sigma)(i, \zeta) : 2 \to 2$ for $\sigma \in \operatorname{supp} \pi$, $(i, \zeta) \in \operatorname{dom} \pi_1(\sigma)$: Let $\pi_1(\sigma)(i, \zeta) = id$ if $\overline{p}(\sigma)(i, \zeta) = \overline{p}'(\sigma)(i, \zeta)$, and $\pi_1(\sigma) \neq id$ in the case that $\overline{p}(\sigma)(i, \zeta) \neq \overline{p}'(\sigma)(i, \zeta)$.

Clearly, $\pi_1\overline{p}_1 = \overline{p}'_1$. Moreover, $\pi \in \bigcap_m Fix(\overline{\eta}_m, \overline{i}_m)$: Let $m < \omega$ and $r \in D_{\pi_1}$ with $\overline{\eta}_m \in \operatorname{supp} r$ and $\overline{i}_m \in \operatorname{dom}_x r(\overline{\eta}_m)$. In the case that $\overline{\eta}_m \in \operatorname{supp} \pi_1$, it follows for any $\zeta \in \operatorname{dom}_y r(\overline{\eta}_m)$ that $(\pi r)(\overline{\eta}_m)(\overline{i}_m, \zeta) = \pi_1(\overline{\eta}_m)(i_m, \zeta)\big(r(\overline{\eta}_m)(\overline{i}_m, \zeta)\big) = r(\overline{\eta}_m)(\overline{i}_m, \zeta)$ by construction of π_1, since we have arranged that $\overline{p}'(\overline{\eta}_m)(\overline{i}_m, \zeta) = \overline{p}(\overline{\eta}_m)(\overline{i}_m, \zeta)$ whenever $(\overline{i}_m, \zeta) \in \operatorname{dom} \overline{p}(\overline{\eta}_m) = \operatorname{dom} \overline{p}'(\overline{\eta}_m) = \operatorname{dom} \pi_1(\overline{\eta}_m)$. If $\overline{\eta}_m \notin \operatorname{supp} \pi_1$, then $(\pi r)(\overline{\eta}_m) = r(\overline{\eta}_m)$ by construction.
Finally, $\pi \in \bigcap_m H_{\overline{k}_m}^{\overline{\lambda}_m}$ follows from the fact that $\operatorname{supp} \pi_1(\lambda) = 0$ for all $\lambda \in \operatorname{supp} \pi_1$.

Hence, the map π has all the desired properties.

This finishes the proof of $X = X'$, and

$$X = X' \in V\big[\prod_{m<\omega} G_{i_m}^{\eta_m} \times \prod_{m<\omega} G_{\overline{i}_m}^{\overline{\eta}_m}\big]$$

follows.

□

It is not difficult to see that with the exception of the maps $G_{\pi_0}(\nu, j)$, all the parameters describing π are given by the requirement that $\pi\overline{p} = \overline{p}'$. We call an isomorphism $\pi \in A$ of this form a *standard isomorphism for* $\pi\overline{p} = \overline{p}'$.

With the same proofs as for Lemma 2.3.2 and 2.3.3, one can show:

Lemma 2.5.3. *Let* $((\sigma_m, i_m) \mid m < \omega)$, $((\overline{\sigma}_m, \overline{i}_m) \mid m < \omega)$ *with* $\sigma_m \in Lim$, $i_m < \alpha_{\sigma_m}$, *and* $\overline{\sigma}_m \in Succ$, $\overline{i}_m < \alpha_{\overline{\sigma}_m}$ *for all* $m < \omega$. *Then* $\prod_{m<\omega} P^{\sigma_m} \times \prod_{m<\omega} P^{\overline{\sigma}_m}$ *preserves cardinals, cofinalities and the* GCH.

Hence, the *Approximation Lemma* 2.5.2 implies:

Corollary 2.5.4. *Cardinals and cofinalities are V-N-absolute.*

We will now take a closer look at the intermediate generic extensions introduced in the *Approximation Lemma* 2.5.2. Firstly, we replace the generic filters $G^{\sigma_m}_{i_m}$ by $G_*(g^{\sigma_m}_{i_m})$, and secondly, we factor at κ_η (or $\kappa_{\eta+1}$).

Definition 2.5.5. For $0 < \eta < \gamma$, we say that $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ is an *η-good pair* if the following hold:

- $(a_m \mid m < \omega)$ is a sequence of pairwise disjoint κ_η-subsets, such that for all $m < \omega$ and $\kappa_{\overline{\nu},\overline{\jmath}} < \kappa_\eta$, it follows that $|a_m \cap [\kappa_{\overline{\nu},\overline{\jmath}}, \kappa_{\overline{\nu},\overline{\jmath}+1})| = 1$,
- for all $m < \omega$, we have $\overline{\sigma}_m \in \mathrm{Succ}$ with $\overline{\sigma}_m \leq \eta$, $\overline{i}_m < \alpha_{\overline{\sigma}_m}$,
- if $m \neq m'$, then $(\overline{\sigma}_m, \overline{i}_m) \neq (\overline{\sigma}_{m'}, \overline{i}_{m'})$.

As in Lemma 2.3.9 and 2.3.10, it follows that for any η-good pair $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$,

$$\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m}$$

is a V-generic filter on $\prod_{m<\omega} (\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m}$.

Proposition 2.5.6. *Let $0 < \eta < \gamma$ and $X \in N$ with $X \subseteq \kappa_\eta$. If $\kappa_{\eta+1} > \kappa_\eta^+$ (or $\kappa_\eta = \kappa_{\overline{\gamma}}$ with $\gamma = \overline{\gamma} + 1$), it follows that there is an η-good pair $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ with*

$$X \in V\big[\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m}\big].$$

Proof. By the *Approximation Lemma* 2.5.2, there are sequences $((\sigma_m, i_m) \mid m < \omega)$, $((\overline{\sigma}_m, \overline{i}_m) \mid m < \omega)$ of pairwise distinct pairs with $\sigma_m \in \mathit{Lim}$, $i_m < \alpha_{\sigma_m}$; $\overline{\sigma}_m \in \mathit{Succ}$, $\overline{i}_m < \alpha_{\overline{\sigma}_m}$ for all $m < \omega$, such that

$$X \in V\big[\prod_{m<\omega} G^{\sigma_m}_{i_m} \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m}\big].$$

The sequence of linking ordinals $(g^{\sigma_m}_{i_m} \mid m < \omega)$ is contained in V, and by the *linking property*, it follows that $V[\prod_{m<\omega} G^{\sigma_m}_{i_m}] = V[\prod_{m<\omega} G_*(g^{\sigma_m}_{i_m})]$.
Hence,

$$X \in V\big[\prod_{m<\omega} G_*(g^{\sigma_m}_{i_m}) \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m}\big].$$

The forcing $\prod_{m<\omega} P^{\sigma_m} \times \prod_{m<\omega} P^{\overline{\sigma}_m}$ can be factored as

$$\Big(\prod_{m<\omega} P^{\sigma_m} \restriction \kappa_\eta \times \prod_{\overline{\sigma}_m \leq \eta} P^{\overline{\sigma}_m}\Big) \times \Big(\prod_{m<\omega} P^{\sigma_m} \restriction [\kappa_\eta, \kappa_{\sigma_m}) \times \prod_{\overline{\sigma}_m > \eta} P^{\overline{\sigma}_m}\Big),$$

where the "lower part" has cardinality $\leq \kappa_\eta^+$ by the GCH in V, and the "upper part" is $\leq \kappa_\eta^+$-closed: If $\kappa_{\eta+1}$ is a limit cardinal, this follows from the fact that $\kappa_{\eta,j+1} \geq \kappa_{\eta,j}^{++}$ for all $j < \mathrm{cf}\, \kappa_{\eta+1}$ by construction (in particular, $\kappa_{\eta,1} \geq \kappa_\eta^{++}$); and if $\kappa_{\eta+1}$ is a successor cardinal, we use our assumption that $\kappa_{\eta+1} > \kappa_\eta^+$. Hence,

$$X \in V\big[\prod_{m<\omega} G_*(g^{\sigma_m}_{i_m} \cap \kappa_\eta) \times \prod_{\overline{\sigma}_m \leq \eta} G^{\overline{\sigma}_m}_{\overline{i}_m}\big].$$

Setting $a_m := g_{i_m}^{\sigma_m} \cap \kappa_\eta$ for $m < \omega$, it follows by the *independence property* that

$$\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega\,,\overline{\sigma}_m \leq \eta}\big)$$

is an η-good pair with

$$X \in V\big[\prod_{m<\omega} G_*(a_m) \times \prod_{\overline{\sigma}_m \leq \eta} G_{\overline{i}_m}^{\overline{\sigma}_m}\big].$$

□

In the case that $\kappa_{\eta+1} = \kappa_\eta^+$, we use our notion of an η-*almost good pair*, which is defined like an η-good pair, with the exception that for an η-*almost good pair* $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$, we have $a_m \subseteq \kappa_{\eta+1}$ for all $m < \omega$.

Definition 2.5.7. For $0 < \eta < \gamma$ with $\kappa_{\eta+1} = \kappa_\eta^+$, we say that $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ is an η-*almost good pair* if the following hold:

- $(a_m \mid m < \omega)$ is a sequence of pairwise disjoint $\kappa_{\eta+1}$-subsets, such that for all $m < \omega$ and $\kappa_{\overline{\nu},\overline{\jmath}} < \kappa_{\eta+1}$, it follows that $|a_m \cap [\kappa_{\overline{\nu},\overline{\jmath}}, \kappa_{\overline{\nu},\overline{\jmath}+1})| = 1$,
- for all m, we have $\overline{\sigma}_m \in \mathrm{Succ}$ with $\overline{\sigma}_m \leq \eta$, and $\overline{i}_m < \alpha_{\overline{\sigma}_m}$,
- if $m \neq m'$, then $(\overline{\sigma}_m, \overline{i}_m) \neq (\overline{\sigma}_{m'}, \overline{i}_{m'})$.

The counterpart of Proposition 2.5.6 states:

Proposition 2.5.8. *Let $0 < \eta < \gamma$ and $X \in N$ with $X \subseteq \kappa_\eta$. In the case that $\kappa_{\eta+1} = \kappa_\eta^+$, there is an η-almost good pair $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ with*

$$X \in V\big[\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G_{\overline{i}_m}^{\overline{\sigma}_m} \times G^{\eta+1}\big].$$

Proof. We follow the proof of Proposition 2.5.6 with a slightly different factorization: Let

$$X \in V\big[\prod_{m<\omega} G_*(g_{i_m}^{\sigma_m}) \times \prod_{m<\omega} G_{\overline{i}_m}^{\overline{\sigma}_m}\big]$$

as before with $\sigma_m \in Lim$, $i_m < \alpha_{\sigma_m}$; $\overline{\sigma}_m \in Succ$, $\overline{i}_m < \alpha_{\overline{\sigma}_m}$ for all $m < \omega$. The forcing $\prod_{m<\omega} P_{i_m}^{\sigma_m} \times \prod_{m<\omega} P_{\overline{i}_m}^{\overline{\sigma}_m}$ can be factored as

$$\big(\prod_{m<\omega} P^{\sigma_m} \upharpoonright \kappa_{\eta+1} \times \prod_{\overline{\sigma}_m \leq \eta+1} P^{\overline{\sigma}_m}\big) \times \big(\prod_{m<\omega} P^{\sigma_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\sigma_m}) \times \prod_{\overline{\sigma}_m > \eta+1} P^{\overline{\sigma}_m}\big),$$

where the "lower part" has cardinality $\leq \kappa_{\eta+1}$ by the GCH in V (since $\kappa_{\eta+1} = \kappa_\eta^+$), and the "upper part" is $\leq \kappa_{\eta+1}$-closed. Hence,

$$X \in V\big[\prod_{m<\omega} G_*(g_{i_m}^{\sigma_m} \cap \kappa_{\eta+1}) \times \prod_{\overline{\sigma}_m \leq \eta+1} G_{\overline{i}_m}^{\overline{\sigma}_m}\big] \subseteq V\big[\prod_{m<\omega} G_*(g_{i_m}^{\sigma_m} \cap \kappa_{\eta+1}) \times \prod_{\overline{\sigma}_m \leq \eta} G_{\overline{i}_m}^{\overline{\sigma}_m} \times G^{\eta+1}\big].$$

With $a_m := g_{i_m}^{\sigma_m} \cap \kappa_{\eta+1}$ for $m < \omega$, it follows that $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega\,,\overline{\sigma}_m \leq \eta}\big)$ is an η-almost good pair with

$$X \in V\big[\prod_{m<\omega} G_*(a_m) \times \prod_{\overline{\sigma}_m \leq \eta} G_{\overline{i}_m}^{\overline{\sigma}_m} \times G^{\eta+1}\big]$$

as desired.

□

2.6 $\forall\eta \ \ \theta^N(\kappa_\eta) = \alpha_\eta$

It remains to make sure that in our ZF-model N, the values $\theta^N(\kappa_\eta)$ are as desired. Firstly, in Chapter 2.6.1, 2.6.2 and 2.6.3, we will show that $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all $0 < \eta < \gamma$. After that, in Chapter 2.6.4 and 2.6.5, we will see that for any cardinal $\lambda \in (\kappa_\eta, \kappa_{\eta+1})$ in a "gap", or $\lambda \geq \kappa_\gamma = \sup\{\kappa_\eta \mid 0 < \eta < \gamma\}$, the value $\theta^N(\lambda)$ is the smallest possible.
By our remarks from Chapter 2.2, this justifies our assumption from the beginning that the sequence $(\alpha_\eta \mid 0 < \eta < \gamma)$ is strictly increasing.

2.6.1 $\forall\eta \ \ \theta^N(\kappa_\eta) \geq \alpha_\eta$

Using the subgroups H^η_k, it is not difficult to see that for all $k < \alpha_\eta$, there exists in N a surjection $s : \wp(\kappa_\eta) \to k$.

Proposition 2.6.1. *Let $0 < \eta < \gamma$. Then $\theta^N(\kappa_\eta) \geq \alpha_\eta$.*

Proof. Let $k < \alpha_\eta$. We construct in N a surjection $s : \wp(\kappa_\eta) \to k$. As already outlined in Chapter 2.4.2, we define around each G^η_i with $i < k$ a "cloud" as follows:

$$\big(\widetilde{G^\eta_i}\big)^{(k)} \ := \ \Big(\big(\dot{\widetilde{G^\eta_i}}\big)^{(k)}\Big)^G,$$

where

$$\big(\dot{\widetilde{G^\eta_i}}\big)^{(k)} := \Big\{\, (\pi\overline{\dot G^\eta_i}^{D_\pi}, \mathbb{1}) \mid [\pi] \in H^\eta_k \Big\};$$

and we take the following canonical name for the i-th generic κ_η-subset:

$$\dot G^\eta_i := \big\{\, (a,p) \;\big|\; p \in \mathbb{P}\,,\ \exists\,\zeta < \kappa_\eta\ \exists\,\epsilon \in \{0,1\}\ :\ a = \mathrm{OR}_{\mathbb{P}}(\check\zeta, \check\epsilon)\ \wedge\ p^\eta_i(\zeta) = \epsilon \big\}.$$

Roughly speaking, $\big(\widetilde{G^\eta_i}\big)^{(k)}$ is the orbit of G^η_i under the $\overline{A}$-subgroup H^η_k; hence, its canonical name $\big(\dot{\widetilde{G^\eta_i}}\big)^{(k)}$ is fixed by all automorphisms in H^η_k.
More precisely:
Let $\sigma \in A$ with $[\sigma] \in H^\eta_k$. Then

$$\overline{(\dot{\widetilde{G^\eta_i}})^{(k)}}^{D_\sigma} = \Big\{\, \Big(\, \overline{\pi\overline{\dot G^\eta_i}^{D_\pi}}^{D_\sigma}, p \,\Big) \mid [\pi] \in H^\eta_k\,,\ p \in D_\sigma \,\Big\}.$$

Moreover, for all π,

$$\overline{\pi\overline{\dot G^\eta_i}^{D_\pi}}^{D_\sigma} = \Big\{\, \Big(\, \overline{a}^{D_\sigma}\,,\ p \,\Big) \mid p \in D_\sigma\,,\ p \Vdash_s a \in \pi\overline{\dot G^\eta_i}^{D_\pi}\,,\ \exists\,\zeta < \kappa_\eta\ \exists\,\epsilon \in \{0,1\}\ :\ a = \mathrm{OR}_{\mathbb{P}}(\check\zeta, \check\epsilon) \,\Big\},$$

since for any $a = \mathrm{OR}_{\mathbb{P}}(\check\zeta, \check\epsilon)$ as above, it follows that $\overline{\pi\overline{a}^{D_\pi}}^{D_\sigma} = \overline{\overline{a}^{D_\pi}}^{D_\sigma} = \overline{a}^{D_\sigma}$.
Now, it is not difficult to see that $p \in D_\sigma$ with $p \Vdash_s a \in \pi\overline{\dot G^\eta_i}^{D_\pi}$ if and only if $p \in D_\sigma$ and for all $q \leq p$ with $q \in D_\pi \cap D_\sigma$ and $\zeta \in \operatorname{dom} q_0$, it follows that $(\pi^{-1}q)^\eta_i(\zeta) = \epsilon$.
Also, $\sigma\overline{a}^{D_\sigma} = \overline{a}^{D_\sigma}$ holds for all σ.

Hence,

$$\overline{\sigma\,\pi\overline{G_i^\eta}^{D_\pi}}^{D_\sigma} = \Big\{ \Big(\,\sigma\overline{a}^{D_\sigma}\ ,\ \sigma p\,\Big) \mid p \in D_\sigma\ ,\ \exists\,\zeta < \kappa_\eta\ \exists\,\epsilon \in \{0,1\}\ \ a = \mathrm{OR}_{\mathbb{P}}(\check{\zeta},\check{\epsilon})\ ,$$
$$\forall\, q \in D_\pi \cap D_\sigma\ \Big((q \le p \wedge \zeta \in \operatorname{dom} q_0) \Rightarrow (\pi^{-1}q)_i^\eta(\zeta) = \epsilon \Big) \Big\}$$

$$= \Big\{ \Big(\,\overline{a}^{D_\sigma}\ ,\ p\,\Big) \mid p \in D_\sigma\ ,\ \exists\,\zeta < \kappa_\eta\ \exists\,\epsilon \in \{0,1\}\ \ a = \mathrm{OR}_{\mathbb{P}}(\check{\zeta},\check{\epsilon})\ ,$$
$$\forall\, q \in D_\pi \cap D_\sigma\ \Big((q \le p \wedge \zeta \in \operatorname{dom} q_0) \Rightarrow (\pi^{-1}\sigma^{-1}q)_i^\eta(\zeta) = \epsilon \Big) \Big\}.$$

Setting $\tau := \sigma\pi$, it follows that

$$\overline{\sigma\pi\overline{G_i^\eta}^{D_\pi}}^{D_\sigma} = \overline{\tau\overline{G_i^\eta}^{D_\tau}}^{D_\sigma}.$$

Now, any element of $\sigma\,\overline{(\widetilde{G_i^\eta})^{(k)}}^{D_\sigma}$ is of the form

$$\Big(\, \overline{\sigma\pi\overline{G_i^\eta}^{D_\pi}}^{D_\sigma}, \sigma p \,\Big)$$

with $[\pi] \in H_k^\eta$ and $p \in D_\sigma$. Since

$$\Big(\, \overline{\sigma\pi\overline{G_i^\eta}^{D_\pi}}^{D_\sigma}, \sigma p \,\Big) = \Big(\, \overline{\tau\overline{G_i^\eta}^{D_\tau}}^{D_\sigma}, \overline{p} \,\Big),$$

where $\tau := \sigma\pi$ and $\overline{p} := \sigma p$ satisfy $[\tau] \in H_k^\eta$ and $\overline{p} \in D_\sigma$, it follows that

$$\Big(\, \overline{\sigma\pi\overline{G_i^\eta}^{D_\pi}}^{D_\sigma}, \sigma p \,\Big) \in \overline{(\dot{\widetilde{G_i^\eta}})^{(k)}}^{D_\sigma}.$$

Hence,

$$\sigma\,\overline{(\widetilde{G_i^\eta})^{(k)}}^{D_\sigma} \subseteq \overline{(\widetilde{G_i^\eta})^{(k)}}^{D_\sigma}.$$

The inclusion "$\supseteq$" is similar.

Thus,

$$\Big((\dot{\widetilde{G_i^\eta}})^{(k)}\ \Big|\ i < k \Big) := \Big\{ \Big(\ \mathrm{OR}_{\mathbb{P}}\Big(\,\check{i}, (\dot{\widetilde{G_i^\eta}})^{(k)}\Big), \mathbb{1}\Big)\ \Big|\ i < k \Big\},$$

is a name for the sequence $\Big((\widetilde{G_i^\eta})^{(k)} \mid i < k\Big)$ that is stabilized by all σ with $[\sigma] \in H_k^\eta$. Hence, $\Big((\widetilde{G_i^\eta})^{(k)} \mid i < k\Big) \in N$.

Now, we can define in N a surjection $s : \wp(\kappa_\eta) \to k$ as follows: For $X \in N$, $X \subseteq \kappa_\eta$, let $s(X) := i$ in the case that $X \in \big(\widetilde{G_i^\eta}\big)^{(k)}$ if such i exists, and $s(x) := 0$, else.

The surjectivity of s is clear, since $G_i^\eta \in N$ for all $i < k$ with $s\big(G_i^\eta\big) = i$. It remains to show that s is well-defined; i.e. for any $i, i' < k$ with $i \neq i'$, it follows that $\big(\widetilde{G_i^\eta}\big)^{(k)} \cap \big(\widetilde{G_{i'}^\eta}\big)^{(k)} = \emptyset$.

First, let $\eta \in Lim$, and take i, $i' < k$ with $i \neq i'$.

The point is that the automorphisms in H^η_k do not permute the vertical lines $P^\eta_i \upharpoonright [\kappa_{\overline{\nu},\overline{\jmath}}, \kappa_\eta)$ and $P^\eta_{i'} \upharpoonright [\kappa_{\overline{\nu},\overline{\jmath}}, \kappa_\eta)$ above some $\kappa_{\overline{\nu},\overline{\jmath}} < \kappa_\eta$. Thus, the orbits of G^η_i and $G^\eta_{i'}$ under H^η_k must be disjoint:

Assume towards a contradiction there was $X \in \big(\widetilde{G^\eta_i}\big)^{(k)} \cap \big(\widetilde{G^\eta_{i'}}\big)^{(k)}$. Then we have

$$\big(\pi\overline{G^\eta_i}^{D_\pi}\big)^G = \big(\tau\overline{G^\eta_{i'}}^{D_\tau}\big)^G$$

for some π, τ with $[\pi] \in H^\eta_k$ and $[\tau] \in H^\eta_k$. Hence, $(\pi^{-1}G)^\eta_i = (\tau^{-1}G)^\eta_{i'}$. Take $\kappa_{\overline{\nu},\overline{\jmath}} < \kappa_\eta$ such that for all $\kappa_{\nu,j} \in [\kappa_{\overline{\nu},\overline{\jmath}}, \kappa_\eta)$ and $l < k$, it follows that $G_{\pi_0}(\nu, j)(\eta, l) = (\eta, l)$ whenever $(\eta, l) \in \operatorname{supp} \pi_0(\nu, j)$, and $G_{\tau_0}(\nu, j)(\eta, l) = (\eta, l)$ whenever $(\eta, l) \in \operatorname{supp} \tau_0(\nu, j)$.
By genericity, take $q \in G$ with $q \in D_\pi \cap D_\tau$ such that there is $\zeta \in \operatorname{dom} q \smallsetminus (\operatorname{dom} \pi_0 \cap \operatorname{dom} \tau_0)$, $\zeta \in [\kappa_{\overline{\nu},\overline{\jmath}}, \kappa_\eta)$ with $q^\eta_i(\zeta) \neq q^\eta_{i'}(\zeta)$.
W.l.o.g., let $q^\eta_i(\zeta) = 1$, $q^\eta_{i'}(\zeta) = 0$. With $r := \pi^{-1}q$, $r' := \tau^{-1}q$, it follows by construction of the isomorphism that $r^\eta_i(\zeta) = q^\eta_i(\eta) = 1$ and $(r')^\eta_{i'}(\zeta) = q^\eta_{i'}(\zeta) = 0$, which would contradict $(\pi^{-1}G)^\eta_i = (\tau^{-1}G)^\eta_{i'}$.

Hence, $s : \wp(\kappa_\eta) \to k$ is a well-defined surjection in N.

The case $\eta \in Succ$ is similar. □

2.6.2 $\forall \eta \ \big(\kappa_{\eta+1} > \kappa^+_\eta \longrightarrow \theta^N(\kappa_\eta) \leq \alpha_\eta\big)$

Let $0 < \eta < \gamma$. Throughout this Chapter 2.6.2, we assume that

$$\boldsymbol{\kappa_{\eta+1} > \kappa^+_\eta.}$$

Then Proposition 2.5.6 can be applied.

In Chapter 2.6.3, we discuss the case that $\kappa_{\eta+1} = \kappa^+_\eta$, where the proof can be structured the very same way; except that the intermediate generic extensions where the κ_η-subsets in N are located are given by Proposition 2.5.8. Thus, we will have to take care of an extra factor $G^{\eta+1}$ in our products describing these intermediate generic extensions, which will lead to a couple of modifications. In Chapter 2.6.3, we take a brief look at each step in the proof presented here, and go through the major changes.

Assume towards a contradiction that there was a surjective function $f : \wp(\kappa_\eta) \to \alpha_\eta$ in N. Let $f = \dot{f}^G$ with $\dot{f} \in HS$, such that $\pi\overline{\dot{f}}^{D_\pi} = \overline{\dot{f}}^{D_\pi}$ holds for all $\pi \in A$ with $[\pi]$ contained in the intersection

$$\bigcap_{m<\omega} Fix(\eta_m, i_m) \cap \bigcap_{m<\omega} H^{\lambda_m}_{k_m} \qquad (I_{\dot{f}}).$$

By Proposition 2.5.6, it follows that any $X \in \operatorname{dom} f$ is of the form

$$X = \dot{X}^{\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m}},$$

where $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ is an η-good pair.

Our proof will be structured as follows: We pick some $\boldsymbol{\beta < \alpha_\eta}$ *large enough for the intersection* (I_f) (we give a definition of this term on the next page) and consider a map f^β, which will be obtained from f by restricting its domain to those X that are contained in a generic extension

$$V\big[\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G_{\bar{i}_m}^{\overline{\sigma}_m}\big]$$

for an η-good pair $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \bar{i}_m)_{m<\omega}\big)$ such that $\boldsymbol{\bar{i}_m < \beta}$ **for all** $\boldsymbol{m < \omega}$.
We wonder if this restricted function f^β could still be surjective onto α_η.

The main steps of our proof can be outlined as follows:

First, we assume that also $f^\beta : \operatorname{dom} f^\beta \to \alpha_\eta$ was surjective onto α_η.

A) We define a forcing notion $\mathbb{P}^\beta \upharpoonright (\eta+1)$, which will be obtained from $\mathbb{P}$ by essentially "cutting off" at height $\eta + 1$ and width β. We show that there is a projection of forcing posets $\rho^\beta : \mathbb{P} \to \mathbb{P}^\beta \upharpoonright (\eta + 1)$. Then the V-generic filter G on $\mathbb{P}$ induces a V-generic filter $G^\beta \upharpoonright (\eta + 1)$ on $\mathbb{P}^\beta \upharpoonright (\eta + 1)$.

B) We show that f^β is contained in an intermediate generic extension similar to $V[G^\beta \upharpoonright (\eta + 1)]$.

C) We prove that the forcing $\mathbb{P}^\beta \upharpoonright (\eta + 1)$ preserves cardinals $\geq \alpha_\eta$.

D) We construct in $V[G^\beta \upharpoonright (\eta+1)]$ a set $\widetilde{\wp}(\kappa_\eta) \supseteq \operatorname{dom} f^\beta$ with an injection $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow \beta$.

Then D) together with B) and C) gives the desired contradiction.

Hence, $f^\beta : \operatorname{dom} f^\beta \to \alpha_\eta$ must *not* be surjective.

E) We consider $\alpha < \alpha_\eta$ with $\alpha \in rg\, f \setminus rg\, f^\beta$, and use an isomorphism argument to obtain a contradiction, again.

We see that either case, whether f^β was surjective or not, leads into a contradiction. Thus, our initial assumption must be wrong, and we can finally conclude:

There is no surjective function $f : \wp(\kappa_\eta) \to \alpha_\eta$.

Before we start with Chapter 2.6.2 A), we first define our term *large enough for the intersection* (I_f):

Definition 2.6.2. A limit ordinal $\widetilde{\beta} < \alpha_\eta$ is *large enough for the intersection* (I_f) if the following hold:

- $\widetilde{\beta} > \kappa_\eta^+$
- $\widetilde{\beta} > \sup\{i_m \mid \eta_m \leq \eta\}$
- $\widetilde{\beta} > \sup\{k_m \mid \lambda_m \leq \eta\}$

(We use that $\alpha_\eta \geq \kappa_\eta^{++}$, and $\operatorname{cf} \alpha_\eta > \omega$.)

Fix a limit ordinal $\widetilde{\beta} < \alpha_\eta$ *large enough for the intersection* $(I_{\dot{f}})$, and let $\beta := \widetilde{\beta} + \kappa_\eta^+$ (addition of ordinals).

The restriction f^β is defined as follows:

Definition 2.6.3.

$$f^\beta := \Big\{ (X, \alpha) \in f \;\Big|\; \exists \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega} \big) \; \eta\text{-good pair} :$$

$$(\forall m \;\; \overline{i}_m < \beta) \wedge \exists \dot{X} \in \operatorname{Name}\big((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m} \big) \;\; X = \dot{X}^{\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m}} \Big\}.$$

First, we assume towards a contradiction that $\boldsymbol{f^\beta : \operatorname{dom} f^\beta \to \alpha_\eta}$ **is surjective**.

A) Constructing $\mathbb{P}^\beta \upharpoonright (\eta + 1)$.

Our aim is to construct a forcing notion $\mathbb{P}^\beta \upharpoonright (\eta+1)$ that is obtained from $\mathbb{P}$ by essentially "cutting off" at height η and width β; i.e. only the cardinals κ_σ for $\sigma \leq \eta$ should be considered, and for any such κ_σ, we add at most β-many new κ_σ-subsets G_i^σ.
Regarding our V-generic filter G on $\mathbb{P}$, we need that the restriction $G^\beta \upharpoonright (\eta + 1) := G \upharpoonright \big(\mathbb{P}^\beta \upharpoonright (\eta + 1)\big)$ is a V-generic filter on $\mathbb{P}^\beta \upharpoonright (\eta + 1)$, which will be guaranteed by making sure that the canonical map $\rho^\beta : \mathbb{P} \to \mathbb{P}^\beta \upharpoonright (\eta+1)$, $p \mapsto p^\beta \upharpoonright (\eta+1)$ is a projection of forcing posets.

A first attempt to define $\mathbb{P}^\beta \upharpoonright (\eta + 1)$ could be the following:

For $p \in \mathbb{P}$, let

$$p^\beta \upharpoonright (\eta + 1) = \big(p_* \upharpoonright \kappa_\eta^2, (p_i^\sigma, a_i^\sigma)_{\sigma \leq \eta, i < \min\{\alpha_\sigma, \beta\}}, (p^\sigma \upharpoonright (\min\{\alpha_\sigma, \beta\} \times \operatorname{dom}_y p^\sigma)_{\sigma \leq \eta} \big)$$

denote the canonical restriction; and set

$$\mathbb{P}^\beta \upharpoonright (\eta + 1) := \{ p^\beta \upharpoonright (\eta + 1) \mid p \in \mathbb{P} \}.$$

But then, $G^\beta \upharpoonright (\eta + 1) := \{ p^\beta \upharpoonright (\eta + 1) \mid p \in G \}$ would not be a V-generic filter on $\mathbb{P}^\beta \upharpoonright (\eta + 1)$: Consider a linking ordinal $\xi \in g_{\overline{i}}^{\overline{\sigma}}$ for some $(\overline{\sigma}, \overline{i})$, such that $\eta < \overline{\sigma} < \gamma$, $\overline{i} < \alpha_{\overline{\sigma}}$ holds; or $\overline{\sigma} \leq \eta$, $\beta \leq \overline{i} < \alpha_{\overline{\sigma}}$. The set $D := \{ p \in \mathbb{P}^\beta \upharpoonright (\eta + 1) \mid \xi \in \bigcup_{\sigma \leq \eta, i < \beta} a_i^\sigma \}$ is dense in $\mathbb{P}^\beta \upharpoonright (\eta + 1)$; but $D \cap G^\beta \upharpoonright (\eta + 1) = \varnothing$ by the *independence property*. Hence, $G^\beta \upharpoonright (\eta + 1)$ can not be a V-generic filter on $\mathbb{P}^\beta \upharpoonright (\eta + 1)$.

This shows that the conditions in $\mathbb{P}^\beta \upharpoonright (\eta + 1)$ should contain some information about which linking ordinals are "forbidden" for $\bigcup_{\sigma \leq \eta, i < \beta} a_i^\sigma$, being already occupied by some index $(\overline{\sigma}, \overline{i})$ with $\overline{\sigma} > \eta$ or $\overline{i} \geq \beta$.
Thus, for $p \in \mathbb{P}$, we add to $p^\beta \upharpoonright (\eta + 1)$ a new coordinate X_p, which is essentially the union of all $a_i^\sigma \cap \kappa_\eta$ for $\sigma > \eta$ or $i \geq \beta$. Then X_p is a subset of κ_η that hits any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ in at most countably many points.

Let $\widetilde{\eta} := \sup\{\sigma < \eta \mid \sigma \in Lim\}$. By closure of the sequence $(\kappa_\sigma \mid 0 < \sigma < \gamma)$, it follows that $\widetilde{\eta} \in Lim$ with $\widetilde{\eta} = \max\{\sigma \leq \eta \mid \eta \in Lim\}$, and $\kappa_{\widetilde{\eta}} = \sup\{\kappa_\sigma \mid \sigma \in Lim\,, \sigma < \widetilde{\eta}\}$.

W.l.o.g. we restrict to the case that

$$\boldsymbol{\beta < \alpha_{\widetilde{\eta}}} \quad \textbf{or} \quad \boldsymbol{Lim \cap (\eta, \gamma) \neq \varnothing}\,;$$

which is the same as requiring that there exist coordinates (σ, i) with $\sigma \in Lim$, and $\sigma > \eta$ or $i \geq \beta$. (Otherwise, the forcing $\mathbb{P}^\beta \upharpoonright (\eta+1)$ already contains all coordinates (σ, i) with $\sigma \in Lim$, and there are no "forbidden" linking ordinals. In that case, we can indeed set $\mathbb{P}^\beta \upharpoonright (\eta+1) := \big\{ \big(p_* \upharpoonright \kappa_\eta^2, (p_i^\sigma, a_i^\sigma)_{\sigma \leq \eta, i < \beta}, (p^\sigma \upharpoonright (\beta \times \operatorname{dom}_y p^\sigma))_{\sigma \leq \eta}\big) \mid p \in \mathbb{P} \big\}$, and obtain that $G^\beta \upharpoonright (\eta+1)$ is a V-generic filter on $\mathbb{P}^\beta \upharpoonright (\eta+1)$.)

For a condition $p \in \mathbb{P}$, let

$$X_p := \bigcup \big\{ \, a_i^\sigma \cap \kappa_{\widetilde{\eta}} \mid \sigma \in Lim \text{ with } (\sigma > \eta \text{ or } i \geq \beta) \, \big\}\,,$$

and

$$p^\beta \upharpoonright (\eta+1) := \big(p_* \upharpoonright \kappa_\eta^2, (p_i^\sigma, a_i^\sigma)_{\sigma \leq \widetilde{\eta}, i < \beta}, (p^\sigma \upharpoonright (\beta \times \operatorname{dom}_y p^\sigma))_{\sigma \leq \eta}, X_p \big).$$

For reasons of homogeneity, we include into $\mathbb{P}^\beta \upharpoonright (\eta+1)$ only those conditions $p^\beta \upharpoonright (\eta+1)$ for which the set X_p hits every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\widetilde{\eta}}$ in countably many points, which is the same as requiring $|\{(\sigma, i) \in \operatorname{supp} p_0 \mid \sigma > \widetilde{\eta} \text{ or } i \geq \beta\}| = \aleph_0$.

Definition 2.6.4. $\mathbb{P}^\beta \upharpoonright (\eta+1) :=$

$$\big\{ \, p^\beta \upharpoonright (\eta+1) \;\big|\; p \in \mathbb{P}\,,\, |\{(\sigma, i) \in \operatorname{supp} p_0 \mid \sigma > \eta \text{ or } i \geq \beta\}| = \aleph_0 \, \big\} \;\cup\; \{\mathbb{1}_{\eta+1}^\beta\},$$

with $\mathbb{1}_{\eta+1}^\beta$ as the maximal element.

For conditions $p^\beta \upharpoonright (\eta+1)$, $q^\beta \upharpoonright (\eta+1)$ in $\mathbb{P}^\beta \upharpoonright (\eta+1) \setminus \{\mathbb{1}_{\eta+1}^\beta\}$, let $q^\beta \upharpoonright (\eta+1) \leq_{\eta+1}^\beta p^\beta \upharpoonright (\eta+1)$ if $X_q \supseteq X_p$, and $\big(q_* \upharpoonright \kappa_\eta^2, (q_i^\sigma, b_i^\sigma)_{\sigma \leq \eta, i < \beta}, (q^\sigma \upharpoonright (\beta \times \operatorname{dom}_y q^\sigma)_{\sigma \leq \eta}\big) \leq \big(p_* \upharpoonright \kappa_\eta^2, (p_i^\sigma, a_i^\sigma)_{\sigma \leq \eta, i < \beta}, (p^\sigma \upharpoonright (\beta \times \operatorname{dom}_y p^\sigma)_{\sigma \leq \eta}\big)$ regarded as conditions in $\mathbb{P}$.

In other words: $\mathbb{P}^\beta \upharpoonright (\eta+1)$ is the collection of all $(p_*, (p_i^\sigma, a_i^\sigma)_{\sigma \leq \eta, i < \beta}, (p^\sigma)_{\sigma \leq \eta}, X_p)$ such that

- $p := \big(p_*, (p_i^\sigma, a_i^\sigma)_{\sigma \leq \eta, i < \beta}, (p^\sigma)_{\sigma \leq \eta}\big)$ is a condition in $\mathbb{P}$ with $\operatorname{dom} p_0 \subseteq \kappa_\eta$, $\operatorname{supp} p_0 \subseteq \{(\sigma, i) \mid \sigma \leq \eta, i < \beta\}$, and $\operatorname{supp} p_1 \subseteq \eta+1$ with $\forall\, \sigma \in \operatorname{supp} p_1 \,:\, \operatorname{dom}_x p^\sigma \subseteq \beta$,
- $X_p \subseteq \kappa_{\widetilde{\eta}}$ with $\forall [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\widetilde{\eta}}$ $|X_p \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})| = \aleph_0$, and $X_p \cap \bigcup_{\sigma \leq \eta\,,\, i < \beta} a_i^\sigma = \varnothing$.

For p, $q \in \mathbb{P}$ with $q \leq p$ and $|\{(\sigma, i) \in \operatorname{supp} p_0 \mid \sigma > \eta \text{ or } i \geq \beta\}| = \aleph_0$, it follows that $q^\beta \upharpoonright (\eta+1) \leq p^\beta \upharpoonright (\eta+1)$.

Definition 2.6.5.

$$G^\beta \upharpoonright (\eta+1) := \big\{ p \in \mathbb{P}^\beta \upharpoonright (\eta+1) \;\big|\; \exists\, \overline{p} \in G \,:\, |\{(\sigma, i) \in \operatorname{supp} \overline{p}_0 \mid \sigma > \eta \text{ or } i \geq \beta\}| = \aleph_0\,,$$

$$\overline{p}^\beta \upharpoonright (\eta+1) \leq_{\eta+1}^\beta p \big\}.$$

We will now show that $G^\beta \upharpoonright (\eta+1)$ is a V-generic filter on $\mathbb{P}^\beta \upharpoonright (\eta+1)$.

Let $\overline{\mathbb{P}} \subseteq \mathbb{P}$ denote the collection of all $p \in \mathbb{P}$ with the property that $|\{(\sigma,i) \in \operatorname{supp} p_0 \mid \sigma > \eta \vee i \geq \beta\}| = \aleph_0$, together with the maximal element $\mathbb{1}$. Then $\overline{\mathbb{P}}$ is a dense subforcing of $\mathbb{P}$.

Proposition 2.6.6. *The map $\rho^\beta : \overline{\mathbb{P}} \to \mathbb{P}^\beta \upharpoonright (\eta+1)$ with $p \mapsto p^\beta \upharpoonright (\eta+1)$ in the case that $|\{(\sigma,i) \in \operatorname{supp} p_0 \mid \sigma > \eta \vee i \geq \beta\}| = \aleph_0$, and $\mathbb{1} \mapsto \mathbb{1}^\beta_{\eta+1}$, is a projection of forcing posets:*

- $\rho^\beta(\mathbb{1}) = \mathbb{1}^\beta_{\eta+1}$,
- *if $\overline{p}$, $\overline{q} \in \overline{\mathbb{P}}$ with $\overline{q} \leq \overline{p}$, it follows that $\rho^\beta(\overline{q}) \leq^\beta_{\eta+1} \rho^\beta(\overline{p})$,*
- *for any $\overline{p} \in \overline{\mathbb{P}}$ and $q \in \mathbb{P}^\beta \upharpoonright (\eta+1)$ with $q \leq^\beta_{\eta+1} \rho^\beta(\overline{p})$, there exists $\overline{q} \in \overline{\mathbb{P}}$ such that $\overline{q} \leq \overline{p}$ and $\rho^\beta(\overline{q}) \leq q$.*

Hence, $G^\beta \upharpoonright (\eta+1)$ is a V-generic filter on $\mathbb{P}^\beta \upharpoonright (\eta+1)$.

Proof. Clearly, the map ρ^β as defined above is order-preserving with $\rho^\beta(\mathbb{1}) = \mathbb{1}^\beta_{\eta+1}$. Consider $\overline{p} = (\overline{p}_*, (\overline{p}^\sigma_i, \overline{a}^\sigma_i)_{\sigma,i}, (\overline{p}^\sigma)_\sigma) \in \overline{\mathbb{P}}$ and $q = (q_* \upharpoonright \kappa^2_\eta, (q^\sigma_i, b^\sigma_i)_{\sigma \leq \eta, i < \beta}, (q^\sigma)_{\sigma \leq \eta}, X_q) \in \mathbb{P}^\beta \upharpoonright (\eta+1)$ with $q \leq^\beta_{\eta+1} \rho^\beta(\overline{p}) = \overline{p}^\beta \upharpoonright (\eta+1)$. Then

$$\begin{aligned} (q_* \upharpoonright \kappa^2_\eta, (q^\sigma_i, a^\sigma_i)_{\sigma \leq \eta, i<\beta}) &\leq_0 (\overline{p}_* \upharpoonright \kappa^2_\eta, (p^\sigma_i, a^\sigma_i)_{\sigma \leq \eta, i < \beta}) && \text{in } \mathbb{P}_0 \ , \\ (q^\sigma)_{\sigma \leq \eta} &\leq_1 (p^\sigma \upharpoonright (\beta \times \operatorname{dom}_y p^\sigma))_{\sigma \leq \eta} && \text{in } \mathbb{P}_1 \ , \quad \textit{and} \\ X_q &\supseteq \bigcup \{\overline{a}^\sigma_i \cap \kappa_{\widetilde{\eta}} \mid \sigma > \eta \vee i \geq \beta\}. \end{aligned}$$

We have to construct $\overline{q} \in \mathbb{P}$, $\overline{q} = (\overline{q}_*, (\overline{q}^\sigma_i, \overline{b}^\sigma_i)_{\sigma,i}, (\overline{q}^\sigma)_\sigma)$, with $\overline{q} \leq \overline{p}$ and $\rho^\beta(\overline{q}) = \overline{q}^\beta \upharpoonright (\eta+1) \leq^\beta_{\eta+1} q$.

We start with $\overline{q}_0$:

- In order to achieve $X_{\overline{q}} \supseteq X_q$, we will enlarge $\operatorname{supp} \overline{p}_0 \cup \operatorname{supp} q_0$ by countably many $((\hat{\eta}, m_k) \mid k < \omega)$, where $\hat{\eta} > \eta$ or $m_k \geq \beta$ for all $k < \omega$, and arrange that any $\xi \in X_q \setminus X_p$ occurs as a linking ordinal in some $\overline{b}^{\hat{\eta}}_{m_k}$.
  More precisely: Let $\operatorname{supp} \overline{q}_0 := \operatorname{supp} \overline{p}_0 \cup \operatorname{supp} q_0 \cup \operatorname{supp}_*$, where $\operatorname{supp}_* := \{(\hat{\eta}, m_k) \mid k < \omega\}$ such that $(\hat{\eta}, m_k) \notin \operatorname{supp} \overline{p}_0 \cup \operatorname{supp} q_0$ for all $k < \omega$, and since we are working in the case that $\beta < \alpha_{\widetilde{\eta}}$ or $(\eta, \gamma) \cap Lim \neq \emptyset$, we can take either $\hat{\eta} := \widetilde{\eta}$ and $m_k \in (\beta, \alpha_{\widetilde{\eta}})$ for all $k < \omega$; or $\hat{\eta} \in (\eta, \gamma) \cap Lim$. Then for all $(\hat{\eta}, m_k)$, it follows that $\hat{\eta} > \eta$ or $m_k \geq \beta$.
- Next, we define the linking ordinals $\overline{b}^\sigma_i$ for $(\sigma, i) \in \operatorname{supp} \overline{q}_0$ such that $X_{\overline{q}} \supseteq X_q$.

  For $(\sigma,i) \in \operatorname{supp} q_0$, we let $\overline{b}^\sigma_i := b^\sigma_i \supseteq \overline{a}^\sigma_i$; and in the case that $(\sigma,i) \in \operatorname{supp} \overline{p}_0 \setminus \operatorname{supp} q_0$, we set $\overline{b}^\sigma_i := \overline{a}^\sigma_i$. Finally, we define $(\overline{b}^{\hat{\eta}}_{m_k} \mid k < \omega)$ with the following properties:

  - as usual, every $\overline{b}^{\hat{\eta}}_{m_k}$ is a subset of $\kappa_{\hat{\eta}}$ that hits every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\hat{\eta}}$ in exactly one point,

– $\bigcup\{\overline{b}^{\hat\eta}_{m_k} \cap \kappa_{\overline{\eta}} \mid k<\omega\} \supseteq X_q \setminus X_{\overline{p}}$,
– $\overline{b}^{\hat\eta}_{m_k} \cap b^{\overline{\sigma}}_{\overline{i}} = \varnothing$ for all $k<\omega$ and $(\overline{\sigma},\overline{i}) \in \operatorname{supp} q_0$,
– $\overline{b}^{\hat\eta}_{m_k} \cap \overline{a}^{\overline{\sigma}}_{\overline{i}} = \varnothing$ for all $k<\omega$ and $(\overline{\sigma},\overline{i}) \in \operatorname{supp} \overline{p}_0 \setminus \operatorname{supp} q_0$ (since $q \leq^{\beta}_{\eta+1} \overline{p}^{\beta} \upharpoonright (\eta+1)$, it follows that in this case, $\overline{\sigma} > \eta$ or $\overline{i} \geq \beta$),
– $\overline{b}^{\hat\eta}_{m_k} \cap \overline{b}^{\hat\eta}_{m_{k'}} = \varnothing$ whenever $k \neq k'$.

This is possible, since $X_q \cap b^{\overline{\sigma}}_{\overline{i}} = \varnothing$ for any $(\overline{\sigma},\overline{i}) \in \operatorname{supp} q$ by construction of $\mathbb{P}^\beta \upharpoonright (\eta+1)$; and whenever $(\overline{\sigma},\overline{i}) \in \operatorname{supp}\overline{p}_0 \setminus \operatorname{supp} q_0$, then $\overline{\sigma} > \eta$ or $\overline{i} \geq \beta$ implies $\overline{a}^{\overline{\sigma}}_{\overline{i}} \subseteq X_{\overline{p}}$; thus $(X_q \setminus X_{\overline{p}}) \cap \overline{a}^{\overline{\sigma}}_{\overline{i}} = \varnothing$.

- We now define $\operatorname{dom}\overline{q}_0$. For any interval $[\kappa_{\nu,j},\kappa_{\nu,j+1}) \subseteq \kappa_\eta$, take $\delta_{\nu,j} \in [\kappa_{\nu,j},\kappa_{\nu,j+1})$ as follows: In the case that $\operatorname{dom} q_0 \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) = \varnothing$, let $\delta_{\nu,j} := \kappa_{\nu,j}$. If $\operatorname{dom} q_0 \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) \neq \varnothing$, we take $\delta_{\nu,j} \in (\kappa_{\nu,j},\kappa_{\nu,j+1})$ such that $\bigcup\{\overline{b}^{\sigma}_i \mid (\sigma,i) \in \operatorname{supp}\overline{q}_0\} \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) \subseteq [\kappa_{\nu,j},\delta_{\nu,j})$ and $\operatorname{dom} q_0 \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) \subseteq [\kappa_{\nu,j},\delta_{\nu,j})$. Since $\operatorname{dom} q_0$ is bounded below all regular cardinals $\kappa_{\overline{\nu},\overline{j}}$, this is also true for $\bigcup\{\,[\kappa_{\nu,j},\delta_{\nu,j}) \mid \kappa_{\nu,j} < \kappa_\eta\,\}$. Let
$$\operatorname{dom}\overline{q}_0 \cap \kappa_\eta := \bigcup\{\,[\kappa_{\nu,j},\delta_{\nu,j}) \mid \kappa_{\nu,j} < \kappa_\eta\,\},$$
and $\operatorname{dom}\overline{q}_0 \cap [\kappa_\eta,\kappa_\gamma) := \operatorname{dom}\overline{p}_0 \cap [\kappa_\eta,\kappa_\gamma)$.

- We take $\overline{q}_* \upharpoonright \kappa_\eta^2 \supseteq q_* \upharpoonright \kappa_\eta^2$ arbitrary on the given domain; and $\overline{q}_* \upharpoonright [\kappa_\eta,\kappa_\gamma)^2 := \overline{p}_* \upharpoonright [\kappa_\eta,\kappa_\gamma)^2$.

- It remains to define $\overline{q}^\sigma_i$ for $(\sigma,i) \in \operatorname{supp}\overline{q}_0$.

  For $(\sigma,i) \in \operatorname{supp} q_0$, we define $\overline{q}^\sigma_i \supseteq q^\sigma_i$ on the given domain $\bigcup_{\kappa_{\nu,j}<\kappa_\sigma}[\kappa_{\nu,j},\delta_{\nu,j})$ according to the *linking property*: Consider an interval $[\kappa_{\nu,j},\kappa_{\nu,j+1})$ with $\delta_{\nu,j} > \kappa_{\nu,j}$. For any $\zeta \in (\operatorname{dom}\overline{q}_0 \setminus \operatorname{dom} q_0) \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$, set $\overline{q}^\sigma_i(\zeta) := \overline{q}_*(\xi,\zeta)$, where $\{\xi\} := b^\sigma_i \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) = \overline{b}^\sigma_i \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$. (Note that $\xi \in \operatorname{dom}\overline{q}_0$ by construction). For $(\sigma,i) \in \operatorname{supp}\overline{p}_0 \setminus \operatorname{supp} q_0$, we set $\overline{q}^\sigma_i \upharpoonright [\kappa_\eta,\kappa_\gamma) := \overline{p}^\sigma_i \upharpoonright [\kappa_\eta,\kappa_\gamma)$, and define $\overline{q}^\sigma_i \upharpoonright \kappa_\eta \supseteq \overline{p}^\sigma_i \upharpoonright \kappa_\eta$ on the given domain according to the *linking property* as before.
  Finally, $\overline{q}^{\hat\eta}_{m_k}$ for $k<\omega$ can be arbitrary on the given domain.

Then $\overline{q}_0 = (\overline{q}_*, (\overline{q}^\sigma_i, \overline{b}^\sigma_i)_{\sigma,i})$ is a condition in $\mathbb{P}_0$. In particular, the *independence property* holds for the linking ordinals $\overline{b}^\sigma_i$: Firstly, by construction of $(\overline{b}^{\hat\eta}_{m_k} \mid k<\omega)$, it follows that $\overline{b}^{\hat\eta}_{m_k} \cap \overline{b}^{\overline{\sigma}}_{\overline{i}} = \varnothing$ for any $(\overline{\sigma},\overline{i}) \in \operatorname{supp} q_0 \cup \operatorname{supp}\overline{p}_0$. Secondly, whenever $(\sigma_0,i_0) \in \operatorname{supp} q_0$ and $(\sigma_1,i_1) \in \operatorname{supp}\overline{p}_0 \setminus \operatorname{supp} q_0$, then $\sigma_1 > \eta$ or $i_1 \geq \beta$; hence, $\overline{b}^{\sigma_1}_{i_1} = \overline{a}^{\sigma_1}_{i_1} \subseteq X_{\overline{p}} \subseteq X_q$. Since $\overline{b}^{\sigma_0}_{i_0} \cap X_q = b^{\sigma_0}_{i_0} \cap X_q = \varnothing$, this implies $\overline{b}^{\sigma_0}_{i_0} \cap \overline{b}^{\sigma_1}_{i_1} = \varnothing$ as desired. Thus, the independence property holds for $\overline{q}_0$.

Moreover, $(\rho^\beta(\overline{q}))_0 = (\overline{q}_* \upharpoonright \kappa_\eta^2, (\overline{q}^\sigma_i, \overline{b}^\sigma_i)_{\sigma\leq\eta, i<\beta}, X_{\overline{q}}) \leq q_0$ by construction; in particular, $X_q \subseteq X_{\overline{q}}$: Consider $\xi \in X_q$. In the case that $\xi \in X_{\overline{p}}$, it follows that $\xi \in \overline{a}^{\overline{\sigma}}_{\overline{i}}$ for some $(\overline{\sigma},\overline{i}) \in \operatorname{supp}\overline{p}_0$ with $\overline{\sigma} > \eta$, or $\overline{i} \geq \beta$. Then $(\overline{\sigma},\overline{i}) \in \operatorname{supp}\overline{p}_0 \setminus \operatorname{supp} q_0$; hence, $\overline{b}^{\overline{\sigma}}_{\overline{i}} = \overline{a}^{\overline{\sigma}}_{\overline{i}}$, and it follows that $\xi \in \overline{b}_{\overline{i}}^{\overline{\sigma}} \subseteq X_{\overline{q}}$ as desired. In the case that $\xi \in X_q \setminus X_{\overline{p}}$, we have $\xi \in \overline{b}_{m_k}^{\widehat{\eta}}$ for some $k < \omega$; so again, $\xi \in X_{\overline{q}}$ as desired.

Finally, $\overline{q}_0 \leq \overline{p}_0$ by construction; and it follows that $\overline{q}_0$ has all the desired properties.

The construction of $\overline{q}_1$ is similar. Thus, the map $\rho^\beta : \overline{\mathbb{P}} \to \mathbb{P}^\beta \upharpoonright (\eta+1)$ as defined above, is indeed a projection of forcing posets.

It follows that $G^\beta \upharpoonright (\eta+1)$ is a V-generic filter on $\mathbb{P}^\beta \upharpoonright (\eta+1)$: For genericity, consider an open dense set $D \subseteq \mathbb{P}^\beta \upharpoonright (\eta+1)$. It suffices to show that the set $\overline{D} := \{\overline{p} \in \overline{\mathbb{P}} \mid \overline{p}^\beta \upharpoonright (\eta+1) \in D\}$ is dense in $\mathbb{P}$. Take a condition $p \in \mathbb{P}$, and let $\overline{p} \leq p$ with $\overline{p} \in \overline{\mathbb{P}}$. Since $D \subseteq \mathbb{P}^\beta \upharpoonright (\eta+1)$ is dense, there exists $q \in \mathbb{P}^\beta \upharpoonright (\eta+1)$ with $q \leq_{\eta+1}^\beta \overline{p}^\beta \upharpoonright (\eta+1)$. By what we have just shown, we there exists $\overline{q} \leq \overline{p}$ with $\overline{q}^\beta \upharpoonright (\eta+1) \leq q$. Then $\overline{q}$ is an extension of p in $\overline{D}$ as desired. □

B) Capturing f^β.

In this section, we will show that the map f^β is contained in a generic extension similar to $V[G^\beta \upharpoonright (\eta+1)]$.

Recall that we are working in the case that $\boldsymbol{\kappa_{\eta+1} > \kappa_\eta^+}$, and $\boldsymbol{\beta < \alpha_{\widetilde{\eta}}}$ **or** $\boldsymbol{(\widetilde{\eta}, \gamma) \cap Lim \neq \emptyset}$, where $\widetilde{\eta} := \max\{\sigma < \eta \mid \sigma \in Lim\}$.

Recall that any $X \in \operatorname{dom} f$ is of the form

$$X = \dot{X}^{\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G_{\overline{i}_m}^{\overline{\sigma}_m}},$$

where $\dot{X} \in \mathrm{Name}((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P_{\overline{i}_m}^{\overline{\sigma}_m})$ and $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ is an η-good pair. Moreover,

$$f^\beta := \Big\{ \, (X, \alpha) \in f \;\Big|\; \exists \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big) \ \eta\text{-good pair} \ : \\ (\forall m \ \ \overline{i}_m < \beta\,) \wedge \exists \dot{X} \in \mathrm{Name}\big((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m}\big) \ \ X = \dot{X}^{\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G_{\overline{i}_m}^{\overline{\sigma}_m}} \ \Big\}.$$

Fix an η-good pair $\varrho = \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$. We use recursion over the $\mathrm{Name}((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m})$-hierarchy to define a map $\tau_\varrho : \mathrm{Name}((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m}) \to \mathrm{Name}(\mathbb{P})$ that maps any name $\dot{Y} \in \mathrm{Name}((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m})$ to a name $\tau_\varrho(\dot{Y}) \in \mathrm{Name}(\mathbb{P})$ such that

$$\dot{Y}^{\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G_{\overline{i}_m}^{\overline{\sigma}_m}} = (\tau_\varrho(\dot{Y}))^G.$$

Definition 2.6.7. For an η-good pair $\varrho = \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$, we define recursively for $\dot{Y} \in \mathrm{Name}((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m})$:

$$\tau_\varrho(\dot{Y}) := \Big\{ \, (\tau_\varrho(\dot{Z}), q) \;\Big|\; q \in \mathbb{P}\,,\ \exists \big(\,\dot{Z}\,,\, \big((p_*(a_m))_{m<\omega}\,,\, (p_{\overline{i}_m}^{\overline{\sigma}_m})_{m<\omega}\big)\big) \in \dot{Y} \ : \\ \forall m \ \ \big(\, q_*(a_m) \supseteq p_*(a_m)\,,\ q_{\overline{i}_m}^{\overline{\sigma}_m} \supseteq p_{\overline{i}_m}^{\overline{\sigma}_m} \,\big) \Big\}.$$

It is not difficult to check that indeed, $\dot{Y}^{\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G_{\overline{i}_m}^{\overline{\sigma}_m}} = (\tau_\varrho(\dot{Y}))^G$ holds for all $\dot{Y} \in \mathrm{Name}((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m})$.

Now, we define a map $(f^\beta)' \supseteq f^\beta$, which is contained in an intermediate generic extension similar to $V[G^\beta \upharpoonright (\eta+1)]$. We will then use an isomorphism argument to show that actually, $(f^\beta)' = f^\beta$.

Recall that $f = \dot{f}^G$, where $\pi \overline{\dot{f}}^{D_\pi} = \overline{\dot{f}}^{D_\pi}$ whenever $[\pi]$ contained in the intersection $\bigcap_{m<\omega} Fix(\eta_m, i_m) \cap \bigcap_{m<\omega} H^{\lambda_m}_{k_m}$ denoted by $(I_{\dot{f}})$.

The idea is that we include into $\mathbb{P}^\beta \upharpoonright (\eta+1)$ the verticals $P^{\eta_m}_{i_m}$ for $\eta_m \in \mathrm{Lim}$, $\eta_m > \eta$. Below κ_η, the linking property will be important, so we also have to include the linking ordinals $a^{\eta_m}_{i_m} \cap \kappa_\eta$.

For a condition $p \in \mathbb{P}$, we set

$$\widetilde{X}_p := \bigcup \{ a^\sigma_i \cap \kappa_{\overline{\eta}} \mid \sigma \in \mathrm{Lim}\ ,\ (\sigma, i) \neq (\eta_m, i_m) \text{ for all } m < \omega\ ,\ (\sigma > \eta \text{ or } i \geq \beta)\}.$$

Then $\widetilde{X}_p$ is similar to X_p, but excludes the linking ordinals $a^{\eta_m}_{i_m}$ for $\eta_m \in Lim$.

For reasons of notational convenience and better clarity, we introduce the following ad-hoc notation:

Let

$$(p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} := \big(p_* \upharpoonright \kappa^2_\eta\ ,\ (p^\sigma_i, a^\sigma_i)_{\sigma \leq \eta, i<\beta}\ ,\ (p^{\eta_m}_{i_m} \upharpoonright \kappa_\eta, a^{\eta_m}_{i_m} \cap \kappa_\eta)_{m<\omega\, , \eta_m > \eta},$$
$$(p^\sigma \upharpoonright (\beta \times \mathrm{dom}_y\, p^\sigma))_{\sigma \leq \eta}, \widetilde{X}_p \big).$$

Then $(p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ can be obtained from $p^\beta \upharpoonright (\eta+1)$ by using $\widetilde{X}_p$ instead of X_p, and including $(p^{\eta_m}_{i_m} \upharpoonright \kappa_\eta, a^{\eta_m}_{i_m} \cap \kappa_\eta)$ for $\eta_m \in Lim$ with $\eta_m > \eta$. (Note that for $\eta_m \leq \eta$, it follows that $i_m < \beta$, so $(p^{\eta_m}_{i_m}, a^{\eta_m}_{i_m})$ is already part of the condition $p^\beta \upharpoonright (\eta+1)$.)

We are now ready to define our forcing notion $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$. The order relation is defined similarly as for the forcing notion $\mathbb{P}^\beta \upharpoonright (\eta+1)$; but additionally, we require for $(q^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \leq (p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ that the linking property below κ_η holds for all (η_m, i_m) with $\eta_m \in \mathrm{Lim}$, $\eta_m > \eta$.

Definition 2.6.8. Let $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ denote the collection of all $(p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ such that $p \in \overline{\mathbb{P}}$ (i.e. $p \in \mathbb{P}$ with $|\{(\sigma, i) \in \mathrm{supp}\, p_0 \mid \sigma > \eta \text{ or } i \geq \beta\}| = \aleph_0$); together with $(\mathbb{1}^\beta_{\eta+1})^{(\eta_m, i_m)_{m<\omega}}$ as the maximal element.

For conditions p, $q \in \overline{\mathbb{P}}$, let $(q^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \leq (p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ if

- $\widetilde{X}_q \supseteq \widetilde{X}_p$,
- $\big(q_* \upharpoonright \kappa^2_\eta, (q^\sigma_i, b^\sigma_i)_{\sigma \leq \eta, i<\beta}, (q^\sigma \upharpoonright (\beta \times \mathrm{dom}_y\, q^\sigma))_{\sigma \leq \eta}\big) \leq \big(p_* \upharpoonright \kappa^2_\eta, (p^\sigma_i, a^\sigma_i)_{\sigma \leq \eta, i<\beta}, (p^\sigma \upharpoonright (\beta \times \mathrm{dom}_y\, p^\sigma))_{\sigma \leq \eta}\big)$ regarded as conditions in $\mathbb{P}$,
- $\forall\, \eta_m > \eta\ :\ q^{\eta_m}_{i_m} \upharpoonright \kappa_\eta \supseteq p^{\eta_m}_{i_m} \upharpoonright \kappa_\eta$,
- $\forall\, \eta_m > \eta\ ,\ (\eta_m, i_m) \in \mathrm{supp}\, p\ :\ b^{\eta_m}_{i_m} = a^{\eta_m}_{i_m}$,
- for all intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$ and $\eta_m > \eta$ with $a^{\eta_m}_{i_m} \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \{\xi\}$, it follows that $q^{\eta_m}_{i_m}(\zeta) = q_*(\xi, \zeta)$ whenever $\zeta \in (\mathrm{dom}\, q \smallsetminus \mathrm{dom}\, p) \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$.

Finally, for constructing our intermediate generic extension for capturing f^β, we also have to include the verticals $P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ for $\eta_m > \eta$.

This gives a product

$$(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m}),$$

which is the set of all

$$\Big((p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\,,\, (p^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m}))_{m<\omega} \Big)$$

such that $p \in \overline{\mathbb{P}}$ (i.e. $p \in \mathbb{P}$ with $|\{(\sigma, i) \in \operatorname{supp} p_0 \mid \sigma > \eta \text{ or } i \geq \beta\}| = \aleph_0$); together with a maximal element $(\overline{\mathbb{1}}^\beta_{\eta+1})^{(\eta_m, i_m)_{m<\omega}}$.

Then

$$(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$$

is the set of all $\Big((p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\,,\, (p^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m}))_{m<\omega} \Big)$ such that there exists $q \in G \cap \overline{\mathbb{P}}$ with $(q^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \leq (p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ and $q^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m}) \supseteq p^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ for all $m < \omega$; together with the maximal element $(\overline{\mathbb{1}}^\beta_{\eta+1})^{(\eta_m, i_m)_{m<\omega}}$.

In order to show that $(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ is a V-generic filter on $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$, we proceed similarly as in Proposition 2.6.6:

Proposition 2.6.9. *The map* $(\rho^\beta)^{(\eta_m, i_m)_{m<\omega}} : \overline{\mathbb{P}} \to (\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$,

$$p \mapsto \Big((p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\,,\, (p^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m}))_{m<\omega} \Big)$$

in the case that $|\{(\sigma, i) \in \operatorname{supp} p_0 \mid \sigma > \eta \vee i \geq \beta\}| = \aleph_0$, *and* $\mathbb{1} \mapsto (\overline{\mathbb{1}}^\beta_{\eta+1})^{(\eta_m, i_m)_{m<\omega}}$, *is a projection of forcing posets.*

Proof. We closely follow the proof of Proposition 2.6.6. Consider $\overline{p} \in \overline{\mathbb{P}}$ with $|\{(\sigma, i) \in \operatorname{supp} \overline{p}_0 \mid \sigma > \eta \vee i \geq \beta\}| = \aleph_0$, and a condition

$$q = \Big(q_* \upharpoonright \kappa_\eta^2\,,\, (q^\sigma_i, b^\sigma_i)_{\sigma \leq \eta, i < \beta}\,,\, (q^{\eta_m}_{i_m} \upharpoonright \kappa_\eta, b^{\eta_m}_{i_m} \cap \kappa_\eta)_{m<\omega, \eta_m > \eta}\,,$$

$$(q^\sigma)_{\sigma \leq \eta}\,,\, \widetilde{X}_q\,,\, (q^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m}))_{m<\omega} \Big)$$

in

$$(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$$

with $q \leq (\rho^\beta)^{(\eta_m, i_m)_{m<\omega}}(\overline{p})$. We have to construct $\overline{q} \leq \overline{p}$, $\overline{q} = (\overline{q}_*, (\overline{q}^\sigma_i, \overline{b}^\sigma_i)_{\sigma, i}, (\overline{q}^\sigma)_\sigma)$, such that $(\rho^\beta)^{(\eta_m, i_m)_{m<\omega}}(\overline{q}) \leq q$.

We start with $\overline{q}_0$.

- Similarly as in Proposition 2.6.6, we construct $\operatorname{supp}_* = \{(\check{\eta}, m_k) \mid k < \omega\}$ such that $\check{\eta} > \eta$ or $m_k \geq \beta$, and $(\check{\eta}, m_k) \notin \operatorname{supp} \overline{p}_0 \cup \operatorname{supp} q_0$ for all $k < \omega$; with the additional property that for all $k < \omega$, we have $(\check{\eta}, m_k) \notin \{(\eta_m, i_m) \mid m < \omega\,,\, \eta_m \in \mathrm{Lim}\}$. We set $\operatorname{supp} \overline{q}_0 = \operatorname{supp} \overline{p}_0 \cup \operatorname{supp} q_0 \cup \operatorname{supp}_* \cup \{(\eta_m, i_m) \mid m < \omega\,,\, \eta_m \in \mathrm{Lim}\}$.

- Next, we define the linking ordinals $\overline{b}_i^\sigma$ for $(\sigma, i) \in \operatorname{supp} \overline{q}_0$, such that $\widetilde{X}_{\overline{q}} \supseteq \widetilde{X}_q$ holds:

  First, we consider the case that $(\sigma, i) \notin \{(\eta_m, i_m) \mid m < \omega\,,\, \eta_m \in \mathrm{Lim}\}$. For $(\sigma, i) \in \operatorname{supp} q_0$, we let $\overline{b}_i^\sigma := b_i^\sigma \supseteq \overline{a}_i^\sigma$, and $\overline{b}_i^\sigma := \overline{a}_i^\sigma$ in the case that $(\sigma, i) \in \operatorname{supp} \overline{p}_0 \setminus \operatorname{supp} q_0$.

  We construct $(\overline{b}_{m_k}^{\check{\eta}} \mid k < \omega)$ as in Proposition 2.6.6.

  After that, we define the linking ordinals $(\overline{b}_{i_m}^{\eta_m} \mid m < \omega\,,\, \eta_m \in \mathrm{Lim})$ with the following properties:

  - As usual, every $\overline{b}_{i_m}^{\eta_m}$ is a subset of κ_{η_m} that hits any interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\eta_m}$ in exactly one point.
  - The $\overline{b}_{i_m}^{\eta_m}$ are pairwise disjoint, and $\overline{b}_{i_m}^{\eta_m} \cap \overline{b}_i^\sigma = \varnothing$ for every $m < \omega$ and $(\sigma, i) \in \operatorname{supp} \overline{q}_0$ with $(\sigma, i) \neq (\eta_m, i_m)$.
  - For every $(\eta_m, i_m) \in \operatorname{supp} \overline{p}_0$, we set $\overline{b}_{i_m}^{\eta_m} := a_{i_m}^{\eta_m}$; for every $(\eta_m, i_m) \in \operatorname{supp} q_0 \setminus \operatorname{supp} \overline{p}_0$ with $\eta_m \leq \eta$, we set $\overline{b}_{i_m}^{\eta_m} := b_{i_m}^{\eta_m}$; and whenever $(\eta_m, i_m) \in \operatorname{supp} q_0 \setminus \operatorname{supp} \overline{p}_0$ with $\eta_m > \eta$, we let $\overline{b}_{i_m}^{\eta_m} \supseteq b_{i_m}^{\eta_m} \cap \kappa_\eta$.

  This concludes our construction of the linking ordinals $\overline{b}_i^\sigma$.

- We define $\operatorname{dom} \overline{q}_0 = \bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j})$ as follows:
  Let $dom := \operatorname{dom} \overline{p}_0 \cup \operatorname{dom} q_0 \cup \bigcup_{\eta_m \in Lim} \operatorname{dom} q_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$. For every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $dom \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \varnothing$, we set $\delta_{\nu,j} := \kappa_{\nu,j}$; and whenever $dom \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \neq \varnothing$, we pick $\delta_{\nu,j} \in (\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with the property that $dom \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq [\kappa_{\nu,j}, \delta_{\nu,j})$, and $\overline{b}_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq [\kappa_{\nu,j}, \delta_{\nu,j})$ for all $(\sigma, i) \in \operatorname{supp} \overline{q}_0$.

  Since $\operatorname{dom} \overline{p}_0$, $\operatorname{dom} q_0$ and the domains $\operatorname{dom} q_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ are bounded below all regular cardinals, this is also true for dom and $\operatorname{dom} \overline{q}_0$.

- We take $\overline{q}_* \upharpoonright \kappa_\eta^2 \supseteq q_* \upharpoonright \kappa_\eta^2$ arbitrary on the given domain.

  The verticals $\overline{q}_i^\sigma \upharpoonright \kappa_\eta$ for $(\sigma, i) \in (\operatorname{supp} q_0 \cup \operatorname{supp} \overline{p}_0) \setminus \{(\eta_m, i_m) \mid m < \omega\,,\, \eta_m \in \mathrm{Lim}\}$ can be defined according to the *linking property* as in Proposition 2.6.6.

  The verticals $\overline{q}_{m_k}^{\hat{\eta}} \upharpoonright \kappa_\eta$ with $(\hat{\eta}, m_k) \in \operatorname{supp}_*$ can be set arbitrarily on the given domain.

  Now, consider (η_m, i_m) with $\eta_m \in \mathrm{Lim}$. In the case that $(\eta_m, i_m) \in \operatorname{supp} q_0$ with $\eta_m \leq \eta$, we can proceed as before, and define $\overline{q}_{i_m}^{\eta_m} \supseteq q_{i_m}^{\eta_m}$ according to the *linking property* as in Proposition 2.6.6.
  Concerning the verticals $\overline{q}_{i_m}^{\eta_m} \upharpoonright \kappa_\eta$ for $(\eta_m, i_m) \in \operatorname{supp} q_0$ with $\eta_m > \eta$, we define $\overline{q}_{i_m}^{\eta_m} \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \supseteq q_{i_m}^{\eta_m} \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ on intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$ according to the *linking property*, and use that we have incorporated the linking ordinals $b_{i_m}^{\eta_m} \cap \kappa_\eta$ into our forcing notion $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$: For $\zeta \in (\operatorname{dom} \overline{q}_0 \setminus \operatorname{dom} q_0) \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$,

we set $\overline{q}^{\eta_m}_{i_m}(\zeta) \coloneqq \overline{q}_*(\xi,\zeta)$, where $\{\xi\} = b^{\eta_m}_{i_m} \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) = \overline{b}^{\eta_m}_{i_m} \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$. (Note that $\xi \in \operatorname{dom}\overline{q}_0$ by construction.)

In the case that $(\eta_m,i_m) \notin \operatorname{supp} q_0$, it follows that also $(\eta_m,i_m) \notin \operatorname{supp}\overline{p}_0$, and we can set $\overline{q}^{\eta_m}_{i_m} \upharpoonright \kappa_\eta$ arbitrarily on the given domain.

- Next, consider an interval $[\kappa_{\nu,j},\kappa_{\nu,j+1}) \subseteq [\kappa_\eta,\kappa_\gamma)$. We first set the verticals $\overline{q}^\sigma_i \upharpoonright [\kappa_{\nu,j},\kappa_{\nu,j+1})$ for $(\sigma,i) \in \operatorname{supp}\overline{q}_0$, $\sigma > \eta$, on the given domain, with the property that $\overline{q}^{\eta_m}_{i_m} \upharpoonright [\kappa_{\nu,j},\kappa_{\nu,j+1}) \supseteq q^{\eta_m}_{i_m} \upharpoonright [\kappa_{\nu,j},\kappa_{\nu,j+1})$ for all $m<\omega$ with $(\eta_m,i_m) \in \operatorname{supp} q_0$, and $\overline{q}^\sigma_i \upharpoonright [\kappa_{\nu,j},\kappa_{\nu,j+1}) \supseteq \overline{p}^\sigma_i \upharpoonright [\kappa_{\nu,j},\kappa_{\nu,j+1})$ whenever $(\sigma,i) \in \operatorname{supp}\overline{p}_0$. After that, we define $\overline{q}_* \upharpoonright [\kappa_{\nu,j},\kappa_{\nu,j+1})^2 \supseteq \overline{p}_* \upharpoonright [\kappa_{\nu,j},\kappa_{\nu,j+1})^2$ according to the *linking property*: Whenever $\zeta \in (\operatorname{dom}\overline{q}_0 \setminus \operatorname{dom}\overline{p}_0) \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$ and $\{\xi\} = \overline{a}^\sigma_i \cap [\kappa_{\nu,j},\kappa_{\nu,j+1}) = \overline{b}^\sigma_i \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$ for some $(\sigma,i) \in \operatorname{supp}\overline{p}_0$, then $\overline{q}_*(\xi,\zeta) \coloneqq \overline{q}^\sigma_i(\zeta)$. Otherwise, $\overline{q}_*(\xi,\zeta)$ can be set arbitrarily.

This defines $\overline{q}_0$. The construction of $\overline{q}_1$ is similar; and it is not difficult to see that $\overline{q} \leq \overline{p}$ with $(\rho^\beta)^{(\eta_m,i_m)_{m<\omega}}(\overline{q}) \leq q$.

Hence, $(\rho^\beta)^{(\eta_m,i_m)_{m<\omega}}$ is a projection of forcing posets.

□

Thus, it follows that $(G^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta,\kappa_{\eta_m})$ is a V-generic filter on the forcing notion $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta,\kappa_{\eta_m})$.

The aim of Chapter 2.6.2 B) is to show that f^β is contained in the intermediate V-generic extension $V[\,(G^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta,\kappa_{\eta_m})\,]$.

Definition 2.6.10. Let $(f^\beta)'$ denote the set of all (X,α) for which there exists an *η-good pair* $\varrho = \big((a_m)_{m<\omega},(\overline{\sigma}_m,\overline{i}_m)_{m<\omega}\big)$ with $\overline{i}_m < \beta$ for all $m<\omega$ such that

$$X = \dot{X}^{\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}},$$

and there is a condition $p \in \mathbb{P}$ with

- $|\{(\sigma,i) \in \operatorname{supp} p_0 \mid \sigma > \eta \text{ or } i \geq \beta\}| = \aleph_0$,
- $p \Vdash_s \big(\tau_\varrho(\dot{X}),\alpha\big) \in \dot{f}$
- $\big((p^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}}\ ,\ (p^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta,\kappa_{\eta_m}))_{m<\omega}\big) \in (G^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta,\kappa_{\eta_m})$,
- $\forall\, \eta_m \in \mathrm{Lim} : (\eta_m,i_m) \in \operatorname{supp} p_0$ with $a^{\eta_m}_{i_m} = g^{\eta_m}_{i_m}$.

Then $(f^\beta)' \in V[\,(G^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta,\kappa_{\eta_m})\,]$, since the sequence $(g^{\eta_m}_{i_m} \mid m<\omega)$ is contained in the ground model V.

We will now use an isomorphism argument and show that $f^\beta = (f^\beta)'$.

Proposition 2.6.11. $f^\beta = (f^\beta)'$.

Proof. By the Forcing Theorem, it follows that $(f^\beta)' \supseteq f^\beta$. Assume towards a contradiction, there was $(X,\alpha) \in (f^\beta)' \smallsetminus f^\beta$. Let

$$X = \dot{X}^{\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m}}$$

for an η-good pair $\varrho = \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ with $\overline{i}_m < \beta$ for all $m<\omega$. Take $p \in \mathbb{P}$ as in Definition 2.6.10 with $p \Vdash_s (\tau_{\varrho(\dot{X})}, \alpha) \in \dot{f}$; and since $(X,\alpha) \notin f^\beta$, we can take $p' \in G$ with $p' \Vdash_s (\tau_\varrho(\dot{X}), \alpha) \notin \dot{f}$ and $(\eta_m, i_m) \in \operatorname{supp} p'_0$ for all $\eta_m \in Lim$.

Our first step will be to extend the conditions p and p' and obtain $\overline{p} \leq p$, $\overline{p}' \leq p'$ such that $\overline{p}$ and $\overline{p}'$ have "the same shape" similarly as in the *Approximation Lemma* 2.5.2; but additionally, $\overline{p}^\beta \upharpoonright (\eta+1) = (\overline{p}')^\beta \upharpoonright (\eta+1)$ holds, and $\overline{p}^{\eta_m}_{i_m} = (\overline{p}')^{\eta_m}_{i_m}$ for all $m<\omega$, and $\overline{a}^{\eta_m}_{i_m} = (\overline{a}')^{\eta_m}_{i_m}$ for all $m<\omega$ with $\eta_m \in Lim$.
After that, we construct an isomorphism π such that firstly, $\pi\overline{p} = \overline{p}'$; secondly, π should not disturb the forcing $\mathbb{P}^\beta \upharpoonright (\eta+1)$ (which will imply $\pi\,\overline{\tau_\varrho(\dot{X})}^{D_\pi} = \overline{\tau_\varrho(\dot{X})}^{D_\pi}$); and thirdly $[\pi]$ should be contained in the intersection $\bigcap_m Fix(\eta_m, i_m) \cap \bigcap_m H^{\lambda_m}_{k_m}$ (which implies $\pi\overline{f}^{D_\pi} = \overline{f}^{D_\pi}$).
Then from $\overline{p} \Vdash_s (\tau_\varrho(\dot{X}), \alpha) \in \dot{f}$ it follows $\pi\overline{p} \Vdash_s (\pi\,\overline{\tau_\varrho(\dot{X})}^{D_\pi}, \alpha) \in \pi\overline{f}^{D_\pi}$. Together with $\overline{p}' \Vdash_s (\overline{\tau_\varrho(\dot{X})}^{D_\pi}, \alpha) \notin \overline{f}^{D_\pi}$, this gives our desired contradiction.

In order to make such an isomorphism π possible, the extensions $\overline{p} \leq p$ and $\overline{p}' \leq p'$ will satisfy the following properties:

- $\overline{\operatorname{supp}}_0 := \operatorname{supp}\overline{p}_0 = \operatorname{supp}\overline{p}'_0$
- $\overline{\operatorname{dom}}_0 := \operatorname{dom}\overline{p}_0 = \operatorname{dom}\overline{p}'_0$
- $\bigcup\overline{a} := \bigcup_{(\sigma,i)\in\overline{\operatorname{supp}}_0} \overline{a}^\sigma_i = \bigcup_{(\sigma,i)\in\overline{\operatorname{supp}}_0} (\overline{a}')^\sigma_i$
- $\forall\,\nu,\,j\ :\ \big(\overline{\operatorname{dom}}_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \neq \emptyset \Rightarrow \bigcup\overline{a} \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \overline{\operatorname{dom}}_0\big)$
- $\overline{\operatorname{supp}}_1 := \operatorname{supp}\overline{p}_1 = \operatorname{supp}\overline{p}'_1$
- $\forall\,\sigma \in \overline{\operatorname{supp}}_1\ :\ \overline{\operatorname{dom}}_1(\sigma) := \operatorname{dom}\overline{p}^\sigma = \operatorname{dom}(\overline{p}')^\sigma$.

Additionally, we want:

- $\forall\,m<\omega\ :\ \overline{p}^{\eta_m}_{i_m} = (\overline{p}')^{\eta_m}_{i_m}$
- $\forall\,m<\omega\,,\ \eta_m \in \mathrm{Lim}\ :\ \overline{a}^{\eta_m}_{i_m} = (\overline{a}')^{\eta_m}_{i_m}$
- $\overline{p}^\beta \upharpoonright (\eta+1) = (\overline{p}')^\beta \upharpoonright (\eta+1)$, i.e.
  - $\overline{p}_* \upharpoonright \kappa^2_\eta = \overline{p}'_* \upharpoonright \kappa^2_\eta$
  - $\forall\,\sigma \in Lim,\ \sigma \leq \eta,\ i < \min\{\alpha_\sigma, \beta\}\ :\ \overline{p}^\sigma_i = (\overline{p}')^\sigma_i,\ \overline{a}^\sigma_i = (\overline{a}')^\sigma_i$
  - $\forall\,\sigma \in Succ\ \sigma \leq \eta\ :\ \overline{p}^\sigma \upharpoonright (\beta \times \operatorname{dom}_y \overline{p}^\sigma) = (\overline{p}')^\sigma \upharpoonright (\beta \times \operatorname{dom}_y (\overline{p}')^\sigma)$.

Then it follows that $\widetilde{X}_{\overline{p}} = \widetilde{X}_{\overline{p}'}$.

Note that $\overline{a}_{i_m}^{\eta_m} = (\overline{a}')_{i_m}^{\eta_m}$ for $\eta_m \in \mathrm{Lim}$ follows automatically, since $a_{i_m}^{\eta_m} = (a')_{i_m}^{\eta_m} = g_{i_m}^{\eta_m}$ by assumption.

Now, we construct the conditions $\overline{p}$ and $\overline{p}'$.

We start with the linking ordinals $\overline{a}_i^\sigma$ and $(\overline{a}')_i^\sigma$, with our aim that $\bigcup_{\sigma,i} \overline{a}_i^\sigma = \bigcup_{\sigma,i} (\overline{a}')_i^\sigma =: \bigcup \overline{a}$. We closely follows our construction from the *Approximation Lemma* 2.5.2; but now, some extra care is needed, since we additionally have to make sure that $\overline{a}_i^\sigma = (\overline{a}')_i^\sigma$ holds for all $\sigma \leq \eta$, $i < \beta$.

Similarly as in the *Approximation Lemma* 2.5.2, let

$$s := \kappa_{\overline{\delta}} := \sup\{\kappa_\sigma \mid \sigma \in \mathrm{Lim}\ ,\ \exists\, i < \alpha_\sigma\ (\sigma, i) \in \operatorname{supp} p_0 \cup \operatorname{supp} p_0'\}.$$

Recall that we are assuming $\boldsymbol{\beta < \alpha_{\widetilde{\eta}}}$ **or** $\boldsymbol{Lim \cap (\widetilde{\eta}, \gamma) \neq \varnothing}$, where $\widetilde{\eta} := \max\{\sigma \leq \eta \mid \sigma \in Lim\}$.

In the case that $\kappa_{\overline{\delta}} = \kappa_\gamma$, we set $\overline{\gamma} := \overline{\delta}$ and take $((\sigma_k, l_k) \mid k < \omega)$ such that $\sup\{\kappa_{\sigma_k} \mid k < \omega\} = \kappa_{\overline{\gamma}} = \kappa_\gamma$, and $(\sigma_k, l_k) \notin \operatorname{supp} p_0 \cup \operatorname{supp} p_0'$ for all $k < \omega$, with the additional property that $\sigma_k > \widetilde{\eta}$ or $l_k \geq \beta$ for all $k < \omega$.
If $\kappa_{\overline{\delta}} < \kappa_\gamma$ and $Lim \cap (\widetilde{\eta}, \gamma) \neq \varnothing$, let $\overline{\gamma} \in \mathrm{Lim} \cap (\widetilde{\eta}, \gamma)$ with $\overline{\gamma} \geq \overline{\delta}$, and take $((\sigma_k, l_k) \mid k < \omega)$ such that $(\sigma_k, l_k) = (\overline{\gamma}, l_k) \notin \operatorname{supp} p_0 \cup \operatorname{supp} p_0'$ for all $k < \omega$.
Finally, if $\kappa_{\overline{\delta}} < \kappa_\gamma$ and $Lim \cap (\widetilde{\eta}, \gamma) = \varnothing$, then $\beta < \alpha_{\widetilde{\eta}}$ follows. In this case, let $\overline{\gamma} := \widetilde{\eta} \geq \overline{\delta}$, and take $((\sigma_k, l_k) \mid k < \omega)$ with $(\sigma_k, l_k) = (\overline{\gamma}, l_k) \notin \operatorname{supp} p_0 \cup \operatorname{supp} p_0'$ for all $k < \omega$; with the additional property that $l_k \geq \beta$ for all $k < \omega$.

Let

$$\overline{\operatorname{supp}}_0 := \operatorname{supp} \overline{p}_0 := \operatorname{supp} \overline{p}_0' := \operatorname{supp} p_0 \cup \operatorname{supp} p_0' \cup \{(\sigma_k, l_k) \mid k < \omega\}.$$

We now construct the linking ordinals $\overline{a}_i^\sigma$. For any $(\sigma, i) \in \operatorname{supp} p_0$, we set $\overline{a}_i^\sigma := a_i^\sigma$; and whenever $(\sigma, i) \in \operatorname{supp} p_0' \setminus \operatorname{supp} p_0$ with $\sigma \leq \eta$, $i < \beta$, then $\overline{a}_i^\sigma := (a')_i^\sigma$.

Now, take a set $Z \subseteq \kappa_{\overline{\gamma}}$ such that for all intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\overline{\gamma}}$, we have $|Z \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})| = \aleph_0$, and $Z \cap \left(\bigcup_{(\sigma,i) \in \operatorname{supp} p_0} a_i^\sigma \cup \bigcup_{(\sigma,i) \in \operatorname{supp} p_0'} (a')_i^\sigma\right) = \varnothing$. Let

$$\overline{Z} := Z \cup \bigcup_{\sigma,i} a_i^\sigma \cup \bigcup_{\sigma,i} (a')_i^\sigma.$$

Our aim is to construct $\overline{p}$ and $\overline{p}'$ with $\bigcup_{\sigma,i} \overline{a}_i^\sigma = \bigcup_{\sigma,i} (\overline{a}')_i^\sigma = \bigcup \overline{a} := \overline{Z}$.

Fix an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\overline{\gamma}}$. Let

$$\overline{Z}_{\nu,j} := \Big(\bigcup \{a_i^\sigma \mid (\sigma, i) \in \operatorname{supp} p_0\} \cup \bigcup \{(a')_i^\sigma \mid (\sigma, i) \in \operatorname{supp} p_0',\ \sigma \leq \eta, i < \beta\} \Big) \cap$$

$$\cap\ [\kappa_{\nu,j}, \kappa_{\nu,j+1})$$

and

$$\{\xi_k(\nu, j) \mid k < \omega\} := \big(\overline{Z} \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})\big) \setminus \overline{Z}_{\nu,j}.$$

This set has cardinality $\aleph_0$ by construction of Z.

Now, let

$$\{(\overline{\sigma}_k, \overline{l}_k) \mid k < \omega\} =: \{(\sigma, i) \in \operatorname{supp} \overline{p}_0 \smallsetminus \operatorname{supp} p_0 \mid \kappa_{\nu,j} < \kappa_\sigma \text{ and } (\sigma > \eta \text{ or } i \geq \beta)\}.$$

This set also has cardinality $\aleph_0$ by construction of $\operatorname{supp} \overline{p}_0$. Now, for any $k < \omega$, we let

$$\overline{a}_{\overline{l}_k}^{\overline{\sigma}_k} \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) := \{\xi_k(\nu, j)\}.$$

We apply the same construction to the linking ordinals $(\overline{a}')_i^\sigma$ for $(\sigma, i) \in \operatorname{supp} \overline{p}_0' = \overline{\operatorname{supp}}_0$. It is not difficult to see that $\bigcup_{\sigma,i} \overline{a}_i^\sigma = \bigcup_{\sigma,i} (\overline{a}')_i^\sigma = \bigcup \overline{a} = \overline{Z}$, the independence property holds, and $\overline{a}_i^\sigma = (\overline{a}')_i^\sigma$ whenever $\sigma \leq \eta$, $i < \beta$.

Next, take $\overline{\operatorname{dom}}_0 := \operatorname{dom} \overline{p}_0 = \operatorname{dom} \overline{p}_0' = \bigcup_{\nu,j} [\kappa_{\nu,j}, \delta_{\nu,j})$ with the property that firstly, $\operatorname{dom} p_0 \cup \operatorname{dom} p_0' \subseteq \overline{\operatorname{dom}}_0$, and secondly, for every interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\gamma$ with $\overline{\operatorname{dom}}_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \neq \varnothing$, it follows that $\overline{Z} \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \overline{\operatorname{dom}}_0$.

It remains to construct $\overline{p}_*$, $\overline{p}_*'$, and $\overline{p}_i^\sigma$, $(\overline{p}')_i^\sigma$ for $(\sigma, i) \in \overline{\operatorname{supp}}_0$.

First, we consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$.

We start with the construction of $\overline{p}_* \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2 = \overline{p}_*' \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2$.

Let ξ, $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap \overline{\operatorname{dom}}_0$.

- In the case that $(\xi, \zeta) \in \operatorname{dom} p_0 \times \operatorname{dom} p_0$, we set $\overline{p}_*'(\xi, \zeta) := \overline{p}_*(\xi, \zeta) := p_*(\xi, \zeta)$.
- If $(\xi, \zeta) \in \operatorname{dom} p_0' \times \operatorname{dom} p_0'$, then $\overline{p}_*'(\xi, \zeta) := \overline{p}_*(\xi, \zeta) := p_*'(\xi, \zeta)$.

  For $(\xi, \zeta) \in (\operatorname{dom} p_0 \times \operatorname{dom} p_0) \cap (\operatorname{dom} p_0' \times \operatorname{dom} p_0')$, this is not a contradiction, since $p_* \upharpoonright \kappa_\eta^2$ and $p_*' \upharpoonright \kappa_\eta^2$ are compatible.
- If $\zeta \in \operatorname{dom} p_0 \smallsetminus \operatorname{dom} p_0'$ and $\xi \notin \operatorname{dom} p_0$, we proceed as follows: In the case that $\{\xi\} = a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for some $(\sigma, i) \in \operatorname{supp} p_0$ with $\sigma \leq \eta$, $i < \beta$ or $(\sigma, i) \in \{(\eta_m, i_m) \mid m < \omega\}$, we set $\overline{p}_*'(\xi, \zeta) := \overline{p}_*(\xi, \zeta) := p_i^\sigma(\zeta)$. Otherwise, we set $\overline{p}_*'(\xi, \zeta) = \overline{p}_*(\xi, \zeta)$ arbitrarily.
- In the case that $\zeta \in \operatorname{dom} p_0' \smallsetminus \operatorname{dom} p_0$ and $\xi \notin \operatorname{dom} p_0'$, we proceed as before: If $\{\xi\} = (a')_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for some $(\sigma, i) \in \operatorname{supp} p_0'$ with $\sigma \leq \eta$, $i < \beta$ or $(\sigma, i) \in \{(\eta_m, i_m) \mid m < \omega\}$, then $\overline{p}_*'(\xi, \zeta) := \overline{p}_*(\xi, \zeta) := (p')_i^\sigma(\zeta)$. Otherwise, we set $\overline{p}_*'(\xi, \zeta) = \overline{p}_*(\xi, \zeta)$ arbitrarily.
- In all other cases, $\overline{p}_*'(\xi, \zeta) = \overline{p}_*(\xi, \zeta)$ can be set arbitrarily.

This defines $\overline{p}_* \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2 = \overline{p}_*' \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2$.

Now, consider $(\sigma, i) \in \overline{\operatorname{supp}}_0$. We define $\overline{p}_i^\sigma$ and $(\overline{p}')_i^\sigma$ on the interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$ as follows:

- For $(\sigma, i) \in \operatorname{supp} p_0$, we define $\overline{p}_i^\sigma \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \supseteq p_i^\sigma \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ according to the *linking property*: Let $\{\xi\} := a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ and consider $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap \overline{\operatorname{dom}}_0$. If $\zeta \in \operatorname{dom} p_0$, we set $\overline{p}_i^\sigma(\zeta) := p_i^\sigma(\zeta)$; and $\overline{p}_i^\sigma(\zeta) := \overline{p}_*(\xi, \zeta)$ in the case that $\zeta \in \overline{\operatorname{dom}}_0 \smallsetminus \operatorname{dom} p_0$. (Note that $\xi \in \overline{\operatorname{dom}}_0$ follows by construction.)

- In the case that $(\sigma, i) \in \operatorname{supp} p_0'$, we define $(\overline{p}')_i^\sigma \restriction [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \supseteq (p')_i^\sigma \restriction [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ according to the *linking property* as before: Let $\{\xi\} := (a')_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$, and consider $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap \overline{\operatorname{dom}}_0$. If $\zeta \in \operatorname{dom} p_0'$, we set $(\overline{p}')_i^\sigma(\zeta) := (p')_i^\sigma(\zeta)$; and $(\overline{p}')_i^\sigma(\zeta) := (\overline{p}_*')(\xi, \zeta)$ in the case that $\zeta \in \overline{\operatorname{dom}}_0 \setminus \operatorname{dom} p_0'$. (Again, $\xi \in \overline{\operatorname{dom}}_0$ by construction.)

- For $(\sigma, i) \in \operatorname{supp} p_0 \setminus \operatorname{supp} p_0'$, let $(\overline{p}')_i^\sigma \restriction [\kappa_{\nu,j}, \kappa_{\nu,j+1}) := \overline{p}_i^\sigma \restriction [\kappa_{\nu,j}, \kappa_{\nu,j+1})$.

- For $(\sigma, i) \in \operatorname{supp} p_0' \setminus \operatorname{supp} p_0$, let $\overline{p}_i^\sigma \restriction [\kappa_{\nu,j}, \kappa_{\nu,j+1}) := (\overline{p}')_i^\sigma \restriction [\kappa_{\nu,j}, \kappa_{\nu,j+1})$.

- If $(\sigma, i) \in \overline{\operatorname{supp}}_0 \setminus (\operatorname{supp} p_0 \cup \operatorname{supp} p_0')$, then $\overline{p}_i^\sigma \restriction [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = (\overline{p}')_i^\sigma \restriction [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ can be set arbitrarily on the given domain.

This defines all $\overline{p}_i^\sigma$ and $(\overline{p}')_i^\sigma$ for $(\sigma, i) \in \overline{\operatorname{supp}}_0$ on intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$.

We now have to verify that $\overline{p}_i^\sigma = (\overline{p}')_i^\sigma$ for any $(\sigma, i) \in \overline{\operatorname{supp}}_0$ with $\sigma \leq \eta$, $i < \beta$. We only have to treat the case that $(\sigma, i) \in \operatorname{supp} p_0 \cap \operatorname{supp} p_0'$.
Consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\sigma \subseteq \kappa_\eta$. Then $p' \in G$ and

$$(p^\beta \restriction (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \in (G^\beta \restriction (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$$

implies that p_i^σ and $(p')_i^\sigma$ are compatible, and $a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = (a')_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) =: \{\xi\}$.
Let $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap \overline{\operatorname{dom}}_0$.

- If $\zeta \in \operatorname{dom} p_0 \cap \operatorname{dom} p_0'$, then $\overline{p}_i^\sigma(\zeta) = p_i^\sigma(\zeta) = (p')_i^\sigma(\zeta) = (\overline{p}')_i^\sigma(\zeta)$.

- For $\zeta \in \overline{\operatorname{dom}}_0 \setminus (\operatorname{dom} p_0 \cup \operatorname{dom} p_0')$, it follows that $\overline{p}_i^\sigma(\zeta) = \overline{p}_*(\xi, \zeta) = \overline{p}_*'(\xi, \zeta) = (\overline{p}')_i^\sigma(\zeta)$ by construction, since we have arranged $\overline{p}_* \restriction \kappa_\eta^2 = \overline{p}_*' \restriction \kappa_\eta^2$.

- Let now $\zeta \in \operatorname{dom} p_0 \setminus \operatorname{dom} p_0'$, $\xi \notin \operatorname{dom} p_0$. Then $\overline{p}_i^\sigma(\zeta) = p_i^\sigma(\zeta)$, and $(\overline{p}')_i^\sigma(\zeta) = \overline{p}_*'(\xi, \zeta)$. Since $\overline{p}_*'(\xi, \zeta) = p_i^\sigma(\zeta)$ by construction of $\overline{p}_*'$, this gives $\overline{p}_i^\sigma(\zeta) = (\overline{p}')_i^\sigma(\zeta)$ as desired.

  The case that $\zeta \in \operatorname{dom} p_0' \setminus \operatorname{dom} p_0$, $\xi \notin \operatorname{dom} p_0'$, can be treated similarly.

- If $\zeta \in \operatorname{dom} p_0 \setminus \operatorname{dom} p_0'$ and $\xi \in \operatorname{dom} p_0$, it follows that $\overline{p}_i^\sigma(\zeta) = p_i^\sigma(\zeta)$ and $(\overline{p}')_i^\sigma(\zeta) = \overline{p}_*'(\xi, \zeta)$ as before; but in this case, we have set $\overline{p}_*'(\xi, \zeta) := p_*(\xi, \zeta)$, so it remains to verify that $p_i^\sigma(\zeta) = p_*(\xi, \zeta)$.

  Since $p' \in G$, $(p^\beta \restriction (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \in (G^\beta \restriction (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$, we can take $q \in G$ with $(q^\beta \restriction (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \leq (p^\beta \restriction (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$, and assume w.l.o.g. that $q \leq p'$. Then $q_i^\sigma(\zeta) = q_*(\xi, \zeta)$ by the linking property for $q \leq p'$, since $(a')_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \{\xi\}$. Moreover, $p_i^\sigma(\zeta) = q_i^\sigma(\zeta)$ and $p_*(\xi, \zeta) = q_*(\xi, \zeta)$, and we are done.

- The remaining case is that $\zeta \in \operatorname{dom} p_0' \setminus \operatorname{dom} p_0$ and $\xi \in \operatorname{dom} p_0'$. Then $\overline{p}_i^\sigma(\zeta) = \overline{p}_*(\xi, \zeta) = p_*'(\xi, \zeta)$ and $(\overline{p}')_i^\sigma(\zeta) = (p')_i^\sigma(\zeta)$, and it remains to verify that $(p')_i^\sigma(\zeta) = p_*'(\xi, \zeta)$. As before, take $q \in G$ with $q \leq p'$ and $(q^\beta \restriction (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \leq (p^\beta \restriction (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$. The latter gives $q_i^\sigma(\zeta) = q_*(\xi, \zeta)$ by the linking property, since $\sigma \leq \eta$, $i < \beta$, $a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \{\xi\}$ and $\zeta \in \operatorname{dom} q_0 \setminus \operatorname{dom} p_0$. Moreover, from $q \leq p'$ it follows that $(p')_i^\sigma(\zeta) = q_i^\sigma(\zeta)$ and $p_*'(\xi, \zeta) = q_*(\xi, \zeta)$; hence, $(p')_i^\sigma(\zeta) = p_*'(\xi, \zeta)$ as desired.

Thus, it follows that $\overline{p}_i^\sigma = (\overline{p}')_i^\sigma$ holds for all $(\sigma, i) \in \overline{\mathrm{supp}}_0$ with $\sigma \leq \eta$, $i < \beta$.

If $m < \omega$ with $\eta_m \leq \eta$, then $i_m < \beta$ follows by construction of β. Hence, $\overline{p}_{i_m}^{\eta_m} = (\overline{p}')_{i_m}^{\eta_m}$. It remains to make sure that whenever $m < \omega$ with $\eta_m > \eta$, then $\overline{p}_{i_m}^{\eta_m} \upharpoonright \kappa_\eta = (\overline{p}')_{i_m}^{\eta_m} \upharpoonright \kappa_\eta$ holds; which can be shown similarly as $\overline{p}_i^\sigma = (\overline{p}')_i^\sigma$ in the case that $\sigma \leq \eta$, $i < \beta$: We use that $a_{i_m}^{\eta_m} = (a')_{i_m}^{\eta_m}$ and $p_{i_m}^{\eta_m}(\zeta) = (p')_{i_m}^{\eta_m}(\zeta)$ for all $m < \omega$ and $\zeta \in \mathrm{dom}\, p_0 \cap \mathrm{dom}\, p_0'$; and now, it is important that we are using the forcing notion $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ instead of $\mathbb{P}^\beta \upharpoonright (\eta+1)$; since we need the linking property below κ_η for the (η_m, i_m) with $\eta_m > \eta$.

It remains to construct $\overline{p}_* \upharpoonright [\kappa_\eta, \kappa_\gamma)^2$, $\overline{p}_*' \upharpoonright [\kappa_\eta, \kappa_\gamma)^2$, and $\overline{p}_i^\sigma \upharpoonright [\kappa_\eta, \kappa_\gamma)$, $(\overline{p}')_i^\sigma \upharpoonright [\kappa_\eta, \kappa_\gamma)$ for all $(\sigma, i) \in \overline{\mathrm{supp}}_0$ with $\sigma > \eta$.

- For (η_m, i_m) with $\eta_m > \eta$, we take $\overline{p}_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m}) \supseteq p_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$, $(\overline{p}')_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m}) \supseteq (p')_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ on the given domain, such that $\overline{p}_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m}) = (\overline{p}')_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$. This is possible, since $p' \in G$ and $(p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \in (G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$; so $p_{i_m}^{\eta_m}$ and $(p')_{i_m}^{\eta_m}$ are compatible for all $m < \omega$.
- For the $(\sigma, i) \in \overline{\mathrm{supp}}_0$ remaining, we set $\overline{p}_i^\sigma \upharpoonright [\kappa_\eta, \kappa_\gamma) \supseteq p_i^\sigma \upharpoonright [\kappa_\eta, \kappa_\gamma)$ and $(\overline{p}')_i^\sigma \upharpoonright [\kappa_\eta, \kappa_\gamma) \supseteq (p')_i^\sigma \upharpoonright [\kappa_\eta, \kappa_\gamma)$ arbitrarily on the given domain.
- Consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq [\kappa_\eta, \kappa_\gamma)$. We define $\overline{p}_* \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2 \supseteq p_* \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2$ according to the *linking property*: Whenever $\zeta \in \overline{\mathrm{dom}}_0 \setminus \mathrm{dom}\, p_0$ and $\{\xi\} = a_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ for some $(\sigma, i) \in \mathrm{supp}\, p_0$, then $\overline{p}_*(\xi, \zeta) := \overline{p}_i^\sigma(\zeta)$.
  The construction of $\overline{p}_*' \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2 \supseteq p_*' \upharpoonright [\kappa_{\nu,j}, \kappa_{\nu,j+1})^2$ is similar.

This completes our construction of $\overline{p}_0 \leq p_0$ and $\overline{p}_0' \leq p_0'$ with all the desired properties.

Similarly, one can construct $\overline{p}_1 \leq p_1$, $\overline{p}_1' \leq p_1'$ such that $\overline{\mathrm{supp}}_1 := \mathrm{supp}\, \overline{p}_1 = \mathrm{supp}\, \overline{p}_1'$, $\overline{\mathrm{dom}}_1(\sigma) := \mathrm{dom}\, \overline{p}_1(\sigma) = \mathrm{dom}\, \overline{p}_1'(\sigma)$ for all $\sigma \in \overline{\mathrm{supp}}_1$; and $\overline{p}_i^\sigma = (\overline{p}')_i^\sigma$ for all $\sigma \leq \eta$, $i < \beta$ with $\sigma \in Succ$, and $\overline{p}_{i_m}^{\eta_m} = (\overline{p}')_{i_m}^{\eta_m}$ for all $m < \omega$ with $\eta_m \in Succ$.

We now proceed similarly as in the *Approximation Lemma* 2.5.2 and construct an isomorphism π such that π a *standard isomorphism for* $\pi\overline{p} = \overline{p}'$. This determines all parameters of π except the maps $G_0(\nu, j) : \mathrm{supp}\, \pi_0(\nu, j) \to \mathrm{supp}\, \pi_0(\nu, j)$, which will be defined as follows: Consider an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1})$. Recall that we have the map $F_{\pi_0}(\nu, j) : \mathrm{supp}\, \pi_0(\nu, j) \to \mathrm{supp}\, \pi_0(\nu, j)$, which is in charge of permuting the linking ordinals: We set $F_{\pi_0}(\nu, j)(\sigma, i) := (\lambda, k)$ for $(\overline{a}')_i^\sigma \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \overline{a}_k^\lambda \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$. We define $G_{\pi_0}(\nu, j) := F_{\pi_0}(\nu, j)$ for all $\kappa_{\nu,j} < \kappa_\eta$, and $G_{\pi_0}(\nu, j) := id$ whenever $\kappa_{\nu,j} \geq \kappa_\eta$.

By construction, it follows that $\pi\overline{p} = \overline{p}'$. We will now check that $[\pi]$ is contained in the intersection $\bigcap_m Fix(\eta_m, i_m) \cap \bigcap_m H_{k_m}^{\lambda_m}$.

- Consider $m < \omega$ with $\eta_m \in Lim$ and $r \in D_\pi$, $r' := \pi r$, with $(\eta_m, i_m) \in \mathrm{supp}\, r_0$.

  For an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\eta_m}$ and $\zeta \in \mathrm{dom}\, \pi_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})$, it follows by construction of the map $\pi_0(\zeta)$ that $(r')_{i_m}^{\eta_m}(\zeta) = r_{i_m}^{\eta_m}(\zeta)$ holds; since $\overline{p}_{i_m}^{\eta_m} = (\overline{p}')_{i_m}^{\eta_m}$.

  In the case that $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1}) \cap (\mathrm{dom}\, r_0 \setminus \mathrm{dom}\, \pi_0)$, it follows that $(r')_{i_m}^{\eta_m}(\zeta) = r_k^\lambda(\zeta)$ with $(\lambda, k) = G_{\pi_0}(\nu, j)(\eta_m, i_m)$. If $\kappa_{\nu,j} < \kappa_\eta$, then $(\lambda, k) = G_{\pi_0}(\nu, j)(\eta_m, i_m) =$

$F_{\pi_0}(\nu,j)(\eta_m,i_m) = (\eta_m,i_m)$, since $\overline{a}_{i_m}^{\eta_m} = (\overline{a}')_{i_m}^{\eta_m}$. In the case that $\kappa_{\nu,j} \geq \kappa_\eta$, we have $G_{\pi_0}(\nu,j) = id$; so again, $(\lambda,k) = (\eta_m,i_m)$.

Hence, $r_{i_m}^{\eta_m}(\zeta) = (r')_{i_m}^{\eta_m}(\zeta)$ holds for all $\zeta \in \operatorname{dom} r_0 \cap \kappa_{\eta_m}$.

This proves $[\pi] \in Fix(\eta_m,i_m)$ in the case that $\eta_m \in Lim$. For $\eta_m \in Succ$, we obtain $[\pi] \in Fix(\eta_m,i_m)$ as in the *Approximation Lemma.*

- Consider $m < \omega$ with $\lambda_m \in Lim$. In the case that $\lambda_m > \eta$, we have $G_{\pi_0}(\nu,j)(\lambda_m,i) = (\lambda_m,i)$ for all $\kappa_{\nu,j} \in [\kappa_\eta,\kappa_{\lambda_m})$, and $[\pi] \in H_{k_m}^{\lambda_m}$ follows. If $\lambda_m \leq \eta$, it follows that $k_m < \beta$ by construction of β. Hence, whenever $\kappa_{\nu,j} < \kappa_{\lambda_m}$ and $i \leq k_m$, we have $G_{\pi_0}(\nu,j)(\lambda_m,i) = F_{\pi_0}(\nu,j)(\lambda_m,i) = (\lambda_m,i)$; since $\overline{a}_i^{\lambda_m} = (\overline{a}')_i^{\lambda_m}$ follows from $\lambda_m \leq \eta$, $i < \beta$.

  In the case that $\lambda_m \in Succ$, we obtain $[\pi] \in H_{k_m}^{\lambda_m}$ as in the *Approximation Lemma.*

Thus, we have shown that $[\pi] \in \bigcap_m Fix(\eta_m,i_m) \cap \bigcap_m H_{k_m}^{\lambda_m}$; which implies $\pi\overline{f}^{D_\pi} = \overline{f}^{D_\pi}$. It remains to make sure that $\pi\overline{\tau_\varrho(\dot X)}^{D_\pi} = \overline{\tau_\varrho(\dot X)}^{D_\pi}$.

Recall that we have an η-good pair $\varrho = ((a_m)_{m<\omega}, (\overline{\sigma}_m,\overline{i}_m)_{m<\omega})$ with $\overline{i}_m < \beta$ for all $m < \omega$, and $\dot X \in \operatorname{Name}\big((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m}\big)$ with

$$\tau_\varrho(\dot X) = \Big\{ \big(\tau_\varrho(\dot Y), q\big) \;\big|\; q \in \mathbb{P}\,,\ \exists \Big(\dot Y\,,\ \big((p_*(a_m))_{m<\omega}\,,\,(p_{\overline{i}_m}^{\overline{\sigma}_m})_{m<\omega}\big)\Big) \in \dot X\ :$$

$$\forall\, m\ \ \Big(q_*(a_m) \supseteq p_*(a_m)\,,\ q_{\overline{i}_m}^{\overline{\sigma}_m} \supseteq p_{\overline{i}_m}^{\overline{\sigma}_m}\Big)\Big\}.$$

Then

$$\overline{\tau_\varrho(\dot X)}^{D_\pi} = \Big\{ \big(\overline{\tau_\varrho(\dot Y)}^{D_\pi}, q\big) \;\big|\; q \in D_\pi\,,\ \dot Y \in \operatorname{dom}\dot X\,,\ q \Vdash_s \tau_\varrho(\dot Y) \in \tau_\varrho(\dot X)\Big\},$$

and

$$\pi\overline{\tau_\varrho(\dot X)}^{D_\pi} = \Big\{ \big(\pi\overline{\tau_\varrho(\dot Y)}^{D_\pi}, \pi q\big) \;\big|\; \pi q \in D_\pi\,,\ \dot Y \in \operatorname{dom}\dot X\,,\ q \Vdash_s \tau_\varrho(\dot Y) \in \tau_\varrho(\dot X)\Big\}.$$

We will now check that π is the identity on $\mathbb{P}^\beta \upharpoonright (\eta+1)$. More precisely: Let $q \in D_\pi$, $q = (q_*, (q_i^\sigma, b_i^\sigma)_{\sigma,i}, (q^\sigma)_\sigma)$ with $\pi q = q' = \big(q'_*, ((q')_i^\sigma, (b')_i^\sigma)_{\sigma,i}, ((q')^\sigma)_\sigma\big)$. We prove that $q'_* \upharpoonright \kappa_\eta^2 = q_* \upharpoonright \kappa_\eta^2$; moreover, $(q')_i^\sigma = q_i^\sigma$, $(b')_i^\sigma = b_i^\sigma$ for all $\sigma \leq \eta$, $i < \beta$ with $\sigma \in Lim$, and $(q')_i^\sigma = q_i^\sigma$ for all $\sigma \leq \eta$, $i < \beta$ with $\sigma \in Succ$.

- Since π is a *standard isomorphism for* $\pi\overline{p} = \overline{p}'$, it follows that $q'_* \upharpoonright \kappa_\eta^2 = q_* \upharpoonright \kappa_\eta^2$ for all $q \in D_\pi$; since firstly, $\overline{p}_* \upharpoonright \kappa_\eta^2 = \overline{p}'_* \upharpoonright \kappa_\eta^2$, and secondly, $G_{\pi_0}(\nu,j) = F_{\pi_0}(\nu,j)$ for all $\kappa_{\nu,j} < \kappa_\eta$. The latter makes sure that $q'_*(\xi_i^\sigma(\nu,j),\zeta) = q_*(\xi_i^\sigma(\nu,j),\zeta)$ whenever $\zeta \in \operatorname{dom} q_0 \smallsetminus \operatorname{dom}\pi_0$, and $\{\xi_i^\sigma(\nu,j)\} := b_i^\sigma \cap [\kappa_{\nu,j},\kappa_{\nu,j+1})$ for some $(\sigma,i) \in \operatorname{supp}\pi_0(\nu,j)$: We have $q'_*(\xi_i^\sigma(\nu,j),\zeta) = q_*(\xi_k^\lambda(\nu,j),\zeta)$ with $(\lambda,k) = G_{\pi_0}(\nu,j) \circ (F_{\pi_0}(\nu,j))^{-1}(\sigma,i)$; so from $G_{\pi_0}(\nu,j) = F_{\pi_0}(\nu,j)$ it follows that $q'_*(\xi_i^\sigma(\nu,j),\zeta) = q_*(\xi_i^\sigma(\nu,j),\zeta)$ as desired.

- Let now $(\sigma, i) \in \operatorname{supp} \pi_0 = \operatorname{supp} \overline{p}_0$ with $\sigma \leq \eta$, $i < \beta$ and $\sigma \in Lim$. Then $\overline{a}_i^\sigma = (\overline{a}')_i^\sigma$; hence, $F_{\pi_0}(\nu, j)(\sigma, i) = (\sigma, i)$ for all $\kappa_{\nu,j} < \kappa_\sigma$. This gives $(b')_i^\sigma = b_i^\sigma$ as desired. For $\zeta \in \operatorname{dom} \pi_0 = \operatorname{dom} \overline{p}_0$, it follows from $\overline{p}_i^\sigma = (\overline{p}')_i^\sigma$ by construction of π_0 that $(q')_i^\sigma(\zeta) = q_i^\sigma(\zeta)$ holds. Finally, if $\zeta \in (\operatorname{dom} q_0 \setminus \operatorname{dom} \pi_0)$, and ζ is contained in an interval $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\sigma$, then $(q')_i^\sigma(\zeta) = q_k^\lambda(\zeta)$ with $(\lambda, k) = G_{\pi_0}(\nu, j)(\sigma, i) = F_{\pi_0}(\nu, j)(\sigma, i) = (\sigma, i)$ as desired. Hence, it follows that $(q')_i^\sigma = q_i^\sigma$ for all $\sigma \leq \eta$, $i < \beta$.
- In the case that $\sigma \leq \eta$, $i < \beta$ with $\sigma \in Succ$, we obtain $(q')_i^\sigma = q_i^\sigma$ from $\overline{p}_i^\sigma = (\overline{p}')_i^\sigma$ as in the *Approximation Lemma* 2.5.2.

Hence, π is the identity on $\mathbb{P}^\beta \upharpoonright (\eta + 1)$.

Now, it is not difficult to prove recursively that for every $\dot{Z} \in \operatorname{Name}((\overline{P}^\eta)^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m})$ the following holds: If H is a V-generic filter on $\mathbb{P}$, then $(\tau_\varrho(\dot{Z}))^{\pi H} = (\tau_\varrho(\dot{Z}))^H = (\tau_\varrho(\dot{Z}))^{\pi^{-1} H}$. This implies $\overline{\tau_\varrho(\dot{X})}^{D_\pi} = \overline{\pi \tau_\varrho(\dot{X})}^{D_\pi}$, since for every $q \in D_\pi$ and $\dot{Y} \in \operatorname{dom} \dot{X}$, we have $q \Vdash_s \tau_\varrho(\dot{Y}) \in \tau_\varrho(\dot{X})$ if and only if $\pi q \Vdash_s \tau_\varrho(\dot{Y}) \in \tau_\varrho(\dot{X})$ holds.

Summing up, this gives our desired contradiction: Since $\overline{p} \Vdash_s \big(\tau_\varrho(\dot{X}), \alpha\big) \in \dot{f}$, it follows that $\pi\overline{p} \Vdash_s \big(\overline{\pi \tau_\varrho(\dot{X})}^{D_\pi}, \alpha\big) \in \overline{\pi \dot{f}}^{D_\pi}$; hence, $\overline{p}' \Vdash_s \big(\overline{\tau_\varrho(\dot{X})}^{D_\pi}, \alpha\big) \in \overline{\dot{f}}^{D_\pi}$. But this contradicts $\overline{p}' \Vdash_s (\tau_\varrho(\dot{X}), \alpha) \notin \dot{f}$.

Thus, our assumption that $(X, \alpha) \in (f^\beta)' \setminus f^\beta$ was wrong, and it follows that $(f^\beta)' = f^\beta$ as desired.

□

Hence, $f^\beta \in V[\,(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})\,]$.

C) $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves cardinals $\geq \alpha_\eta$.

The next step is to show that cardinals $\geq \alpha_\eta$ are absolute between V and $V[G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G_{i_m}^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$.

Recall that we are assuming GCH in our ground model V, which will be used implicitly throughout this Chapter 2.6.2 C): When we claim that a particular forcing notion preserves cardinals, then we mean it preserves cardinals under the assumption that GCH holds, if not stated differently.

First, we have a look at the cardinality of $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$. Recall that β was an ordinal *large enough for the intersection* $(I_{\dot{f}})$ with $\kappa_\eta^+ < \beta < \alpha_\eta$.

Lemma 2.6.12. $|(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}| \leq |\beta|^+$.

Proof. The forcing notion $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ is the set of all

$$\big(p_* \upharpoonright \kappa_\eta\ ,\ (p_i^\sigma, a_i^\sigma)_{\sigma \leq \eta, i<\beta}, (p^\sigma \upharpoonright (\beta \times \operatorname{dom}_y p^\sigma))_{\sigma \leq \eta}\ ,\ (p_{i_m}^{\eta_m} \upharpoonright \kappa_\eta, a_{i_m}^{\eta_m} \cap \kappa_\eta)_{m<\omega\,, \eta_m > \eta}\ ,\ \widetilde{X}_p\big)$$

for $p \in \mathbb{P}$ with $|\{(\sigma, i) \in \operatorname{supp} p_0 \mid \sigma > \eta \vee i \geq \beta\}| = \aleph_0$, together with the maximal element $(\mathbb{1}_{\eta+1}^\beta)^{(\eta_m, i_m)_{m<\omega}}$. Since $\widetilde{X}_p \subseteq \kappa_\eta$, there are only $\kappa_\eta^+ \leq |\beta|$-many possibilities for $\widetilde{X}_p$; and there are only $\leq \kappa_\eta^+ \leq |\beta|$-many possibilities for $p_* \upharpoonright \kappa_\eta^2$ and $(p_{i_m}^{\eta_m} \upharpoonright \kappa_\eta, a_{i_m}^{\eta_m} \cap \kappa_\eta)_{m<\omega}$. Concerning

$(p_i^\sigma, a_i^\sigma)_{\sigma \leq \eta, i < \beta}$, there are $|\beta|^{\aleph_0} \leq |\beta|^+$-many possibilities for the countable support; and with the support fixed, we have $(2^{\kappa_\eta})^{\aleph_0} \leq \kappa_\eta^+ \leq |\beta|$-many possibilities for countably many (p_i^σ, a_i^σ) with $\sigma \leq \eta$, $i < \beta$. Finally, for $(p^\sigma \upharpoonright (\beta \times \mathrm{dom}_y\, p^\sigma))_{\sigma \leq \eta}$, there are only $|\eta|^\omega \leq \kappa_\eta^+ \leq |\beta|$-many possibilities for the countable support; and with the countable support fixed, there are $\leq (2^{|\beta| \cdot \kappa_\eta})^{\aleph_0} = |\beta|^+$-many possibilities for countably many p^σ with $\mathrm{dom}\, p^\sigma \subseteq \beta \times \kappa_\sigma \subseteq \beta \times \kappa_\eta$. Hence, it follows that the forcing notion $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ has cardinality $\leq |\beta|^+$. $\square$

Corollary 2.6.13. *If $|\beta|^+ < \alpha_\eta$, then $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves cardinals $\geq \alpha_\eta$.*

Proof. With the same arguments as in Lemma 2.3.2, one can show that the forcing $\prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves all cardinals. By Lemma 2.6.12 above, the forcing $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ has cardinality $\leq |\beta|^+$ (in V; and hence, also in any $\prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$-generic extension). It follows that the product $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves all cardinals $\geq |\beta|^{++}$. $\square$

It remains to consider the case that $|\beta|^+ = \alpha_\eta$. Then by our assumptions on the sequence $(\alpha_\eta \mid 0 < \eta < \gamma)$ (cf. Chapter 2), it follows that $cf\, |\beta| > \omega$. Hence, GCH gives $|\beta|^{\aleph_0} = |\beta| < \alpha_\eta$; and by our proof of Lemma 2.6.12, it follows that all components of $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ have cardinality $\leq |\beta| < \alpha_\eta$; with the exception of $(p^\sigma \upharpoonright (\beta \times \mathrm{dom}_y\, p^\sigma))_{\sigma \leq \eta}$, where there might be $(2^{|\beta| \cdot \kappa_\eta})^{\aleph_0} = |\beta|^+ = \alpha_\eta$-many possibilities.

We now have to distinguish several cases depending on whether η is a limit ordinal or not, and depending on whether κ_η is a limit cardinal or a successor cardinal (i.e. $\eta \in Lim$ or $\eta \in Succ$).
We will have to separate one or two components $P^\sigma \upharpoonright (\beta \times [\overline{\kappa_\sigma}, \kappa_\sigma))$, where $\sigma \in Succ$, $\sigma \leq \eta$, $\kappa_\sigma = \overline{\kappa_\sigma}^+$, from the forcing notion $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$; and obtain a forcing $\left((\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\right)'$ which has cardinality $< \alpha_\eta$, while the product of the remaining $P^\sigma \upharpoonright (\beta \times [\overline{\kappa_\sigma}, \kappa_\sigma))$ and $\prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves cardinals.

Proposition 2.6.14. *The forcing notion $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves all cardinals $\geq \alpha_\eta$.*

Proof. By Corollary 2.6.13, we only have to treat the case that $\alpha_\eta = |\beta|^+$. Then $\mathrm{cf}\, |\beta| > \omega$ and $|\beta|^{\aleph_0} = |\beta|$.
First, we assume that **η is a limit ordinal**. Then by closure of the sequence $(\kappa_\sigma \mid 0 < \sigma < \gamma)$, it follows that **$\eta \in$ Lim**, i.e. $\kappa_\eta = \sup\{\kappa_\sigma \mid 0 < \sigma < \eta\}$ is a limit cardinal.

Since the sequence $(\alpha_\sigma \mid 0 < \sigma < \gamma)$ is strictly increasing (cf. Chapter 2.2), it follows that $\alpha_\sigma < |\beta|$ for all $\sigma < \eta$. Hence, for any $\sigma \in Succ$ with $\sigma < \eta$, the forcing notion $P^\sigma \upharpoonright (\beta \times [\overline{\kappa_\sigma}, \kappa_\sigma)) = P^\sigma \upharpoonright (\alpha_\sigma \times [\overline{\kappa_\sigma}, \kappa_\sigma))$ has cardinality $\leq \alpha_\sigma^+ \leq |\beta|$; and we conclude that there are only $\leq |\eta|^{\aleph_0} \cdot |\beta|^{\aleph_0} = |\beta|$-many possibilities for $(p^\sigma \upharpoonright (\beta \times \mathrm{dom}_y\, p^\sigma))_{\sigma \leq \eta}$.
Hence, by the proof of Lemma 2.6.12, it follows that $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ has cardinality $\leq |\beta| < \alpha_\eta$. Like in Corollary 2.6.13, this implies that the product $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves all cardinals $\geq |\beta|^+ = \alpha_\eta$ as desired.

The remaining case is that **η is a successor ordinal**. Let $\boldsymbol{\eta = \overline{\eta} + 1}$. We now have to distinguish four cases, depending on whether κ_η and $\kappa_{\overline{\eta}}$ are successor cardinals or limit cardinals.

If $\boldsymbol{\eta \in Lim}$ **and** $\boldsymbol{\overline{\eta} \in Lim}$, it follows for any $P^\sigma \upharpoonright (\beta \times [\overline{\kappa_\sigma}, \kappa_\sigma))$ with $\sigma \leq \eta$, $\sigma \in Succ$ that $\sigma < \overline{\eta}$ must hold; hence, $\alpha_\sigma < \alpha_{\overline{\eta}} < \alpha_\eta = |\beta|^+$, which implies $\alpha_\sigma < |\beta|$. Thus, the corresponding forcing notion $P^\sigma \upharpoonright (\beta \times [\overline{\kappa_\sigma}, \kappa_\sigma)) = P^\sigma \upharpoonright (\alpha_\sigma \times [\overline{\kappa_\sigma}, \kappa_\sigma))$ has cardinality $\leq \alpha_\sigma^+ \leq |\beta|$; and as before, it follows that the forcing $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ has cardinality $\leq |\beta|^{\aleph_0} = |\beta|$. Like in Corollary 2.6.13, this implies that the product $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves all cardinals $\geq |\beta|^+ = \alpha_\eta$ as desired.

If $\boldsymbol{\overline{\eta} \in Lim}$ **and** $\boldsymbol{\eta \in Succ}$, we consider the forcing notion $\big((\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\big)'$, which is obtained from $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ by excluding $P^\eta \upharpoonright (\beta \times [\overline{\kappa_\eta}, \kappa_\eta))$; i.e. we consider

$$(p^\sigma \upharpoonright (\beta \times \operatorname{dom}_y p^\sigma))_{\sigma<\eta} = (p^\sigma \upharpoonright (\beta \times \operatorname{dom}_y p^\sigma))_{\sigma<\overline{\eta}}$$

instead of $(p^\sigma \upharpoonright (\beta \times \operatorname{dom}_y p^\sigma))_{\sigma\leq\eta}$. Then $\big((\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\big)'$ has cardinality $\leq |\beta|$ as before; and it suffices to check that the remaining product

$$P^\eta \upharpoonright (\beta \times [\overline{\kappa_\eta}, \kappa_\eta)) \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$$

preserves all cardinals.
The forcing notion $\prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves cardinals. Moreover, $\prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ is $\leq \kappa_\eta$-closed. Hence, in any V-generic extension by $\prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ the following holds: Firstly, $P^\eta \upharpoonright (\beta \times [\overline{\kappa_\eta}, \kappa_\eta))$ is the same forcing notion as in V; and secondly, $P^\eta \upharpoonright (\beta \times [\overline{\kappa_\eta}, \kappa_\eta))$ preserves cardinals, since $2^{<\kappa_\eta} = \kappa_\eta$. Thus, it follows that the product $P^\eta \upharpoonright (\beta \times [\overline{\kappa_\eta}, \kappa_\eta)) \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves all cardinals as desired.

If $\boldsymbol{\overline{\eta} \in}$ **Succ and** $\boldsymbol{\eta \in}$ **Lim**, we proceed similarly, but exclude $P^{\overline{\eta}} \upharpoonright (\beta \times [\overline{\kappa_{\overline{\eta}}}, \kappa_{\overline{\eta}}))$ instead of $P^\eta \upharpoonright (\beta \times [\overline{\kappa_\eta}, \kappa_\eta))$.

If $\boldsymbol{\eta \in Succ}$ **and** $\boldsymbol{\overline{\eta} \in Succ}$, then both $P^\eta \upharpoonright (\beta \times [\overline{\kappa_\eta}, \kappa_\eta))$ and $P^{\overline{\eta}} \upharpoonright (\beta \times [\overline{\kappa_{\overline{\eta}}}, \kappa_{\overline{\eta}}))$ have to be parted from $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$. As before, it follows that firstly, the remaining forcing notion, denoted by $\big((\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\big)''$, has cardinality $\leq |\beta|$; and secondly, the remaining product

$$P^{\overline{\eta}} \upharpoonright (\beta \times [\overline{\kappa_{\overline{\eta}}}, \kappa_{\overline{\eta}})) \times P^\eta \upharpoonright (\beta \times [\overline{\kappa_\eta}, \kappa_\eta)) \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$$

preserves all cardinals.

It follows that $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves all cardinals $\geq \alpha_\eta$.

This concludes our proof by cases.

□

D) A set $\widetilde{\mathcal{P}}(\kappa_\eta) \supseteq \operatorname{dom} f^\beta$ with an injection $\iota : \widetilde{\mathcal{P}}(\kappa_\eta) \hookrightarrow |\beta|^{\aleph_0}$.

In this section, we construct in $V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$ a set $\widetilde{\mathcal{P}}(\kappa_\eta)$ with $\widetilde{\mathcal{P}}(\kappa_\eta) \supseteq \operatorname{dom} f^\beta$, together with an injective function $\iota : \widetilde{\mathcal{P}}(\kappa_\eta) \hookrightarrow (|\beta|^{\aleph_0})^V < \alpha_\eta$.

Since f^β is contained in $V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$ by Definition 2.6.10 and Proposition 2.6.11, and $(\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ preserves cardinals $\geq \alpha_\eta$ by Proposition 2.6.14, this will contradict our initial assumption that $f^\beta : \operatorname{dom} f^\beta \to \alpha_\eta$ was surjective.

Fix an η-good pair $\varrho = \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$. Then $\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}$ is a V-generic filter on $\prod_m \overline{P}^\eta \times \prod_m P^{\overline{\sigma}_m}$; and as in Lemma 2.3.2, it follows that this forcing preserves cardinals and the GCH. Hence, there is an injection $\chi : \wp(\kappa_\eta) \hookrightarrow (\kappa_\eta^+)^V$ in $V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}]$.

Let M_β be the set of all η-good pairs $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ in V with the property that $\overline{i}_m < \beta$ for all $m < \omega$. Then M_β has cardinality $\leq (2^{\kappa_\eta})^{\aleph_0} \cdot |\eta|^{\aleph_0} \cdot |\beta|^{\aleph_0} \leq |\beta|^{\aleph_0}$.

First, we consider the case that $\boldsymbol{|\beta|^+ = \alpha_\eta}$. Then $cf\,|\beta| > \omega$; hence, GCH gives $|\beta|^{\aleph_0} = |\beta|$ and there is an injection $\psi : M_\beta \hookrightarrow |\beta|$ in V.

By construction of f^β (cf. Definition 2.6.3), it follows that any $X \subseteq \kappa_\eta$ with $X \in dom\, f^\beta$ is contained in a model $V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}]$ for some η-good pair

$$\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big) \in M_\beta.$$

Hence, $dom\, f^\beta$ is a subset of

$$\widetilde{\wp}(\kappa_\eta) := \bigcup \Big\{ \wp(\kappa_\eta) \cap V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}] \;\Big|\; \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big) \in M_\beta \Big\}.$$

The set $\widetilde{\wp}(\kappa_\eta)$ can be defined in $V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$, since for any $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big) \in M_\beta$, we have $a_m \subseteq \kappa_\eta$, and $\overline{\sigma}_m \leq \eta$, $\overline{i}_m < \beta$ for all $m < \omega$.

For the rest of this section, we work in $V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$, and construct there an injective function $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow |\beta|^V$.

For a set $X \in \widetilde{\wp}(\kappa_\eta)$, let

$$\widetilde{\iota}(X) := ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$$

if $((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}) \in M_\beta$ with $X \in \wp(\kappa_\eta) \cap V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}]$, and $\psi((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$ is least with this property.
Now, we use the Axiom of Choice in $V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m,i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$, and choose for all $((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}) \in M_\beta$ an injection

$$\chi_{((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})} \;:\; \Big(\wp(\kappa_\eta) \cap V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}] \Big) \hookrightarrow (\kappa_\eta^+)^V.$$

Now, we can define $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow (\kappa_\eta^+)^V \cdot |\beta|^V$ as follows: For $X \in \widetilde{\wp}(\kappa_\eta)$, let

$$\iota(X) := \big(\chi_{\widetilde{\iota}(X)}(X), \psi(\widetilde{\iota}(X)) \big).$$

Since ψ and the maps $\chi_{((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})}$ for $((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}) \in M_\beta$ are injective, it follows that also ι is injective; which finishes our construction in the case that $(|\beta|^+)^V = \alpha_\eta$.

If $|\boldsymbol{\beta}|^+ < \boldsymbol{\alpha_\eta}$ in V, we can take an injection $\psi : M_\beta \hookrightarrow (|\beta|^+)^V$, and construct an injective function $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow (\kappa_\eta^+)^V \cdot (|\beta|^+)^V$ in $V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$ similarly as before.

This gives the following proposition:

Proposition 2.6.15. *If $(|\beta|^+)^V = \alpha_\eta$, then there is in $V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$ an injection $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow |\beta|^V$, where*

$$\widetilde{\wp}(\kappa_\eta) := \bigcup \Big\{ \wp(\kappa_\eta) \cap V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}] \ \Big| \ ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}) \in M_\beta \Big\}.$$

If $(|\beta|^+)^V < \alpha_\eta$, there is in $V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$ an injection $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow (|\beta|^+)^V$.

This leads to our desired contradiction: We assumed that $f^\beta : \operatorname{dom} f^\beta \to \alpha_\eta$ was surjective. By Chapter 2.6.2 B), Definition 2.6.10 and Proposition 2.6.11, it follows that $f^\beta \in V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$; where $(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$ is a V-generic filter on the forcing notion $((\mathbb{P}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m P^{\eta_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})$, which preserves cardinals $\geq \alpha_\eta$ by Chapter 2.6.2 C), Proposition 2.6.14.
However, since $\operatorname{dom} f^\beta \subseteq \widetilde{\wp}(\kappa_\eta)$ and $|\beta|^V < \alpha_\eta$, it follows that f^β together with the map ι from Proposition 2.6.15 above, collapses the cardinal α_η in $V[(G^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m G^{\eta_m}_{i_m} \upharpoonright [\kappa_\eta, \kappa_{\eta_m})]$. Contradiction.

Thus, we have shown that our initial assumption that $f^\beta : \operatorname{dom} f^\beta \to \alpha_\eta$ was surjective, was wrong.

Hence, there must be $\alpha < \alpha_\eta$ with $\alpha \notin \operatorname{rg} f^\beta$.

E) We use an isomorphism argument and obtain a contradiction.

We fix an ordinal $\alpha < \alpha_\eta$ with $\alpha \notin \operatorname{rg} f^\beta$. By surjectivity of f, there must be $X \subseteq \kappa_\eta$, $X \in N$, with $f(X) = \alpha$. Hence, there is an η-good pair $\varrho = ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$ with $X \in V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}]$; but since $X \notin \operatorname{dom} f^\beta$, there must be at least one index $m < \omega$ with $\overline{i}_m \geq \beta$. Let S_0 denote the set of all $(\overline{\sigma}_m, \overline{i}_m)$ with $\overline{i}_m < \beta$, and let S_1 be the set of all $(\overline{\sigma}_m, \overline{i}_m)$ with $\overline{i}_m \geq \beta$. Then $|S_1| \geq 1$.
For better clarity, we now switch to a slightly different notation, and write $(\overline{\lambda}_m, \overline{k}_m) := (\overline{\sigma}_m, \overline{i}_m)$ in the case that $m \in S_1$. We denote our η-good pair ϱ by

$$\varrho = \big((a_m)_{m<\omega}, \big((\overline{\sigma}_m, \overline{i}_m)_{m\in S_0}, (\overline{\lambda}_m, \overline{k}_m)_{m\in S_1}\big)\big).$$

Then

$$X = \dot{X}^{\prod_m G_*(a_m) \times \prod_{m\in S_0} G^{\overline{\sigma}_m}_{\overline{i}_m} \times \prod_{m\in S_1} G^{\overline{\lambda}_m}_{\overline{k}_m}}$$

for some $\dot{X} \in \operatorname{Name}((\overline{P}^\eta)^\omega \times \prod_{m\in S_0} P^{\overline{\sigma}_m} \times \prod_{m\in S_1} P^{\overline{\lambda}_m})$, such that the following holds:

- $(a_m \mid m < \omega)$ is a sequence of pairwise disjoint κ_η-subsets, such that for all $m < \omega$ and $\kappa_{\overline{\nu},\overline{\jmath}} < \kappa_\eta$, it follows that $|a_m \cap [\kappa_{\overline{\nu},\overline{\jmath}}, \kappa_{\overline{\nu},\overline{\jmath}+1})| = 1$,

- $S_0 \subseteq \omega$, and for all $m \in S_0$, we have $\overline{\sigma}_m \in Succ$ with $\overline{\sigma}_m \leq \eta$, $\bar{i}_m < \min\{\alpha_{\overline{\sigma}_m}, \beta\}$,
- if m, $m' \in S_0$ with $m \neq m'$, then $(\overline{\sigma}_m, \bar{i}_m) \neq (\overline{\sigma}_{m'}, \bar{i}_{m'})$,
- $\varnothing \neq S_1 \subseteq \omega$, and for all $m \in S_1$, we have $\overline{\lambda}_m \in Succ$ with $\overline{\lambda}_m \leq \eta$, $\overline{k}_m \in [\beta, \alpha_{\overline{\lambda}_m})$,
- if m, $m' \in S_1$ with $m \neq m'$, then $(\overline{\lambda}_m, \overline{k}_m) \neq (\overline{\lambda}_{m'}, \overline{k}_{m'})$.

Since $(X, \alpha) \in f$, take $p \in G$ with

$$p \Vdash_s (\tau_\varrho(\dot{X}), \alpha) \in \dot{f}.$$

Since we are using countable support, we can asssume w.l.o.g. that $\overline{\sigma}_m \in \operatorname{supp} p_1$, $\bar{i}_m \in \operatorname{dom}_x p_1(\overline{\sigma}_m)$ for all $m \in S_0$; and $\overline{\lambda}_m \in \operatorname{supp} p_1$, $\overline{k}_m \in \operatorname{dom}_x p_1(\overline{\lambda}_m)$ for all $m \in S_1$.

The idea can roughly be explained as follows: Recall that we have $\beta = \widetilde{\beta} + \kappa_\eta^+$ (addition of ordinals), where the ordinal $\widetilde{\beta}$ is *large enough for* $(I_{\dot{f}})$. In particular, $\kappa_\eta^+ < \widetilde{\beta} < \beta < \alpha_\eta$. We will now extend p and obtain a condition $q \in G$, $q \leq p$, such that there is a sequence $(\bar{l}_m \mid m \in S_1)$ with $\bar{l}_m \in (\widetilde{\beta}, \beta)$ for all $m \in S_1$, such that $q_{\overline{k}_m}^{\overline{\lambda}_m} = q_{\bar{l}_m}^{\overline{\lambda}_m}$ for all $m \in S_1$. Then we construct an isomorphism $\pi \in A$ that swaps any $(\overline{\lambda}_m, \overline{k}_m)$-coordinate with the according $(\overline{\lambda}_m, \bar{l}_m)$-coordinate.
Then $\pi q = q$; and we will see that $\pi \in \bigcap_m Fix(\eta_m, i_m) \cap \bigcap_m H_{k_m}^{\lambda_m}$, since $\widetilde{\beta}$ is *large enough for* $(I_{\dot{f}})$. Hence, $\pi \overline{\dot{f}}^{D_\pi} = \overline{\dot{f}}^{D_\pi}$; so from $q \Vdash_s (\tau_\varrho(\dot{X}), \alpha) \in \dot{f}$, we obtain that $q \Vdash_s (\pi \overline{\tau_\varrho(\dot{X})}^{D_\pi}, \alpha) \in \overline{\dot{f}}^{D_\pi}$. Setting

$$Y := \left(\pi \overline{\tau_\varrho(\dot{X})}^{D_\pi}\right)^G,$$

it follows that

$$(Y, \alpha) \in f.$$

However, we will see that $Y = \dot{X}^{\prod_m G_*(a_m) \times \prod_{m \in S_0} G_{\bar{i}_m}^{\overline{\sigma}_m} \times \prod_{m \in S_1} G_{\bar{l}_m}^{\overline{\lambda}_m}}$; where $\bar{i}_m < \beta$ for all $m \in S_0$, but also $\bar{l}_m < \beta$ for all $m \in S_1$. But then, the η-good pair

$$\varrho' = ((a_m)_{m<\omega}, ((\overline{\sigma}_m, \bar{i}_m)_{m \in S_0}, (\overline{\lambda}_m, \bar{l}_m)_{m \in S_1}))$$

is an element of M_β, and it follows that

$$(Y, \alpha) = \left(\dot{X}^{\prod_m G_*(a_m) \times \prod_{m \in S_0} G_{\bar{i}_m}^{\overline{\sigma}_m} \times \prod_{m \in S_1} G_{\bar{l}_m}^{\overline{\lambda}_m}}, \alpha\right) \in f^\beta.$$

But this would be a contradiction towards $\alpha \notin \operatorname{rg} f^\beta$.

We start our proof with the following lemma:

Lemma 2.6.16. *Let D be the set of all $q \in \mathbb{P}$ for which there exists a sequence of pairwise distinct ordinals $(\bar{l}_m \mid m \in S_1)$ with $\bar{l}_m \in (\widetilde{\beta}, \beta) \setminus \{\bar{i}_m \mid m \in S_0\}$ for all $m \in S_1$, such that $q_{\overline{k}_m}^{\overline{\lambda}_m} = q_{\bar{l}_m}^{\overline{\lambda}_m}$ holds for all $m \in S_1$. Then D is dense below p.*

Proof. Consider $q \in \mathbb{P}$ with $q \leq p$. We have to construct $\overline{q} \leq q$ with $q \in D$. The idea is that for every $m \in S_1$, we enlarge $\mathrm{dom}_x\, q(\overline{\lambda}_m)$ by some suitable $\overline{k}_m$, and set $\overline{q}(\overline{\lambda}_m)(\overline{k}_m, \zeta) := \overline{q}(\overline{\lambda}_m)(\overline{l}_m, \zeta) = q(\overline{\lambda}_m)(\overline{l}_m, \zeta)$ for all $\zeta \in \mathrm{dom}_y\, \overline{q}(\overline{\lambda}_m) = \mathrm{dom}_y\, q(\overline{\lambda}_m)$.

Note that for every $m \in S_1$, we have $\overline{\lambda}_m \in \mathrm{supp}\, q_1$ with $|\mathrm{dom}_x\, q_1(\overline{\lambda}_m)| < \kappa_{\overline{\lambda}_m} \leq \kappa_\eta$, since $\overline{\lambda}_m \leq \eta$. Hence, it follows that $|\bigcup_{m \in S_1} \mathrm{dom}_x\, q(\overline{\lambda}_m)| \leq \kappa_\eta < \kappa_\eta^+$; and similarly, $|\bigcup_{m \in S_0} \mathrm{dom}_x\, q(\overline{\sigma}_m)| \leq \kappa_\eta < \kappa_\eta^+$. Thus, the set

$$\Delta := (\widetilde{\beta}, \beta) \smallsetminus \Big(\bigcup_{m \in S_1} \mathrm{dom}_x\, q(\overline{\lambda}_m) \cup \bigcup_{m \in S_0} \mathrm{dom}_x\, q(\overline{\sigma}_m) \Big)$$

has cardinality κ_η^+.

Recall that for every $m \in S_0$, we have assumed that $\overline{i}_m \in \mathrm{dom}_x\, p(\overline{\sigma}_m) \subseteq \mathrm{dom}_x\, q(\overline{\sigma}_m)$; hence $\overline{i}_m \notin \Delta$.
For $m \in S_1$, we have $\overline{k}_m \in [\beta, \alpha_{\overline{\lambda}_m})$; hence, $\beta < \alpha_{\overline{\lambda}_m}$ and $\Delta \subseteq (\widetilde{\beta}, \beta) \subseteq \alpha_{\overline{\lambda}_m}$ follows.

We take a sequence of pairwise distinct ordinals $(\overline{l}_m \mid m \in S_1)$ in Δ (then $\{\overline{l}_m \mid m \in S_1\} \subseteq (\widetilde{\beta}, \beta) \smallsetminus \{\overline{i}_m \mid m \in S_0\}$), and define the extension $\overline{q} \leq q$ as follows:

Set $\overline{q}_0 := q_0$, and $\mathrm{supp}\, \overline{q}_1 = \mathrm{supp}\, q_1$. (From $q \leq p$ it follows that $\overline{\lambda}_m \in \mathrm{supp}\, q_1$ for all $m \in S_1$.) For $\sigma \in \mathrm{supp}\, \overline{q}_1$ with $\sigma \notin \{\overline{\lambda}_m \mid m \in S_1\}$, we set $\overline{q}(\sigma) := q(\sigma)$. For $\sigma \in \{\overline{\lambda}_m \mid m \in S_1\}$, we proceed as follows: Let $S_1(\sigma) := \{m \in S_1 \mid \sigma = \overline{\lambda}_m\}$. We set $\mathrm{dom}_y\, \overline{q}(\sigma) := \mathrm{dom}_y\, q(\sigma)$, and $\mathrm{dom}_x\, \overline{q}(\sigma) := \mathrm{dom}_x\, q(\sigma) \cup \{\overline{l}_m \mid m \in S_1(\sigma)\}$. Note that by construction of Δ this union is disjoint, since $\overline{l}_m \notin \mathrm{dom}_x\, q(\sigma) = \mathrm{dom}_x\, q(\overline{\lambda}_m)$ for all $m \in S_1(\sigma)$.

Note that for every $m \in S_1(\sigma)$, we have $\overline{k}_m \in \mathrm{dom}_x\, p(\sigma) \subseteq \mathrm{dom}_x\, q(\sigma) \subseteq \mathrm{dom}_x\, \overline{q}(\sigma)$.

We let $\overline{q}(\sigma)(i, \zeta) := q(\sigma)(i, \zeta)$ whenever $(i, \zeta) \in \mathrm{dom}_x\, q(\sigma) \times \mathrm{dom}_y\, q(\sigma)$. If $(i, \zeta) \in \mathrm{dom}\, \overline{q}(\sigma) \smallsetminus \mathrm{dom}\, q(\sigma)$, then $\zeta \in \mathrm{dom}_y\, q(\sigma)$ and $i = \overline{l}_n$ for some $n \in S_1(\sigma)$, i.e. $n \in S_1$ with $\sigma = \overline{\lambda}_n$. In this case, we set $\overline{q}(\sigma)(i, \zeta) = \overline{q}(\overline{\lambda}_n)(\overline{l}_n, \zeta) := q(\overline{\lambda}_n)(\overline{k}_n, \zeta) = q(\sigma)(\overline{k}_n, \zeta)$.

This defines $\overline{q} \leq q$ with the property that $\overline{q}^{\overline{\lambda}_m}_{\overline{k}_m} = \overline{q}^{\overline{\lambda}_m}_{\overline{l}_m}$ holds for all $m \in S_1$.

Thus, it follows that D is dense below p.

□

Since $p \in G$, we can now take $q \in G$, $q \leq p$ with $q \in D$. Take $(\overline{l}_m \mid m \in S_1)$ as in the definition of D, with $\overline{l}_m \in (\widetilde{\beta}, \beta) \smallsetminus \{\overline{i}_m \mid m \in S_0\}$ and $q^{\overline{\lambda}_m}_{\overline{k}_m} = q^{\overline{\lambda}_m}_{\overline{l}_m}$ for all $m \in S_1$. Then the sets $\{(\overline{\lambda}_m, \overline{l}_m) \mid m \in S_1\}$ and $\{(\overline{\sigma}_m, \overline{i}_m) \mid m \in S_0\}$ are disjoint.

Since $q \leq p$, we have

$$q \Vdash_s (\tau_\varrho(\dot{X}), \alpha) \in \dot{f}.$$

The next step is to construct an isomorphism π that swaps every $(\overline{\lambda}_m, \overline{k}_m)$-coordinate with the according $(\overline{\lambda}_m, \overline{l}_m)$-coordinate for $m \in S_1$, and does nothing else.

Definition 2.6.17. We define an isomorphism $\pi \in A$ as follows:

- The map π_0 is the identity on $D_{\pi_0} = \mathbb{P}_0$.

- We set $\operatorname{supp}\pi_1 := \operatorname{supp} q_1$, and for every $\sigma \in \operatorname{supp} q_1$, we let $\operatorname{dom}\pi_1(\sigma) := \operatorname{dom} q_1(\sigma)$.

  Then for all $m \in S_1$, it follows that $\overline{\lambda}_m \in \operatorname{supp} p_1 \subseteq \operatorname{supp} q_1 = \operatorname{supp}\pi_1$; and $\overline{k}_m \in \operatorname{dom}_x p_1(\overline{\lambda}_m) \subseteq \operatorname{dom}_x q_1(\overline{\lambda}_m) = \operatorname{dom}_x \pi_1(\overline{\lambda}_m)$, $\overline{l}_m \in \operatorname{dom}_x q_1(\overline{\lambda}_m) = \operatorname{dom}_x \pi_1(\overline{\lambda}_m)$.

- Consider $\sigma \in \operatorname{supp}\pi_1$ with $\kappa_\sigma = \overline{\kappa_\sigma}^+$. In the case that $\sigma \notin \{\overline{\lambda}_m \mid m \in S_1\}$, we set $\operatorname{supp}\pi_1(\sigma) := \varnothing$, and let $\pi_1(\sigma)(i,\zeta) : 2 \to 2$ be the identity map for all $(i,\zeta) \in \alpha_\sigma \times [\overline{\kappa_\sigma}, \kappa_\sigma)$.

- For $\sigma \in \{\overline{\lambda}_m \mid m \in S_1\}$, consider the set $S_1(\sigma) := \{m \in S_1 \mid \sigma = \overline{\lambda}_m\}$, and let $\operatorname{supp}\pi_1(\sigma) := \{\overline{k}_m \mid m \in S_1(\sigma)\} \cup \{\overline{l}_m \mid m \in S_1(\sigma)\}$. Then $\operatorname{supp}\pi_1(\sigma)$ is a subset of $\operatorname{dom}_x \pi_1(\sigma)$.

  The map $f_{\pi_1}(\sigma) : \operatorname{supp}\pi_1(\sigma) \to \operatorname{supp}\pi_1(\sigma)$ is defined as follows: Let $f_{\pi_1}(\sigma)(\overline{k}_m) = \overline{l}_m$, and $f_{\pi_1}(\sigma)(\overline{l}_m) = \overline{k}_m$ for all $m \in S_1(\sigma)$.

  Then $f_{\pi_1}(\sigma)$ is well-defined and bijective, since $\overline{k}_m \geq \beta$ for all $m \in S_1$, and $\overline{l}_m < \beta$ for all $m \in S_1$.

  It remains to define the maps $\pi_1(\zeta) : 2^{\operatorname{supp}\pi_1(\sigma)} \to 2^{\operatorname{supp}\pi_1(\sigma)}$ for $\zeta \in \operatorname{dom}_y \pi_1(\sigma)$: Let $\pi_1(\zeta)(\epsilon_i \mid i \in \operatorname{supp}\pi_1(\sigma)) := (\widetilde{\epsilon}_i \mid i \in \operatorname{supp}\pi_1(\sigma))$, where $\widetilde{\epsilon}_{\overline{k}_m} := \epsilon_{\overline{l}_m}$, $\widetilde{\epsilon}_{\overline{l}_m} := \epsilon_{\overline{k}_m}$ for all $m \in S_1(\sigma)$.

  Finally, for every $(i,\zeta) \in \alpha_\sigma \times [\overline{\kappa_\sigma}, \kappa_\sigma)$, we let $\pi_1(\sigma)(i,\zeta) : 2 \to 2$ be the identity.

This defines our automorphism $\pi \in A$.

Lemma 2.6.18. *For $Y := \left(\overline{\pi\tau_\varrho(\dot{X})}^{D_\pi}\right)^G$, it follows that $(Y,\alpha) \in f$.*

Proof. By construction of π it follows that whenever r is a condition in D_π with $r' := \pi r$, then the following holds: Firstly, for all $m \in S_1$, we have $(r')^{\overline{\lambda}_m}_{\overline{k}_m} = r^{\overline{\lambda}_m}_{\overline{l}_m}$ and $(r')^{\overline{\lambda}_m}_{\overline{l}_m} = r^{\overline{\lambda}_m}_{\overline{k}_m}$. Secondly, whenever $\sigma \in \operatorname{supp} r_1$, $i \in \operatorname{dom}_x r(\sigma)$ with $(\sigma, i) \notin \{(\overline{\lambda}_m, \overline{k}_m) \mid m \in S_1\} \cup \{(\overline{\lambda}_m, \overline{l}_m) \mid m \in S_1\}$, then $(r')^\sigma_i = r^\sigma_i$.
In particular, $(r')^{\overline{\sigma}_{m'}}_{\overline{i}_{m'}} = r^{\overline{\sigma}_{m'}}_{\overline{i}_{m'}}$ holds for all $m' \in S_0$:
On the one hand, we have $(\overline{\sigma}_{m'}, \overline{i}_{m'}) \notin \{(\overline{\lambda}_m, \overline{k}_m) \mid m \in S_1\}$ for all $m' \in S_0$, since $\overline{i}_{m'} < \beta$; but $\overline{k}_m \geq \beta$ for all $m \in S_1$. On the other hand, $(\overline{\sigma}_{m'}, \overline{i}_{m'}) \notin \{(\overline{\lambda}_m, \overline{l}_m) \mid m \in S_1\}$ for all $m' \in S_0$ follows by construction of the set D.

In other words: The map π swaps for all $m \in S_1$ the $(\overline{\lambda}_m, \overline{k}_m)$-coordinate with the according $(\overline{\lambda}_m, \overline{l}_m)$-coordinate, and does nothing else.
Hence, it follows that $\pi q = q$; since $q^{\overline{\lambda}_m}_{\overline{k}_m} = q^{\overline{\lambda}_m}_{\overline{l}_m}$ for all $m \in S_1$.

Next, we want to show that $\pi \in \bigcap_m Fix(\eta_m, i_m) \cap \bigcap_m H^{\lambda_m}_{k_m}$. Then $\pi\overline{\dot{f}}^{D_\pi} = \overline{\dot{f}}^{D_\pi}$ follows. Regarding $\pi \in \bigcap_m Fix(\eta_m, i_m)$, it suffices to make sure that for all $m < \omega$, we have $(\eta_m, i_m) \notin \{(\overline{\lambda}_{m'}, \overline{k}_{m'}) \mid m' \in S_1\} \cup \{(\overline{\lambda}_{m'}, \overline{l}_{m'}) \mid m' \in S_1\}$. But this follows from the fact that $\overline{\lambda}_{m'} \leq \eta$ and $\overline{k}_{m'} \geq \beta > \widetilde{\beta}$, $\overline{l}_{m'} > \widetilde{\beta}$ for all $m' \in S_1$; but $\widetilde{\beta}$ is *large enough for* $(I_{\dot{f}})$, so for any η_m with $\eta_m \leq \eta$, it follows that $i_m < \widetilde{\beta}$. This implies $(\eta_m, i_m) \notin \{(\overline{\lambda}_{m'}, \overline{k}_{m'}) \mid m' \in S_1\} \cup \{(\overline{\lambda}_{m'}, \overline{l}_{m'}) \mid m' \in S_1\}$ for all $m < \omega$ as desired.

Hence, $\pi \in \bigcap_m Fix(\eta_m, i_m)$.

Regarding $\pi \in \bigcap_m H^{\lambda_m}_{k_m}$, we have to make sure that whenever $\lambda_m = \overline{\lambda}_{m'}$ for some $m < \omega$ and $m' \in S_1$, then $\operatorname{supp} \pi_1(\lambda_m) = \operatorname{supp} \pi_1(\overline{\lambda}_{m'}) \subseteq (k_m, \alpha_{\lambda_m})$ holds; i.e. $\overline{k}_{m'} > k_m$ and $\overline{l}_{m'} > k_m$. Again, this follows from the fact that $\overline{\lambda}_{m'} \leq \eta$ and $\overline{k}_{m'} \geq \beta > \widetilde{\beta}$, $\overline{l}_{m'} > \widetilde{\beta}$ for all $m' \in S_1$; and $\widetilde{\beta}$ is *large enough for* $(I_{\dot{f}})$, so whenever $\lambda_m \leq \eta$, then $k_m < \widetilde{\beta}$ follows. Hence, $\pi \in \bigcap_m H^{\lambda_m}_{k_m}$. Thus, it follows that $\pi \overline{\dot{f}}^{D_\pi} = \overline{\dot{f}}^{D_\pi}$.

Now, from $q \Vdash_s (\tau_\varrho(\dot{X}), \alpha) \in \dot{f}$, we obtain $\pi q \Vdash_s (\pi \overline{\tau_\varrho(\dot{X})}^{D_\pi}, \alpha) \in \pi \overline{\dot{f}}^{D_\pi}$; hence, $q \Vdash_s (\pi \overline{\tau_\varrho(\dot{X})}^{D_\pi}, \alpha) \in \overline{\dot{f}}^{D_\pi}$. With

$$Y := \big(\pi \overline{\tau_\varrho(\dot{X})}^{D_\pi}\big)^G,$$

it follows from $q \in G$ that $(Y, \alpha) \in f$ as desired.

□

We will now show that $(Y, \alpha) \in f$ implies that also $(Y, \alpha) \in f^\beta$ must hold. This finally gives our desired contradiction, since $\alpha \notin \operatorname{rg} f^\beta$.

Indeed, we will prove that

$$Y = \dot{X}^{\prod_m G_*(a_m) \times \prod_{m \in S_0} G^{\overline{\sigma}_m}_{\overline{i}_m} \times \prod_{m \in S_1} G^{\overline{\lambda}_m}_{\overline{l}_m}}.$$

Since $\overline{i}_m < \beta$ for all $m \in S_0$ and $\overline{l}_m < \beta$ for all $m \in S_1$, it follows that the η-good pair

$$\varrho' := \big((a_m)_{m<\omega}, ((\overline{\sigma}_m, \overline{i}_m)_{m \in S_0}, (\overline{\lambda}_m, \overline{l}_m)_{m \in S_1})\big)$$

is an element of M_β. Hence, $(Y, \alpha) \in f$ would then imply that also $(Y, \alpha) \in f^\beta$ must hold, and we are done.

Recall that

$$\varrho := \big((a_m)_{m<\omega}, ((\overline{\sigma}_m, \overline{i}_m)_{m \in S_0}, (\overline{\lambda}_m, \overline{k}_m)_{m \in S_1})\big),$$

$\dot{X} \in \operatorname{Name}((\overline{P}^\eta)^\omega \times \prod_{m \in S_0} P^{\overline{\sigma}_m} \times \prod_{m \in S_1} P^{\overline{\lambda}_m})$, and $\tau_\varrho(\dot{X})$ is the canonical extension of $\dot{X}$ to a name for $\mathbb{P}$ (see Definition 2.6.7).

We will show recursively:

Lemma 2.6.19. *For every* $\dot{Y} \in \operatorname{Name}\big((\overline{P}^\eta)^\omega \times \prod_{m \in S_0} P^{\overline{\sigma}_m} \times \prod_{m \in S_1} P^{\overline{\lambda}_m}\big)$*, it follows that*

$$\pi \, \overline{\tau_\varrho(\dot{Y})}^{D_\pi} = \overline{\tau_{\varrho'}(\dot{Y})}^{D_\pi}.$$

Proof. Consider $\dot{Y} \in \operatorname{Name}_{\alpha+1}((\overline{P}^\eta)^\omega \times \prod_{m \in S_0} P^{\overline{\sigma}_m} \times \prod_{m \in S_1} P^{\overline{\lambda}_m})$, and assume recursively that the claim was true for all $\dot{Z} \in \operatorname{Name}_\alpha((\overline{P}^\eta)^\omega \times \prod_{m \in S_0} P^{\overline{\sigma}_m} \times \prod_{m \in S_1} P^{\overline{\lambda}_m})$.
First,

$$\overline{\tau_\varrho(\dot{Y})}^{D_\pi} = \big\{ \big(\overline{\tau_\varrho(\dot{Z})}^{D_\pi}, r \big) \ \big| \ r \in D_\pi \, , \, \dot{Z} \in \operatorname{dom} \dot{Y} \, , \, r \Vdash_s \tau_\varrho(\dot{Z}) \in \tau_\varrho(\dot{Y}) \big\},$$

and

$$\overline{\pi\tau_{\varrho}(\dot{Y})}^{D_\pi} = \big\{ \big(\overline{\pi\tau_{\varrho}(\dot{Z})}^{D_\pi}, \pi r \big) \;\big|\; r \in D_\pi\,,\ \dot{Z} \in \operatorname{dom} \dot{Y}\,,\ r \Vdash_s \tau_{\varrho}(\dot{Z}) \in \tau_{\varrho}(\dot{Y}) \big\}.$$

Now, for any H a V-generic filter on $\mathbb{P}$ and $\dot{Z}_0 \in \mathrm{Name}((\overline{P}^{\eta})^{\omega} \times \prod_{m \in S_0} P^{\overline{\sigma}_m} \times \prod_{m \in S_1} P^{\overline{\lambda}_m})$, it follows by construction of the map π that

$$\begin{aligned} \big(\tau_{\varrho}(\dot{Z}_0)\big)^H &= \dot{Z}_0^{\prod_{m<\omega} H_*(a_m) \times \prod_{m \in S_0} H^{\overline{\sigma}_m}_{\overline{i}_m} \times \prod_{m \in S_1} H^{\overline{\lambda}_m}_{\overline{k}_m}} \\ &= \dot{Z}_0^{\prod_{m<\omega} (\pi H)_*(a_m) \times \prod_{m \in S_0} (\pi H)^{\overline{\sigma}_m}_{\overline{i}_m} \times \prod_{m \in S_1} (\pi H)^{\overline{\lambda}_m}_{\overline{l}_m}} \\ &= \big(\tau_{\varrho'}(\dot{Z}_0)\big)^{\pi H}, \end{aligned}$$

since π swaps any $(\overline{\lambda}_m, \overline{k}_m)$-coordinate with the according $(\overline{\lambda}_m, \overline{l}_m)$-coordinate, and does nothing else.

Hence, whenever $r \in D_\pi$, then $r \Vdash_s \tau_{\varrho}(\dot{Z}) \in \tau_{\varrho}(\dot{Y})$ if and only if $\pi r \Vdash_s \tau_{\varrho'}(\dot{Z}) \in \tau_{\varrho'}(\dot{Y})$. Thus, by our recursive assumption,

$$\begin{aligned} \overline{\pi\tau_{\varrho}(\dot{Y})}^{D_\pi} &= \big\{ \big(\overline{\pi\tau_{\varrho}(\dot{Z})}^{D_\pi}, \pi r \big) \;\big|\; \pi r \in D_\pi\,,\ \dot{Z} \in \operatorname{dom} \dot{Y}\,,\ \pi r \Vdash_s \tau_{\varrho'}(\dot{Z}) \in \tau_{\varrho'}(\dot{Y}) \big\} \\ &= \big\{ \big(\overline{\tau_{\varrho'}(\dot{Z})}^{D_\pi}, r \big) \;\big|\; r \in D_\pi\,,\ \dot{Z} \in \operatorname{dom} \dot{Y}\,,\ r \Vdash_s \tau_{\varrho'}(\dot{Z}) \in \tau_{\varrho'}(\dot{Y}) \big\} \\ &= \overline{\tau_{\varrho'}(\dot{Y})}^{D_\pi}. \end{aligned}$$

□

Hence,

$$Y = \big(\overline{\pi\tau_{\varrho}(\dot{X})}^{D_\pi}\big)^G = \big(\overline{\tau_{\varrho'}(\dot{X})}^{D_\pi}\big)^G = \big(\tau_{\varrho'}(\dot{X})\big)^G =$$
$$= \dot{X}^{\prod_m G_*(a_m) \times \prod_{m \in S_0} G^{\overline{\sigma}_m}_{\overline{i}_m} \times \prod_{m \in S_1} G^{\overline{\lambda}_m}_{\overline{l}_m}}.$$

Hence, by Lemma 2.6.18 above, it follows that

$$(\dot{X}^{\prod_m G_*(a_m) \times \prod_{m \in S_0} G^{\overline{\sigma}_m}_{\overline{i}_m} \times \prod_{m \in S_1} G^{\overline{\lambda}_m}_{\overline{l}_m}}, \alpha) \in f.$$

But $\overline{i}_m < \beta$ for all $m \in S_0$ and $\overline{l}_m < \beta$ for all $m \in S_1$; hence,

$$(\dot{X}^{\prod_m G_*(a_m) \times \prod_{m \in S_0} G^{\overline{\sigma}_m}_{\overline{i}_m} \times \prod_{m \in S_1} G^{\overline{\lambda}_m}_{\overline{l}_m}}, \alpha) \in f^{\beta}.$$

But this contradicts our choice of $\alpha \notin \operatorname{rg} f^{\beta}$.

Thus, in either case our assumption of a surjective function $f : \wp^N(\kappa_\eta) \to \alpha_\eta$ in N has lead to a contradiction, and it follows that indeed, $\theta^N(\kappa_\eta) \leq \alpha_\eta$.

Recall that we have assumed throughout our proof that $\kappa_{\eta+1} > \kappa_\eta^+$. In the next Chapter 6.3, we will treat the case that $\kappa_{\eta+1} = \kappa_\eta^+$, and discuss where the arguments from Chapter 2.6.2 have to be modified.

2.6.3 $\forall\, \eta\ \big(\kappa_{\eta+1} = \kappa_\eta^+ \longrightarrow \theta^N(\kappa_\eta) \leq \alpha_\eta \big)$

If $\kappa_{\eta+1} = \kappa_\eta^+$, we need the notion of an *η-almost good pair* (cf. Definition 2.5.7 and Proposition 2.5.8): For any $X \in N$, $X \subseteq \kappa_\eta$, there exists an *η-almost good pair* $((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$ such that $X \in V[\prod_m G_*(a_m) \times \prod_m G_{\overline{i}_m}^{\overline{\sigma}_m} \times G^{\eta+1}]$.

Throughout this Chapter 2.6.3, we assume that

$$\boldsymbol{\kappa_{\eta+1} = \kappa_\eta^+}.$$

As before in Chapter 2.6.2, we assume towards a contradiction that there was a surjective function $f : \wp^N(\kappa_\eta) \to \alpha_\eta$ in N with $\pi \overline{f}^{D_\pi} = \overline{f}^{D_\pi}$ for all $\pi \in A$ with $[\pi]$ contained in the intersection

$$\bigcap_{m<\omega} Fix(\eta_m, i_m) \cap \bigcap_{m<\omega} H_{k_m}^{\lambda_m} \qquad (I_{\dot f}).$$

We take $\widetilde{\beta}$ *large enough for* $(I_{\dot f})$ as in Chapter 2.6.2, Definition 2.6.2, and set $\beta := \widetilde{\beta} + \kappa_\eta^+$ (addition of ordinals).

Now, we can adapt our definition of f^β to η-almost good pairs, and obtain:

$$f^\beta := \Big\{ (X, \alpha) \in f \ \Big| \ \exists\, ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}) \ \ \eta\text{-almost good pair} : \ (\forall m \ \overline{i}_m < \beta) \wedge$$
$$\exists\, \dot X \in \mathrm{Name}\big((\overline{P}^\eta)^\omega \times \prod_m P^{\overline{\sigma}_m} \times P^{\eta+1}\big) \ X = \dot X^{\prod_m G_*(a_m) \times \prod_m G_{\overline{i}_m}^{\overline{\sigma}_m} \times G^{\eta+1}} \Big\}.$$

First, we assume towards a contradiction that $\boldsymbol{f^\beta : \mathrm{dom}\, f^\beta \to \alpha_\eta}$ **is surjective**.

A) Constructing $\widetilde{\mathbb{P}}^\beta \restriction (\eta + 1)$.

As before, we only treat the case that

$$\boldsymbol{\beta < \alpha_{\widetilde{\eta}}} \quad \textbf{or} \quad \boldsymbol{Lim \cap (\eta, \gamma) \neq \emptyset},$$

where $\widetilde{\eta} := \sup\{\sigma < \eta \mid \sigma \in Lim\}$, i.e. we presume that there exist (σ, i) with $\sigma \in Lim$ and $i \geq \beta$ or $\sigma > \eta$.

This time, we construct a forcing notion $\widetilde{\mathbb{P}}^\beta \restriction (\eta+1)$ instead of $\mathbb{P}^\beta \restriction (\eta+1)$; which should be like $\mathbb{P}^\beta \restriction (\eta + 1)$, except that firstly, we use restrictions $p_* \restriction \kappa_{\eta+1}^2$ instead of $p_* \restriction \kappa_\eta^2$, and secondly, we include $P^{\eta+1}$.

Definition 2.6.20. For $p \in \mathbb{P}$, let

$$\widetilde{p}^\beta \restriction (\eta+1) := \big(p_* \restriction \kappa_{\eta+1}^2, (p_i^\sigma, a_i^\sigma)_{\sigma\leq\eta, i<\beta}, (p^\sigma \restriction (\beta \times \mathrm{dom}_y\, p^\sigma))_{\sigma\leq\eta}, p^{\eta+1}, X_p \big),$$

and denote by $\widetilde{\mathbb{P}}^\beta \restriction (\eta+1)$ the collection of all $\widetilde{p}^\beta \restriction (\eta+1)$ such that $p \in \overline{\mathbb{P}}$ (i.e. $p \in \mathbb{P}$ with $|\{(\sigma,i) \in \mathrm{supp}\, p_0 \mid \sigma > \eta \vee i \geq \beta\}| = \aleph_0$); together with the maximal element $\widetilde{\mathbb{1}}_{\eta+1}^\beta$. The order relation $\widetilde{\leq}_{\eta+1}^\beta$ is defined as in Definition 2.6.4.

Like in Chapter 2.6.2 A), one can write down a projection of forcing posets $\widetilde{\rho}_{\eta+1}^\beta : \overline{\mathbb{P}} \to \widetilde{\mathbb{P}}^\beta \restriction (\eta+1)$ and conclude that

$$\widetilde{G}^\beta \restriction (\eta+1) := \big\{ p \in \widetilde{\mathbb{P}}^\beta \restriction (\eta+1) \ \big| \ \exists\, q \in G \cap \overline{\mathbb{P}} \ \ \widetilde{q}^\beta \restriction (\eta+1) \ \widetilde{\leq}_{\eta+1}^\beta\ p \big\}$$

is a V-generic filter on $\widetilde{\mathbb{P}}^\beta \restriction (\eta+1)$.

B) Capturing f^β.

We define a forcing notion $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$, which will be obtained from $\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1)$ by using $\widetilde{X}_p$ instead of X_p (cf. Chapter 2.6.2 B)), and including for $\eta_m \in \mathrm{Lim}$, $\eta_m > \eta$ the verticals $p^{\eta_m}_{i_m} \upharpoonright \kappa_{\eta+1}$, and also $a^{\eta_m}_{i_m} \cap \kappa_{\eta+1}$, the according linking ordinals up to $\kappa_{\eta+1}$.

The restriction $(\widetilde{p}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ for $p \in \mathbb{P}$ is defined as follows:

$$(\widetilde{p}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} := \big(p_* \upharpoonright \kappa^2_{\eta+1}\ ,\ (p^\sigma_i, a^\sigma_i)_{\sigma\leq\eta, i<\beta}\ ,\ (p^{\eta_m}_{i_m} \upharpoonright \kappa_{\eta+1}, a^{\eta_m}_{i_m} \cap \kappa_{\eta+1})_{m<\omega\,,\,\eta_m>\eta},$$
$$(p^\sigma \upharpoonright (\beta \times \mathrm{dom}_y\, p^\sigma))_{\sigma\leq\eta}\ ,\ \widetilde{X}_p\ ,\ p^{\eta+1} \big).$$

Roughly speaking, the difference with the restrictions $(p^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ introduced in Chapter 2.6.2 B) is, that we are now reaching up to $\kappa_{\eta+1} = \kappa^+_\eta$ instead of κ_η.

We denote by $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ the collection of all $(\widetilde{p}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ for $p \in \overline{\mathbb{P}}$ together with the maximal element $(\widetilde{\mathbb{1}}^\beta_{\eta+1})^{(\eta_m, i_m)_{m<\omega}}$. The order relation "$\leq$" is defined like in Definition 2.6.8.

Finally, we include the verticals $P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$ for $\eta_m > \eta+1$, which gives the product

$$(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m}).$$

Let

$$(\widetilde{G}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$$

denote the collection of all

$$\big((\widetilde{p}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}, (p^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})_{m<\omega})\big)$$

such that there exists $q \in G \cap \overline{\mathbb{P}}$ with $(\widetilde{q}^\beta \upharpoonright (\eta+1))_{(\eta_m, i_m)_{m<\omega}} \leq (\widetilde{p}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ and $q^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m}) \supseteq p^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$ for all $m < \omega$.

As in Proposition 2.6.9, one can construct a projection of forcing posets

$$(\widetilde{\rho}^\beta)^{(\eta_m, i_m)_{m<\omega}} : \overline{\mathbb{P}} \to (\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m}),$$

and it follows that $(\widetilde{G}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$ is a V-generic filter on $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$.

Like in Chapter 2.6.2 B), we want to define a map $(f^\beta)'$ contained in $V[\widetilde{G}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})]$, and then use an isomorphism argument to show that $f^\beta = (f^\beta)'$.

Before that, we have to modify our transformations of names τ_ϱ (where ϱ is an η-good pair), and define transformations $\widetilde{\tau}_\varrho$(where ϱ is an η-almost good pair) with

$$\widetilde{\tau}_\varrho : \mathrm{Name}((\overline{P}^{\eta+1})^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m} \times P^{\eta+1}) \to \mathrm{Name}(\mathbb{P})$$

as follows (cf. Definition 2.6.7):

Definition 2.6.21. For an η-almost good pair $\varrho = \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$, define recursively for $\dot{Y} \in \mathrm{Name}((\overline{P}^{\eta+1})^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m} \times P^{\eta+1})$:

$$\widetilde{\tau}_\varrho(\dot{Y}) := \big\{ (\widetilde{\tau}_\varrho(\dot{Z}), q) \ \big| \ q \in \mathbb{P} \ , \ \exists \big(\dot{Z} \, , \, \big((p_*(a_m))_{m<\omega} \, , \, (p^{\overline{\sigma}_m}_{\overline{i}_m})_{m<\omega} \, , \, p^{\eta+1} \big) \big) \in \dot{Y} \ :$$

$$\forall m \ \big(q_*(a_m) \supseteq p_*(a_m) \ , \ q^{\overline{\sigma}_m}_{\overline{i}_m} \supseteq p^{\overline{\sigma}_m}_{\overline{i}_m} \big) \ , \ q^{\eta+1} \supseteq p^{\eta+1} \big\}.$$

Then $\dot{Y}^{\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m} \times G^{\eta+1}} = (\widetilde{\tau}_\varrho(\dot{Y}))^G$ holds for all $\dot{Y} \in \mathrm{Name}((\overline{P}^{\eta+1})^\omega \times \prod_{m<\omega} P^{\overline{\sigma}_m} \times P^{\eta+1})$.

Definition 2.6.22. Let $(f^\beta)'$ denote the set of all (X, α) for which there exists an η-*almost good pair* $\varrho = ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$ with $\overline{i}_m < \beta$ for all $m < \omega$, such that

$$X = \dot{X}^{\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m} \times G^{\eta+1}},$$

and there is a condition $p \in \mathbb{P}$ with the following properties:

- $|\{(\sigma, i) \in \operatorname{supp} p_0 \mid \sigma > \eta \text{ or } i \geq \beta\}| = \aleph_0$,
- $p \Vdash_s \big(\widetilde{\tau}_\varrho(\dot{X}), \alpha\big) \in \dot{f}$,
- $\big((\widetilde{p}^\beta \upharpoonright (\eta{+}1))^{(\eta_m, i_m)_{m<\omega}}, (p^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m}))_{m<\omega} \big) \in (\widetilde{G}^\beta \upharpoonright (\eta{+}1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$,
- $\forall\, \eta_m \in \mathrm{Lim} \ : \ (\eta_m, i_m) \in \operatorname{supp} p_0$ with $a^{\eta_m}_{i_m} = g^{\eta_m}_{i_m}$.

Then $(f^\beta)' \in V[(\widetilde{G}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})]$.

Proposition 2.6.23. $f^\beta = (f^\beta)'$.

Proof. We briefly outline where the isomorphism argument form Proposition 2.6.11 has to be modified. We start with $(X, \alpha) \in (f^\beta)' \setminus f^\beta$, $X = \dot{X}^{\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m} \times G^{\eta+1}}$, for an η-almost good pair $\varrho = ((a_m)_{m<\omega}, ((\overline{\sigma}_m, \overline{i}_m)_{m<\omega}))$. Take p as in the definition of $(f^\beta)'$ with $p \Vdash_s (\widetilde{\tau}_\varrho(\dot{X}), \alpha) \in \dot{f}$, and $p' \in G$ with $p' \Vdash_s (\widetilde{\tau}_\varrho(\dot{X}), \alpha) \notin \dot{f}$.

The first step is the construction of extensions $\overline{p} \leq p$, $\overline{p}' \leq p'$ such that $\overline{p}$ and $\overline{p}'$ have "the same shape", agree on $\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1)$; and $\overline{p}^{\eta_m}_{i_m} = (\overline{p}')^{\eta_m}_{i_m}$ holds for all $m < \omega$, and $\overline{a}^{\eta_m}_{i_m} = (\overline{a}')^{\eta_m}_{i_m}$ holds for all $m < \omega$ with $\eta_m \in \mathit{Lim}$.

We proceed as in Proposition 2.6.11, with the following modifications:

- The construction of $\overline{p}_*$, $\overline{p}'_*$ that we used in the Proposition 2.6.11 for intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$, has to be applied to all intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\eta+1}$ now, since we need $\overline{p}_*$ and $\overline{p}'_*$ agree on $\kappa^2_{\eta+1}$.
- Analogously, the construction of $\overline{p}^\sigma_i$, $(\overline{p}')^\sigma_i$ for $\sigma \in \mathit{Lim}$, $i < \alpha_\sigma$ for intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$, has to be applied to all intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\eta+1}$ now, in the case that $\sigma > \eta{+}1$.

- Additionally, we have to make sure that $\overline{p}^{\eta+1} = (\overline{p}')^{\eta+1}$.

The next step is the construction of an isomorphism π such that $\pi\overline{p} = \overline{p}'$, $\pi\overline{f}^{D_\pi} = \overline{f}^{D_\pi}$, and $\pi\overline{\widetilde{\tau}_\varrho(\dot{X})}^{D_\pi} = \overline{\widetilde{\tau}_\varrho(\dot{X})}^{D_\pi}$. Again, we take for π a *standard isomorphism for* $\pi\overline{p} = \overline{p}'$; but this time, we set $G_{\pi_0}(\nu, j) := F_{\pi_0}(\nu, j)$ for all intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_{\eta+1}$ (instead of only intervals $[\kappa_{\nu,j}, \kappa_{\nu,j+1}) \subseteq \kappa_\eta$), and $G_{\pi_0}(\nu, j) = id$ for all $\kappa_{\nu,j} \geq \kappa_{\eta+1}$ (instead of all $\kappa_{\nu,j} \geq \kappa_\eta$). Then as before, it follows that $\pi \in \bigcap_m Fix(\eta_m, i_m) \cap \bigcap_m H^{\lambda_m}_{k_m}$.

For verifying $\pi\overline{\widetilde{\tau}_\varrho(\dot{X})}^{D_\pi} = \overline{\widetilde{\tau}_\varrho(\dot{X})}^{D_\pi}$, we now additionally have to make sure that π is the identity on $P^{\eta+1}$. But since we have arranged $\overline{p}^{\eta+1} = (\overline{p}')^{\eta+1}$, this is clear by construction of π_1.

Now, it follows from $\overline{p} \Vdash_s \big(\widetilde{\tau}_\varrho(\dot{X}), \alpha\big) \in \dot{f}$ that $\pi\overline{p} \Vdash_s \big(\pi\overline{\widetilde{\tau}_\varrho(\dot{X})}^{D_\pi}, \alpha\big) \in \pi\overline{f}^{D_\pi}$. Hence, $\overline{p}' \Vdash_s \big(\overline{\widetilde{\tau}_\varrho(\dot{X})}^{D_\pi}, \alpha\big) \in \overline{f}^{D_\pi}$, which is a contradiction towards $\overline{p}' \Vdash_s \big(\widetilde{\tau}_\varrho(\dot{X}), \alpha\big) \notin \dot{f}$. □

Thus, $f^\beta = (f^\beta)' \in V[(\widetilde{G}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})]$ as desired.

C) $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$ preserves cardinals $\geq \alpha_\eta$.

Now, we will show that cardinals $\geq \alpha_\eta$ are absolute between V and $V[(\widetilde{G}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})]$.

As in Chapter 2.6.2 C), we are using that GCH holds in our ground model V, and when we argue that a particular forcing notion preserves cardinals, we mean that it preserves cardinals under GCH, if not stated differently.

Lemma 2.6.24. *If* $|\beta|^+ < \alpha_\eta$, *then* $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$ *preserves cardinals* $\geq \alpha_\eta$.

Proof. We closely follow the proof of Lemma 2.6.12 and Corollary 2.6.13.

The forcing notion $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ is the set of all

$$\big(p_* \upharpoonright \kappa^2_{\eta+1}\ ,\ (p^\sigma_i, a^\sigma_i)_{\sigma\leq\eta, i<\beta}\ ,\ (p^{\eta_m}_{i_m} \upharpoonright \kappa_{\eta+1}, a^{\eta_m}_{i_m} \cap \kappa_{\eta+1})_{m<\omega, \eta_m>\eta},$$

$$(p^\sigma \upharpoonright (\beta \times \operatorname{dom}_y p^\sigma))_{\sigma\leq\eta}\ ,\ \widetilde{X}_p\ ,\ p^{\eta+1}\big),$$

where $p \in \mathbb{P}$ with $|\{(\sigma, i) \in \operatorname{supp} p_0 \mid \sigma > \eta \vee i \geq \beta\}| = \aleph_0$.

Since $\kappa_{\eta+1} = \kappa^+_\eta$, it follows that the $p_* \upharpoonright \kappa^2_{\eta+1}$, as well as $(p^{\eta_m}_{i_m} \upharpoonright \kappa_{\eta+1}, a^{\eta_m}_{i_m} \cap \kappa_{\eta+1})$ for $m < \omega$ are bounded below $\kappa_{\eta+1}$; which gives only $\leq (\kappa_{\eta+1} \cdot 2^{\kappa_\eta})^\omega = \kappa_{\eta+1} = \kappa^+_\eta \leq |\beta|$-many possibilities.
Since $\widetilde{X}_p \subseteq \kappa_\eta$, there are only $\leq \kappa^+_\eta \leq |\beta|$-many possibilities for $\widetilde{X}_p$, as well. Regarding $(p^\sigma_i, a^\sigma_i)_{\sigma\leq\eta, i<\beta}$ and $(p^\sigma \upharpoonright (\operatorname{dom}_x p^\sigma \times \beta))_{\sigma\leq\eta}$, it follows as in Lemma 2.6.12 that there are only $\leq |\beta|^+ \cdot \kappa^+_\eta = |\beta|^+$-many possibilities.

We denote by $\big((\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\big)'$ the forcing notion that is obtained from $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ by excluding $P^{\eta+1}$. Then $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ is isomorphic to the product $\big((\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\big)' \times P^{\eta+1}$.

By what we have just argued, it follows that the forcing notion $\big((\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\big)'$ has cardinality $\leq |\beta|^+$; and the remaining product $P^{\eta+1} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$ preserves all cardinals by similar arguments as in Proposition 2.6.14. Hence, it follows that $\big((\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\big)' \times P^{\eta+1} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$ preserves all cardinals $\geq |\beta|^{++}$.

□

Proposition 2.6.25. *The forcing* $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$ *preserves cardinals* $\geq \alpha_\eta$.

Proof. We only have to treat the case that $\alpha_\eta = |\beta|^+$. Then $\operatorname{cf}|\beta| > \omega$, and GCH gives $|\beta|^{\aleph_0} = |\beta|$. The proof is similar as for Proposition 2.6.14: We distinguish several cases, and construct $\big((\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\big)''$ from $(\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ by splitting up $P^{\eta+1}$, and also one or two factors $P^\sigma \upharpoonright (\beta \times [\overline{\kappa_\sigma}, \kappa_\sigma))$ for $\sigma \in Succ$ with $\sigma = \eta$, or $\sigma = \overline{\eta}$ in the case that η is a successor ordinal with $\eta = \overline{\eta} + 1$. Then as in the proof of Proposition 2.6.14, it follows that $\big((\widetilde{\mathbb{P}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}\big)''$ has cardinality $\leq |\beta| < \alpha_\eta$, and the product of the remaining $P^\sigma \upharpoonright (\beta \times [\overline{\kappa_\sigma}, \kappa_\sigma))$, $P^{\eta+1}$ and $\prod_m P^{\eta_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})$ preserves all cardinals. □

D) A set $\widetilde{\wp}(\kappa_\eta) \supseteq \operatorname{dom} f^\beta$ with an injection $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow |\beta|^{\aleph_0}$.

For an η-almost good pair $\varrho = ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$, it follows that $\prod_m G_*(a_m) \times \prod_m G^{\eta_m}_{i_m} \times G^{\eta+1}$ is a V-generic filter on $\prod_m (\overline{P}^\eta)^\omega \times \prod_m P^{\overline{\sigma}_m} \times P^{\eta+1}$, and

$$\big(2^\alpha\big)^{\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m} \times G^{\eta+1}} = (\alpha^+)^V$$

holds for all $\alpha \leq \kappa_\eta$ by the same proof as for Lemma 2.3.2, since $P^{\eta+1}$ ist $\leq \kappa_\eta$-closed. Thus, there is an injection $\chi : \wp(\kappa_\eta) \hookrightarrow (\kappa_\eta^+)^V$ in $V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m} \times G^{\eta+1}]$.

Let $\widetilde{M_\beta}$ denote the set of all η-almost good pairs $\varrho = ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$ in V with the property that $\overline{i}_m < \beta$ for all $m < \omega$. Then $\widetilde{M_\beta}$ has cardinality $\leq \kappa_{\eta+1} \cdot |\eta|^{\aleph_0} \cdot |\beta|^{\aleph_0} = |\beta|^{\aleph_0}$.

Moreover, *dom* f^β is a subset of $\widetilde{\wp}(\kappa_\eta) :=$

$$\bigcup \Big\{ \wp(\kappa_\eta) \cap V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m} \times G^{\eta+1}] \;\Big|\; \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big) \in \widetilde{M_\beta} \Big\}.$$

Now, we can proceed as in Chapter 2.6.2 D) and construct in $V[(\widetilde{G}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta+1}, \kappa_{\eta_m})]$ an injection $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow |\beta|^V$ in the case that $\alpha_\eta = (|\beta|^+)^V$ and an injection $\iota : \widetilde{\wp}(\kappa_\eta) \hookrightarrow (|\beta|^+)^V$ in the case that $\alpha_\eta > (|\beta|^+)^V$. Together with Chapter 2.6.3 B) and 2.6.3 C), this gives the desired contradiction.

Thus, we have shown that the map $f^\beta : \operatorname{dom} f^\beta \to \alpha_\eta$ must not be surjective.

E) We use an isomorphism argument and obtain a contradiction.

The arguments for this part are the very same as in the case that $\kappa_{\eta+1} > \kappa_\eta^+$, except that we are now working with *η-almost good pairs* $\varrho = ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$ instead of η-good pairs.

Thus, also in the case that $\kappa_{\eta+1} = \kappa_\eta^+$, it follows that $\theta^N(\kappa_\eta) = \alpha_\eta$.

2.6.4 The Remaining Cardinals in the "Gaps" $\lambda \in (\kappa_\eta, \kappa_{\eta+1})$

So far, we have shown that $\theta^N(\kappa_\eta) = \alpha_\eta$ holds for all $0 < \eta < \gamma$. Recall that in the very beginning (see Chapter 2.2), we started by "thinning out" our sequence $(\kappa_\eta \mid 0 < \eta < \gamma)$ and assuming w.l.o.g. that $(\alpha_\eta \mid 0 < \eta < \gamma)$ is strictly increasing. Thus, it remains make sure that for all cardinals $\lambda \in (\kappa_\eta, \kappa_{\eta+1})$ in the "gaps", $\theta^N(\lambda)$ gets the smallest possible value, i.e. $\theta^N(\lambda) = \max\{\alpha_\eta, \lambda^{++}\}$. This will be our aim for this Chapter 2.6.4.
After that, in Chapter 2.6.5, we make sure that also for all cardinals $\lambda \geq \kappa_\gamma$, the value $\theta^N(\lambda)$ will be the smallest possible.

We consider a cardinal λ in a "gap" $\lambda \in (\kappa_\eta, \kappa_{\eta+1})$ (then $\kappa_{\eta+1} > \kappa_\eta^+$), and set $\alpha(\lambda) := \max\{\lambda^{++}, \alpha_\eta\}$. Then $\theta^N(\lambda) \geq \alpha(\lambda)$ is clear, and it remains to make sure that there is no surjective function $f : \wp^N(\lambda) \to \alpha(\lambda)$ in N.

First, we want to describe the intermediate generic extensions where the λ-subsets $X \in \wp^N(\lambda)$ are located.

Let $\lambda \in [\kappa_{\eta,\bar{\jmath}}, \kappa_{\eta,\bar{\jmath}+1})$, where $\bar{\jmath} < cf\,\kappa_{\eta+1}$ in the case that $\eta+1 \in Lim$, and $\bar{\jmath} = 0$ with $\lambda \in (\kappa_{\eta,0}, \kappa_{\eta,1}) = (\kappa_\eta, \kappa_{\eta+1})$ in the case that $\eta + 1 \in Succ$.

We will modify our definition of an η-good pair and obtain the notion of an *η-good pair for λ*, which will be used to describe the intermediate generic extensions where the sets $X \in \wp^N(\lambda)$ are located:

Definition 2.6.26. We say that $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ is an *η-good pair for* λ, if the following hold:

- $(a_m \mid m < \omega)$ is a sequence of pairwise disjoint subsets of $\kappa_{\eta,\bar{\jmath}}$, such that for all $\kappa_{\overline{\nu},\bar{l}} < \kappa_{\eta,\bar{\jmath}}$, it follows that $|a_m \cap [\kappa_{\overline{\nu},\bar{l}}, \kappa_{\overline{\nu},\bar{l}+1})| = 1$,
- for all $m < \omega$, we have $\overline{\sigma}_m \in Succ$ with $\overline{\sigma}_m \leq \eta$, and $\overline{i}_m < \alpha_{\overline{\sigma}_m}$,
- if $m \neq m'$, then $(\overline{\sigma}_m, \overline{i}_m) \neq (\overline{\sigma}_{m'}, \overline{i}_{m'})$.

Similarly as in Proposition 2.5.6, we obtain:

Proposition 2.6.27. *For every $X \in N$, $X \subseteq \lambda$, there is an η-*good pair for λ*, denoted by $\varrho = \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$, such that $X \in V[\prod_{m<\omega} G_*(a_m) \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m}]$.*

Proof. As in Proposition 2.5.6, it follows by the *Approximation Lemma* 2.5.2 that any $X \in N$, $X \subseteq \lambda$ is contained in a generic extension

$$X \in V[\prod_{m<\omega} G_*(g^{\sigma_m}_{i_m}) \times \prod_{m<\omega} G^{\overline{\sigma}_m}_{\overline{i}_m}],$$

where $((\sigma_m, i_m) \mid m < \omega)$ and $((\overline{\sigma}_m, \overline{i}_m) \mid m < \omega)$ are sequences of pairwise distinct pairs with $\sigma_m \in Lim$, $i_m < \alpha_{\sigma_m}$, and $\overline{\sigma}_m \in Succ$, $\overline{i}_m < \alpha_{\overline{\sigma}_m}$ for all $m < \omega$.

The forcing $\prod_{m<\omega} P^{\sigma_m} \times \prod_{m<\omega} P^{\overline{\sigma}_m}$ can be factored as

$$\Big(\prod_{m<\omega} P^{\sigma_m} \upharpoonright \kappa_{\eta,\bar{\jmath}} \times \prod_{\overline{\sigma}_m \leq \eta} P^{\overline{\sigma}_m}\Big) \times \Big(\prod_{m<\omega} P^{\sigma_m} \upharpoonright [\kappa_{\eta,\bar{\jmath}}, \kappa_{\eta_m}) \times \prod_{\overline{\sigma}_m > \eta} P^{\overline{\sigma}_m}\Big).$$

In the case that $\lambda \in (\kappa_{\eta,\overline{\jmath}}, \kappa_{\eta,\overline{\jmath}+1})$, it follows that the "lower part" has cardinality $\leq \kappa_{\eta,\overline{\jmath}}^+ \leq \lambda$, and the "upper part" is $\leq \lambda$-closed.
If $\lambda = \kappa_{\eta,\overline{\jmath}}$, then firstly, the "lower part" has cardinality $\leq \kappa_{\eta,\overline{\jmath}}^+ = \lambda^+$, and secondly, it follows that $\overline{\jmath} > 0$ and $\kappa_{\eta+1} \in Lim$, so $\kappa_{\eta,\overline{\jmath}+1} \geq \kappa_{\eta,\overline{\jmath}}^{++}$ by construction. Hence, the "upper part" is $\leq \lambda^+$-closed.
In either case, we obtain

$$X \in V[\prod_{m<\omega} G_*(g_{i_m}^{\sigma_m} \cap \kappa_{\eta,\overline{\jmath}}) \times \prod_{\overline{\sigma}_m \leq \eta} G_{\overline{i}_m}^{\overline{\sigma}_m}].$$

With $a_m := g_{i_m}^{\sigma_m} \cap \kappa_{\eta,\overline{\jmath}}$ for all $m < \omega$, it follows by the *independence property* that $\big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big)$ is an *η-good pair for λ* with

$$X \in V[\prod_{m<\omega} G_*(a_m) \times \prod_{\overline{\sigma}_m \leq \eta} G_{\overline{i}_m}^{\overline{\sigma}_m}].$$

□

(Note that $\prod_m G_*(a_m) \times \prod_{\overline{\sigma}_m \leq \eta} G_{\overline{i}_m}^{\overline{\sigma}_m}$ is a V-generic filter on the forcing $(\overline{P}^{\eta+1} \restriction \kappa_{\eta,\overline{\jmath}})^\omega \times \prod_{\overline{\sigma}_m \leq \eta} P^{\overline{\sigma}_m})$.

As before, we assume towards a contradiction that there was a surjective function $f : \wp^N(\lambda) \to \alpha(\lambda)$ in N, where $\pi \overline{\dot{f}}^{D_\pi} = \overline{\dot{f}}^{D_\pi}$ holds for all $\pi \in A$ with $[\pi]$ contained in the intersection

$$\bigcap_{m<\omega} Fix(\eta_m, i_m) \cap \bigcap_{m<\omega} H_{k_m}^{\lambda_m} \qquad (I_{\dot{f}}).$$

We take $\widetilde{\beta}$ *large enough for the intersection* $(I_{\dot{f}})$ as in Chapter 2.6.2, Definition 2.6.2, and set $\beta := \widetilde{\beta} + \kappa_\eta^+$ (addition of ordinals).

Let

$$f^\beta := \Big\{ (X, \alpha) \in f \;\Big|\; \exists \big((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}\big) \text{ η-good pair for λ} \;:\; (\forall m \;\; \overline{i}_m < \beta) \wedge$$
$$\exists \dot{X} \in \mathrm{Name}\big((\overline{P}^{\eta+1} \restriction \kappa_{\eta,\overline{\jmath}})^\omega \times \prod_{\overline{\sigma}_m \leq \eta} P^{\overline{\sigma}_m}\big) \;\; X = \dot{X}^{\prod_m G_*(a_m) \times \prod_m G_{\overline{i}_m}^{\overline{\sigma}_m}} \Big\}.$$

First, we assume towards a contradiction that $\boldsymbol{f^\beta : \mathrm{dom}\, f^\beta \to \alpha(\lambda)}$ **is surjective**.

A) Constructing $\widetilde{\widetilde{\mathbb{P}}}^\beta \restriction (\eta + 1)$.

We proceed as in Chapter 2.6.2 A) and 2.6.3 A), except that we have to use $\mathbb{P}_* \restriction \kappa_{\eta,\overline{\jmath}}^2$ instead of $\mathbb{P}_* \restriction \kappa_\eta^2$, and do *not* include $P^{\eta+1}$:
For $p \in \mathbb{P}$, we set

$$\widetilde{\widetilde{p}}^\beta \restriction (\eta + 1) := \big(p_* \restriction \kappa_{\eta,\overline{\jmath}}^2 \,,\, (p_i^\sigma, a_i^\sigma)_{\sigma \leq \eta, i < \beta} \,,\, (p^\sigma \restriction (\beta \times \mathrm{dom}_y\, p^\sigma))_{\sigma \leq \eta} \,,\, X_p \big),$$

and denote by $\widetilde{\widetilde{\mathbb{P}}}^\beta \restriction (\eta + 1)$ the collection of all $\widetilde{\widetilde{p}}^\beta \restriction (\eta + 1)$, where $p \in \overline{\mathbb{P}}$ (i.e. $p \in \mathbb{P}$ with $|\{(\sigma, i) \in \mathrm{supp}\, p_0 \mid \sigma > \eta \vee i \geq \beta\}| = \aleph_0$); together with the maximal element $\widetilde{\widetilde{\mathbb{1}}}_{\eta+1}^\beta$, and the order relation $\widetilde{\widetilde{\le}}^{\beta}_{\eta+1}$ defined similarly as in Definition 2.6.4.

We denote by $\widetilde{\widetilde{G}}^{\beta} \upharpoonright (\eta+1)$ the set of all $p \in \widetilde{\widetilde{\mathbb{P}}}^{\beta} \upharpoonright (\eta+1)$ such that there exists $q \in G \cap \overline{\mathbb{P}}$ with $\widetilde{\widetilde{q}}^{\beta} \upharpoonright (\eta+1) \,\widetilde{\widetilde{\le}}^{\beta}_{\eta+1} p$. Then as in Chapter 2.6.2 A), Proposition 2.6.6, it follows that $\widetilde{\widetilde{G}}^{\beta} \upharpoonright (\eta+1)$ is a V-generic filter on $\widetilde{\widetilde{\mathbb{P}}}^{\beta} \upharpoonright (\eta+1)$.

B) Capturing f^{β}.

For $p \in \mathbb{P}$, the restriction $(\widetilde{\widetilde{p}}^{\beta} \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ is defined as follows:

$$(\widetilde{\widetilde{p}}^{\beta} \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} := \big(p_* \upharpoonright \kappa^2_{\eta,\bar{\jmath}} \,,\, (p_i^{\sigma}, a_i^{\sigma})_{\sigma \le \eta, i<\beta} \,,\, (p^{\eta_m}_{i_m} \upharpoonright \kappa_{\eta,\bar{\jmath}}, a^{\eta_m}_{i_m} \cap \kappa_{\eta,\bar{\jmath}})_{m<\omega\,,\,\eta_m>\eta} \,,$$

$$(p^{\sigma} \upharpoonright (\beta \times \mathrm{dom}_y\, p^{\sigma}))_{\sigma \le \eta} \,,\, \widetilde{X}_p \big).$$

We define $(\widetilde{\widetilde{\mathbb{P}}}^{\beta} \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ and $(\widetilde{\widetilde{G}}^{\beta} \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}}$ as in Chapter 2.6.2 B) and 2.6.3 B). Then

$$(\widetilde{\widetilde{G}}^{\beta} \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta,\bar{\jmath}}, \kappa_{\eta_m})$$

is a V-generic filter on

$$(\widetilde{\widetilde{\mathbb{P}}}^{\beta} \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_{\eta,\bar{\jmath}}, \kappa_{\eta_m}).$$

The construction of $(f^{\beta})'$ as well as the proof of $f^{\beta} = (f^{\beta})'$ are as in Chapter 2.6.2 B) and 2.6.3 B); except that this time, the isomorphism π from the proof of Proposition 2.6.11 has to be the identity on $\mathbb{P}_* \upharpoonright \kappa^2_{\eta,\bar{\jmath}}$ (not only on $\mathbb{P}_* \upharpoonright \kappa^2_{\eta}$). This can be achieved by the following modifications: Firstly, we demand that p_* and p'_* cohere on $\mathbb{P}_* \upharpoonright \kappa^2_{\eta,\bar{\jmath}}$ (not only $\mathbb{P}_* \upharpoonright \kappa^2_{\eta}$); secondly, we arrange $\overline{p}_* \upharpoonright \kappa^2_{\eta,\bar{\jmath}} = \overline{p}'_* \upharpoonright \kappa^2_{\eta,\bar{\jmath}}$ (instead of just $\overline{p}_* \upharpoonright \kappa^2_{\eta} = \overline{p}'_* \upharpoonright \kappa^2_{\eta}$); and thirdly, when constructing the isomorphism π, we set $G_{\pi_0}(\nu, j) := F_{\pi_0}(\nu, j)$ for all $\kappa_{\nu,j} < \kappa_{\eta,\bar{\jmath}}$ now, and $G_{\pi_0}(\nu, j) = id$ whenever $\kappa_{\nu,j} \ge \kappa_{\eta,\bar{\jmath}}$.

It follows that $f^{\beta} = (f^{\beta})' \in V[(\widetilde{\widetilde{G}}^{\beta} \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta,\bar{\jmath}}, \kappa_{\eta_m})]$.

C) $(\widetilde{\widetilde{\mathbb{P}}}^{\beta} \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_{m<\omega} P^{\eta_m} \upharpoonright [\kappa_{\eta,\bar{\jmath}}, \kappa_{\eta_m})$ preserves cardinals $\ge \alpha(\lambda) = \max\{\lambda^{++}, \alpha_{\eta}\}$.

The arguments here are similar as in Chapter 2.6.2 C) and 2.6.3 C), since there are only $\le (2^{\kappa_{\eta,\bar{\jmath}}})^{\aleph_0} = \kappa^+_{\eta,\bar{\jmath}} \le \lambda^+ < \alpha(\lambda)$-many possibilities for $p_* \upharpoonright \kappa^2_{\eta,\bar{\jmath}}$ and $(p^{\eta_m}_{i_m} \upharpoonright \kappa_{\eta,\bar{\jmath}}, a^{\eta_m}_{i_m} \cap \kappa_{\eta,\bar{\jmath}})_{m<\omega}$.

D) A set $\widetilde{\wp}(\lambda) \supseteq \mathrm{dom}\, f^{\beta}$ with an injection $\iota : \widetilde{\wp}(\lambda) \hookrightarrow \lambda^+ \cdot |\beta|^{\aleph_0}$.

We proceed as in Chapter 2.6.2 D) and 2.6.3 D). Whenever $((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$ is an η-good pair for λ, it follows that $\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}$ is a V-generic filter on $(\overline{P}^{\eta+1} \upharpoonright \kappa_{\eta,\bar{\jmath}})^{\omega} \times \prod_m P^{\overline{\sigma}_m}$; and

$$\left(2^{\alpha}\right)^{V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}]} = (\alpha^+)^V$$

holds for all cardinals α by the same proof as in Lemma 2.3.2. Hence, it follows that in $V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}]$, there is an injection $\chi : \wp(\lambda) \hookrightarrow (\lambda^+)^V$.

Let $\widetilde{\widetilde{\widetilde{M}}}_\beta$ be the set of all $\varrho = ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega})$ in V such that ϱ is an η-good pair for λ with the property that $\overline{i}_m < \beta$ for all $m < \omega$. Then $\widetilde{\widetilde{\widetilde{M}}}_\beta$ has cardinality $\leq (\kappa^+_{\eta,\overline{\jmath}})^{\aleph_0} \cdot |\beta|^{\aleph_0} \leq \lambda^+ \cdot |\beta|^{\aleph_0}$.

By construction, it follows that $\operatorname{dom} f^\beta$ is a subset of

$$\widetilde{\wp}(\lambda) := \bigcup \Big\{ \wp(\lambda) \cap V[\prod_m G_*(a_m) \times \prod_m G^{\overline{\sigma}_m}_{\overline{i}_m}] \;\Big|\; ((a_m)_{m<\omega}, (\overline{\sigma}_m, \overline{i}_m)_{m<\omega}) \in \widetilde{\widetilde{\widetilde{M}}}_\beta \Big\}.$$

As in Chapter 2.6.2 D), we can now work in $V[(\widetilde{\widetilde{G}}^\beta \upharpoonright (\eta+1))^{(\eta_m, i_m)_{m<\omega}} \times \prod_m G^{\eta_m}_{i_m} \upharpoonright [\kappa_{\eta,\overline{\jmath}}, \kappa_{\eta_m})]$ and construct there an injection $\iota : \widetilde{\wp}(\lambda) \hookrightarrow (\lambda^+)^V \cdot |\beta|^V$ in the case that $\alpha_\eta = (|\beta|^+)^V$, and an injection $\iota : \widetilde{\wp}(\lambda) \hookrightarrow (\lambda^+)^V \cdot (|\beta|^+)^V$ in the case that $\alpha_\eta > (|\beta|^+)^V$. Together with Chapter 2.6.4 B) and 2.6.4 C), this gives the desired contradiction.

Hence, it follows that there must be $\alpha < \alpha(\lambda)$ with $\alpha \notin \operatorname{rg} f^\beta$.

E) We use an isomorphism argument and obtain a contradiction.

The arguments for this part are the same as in Chapter 2.6.2 E); except that we are working with *η-good pairs for λ* now (instead of η-good pairs).

Thus, we have shown that for all cardinals $\lambda \in (\kappa_\eta, \kappa_{\eta+1})$ in a "gap", the value $\theta^N(\lambda)$ is the smallest possible: $\theta^N(\lambda) = \alpha(\lambda) = \max\{\lambda^{++}, \alpha_\eta\}$.

It remains to consider the cardinals $\lambda \geq \kappa_\gamma := \sup\{\kappa_\eta \mid 0 < \eta < \gamma\}$. We prove that for all $\lambda \geq \kappa_\gamma$, again, $\theta^N(\lambda)$ takes the smallest possible value.

This will be the aim of the next Chapter 2.6.5.

2.6.5 The Cardinals $\lambda \geq \kappa_\gamma := \sup\{\kappa_\eta \mid 0 < \eta < \gamma\}$

Let $\alpha_\gamma := \sup\{\alpha_\eta \mid 0 < \eta < \gamma\}$, and consider a cardinal $\lambda \geq \kappa_\gamma$. We want to show that $\theta^N(\lambda)$ takes the smallest possible value $\alpha(\lambda)$, defined as follows:

- In the case that $\operatorname{cf} \alpha_\gamma = \omega$, we set $\alpha(\lambda) = \max\{\alpha_\gamma^{++}, \lambda^{++}\}$.
- In the case that $\alpha_\gamma = \alpha^+$ for some α with $\operatorname{cf} \alpha = \omega$, we set $\alpha(\lambda) = \max\{\alpha_\gamma^+, \lambda^{++}\}$.
- In other cases, we set $\alpha(\lambda) := \max\{\alpha_\gamma, \lambda^{++}\}$.

Then by our remarks from Chapter 2.2, it follows that indeed, $\theta^N(\lambda) \geq \alpha(\lambda)$ holds for all $\lambda \geq \kappa_\gamma$.

First, we assume that

$$\boldsymbol{\alpha(\lambda) > \alpha_\gamma}.$$

It remains to prove that there is no surjective function $f : \wp^N(\lambda) \to \alpha(\lambda)$ in N.

We start with the following observation (again, we use that $V \vDash \mathsf{GCH}$):

Lemma 2.6.28. *Let $\lambda \geq \kappa_\gamma$ with $\alpha(\lambda) > \alpha_\gamma$. Then $\mathbb{P}$ preserves cardinals $\geq \alpha(\lambda)$.*

Proof. For every $p \in \mathbb{P}$, $p = (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma<\gamma, i<\alpha_\sigma}, (p^\sigma)_{\sigma<\gamma})$, there are

- $\leq \kappa_\gamma^+$-many possibilities for p_*,
- $\leq \alpha_\gamma^{\aleph_0}$-many possibilities for the countable support of $(p_i^\sigma, a_i^\sigma)_{\sigma<\gamma, i<\alpha_\sigma}$,
- $\leq \kappa_\gamma^+$-many possibilities for $(p_i^\sigma, a_i^\sigma)_{\sigma<\gamma, i<\alpha_\sigma}$ when the support is given.

In the case that **γ is a limit ordinal**, it follows by the strict monotonicity of the sequence $(\alpha_\sigma \mid 0 < \sigma < \gamma)$ that $\alpha_\sigma < \alpha_\gamma$ holds for all $0 < \sigma < \gamma$. Hence, for any $\sigma \in Succ$, the forcing notion P^σ has cardinality $\leq \alpha_\sigma^+ \leq \alpha_\gamma$; and it follows by countable support that we have $\leq |\gamma|^{\aleph_0} \cdot \alpha_\gamma^{\aleph_0} = \alpha_\gamma^{\aleph_0}$-many possibilities for $(p^\sigma)_{\sigma<\gamma}$. Hence, the forcing $\mathbb{P}$ has cardinality $\leq \kappa_\gamma^+ \cdot \alpha_\gamma^{\aleph_0} \leq \lambda^+ \cdot \alpha_\gamma^{\aleph_0}$. If $cf\,\alpha_\gamma > \omega$, GCH gives $|\mathbb{P}| \leq \lambda^+ \cdot \alpha_\gamma$, and $\alpha(\lambda) = \max\{\lambda^{++}, \alpha_\gamma^+\}$. Hence, $\mathbb{P}$ preserves cardinals $\geq \alpha(\lambda)$ as desired. If $cf\,\alpha_\gamma = \omega$, then $\alpha(\lambda) = \max\{\lambda^{++}, \alpha_\gamma^{++}\} \geq |\mathbb{P}|^+$; and again, it follows that $\mathbb{P}$ preserves cardinals $\geq \alpha(\lambda)$.
It remains to consider the case that **$\gamma = \overline{\gamma} + 1$ is a successor ordinal**. Then our sequences $(\kappa_\sigma \mid 0 < \sigma < \gamma) = (\kappa_\sigma \mid 0 < \sigma \leq \overline{\gamma})$ and $(\alpha_\sigma \mid 0 < \sigma < \gamma) = (\alpha_\sigma \mid 0 < \sigma \leq \overline{\gamma})$ have a maximal element, and $\kappa_\gamma = \kappa_{\overline{\gamma}}$, $\alpha_\gamma = \alpha_{\overline{\gamma}}$.
If $\overline{\gamma} \in \boldsymbol{Lim}$, i.e. $\kappa_{\overline{\gamma}}$ is a limit cardinal, it follows that for any $\sigma \in Succ$, we have $\sigma < \overline{\gamma}$; hence, $\alpha_\sigma^+ \leq \alpha_{\overline{\gamma}} = \alpha_\gamma$. This gives $|\mathbb{P}| \leq \kappa_\gamma^+ \cdot \alpha_\gamma^{\aleph_0} \leq \lambda^+ \cdot \alpha_\gamma^{\aleph_0}$ as before, and $\alpha(\lambda) \geq |\mathbb{P}|^+$.
If $\overline{\gamma} \in \boldsymbol{Succ}$, i.e. $\kappa_{\overline{\gamma}}$ is a successor cardinal, then $P^{\overline{\gamma}}$ has to be treated separately. We factor $\mathbb{P} \cong \mathbb{P}' \times P^{\overline{\gamma}}$ with $\mathbb{P}' := \{ (p_*, (p_i^\sigma, a_i^\sigma)_{\sigma \leq \overline{\gamma}, i<\alpha_\sigma}, (p^\sigma)_{\sigma<\overline{\gamma}}) \mid p \in \mathbb{P} \}$. Then $P^{\overline{\gamma}}$ preserves cardinals, and $\mathbb{P}'$ has cardinality $\leq (\lambda^+)^V \cdot (\alpha_\gamma^{\aleph_0})^V$ as before (in V, and hence, also in any $P^{\overline{\gamma}}$-generic extension); where $\alpha(\lambda) \geq |\mathbb{P}'|^+$. Hence, the forcing $\mathbb{P} \cong \mathbb{P}' \times P^{\overline{\gamma}}$ preserves cardinals $\geq \alpha(\lambda)$ as desired.

□

Now, we assume towards a contradiction that there was a surjective function $f : \wp^N(\lambda) \to \alpha(\lambda)$ in N.
By the *Approximation Lemma* 2.5.2, it follows that any $X \in N$, $X \subseteq \lambda$, is contained in an intermediate generic extension $V[\prod_{m<\omega} G_{i_m}^{\sigma_m}]$, with a sequence $((\sigma_m, i_m) \mid m < \omega)$ of pairwise distinct pairs in V such that $0 < \sigma_m < \gamma$, $i_m < \alpha_{\sigma_m}$ for all $m < \omega$. Denote by M the collection of these $((\sigma_m, i_m) \mid m < \omega)$. Then $|M| \leq \alpha_\gamma^{\aleph_0}$ in V; and $\alpha_\gamma^{\aleph_0} < \alpha(\lambda)$ as argued before.
The product $\prod_m P^{\sigma_m}$ preserves cardinals and the GCH. Hence, it follows that in any generic extension $V[\prod_m G_{i_m}^{\sigma_m}]$, there is an injection $\chi : \wp(\lambda) \hookrightarrow (\lambda^+)^V$. Now, one can argue as in Chapter 6.2 D), and define in $V[G]$ a set $\widetilde{\wp}(\lambda) \supseteq \wp^N(\lambda)$ with an injection $\iota : \widetilde{\wp}(\lambda) \hookrightarrow (\lambda^+)^V \cdot \alpha_\gamma$, or $\iota : \widetilde{\wp}(\lambda) \hookrightarrow (\lambda^+)^V \cdot (\alpha_\gamma^+)^V$ in the case that $\alpha(\lambda) \geq (\alpha_\gamma^{++})^V$. Together with Lemma 2.6.28, this gives the desired contradiction.

Thus, we have shown that in the case that $\alpha(\lambda) > \alpha_\gamma$, there can not be a surjective function $f : \wp(\lambda) \to \alpha(\lambda)$ in N.

It remains to consider the case that

$$\boldsymbol{\alpha(\lambda) = \alpha_\gamma}.$$

Then $\lambda^+ < \alpha_\gamma$, $\mathrm{cf}\, \alpha_\gamma > \omega$; and if $\alpha_\gamma = \alpha^+$ for some α, then $cf\, \alpha > \omega$.

Assume towards a contradiction that there was a surjective function $f : \wp^N(\lambda) \to \alpha(\lambda)$ in N, $f = \dot{f}^G$ with $\pi \overline{\dot{f}}^{D_\pi} = \overline{\dot{f}}^{D_\pi}$ for all $\pi \in A$ with $[\pi]$ contained in the intersection

$$\bigcap_{m<\omega} Fix(\eta_m, i_m) \cap \bigcap_{m<\omega} H_{k_m}^{\lambda_m} \qquad (I_{\dot{f}}).$$

Similary as before, we take $\widetilde{\beta} < \alpha(\lambda)$ *large enough for the intersection* $(I_{\dot{f}})$, i.e. $\widetilde{\beta} > \lambda^+$ with $\widetilde{\beta} > \sup\{i_m \mid m < \omega\} \cup \sup\{k_m \mid m < \omega\}$ (this is possible, since $cf\, \alpha(\lambda) > \omega$). Let $\beta := \widetilde{\beta} + \kappa_\gamma^+$ (addition of ordinals). Then $\kappa_\gamma^+ \leq \lambda^+ < \alpha(\lambda)$ gives $\lambda^+ < \beta < \alpha(\lambda)$.

By the *Approximation Lemma* 2.5.2, it follows as in Proposition 2.5.6 that any $X \in N$, $X \subseteq \lambda$, is contained in an intermediate generic extension $V[\prod_m G_*(a_m) \times \prod_m G_{\bar{i}_m}^{\overline{\sigma}_m}]$, where $((a_m)_{m<\omega}, (\overline{\sigma}_m, \bar{i}_m)_{m<\omega})$ is a *good pair for* κ_γ, i.e.

- $(a_m \mid m < \omega)$ is a sequence of pairwise disjoint subsets of κ_γ, such that for all $\kappa_{\overline{\nu},\bar{\jmath}} < \kappa_\gamma$ and $m < \omega$, it follows that $|a_m \cap [\kappa_{\overline{\nu},\bar{\jmath}}, \kappa_{\overline{\nu},\bar{\jmath}+1})| = 1$,
- for all $m < \omega$, we have $\overline{\sigma}_m \in Succ$, $0 < \sigma < \gamma$, and $\bar{i}_m < \alpha_{\overline{\sigma}_m}$,
- if $m \neq m'$, then $(\overline{\sigma}_m, \bar{i}_m) \neq (\overline{\sigma}_{m'}, \bar{i}_{m'})$.

As before, let

$$f^\beta := \Big\{ (X, \alpha) \in f \;\Big|\; \exists\, ((a_m)_{m<\omega}, (\overline{\sigma}_m, \bar{i}_m)_{m<\omega}) \text{ good pair for } \kappa_\gamma \;:\; (\forall m \;\; \bar{i}_m < \beta) \wedge$$

$$\wedge\, \exists\, \dot{X} \in \mathrm{Name}\big((\overline{P}^\gamma)^\omega \times \prod_m P^{\overline{\sigma}_m}\big) \;\; X = \dot{X}^{\prod_m G_*(a_m) \times \prod_m G_{\bar{i}_m}^{\overline{\sigma}_m}} \Big\}.$$

First, we assume towards a contradiction that $\boldsymbol{f^\beta : \mathrm{dom}\, f^\beta \to \alpha(\lambda)}$ **is surjective**.

A) + B) Constructing $\mathbb{P}^\beta$ and capturing f^β.

For a condition $p \in \mathbb{P}$, let

$$p^\beta := (\, p_*, (p_i^\sigma, a_i^\sigma)_{\sigma \in \mathrm{Lim}\,,\, i<\beta}, (p^\sigma \upharpoonright (\beta \times \mathrm{dom}_x\, p^\sigma))_{\sigma \in \mathrm{Succ}}, X_p\,),$$

where

$$X_p := \bigcup \{a_i^\sigma \mid \sigma \in \mathrm{Lim}\,,\, i \geq \beta\}.$$

We define $\mathbb{P}^\beta$ and G^β as before.

The construction of $(f^\beta)' \in V[G^\beta]$ and the isomorphism argument for $f^\beta = (f^\beta)'$ are as in Chapter 6.2 and 6.3; except that when constructing the isomorphism π, we now have to set $G_{\pi_0}(\nu, j) := F_{\pi_0}(\nu, j)$ for all $\kappa_{\nu,j} < \kappa_\gamma$.

C) $\mathbb{P}^\beta$ preserves cardinals $\geq \alpha(\lambda) = \alpha_\gamma = \sup\{\alpha_\eta \mid 0 < \eta < \gamma\}$.

The arguments here are similar as in Chapter 2.6.2 C): If $\alpha_\gamma > |\beta|^+$, it follows as in Lemma 2.6.12 that $|\mathbb{P}^\beta| \leq \kappa_\gamma^+ \cdot |\beta|^{\aleph_0} \leq \lambda^+ \cdot |\beta|^+ < \alpha_\gamma$. In the case that $\alpha_\gamma = |\beta|^+$, it follows that $\mathrm{cf}\,|\beta| > \omega$, and as before, we distinguish several cases, whether γ is a limit ordinal or $\gamma = \overline{\gamma} + 1$, and in the latter case, whether $\overline{\gamma} \in Lim$, or $\overline{\gamma} \in Succ$ with $\overline{\gamma} = \overline{\overline{\gamma}} + 1$ etc. We separate $P^{\overline{\gamma}}$ (or $P^{\overline{\overline{\gamma}}}$, or both), and obtain that $P^{\overline{\gamma}}$ (or $P^{\overline{\overline{\gamma}}}$, or the product $P^{\overline{\overline{\gamma}}} \times P^{\overline{\gamma}}$) preserves cardinals, while the remaining forcing is now sufficiently small.

D) A set $\widetilde{\wp}(\lambda) \supseteq \mathrm{dom}\, f^\beta$ with an injection $\iota : \widetilde{\wp}(\lambda) \hookrightarrow \lambda^+ \cdot |\beta|^{\aleph_0}$.

As in Chapter 2.6.2 D) and 2.6.4 D), we construct in $V[G^\beta]$ a set $\widetilde{\wp}(\lambda) \supseteq \mathrm{dom}\, f^\beta$ with an injection $\iota : \widetilde{\wp}(\lambda) \hookrightarrow (\lambda^+)^V \cdot (|\beta|^+)^V$ in the case that $(|\beta|^+)^V < \alpha(\lambda)$ and an injection $\iota : \widetilde{\wp}(\lambda) \hookrightarrow (\lambda^+)^V \cdot |\beta|^V$ in the case that $(|\beta|^+)^V = \alpha(\lambda)$.

Together with Chapter 2.6.5 B) and 2.6.5 C), this gives the desired contradiction.

Hence, it follows that there must be some $\alpha < \alpha(\lambda)$ with $\alpha \notin \mathrm{rg}\, f^\beta$.

E) We use an isomorphism argument and obtain a contradiction.

With the same isomorphism argument as in Chapter 2.6.2 E), it follows that $\theta^N(\lambda) = \alpha(\lambda)$ as desired.

Thus, we have shown that also for all cardinals $\lambda \geq \kappa_\gamma$, $\theta^N(\lambda)$ takes the smallest possible value.

This was the last step in the proof of our main theorem.

2.7 Discussion and Remarks

• Our result gives an answer to Shelah's question from [She16, §0.2 1)] (*"Can we bound* $\mathrm{hrtg}\,(\wp(\mu))$ $[= \theta(\mu)]$ *for μ singular?"* No, we can not), and confirms his thesis from [She10, p.2] that in ZF + DC + AX_4 it is "better" to look at $\big([\kappa]^{\aleph_0} \mid \kappa \in \mathrm{Card} \big)$ rather than $\big(\wp(\kappa) \mid \kappa \in \mathrm{Card} \big)$, in the sense that by what we have shown, the only restrictions that can be imposed on the θ-function on a set of cardinals in ZF + DC + AX_4, are the obvious ones.

In [She10, §0 (A)], Shelah suggests to investigate possible cardinalities of $\big(\kappa^{\aleph_0} \mid \kappa \in \mathrm{Card}\big)$. From Theorem 1 in [AK10], it follows that increasing the surjective size of $[\aleph_\omega]^{\aleph_0}$ together with preserving GCH below $\aleph_\omega$, requires a measurable cardinal, which indicates how differently $\wp(\aleph_\omega)$ and $[\aleph_\omega]^{\aleph_0}$ behave without the Axiom of Choice.
In further investigation, one might look at the cardinal arithmetic in our constructed model, such as possible surjective sizes of $([\kappa]^\lambda \mid \kappa \in \mathrm{Card})$ for $\lambda << \kappa$.

• We now look at the following requirement that we put on the sequences $(\kappa_\eta \mid 0 < \eta < \gamma)$, $(\alpha_\eta \mid 0 < \eta < \gamma)$:

$$\forall\, \eta\ (\alpha_\eta = \alpha^+ \rightarrow \mathrm{cf}\,\alpha > \omega).$$

In Chapter 2.2, we mentioned that this condition is necessary under $\mathsf{ZF} + \mathsf{DC} + \mathsf{AX}_4$. Moreover, we proved that whenever we start from a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$, and construct a symmetric extension $N \supseteq V$ with $N \vDash \mathsf{ZF} + \mathsf{DC}$ such that V and N have the same cardinals and cofinalities, then the following holds:

$$\textit{If } \kappa,\ \alpha \in \mathrm{Card} \textit{ with } \theta^N(\kappa) = \alpha^+, \textit{ then } \mathrm{cf}^N(\alpha) > \omega.$$

One could ask what happens if we drop the requirement that N should extend a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ cardinal- and cofinality-preservingly:

Can there be any inner model $N \vDash \mathsf{ZF} + \mathsf{DC}$ with cardinals κ, α such that $\theta^N(\kappa) = \alpha^+$ and $\mathrm{cf}^N(\alpha) = \omega$?

Let $s : 2^\kappa \to \alpha$ denote a surjective function in N. Then with DC, it follows as in Chapter 2.2 that there is also a surjection $s_1 : (2^\kappa)^\omega \to \alpha^\omega$ in N; and we also have a surjective function $s_0 : 2^\kappa \to (2^\kappa)^\omega$.
In Chapter 2.2, we then took a surjection $\widetilde{s}_2 : (\alpha^\omega)^V \to (\alpha^+)^V$ from our ground model V, which gave a surjection $s_2 : (\alpha^\omega)^N \to (\alpha^+)^N$ in N. Then $s_2 \circ s_1 \circ s_0 : 2^\kappa \to \alpha^+$ was a surjective function in N, hence, $\theta^N(\kappa) \geq \alpha^{++}$.

In a more general setting, where we can not refer to a ground model V, we try to use the constructible universe $L = L^N$ instead. Under the assumption $\neg 0^\sharp$, it follows by *Jensen's Covering Theorem* ([DJ75]) that L does not differ drastically from N: In particular, L and N have the same successors of singular cardinals; so if $\mathrm{cf}^N(\alpha) = \omega$, then $(\alpha^+)^L = (\alpha^+)^N$.

This yields the following lemma:

Lemma 2.7.1. *Let N be an inner model of* $\mathsf{ZF} + \mathsf{DC}$ *with $N \vDash \neg\exists 0^\sharp$, and $\alpha \in \mathrm{Card}^N$ with $\mathrm{cf}^N(\alpha) = \omega$. Then there exists a surjective function $s_2 : (\alpha^\omega)^N \to (\alpha^+)^N$ in N.*

Proof. Let $(\alpha_i \mid i < \omega)$ denote a strictly increasing sequence in N that is cofinal in α. First, we construct in N an injection $\iota : (2^\alpha)^L \hookrightarrow (\alpha^\omega)^N$, $\iota = \iota_2 \circ \iota_1 \circ \iota_0$, as follows:

- Let $\iota_0 : (2^\alpha)^L \to \prod_{i<\omega} (2^{\alpha_i})^L$ denote the injection that maps any $g : \alpha \to 2$, $g \in L$, to the sequence of its restrictions $\big((g \upharpoonright \alpha_i) \mid i < \omega\big)$.
- For any $i < \omega$, there is in L an injection $\gamma : (2^{\alpha_i})^L \hookrightarrow (\alpha_i^+)^L$; so with DC in N, we can choose a sequence of injective maps $(\gamma_i \mid i < \omega)$ such that $\gamma_i : (2^{\alpha_i})^L \hookrightarrow (\alpha_i^+)^L$ for all $i < \omega$. Then we define in N an injection $\iota_1 : \prod_{i<\omega} (2^{\alpha_i})^L \to \prod_{i<\omega} (\alpha_i^+)^L$ by setting $\iota_1(X_i \mid i < \omega) := (\gamma_i(X_i) \mid i < \omega)$.
- Finally, since $(\alpha_i^+)^L \leq (\alpha_i^+)^N < \alpha$ for all $i < \omega$, it follows that there is in N an injective map $\iota_2 : \prod_{i<\omega} (\alpha_i^+)^L \hookrightarrow (\alpha^\omega)^N$.

Thus, $\iota := \iota_2 \circ \iota_1 \circ \iota_0 : (2^\alpha)^L \hookrightarrow (\alpha^\omega)^N$ is an injection in N; which yields a surjection $s : (\alpha^\omega)^N \to (2^\alpha)^L$, or $\overline{s} : (\alpha^\omega)^N \to (\alpha^+)^L$.

Since we have assumed that $N \vDash \neg\exists 0^\sharp$ and $\mathrm{cf}^N(\alpha) = \omega$, it follows by *Jensen's Covering Lemma* in N that $(\alpha^+)^L = (\alpha^+)^N$.

This gives our surjecion $s_2 : (\alpha^\omega)^N \to (\alpha^+)^N$ in N as desired.

□

Corollary 2.7.2. *Let N be an inner model of* $\mathsf{ZF} + \mathsf{DC}$ *with $N \vDash \neg 0^\sharp$ and cardinals κ, α such that $\theta^N(\kappa) = \alpha^+$. Then $\mathrm{cf}^N(\alpha) > \omega$.*

Proof. Let $s : 2^\kappa \to \alpha$ denote a surjective function in N, and assume towards a contradiction that $\mathrm{cf}^N(\alpha) = \omega$. As mentioned before, we have surjections $s_0 : 2^\kappa \to (2^\kappa)^\omega$ and $s_1 : (2^\kappa)^\omega \to \alpha^\omega$. By Lemma 2.7.1, it follows that there is also a surjection $s_2 : \alpha^\omega \to \alpha^+$ in N. Setting $s := s_2 \circ s_1 \circ s_0$, we obtain in N a surjective function $s : 2^\kappa \to \alpha^+$. Contradiction. □

Thus, without large cardinal assumptions, it is not possible to obtain a model $N \vDash \mathsf{ZF} + \mathsf{DC}$ such that $\theta^N(\kappa) = \alpha^+$ for cardinals κ, α with $\mathrm{cf}^N(\alpha) = \omega$.

• Another question to ask is, under what circumstances certain $\neg$AC-large cardinal properties are preserved in our symmetric extension N. As an example, we will briefly look at the question whether an inaccessible cardinal κ from the ground model V could remain inaccessible in N.

The notion of inaccessibility in ZFC reads as follows: *A cardinal κ is **inaccessible** (or **strongly inaccessible**) if κ is regular and $2^\lambda < \kappa$ holds for all cardinals $\lambda < \kappa$.*
Hence, it can not be transferred directly to the $\neg$AC-context, since the power sets $\wp(\lambda)$ for $\lambda < \kappa$ are usually not well-ordered. In [BDL07, Chapter 2], there are several characterizations how inaccessibility can be defined in ZF:

Definition 2.7.3 ([BDL07])**.** (i) A regular uncountable cardinal κ is **i-inaccessible** if for all $\lambda < \kappa$, there is an ordinal $\alpha < \kappa$ with an injection $\iota : \wp(\lambda) \hookrightarrow \alpha$.
(ii) A regular uncountable cardinal κ is **v-inaccessible** if for all $\lambda < \kappa$, there is no surjection $s : V_\lambda \to \kappa$.
(iii) A regular uncountable cardinal κ is **$\overline{\text{s}}$-inaccessible** if for all $\lambda < \kappa$, there is no surjection $s : \wp(\lambda) \to \kappa$.

Note that i-inaccessibility implies v-inaccessibility, and v-inaccessibility implies $\overline{s}$-inaccessibility. It is not difficult to see that a cardinal κ is v-inaccessible if and only if V_κ is a model of second-order ZF (see [BDL07, Chapter 2]).

Let now κ be an inaccessible cardinal in the setting of our theorem: $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with sequences $(\kappa_\eta \mid 0 < \eta < \gamma)$, $(\alpha_\eta \mid 0 < \eta < \gamma)$ as before, with the additional property that for all $\kappa_\eta < \kappa$, it follows that also $\alpha_\eta < \kappa$. Then by construction, it follows that κ is $\overline{s}$-inaccessible in N, while i-inaccessibility of κ is out of reach, since we do not have our power set well-ordered.

The question remains whether κ is v-inaccessible in N.

By induction over λ, we could prove (using several isomorphism and factoring arguments similar to those in Chapter 2.6):

Proposition 2.7.4. *Let V be a ground model of* $\mathsf{ZFC} + \mathsf{GCH}$ *with $\gamma \in \mathrm{Ord}$, and sequences of uncountable cardinals $(\kappa_\eta \mid 0 < \eta < \gamma)$ and $(\alpha_\eta \mid 0 < \eta < \gamma)$ with the properties listed in Chapter 2.2. Moreover, let $N \supseteq V$ denote the symmetric extension constructed in Chapter 2.3, 2.4 and 2.5.*

*If κ is an inaccessible cardinal in V with the property that for all $\kappa_\eta < \kappa$ it follows that $\alpha_\eta < \kappa$, then κ is υ-*inaccessible *in N: For $\lambda < \kappa$, there can not be a surjective function $s : V_\lambda \to \kappa$ in N.*

In our inductive proof, we show that for any cardinal $\lambda < \kappa$, there exists $\kappa_{\overline{\nu},\overline{\jmath}}(\lambda) < \kappa$ and a cardinal $\beta_\lambda < \kappa$ with an injection $\iota : V_\lambda^N \hookrightarrow \beta_\lambda$ in $V[G \upharpoonright \kappa_{\overline{\nu},\overline{\jmath}}(\lambda)]$.

- Finally, we remark that our theorem gives a result about possible behaviors of the θ-function on a *set* of uncountable cardinals. A straightforward generalization of our forcing notion to ordinal length sequences $(\kappa_\eta \mid \eta \in \mathrm{Ord})$, $(\alpha_\eta \mid \eta \in \mathrm{Ord})$ does not result in a ZF-model:

Denote by $\overline{\mathbb{P}}$ the class forcing which canonically generalizes our forcing notion $\mathbb{P}$ to sequences $(\kappa_\eta \mid \eta \in \mathrm{Ord})$, $(\alpha_\eta \mid \eta \in \mathrm{Ord})$ of ordinal length, denote by $\overline{G}$ a V-generic filter on $\overline{\mathbb{P}}$, and let $\overline{N} := V(\overline{G})$. Then $\overline{N} \nvDash$ *Power Set*: Assume towards a contradiction that $Z := \wp^{\overline{N}}(\aleph_1) \in \overline{N}$. Then there would be an ordinal γ and a symmetric name $\dot{Z} \in HS \cap \mathrm{Name}(\overline{\mathbb{P}} \upharpoonright \gamma)$ with $Z = \dot{Z}^{\overline{G} \upharpoonright \gamma}$, where $\overline{\mathbb{P}} \upharpoonright \gamma$ denotes the initial part of $\overline{\mathbb{P}}$ up to κ_γ. Now, by an isomorphism argument similar as in the *Approximation Lemma* 2.5.2, one can show that any set $X \in \wp^{\overline{N}}(\aleph_1)$ is contained in an intermediate generic extension $V[\prod_{m<\omega} G_{i_m}^{\eta_m}]$ with $\eta_m < \gamma$, $i_m < \alpha_{\eta_m}$ for all $m < \omega$. Consider $X := G_i^{\gamma+1} \upharpoonright \aleph_1$ for some $i < \alpha_{\gamma+1}$. Then $X \in \wp^{\overline{N}}(\aleph_1)$; hence, $X = G_i^{\gamma+1} \upharpoonright \aleph_1 \in V[\prod_{m<\omega} G_{i_m}^{\eta_m}]$ for a sequence $((\eta_m, i_m) \mid m < \omega)$ with $\eta_m < \gamma$, $i_m < \alpha_{\eta_m}$ for all $m < \omega$. But this is not possible, since $G_i^{\gamma+1}$ is $V[\prod_{m<\omega} G_{i_m}^{\eta_m}]$-generic on $P^{\gamma+1}$.

Broadly speaking, the point is that a class-sized version of our forcing construction never stops adding new subsets of $\aleph_1$ (or any other uncountable cardinal). Although we can try and keep control over the surjective size of $\wp^N(\aleph_1)$, it is not possible to capture $\wp^N(\aleph_1)$ in an appropriate set-sized intermediate generic extension; and it remains a future project to settle this problem and find a countably closed forcing notion that is also suitable for sequences $(\kappa_\eta \mid \eta \in \mathrm{Ord})$, $(\alpha_\eta \mid \eta \in \mathrm{Ord})$ of ordinal length.

Chapter 3

An Easton-like Theorem for all Cardinals in ZF

In this chapter, we show that in the theory ZF, the θ-function can take almost arbitrary values on all cardinals.

More precisely, we prove the following theorem (see [FK16]):

Theorem. *Let V be a ground model of* ZFC + GCH *with a function $F : \mathrm{Card} \to \mathrm{Card}$ such that the following properties hold:*

- $\forall \kappa \ F(\kappa) \geq \kappa^{++}$
- $\forall \kappa, \lambda \ \big(\kappa \leq \lambda \to F(\kappa) \leq F(\lambda)\big)$.

Then there is a cardinal-preserving extension $N \supseteq V$ with $N \vDash$ ZF *such that $\theta^N(\kappa) = F(\kappa)$ for all $\kappa \in \mathrm{Card}$.*

In other words: In the theory ZF, an analogue of Easton's Theorem holds for regular *and* singular cardinals. The only constraints on the θ-function are the obvious ones: weak monotonicity, and $\theta(\kappa) \geq \kappa^{++}$ for all cardinals κ.

This is in sharp contrast to the situation in ZFC, where Easton's Theorem includes only *regular* cardinals, while possible values of 2^κ for *singular* κ strongly depend on the behavior of the Continuum Function below.
Recall that in Chapter 2, we additionally retained DC in our symmetric extension. However, the forcing notion introduced there could not be turned into a class forcing, and therefore merely allowed for setting the θ-values of *set*-many cardinals.

We now complement our results from Chapter 2 by dropping the restriction that only set many cardinals can be considered (but in return losing DC in the constructed model N.)

Let $F : \mathrm{Card} \to \mathrm{Card}$ be a function on the class of infinite cardinals with the properties stated above: F is weakly monotone, and $F(\kappa) \geq \kappa^{++}$ for all $\kappa \in \mathrm{Card}$. We introduce a class-sized forcing notion $\mathbb{P}$ (completely different from the forcing notion from Chapter 2.3) that allows for treating class many cardinals at the same time.

The conditions p are 0-1-functions on trees with finitely many maximal points. The trees' levels are indexed by cardinals, and the vertices on any level κ are denoted by pairs (κ, i) where $i < F(\kappa)$. For a successor cardinal κ^+ and (κ^+, i) a vertex in the tree $t(p)$ associated with p, the value $p(\kappa^+, i)$ is a partial 0-1-function on the interval $[\kappa, \kappa^+)$, bounded below κ^+. Thus, below any vertex (κ, i), the generic filter G adds a new κ-subset. For $i,\ j < F(\kappa)$ with $i \neq j$, the κ-subsets below the vertices (κ, i) and (κ, j) agree on some interval $[0, \alpha)$, where α denotes the level where the branches below (κ, i) and (κ, j) split. We do not allow branches to split at limit levels, thus making sure that the forcing indeed adds $F(\kappa)$-many pairwise distinct κ-subsets for every cardinal κ.

In Chapter 3.1, we define our class forcing $\mathbb{P}$. Like in Chapter 2.3, $\mathbb{P}$ will be a product $\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_0$ (a forcing notion consisting of partial 0-1-functions on finitary trees as described above) is in charge of setting the θ-values of the limit cardinals, while $\mathbb{P}_1$ (a finite support product of Cohen-like forcing notions) is in charge of setting the θ-values of the successor cardinals. We will see that $\mathbb{P}$ has a *nice hierarchy* (cf. Definition 1.4.2 and Lemma 3.1.6), so our methods from Chapter 1.4 can be applied.

In Chapter 3.2, we first construct our almost-group $A = A_0 \times A_1$ of partial $\mathbb{P}$-automorphisms. Any automorphisms π in A_0 has a *height* $\mathrm{ht}\,\pi \in \mathrm{Card}$, and for all $\kappa < \mathrm{ht}\,\pi$ basically "renames" the vertices on level κ. The partial automorphisms in A_1 are similar to the partial $\mathbb{P}_1$-automorphisms introduced in Chapter 2.4.

We proceed as described in Chapter 1.4 and use a method similar to *Scott's Trick* to turn A into a group $\overline{A}$. We define the following $\overline{A}$-subgroups that will yield our notion of symmetry: Firstly, for $\kappa \in \mathrm{Card}$ and $i < F(\kappa)$, a subgroup $Fix(\kappa, i)$ will be included into our symmetric system in order to make sure that the i-th generic κ-subset G_i^κ has a symmetric name. Secondly, for $\kappa \in \mathrm{Card}$ and $\alpha < F(\kappa)$, including a subgroup $Small(\kappa, [0, \alpha))$ ensures that there exists a surjective function $s : \wp(\kappa) \to \alpha$ in the eventual symmetric extension N (hence, $\theta^N(\kappa) \geq F(\kappa)$). We verify that the collection of these subgroups satisfies the *normality property* (cf. 1.4.13 c)), and hence yields a *finitely generated symmetric system* $\mathcal{S}$ (cf. 1.4.13).

We take a V-generic filter G on $\mathbb{P}$ and denote by $N := V(G)$ the symmetric extension by $\mathcal{S}$ and G.

Although due to its finiteness properties, the class forcing $\mathbb{P}$ adds a cofinal function $f : \omega \to \mathrm{Ord}$ (see Proposition 3.1.3), the symmetric extension N satisfies all axioms of ZF. This will be shown in Chapter 3.3.

We will also see that an *Approximation Lemma* holds (cf. Lemma 3.3.6): Any set of ordinals located in N can be captured in a "mild" V-generic extension that preserves cardinals and the GCH.

In Chapter 3.4, we finally prove that indeed, $\theta^N(\kappa) = F(\kappa)$ holds for all $\kappa \in \mathrm{Card}$.

The direction "$\theta^N(\kappa) \geq F(\kappa)$" is rather immediate by construction of the subgroups $Small(\kappa, [0, \alpha))$. Regarding "$\theta^N(\kappa) \leq F(\kappa)$", we use a similar proof structure as in Chapter 2.6.2 A) - E) (cf. p. 108):

We assume towards a contradiction that there was a surjective function $S : \wp(\kappa) \to F(\kappa)$ in N. First, we define a restriction $S^\beta : \operatorname{dom} S^\beta \to F(\kappa)$, which is obtained from S by roughly allowing only κ-subsets contained in those intermediate generic extensions from the *Approximation Lemma* that use merely branches below indices (κ^+, i) with $i < \beta$. In Proposition 3.4.2 we prove that whenever $\beta < F(\kappa)$ is "large enough", it follows by surjectivity of S that the map S^β must be surjective, as well. We construct an involved intermediate generic extension that preserves all cardinals $\geq F(\kappa)$ (cf. Lemma 3.4.7), and then use an isomorphism argument to show that this intermediate generic extension must contain the map $S^\beta : \operatorname{dom} S^\beta \to F(\kappa)$ (cf. Proposition 3.4.9). In Proposition 3.4.10, we prove that our intermediate generic extension also contains an injection $\iota_\beta : \operatorname{dom} S^\beta \to \beta$; which finally gives the desired contradiction.

3.1 The Forcing

We start from a ground model $V \vDash \mathsf{ZFC} + \mathsf{GCH}$ with a function $F : \mathrm{Card} \to \mathrm{Card}$ on the class of infinite cardinals such that the following properties hold:

- $\forall \kappa \; F(\kappa) \geq \kappa^{++}$
- $\forall \kappa, \lambda \; \big(\kappa \leq \lambda \to F(\kappa) \leq F(\lambda)\big)$.

In this section, we define our class forcing $\mathbb{P}$ and give some basic properties.

We will have to treat limit cardinals and successor cardinals separately: $\mathbb{P}$ is a product $\mathbb{P} := \mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_0$ will blow up the power sets of all limit cardinals κ, and $\mathbb{P}_1$ is in charge of the successor cardinals κ^+.

The conditions in $\mathbb{P}_0$ will be functions on trees with finitely many maximal points.

For constructing $\mathbb{P}_0$, our function F has to be modified as follows: For all limit cardinals κ, let $F_{\lim}(\kappa) := F(\kappa)$, and for any successor cardinal $\kappa^+ > \aleph_\omega$, let $F_{\lim}(\kappa^+) := F(\overline{\kappa})$, where $\overline{\kappa} := \sup\{\lambda < \kappa^+ \mid \lambda \text{ is a limit cardinal }\}$. For $n < \omega$, set $F_{\lim}(\aleph_n) := F(\aleph_0)$. Moroever, let $F_{\lim}(0) := \{0\}$.

Our trees' levels will be indexed by cardinals, and on any level κ, the trees contain finitely many vertices (κ, i) with $i < F_{\lim}(\kappa)$.

Definition 3.1.1. A partial order $(t, \leq_t)$ is an $F_{\lim}$-*tree*, if

$$t \subseteq \bigcup_{\kappa \in \mathrm{Card}} \{\kappa\} \times F_{\lim}(\kappa) \;\cup\; \{(0,0)\}$$

with the following properties:

- If $(\kappa, i) \leq_t (\lambda, j)$, then $\kappa \leq \lambda$.
- For any $(\lambda, j) \in t$ and $\kappa < \lambda$, there exists exactly one $i < F_{\lim}(\kappa)$ with $(\kappa, i) \leq_t (\lambda, j)$.
- The tree t has finitely many *maximal points*, i.e. there are finitely many (κ_0, i_0), ... , (κ_{n-1}, i_{n-1}) with $t = \{(\kappa, i) \mid \exists\, m < n \; (\kappa, i) \leq_t (\kappa_m, i_m)\}$.

- There is no *splitting at limits*, i.e. for any limit level κ and $(\kappa, i), (\kappa, i') \in t$ with $\{(\lambda, j) \in t \mid (\lambda, j) \leq_t (\kappa, i)\} = \{(\lambda, j) \in t \mid (\lambda, j) \leq_t (\kappa, i')\}$, it follows that $i = i'$.

If $(t, \leq_t)$ is an $F_{\lim}$-tree with maximal points $(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})$, we call $\operatorname{ht} t := \max\{\kappa_0, \ldots, \kappa_{n-1})$ the *height* of t.

The first and second conditions make sure that for any $F_{\lim}$-tree $(t, \leq_t)$, the predecessors of any $(\kappa, i) \in t$ with $\kappa = \aleph_\alpha$ are linearly ordered by $\leq_t$ in order type α (or $\alpha + 1$ in the case that α is finite), and for any $(\kappa, i) \in t$, it follows that $(0, 0) \leq_t (\kappa, i)$.

There is a canonical partial order on the class of $F_{\lim}$-trees: Let $(s, \leq_s) \leq_{F_{\lim}-tree} (t, \leq_t)$ iff $s \supseteq t$ and $\leq_s \supseteq \leq_t$.

The conditions in our forcing $\mathbb{P}_0$ will be functions $p : (t(p), \leq_{t(p)}) \to V$ whose domain $(t(p), \leq_{t(p)})$ is an $F_{\lim}$-tree.
The functional values of p below any maximal point $(\kappa, i) \in t(p)$ will make up a partial 0-1-function on κ. If (κ, i) and (λ, j) are the maximal points of two branches meeting at level ν, then the according 0-1-functions coincide up to ν.
Hence, a $\mathbb{P}_0$-generic filter will add a new κ-subset $G_{(\kappa,i)}$ below any vertex (κ, i) with $i < F_{\lim}(\kappa)$. The fourth condition in Definition 3.1.1 makes sure that for any $i, i' < F_{\lim}(\kappa)$ with $i \neq i'$, the according κ-subsets $G_{(\kappa,i)}$ and $G_{(\kappa,i')}$ given by the branches below (κ, i) and (κ, i') are distinct. Hence, our forcing adds $F_{\lim}(\kappa)$-many pairwise distinct κ-subsets for any cardinal κ.

For a set X, we denote by $Fn(X, 2, \kappa)$ the collection of all functions $f : \operatorname{dom} f \to 2$ with $\operatorname{dom} f \subseteq X$ and $|\operatorname{dom} f| < \kappa$.

Definition 3.1.2. The class forcing $(\mathbb{P}_0, \leq_0)$ consists of all functions $p : (t(p), \leq_{t(p)}) \to V$ such that $(t(p), \leq_{t(p)})$ is an $F_{\lim}$-tree, and

- $p(\kappa^+, i) \in Fn([\kappa, \kappa^+), 2, \kappa^+)$ for all $(\kappa^+, i) \in t(p)$ with κ^+ a successor cardinal,
- $p(\aleph_0, i) \in Fn(\aleph_0, 2, \aleph_0)$ for all $(\aleph_0, i) \in t(p)$, and
- $p(\kappa, i) = \emptyset$ for all $(\kappa, i) \in t(p)$ with κ a limit cardinal or $\kappa = 0$.
- For $(\kappa, i) \in t(p)$, let
$$p_{(\kappa,i)} := \bigcup \{p(\nu^+, j) \mid (\nu^+, j) \leq_{t(p)} (\kappa, i)\}.$$
We require that $|p_{(\kappa,i)}| < \kappa$ for all $i < F_{\lim}(\kappa)$ whenever κ is a regular limit cardinal.

For $p : (t(p), \leq_{t(p)}) \to V$, $q : (t(q), \leq_{t(q)}) \to V$ conditions in $\mathbb{P}_0$, let $q \leq_0 p$ iff

- $(t(q), \leq_{t(q)}) \leq_{F_{\lim}-tree} (t(p), \leq_{t(p)})$,
- $q(\kappa, i) \supseteq p(\kappa, i)$ for all $(\kappa, i) \in t(p)$.

Let $\mathbb{1}_0 := \emptyset$.

For a condition $p \in \mathbb{P}_0$, $p : (t(p), \leq_{t(p)}) \to V$, we call $\operatorname{ht} p := \operatorname{ht} t(p)$ the *height* of p. Let $\operatorname{ht} \mathbb{1}_0 := 0$.

Let λ be a cardinal. We denote by $p \upharpoonright (\lambda + 1) : t(p) \upharpoonright (\lambda + 1) \to V$ the restriction of p to the subtree $t(p) \upharpoonright (\lambda+1) := \{(\kappa, i) \in t(p) \mid \kappa \leq \lambda\}$, $\leq_{t(p)\upharpoonright(\lambda+1)} := \leq_{t(p)} \cap (t(p) \upharpoonright (\lambda+1))$. Then $p \upharpoonright (\lambda + 1) \in \mathbb{P}_0$ with $p \leq_0 p \upharpoonright (\lambda + 1)$. Let $\mathbb{P}_0 \upharpoonright (\lambda + 1) := \{p \upharpoonright (\lambda + 1) \mid p \in \mathbb{P}_0\}$.

Similarly, we define $p \upharpoonright [\lambda, \infty) : t(p) \upharpoonright [\lambda, \infty) \to V$ (which is not a condition in $\mathbb{P}_0$), with $t(p) \upharpoonright [\lambda, \infty) := \{(\kappa, i) \in t(p) \mid \kappa \geq \lambda\}$. Let $\big(p \upharpoonright [\lambda, \infty)\big)(\kappa, i) := p(\kappa, i)$ for all $(\kappa, i) \in t(p)$ with $\kappa > \lambda$, and $\big(p \upharpoonright [\lambda, \infty)\big)(\lambda, i) := \varnothing$ for any $(\lambda, i) \in t(p)$. Set $\mathbb{P}_0 \upharpoonright [\lambda, \infty) := \{p \upharpoonright [\lambda, \infty) \mid p \in \mathbb{P}_0\}$.

The forcing $\mathbb{P}_0$ is dense in the product $\mathbb{P}_0 \upharpoonright (\lambda + 1) \times \mathbb{P}_0 \upharpoonright [\lambda, \infty)$.

Similarly, for cardinals μ, λ with $\mu < \lambda$, we define $p \upharpoonright [\mu, \lambda + 1) : t(p) \upharpoonright [\mu, \lambda + 1) \to V$ with $t(p) \upharpoonright [\mu, \lambda + 1) := \{(\kappa, i) \in t(p) \mid \mu \leq \kappa \leq \lambda\}$. Let $\big(p \upharpoonright [\mu, \lambda + 1)\big)(\kappa, i) := p(\kappa, i)$ for all $(\kappa, i) \in t(p)$ with $\kappa > \mu$, and $\big(p \upharpoonright [\mu, \lambda + 1)\big)(\mu, i) := \varnothing$ for any $(\mu, i) \in t(p)$. We set $\mathbb{P}_0 \upharpoonright [\mu, \lambda + 1) := \{p \upharpoonright [\mu, \lambda + 1) \mid p \in \mathbb{P}_0\}$.

For conditions $p, q \in \mathbb{P}_0$ with $p \parallel q$, it follow that $t(p) \cup t(q)$ with the order relation $\leq_{t(p)} \cup \leq_{t(q)}$ is an $F_{\lim}$-tree as well, and we can define a *maximal common extension* $p \cup q$ of p and q as follows: Let $t(p \cup q) := t(p) \cup t(q)$, $\leq_{t(p \cup q)} := \leq_{t(p)} \cup \leq_{t(q)}$ with $(p \cup q)(\kappa, i) := p(\kappa, i) \cup q(\kappa, i)$ for any $(\kappa, i) \in t(p) \cap t(q)$, $(p \cup q)(\kappa, i) := p(\kappa, i)$ whenever $(\kappa, i) \in t(p) \setminus t(q)$ and $(p \cup q)(\kappa, i) := q(\kappa, i)$ for all $(\kappa, i) \in t(q) \setminus t(p)$.

Surely, the class forcing $\mathbb{P}_0$ does not preserve ZFC:

Proposition 3.1.3. *Let G_0 be a V-generic filter on $\mathbb{P}_0$. There is a cofinal function $f : \omega \to \mathrm{Ord}$ in $\langle V[G_0], \in, V, G_0\rangle$. In particular, the Axiom of Replacement fails in $\langle V[G_0], \in, V, G_0\rangle$.*

Proof. We work in $\langle V[G_0], \in, V, G_0\rangle$. For any cardinal λ and $i < F(\lambda^+)$, note that (λ^+, i) is a vertex in the generic tree with $G_0(\lambda^+, i) : [\lambda, \lambda^+) \to 2$.

We define a function $f : \omega \to \mathrm{Ord}$ as follows: Let $n < \omega$. Let $f(n) := \lambda$ if λ is the least cardinal with the property that $G(\lambda^+, m)(\lambda) = 0$ for all $m < n$, but $G(\lambda^+, n)(\lambda) = 1$.

In order to make sure that f is well-defined, we check that for any $n < \omega$, the following set is dense in $\mathbb{P}_0$:

$$D_n := \Big\{ p \in \mathbb{P}_0 \;\Big|\; \exists\, \lambda \in \mathrm{Card}\; \Big(\forall\, m < n\;\; p(\lambda^+, m)(\lambda) = 0 \wedge p(\lambda^+, n)(\lambda) = 1 \Big) \Big\}.$$

Fix $n < \omega$, and consider a condition $p \in \mathbb{P}$. Let $\operatorname{ht} t(p) =: \overline{\lambda}$, and take $\lambda > \overline{\lambda}$ arbitrary. We define an extension $\overline{p} \leq p$ as follows: $t(\overline{p})$ is obtained from $t(p)$ by adding $(n + 1)$ - many new branches disjoint from $t(p)$ with maximal points $(\lambda^+, 0), \ldots, (\lambda^+, n)$. We set $\overline{p}(\lambda^+, m)(\lambda) := 0$ for all $m < n$, $\overline{p}(\lambda^+, n)(\lambda) := 1$, and the remaining values $\overline{p}(\kappa^+, i)(\zeta)$ for $(\kappa^+, i) \in t(\overline{p})$, $\zeta \in [\kappa, \kappa^+)$ arbitrary with the property that $\overline{p}(\kappa^+, i) := p(\kappa^+, i)$ whenever $(\kappa^+, i) \in t(p)$. Then $\overline{p}$ is an extension of p in D_n.

It follows that D_n is dense in $\mathbb{P}_0$; and we can pick $p \in G \cap D_n$. By definition on D_n, there exists $\lambda_n \in \mathrm{Card}$ with the property that $G_0(\lambda_n^+, m)(\lambda_n) = p(\lambda_n^+, m)(\lambda_n) = 0$ for all $m < n$, and $G_0(\lambda_n^+, n)(\lambda_n) = p(\lambda_n^+, n)(\lambda_n) = 1$. Hence, $f(n)$ is well-defined with $f(n) \leq \lambda_n$.

It remains to make sure that the function f is cofinal in Ord. Assume towards a contradiction that $\operatorname{rg} f$ was bounded below some cardinal κ. We claim that the following set is dense in $\mathbb{P}_0$:

$$D := \{ p \in \mathbb{P}_0 \mid \exists\, n < \omega\ \forall\, \lambda < \kappa\ \exists\, m < n\ p(\lambda^+, m)(\lambda) = 1 \}.$$

Consider $p \in \mathbb{P}$, and let $l < \omega$ denote the number of maximal points of $t(p)$. Then $t(p)$ has $\leq l$-many vertices on any level $\lambda \leq \operatorname{ht} t(p)$. Let $n := l + 1$. For any $\lambda < \kappa$, there exists $m < n$, i.e. $m \in \{0, 1, \ldots, l\}$, with the property that $(\lambda, m) \notin t(p)$. We define an extension $\overline{p} \leq p$ as follows: $t(\overline{p})$ is obtained from $t(p)$ by adding a new branch $\{(\lambda, m(\lambda)) \mid 0 < \lambda \leq \kappa\} \cup \{(0,0)\}$ such that any $m(\lambda)$ is the least $m \leq l$ with the property that $(\lambda, m) \notin t(p)$. For $\lambda^+ \leq \kappa$, set $\overline{p}(\lambda^+, m(\lambda^+))(\lambda) := 1$, and the remaining values $\overline{p}(\lambda^+, m(\lambda^+))(\zeta)$ for $\zeta \in (\lambda, \lambda^+)$ arbitrary. Moreover, $\overline{p}(\lambda^+, i) := p(\lambda^+, i)$ whenever $(\lambda^+, i) \in t(p)$. Then $\overline{p}$ is an extension of p in D; which proves the density of $D \subseteq \mathbb{P}_0$.

Now, take $q \in D \cap G_0$ and $n < \omega$ as in the definition of D. By assumption, $\mu := f(n) < \kappa$, so $G_0(\mu^+, m)(\mu) = 0$ for all $m < n$. This contradicts $q \in D$.

Thus, it follows that the function $f : \omega \to \mathrm{Ord}$ can not be bounded below any cardinal κ. □

Note that for successor cardinals κ^+, the forcing $\mathbb{P}_0$ only adds $F_{\lim}(\kappa^+)$-many κ^+-subsets, which might be less than the desired $F(\kappa^+)$. Hence, we need a second forcing $\mathbb{P}_1$ to blow up the power sets $\mathcal{P}(\kappa^+)$.
(The reason why we use for $\mathbb{P}_0$ the function $F_{\lim}$ instead of F is that for singular limit cardinals κ, we will have to use the forcing notion $\mathbb{P}_0 \upharpoonright (\kappa^+ + 1)$ instead of $\mathbb{P}_0 \upharpoonright (\kappa + 1)$ for capturing κ-subsets in N in our proof of $\theta^N(\kappa) \leq F(\kappa)$; and we will need that $F_{\lim}(\kappa^+) = F(\kappa)$ to make sure that $\mathbb{P}_0 \upharpoonright (\kappa^+ + 1)$ only has size $F(\kappa)$.)

Now, we turn to $\mathbb{P}_1$. The forcing $\mathbb{P}_1$ will be a variant of Easton forcing with finite support: We will have a finite support-product of forcings $Fn([\kappa, \kappa^+) \times F(\kappa^+), 2, \kappa^+)$, where a successor cardinal κ^+ shall only be included into the forcing $\mathbb{P}_1$ if $F(\kappa^+)$ is strictly greater than all $F(\nu^+)$ for $\nu < \kappa$.

Definition 3.1.4. Let Succ$'$ denote the class of all successor cardinals κ^+ with the property that $F(\kappa^+) > F(\nu^+)$ for all $\nu^+ < \kappa^+$. The forcing $(\mathbb{P}_1, \leq_1, \mathbb{1}_1)$ consists of all conditions $p : \operatorname{supp} p \to V$ with $\operatorname{supp} p \subseteq \mathrm{Succ}'$ finite and

$$p(\kappa^+) \in Fn([\kappa, \kappa^+) \times F(\kappa^+), 2, \kappa^+)$$

for all $\kappa^+ \in \operatorname{supp} p$; such that $\operatorname{dom} p(\kappa^+)$ is *rectangular*, i.e. $\operatorname{dom} p(\kappa^+) = \operatorname{dom}_x p(\kappa^+) \times \operatorname{dom}_y p(\kappa^+)$ for some $\operatorname{dom}_x p(\kappa^+) \subseteq [\kappa, \kappa^+)$ and $\operatorname{dom}_y p(\kappa^+) \subseteq F(\kappa^+)$.
The conditions in $\mathbb{P}_1$ are ordered by reverse inclusion: Let $q \leq_1 p$ iff $\operatorname{supp} q \supseteq \operatorname{supp} p$ with $q(\kappa^+) \supseteq p(\kappa^+)$ for all $\kappa^+ \in \operatorname{supp} p$. The maximal element is $\mathbb{1}_1 := \varnothing$.

For a cardinal λ and $p \in \mathbb{P}_1$, we denote by $p \upharpoonright (\lambda + 1)$ the restriction of p to the domain $\{\kappa^+ \in \operatorname{supp} p \mid \kappa^+ \leq \lambda\}$. Similarly, we write $p \upharpoonright [\lambda, \infty)$ for the restriction of p to $\{\kappa^+ \in \operatorname{supp} p \mid \kappa^+ > \lambda\}$.

Let $\mathbb{P}_1 \upharpoonright (\lambda+1) := \{p_1 \upharpoonright (\lambda+1) \mid p_1 \in \mathbb{P}_1\}$, and $\mathbb{P}_1 \upharpoonright [\lambda,\infty) := \{p_1 \upharpoonright [\lambda,\infty) \mid p_1 \in \mathbb{P}_1\}$. Then $\mathbb{P}_1$ is isomorphic to the product $\mathbb{P}_1 \upharpoonright (\lambda+1) \times \mathbb{P}_1 \upharpoonright [\lambda,\infty)$.

For a successor cardinal $\kappa^+ \in \mathrm{Succ}'$, we set $\mathbb{P}_1(\kappa^+) := \{p(\kappa^+) \mid p \in \mathbb{P}_1\}$, which is the forcing notion $Fn([\kappa,\kappa^+) \times F(\kappa^+), 2, \kappa^+)$. If G_1 is a V-generic filter on $\mathbb{P}_1$, it follows that $G_1(\kappa^+) := \{p(\kappa^+) \mid p \in G_1\}$ is V-generic on $\mathbb{P}_1(\kappa^+)$.

Definition 3.1.5.

$$(\mathbb{P}, \leq) := (\mathbb{P}_0 \times \mathbb{P}_1, \leq_{\mathbb{P}_0 \times \mathbb{P}_1}).$$

For a condition $p = (p_0, p_1) \in \mathbb{P}$ and a cardinal λ, let $p \upharpoonright (\lambda+1) := (p_0 \upharpoonright (\lambda+1), p_1 \upharpoonright (\lambda+1))$, and $p \upharpoonright [\lambda,\infty) := (p_0 \upharpoonright [\lambda,\infty), p_1 \upharpoonright [\lambda,\infty))$. Let $\eta(p) := \min\{\lambda \mid p \upharpoonright (\lambda+1) = p\}$.
Moreover, $\mathbb{P} \upharpoonright (\lambda+1) := \{p \upharpoonright (\lambda+1) \mid p \in \mathbb{P}\}$, and $\mathbb{P} \upharpoonright [\lambda,\infty) := \{p \upharpoonright [\lambda,\infty) \mid p \in \mathbb{P}\}$.

Lemma 3.1.6. *$\mathbb{P}$ has a nice hierarchy.*

Proof. For $\alpha \in \mathrm{Ord}$, let $(\mathbb{P}_0)_\alpha := \mathbb{P}_0 \upharpoonright (\aleph_\alpha+1) = \{p \in \mathbb{P}_0 \mid \mathrm{ht}\, p \leq \aleph_\alpha\}$, with $(\leq_0)_\alpha$ the ordering on $(\mathbb{P}_0)_\alpha$ induced by $\leq_0$, and $(\mathbb{1}_0)_\alpha := \varnothing = \mathbb{1}_0$. Moreover, let $(\mathbb{P}_1)_\alpha := \mathbb{P}_1 \upharpoonright (\aleph_\alpha+1) = \{p \in \mathbb{P}_1 \mid \mathrm{supp}\, p \subseteq \aleph_\alpha + 1\}$, with $(\leq_1)_\alpha$ the ordering induced by $\leq_1$, and $(\mathbb{1}_1)_\alpha := \varnothing = \mathbb{1}_1$.

Setting

$$\mathbb{P}_\alpha := (\mathbb{P}_0)_\alpha \times (\mathbb{P}_1)_\alpha$$

for $\alpha \in \mathrm{Ord}$, it follows that $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ is an increasing chain of set-sized forcing notions.

Let now $\alpha < \beta$. We have to make sure that $\mathbb{P}_\alpha$ is a complete suborder of $\mathbb{P}_\beta$. For A a maximal antichain in $\mathbb{P}_\alpha$, we have to show that A is also maximal in $\mathbb{P}$. Assume towards a contradiction there was $p = (p_0, p_1) \in \mathbb{P}_\beta$ with $p \perp q$ for all $q \in A$. Take $\overline{q} \in A$ with $\overline{q} \parallel (p \upharpoonright (\aleph_\alpha + 1))$, and denote by $r = (r_0, r_1)$ a common extension of $p \upharpoonright (\aleph_\alpha+1)$ and $\overline{q}$ in $\mathbb{P}_\alpha$. Then $\overline{r} := (\overline{r}_0, \overline{r}_1)$ with $\overline{r}_0 := r_0 \cup p_0$ (i.e. $\overline{r} \upharpoonright (\aleph_\alpha+1) = r_0$, $\overline{r} \upharpoonright [\aleph_\alpha,\infty) = p_0 \upharpoonright [\aleph_\alpha,\infty) \cup \{((\aleph_\alpha, i), \varnothing) \mid (\aleph_\alpha, i) \in t(r_0) \smallsetminus t(p_0)\}$) and $\overline{r}_1 := r_1 \cup p_1$ is a common extension of p and $\overline{q}$. Contradiction.

Thus, we have shown that $\mathbb{P} = \bigcup_{\alpha \in \mathrm{Ord}} \mathbb{P}_\alpha$ is an increasing chain of set-sized complete forcing notions, and clearly, each $\mathbb{P}_\alpha$ is upwards closed.

We now go through Definition 1.4.2:

a) For $\alpha \in \mathrm{Ord}$, define $\rho_\alpha : \mathbb{P} \to \mathbb{P}_\alpha$ by setting $\rho_\alpha(p) := p \upharpoonright (\aleph_\alpha+1)$ for all $p \in \mathbb{P}$. The properties (i) - (v) are not difficult to verify. Regarding (iv), consider $p = (p_0, p_1) \in \mathbb{P}$ and $q = (q_0, q_1) \in \mathbb{P}_\alpha$ with $q \leq \rho_\alpha(p)$. Let $p' := (p_0', p_1')$ with $p_0' := q_0 \cup p_0$, $p_1' := q_1 \cup p_1$. Then $p' \leq p$, and $\rho_\alpha(p') = (q_0, q_1) = q$ as desired.
b) For $\alpha \in \mathrm{Ord}$, let $\mathbb{P}_{[\alpha,\infty)} := \mathbb{P} \upharpoonright [\aleph_\alpha,\infty)$ with the projection $\rho_{[\alpha,\infty)} : \mathbb{P} \to \mathbb{P}_{[\alpha,\infty)}$, $p \mapsto p \upharpoonright [\aleph_\alpha,\infty)$.

   We define: $\overline{\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}} :=$

   $$\Big\{ \big((p_0,p_1),(q_0,q_1)\big) \in \mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)} \;\Big|\; \{(\kappa,i) \in t(q_0) \mid \kappa = \aleph_\alpha\} \subseteq \{(\kappa,i) \in t(p_0) \mid \kappa = \aleph_\alpha\} \Big\},$$

   i.e. $\overline{\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}}$ consists of all those conditions $((p_0,p_1),(q_0,q_1))$ in $\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$ for which $p_0 \cup q_0$ has no "extra roots" at level α. Then $\overline{\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}}$ is dense in $\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}$

(one can extend p_0 if necessary).

For any $((p_0,p_1),(q_0,q_1)) \in \overline{\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}}$, we can regard (p_0,q_0) as a condition in $\mathbb{P}_0$ by identifying the pair (p_0,q_0) with the condition $p_0 \cup q_0$ defined as follows: $t(p_0 \cup q_0) := t(p_0) \cup t(q_0)$ with the order relation $\leq_{t(p_0 \cup q_0)} := \leq_{t(p_0)} \cup \leq_{t(q_0)}$; $(p_0 \cup q_0)(\kappa,i) := p_0(\kappa,i)$ in the case that $\kappa \leq \aleph_\alpha$, and $(p_0 \cup q_0)(\kappa,i) := q_0(\kappa,i)$ for $\kappa > \aleph_\alpha$. Similarly, we can regard (p_1,q_1) as a condition in $\mathbb{P}_1$ by identifying the pair (p_1,q_1) with $p_1 \cup q_1$.
It follows that the map $b_\alpha : \mathbb{P} \to \overline{\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}}$, $p \mapsto (\rho_\alpha(p), \rho_{[\alpha,\infty)}(p))$ is an isomorphism of forcings with inverse $(b_\alpha)^{-1} : \overline{\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}} \to \mathbb{P}$, $((p_0,p_1),(q_0,q_1)) \mapsto (p_0 \cup q_0, p_1 \cup q_1)$.

We will confuse any $p \in \mathbb{P}$ with its image $b_\alpha(p)$ in $\overline{\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}}$.

c) The properties (i) - (iii) are not difficult to verify. For (iii), consider $p \in \mathbb{P}$ and $q_{[\alpha,\infty)} \in \mathbb{P}_{[\alpha,\infty)}$ with $q_{[\alpha,\infty)} \leq_{[\alpha,\infty)} \rho_{[\alpha,\infty)}(p)$. By density, take $p' \in \overline{\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}}$ with $p' \leq (\rho_\alpha(p), q_{[\alpha,\infty)})$. Then $p' \leq (\rho_\alpha(p), \rho_{[\alpha,\infty)}(p)) = p$ with $\rho_{[\alpha,\infty)}(p') \leq_{[\alpha,\infty)} q_{[\alpha,\infty)}$.

d) The properties (i) and (ii) are clear, identifying each $p \in \mathbb{P}$ with its image $b_\alpha(p) = (\rho_\alpha(p), \rho_{[\alpha,\infty)}(p))$ in $\overline{\mathbb{P}_\alpha \times \mathbb{P}_{[\alpha,\infty)}}$.

Hence, it follows that $\mathbb{P}$ has a nice hierarchy.

$\square$

We conclude tat $\mathbb{P}$ satisfies the Forcing Theorem. In particular, the forcing relation $\Vdash_{\mathbb{P}}^V$ is definable.

Moreover, for any ordinal α and G a V-generic filter on $\mathbb{P}$, it follows that $G_\alpha := G \cap \mathbb{P}_\alpha$ is a V-generic filter on $\mathbb{P}_\alpha$.

By the same arguments as above, it follows that also $\mathbb{P}_0 = \bigcup_{\alpha \in \mathrm{Ord}} (\mathbb{P}_0)_\alpha$ has a nice hierarchy with projections $(\rho_0)_\alpha : \mathbb{P}_0 \to (\mathbb{P}_0)_\alpha$, $p \mapsto p \upharpoonright (\aleph_\alpha + 1)$ and $(\rho_0)_{[\alpha,\infty)} : \mathbb{P}_0 \to (\mathbb{P}_0)_{[\alpha,\infty)}$, $p \mapsto p \upharpoonright [\aleph_\alpha, \infty)$; and similarly, $\mathbb{P}_1 = \bigcup_{\alpha \in \mathrm{Ord}} (\mathbb{P}_1)_\alpha$ has a nice hierarchy with projections $(\rho_1)_\alpha : \mathbb{P}_1 \to (\mathbb{P}_1)_\alpha$ and $(\rho_1)_{[\alpha,\infty)} : \mathbb{P}_1 \to (\mathbb{P}_1)_{[\alpha,\infty)}$.

If G_0 is a V-generic filter on $\mathbb{P}_0$ and G_1 is $V[G_0]$-generic on $\mathbb{P}_0$, then $G := G_0 \times G_1$ is a V-generic filter on $\mathbb{P}$. By the definability of $\Vdash_{\mathbb{P}_0}^V$, it follows that the converse is true, as well (cf. Lemma 1.3.14).

Our eventual symmetric submodel $N \subseteq V[G]$ will have the crucial property that sets of ordinals $X \subseteq \alpha$ with $X \in N$ can be captured in "mild" V-generic extensions of the following form:

Definition/Lemma 3.1.7. For $p,q \in \mathbb{P}_0$ with $(t(q), \leq_{t(q)}) \leq_{F_{\lim}-tree} (t(p), \leq_{t(p)})$, we denote by $q \upharpoonright t(p)$ the restriction of q to the domain $t(p)$. Let

$$\mathbb{P}_0 \upharpoonright t(p) := \{q \upharpoonright t(p) \mid q \in \mathbb{P}_0, t(q) \leq t(p)\},$$

with the partial order induced by $\leq_0$, and the maximal element $\mathbb{1}_{\mathbb{P}_0 \upharpoonright t(p)} : t(p) \to V$ with $\mathbb{1}_{\mathbb{P}_0 \upharpoonright t(p)}(\kappa,i) = \varnothing$ for all $(\kappa,i) \in t(p)$.

For G_0 a V-generic filter on $\mathbb{P}_0$ and $p \in G_0$, it follows that

$$G_0 \upharpoonright t(p) := \{q \upharpoonright t(p) \mid q \in G_0, t(q) \leq_{F_{\lim}-tree} t(p)\} = \{q \in G_0 \mid t(q) = t(p)\}$$

is a V-generic filter on $\mathbb{P}_0 \upharpoonright t(p)$.

Proof. Consider a dense set $D \subseteq \mathbb{P}_0 \upharpoonright t(p)$. It suffices to show that $\overline{D} := \{q \in \mathbb{P}_0 \mid q \upharpoonright t(p) \in D\}$ is dense in $\mathbb{P}_0$ below p.
Take $q \in \mathbb{P}_0$ with $q \leq_0 p$. There exists $r \in \mathbb{P}_0 \upharpoonright t(p)$, $r \in D$, with $r \leq_0 q \upharpoonright t(p)$. We define a condition $\overline{q} \in \mathbb{P}_0$ as follows: $(t(\overline{q}), \leq_{t(\overline{q})}) := (t(q), \leq_{t(q)})$ with $\overline{q}(\kappa, i) := r(\kappa, i)$ for $(\kappa, i) \in t(p)$, and $\overline{q}(\kappa, i) := q(\kappa, i)$, else. Then $\overline{q} \leq_0 q$ with $\overline{q} \upharpoonright t(p) = r \in D$ as desired. □

For finitely many $(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1}) \in t(p)$, we denote by $t(p) \upharpoonright \{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\}$ the subtree $\{(\kappa, i) \in t(p) \mid \exists\, m < n\ (\kappa, i) \leq_{t(p)} (\kappa_m, i_m)\}$ with the ordering induced by $\leq_{t(p)}$. We write $p \upharpoonright \{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\}$ for the restriction of p to the subtree $t(p) \upharpoonright \{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\}$.
If the set $\{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\}$ contains all maximal points of p, i.e. for any $(\kappa, i) \in t(p)$ there is $l < n$ with $(\kappa, i) \leq_{t(p)} (\kappa_l, i_l)$, then we sometimes use the notation $G_0 \upharpoonright \{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\}$ instead of $G_0 \upharpoonright t(p)$.

We have similar restrictions for $\mathbb{P}_1$:

Definition/Lemma 3.1.8. Consider finitely many cardinals $\overline{\kappa}_0, \ldots, \overline{\kappa}_{\overline{n}-1} \in \mathrm{Succ}'$, and $\overline{\imath}_0 < F(\overline{\kappa}_0), \ldots, \overline{\imath}_{\overline{n}-1} < F(\overline{\kappa}_{\overline{n}-1})$. For a condition $p_1 \in \mathbb{P}_1$, we define $p_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$ as follows:

$$\mathrm{dom}\ p_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\} :=$$

$$\big(\mathrm{dom}_x\, p(\overline{\kappa}_0) \times \{\overline{\imath}_0\}\big) \cup \cdots \cup \big(\mathrm{dom}_x\, p(\overline{\kappa}_{\overline{n}-1}) \times \{\overline{\imath}_{\overline{n}-1}\}\big) =$$
$$= \{(\alpha, i) \in \mathrm{dom}\, p \mid \exists\, l < \overline{n}\ \ i = \overline{\imath}_l\},$$

and for any $(\alpha, \overline{\imath}_l) \in \mathrm{dom}\ p_1(\overline{\kappa}_l)$,

$$\big(p_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}\big)(\alpha, \overline{\imath}_l) := p_1(\overline{\kappa}_l)(\alpha, \overline{\imath}_l).$$

Let

$$\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\} := \{p_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\} \mid p_1 \in \mathbb{P}_1\}.$$

For G_1 a V-generic filter on $\mathbb{P}_1$, it follows that

$$G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\} := \{\, p_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\} \mid p_1 \in G_1 \,\}$$

is a V-generic filter on $\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$.

In other words, for any $l < \overline{n}$ with $\overline{\kappa}_l = \widetilde{\kappa}_l^+$, it follows that $\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$ adds a new Cohen-subset to the interval $[\widetilde{\kappa}_l, \widetilde{\kappa}_l^+)$.

Proof. If D is a dense subset of $\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$, it follows that

$$\overline{D} := \{p_1 \in \mathbb{P}_1 \mid p_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\} \in D\}$$

is dense in $\mathbb{P}_1$. □

Hence, if $G = G_0 \times G_1$ is a V-generic filter on $\mathbb{P}$ with $(\kappa_0, i_0), \dots, (\kappa_{n-1}, i_{n-1})$, $(\overline{\kappa}_0, \overline{\imath}_0), \dots,$ $(\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})$ as before and $p \in G_0$, $p : t(p) \to V$ such that $\{(\kappa_0, i_0), \dots, (\kappa_{n-1}, i_{n-1})\} \subseteq t(p)$ contains all maximal points of $t(p)$, it follows that

$$G_0 \upharpoonright t(p) \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$$

is a V-generic filter on $\mathbb{P}_0 \upharpoonright t(p) \times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$.

We will now see that these forcings preserves all cardinals.

Proposition 3.1.9. *Consider a condition $p \in \mathbb{P}_0$ such that $\{(\kappa_0, i_0), \dots, (\kappa_{n-1}, i_{n-1})\} \subseteq t(p)$ contains all maximal points of $t(p)$; moreover, finitely many $(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})$ with $\overline{\kappa}_0, \dots, \overline{\kappa}_{\overline{n}-1} \in \mathrm{Succ}'$, $\overline{\imath}_0 < F(\overline{\kappa}_0), \dots, \overline{\imath}_{\overline{n}-1} < F(\overline{\kappa}_{\overline{n}-1})$.*
The forcing

$$\mathbb{P}_0 \upharpoonright t(p) \times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$$

preserves cardinals and the GCH.

Proof. We show that for all cardinals λ,

$$\left(2^{\lambda}\right)^{V[G_0 \upharpoonright t(p) \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}]} = (\lambda^+)^V.$$

First, consider the case that $\lambda = \overline{\lambda}^+$ is a successor cardinal. Let $(\mathbb{P}_0 \upharpoonright t(p)) \upharpoonright (\lambda + 1) := \{q \upharpoonright (\lambda + 1) \mid q \in \mathbb{P}_0 \upharpoonright t(p)\}$ and $(\mathbb{P}_0 \upharpoonright t(p)) \upharpoonright [\lambda, \infty) := \{q \upharpoonright [\lambda, \infty) \mid q \in \mathbb{P}_0 \upharpoonright t(p)\}$.
Similarly, let $(\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}) \upharpoonright (\lambda + 1) := \{(p \upharpoonright (\lambda + 1)) \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\} \mid p \in \mathbb{P}_1\}$ denote the lower part, and $(\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}) \upharpoonright [\lambda, \infty) := \{(p \upharpoonright [\lambda, \infty)) \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\} \mid p \in \mathbb{P}_1\}$ the upper part of the forcing $\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}$.

Then $\mathbb{P}_0 \upharpoonright t(p) \times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$ can be factored as

$$\Big((\mathbb{P}_0 \upharpoonright t(p)) \upharpoonright (\lambda + 1) \times (\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}) \upharpoonright (\lambda + 1)\Big) \times$$
$$\Big((\mathbb{P}_0 \upharpoonright t(p)) \upharpoonright [\lambda, \infty) \times (\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}) \upharpoonright [\lambda, \infty)\Big),$$

where the first factor has cardinality $\leq \lambda$, since $\lambda = \overline{\lambda}^+$ is a successor cardinal, and the second factor is $\leq \lambda$ - closed. Thus, it follows that

$$\left(2^{\lambda}\right)^{V[G_0 \upharpoonright t(p) \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}]} \leq |\mathcal{P}(\lambda)|^V = (\lambda^+)^V$$

as desired.

If λ is a regular limit cardinal, the same argument applies.

It remains to show that

$$\left(2^{\lambda}\right)^{V[G_0 \upharpoonright t(p) \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}]} = (\lambda^+)^V$$

in the case that λ is a singular limit cardinal. Assume the contrary and take λ least such that $\eta := \operatorname{cf}\lambda < \lambda$ and

$$\left(2^{\lambda}\right)^{V[G_0 \restriction t(p) \times G_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}]} > (\lambda^+)^V.$$

Let $(\lambda_i \mid i < \eta)$ denote a cofinal sequence in λ. By assumption, it follows that

$$\left(2^{\overline{\lambda}}\right)^{V[G_0 \restriction t(p) \times G_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}]} = (\overline{\lambda}^+)^V$$

for all $\overline{\lambda} < \lambda$. Thus,

$$2^{\lambda} \leq \prod_{i<\eta} 2^{\lambda_i} \leq \left(2^{<\lambda}\right)^{\eta} = \lambda^{\eta} \leq \lambda^{\lambda} = 2^{\lambda}$$

holds true in V and $V[G_0 \restriction t(p) \times G_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}]$. Since η is regular, we have

$$\left| (\mathbb{P}_0 \restriction t(p)) \restriction (\eta + 1) \times (\mathbb{P}_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}) \restriction (\eta + 1) \right| \leq \eta,$$

and

$$(\mathbb{P}_0 \restriction t(p)) \restriction [\eta, \infty) \times (\mathbb{P}_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}) \restriction [\eta, \infty)$$

is $\leq \eta$ - closed. Thus,

$$(2^{\lambda})^{V[G_0 \restriction t(p) \times G_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}]} = (\lambda^{\eta})^{V[G_0 \restriction t(p) \times G_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}]} \leq$$

$$\leq (\lambda^{\eta})^{V[(G_0 \restriction t(p)) \restriction (\eta+1) \times (G_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}) \restriction (\eta+1)]} \leq (2^{\lambda})^{V[(G_0 \restriction t(p)) \restriction (\eta+1) \times (G_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}) \restriction (\eta+1)]} \leq$$

$$\leq |\wp(\lambda \times \eta)|^V \leq (2^{\lambda})^V = (\lambda^+)^V,$$

which gives the desired contradiction.

□

We will see that any set of ordinals in our eventual symmetric submodel N can be captured in a generic extension by one of these forcings $\mathbb{P}_0 \restriction t(p) \times \mathbb{P}_1 \restriction \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$.

Hence, N preserves all cardinals.

3.2 Symmetric Names

For defining our symmetric submodel N, we first we need an *almost-group A of partial $\mathbb{P}$-automorphisms* (cf. Definition 1.4.7). We will have $A = A_0 \times A_1$, where A_0 is a group of $\mathbb{P}_0$-automorphisms each of which is nicely level-preserving and can be described below some ordinal α, and A_1 is an almost-group of partial $\mathbb{P}_1$-automorphisms. It is not difficult to check that in this setting, it follows that A is an almost-group of partial $\mathbb{P}$-automorphisms.

We start with the construction of A_0.

Definition 3.2.1. Denote by $A_0(\mathit{levels})$ the collection of all $\pi = (\pi(\kappa) \mid \kappa \in \mathrm{Card}, \kappa < \operatorname{ht}\pi)$ with $\operatorname{ht}\pi$, the *height* of π, a cardinal, such that each $\pi(\kappa) : \{(\kappa, i) \mid i < F_{\lim}(\kappa)\} \to \{(\kappa, i) \mid i < F_{\lim}(\kappa)\}$ is a bijection with finite support $\operatorname{supp}\pi(\kappa) := \{(\kappa, i) \mid \pi(\kappa)(\kappa, i) \neq (\kappa, i)\}$.

A map $\pi \in A_0(levels)$ induces an automorphism π_{tree} on the class of $F_{\lim}$-trees as follows: Set $\pi_{tree}(t, \leq_t) := (s, \leq_s)$ with $s := \pi[t] := \{\pi(\kappa)(\kappa,i) \mid (\kappa,i) \in t\}$, where for $\kappa \geq \operatorname{ht}\pi$, we take for $\pi(\kappa)$ the identity on $\{(\kappa,i) \mid i < F_{\lim}(\kappa)\}$. Let $\leq_s := \pi[\leq_t] := \{(\pi(\kappa)(\kappa,i), \pi(\lambda)(\lambda,k)) \mid (\kappa,i) \leq_t (\lambda,k)\}$.
Moreover, π induces an automorphism $\overline{\pi} : \mathbb{P}_0 \to \mathbb{P}_0$: For $p \in \mathbb{P}_0$, $p : t(p) \to V$, let $\overline{\pi}(p) : \pi_{tree}(t(p), \leq_{t(p)}) \to V$ with $\overline{\pi}(p)\big(\pi(\kappa)(\kappa,i)\big) = p(\kappa,i)$ for all $(\kappa,i) \in t(p)$.
Let

$$A_0 := \{\overline{\pi} \mid \pi \in A_0(levels)\}.$$

We will often confuse an automorphism π with its extensions π_{tree} and $\overline{\pi}$.

Note that for an $F_{\lim}$-tree $t(p)$, it follows that $\pi(t(p))$ is essentially the same tree, where only the vertices (κ,i) have now different "names" $\pi(\kappa)(\kappa,i)$.

It is not difficult to see that any $\pi \in A_0$ can be described below $\alpha := \operatorname{ht}\pi + 1$; and any automorphism $\pi \in A_0$ is nicely level-preserving (cf Definition 1.4.3).

As usual, every $\pi \in A_0$ can be extended to an automorphism on $\operatorname{Name}(\mathbb{P}_0)$, which will be denoted by the same letter.

Let κ be a cardinal and G_0 a V-generic filter on $\mathbb{P}_0$. For every $i < F_{\lim}(\kappa)$, the forcing $\mathbb{P}_0$ adjoins a new κ-subset $(G_0)_{(\kappa,i)}$ given by the branch through (κ,i):

$$(G_0)_{(\kappa,i)} = \{\zeta < \kappa \mid \exists\, p \in G_0\ \exists\, (\lambda,j) \leq_{t(p)} (\kappa,i) :\ p(\lambda,j)(\zeta) = 1\}.$$

Then $(G_0)_{(\kappa,i)}$ has a canonical $\mathbb{P}_0$-name

$$(\ddot{G}_0)_{(\kappa,i)} := \{(\zeta,p) \mid \zeta < \kappa\,,\ p \in \mathbb{P}_0 \restriction (\kappa+1)\,,\ \exists\, (\lambda,j) \leq_{t(p)} (\kappa,i) :\ p(\lambda,j)(\zeta) = 1\},$$

For any $\pi \in A_0$, it follows that $\pi\left((\ddot{G}_0)_{(\kappa,i)}\right) = (\ddot{G}_0)_{\pi(\kappa)(\kappa,i)}$. Thus, the automorphisms in A_0 allow for swapping the generic subsets.

(We use the notation $(\ddot{G}_0)_{(\kappa,i)}$ here, because later on, $(\dot{G}_0)_{(\kappa,i)}$ will be used for the canonical $\mathbb{P}$-name.)

We call an automorphism $\pi \in A_0$ *small* if it satisfies the following property:

For all (κ,i), it follows that $\pi(\kappa)(\kappa,i) = (\kappa,j)$ such that there is a limit ordinal $\gamma(i)$ with $i,j \in [\gamma(i), \gamma(i)+\omega)$.

It is not difficult to see that for any pair of conditions $p,q \in \mathbb{P}_0$, there is a small automorphism $\pi \in A_0$ with $\pi p \parallel q$. Indeed, by finiteness of the trees, it is possible to arrange that for any $(\kappa,i) \in t(p)$, we have $\pi(\kappa)(\kappa,i) \notin t(p) \cup t(q)$.

Now, we turn to $\mathbb{P}_1$. We first outline the basic ideas about how our almost-group A_1 of partial $\mathbb{P}$-automorphisms shall look like.

If G_1 is V-generic on $\mathbb{P}_1$, then for any $\kappa^+ \in Succ'$, $i < F(\kappa^+)$, the generic κ^+-subset

$$(G_1)_{(\kappa^+,i)} := \{\zeta \in [\kappa,\kappa^+) \mid \exists\, p \in G_1\ p(\kappa^+)(\zeta,i) = 1\}$$

has the canonical $\mathbb{P}_1$-name

$$(\ddot{G}_1)_{(\kappa^+,i)} := \{(\zeta, p) \mid \zeta \in [\kappa, \kappa^+), p \in \mathbb{P}_1 \upharpoonright (\kappa^+ + 1), p(\kappa^+)(\zeta, i) = 1\}.$$

(Again, we use the notation $(\ddot{G}_1)_{(\kappa,i)}$ here, because later on, $(\dot{G}_1)_{(\kappa,i)}$ will be used for the canonical $\mathbb{P}$-name.)

Firstly, we want that for any two generic κ^+-subsets $(G_1)_{(\kappa^+,i)}$ and $(G_1)_{(\kappa^+,i')}$, there is an automorphism $\pi \in A_1$ interchanging them. In other words: We want to include into A_1 the collection of all $\pi = (\pi(\kappa^+) \mid \kappa^+ \in \operatorname{supp} \pi)$ with finite support *supp* π, such that for every $\kappa^+ \in \operatorname{supp} \pi$, there is a bijection $f_\pi(\kappa^+)$ on a finite set *supp* $\pi(\kappa^+) \subseteq F(\kappa^+)$ with $\pi(\ddot{G}_1)_{(\kappa^+,i)} = (\ddot{G}_1)_{(\kappa^+, f_\pi(\kappa^+)(i))}$ for all $i \in \operatorname{supp} \pi(\kappa^+)$.
For these automorphisms π, we will have $\pi p(\kappa^+)(\zeta, i) = p(\kappa^+)\big(\zeta, f_\pi(\kappa^+)(i)\big)$ whenever $p \in \mathbb{P}_1$ with $\zeta \in [\kappa, \kappa^+)$, $i \in \operatorname{supp} \pi(\kappa^+)$. For all the remaining κ^+ and (ζ, i), we will have $\pi p(\kappa^+)(\zeta, i) = p(\kappa^+)(\zeta, i)$.

Moreover, we want that for any $p, q \in \mathbb{P}_1$, there is an automorphism $\pi \in A_1$ with $\pi p \parallel q$. These π will be of the form $\pi = (\pi(\kappa^+) \mid \kappa^+ \in \operatorname{supp} \pi)$ with finite support $\operatorname{supp} \pi$, such that for any $\kappa^+ \in \operatorname{supp} \pi$, there is $\operatorname{dom} \pi(\kappa^+) = \operatorname{dom}_x \pi(\kappa^+) \times \operatorname{dom}_y \pi(\kappa^+) \subseteq [\kappa, \kappa^+) \times F(\kappa^+)$ with $|\operatorname{dom} \pi(\kappa^+)| < \kappa^+$, and a collection

$$\big(\pi(\kappa^+)(\zeta, i) \mid (\zeta, i) \in \operatorname{dom} \pi(\kappa^+)\big) \in 2^{\operatorname{dom} \pi(\kappa^+)},$$

such that π changes the values $p(\kappa^+)(\zeta, i)$ if and only if $\pi(\kappa^+)(\zeta, i) = 1$. In other words: $\pi p(\kappa^+)(\zeta, i) \neq p(\kappa^+)(\zeta, i)$ whenever $\pi(\kappa^+)(\zeta, i) = 1$, and $\pi p(\kappa^+)(\zeta, i) = p(\kappa^+)(\zeta, i)$ in the case that $\pi(\kappa^+)(\zeta, i) = 0$ or $(\zeta, i) \notin \operatorname{dom} \pi(\kappa^+)$.

A_1 will be generated by those two types of automorphisms.

All the (ζ, i) with $(\zeta, i) \in \operatorname{dom} \pi(\kappa^+)$ *and* $i \in \operatorname{supp} \pi(\kappa^+)$ will have to be treated seperately: Namely, for any $\zeta \in \operatorname{dom}_x \pi(\kappa^+)$, we will have a bijection $\pi(\kappa^+)(\zeta)$ which maps the sequence $(p(\zeta, i) \mid i \in \operatorname{supp} \pi(\kappa^+))$ to $((\pi p)(\zeta, i) \mid i \in \operatorname{supp} \pi(\kappa^+))$.
These bijections $\pi(\kappa^+)(\zeta)$ will be necessary to retain a group structure.

We will now define the collection of dense classes D_s that will serve as the domains of the partial automorphisms $\pi : D_s \overset{\cong}{\to} D_s$ in A_1. We need a class of parameters S and a formula φ, such that any D_s is of the form $D_s = \{p \in \mathbb{P}_1 \mid \varphi(p, s)\}$ (cf. Definition 1.4.7).

Definition/Proposition 3.2.2. Let S be the class of all $s = (\operatorname{dom} s(\kappa^+) \mid \kappa^+ \in \operatorname{supp} s)$ such that $\operatorname{supp} s \subseteq Succ'$ is finite, and any $\operatorname{dom} s(\kappa^+)$ is of the form $\operatorname{dom} s(\kappa^+) = \operatorname{dom}_x s(\kappa^+) \times \operatorname{dom}_y s(\kappa^+)$ with $\operatorname{dom}_x s(\kappa^+) \subseteq [\kappa, \kappa^+)$, $\operatorname{dom}_y s(\kappa^+) \subseteq F(\kappa^+)$, and $|\operatorname{dom} s(\kappa^+)| < \kappa^+$.

Let $\varphi(p, s)$ be the formula

$$p \in \mathbb{P}_1 \wedge \forall\, \kappa^+ \in (\operatorname{supp} p \cap \operatorname{supp} s) \;\; \operatorname{dom} p(\kappa^+) \supseteq \operatorname{dom} s(\kappa^+).$$

Then a), b) and c) from Definition 1.4.7 hold.

Proof. We observe that S consists of all finite sequences $s = (\operatorname{dom} s(\kappa^+) \mid \kappa^+ \in \operatorname{supp} p)$ which are allowed as domains for conditions in $\mathbb{P}_1$, and $\varphi(p, s)$ states that p is a condition in $\mathbb{P}_1$, the domains of which extend the domains given by s.

a) For every $s \in S$, the set

$$D_s = \{p \in \mathbb{P}_1 \mid \forall\, \kappa^+ \in (\operatorname{supp} p \cap \operatorname{supp} s) \ \ \operatorname{dom} p(\kappa^+) \supseteq \operatorname{dom} s(\kappa^+)\}$$

is dense in $\mathbb{P}_1$, and D_s can be described below $\alpha(s) := \max\{\kappa^+ \in \operatorname{supp} s \mid \operatorname{dom} s(\kappa^+) \neq \emptyset\}$.

b) Let $s_0,\ s_1 \in S$. Then we can construct $s_2 \in S$ with $D_{s_2} = D_{s_0} \cap D_{s_1}$ as follows: Let $\operatorname{supp} s_2 = \operatorname{supp} s_1 \cup \operatorname{supp} s_0$, and $\operatorname{dom}_x s_2(\kappa^+) := \operatorname{dom}_x s_0(\kappa^+) \cup \operatorname{dom}_x s_1(\kappa^+)$, $\operatorname{dom}_y s_2(\kappa^+) = \operatorname{dom}_y s_0(\kappa^+) \cup \operatorname{dom}_y s_1(\kappa^+)$ for all $\kappa^+ \in \operatorname{supp} s_2$. Then $D_{s_2} = \{p \in \mathbb{P}_1 \mid \varphi(p, s_2)\} = \{p \in \mathbb{P}_1 \mid \varphi(p, s_0) \wedge \varphi(p, s_1)\} = D_{s_0} \cap D_{s_1}$, with $\alpha(s_2) = \max\{\alpha(s_0), \alpha(s_1)\}$.

c) Setting $s_{\max} := \emptyset$, it follows that $D_{s_{\max}} = \mathbb{P}_1$ with $D_{s_{\max}} \supseteq D_s$ for all $s \in S$.

□

We can now define our almost-group A_1:

Definition 3.2.3. Let A_1 consist of all automorphisms $\pi : D_\pi \to D_\pi$, $\pi = (\pi(\kappa^+) \mid \kappa^+ \in \operatorname{supp}\pi)$ with finite support $\operatorname{supp}\pi \subseteq Succ'$ such that for all $\kappa^+ \in \operatorname{supp}\pi$, there are

- a finite set $\operatorname{supp}\pi(\kappa^+) \subseteq F(\kappa^+)$ with a bijection $f_\pi(\kappa^+) : \operatorname{supp}\pi(\kappa^+) \to \operatorname{supp}\pi(\kappa^+)$,
- a domain $\operatorname{dom}\pi(\kappa^+) = \operatorname{dom}_x \pi(\kappa^+) \times \operatorname{dom}_y \pi(\kappa^+) \subseteq [\kappa, \kappa^+) \times F(\kappa^+)$ with $|\operatorname{dom}\pi(\kappa^+)| < \kappa^+$ such that $\operatorname{supp}\pi(\kappa^+) \subseteq \operatorname{dom}_y \pi(\kappa^+)$, and a collection $\big(\pi(\kappa^+)(\zeta, i) \mid (\zeta, i) \in [\kappa, \kappa^+) \times F(\kappa^+)\big)$ with $\pi(\kappa^+)(\zeta, i) \in 2$ for all (ζ, i) and $\pi(\kappa^+)(\zeta, i) = 0$ whenever $(\zeta, i) \notin \operatorname{dom}\pi(\kappa^+)$, and
- for any $\zeta \in \operatorname{dom}_x \pi(\kappa^+)$, a bijection $\pi(\kappa^+)(\zeta) : 2^{\operatorname{supp}\pi(\kappa^+)} \to 2^{\operatorname{supp}\pi(\kappa^+)}$,

such that setting $s(\pi) := (\operatorname{dom}\pi(\kappa^+) \mid \kappa^+ \in \operatorname{supp}\pi)$, it follows that $s(\pi) \in S$ with

$$D_\pi = D_{s(\pi)} = \{p \in \mathbb{P}_1 \mid \forall \kappa^+ \in (\operatorname{supp} p \cap \operatorname{supp}\pi) \ \ \operatorname{dom} p(\kappa^+) \supseteq \operatorname{dom}\pi(\kappa^+)\};$$

and for any $p \in D_\pi$, the condition πp is defined as follows:

We will have $\operatorname{supp}(\pi p) = \operatorname{supp} p$ with $\pi p(\kappa^+) = p(\kappa^+)$ whenever $\kappa^+ \in \operatorname{supp} p \setminus \operatorname{supp}\pi$. Let now $\kappa^+ \in \operatorname{supp} p \cap \operatorname{supp}\pi$.

- For any $i \in \operatorname{supp}\pi(\kappa^+)$ and $\zeta \notin \operatorname{dom}_x \pi(\kappa^+)$, we have

$$\pi p(\kappa^+)(\zeta, i) = p(\kappa^+)\big(\zeta, f_\pi(\kappa^+)(i)\big).$$

- For $\zeta \in \operatorname{dom}_x \pi(\kappa^+)$,

$$\big(\pi p(\kappa^+)(\zeta, i) \mid i \in \operatorname{supp}\pi(\kappa^+)\big) = \pi(\kappa^+)(\zeta)\big(p(\kappa^+)(\zeta, i) \mid i \in \operatorname{supp}\pi(\kappa^+)\big).$$

- Whenever $i \notin \operatorname{supp}\pi(\kappa^+)$, then $\pi p(\kappa^+)(\zeta, i) := p(\kappa^+)(\zeta, i)$ if $\pi(\kappa^+)(\zeta, i) = 0$, and $\pi p(\kappa^+)(\zeta, i) \neq p(\kappa^+)(\zeta, i)$ in the case that $\pi(\kappa^+)(\zeta, i) = 1$.

In other words: Outside the domain $\operatorname{dom}\pi(\kappa^+)$, we have a swap of the horizontal lines $p(\kappa^+)(\cdot, i)$ for $i \in \operatorname{supp}\pi(\kappa^+)$ according to $f_\pi(\kappa^+)$.
Inside $\operatorname{dom}\pi(\kappa^+)$, the values $\pi p(\kappa^+)(\zeta, i)$ for $i \in \operatorname{supp}\pi(\kappa^+)$ are determined by the maps $\pi(\kappa^+)(\zeta)$; while any of the remaining values $\pi p(\kappa^+)(\zeta, i)$ with $i \notin \operatorname{supp}\pi(\kappa^+)$ is equal to $p(\kappa^+)(\zeta, i)$ if and only if $\pi(\kappa^+)(\zeta, i) = 1$.

We need the dense sets D_π to make sure that $\operatorname{dom} p(\kappa^+)$ is not mixed up by π.
For notational convenience, we write D_π rather than $D_{s(\pi)}$, but keep in mind that any D_π is of the form $D_\pi = D_{s(\pi)} = \{p \in \mathbb{P}_1 \mid \varphi(p, s(\pi))\}$, where $s(\pi) = (\operatorname{dom}\pi(\kappa^+) \mid \kappa^+ \in \operatorname{supp}\pi) \in S$.

Lemma 3.2.4. *A_1 is an almost-group of partial $\mathbb{P}$-automorphisms for φ and S.*

Proof. We go through Definition 1.4.7. By 3.2.2, it remains to make sure that d), e) and f) hold.

d) It is not difficult to verify that any $\pi \in A_1$ is order preserving. The inverse map $\pi^{-1} \in A_1$ can be written down explicitly, using Definition 3.2.3. Moreover, it follows from Definition 3.2.3 that any $\pi \in A_1$ is nicely level-preserving and can be described below some ordinal α (take for α the maximal element of $\operatorname{supp}\pi$), i.e. there exists an automorphism $\pi_\alpha : D_\pi \cap \mathbb{P}_\alpha \to D_\pi \cap \mathbb{P}_\alpha$ with $\pi = \overline{\pi_\alpha}$.

e) Let $s \in S$, and α an ordinal with $\alpha \geq \alpha(s)$. We have to make sure that

$$(A_1)_{(s,\alpha)} := \{\pi = \overline{\pi_\alpha} \in A_1 \mid \pi_\alpha : D_s \cap \mathbb{P}_\alpha \overset{\cong}{\to} D_s \cap \mathbb{P}_\alpha\}$$

is a group. Let $\pi = \overline{\pi_\alpha}$, $\sigma = \overline{\sigma_\alpha} \in (A_1)_{(s,\alpha)}$. Firstly, it is not difficult to write down a map $\nu \in (A_1)_{(s,\alpha)}$, $\nu = \overline{\nu_\alpha}$ such that $\nu_\alpha = (\pi_\alpha)^{-1}$. Then ν is the inverse of π. Secondly, using Definition 3.2.3, one can write down a map $\tau \in (A_1)_{(s,\alpha)}$, $\tau = \overline{\tau_\alpha}$ such that $\tau_\alpha(p) = \pi_\alpha(\sigma_\alpha(p))$ for all $p \in \mathbb{P}_\alpha \cap D_s$. Then $\tau = \pi \circ \sigma$. Thirdly, $(A_1)_{(s,\alpha)}$ contains the identity element id_{D_s} (the identity map on D_s), since $\mathrm{id}_{D_s} = \overline{\mathrm{id}_{D_s \cap \mathbb{P}_\alpha}}$. Hence, $(A_1)_{(s,\alpha)}$ is indeed a group.

f) Let s, $s' \in S$ with $D_s \subseteq D_{s'}$, and $\alpha' \geq \alpha(s')$. For any $\pi \in (A_1)_{(s',\alpha')}$, it follows that $\pi[D_s] = D_s$, since the maps π in A_1 do not change the support and domains of the conditions. Moreover, $\pi \upharpoonright D_s \in (A_1)_{(s,\alpha)}$ for every $\alpha \geq \max\{\alpha', \alpha(s)\}$ follows from Definition 3.2.3.

□

Definition 3.2.5. Let $A := A_0 \times A_1$, i.e. any $\pi \in A$ is of the form $\pi = (\pi_0, \pi_1)$, where $\pi_0 \in A_0$, $\pi_0 : \mathbb{P}_0 \to \mathbb{P}_0$; and $\pi_1 \in A_1$, $\pi_1 : D_{\pi_1} \to D_{\pi_1}$ is a partial $\mathbb{P}_1$-automorphism.

By what we have just shown, it follows that A is an almost-group of partial $\mathbb{P}$-automorphisms.

Let $\overline{A}$ denote the *group of partial $\mathbb{P}$-automorphisms derived from A* as in Definition 1.4.8:

For π, $\pi' \in A$, $\pi : D_\pi \to D_\pi$, $\pi' : D_{\pi'} \to D_{\pi'}$, we set

$$\pi \sim \pi' \;:\Leftrightarrow\; \pi \upharpoonright (D_\pi \cap D_{\pi'}) = \pi' \upharpoonright (D_\pi \cap D_{\pi'}),$$

and use a method similar to *Scott's Trick* to obtain set-sized equivalence classes $[\pi] = [\pi]_\sim$ (cf. p. 60 and 61). Then we set $\overline{A} := \{[\pi] \mid \pi \in A\}$. Concatenation in $\overline{A}$ is given by concatenation in A, and the group structure of the $A_{(s,\alpha)}$ gives a group structure on $\overline{A}$ as described in 1.4.8.

Now, we are ready to introduce a finitely generated symmetric system $\mathcal{S}$ for $\overline{A}$. For this, we introduce formulas $\psi_0(x,y), \dots, \psi_3(x,y)$, and $\chi_0(y), \dots, \chi_3(y)$ such that that for each $i \in \{0,1,2,3\}$ the following holds (cf. Definition 1.4.13): Firstly, whenever π, $\pi' \in A$, then $\psi_i(\pi, y) \Leftrightarrow \psi_i(\pi', y)$ for all y with $\chi_i(y)$. Secondly, for all y with $\chi_i(y)$, it follows that

$$\overline{A}_i(y) := \{[\pi] \in \overline{A} \mid \psi_i(\pi, y)\}$$

is a subgroup of $\overline{A}$. And thirdly, the *normality property* holds for the $\overline{A}_i(y)$, see 1.4.13 c).

Then a subgroup $B \subseteq \overline{A}$ *gives rise to symmetry* if there are finitely many $\overline{A}_{i_0}(y_0), \dots, \overline{A}_{i_{n-1}}(y_{n-1})$, where $i_0, \dots, i_{n-1} \in \{0,1,2,3\}$ and $\chi_{i_0}(y_0), \dots, \chi_{i_{n-1}}(y_{n-1})$ hold, with

$$B \supseteq \overline{A}_{i_0}(y_0) \cap \dots \cap \overline{A}_{i_{n-1}}(y_{n-1}).$$

A name $\dot{x}$ is *symmetric*, if the stabilizer group

$$\operatorname{sym}^{\overline{A}}(\dot{x}) = \left\{ [\pi] \in \overline{A} \mid \pi \overline{x}^{D_\pi} = \overline{x}^{D_\pi} \right\}$$

gives rise to symmetry.

We now go through the four types of $\overline{A}$-subgroups that we want to include into our finitely generated symmetric system $\mathcal{S}$, and give some motivation.

Firstly, for any $\kappa \in \mathrm{Card}, i < F_{\lim}(\kappa)$, we want to include

$$Fix_0(\kappa, i) := \left\{ [\pi] = [(\pi_0, \pi_1)] \in \overline{A} \;\middle|\; \pi_0(\kappa)(\kappa, i) = (\kappa, i) \right\},$$

which is well-defined (since for any π, $\pi' \in A$ with $\pi \sim \pi'$, it follows that $\pi \in \mathrm{Fix}_0(\kappa, i) \Leftrightarrow \pi' \in \mathrm{Fix}_0(\kappa, i)$), and $Fix_0(\kappa, i)$ is an $\overline{A}$-subgroup.

By this, we make sure that any canonical name

$$(\dot{G}_0)_{(\kappa,i)} := \{(\zeta, p) \mid \zeta < \kappa\,,\, p \in \mathbb{P} \upharpoonright (\kappa+1)\,,\, \exists\, (\lambda, j) \leq_{t(p)} (\kappa, i) :\; p(\lambda, j)(\zeta) = 1\}$$

is hereditarily symmetric; since

$$\pi\, \overline{(G_0)_{(\kappa,i)}}^{D_\pi} = \overline{(G_0)_{(\kappa,i)}}^{D_\pi}$$

whenever $[\pi] \in Fix_0(\kappa, i)$. Thus, our model N will contain any of the adjoined κ-subsets $(G_0)_{(\kappa,i)}$ given by the branches through the generic tree.

Now, we turn to the second type of $\overline{A}$-subgroup for our finitely generated symmetric system $\mathcal{S}$: For any cardinal κ and $\alpha < F_{\lim}(\kappa)$, we want in N a surjection $s : \wp(\kappa) \to \alpha$; which gives $\theta^N(\kappa) \geq F(\kappa)$ for all limit cardinals κ. However, we have to make sure that $\theta^N(\kappa) < F(\kappa)^+$; so the sequence $\big((G_0)_{(\kappa,i)} \mid i < F(\kappa)\big)$ must **not** be contained in N.

Therefore, for cardinals κ and $\alpha < F_{\lim}(\kappa)$ a limit ordinal, we consider the subgroup $Small_0(\kappa, [0,\alpha))$ containing all $[\pi] = [(\pi_0, \pi_1)]$ with the property that $\pi_0(\kappa)$ is *small below* α, i.e. for any $i < \alpha$, it follows that $\pi_0(\kappa)(\kappa, i) = (\kappa, j)$ such that $i, j \in [\gamma(i), \gamma(i) + \omega)$ for some limit ordinal $\gamma(i)$:

$$Small_0(\kappa, [0,\alpha)) := \Big\{ [\pi] = [(\pi_0, \pi_1)] \in \overline{A} \;\Big|\; \forall\, i < \alpha,\ i \in [\gamma(i), \gamma(i) + \omega) \text{ with } \gamma(i) \text{ a limit}$$

$$\text{ordinal} \;:\; \big(\pi_0(\kappa)(\kappa, i) = (\kappa, j) \text{ for some } j \in [\gamma(i), \gamma(i) + \omega)\ \big) \Big\}.$$

Then $Small_0(\kappa, [0,\alpha))$ is well-defined, since for any π, $\pi' \in A$ with $\pi \sim \pi'$, it follows that $\pi(\kappa)(\kappa, i) = \pi'(\kappa)(\kappa, i)$. Moreover, $Small_0(\kappa, [0,\alpha))$ is a subgroup of $\overline{A}$.

Now, for any limit ordinal $i < \alpha$, we can define a "cloud" around $(\dot{G}_0)_{(\kappa,i)}$ as follows:

$$(\dot{\widetilde{G_0}})^{\alpha}_{(\kappa,i)} := \Big\{ \big(\pi\, \overline{(G_0)_{(\kappa,i)}}^{D_\pi}, \mathbb{1}\big) \;\Big|\; [\pi] \in Small_0(\kappa, [0,\alpha)) \Big\} =$$

$$= \Big\{ \big(\overline{(G_0)_{(\kappa,i+n)}}^{D_\pi}, \mathbb{1} \big) \;\Big|\; n < \omega \Big\}.$$

Then $(\widetilde{G_0})^{\alpha}_{(\kappa,i)} := \big((\dot{\widetilde{G_0}})^{\alpha}_{(\kappa,i)}\big)^G$ is the set of all $(G_0)_{(\kappa,i+n)}$ for $n < \omega$; hence, any two distinct clouds $(\widetilde{G_0})^{\alpha}_{(\kappa,i)}$ and $(\widetilde{G_0})^{\alpha}_{(\kappa,j)}$ for limit ordinals $i, j < \alpha$ are disjoint. It follows that the sequence

$$\big((\widetilde{G_0})^{\alpha}_{(\kappa,i)} \mid i < \alpha \text{ a limit ordinal} \big),$$

which has a canonical symmetric name stabilized by all $\pi \in Small_0(\kappa, [0,\alpha))$, yields a surjection $s : \wp(\kappa) \to \alpha$ in N.

This argument is carried out in more detail in Proposition 3.4.1.

Moreover, for any $\kappa \in Succ'$, $\kappa = \overline{\kappa}^+$ and $i < F(\kappa)$, we include into our finitely generated symmetric system $\mathcal{S}$:

$$Fix_1(\kappa, i) := \big\{ [\pi] = [(\pi_0, \pi_1)] \in \overline{A} \;\big|\; \forall p \in D_\pi \;:\; (\pi p) \upharpoonright \{(\kappa, i)\} = p \upharpoonright \{(\kappa, i)\} \big\}.$$

As before, $\mathrm{Fix}_1(\kappa, i)$ is a well-defined subgroup of $\overline{A}$.

By this, we make sure that any generic κ-subset $(G_1)_{(\kappa,i)}$ is contained in our eventual symmetric submodel N; since with the canonical name

$$(\dot{G}_1)_{(\kappa,i)} := \{ (\zeta, p) \mid \zeta \in [\kappa, \kappa^+)\,,\, p \in \mathbb{P} \upharpoonright (\kappa^+ + 1)\,,\, p(\kappa^+)(\zeta, i) = 1 \},$$

it follows that $\pi\overline{(G_1)_{(\kappa,i)}}^{D_\pi} = \overline{(G_1)_{(\kappa,i)}}^{D_\pi}$ for all $\pi \in Fix_1(\kappa, i)$.

Again, we have to make sure that the sequence $\big((G_1)_{(\kappa,i)} \mid i < F(\kappa)\big)$ is **not** contained in N, in order to achieve $\theta^N(\kappa) \le F(\kappa)$. On the other hand, we need surjections $s : \wp(\kappa) \to \alpha$ for all $\alpha < F(\kappa)$. Thus, we include into our finitely generated symmetric system for $\kappa \in Succ'$, $\alpha < F(\kappa)$:

$$Small_1(\kappa, [0,\alpha)) := \big\{ [\pi] = [(\pi_0, \pi_1)] \in \overline{A} \;\big|\; \forall\, i < \alpha \;\; (i \notin \operatorname{supp} \pi_1(\kappa) \vee f_{\pi_1}(\kappa)(i) = i) \big\}.$$

Again, $Small_1(\kappa,[0,\alpha))$ is a well-defined $\overline{A}$-subgroup.

Moreover, $Small_1(\kappa,[0,\alpha))$ does not contain any of those automorphisms that interchange some $(\dot{G}_1)_{(\kappa,i)}$ and $(\dot{G}_1)_{(\kappa,j)}$ for $i,\ j<\alpha$. Thus, for any $i<\alpha$, we can define a "cloud" $(\widetilde{G_1})^{\alpha}_{(\kappa,i)}$ around $(G_1)_{(\kappa,i)}$ with the symmetric name

$$(\dot{\widetilde{G_1}})^{\alpha}_{(\kappa,i)} := \big\{ \big(\pi\overline{(G_1)_{(\kappa,i)}}^{D_\pi}, \mathbb{1}\big) \mid [\pi] \in Small_1(\kappa,[0,\alpha)) \big\}$$

such that with $(\widetilde{G_1})^{\alpha}_{(\kappa,i)} := \big((\dot{\widetilde{G_1}})^{\alpha}_{(\kappa,i)}\big)^G$, it follows that any two distinct clouds $(\widetilde{G_1})_{(\kappa,i)}$ and $(\widetilde{G_1})_{(\kappa,j)}$ are disjoint. Hence, the sequence $\big((\widetilde{G_1})_{(\kappa,i)} \mid i<\alpha\big)$, which has a canonical symmetric name stabilized by all $\pi\in Small_1(\kappa,[0,\alpha))$, gives a surjection $s:\mathcal{P}(\kappa)\to\alpha$ in N.

This concludes the introduction of our finitely generated system $\mathcal{S}$ for $\overline{A}$, and we have already checked 1.4.13 a) and b). Regarding 1.4.13 c), it is not difficult to verify:

Lemma 3.2.6. • *For all $\pi\in A$ and $\kappa\in\mathrm{Card}$, $i<F_{\lim}(\kappa)$,*

$$[\pi]Fix_0(\kappa,i)[\pi]^{-1} \supseteq Fix_0(\kappa,i)\cap\bigcap\{Fix_0(\kappa,j)\mid(\kappa,j)\in\operatorname{supp}\pi_0(\kappa)\}.$$

- *For $\pi\in A$ and $\kappa\in\mathrm{Succ}'$, $i<F(\kappa)$,*

$$[\pi]Fix_1(\kappa,i)[\pi]^{-1} \supseteq Fix_1(\kappa,i)\cap\bigcap\{Fix_1(\kappa,j)\mid j\in\operatorname{supp}\pi_1(\kappa)\}.$$

- *For $\pi\in A$ and $\kappa\in\mathrm{Card}$, $\alpha<F_{\lim}(\kappa)$ a limit ordinal,*

$$[\pi]Small_0(\kappa,[0,\alpha))[\pi]^{-1} \supseteq Small_0(\kappa,[0,\alpha))\cap\bigcap\{Fix_0(\kappa,j)\mid(\kappa,j)\in\operatorname{supp}\pi_0(\kappa)\}.$$

- *For $\pi\in A$ and $\kappa\in\mathrm{Succ}'$, $\alpha<F(\kappa)$,*

$$[\pi]Small_1(\kappa,[0,\alpha))[\pi]^{-1} \supseteq Small_1(\kappa,[0,\alpha))\cap\bigcap\{Fix_1(\kappa,j)\mid j\in\operatorname{supp}\pi_1(\kappa)\}.$$

We conclude:

Definition/Proposition 3.2.7. The $\overline{A}$-subgroups

- $Fix_0(\kappa,i)$ for $\kappa\in\mathrm{Card}$, $i<F_{\lim}(\kappa)$
- $Small_0(\kappa,[0,\alpha))$ for $\kappa\in\mathrm{Card}$, $\alpha<F_{\lim}(\kappa)$ a limit ordinal
- $Fix_1(\kappa,i)$ for $\kappa\in Succ'$, $i<F(\kappa)$, and
- $Small_1(\kappa,[0,\alpha))$ for $\kappa\in Succ'$, $\alpha<F(\kappa)$

yield a finitely generated symmetric system as in Definition 1.4.13, denoted by $\mathcal{S}$.

The following Definition corresponds to Definition 1.4.14:

Definition 3.2.8. A subgroup $B\subseteq\overline{A}$ *gives rise to symmetry with respect to* $\mathcal{S}$ if there are $n,m,\overline{n},\overline{m}<\omega$ and

- $\kappa_0, \dots, \kappa_{n-1} \in \text{Card}$, $i_0 < F_{\lim}(\kappa_0), \dots, i_{n-1} < F_{\lim}(\kappa_{n-1})$,
- $\lambda_0, \dots, \lambda_{m-1} \in \text{Card}$, $\alpha_0 < F_{\lim}(\lambda_0), \dots, \alpha_{m-1} < F_{\lim}(\lambda_{m-1})$ limit ordinals,
- $\overline{\kappa}_0, \dots, \overline{\kappa}_{\overline{n}-1} \in \text{Succ}'$, $\overline{\imath}_0 < F(\overline{\kappa}_0), \dots, \overline{\imath}_{\overline{n}-1} < F(\overline{\kappa}_{\overline{n}-1})$, *and*
- $\overline{\lambda}_0, \dots, \overline{\lambda}_{\overline{m}-1} \in \text{Succ}', \overline{\alpha}_0 < F(\overline{\lambda}_0), \dots, \overline{\alpha}_{\overline{m}-1} < F(\overline{\lambda}_{\overline{m}-1})$

such that B is a superset of the following intersection:

$$Fix_0(\kappa_0, i_0) \cap \dots \cap Fix_0(\kappa_{n-1}, i_{n-1}) \cap Small_0(\lambda_0, [0, \alpha_0)) \cap \dots \cap Small_0(\lambda_{m-1}, [0, \alpha_{m-1})) \cap$$

$$\cap Fix_1(\overline{\kappa}_0, \overline{\imath}_0) \cap \dots \cap Fix_1(\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1}) \cap Small_1(\overline{\lambda}_0, [0, \overline{\alpha}_0)) \cap \dots \cap Small_1(\overline{\lambda}_{\overline{m}-1}, [0, \overline{\alpha}_{\overline{m}-1})).$$

We define corresponding to Definition 1.4.17:

Definition 3.2.9. A $\mathbb{P}$-name $\dot{x}$ is *symmetric for $\mathcal{S}$* if the stabilizer group

$$\text{sym}^{\overline{A}}(\dot{x}) := \left\{ [\pi] \in \overline{A},\, \pi : D_\pi \to D_\pi \;\middle|\; \pi \overline{x}^{D_\pi} = \overline{x}^{D_\pi} \right\}$$

gives rise to symmetry with respect to $\mathcal{S}$. Recursively, a name $\dot{x}$ is *hereditarily symmetric*, $\dot{x} \in HS^{\mathcal{S}}$, if $\dot{x}$ is symmetric and $\dot{y}$ is hereditarily symmetric for all $\dot{y} \in \operatorname{dom} \dot{x}$.

3.3 The Symmetric Submodel

Fix a V-generic filter G on $\mathbb{P}$.

Definition 3.3.1. The *symmetric extension by $\mathcal{S}$* and G is

$$N := V(G)^{\mathcal{S}} := \{\dot{x}^G \mid \dot{x} \in HS^{\mathcal{S}}\}.$$

We claim that N satifies the statement from our theorem, i.e. $N \vDash \mathsf{ZF}$, N preserves all V-cardinals, and $\theta^N(\kappa) = F(\kappa)$ for all κ.

We will work with the structure $\langle N, \in, V \rangle = \langle V(G)^{\mathcal{S}}, \in, V \rangle$, where we have a unary predicate symbol for the ground model.

Since $\mathbb{P}$ is approachable by projections, it follows that the Forcing Theorem holds for $\Vdash^V_{\mathbb{P}}$, ans also for the symmetric forcing relation $(\Vdash_s)^V_{\mathbb{P}, \mathcal{S}}$.
In this chapter, we will verify that N is indeed a model of ZF, although the class forcing $\mathbb{P}$ does not preserve ZFC. Later on, we will see that any set of ordinals located in N can be captured in a "mild" V-generic extension by set forcing that preserves cardinals and the GCH.

By Proposition 1.4.21, it follows that N is satisfies the axioms of *Extensionality*, *Foundation*, *Pairing*, *Weak Union* and *Infinity*.

Proposition 3.3.2. *The Axiom of Separation holds in $\langle V[G], \in, V \rangle$ and $\langle N, \in, V \rangle$ for every $\mathcal{L}^A_\in$-formula φ.*

Proof. We first consider $V[G]$. Let $a \in V[G]$ and $\varphi \in \mathcal{L}_\in^A$.
W.l.o.g. assume $n = 1$ and take a parameter $z := z_0$ in $V[G]$. We have to show that there is $b \in V[G]$ with

$$b = \{x \in a \mid \langle V[G], \in, V\rangle \vDash \varphi(x, z, V)\}.$$

Take a cardinal λ large enough such that there are names $\dot{a}, \dot{z} \in \mathrm{Name}(\mathbb{P} \upharpoonright (\lambda + 1))$ with $a = \dot{a}^{G\upharpoonright(\lambda+1)}$, $z = \dot{z}^{G\upharpoonright(\lambda+1)}$.
Let

$$\dot{b} := \{(\dot{x}, p) \mid \dot{x} \in \mathrm{dom}\ \dot{a},\, p \in \mathbb{P} \upharpoonright (\lambda+1), p \Vdash_{\mathbb{P}}^V (\dot{x} \in \dot{a} \wedge \varphi(\dot{x}, \dot{z}, \check{V}))\}.$$

We claim that $\dot{b}^G = b$. The direction "$\subseteq$" is clear. Concerning "$\supseteq$", consider $x \in b$. Let $\dot{x} \in \mathrm{dom}\, \dot{a}$ with $x = \dot{x}^G$ and $\overline{p} \in G$ with

$$\overline{p} \Vdash_{\mathbb{P}}^V (\dot{x} \in \dot{a} \wedge \varphi(\dot{x}, \dot{z}, \check{V})).$$

Let $p := \overline{p} \upharpoonright (\lambda + 1)$. It suffices to verify also $p \Vdash_{\mathbb{P}}^V (\dot{x} \in \dot{a} \wedge \varphi(\dot{x}, \dot{z}, \check{V}))$. If not, there would be $q \in \mathbb{P}$, $q \leq p$ with

$$q \Vdash_{\mathbb{P}}^V \neg(\dot{x} \in \dot{a} \wedge \varphi(\dot{x}, \dot{z}, \check{V})).$$

We construct a $\mathbb{P}$-automorphism π with $\pi\overline{p} \parallel q$ such that π is the identity on $\mathbb{P} \upharpoonright (\lambda + 1)$. Then $\pi\overline{x}^{D_\pi} = \overline{x}^{D_\pi}$, $\pi\overline{a}^{D_\pi} = \overline{a}^{D_\pi}$ and $\pi\overline{z}^{D_\pi} = \overline{z}^{D_\pi}$; hence,

$$\pi\overline{p} \Vdash_{\mathbb{P}}^V (\overline{x}^{D_\pi} \in \overline{a}^{D_\pi} \wedge \varphi(\overline{x}^{D_\pi}, \overline{z}^{D_\pi}, \check{V})),$$

contradicting that $\pi\overline{p} \parallel q$.

We start with π_0. Let $\mathrm{ht}\, \pi_0 := \max\{\eta(\overline{p}), \eta(q)\}$. For $\alpha \leq \lambda$, let $\pi_0(\alpha)$ be the identity. For $\lambda^+ \leq \alpha \leq \mathrm{ht}\, \pi_0$, take for $\pi_0(\alpha)$ a bijection on $\{(\alpha, i) \mid i < F_{\lim}(\alpha)\}$ with finite support such that for any $(\alpha, i) \in t(\overline{p})$, it follows that $\pi_0(\alpha)(\alpha, i) = (\alpha, j)$ for some $(\alpha, j) \notin t(\overline{p}) \cup t(q)$. Then from $q \leq \overline{p} \upharpoonright (\lambda + 1)$ it follows that $\pi_0\overline{p}_0 \parallel q_0$.

Now, we turn to π_1. Let $\mathrm{supp}\, \pi_1 := \mathrm{supp}\, \overline{p}_1 \cup \mathrm{supp}\, q_1$. For $\alpha^+ \in \mathrm{supp}\, \pi_1$ with $\alpha^+ \leq \lambda$, let $\pi_1(\alpha^+)$ be the identity. For $\alpha^+ \in \mathrm{supp}\, \pi_1$ with $\alpha^+ > \lambda$, we define $\pi_1(\alpha^+)$ as follows: Let $\mathrm{dom}\, \pi_1(\alpha^+) := \mathrm{dom}\, \overline{p}_1(\alpha^+) \cap \mathrm{dom}\, q_1(\alpha^+)$ and $\mathrm{supp}\, \pi_1(\alpha^+) = \varnothing$; then we only need to define $\pi_1(\alpha^+)(\zeta, i)$ for $\zeta \in \mathrm{dom}_x\, \overline{p}_1(\alpha^+) \cap \mathrm{dom}_x\, q_1(\alpha^+)$, $i \in \mathrm{dom}_y\, \overline{p}_1(\alpha^+) \cap \mathrm{dom}_y\, q_1(\alpha^+)$. Let $\pi_1(\alpha^+)(\zeta, i) = 0$ if $\overline{p}_1(\alpha^+)(\zeta, i) = q_1(\alpha^+)(\zeta, i)$, and $\pi_1(\alpha^+)(\zeta, i) = 1$ in the case that $\overline{p}_1(\alpha^+)(\zeta, i) \neq q_1(\alpha^+)(\zeta, i)$. Then $\pi_1\overline{p}_1 \parallel q_1$.

Hence, our automorphism $\pi = (\pi_0, \pi_1)$ is as desired.

This proves Separation in $\langle V[G], \in, V\rangle$ for any $\mathcal{L}_\in^A$-formula φ.

The proof for N is similar, using symmetric names and the symmetric forcing relation $(\Vdash_s)_{\mathbb{P}}^V$.

□

Now, in order to show that Replacement holds in N, it is enough to verify the Axiom Scheme of *Collection* (and then use Separation):

Proposition 3.3.3. *For any $\mathcal{L}_\in^A$-formula $\varphi(x, y, v_0, \ldots, v_{n-1})$ and a, $z_0, \ldots, z_{n-1} \in N$ such that*

$$\langle N, \in, V\rangle \vDash \forall x \in a\ \exists y\ \varphi(x, y, z_0, \ldots, z_{n-1}, V),$$

there exists $b \in N$ *with the property that*

$$\langle N, \in, V \rangle \ \vDash \ \forall x \in a \ \exists y \in b \ \varphi(x, y, z_0, \dots, z_{n-1}, V).$$

Proof. For an ordinal α and the set forcing $\mathbb{P}_\alpha$ as above, the $\mathrm{Name}_\beta(\mathbb{P}_\alpha)^V$-hierarchy is defined recursively (in V) as usual: $\dot{x} \in \mathrm{Name}_{\beta+1}(\mathbb{P}_\alpha)^V$ iff $\dot{x} \subseteq \mathrm{Name}_\beta(\mathbb{P}_\alpha)^V \times \mathbb{P}_\alpha$, and for λ a limit ordinal, $\dot{x} \in \mathrm{Name}_\lambda(\mathbb{P}_\alpha)^V$ iff $\dot{x} \in \mathrm{Name}_\beta(\mathbb{P}_\alpha)^V$ for some $\beta < \lambda$.
We are going to use the following "diagonal hierarchy": For $\alpha \in \mathrm{Ord}$, let

$$N_\alpha := \{\dot{x}^{G_\alpha} \mid \dot{x} \in HS \cap \mathrm{Name}_{\alpha+1}(\mathbb{P}_\alpha)^V\}.$$

One has to check that this hierarchy is indeed definable in the structure $\langle V[G], \in, V, G\rangle$, i.e. there is an $\mathcal{L}_\in^{A,B}$-formula τ such that $\langle V[G], \in, V, G\rangle \ \vDash \tau(x, \alpha, V, G)$ iff $\alpha = \min\{\beta \mid x \in N_\beta\}$.
Therefore, one first has to make sure that the interpretation function $(\cdot)^G$ is definable within $\langle V[G], \in, V, G\rangle$, where some extra care is needed, since the recursion theorem can only be applied very carefully (we do not have replacement in $V[G]$).
This issue is adressed in [Git80] and Proposition 1.3.17, where we show: There is a function f in $\langle V[G], \in, V, G\rangle$ with $f(\dot{x}, \alpha, V, G) = x$ if and only if $\dot{x} \in \mathrm{Name}_{\alpha+1}(\mathbb{P}_\alpha)^V$ and $x = \dot{x}^{G_\alpha}$.

This function f can be used to define our N_α-hierarchy: Let $\tau(x, \alpha, V, G)$ be the formula

$$\alpha = \min\{\beta \mid \exists \dot{x} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V \ \ x = f(\dot{x}, \beta, V, G)\}.$$

Then $\langle V[G], \in, V, G\rangle \vDash \tau(x, \alpha, V, G)$ if and only if $\alpha = \min\{\beta \mid x \in N_\beta\}$.

Now, consider $a \in N$ and an $\mathcal{L}_\in^{A}$-formula φ with

$$\langle N, \in V\rangle \vDash \forall x \in a \ \exists y \ \varphi(x, y, V).$$

(We suppress the parameters $z_0, \dots, z_{n-1}$ for simplicity.) We have to show that there exists $b \in N$ with the property that

$$\forall x \in a \ \exists y \in b \ \langle N, \in, V\rangle \ \vDash \varphi(x, y, V).$$

First, we use structural induction over the formula φ to construct an $\mathcal{L}_\in^{A,B}$-formula $\overline{\varphi}$ such that for all $x \in a$ and y,

$$\langle V[G], \in, V, G\rangle \vDash \overline{\varphi}(x, y, V, G)$$

if and only if

$$\langle N, \in, V\rangle \ \vDash \varphi(x, y, V).$$

Then we define in $\langle V[G], \in, V, G\rangle$:

$$M := \Big\{ (x, \alpha) \ \Big| \ x \in a \ \wedge \ \alpha = \min\{ \beta \mid \exists y \ \exists \dot{y} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V \ : \ y = f(\dot{y}, \beta, V, G) \wedge \overline{\varphi}(x, y, V, G) \} \Big\}.$$

Then $M = \big\{ (x, \alpha) \mid x \in a \wedge \alpha = \min\{\beta \mid \exists y \in N_\beta \ \langle N, \in, V\rangle \vDash \varphi(x, y, V)\} \big\}$.

It suffices to show that there exists δ with rg $M \subseteq \delta$, since this would imply that for all $x \in a$, there exists $y \in N_\delta$ with $\langle N, \in, V\rangle \vDash \varphi(x, y, V)$.

Take λ large enough such that there is $\dot{a} \in HS \cap \mathrm{Name}(\mathbb{P} \upharpoonright (\lambda+1))^V$ with $a = \dot{a}^{G \upharpoonright (\lambda+1)}$. We claim that $M \in V[G \upharpoonright (\lambda+1)]$.

Let

$$M' := \big\{ (\dot{x}^{G\upharpoonright(\lambda+1)}, \alpha) \;\big|\; \dot{x} \in \mathrm{dom}\, \dot{a}, \alpha = \min\{\beta \mid \exists\, \dot{y} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V \\ \exists\, p : \; p \Vdash^V_{\mathbb{P}} \big(\dot{x} \in \dot{a} \wedge \overline{\varphi}(\dot{x}, \dot{y}, \check{V}, \dot{G})\big), p \upharpoonright (\lambda+1) \in G \upharpoonright (\lambda+1)\} \big\}.$$

Then $M' \in V[G \upharpoonright (\lambda+1)]$. It remains to prove that $M = M'$.

Therefore, it suffices to show that in $\langle V[G], \in, V, G\rangle$, for any $\dot{x} \in \mathrm{dom}\, \dot{a}$ and $\beta \in \mathrm{Ord}$ the following are equivalent:

(I) $\dot{x}^{G\upharpoonright(\lambda+1)} \in a \wedge \exists\, \dot{y} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V \;\exists\, y \;:\; y = f(\dot{y}, \beta, V, G) \wedge \overline{\varphi}(\dot{x}^{G\upharpoonright(\lambda+1)}, y, V, G)$

(II) $\exists\, \dot{y} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V \;\exists\, p : \; p \Vdash^V_{\mathbb{P}} \big(\dot{x} \in \dot{a} \wedge \overline{\varphi}(\dot{x}, \dot{y}, \check{V}, \dot{G})\big)$,
$p \upharpoonright (\lambda+1) \in G \upharpoonright (\lambda+1)$.

The direction "(I) $\Rightarrow$ (II)" is clear. Concerning "(II) $\Rightarrow$ (I)", assume towards a contradiction that there was $\dot{x} \in \mathrm{dom}\, \dot{a}$, $\beta \in \mathrm{Ord}$ and $\dot{y} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V$ with $p \Vdash^V_{\mathbb{P}} \big(\dot{x} \in \dot{a} \wedge \overline{\varphi}(\dot{x}, \dot{y}, \check{V}, \dot{G})\big)$ for some $p \in \mathbb{P}$ with $p \upharpoonright (\lambda+1) \in G \upharpoonright (\lambda+1)$, but (I) fails.
From $p \Vdash^V_{\mathbb{P}} \dot{x} \in \dot{a}$ with $p \upharpoonright (\lambda+1) \in G \upharpoonright (\lambda+1)$ and $\dot{x}, \dot{a} \in \mathrm{Name}(\mathbb{P} \upharpoonright (\lambda+1))^V$, it follows that $\dot{x}^{G\upharpoonright(\lambda+1)} \in \dot{a}^{G\upharpoonright(\lambda+1)} = a$; hence,

$$\langle V[G], V, \in, G\rangle \vDash \neg\big(\, \exists\, \dot{y} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V \;\exists\, y \;:\; y = f(\dot{y}, \beta, V, G) \wedge \overline{\varphi}(\dot{x}^{G\upharpoonright(\lambda+1)}, y, V, G)\,\big).$$

Take $q \in G$ such that

$$q \Vdash^V_{\mathbb{P}} \; \forall \dot{y} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V \;\forall\, y\; \big(\, y = f(\dot{y}, \beta, V, G) \longrightarrow \neg\overline{\varphi}(\dot{x}, y, V, G)\,\big).$$

As in Proposition 3.3.2, we can construct an automorphism π such that $\pi p \,\|\, q$, and π is the identity on $\mathbb{P} \upharpoonright (\lambda+1)$. Then $\pi \overline{x}^{D_\pi} = \overline{x}^{D_\pi}$; hence,

$$\pi p \Vdash^V_{\mathbb{P}} \overline{\varphi}(\dot{x}, \pi \overline{y}^{D_\pi}, \check{V}, \pi \dot{G}).$$

By structural induction over the formula $\overline{\varphi}$, one can use an isomorphism argument to show that for any condition $r \in \mathbb{P}$, it follows that $r \Vdash^V_{\mathbb{P}} \overline{\varphi}(\dot{x}, \pi \overline{y}^{D_\pi}, \check{V}, \dot{G})$ if and only if $r \Vdash^V_{\mathbb{P}} \overline{\varphi}(\dot{x}, \pi \overline{y}^{D_\pi}, \check{V}, \pi \dot{G})$. The induction step regarding the existential quantifier follows from the fact that for any $\dot{v} \in HS \cap \mathrm{Name}_{\alpha+1}(\mathbb{P}_\alpha)^V$ and $\pi \in A$, also $\pi \overline{v}^{D_\pi} \in HS \cap \mathrm{Name}_{\alpha+1}(\mathbb{P}_\alpha)^V$; and $\dot{v}^H = (\pi \overline{v}^{D_\pi})^{\pi H}$ for any V-generic filter H on $\mathbb{P}$.

Hence, it follows that also

$$\pi p \Vdash^V_{\mathbb{P}} \overline{\varphi}(\dot{x}, \pi \overline{y}^{D_\pi}, \check{V}, \dot{G}).$$

But $\pi \overline{y}^{D_\pi} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V$, which contradicts $\pi p \,\|\, q$.

Thus, (I) and (II) are equivalent, which implies $M = M'$ as desired. Now, since $M \in V[G \upharpoonright (\lambda + 1)]$, we can apply *Replacement* in the ZFC-model $V[G \upharpoonright (\lambda + 1)]$ and obtain that rg $M \subseteq \delta$ for some ordinal δ. Therefore,

$$\forall x \in a \ \exists y \in N_\delta \ \langle N, \in, V \rangle \vDash \varphi(x, y, V).$$

Since $N_\delta \in N$ (the canonical name $\dot{N}_\delta := \{(\dot{x}, \mathbb{1}) \mid \dot{x} \in HS \cap \mathrm{Name}_{\alpha+1}(\mathbb{P}_\alpha)^V\}$ is symmetric), this finishes the proof.

□

Similarly, one can show that the Axiom of Replacement holds true in $V[G]$ as long the formula φ does not make use of the parameter G for the generic filter:

Proposition 3.3.4. *For any $\mathcal{L}_\in^A$-formula $\varphi(x, y, v_0, \dots, v_{n-1})$ and $a, z_0, \dots, z_{n-1} \in V[G]$ such that*

$$\langle V[G], \in, V, G \rangle \vDash \forall x \in a \ \exists y \ \varphi(x, y, z_0, \dots, z_{n-1}, V),$$

it follows that there exists $b \in V[G]$ with the property that

$$\langle V[G], \in, V, G \rangle \vDash \forall x \in a \ \exists y \in b \ \varphi(x, y, z_0, \dots, z_{n-1}, V).$$

One can use basically the same proof, but with the hierarchy $\big((V[G])_\alpha \mid \alpha \in \mathrm{Ord}\big)$ instead of $(N_\alpha \mid \alpha \in \mathrm{Ord})$, where $(V[G])_\alpha := \{\dot{x}^{G_\alpha} \mid \dot{x} \in \mathrm{Name}_{\alpha+1}(\mathbb{P}_\alpha)\}$.

Proposition 3.3.5. *The Axiom of Power Set holds in N.*

Proof. Consider a set $Y \in N$. We first show:

$$\exists \overline{\lambda} \in \mathrm{Card} \ \wp^N(Y) \subseteq V[G \upharpoonright (\overline{\lambda} + 1)] \qquad (*).$$

Take a cardinal μ large enough such that $Y \in V[G \upharpoonright (\mu + 1)]$ and $|Y|^{V[G \upharpoonright (\mu+1)]} \leq \mu$, i.e. there exists an injection $\iota : Y \hookrightarrow \mu$ in $V[G \upharpoonright (\mu + 1)]$. Take $\dot{Y} \in \mathrm{Name}(\mathbb{P} \upharpoonright (\mu + 1))^V$ with $Y = \dot{Y}^{G \upharpoonright (\mu+1)}$. Let $\lambda := F(\mu)^+$; then $|\mathbb{P} \upharpoonright (\mu + 1)| \leq \lambda$.
We claim that $\wp^N(Y) \subseteq V[G \upharpoonright (\lambda + 1)]$.

Consider $Z \in \wp^N(Y)$, $Z = \dot{Z}^G$ with $\dot{Z} \in HS$ such that $\pi \overline{\dot{Z}}^{D_\pi} = \overline{\dot{Z}}^{D_\pi}$ for all π which are contained in the intersection

$$Fix_0(\kappa_0, i_0) \cap \dots \cap Fix_0(\kappa_{n-1}, i_{n-1}) \cap Small_0(\lambda_0, [0, \alpha_0)) \cap \dots \cap Small_0(\lambda_{m-1}, [0, \alpha_{m-1})) \cap$$

$$\cap Fix_1(\overline{\kappa}_0, \overline{\imath}_0) \cap \dots \cap Fix_1(\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1}) \cap \dots \cap Small_1(\overline{\lambda}_0, [0, \overline{\alpha}_0)) \cap Small_1(\overline{\lambda}_{\overline{m}-1}, [0, \overline{\alpha}_{\overline{m}-1})).$$

Take a condition $r \in G$ such that $t(r)$ contains the vertices $(\kappa_0, i_0), \dots, (\kappa_{n-1}, i_{n-1})$ and all $t(r)$-branches have height $\geq \mu$.
Then $G_0 \upharpoonright (\mu + 1) \times (G_0 \upharpoonright t(r)) \upharpoonright [\mu, \infty) \times G_1 \upharpoonright (\mu + 1) \times \big(G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{m}-1}, \overline{\imath}_{\overline{m}-1})\}\big) \upharpoonright [\mu, \infty)$ is a V-generic filter on $\mathbb{P}_0 \upharpoonright (\mu + 1) \times (\mathbb{P}_0 \upharpoonright t(r)) \upharpoonright [\mu, \infty) \times \mathbb{P}_1 \upharpoonright (\mu + 1) \times \big(\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{m}-1}, \overline{\imath}_{\overline{m}-1})\}\big) \upharpoonright [\mu, \infty)$.

We want to show that Z is contained in the intermediate generic extension

$$V\big[G_0 \upharpoonright (\mu{+}1) \times (G_0 \upharpoonright t(r)) \upharpoonright [\mu, \infty) \times G_1 \upharpoonright (\mu{+}1) \times \big(G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots, (\overline{\kappa}_{\overline{m}-1}, \overline{\imath}_{\overline{m}-1})\}\big) \upharpoonright [\mu, \infty)\big].$$

Let Z' be the set of all $\dot{y}^{G \upharpoonright (\mu+1)}$ with $\dot{y} \in \mathrm{dom}\, \dot{Y}$ such that there exists $p \in \mathbb{P}$, $p_0 \leq r$, with $p \Vdash_{\mathbb{P}}^V \dot{y} \in \dot{Z}$ such that:

- $p_0 \upharpoonright (\mu+1) \in G_0 \upharpoonright (\mu+1)$,
- $(p_0 \upharpoonright t(r)) \upharpoonright [\mu, \infty) \in (G_0 \upharpoonright t(r)) \upharpoonright [\mu, \infty)$,
- $p_1 \upharpoonright (\mu+1) \in G_1 \upharpoonright (\mu+1)$,
- $(p_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}) \upharpoonright [\mu, \infty) \in (G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}) \upharpoonright [\mu, \infty)$.

It suffices to show that $Z = Z'$. The direction "$\subseteq$" follows from the Forcing Theorem. For "$\supseteq$", we use an isomorphism argument similarly as before: Assume there was $\dot{y}^{G \upharpoonright (\mu+1)} \in Z' \smallsetminus Z$ with $\dot{y} \in \operatorname{dom} \dot{Y}$ and p with $p \Vdash \dot{y} \in \dot{Z}$ as in the definition of Z'.
Take $q \in G$ such that $q_0 \leq r$ and $q \Vdash \dot{y} \notin \dot{Z}$. We will construct an automorphism π with $\pi p \parallel q$ such that π restricted to $\mathbb{P} \upharpoonright (\mu+1)$ is the identity, and additionally,

$$\pi \in Fix_0(\kappa_0, i_0) \cap \dots \cap Small_0(\lambda_0, [0, \alpha_0)) \cap \dots \cap Fix_1(\overline{\kappa}_0, \overline{\imath}_0) \cap \dots$$

$$\dots \cap Small_1(\overline{\lambda}_0, [0, \overline{\alpha}_0)) \cap \dots .$$

But then, from $\pi p \Vdash \pi \overline{y}^{D_\pi} \in \pi \overline{Z}^{D_\pi}$ and $\pi \overline{Z}^{D_\pi} = \overline{Z}^{D_\pi}$, $\pi \overline{y}^{D_\pi} = \overline{y}^{D_\pi}$, it follows that $\pi p \Vdash \overline{y}^{D_\pi} \in \overline{Z}^{D_\pi}$. Together with $\pi p \parallel q$ and $q \Vdash \dot{y} \notin \dot{Z}$, this gives the desired contradiction.

We start with the construction of π_0. Let $\operatorname{ht} \pi := \max\{\eta(p), \eta(q)\}$. For $\alpha \leq \mu$, let $\pi_0(\alpha)$ be the identity. In the case that $\alpha \in [\mu^+, \operatorname{ht} \pi]$, we take for $\pi_0(\alpha)$ a bijection on $\{(\alpha, i) \mid i < F_{\lim}(\alpha)\}$ with finite support such that:

- for any $(\alpha, i) \in t(r)$, we have $\pi_0(\alpha)(\alpha, i) = (\alpha, i)$,
- for any $(\alpha, i) \in t(p) \smallsetminus t(r)$, we have $\pi_0(\alpha)(\alpha, i) = (\alpha, j)$ for some $j < F_{\lim}(\alpha)$ with $(\alpha, j) \notin t(p) \cup t(q)$,
- for any $i < F_{\lim}(\alpha)$ with $i \in [\gamma(i), \gamma(i) + \omega)$ for γ a limit ordinal, we have $\pi_0(\alpha)(\alpha, i) = (\alpha, i')$ such that also $i' \in [\gamma(i), \gamma(i) + \omega)$.

Then π_0 is the identity on $\mathbb{P}_0 \upharpoonright (\mu+1)$, and $\pi_0 \in Fix_0(\kappa_0, i_0) \cap \dots \cap Fix_0(\kappa_{n-1}, i_{n-1})$, since $\pi_0(\alpha)(\alpha, i) = (\alpha, i)$ for all $(\alpha, i) \in t(r)$. Moreover, $\pi_0 \in Small_0[\lambda_0, [0, \alpha_0)) \cap \dots \cap Small_0(\lambda_{m-1}, [0, \alpha_{m-1}))$, since we only use small permutations. By construction, it follows that $\pi_0 p_0 \parallel q_0$.

The map π_1 can be constructed as in the proof of Proposition 3.3.2. Then $\pi_1 p_1 \parallel q_1$, π_1 restricted to $\mathbb{P}_1 \upharpoonright (\mu+1)$ is the identity, $\pi_1 \in Fix_1(\overline{\kappa}_0, \overline{\imath}_0) \cap \dots \cap Fix_1(\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})$ since p_1 and q_1 agree on $\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}$, and $\pi_1 \in Small_1(\overline{\lambda}_0, [0, \overline{\alpha}_0)) \cap \dots \cap Small_1(\lambda_{\overline{m}-1}, [0, \alpha_{\overline{m}-1}))$, since $\operatorname{supp} \pi_1(\alpha^+) = \emptyset$ for all $\alpha^+ \in \mathrm{Succ}'$.

Hence, our automorphism π has all the desired properties, which implies $Z = Z'$; so

$$Z \in V\big[\, G_0 \upharpoonright (\mu+1) \times (G_0 \upharpoonright t(r)) \upharpoonright [\mu, \infty) \times G_1 \upharpoonright (\mu+1) \times \big(G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}\big) \upharpoonright [\mu, \infty) \,\big].$$

Recall that we have an injection $\iota : Y \hookrightarrow \mu$ in $V[G \upharpoonright (\mu+1)]$; so using the parameter Z, we can construct in $V[G_0 \upharpoonright (\mu+1) \times (G_0 \upharpoonright t(r)) \upharpoonright [\mu, \infty) \times G_1 \upharpoonright (\mu+1) \times (G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \dots\}) \upharpoonright$

$[\mu, \infty)]$ a function $\iota_Z : \mu \to 2$ with $\iota_Z(\alpha) = 1$ iff $\alpha \in \mathrm{im}(\iota)$ with $\iota^{-1}(\alpha) \in Z$, and $\iota_Z(\alpha) = 0$, else.

The forcing

$$\mathbb{P}_0 \upharpoonright (\mu+1) \times (\mathbb{P}_0 \upharpoonright t(r)) \upharpoonright [\mu, \infty) \times \mathbb{P}_1 \upharpoonright (\mu+1) \times (\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots\}) \upharpoonright [\mu, \infty)$$

can be factored as

$$\Big(\mathbb{P}_0 \upharpoonright (\mu+1) \times (\mathbb{P}_0 \upharpoonright t(r)) \upharpoonright [\mu, \lambda+1) \times \mathbb{P}_1 \upharpoonright (\mu+1) \times (\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots\}) \upharpoonright [\mu, \lambda+1) \Big) \times$$
$$\times \Big((\mathbb{P}_0 \upharpoonright t(r)) \upharpoonright [\lambda, \infty) \times (\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots\}) \upharpoonright [\lambda, \infty) \Big),$$

where the "lower part"

$$\mathbb{P}_0 \upharpoonright (\mu+1) \times (\mathbb{P}_0 \upharpoonright t(r)) \upharpoonright [\mu, \lambda+1) \times \mathbb{P}_1 \upharpoonright (\mu+1) \times (\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots\}) \upharpoonright [\mu, \lambda+1)$$

has cardinality $\leq F_{\lim}(\mu) \cdot \lambda \cdot F(\mu)^+ \cdot \lambda = F(\mu)^+ = \lambda$, and the "upper part"

$$(\mathbb{P}_0 \upharpoonright t(r)) \upharpoonright [\lambda, \infty) \times (\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots\}) \upharpoonright [\lambda, \infty)$$

is $\leq \lambda$-closed.

Hence,

$$\iota_Z \in V\big[G_0 \upharpoonright (\mu+1) \times (G_0 \upharpoonright t(r)) \upharpoonright [\mu, \lambda+1) \times G_1 \upharpoonright (\mu+1) \times (G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots\}) \upharpoonright [\mu, \lambda+1)\big];$$

so $\iota_Z \in V[G \upharpoonright (\lambda+1)]$, which implies that also $Z \in V[G \upharpoonright (\lambda+1)]$.

Since $Z \in \wp^N(Y)$ was arbitrary, it follows that $\wp^N(Y) \subseteq V[G \upharpoonright (\lambda+1)]$ as desired. This proves $(*)$.

Now, let $a := \wp^{V[G \upharpoonright (\lambda+1)]}(Y) \in V[G \upharpoonright (\lambda+1)]$. Then $\wp^N(Y) \subseteq a$. Take $\dot{a} \in \mathrm{Name}(\mathbb{P} \upharpoonright (\lambda+1))^V$ with $a = \dot{a}^G = \dot{a}^{G \upharpoonright (\lambda+1)}$.

Inside the structure $\langle V[G], \in, V, G \rangle$, we define a function $F : a \to Ord$ as follows:

For $z \in a$, let $F(z) = \alpha$ if $\alpha = \min\{\beta \mid z \in N_\beta\}$ if such an α exists. Let $F(z) = 0$, else.

Now, we will use the function f from Proposition 3.3.3 with the property that $\langle V[G], \in, V, G \rangle \models f(\dot{x}, \alpha, V, G) = x$ iff $\dot{x} \in \mathrm{Name}_{\alpha+1}(\mathbb{P}_\alpha)^V$ with $x = \dot{x}^{G_\alpha}$.

Let $\eta(z, \beta, V, G)$ denote the statement

$$\exists\, \dot{x} \in HS \cap \mathrm{Name}_{\beta+1}(\mathbb{P}_\beta)^V \quad z = f(\dot{x}, \beta, V, G).$$

Then

$$F = \Big\{ (\dot{z}^{G \upharpoonright (\lambda+1)}, \alpha) \mid \dot{z} \in \mathrm{dom}\, \dot{a} \wedge \dot{z}^{G \upharpoonright (\lambda+1)} \in \dot{a}^{G \upharpoonright (\lambda+1)} \wedge \alpha = \min\{\beta \mid \eta(\dot{z}^{G \upharpoonright (\lambda+1)}, \beta, V, G)\} \Big\} \cup$$
$$\cup \Big\{ (\dot{z}^{G \upharpoonright (\lambda+1)}, 0) \mid \dot{z} \in \mathrm{dom}\, \dot{a} \wedge \dot{z}^{G \upharpoonright (\lambda+1)} \in \dot{a}^{G \upharpoonright (\lambda+1)} \wedge \neg \exists\, \beta\ \eta(\dot{z}^{G \upharpoonright (\lambda+1)}, \beta, V, G) \Big\}.$$

We claim that $F \in V[G \upharpoonright (\lambda+1)]$.

Let

$$\widetilde{F} := \Big\{ (\dot z^{G\restriction(\lambda+1)}, \alpha) \mid \dot z \in \operatorname{dom} \dot a, \dot z^{G\restriction(\lambda+1)} \in \dot a^{G\restriction(\lambda+1)}, \exists p :\ p \Vdash^V_{\mathbb{P}} \ \alpha = \min\{\beta \mid \eta(\dot z, \beta, \check V, \dot G)\},$$

$$p \restriction (\lambda+1) \in G \restriction (\lambda+1) \Big\} \ \cup$$

$$\cup \Big\{ (\dot z^{G\restriction(\lambda+1)}, 0) \mid \dot z \in \operatorname{dom} \dot a, \dot z^{G\restriction(\lambda+1)} \in \dot a^{G\restriction(\lambda+1)}, \exists p :\ p \Vdash^V_{\mathbb{P}} \ \neg\exists \beta \ \eta(\dot z, \beta, \check V, \dot G),$$

$$p \restriction (\lambda+1) \in G \restriction (\lambda+1) \Big\}.$$

It suffices to show that $F = \widetilde{F}$. The direction "$\subseteq$" follows from the Forcing Theorem. Concerning "$\supseteq$", we proceed as in the proof of Proposition 3.3.3:
Assume towards a contradiction, there was $(\dot z^{G\restriction(\lambda+1)}, \alpha) \in \widetilde{F} \setminus F$ with $\dot z \in \operatorname{dom} \dot a$, $\dot z^{G\restriction(\lambda+1)} \in \dot a^{G\restriction(\lambda+1)}$. W.l.o.g., let $\alpha > 0$.
Take $p \in \mathbb{P}$ with

$$p \Vdash^V_{\mathbb{P}} \ \alpha = \min\{\beta \mid \eta(\dot z, \beta, \check V, \dot G)\}$$

and $p \restriction (\lambda+1) \in G \restriction (\lambda+1)$. Since $(\dot z^{G\restriction(\lambda+1)}, \alpha) \notin F$, there must be $q \in G$ with

$$q \Vdash^V_{\mathbb{P}} \ \neg\,(\alpha = \min\{\beta \mid \eta(\dot z, \beta, \check V, \dot G)\}).$$

As in the proof of Proposition 3.3.2, we construct an automorphism π with $\pi p \parallel q$ such that π restricted to $\mathbb{P} \restriction (\lambda+1)$ is the identity. Then $\pi \overline{z}^{D_\pi} = \overline{z}^{D_\pi}$; so

$$\pi p \Vdash^V_{\mathbb{P}} \ \alpha = \min\{\beta \mid \eta(\dot z, \beta, \check V, \pi \dot G)\}.$$

Now, for any condition $r \in \mathbb{P}$ and β an ordinal, we have $r \Vdash^V_{\mathbb{P}} \eta(\dot z, \beta, \check V, \dot G)$ if and only if $r \Vdash^V_{\mathbb{P}} \eta(\dot z, \beta, \check V, \pi \dot G)$, similarly as in the proof of Proposition 3.3.3. Hence,

$$\pi p \Vdash^V_{\mathbb{P}} \alpha = \min\{\beta \mid \eta(\dot z, \beta, \check V, \dot G)\},$$

contradicting that $\pi p \parallel q$.

The case $\alpha = 0$ is similar. Hence, $F = \widetilde{F} \in V[G \restriction (\lambda+1)]$ as desired.

Now, by *Replacement* in $V[G \restriction (\lambda+1)]$, it follows that $\operatorname{rg} F$ is bounded by some ordinal δ. Then any $z \in \wp^N(Y) \subseteq a$ is contained in some N_α for $\alpha < \delta$; hence, $\wp^N(Y) \subseteq N_\delta$. By the Axiom of Separation, this implies $\wp^N(Y) \in N$ as desired.

□

Thus, we have shown that the symmetric extension N is indeed a model of ZF.

We will now see that N preserves all V-cardinals, which follows from the fact that any set of ordinals $X \in N$, $X \subseteq \alpha$ can be captured in a "mild" V-generic extension by a forcing as in Proposition 3.1.9:

Lemma 3.3.6 (Approximation Lemma)**.** *Let $X \in N$, $X \subseteq \alpha$ with $X = \dot X^G$ such that $\pi \overline{X}^{D_\pi} = \overline{X}^{D_\pi}$ for all π which are contained in the intersection*

$Fix_0(\kappa_0, i_0) \cap \dots \cap Fix_0(\kappa_{n-1}, i_{n-1}) \cap Small_0(\lambda_0, [0, \alpha_0)) \cap \dots \cap Small_0(\lambda_{m-1}, [0, \alpha_{m-1})) \cap$

$\cap\, Fix_1(\overline{\kappa}_0, \overline{\imath}_0) \cap \cdots \cap Fix_1(\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1}) \cap Small_1(\overline{\lambda}_0, [0, \overline{\alpha}_0)) \cap \cdots \cap Small_1(\overline{\lambda}_{\overline{m}-1}, [0, \overline{\alpha}_{\overline{m}-1}))$. *Let $r \in G_0$ such that $\{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\} \subseteq t(r)$ contains all maximal points of $t(r)$. Then*

$$X \in V[G_0 \upharpoonright t(r) \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}] =$$
$$= V[G_0 \upharpoonright \{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\} \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}].$$

Proof. Define

$$X' := \{\beta < \alpha \mid \exists\, q = (q_0, q_1) : \ q_0 \leq_0 r\,,\ q \Vdash \beta \in \dot{X}\,,\ q_0 \upharpoonright t(r) \in G_0 \upharpoonright t(r),$$
$$q_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots\} \in G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots\}\}.$$

Then $X = X'$ follows by an isomorphism argument as before.

□

From Lemma 3.3.6 and Proposition 3.1.9 we obtain:

Corollary 3.3.7. *Cardinals are N-V-absolute.*

A factoring argument shows that for $X \subseteq \kappa$ with κ a cardinal, the according forcings in the statement of Lemma 3.3.6 can be cut off at level κ^+:

Corollary 3.3.8. *Let $X \in N$, $X \subseteq \kappa$ with κ a limit cardinal. Then there are $n, n' < \omega$, $j_0, \ldots, j_{n-1} < F_{\lim}(\kappa^+) = F(\kappa)$, and $\overline{\kappa}_0, \ldots, \overline{\kappa}_{n'-1} \in Succ'$ with $\overline{\kappa}_0 < \kappa, \ldots, \overline{\kappa}_{n'-1} < \kappa$; $\overline{\imath}_0 < F(\overline{\kappa}_0), \ldots, \overline{\imath}_{n'-1} < F(\overline{\kappa}_{n'-1})$ such that*

$$X \in V[G_0 \upharpoonright \{(\kappa^+, j_0), \ldots, (\kappa^+, j_{n-1})\} \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{n'-1}, \overline{\imath}_{n'-1})\} \times G_1(\kappa^+)].$$

For a successor cardinal κ^+ and $X \in N$, $X \subseteq \kappa^+$, there are $n, n' < \omega$, $j_0, \ldots, j_{n-1} < F_{\lim}(\kappa^+)$, and $\overline{\kappa}_0, \ldots, \overline{\kappa}_{n'-1} \in Succ'$ with $\overline{\kappa}_0 \leq \kappa^+, \ldots, \overline{\kappa}_{n'-1} \leq \kappa^+$; $\overline{\imath}_0 < F(\overline{\kappa}_0), \ldots, \overline{\imath}_{n'-1} < F(\overline{\kappa}_{n'-1})$ such that

$$X \in V[G_0 \upharpoonright \{(\kappa^+, j_0), \ldots, (\kappa^+, j_{n-1})\} \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{n'-1}, \overline{\imath}_{n'-1})\}].$$

Proof. First, we consider the case that κ is a limit cardinal. From Lemma 3.3.6, it follows that there are finitely many cardinals $\kappa_0, \ldots, \kappa_{n-1}$, and $i_0 < F_{\lim}(\kappa_0), \ldots, i_{n-1} < F_{\lim}(\kappa_{n-1})$; moreover, finitely many $\overline{\kappa}_0, \ldots, \overline{\kappa}_{\overline{n}-1} \in Succ'$ and $\overline{\imath}_0 < F(\overline{\kappa}_0), \ldots, \overline{\imath}_{\overline{n}-1} < F(\overline{\kappa}_{\overline{n}-1})$ with

$$X \in V[G_0 \upharpoonright \{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\} \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}].$$

W.l.o.g. we can assume $\kappa_0, \ldots, \kappa_{n-1} \geq \kappa^+$. Take a condition $r \in G_0$ such that $\{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\} \subseteq t(r)$ contains all maximal points of $t(r)$. Then

$$G_0 \upharpoonright \{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\} \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}$$

is a V-generic filter on the forcing

$$\mathbb{P}_0 \upharpoonright t(r) \times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1}\}),$$

which can be factored in a "lower part"

$$\big((\mathbb{P}_0 \upharpoonright t(r)) \upharpoonright (\kappa^+ + 1)\big) \times \big((\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}) \upharpoonright (\kappa^+ + 1)\big),$$

with cardinality $\leq \kappa^+$, and an "upper part"

$$\big((\mathbb{P}_0 \upharpoonright t(r)) \upharpoonright [\kappa^+, \infty)\big) \times \big((\mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1})\}) \upharpoonright [\kappa^+, \infty)\big),$$

which is $\leq \kappa^+$-closed. Thus, X is contained in the generic extension by the lower part: Let $(\kappa^+, j_0), \ldots, (\kappa^+, j_{n-1})$ denote the $\leq_{t(r)}$-predecessors of $(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})$ respectively, on level κ^+. Moreover, assume w.l.o.g. that $0 \leq n' \leq n'' \leq \overline{n}$ with $\overline{\kappa}_0, \ldots, \overline{\kappa}_{n'-1} < \kappa$; $\overline{\kappa}_{n'}, \ldots, \overline{\kappa}_{n''-1} = \kappa^+$, and $\overline{\kappa}_{n''}, \ldots, \overline{\kappa}_{\overline{n}-1} > \kappa^+$. Then

$$X \in V[G_0 \upharpoonright \{(\kappa^+, j_0), \ldots, (\kappa^+, j_{n-1})\} \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{n''-1}, \overline{\imath}_{n''-1})\}] \subseteq$$

$$\subseteq V[G_0 \upharpoonright \{(\kappa^+, j_0), \ldots (\kappa^+, j_{n-1})\} \times G_1 \upharpoonright \{(\overline{\kappa}_0, \overline{\imath}_0), \ldots, (\overline{\kappa}_{n'-1}, \overline{\imath}_{n'-1})\} \times G_1(\kappa^+)]$$

as desired.

The case $X \subseteq \kappa^+$ is similar.

□

3.4 $\forall\, \kappa \in$ Card $\theta^N(\kappa) = \mathrm{F}(\kappa)$

Firstly, using the subgroups $Small_0(\kappa, [0, \alpha))$ or $Small_1(\kappa, [0, \alpha))$, it is not difficult to see that $\theta^N(\kappa) \geq F(\kappa)$ for all cardinals κ; i.e. for any $\alpha < F(\kappa)$, there exists in N a surjection $s : \mathcal{P}(\kappa) \to \alpha$:

Proposition 3.4.1. $\forall\, \kappa \in$ Card $\theta^N(\kappa) \geq F(\kappa)$.

Proof. First, we consider the case that κ is a limit cardinal. Fix some cardinal $\alpha < F_{\lim}(\kappa) = F(\kappa)$; we construct in N a surjection $s : \mathcal{P}(\kappa) \to \alpha$.
As already mentioned in Chapter 3.2, we define for any limit ordinal $i < \alpha$ a "cloud" around $(\dot{G}_0)_{(\kappa,i)}$ as follows:

$$(\widetilde{\dot{G}_0})^{\alpha}_{(\kappa,i)} := \big\{ \big(\pi \overline{(\dot{G}_0)_{(\kappa,i)}}^{D_\pi}, \mathbb{1}\big) \mid [\pi] \in Small_0(\kappa, [0, \alpha)) \big\} =$$

$$= \big\{ \big(\overline{((\dot{G}_0)_{(\kappa,i+n)}}^{D_\pi}, \mathbb{1}\big) \mid n < \omega \big\}.$$

Then

$$(\widetilde{G_0})^{\alpha}_{(\kappa,i)} := \big((\widetilde{\dot{G}_0})^{\alpha}_{(\kappa,i)}\big)^G = \big\{ (G_0)_{(\kappa,i+n)} \mid n < \omega \big\} \in N$$

for any limit ordinal $i < \alpha$, since the name $(\widetilde{\dot{G}_0})^{\alpha}_{(\kappa,i)}$ is fixed by all π with $[\pi] \in Small_0(\kappa, [0, \alpha))$. Moreover, any two distinct clouds $(\widetilde{G_0})^{\alpha}_{(\kappa,i)}$ and $(\widetilde{G_0})^{\alpha}_{(\kappa,j)}$ for limit ordinals i and j are disjoint – here, we have to use that splitting at limits is not allowed in our tree forcing; so for $j, j' < F_{\lim}(\kappa)$ with $j \neq j'$ it follows by genericity that indeed, $(G_0)_{(\kappa,j)} \neq (G_0)_{(\kappa,j')}$.

Recall that for $\mathbb{P}$-names $\dot{x}$, $\dot{y}$, we denote by $\mathrm{OR}_{\mathbb{P}}(\dot{x},\dot{y})$ the canonical name for the ordered pair $(\dot{x}^G,\dot{y}^G)$. The sequence $\big((\widetilde{G_0})^{\alpha}_{(\kappa,i)} \mid i<\alpha \text{ a limit ordinal}\big)$ is contained in N as well, since its name

$$\big\{\big(\ \mathrm{OR}_{\mathbb{P}}(i,(\widetilde{\dot{G}_0})^{\alpha}_{(\kappa,i)}),\mathbb{1}\big) \mid i<\alpha \text{ a limit ordinal}\big\}$$

is fixed by all $\pi\in Small_0(\kappa,[0,\alpha))$.

This gives in N a well-defined surjection $\overline{s}:\wp(\kappa)\to\{i<\alpha \mid i \text{ is a limit ordinal}\}$, by setting $\overline{s}(X):=i$ whenever $X\in(\widetilde{G_0})^{\alpha}_{(\kappa,i)}$ for some $i<\alpha$, and $\overline{s}(X):=0$, else.
Also without the Axiom of Choice, $\overline{s}$ can be turned into a surjection $s:\wp(\kappa)\to\alpha$.

Concerning successor cardinals, it suffices to show that $\theta^N(\kappa^+)\geq F(\kappa^+)$ for all $\kappa^+\in Succ'$. Let $\alpha<F(\kappa^+)$. We proceed similarly as before, setting for $i<\alpha$:

$$(\widetilde{\dot{G}_1})^{\alpha}_{(\kappa^+,i)}:=\big\{\big(\pi\overline{(G_1)_{(\kappa^+,i)}}^{D_\pi},\mathbb{1}\big) \mid [\pi]\in Small_1(\kappa^+,[0,\alpha))\big\}.$$

With $\pi\,(G_1)_{(\kappa^+,i)}:=\big(\pi\overline{(G_1)_{(\kappa^+,i)}}^{D_\pi}\big)^G$, we obtain

$$(\widetilde{G_1})^{\alpha}_{(\kappa^+,i)}:=\big((\widetilde{\dot{G}_1})^{\alpha}_{(\kappa^+,i)}\big)^G=\big\{\pi\,(G_1)^{\alpha}_{(\kappa^+,i)} \mid [\pi]\in Small_1(\kappa^+,[0,\alpha))\big\}.$$

As before, it follows that the sequence $\big((\widetilde{G_1})^{\alpha}_{(\kappa^+,i)} \mid i<\alpha\big)$ is contained in N, so it suffices to check that two distinct "clouds" $(\widetilde{G_1})^{\alpha}_{(\kappa^+,i)}$ and $(\widetilde{G_1})^{\alpha}_{(\kappa^+,j)}$ are indeed disjoint. Assume towards a contradiction, there were π, $\sigma\in Small_1(\kappa^+,[0,\alpha))$ with $\pi\,(G_1)^{\alpha}_{(\kappa^+,i)}=\sigma\,(G_1)^{\alpha}_{(\kappa^+,j)}$. By genericity, take $\zeta\in[\kappa,\kappa^+)\smallsetminus(\mathrm{dom}_x\,\pi(\kappa^+)\cup\mathrm{dom}_x\,\sigma(\kappa^+))$ with $(G_1)_{(\kappa^+,i)}(\zeta)\neq(G_1)_{(\kappa^+,j)}(\zeta)$. Since $i,j<\alpha$ and $\pi,\sigma\in Small_1(\kappa^+,\ [0,\alpha))$, it follows that $\pi\,(G_1)_{(\kappa^+,i)}(\zeta)=(G_1)_{(\kappa^+,i)}(\zeta)$ and $\sigma\,(G_1)_{(\kappa^+,j)}(\zeta)=(G_1)_{(\kappa^+,j)}(\zeta)$. Contradiction.
Hence, the sequence $\big((\widetilde{G_1})^{\alpha}_{(\kappa^+,i)} \mid i<\alpha\big)$ gives in N a surjective function $s:\wp(\kappa^+)\to\alpha$ as desired.

□

It remains to show that $\theta^N(\kappa)\leq F(\kappa)$ for all cardinals κ.

First, we consider the case that

κ is a limit cardinal.

Assume towards a contradiction that there was a surjection $S:\wp(\kappa)\to F(\kappa)$ in N. For the rest of this section, fix such a surjection S.
Let $\dot{S}\in HS$ with $S=\dot{S}^G$ such that $\pi\overline{\dot{S}}^{D_\pi}=\overline{\dot{S}}^{D_\pi}$ for all π that are contained in the intersection

$$Fix_0(\kappa_0,i_0)\cap\dots\cap Fix_0(\kappa_{n-1},i_{n-1})\cap Small_0(\lambda_0,[0,\alpha_0))\cap\dots\cap Small_0(\lambda_{m-1},[0,\alpha_{m-1}))\cap$$

$$\cap Fix_1(\overline{\kappa}_0,\overline{\imath}_0)\cap\dots\cap Fix_1(\overline{\kappa}_{\overline{n}-1},\overline{\imath}_{\overline{n}-1})\cap Small_1(\overline{\lambda}_0,[0,\overline{\alpha}_0))\cap\dots\cap Small_1(\overline{\lambda}_{\overline{m}-1},[0,\overline{\alpha}_{\overline{m}-1})),$$

which will be abbreviated by $(I_{\dot{S}})$.

We know from Corollary 3.3.8 that any $X\in N$, $X\subseteq\kappa$ is contained in a generic extension of the form

$$V[G_0\restriction\{(\kappa^+,j_0),\dots,(\kappa^+,j_{k-1})\}\times G_1\restriction\{(\mu_0,\overline{\jmath}_0),\dots,(\mu_{\overline{k}-1},\overline{\jmath}_{\overline{k}-1})\}\times G_1(\kappa^+)],$$

where $k, \overline{k} < \omega$, $j_0, \ldots, j_{k-1} < F_{\lim}(\kappa^+) = F(\kappa)$, and $\mu_0, \ldots, \mu_{\overline{k}-1} < \kappa$, $\overline{\jmath}_0 < F(\mu_0), \ldots, \overline{\jmath}_{\overline{k}-1} < F(\mu_{\overline{k}-1})$.

For a limit ordinal $\beta < F(\kappa)$ *large enough for* $(I_{\dot{S}})$ (we give a precise definition of this term later), we want to consider a map $S^\beta \subseteq S$, which will be the restriction of S to all X that are contained in a generic extension

$$V[G_0 \upharpoonright \{(\kappa^+, j_0), \ldots, (\kappa^+, j_{k-1})\} \times G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times G_1(\kappa^+)],$$

where $j_0, \ldots, j_{k-1} < \beta$ and $\overline{\jmath}_0, \ldots, \overline{\jmath}_{\overline{k}-1} < \beta$.

Let M denote the collection of all tuples $(s, (\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1}))$ such that $\overline{k} < \omega$, $\mu_0, \ldots, \mu_{\overline{k}-1} \in \kappa \cap Succ'$, $\overline{\jmath}_0 < F(\mu_0), \ldots, \overline{\jmath}_{\overline{k}-1} < F(\mu_{\overline{k}-1})$, and s is a condition in $\mathbb{P}_0$ with finitely many maximal points $(\kappa^+, j_0), \ldots, (\kappa^+, j_{k-1})$ with $j_0, \ldots, j_{k-1} < F_{\lim}(\kappa^+) = F(\kappa)$. For $\beta < F(\kappa)$, we denote by M_β the collection of all tuples $(s, (\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in M$ such that additionally, $\overline{\jmath}_0 < \beta, \ldots, \overline{\jmath}_{\overline{k}-1} < \beta$, and s has maximal points $(\kappa^+, j_0), \ldots, (\kappa^+, j_{k-1})$ with $j_0, \ldots, j_{k-1} < \beta$.

Proposition 3.4.2. *There is a limit ordinal $\beta < F(\kappa)$ such that the restriction*

$$S^\beta := S \upharpoonright \Big\{ X \subseteq \kappa \mid \exists\, (s, (\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in M_\beta : \ s \in G_0 \upharpoonright (\kappa^+ + 1),$$
$$X \in V[G_0 \upharpoonright t(s) \times G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times G_1(\kappa^+)] \Big\}$$

is surjective onto $F(\kappa)$, as well.

Later on, we will lead this into a contradiction by showing that any such S^β must be contained in an intermediate generic extension which preserves cardinals $\geq F(\kappa)$, but also contains an injection $\iota : \operatorname{dom} S^\beta \hookrightarrow \beta$.

We now define what we mean by *large enough for* $(I_{\dot{S}})$: Fix a condition $r \in G_0$ such that $\{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\} \subseteq t(r)$ contains all maximal points of $t(r)$, and an extension $\overline{r} \leq_0 r$, $\overline{r} \in G_0$ such that all $t(\overline{r})$-branches have height $\geq \kappa^+$. For $l < n$ with $\kappa_l \geq \kappa^+$, let (κ^+, i'_l) be the $t(\overline{r})$-predecessor of (κ_l, i_l) on level κ^+; in the case that $\kappa_l < \kappa^+$, let (κ^+, i'_l) denote some $t(\overline{r})$-successor of (κ_l, i_l) on level κ^+.
We say that a limit ordinal $\widetilde{\beta} < F_{\lim}(\kappa^+) = F(\kappa)$ is *large enough for* $(I_{\dot{S}})$ if the following hold:

- $\widetilde{\beta} > i'_0, \ldots, i'_{n-1}$,
- $\widetilde{\beta} > \alpha_l$ for all $l < m$ with $\lambda_l \leq \kappa^+$,
- $\widetilde{\beta} > \overline{\imath}_l$ for all $l < \overline{n}$ with $\overline{\kappa}_l < \kappa$,
- $\widetilde{\beta} > \overline{\alpha}_l$ for all $l < \overline{m}$ with $\overline{\lambda}_l < \kappa$.

We will refer to these conditions r, $\overline{r}$ later on.

We want to show that whenever a limit ordinal $\widetilde{\beta} < F(\kappa)$ is *large enough for* $(I_{\dot{S}})$ and $\beta := \widetilde{\beta} + \kappa^+$ (addition of ordinals), then S^β must be surjective onto $F(\kappa)$, as well.

For any tuple $(s, (\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in M$ and

$$\dot{x} \in \operatorname{Name}(\mathbb{P}_0 \upharpoonright t(s) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+)),$$

we define a canonical extension $\widetilde{\dot{x}} \in \operatorname{Name}(\mathbb{P})$ as follows:

Recursively, set

$$\widetilde{\dot{x}} := \big\{ (\widetilde{\dot{y}}, \overline{v}) \;\big|\; \exists \big(\dot{y}, (v_0 \upharpoonright t(s), v_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots\}, v_1(\kappa^+))\big) \in \dot{x} \;:\; \overline{v}_0 = v_0 \upharpoonright t(s),$$

$$\operatorname{supp} \overline{v}_1 \subseteq \kappa^+ + 1, \overline{v}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots\} = v_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots\}, \overline{v}_1(\kappa^+) = v_1(\kappa^+) \big\}.$$

If $s \in G_0 \upharpoonright (\kappa^+ + 1)$, it follows that

$$\widetilde{\dot{x}}^G = \dot{x}^{G_0 \upharpoonright t(s) \times G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots\} \times G_1(\kappa^+)}.$$

Sometimes, this name $\widetilde{\dot{x}}$ will be extended further to a name $\overline{\widetilde{\dot{x}}}^{D_\pi} \in \overline{\operatorname{Name}(\mathbb{P})}^{D_\pi}$. In order to simplify notation, this extension will be denoted by $\widetilde{\dot{x}}^{D_\pi}$.

We now give a proof of Proposition 3.4.2.

Proof. Assume towards a contradiction that a limit ordinal $\widetilde{\beta} < F(\kappa)$ is large enough and $\beta := \widetilde{\beta} + \kappa^+$ (addition of ordinals), but S^β is not surjective. Let $\alpha < F(\kappa)$ with $\alpha \notin \operatorname{rg} S^\beta$. Fix some cardinal λ with $\lambda > \max\{\kappa^+, \kappa_0, \ldots, \kappa_{n-1}, \lambda_0, \ldots, \lambda_{m-1}, \overline{\kappa}_0, \ldots, \overline{\kappa}_{\overline{n}-1}, \overline{\lambda}_0, \ldots, \overline{\lambda}_{\overline{m}-1}\}$ such that $\dot{S} \in \operatorname{Name}(\mathbb{P} \upharpoonright (\lambda + 1))^V$. Then $S \in V[G \upharpoonright (\lambda + 1)]$, and we can define a canonical $\mathbb{P} \upharpoonright (\lambda + 1)$-name for S^β as follows:

$$\dot{S}^\beta := \big\{ \big(\operatorname{OR}_{\mathbb{P} \upharpoonright (\lambda+1)}(\widetilde{\dot{X}}, \alpha), \overline{p} \big) \mid \exists\, (s, (\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in M_\beta :$$

$$\dot{X} \in \operatorname{Name}(\mathbb{P}_0 \upharpoonright t(s) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+)), \overline{p} = (\overline{p}_0, \overline{p}_1) \in \mathbb{P} \upharpoonright (\lambda + 1),$$

$$\overline{p}_0 \leq s, \overline{p} \Vdash_{\mathbb{P} \upharpoonright (\lambda+1)} \operatorname{OR}_{\mathbb{P} \upharpoonright (\lambda+1)}(\widetilde{\dot{X}}, \alpha) \in \dot{S} \big\}.$$

It is not difficult to check that indeed, $(\dot{S}^\beta)^{G \upharpoonright (\lambda+1)} = S^\beta$.

Since $S : \wp^N(\kappa) \to F(\kappa)$ is surjective, there must be $X \in \wp^N(\kappa)$ with $(X, \alpha) \in S$. By Corollary 3.3.8, take $(s, (\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in M$ such that $s \in G_0 \upharpoonright (\kappa^+ + 1)$ has maximal points $(\kappa^+, j_0), \ldots, (\kappa^+, j_{k-1})$, and

$$X = \dot{X}^{G_0 \upharpoonright t(s) \times G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times G_1(\kappa^+)}$$

for some $\dot{X} \in \operatorname{Name}(\mathbb{P}_0 \upharpoonright t(s) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+))$. W.l.o.g. we can assume that the sequences $(j_l \mid l < k)$ and $(\overline{\jmath}_l \mid l < \overline{k})$ are both increasing.

Now, $(X, \alpha) = (\widetilde{\dot{X}}^{G \upharpoonright (\lambda+1)}, \alpha) \in \dot{S}^{G \upharpoonright (\lambda+1)}$, but $\alpha \notin \operatorname{rg} (\dot{S}^\beta)^{G \upharpoonright (\lambda+1)}$, so we can take $p \in G \upharpoonright (\lambda + 1)$ such that

$$p \Vdash_{\mathbb{P} \upharpoonright (\lambda+1)} \operatorname{OR}_{\mathbb{P} \upharpoonright (\lambda+1)}(\widetilde{\dot{X}}, \alpha) \in \dot{S} \quad , \quad p \Vdash_{\mathbb{P} \upharpoonright (\lambda+1)} \alpha \notin \operatorname{rg} \dot{S}^\beta.$$

W.l.o.g., let $p_0 \leq \overline{r}$, $p_0 \leq s$ and $\operatorname{ht} p \geq \kappa^+$.

Now, take $h \leq k$ such that $j_0, \ldots, j_{h-1} < \beta$, $j_h, \ldots, j_{k-1} \geq \beta$, and $\overline{h} \leq \overline{k}$ with $\overline{\jmath}_0, \ldots, \overline{\jmath}_{\overline{h}-1} < \beta$, $\overline{\jmath}_{\overline{h}}, \ldots, \overline{\jmath}_{\overline{k}-1} \geq \beta$. Then $(X, \alpha) \notin S^\beta$ implies that $h < k$ or $\overline{h} < \overline{k}$.

Pick pairwise distinct ordinals $\pi_0 j_h, \ldots, \pi_0 j_{k-1}$ in $(\widetilde{\beta}, \beta) \smallsetminus \{j_0, \ldots, j_{h-1}\}$ such that $\{(\kappa^+, \pi_0 j_h), \ldots, (\kappa^+, \pi_0 j_{k-1})\} \cap t(p_0) = \emptyset$.
We want to construct a $\mathbb{P}_0$-automorphism π_0 which is the identity below level κ^+, and swaps for any $l \in [h,k)$ the vertex (κ^+, j_l) with the vertex $(\kappa^+, \pi_0 j_l)$; i.e. for any $q_0 \in \mathbb{P}_0$ and $l \in [h,k)$ with $(\kappa^+, j_l), (\kappa^+, \pi_0 j_l) \in t(q_0)$, we want that $(\pi_0 q_0) \upharpoonright \{(\kappa^+, \pi_0 j_l)\} = q_0 \upharpoonright \{(\kappa^+, j_l)\}$, and $(\pi_0 q_0) \upharpoonright \{(\kappa^+, j_l)\} = q_0 \upharpoonright \{(\kappa^+, \pi_0 j_l)\}$. Since $\{(\kappa^+, \pi_0 j_h), \ldots, (\kappa^+, \pi_0 j_{k-1})\} \cap t(p_0) = \emptyset$, we can assure that at the same time, $\pi_0 p_0 \parallel p_0$.

For $\mathbb{P}_1$ we proceed similarly, but in order to achieve $\pi_1 p_1 \parallel p_1$, we first have to extend p to condition $\overline{p} = (p_0, \overline{p}_1) \leq (p_0, p_1)$ with $\overline{p} \in G \upharpoonright (\lambda + 1)$ such that the following holds:

For any $\mu^+ \in \mathrm{Succ}'$ with $\{l \in [\overline{h}, \overline{k}) \mid \mu_l = \mu^+\} = \{l_0, \ldots, l_{z-1}\}$ for some $1 \leq z < \omega$ (i.e. , $\mu_{l_0} = \cdots = \mu_{l_{z-1}} = \mu^+$, so $l_0, \ldots, l_{z-1} \in [\overline{h}, \overline{k})$ implies that $\overline{\jmath}_{l_0}, \ldots, \overline{\jmath}_{l_{z-1}} \geq \beta$), it follows that $\mu^+ \in \operatorname{supp} \overline{p}_1$ with $\overline{\jmath}_{l_0}, \ldots, \overline{\jmath}_{l_{z-1}} \in \operatorname{dom}_y \overline{p}_1(\mu^+)$, and there are $\pi_1 \overline{\jmath}_{l_0}, \ldots, \pi_1 \overline{\jmath}_{l_{z-1}} \in (\widetilde{\beta}, \beta) \smallsetminus \{\overline{\jmath}_0, \ldots, \overline{\jmath}_{\overline{h}-1}\}$ with $\overline{p}_1 \upharpoonright \{(\mu^+, \overline{\jmath}_{l_0})\} = \overline{p}_1 \upharpoonright \{(\mu^+, \pi_1 \overline{\jmath}_{l_0})\}, \ldots, \overline{p}_1 \upharpoonright \{(\mu^+, \overline{\jmath}_{l_{z-1}})\} = \overline{p}_1 \upharpoonright \{(\mu^+, \pi_1 \overline{\jmath}_{l_{z-1}})\}$.

Since $\beta = \widetilde{\beta} + \kappa^+$, and $\operatorname{dom}_y p_1(\mu^+)$ has cardinality $\leq \mu < \kappa$, this is possible by a density argument.

Now, it is possible to construct a $\mathbb{P}_1$-automorphism that exchanges for every $l \in [\overline{h}, \overline{k})$ the $(\mathbb{P}_1)_{(\mu_l, \overline{\jmath}_l)}$-coordinate with the according $(\mathbb{P}_1)_{(\mu_l, \pi_1 \overline{\jmath}_l)}$-coordinate; so for any $q_1 \in D_\pi$, we will have $(\pi_1 q_1)_{(\mu_l, \pi_1 \overline{\jmath}_l)} = (q_1)_{(\mu_l, \overline{\jmath}_l)}$, $(\pi_1 q_1)_{(\mu_l, \overline{\jmath}_l)} = (q_1)_{(\mu_l, \pi_1 \overline{\jmath}_l)}$. By our preparations about $\overline{p}$, we can also assure $\pi_1 \overline{p}_1 \parallel \overline{p}_1$.

Moreover, we will have $\pi \overline{S}^{D_\pi} = \overline{S}^{D_\pi}$: Recall that $\widetilde{\beta}$ was *large enough for* $(I_{\dot{S}})$, and for both π_0 and π_1 we do not disturb indices below $\widetilde{\beta}$; so $\pi \in Fix_0(\kappa_0, i_0) \cap \cdots \cap Small_0(\lambda_0, [0, \alpha_0)) \cap \cdots \cap Fix_1(\overline{\kappa}_0, \overline{\imath}_0) \cap \cdots \cap Small_1(\overline{\lambda}_0, [0, \overline{\alpha}_0)) \cap \cdots$.
For a condition $\overline{q} \leq \overline{p}, \pi \overline{p}$ and H a V-generic filter on $\mathbb{P}$ with $\overline{q} \in H$, it follows that $\alpha \notin \operatorname{rg}(\dot{S}^\beta)^H$, but at the same time

$$\big(\big(\pi \widetilde{\dot{X}}^{D_\pi} \big)^H, \alpha \big) \in \big(\pi \overline{S}^{D_\pi} \big)^H = \dot{S}^H.$$

We will see that this is a contradiction, since $\big(\pi \widetilde{\dot{X}}^{D_\pi} \big)^H$ will be equal to some $\big(\widetilde{\ddot{X}}^{D_\pi} \big)^H$, where $\ddot{X}$ is a name for the forcing

$$\mathbb{P}_0 \upharpoonright t(\pi_0 s) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, j_0), \ldots, (\mu_{\overline{h}-1}, \overline{\jmath}_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1 \overline{\jmath}_{\overline{h}}), \ldots, (\mu_{\overline{k}-1}, \pi_1 \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+),$$

where $\pi_0 s \geq \pi_0 \overline{p} \geq q$ has maximal points $(\kappa^+, j_0), \ldots, (\kappa^+, j_{h-1}), (\kappa^+, \pi_0 j_h), \ldots, (\kappa^+, \pi_0 j_k)$; thus, $(\pi_0 s, (\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{h}-1}, \overline{\jmath}_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1 \overline{\jmath}_{\overline{h}}), \ldots, (\mu_{\overline{k}}, \pi_1 \overline{\jmath}_{\overline{k}})) \in M_\beta$. This will imply

$$\big(\big(\widetilde{\ddot{X}}^{D_\pi} \big)^H, \alpha \big) \in (\dot{S}^\beta)^H,$$

contradicting that $\alpha \notin \operatorname{rg}(\dot{S}^\beta)^H$.

We start with defining π_0. Let $\operatorname{ht} \pi_0 := \eta(\overline{p})$. For any $\alpha < \kappa^+$, $\pi_0(\alpha)$ will be the identity on $\{(\alpha, i) \mid i < F_{\lim}(\alpha)\}$. Regarding level κ^+, let $\operatorname{supp} \pi_0(\kappa^+) := \{(\kappa^+, j_h), \ldots, (\kappa^+, j_{k-1}), (\kappa^+, \pi_0 j_h), \ldots, (\kappa^+, \pi_0 j_{k-1})\}$ with $\pi_0(\kappa^+)(\alpha, j_l) = (\alpha, \pi_0 j_l)$, $\pi_0(\kappa^+)(\alpha, \pi_0 j_l) = (\alpha, j_l)$ for

all $l \in [h,k)$. For $\kappa^{++} \leq \alpha \leq \mathrm{ht}\,\pi_0$, the map $\pi_0(\alpha)$ is constructed as follows: Let $\{(\alpha,\delta_0),\ldots,(\alpha,\delta_{n(\alpha)-1})\}$ denote the collection of all $(\alpha,\delta) \in t(p_0)$ which have a $t(p_0)$-predecessor in $\{(\kappa^+,j_h),\ldots,(\kappa^+,j_{k-1})\}$. Pick $\widetilde{\delta}_0,\ldots,\widetilde{\delta}_{n(\alpha)-1} < F_{\lim}(\alpha)$ pairwise distinct with $\{(\alpha,\widetilde{\delta}_0),\ldots,(\alpha,\widetilde{\delta}_{n(\alpha)-1})\} \cap t(p_0) = \varnothing$ such that for all $i < n(\alpha)$, there is a limit ordinal γ with $\delta_i, \widetilde{\delta}_i \in [\gamma, \gamma{+}\omega)$. Let $\mathrm{supp}\,\pi_0(\alpha) := \{(\alpha,\delta_0),\ldots,(\alpha,\delta_{n(\alpha)-1}),(\alpha,\widetilde{\delta}_0),\ldots,(\alpha,\widetilde{\delta}_{n(\alpha)-1})\}$ with $\pi_0(\alpha)(\alpha,\delta_l) = (\alpha,\widetilde{\delta}_l)$, $\pi_0(\alpha)(\alpha,\widetilde{\delta}_l) = (\alpha,\delta_l)$ for all $l < n(\alpha)$.
This defines π_0.

First, we have to check whether $\pi_0 \in Fix_0(\kappa_0,i_0) \cap \cdots \cap Fix_0(\kappa_{n-1},i_{n-1})$. Consider $l < n$. Then $\pi_0 \in Fix_0(\kappa_l,i_l)$ is clear in the case that $\kappa_l < \kappa^+$. If $\kappa_l = \kappa^+$, then $(\kappa_l,i_l) \notin \mathrm{supp}\,\pi_0(\kappa_l)$ follows from $\widetilde{\beta} > i'_l = i_l$. In the case that $\kappa_l > \kappa^+$, let $\mathrm{supp}\,\pi_0(\kappa_l) = \{(\kappa_l,\delta_0),\ldots,(\kappa_l,\delta_{n(\kappa_l)-1}),(\kappa_l,\widetilde{\delta}_0),\ldots,(\kappa_l,\widetilde{\delta}_{n(\kappa_l)-1})\}$ as before. Recall that we denote by (κ^+,i'_l) the $t(\overline{r})$-predecessor of (κ_l,i_l) on level κ^+ (which is also its $t(p_0)$-predecessor). Since $\widetilde{\beta}$ is *large enough for* $(I_{\dot{S}})$, it follows that $i'_l < \widetilde{\beta}$, so $(\kappa^+,i'_l) \notin \{(\kappa^+,j_h),\ldots,(\kappa^+,j_{k-1})\}$; thus, $(\kappa_l,i_l) \notin \{(\kappa_l,\delta_0),\ldots,(\kappa_l,\delta_{n(\kappa_l)-1})\}$.
Also, $(\kappa_l,i_l) \in t(r) \subseteq t(p_0)$ gives $(\kappa_l,i_l) \notin \{(\kappa_l,\widetilde{\delta}_0),\ldots,(\kappa_l,\widetilde{\delta}_{n(\kappa_l)-1})\}$. Hence, $(\kappa_l,i_l) \notin \mathrm{supp}\,\pi_0(\kappa_l)$ as desired; and it follows that $\pi_0 \in Fix_0(\kappa_0,i_0) \cap \cdots \cap Fix_0(\kappa_{n-1},i_{n-1})$.

Also, $\pi_0 \in Small_0(\lambda_0,[0,\alpha_0)) \cap \cdots \cap Small_0(\lambda_{m-1},[0,\alpha_{m-1}))$: Let $l < m$. For $\lambda_l \neq \kappa^+$, there is nothing to show. In the case that $\lambda_l = \kappa^+$, we use again that $\widetilde{\beta}$ is *large enough for* $(I_{\dot{S}})$; so $\widetilde{\beta} > \alpha_0,\ldots,\alpha_{m-1}$, which implies $\mathrm{supp}\,\pi_0(\kappa^+) \cap \{(\kappa^+,i) \mid i < \alpha_l\} = \varnothing$.

Finally, $\pi p_0 \parallel p_0$ by construction.

Now, we turn to π_1. Let $\mathrm{supp}\,\pi_1 := \{\mu_{\overline{h}},\ldots,\mu_{\overline{k}-1}\}$.

Consider $\mu^+ \in Succ'$ with $\{l \in [\overline{h},\overline{k}) \mid \mu_l = \mu^+\} = \{l_0,\ldots,l_{z-1}\}$ for some $1 \leq z < \omega$. (Then $\mu_{l_0} = \cdots = \mu_{l_{z-1}} = \mu^+$, and $l_0,\ldots,l_{z-1} \in [\overline{h},\overline{k})$ implies $\overline{\jmath}_{l_0},\ldots,\overline{\jmath}_{l_{z-1}} \geq \beta$.) Recall that we have $\pi_1\overline{\jmath}_{l_0},\ldots,\pi_1\overline{\jmath}_{l_{z-1}} \in (\widetilde{\beta},\beta) \smallsetminus \{\overline{\jmath}_0,\ldots,\overline{\jmath}_{\overline{h}-1}\}$ with $\overline{p}_1 \upharpoonright \{(\mu_{l_0},\overline{\jmath}_{l_0})\} = \overline{p}_1 \upharpoonright \{(\mu_{l_0},\pi_1\overline{\jmath}_{l_0})\},\ldots,\overline{p}_1 \upharpoonright \{(\mu_{l_{z-1}},\overline{\jmath}_{l_{z-1}})\} = \overline{p}_1 \upharpoonright \{(\mu_{l_{z-1}},\pi_1\overline{\jmath}_{l_{z-1}})\}$.
Let $\mathrm{dom}\,\pi_1(\mu^+) = \mathrm{dom}_x\,\pi_1(\mu^+) \times \mathrm{dom}_y\,\pi_1(\mu^+) := \mathrm{dom}_x\,\overline{p}_1(\mu^+) \times \mathrm{dom}_y\,\overline{p}_1(\mu^+)$, and $\mathrm{supp}\,\pi_1(\mu^+) := \{\overline{\jmath}_{l_0},\ldots,\overline{\jmath}_{l_{z-1}},\pi_1\overline{\jmath}_{l_0},\ldots,\pi_1\overline{\jmath}_{l_{z-1}}\}$. The map $f_{\pi_1}(\mu^+) : \mathrm{supp}\,\pi_1(\mu^+) \to \mathrm{supp}\,\pi_1(\mu^+)$ will be defined as follows: $f_{\pi_1}(\mu^+)(\overline{\jmath}_l) = \pi_1\overline{\jmath}_l$, $f_{\pi_1}(\mu^+)(\pi_1\overline{\jmath}_l) = \overline{\jmath}_l$ for all $l \in \{l_0,\ldots,l_{z-1}\}$.
For $\zeta \in \mathrm{dom}_x\,\pi_1(\mu^+)$, we need a bijection $\pi_1(\mu^+)(\zeta) : 2^{\mathrm{supp}\,\pi_1(\mu^+)} \to 2^{\mathrm{supp}\,\pi_1(\mu^+)}$. Again, we swap any $\overline{\jmath}_l$-coordinate with the according $\pi_1\overline{\jmath}_l$-coordinate:
$\big(\pi_1(\mu^+)(\zeta)\big(\epsilon_i \mid i \in \mathrm{supp}\,\pi_1(\mu^+)\big)\big)_{\overline{\jmath}_l} := \epsilon_{\pi_1\overline{\jmath}_l}$, $\big(\pi_1(\mu^+)(\zeta)\big(\epsilon_i \mid i \in \mathrm{supp}\,\pi_1(\mu^+)\big)\big)_{\pi_1\overline{\jmath}_l} := \epsilon_{\overline{\jmath}_l}$ for $l \in \{l_0,\ldots,l_{z-1}\}$.
Finally, for $(\zeta,i) \in [\mu,\mu^+) \times F(\mu^+)$, let $\pi_1(\mu^+)(\zeta,i) = 0$.

This defines π_1, with $D_{\pi_1} = \{q \in \mathbb{P}_1 \mid \forall\, \mu^+ \in \mathrm{supp}\,q \cap \{\mu_{\overline{h}},\ldots,\mu_{\overline{k}-1}\}\ \mathrm{dom}\,q(\mu^+) \supseteq \mathrm{dom}\,\pi_1(\mu^+)\}$.

For any such $q \in D_{\pi_1}$ and $\mu^+ \in \mathrm{supp}\,q$ with $\mu^+ = \mu_{l_0} = \cdots = \mu_{l_{z-1}}$ for ${}_0,\ldots,l_{z-1}$ as above, we have $\{\overline{\jmath}_{l_0},\ldots,\overline{\jmath}_{l_{z-1}},\pi_1\overline{\jmath}_{l_0},\ldots\pi_1\overline{\jmath}_{l_{z-1}}\} \subseteq \mathrm{dom}_y\,\overline{p}(\mu^+) = \mathrm{dom}_y\,\pi_1(\mu^+) \subseteq \mathrm{dom}_y\,q_1(\mu^+)$, and $(\pi_1 q)(\mu^+)(\zeta,\overline{\jmath}_l) = q(\mu^+)(\zeta,\pi_1\overline{\jmath}_l)$, $(\pi_1 q)(\mu^+)(\zeta,\pi_1\overline{\jmath}_l) = q(\mu^+)(\zeta,\overline{\jmath}_l)$ for all $l \in \{l_0,\ldots,l_{z-1}\}$, $\zeta \in \mathrm{dom}_x\,q(\mu^+)$. Moreover, $(\pi_1 q)(\mu^+)(\zeta,i) = q(\mu^+)(\zeta,i)$ for all the $\zeta \in \mathrm{dom}_x\,q(\mu^+)$, $i < F(\mu^+)$ remaining with $i \notin \mathrm{supp}\,\pi_1(\mu^+) = \{\overline{\jmath}_{l_0},\ldots,\overline{\jmath}_{l_{z-1}},\ \pi_1\overline{\jmath}_{l_0},\ldots,\pi_1\overline{\jmath}_{l_{z-1}}\}$.
Since we have arranged that $\overline{p}_1 \upharpoonright \{(\mu_{l_0},\overline{\jmath}_{l_0})\} = \overline{p}_1 \upharpoonright \{(\mu_{l_0},\pi_1\overline{\jmath}_{l_0})\},\ \ldots,\overline{p}_1 \upharpoonright \{(\mu_{l_{z-1}},\overline{\jmath}_{l_{z-1}})\} = \overline{p}_1 \upharpoonright \{(\mu_{l_{z-1}},\pi_1\overline{\jmath}_{l_{z-1}})\}$, it follows that $(\pi_1\overline{p}_1)(\mu^+) = \overline{p}_1(\mu^+)$.

Hence, $\pi_1\overline{p}_1 \parallel \overline{p}_1$.

It remains to check that $\pi_1 \in Fix_1(\overline{\kappa}_0, \overline{\imath}_0) \cap \dots \cap Fix_1(\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1}) \cap Small_1(\overline{\lambda}_0, [0, \alpha_0)) \cap \dots \cap Small_1(\overline{\lambda}_{\overline{m}-1}, [0, \alpha_{\overline{m}-1}))$. For $l < \overline{n}$ and $\overline{\kappa}_l < \kappa$, we have $\overline{\imath}_l < \widetilde{\beta}$, since $\widetilde{\beta}$ is *large enough*, so $\overline{\imath}_l \notin \operatorname{supp} \pi_1(\overline{\kappa}_l)$. Hence, $\pi_1 \in Fix_1(\overline{\kappa}_l, \overline{\imath}_l)$. In the case that $\overline{\kappa}_l \geq \kappa$, it follows that $\overline{\kappa}_l \notin \operatorname{supp} \pi_1$, so again, $\pi_1 \in Fix_1(\overline{\kappa}_l, \overline{\imath}_l)$ as desired. Similarly, $\pi_1 \in Small_1(\overline{\lambda}_0, [0, \alpha_0)) \cap \dots \cap Small_1(\overline{\lambda}_{\overline{m}-1}, [0, \alpha_{\overline{m}-1}))$.

Thus, we have constructed an automorphism $\pi = (\pi_0, \pi_1)$ with $\pi\overline{p} \parallel \overline{p}$ and $\pi \in Fix_0(\kappa_0, i_0) \cap \dots \cap Small_0(\lambda_0, [0, \alpha_0)) \cap \dots \cap Fix_1(\overline{\kappa}_0, \overline{\imath}_0) \cap \dots \cap Small_1(\overline{\lambda}_0, [0, \overline{\alpha}_0)) \cap \cdots$. This gives $\pi\overline{S}^{D_\pi} = \overline{S}^{D_\pi}$.

Since $p \Vdash_{\mathbb{P}\restriction(\lambda+1)} \mathrm{OR}_{\mathbb{P}\restriction(\lambda+1)}(\widetilde{\dot{X}}, \alpha) \in \dot{S}$, it follows that $\pi\overline{p} \Vdash_{\mathbb{P}\restriction(\lambda+1)} \mathrm{OR}_{\mathbb{P}\restriction(\lambda+1)}(\pi\widetilde{\dot{X}}^{D_\pi}, \alpha) \in \pi\overline{S}^{D_\pi}$; hence,

$$\pi\overline{p} \Vdash_{\mathbb{P}\restriction(\lambda+1)} \mathrm{OR}_{\mathbb{P}\restriction(\lambda+1)}(\pi\widetilde{\dot{X}}^{D_\pi}, \alpha) \in \dot{S}\,.$$

Take $q \in \mathbb{P}\restriction(\lambda+1)$ with $q \leq \overline{p}, \pi\overline{p}$. Then $q \Vdash_{\mathbb{P}\restriction(\lambda+1)} \alpha \notin \operatorname{rg} \dot{S}^\beta$, and

$$q \Vdash_{\mathbb{P}\restriction(\lambda+1)} \mathrm{OR}_{\mathbb{P}\restriction(\lambda+1)}(\pi\widetilde{\dot{X}}^{D_\pi}, \alpha) \in \dot{S}.$$

We will lead this into a contradiction.

As already indicated, $\pi\widetilde{\dot{X}}^{D_\pi}$ will be equal to some $\widetilde{\ddot{X}}^{D_\pi}$, where $\ddot{X}$ is a name for the forcing

$$\mathbb{P}_0\restriction t(\pi_0 s) \times \mathbb{P}_1\restriction\{(\mu_0, j_0), \dots, (\mu_{\overline{h}-1}, \overline{\jmath}_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1\overline{\jmath}_{\overline{h}}), \dots, (\mu_{\overline{k}-1}, \pi_1\overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+),$$

where $\pi_0 s \in \mathbb{P}_0$ has maximal points $(\kappa^+, j_0), \dots, (\kappa^+, j_{h-1}), (\kappa^+, \pi_0 j_h), \dots, (\kappa^+, \pi_0 j_{k-1})$.

More generally, for a name $\dot{x} \in \mathrm{Name}(\mathbb{P}_0\restriction t(s) \times \mathbb{P}_1\restriction\{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+))$, we cannot apply π directly to obtain $\pi\dot{x}$, but have transform $\dot{x}$ into a $\mathbb{P}$-name $\widetilde{\dot{x}}$ first, and then consider the extension $\widetilde{\dot{x}}^{D_\pi}$.

However, the map π induces a canonical isomorphism $T_\pi : \mathbb{P}_0\restriction t(s) \times \mathbb{P}_1\restriction\{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+) \to \mathbb{P}_0\restriction t(\pi_0 s) \times \mathbb{P}_1\restriction\{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{h}-1}, \overline{\jmath}_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1\overline{\jmath}_{\overline{h}}), \dots, (\mu_{\overline{k}-1}, \pi_1\overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1\restriction(\kappa^+)$, which extends to the name space, such that for all $\dot{x} \in \mathrm{Name}(\mathbb{P}_0\restriction t(s) \times \mathbb{P}_1\restriction\{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+))$, we have

$$\widetilde{T_\pi\dot{x}}^{D_\pi} = \pi\widetilde{\dot{x}}^{D_\pi}.$$

This transformation T_π can be defined as follows:
Recall that s is a condition in $\mathbb{P}_0$ with maximal points $(\kappa^+, j_0), \dots, (\kappa^+, j_{k-1})$, so the condition $\pi_0 s$ has maximal points $(\kappa^+, j_0), \dots, (\kappa^+, j_{h-1})$, $(\kappa^+, \pi_0 j_h), \dots, (\kappa^+, \pi_0 j_{k-1})$ with $(\pi_0 s)\restriction\kappa^+ = s\restriction\kappa^+$, and for any $l < h$, it follows that $\pi_0 s$ has the same branch below (κ^+, j_l) as s; but for $l \in [h, k)$, the $\pi_0 s$-branch below $(\kappa^+, \pi_0 j_l)$ coincides with the s-branch below (κ^+, j_l).

For a condition

$$\big(v_0\restriction t(s), v_1\restriction\{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}, v_1(\kappa^+)\big)$$

in $\mathbb{P}_0 \restriction t(s) \times \mathbb{P}_1 \restriction \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+)$, let

$$T_\pi\big(v_0 \restriction t(s), v_1 \restriction \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}, v_1(\kappa^+)\big)$$

be the condition

$$\big(\, v_0' \restriction t(\pi_0 s), v_1' \restriction \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{h}-1}, \overline{\jmath}_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1 \overline{\jmath}_{\overline{h}}), \dots, (\mu_{\overline{k}-1}, \pi_1 \overline{\jmath}_{\overline{k}-1})\}, v_1'(\kappa^+)\,\big)$$

with

- $v_0' \restriction t(\pi_0 s) = \pi_0(v_0 \restriction t(s))$,
- $v_1' \restriction \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{h}-1}, \overline{\jmath}_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1 \overline{\jmath}_{\overline{h}}), \dots, (\mu_{\overline{k}-1}, \pi_1 \overline{\jmath}_{\overline{k}-1})\} \in \mathbb{P}_1 \restriction \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{h}-1}, \overline{\jmath}_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1 \overline{\jmath}_{\overline{h}}), \dots, (\mu_{\overline{k}-1}, \pi_1 \overline{\jmath}_{\overline{k}-1})\}$ is obtained from $v_1 \restriction \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}$ by swapping any $(\mu_l, \overline{\jmath}_l)$-coordinate for $l \in [\overline{h}, \overline{k})$ with the according $(\mu_l, \pi_1 \overline{\jmath}_l)$-coordinate,
- $v_1'(\kappa^+) = v_1(\kappa^+)$.

Then T_π induces a canonical transformation of names $T_\pi : \mathrm{Name}\big(\mathbb{P}_0 \restriction t(s) \times \mathbb{P}_1 \restriction \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+)\big) \to \mathrm{Name}\big(\mathbb{P}_0 \restriction t(\pi_0 s) \times \mathbb{P}_1 \restriction \{(\mu_0, j_0), \dots, (\mu_{\overline{h}-1}, j_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1 j_{\overline{h}}), \dots, (\mu_{\overline{k}-1}, \pi_1 j_{\overline{k}-1})\} \times \mathbb{P}_1 \restriction (\kappa^+)\big)$, which will be denoted by the same letter.
Recursively, it is not difficult to check that indeed, $\pi \widetilde{\dot{x}}^{D_\pi} = \widetilde{T_\pi \dot{x}}^{D_\pi}$.

Thus, from

$$q \Vdash_{\mathbb{P} \restriction (\lambda+1)} \mathrm{OR}_{\mathbb{P} \restriction (\lambda+1)}(\pi \widetilde{\dot{X}}^{D_\pi}, \alpha) \in \dot{S}$$

it follows that

$$q \Vdash_{\mathbb{P} \restriction (\lambda+1)} \mathrm{OR}_{\mathbb{P} \restriction (\lambda+1)}(\widetilde{T_\pi \dot{X}}^{D_\pi}, \alpha) \in \dot{S}.$$

Now, $T_\pi \dot{X} \in \mathrm{Name}\big(\mathbb{P}_0 \restriction t(\pi_0 s) \times \mathbb{P}_1 \restriction \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{h}-1}, \overline{\jmath}_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1 \overline{\jmath}_{\overline{h}}), \dots, (\mu_{\overline{k}-1}, \pi_1 \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+)\big)$, where $\pi_0 s$ has maximal points $(\kappa^+, j_0), \dots, (\kappa^+, j_{h-1}), (\kappa^+, \pi_0 j_h), \dots, (\kappa^+, \pi_0 j_{k-1})$ with $j_0 < \beta, \dots, j_{h-1} < \beta$, and $\pi_0 j_h < \beta, \dots, \pi_0 j_{k-1} < \beta$ by construction. Also, $\overline{\jmath}_0 < \beta, \dots, \overline{\jmath}_{\overline{h}-1} < \beta$, and $\pi_1 \overline{\jmath}_{\overline{h}} < \beta, \dots, \pi_1 \overline{\jmath}_{\overline{k}-1} < \beta$ by construction. Thus, $(\pi_0 s, (\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{h}-1}, \overline{\jmath}_{\overline{h}-1}), (\mu_{\overline{h}}, \pi_1 \overline{\jmath}_{\overline{h}}), \dots, (\mu_{\overline{k}-1}, \pi_1 \overline{\jmath}_{\overline{k}-1})) \in M_\beta$.
Since $q_0 \leq \pi_0 \overline{p}_0 \leq \pi_0 s$ and $q \Vdash_{\mathbb{P} \restriction (\lambda+1)} \mathrm{OR}_{\mathbb{P} \restriction (\lambda+1)}(\widetilde{T_\pi \dot{X}}, \alpha) \in \dot{S}$, it follows that

$$\big(\, \mathrm{OR}_{\mathbb{P} \restriction (\lambda+1)}(\widetilde{T_\pi \dot{X}}^{D_\pi}, \alpha), q\big) \in \dot{S}^\beta.$$

contradicting that also $q \Vdash_{\mathbb{P} \restriction (\lambda+1)} \alpha \notin \mathrm{rg}\, \dot{S}^\beta$.

Hence, S^β must be surjective, which finishes the proof. □

Thus, we have shown that for any $\widetilde{\beta} < F(\kappa)$ large enough and $\beta = \widetilde{\beta} + \kappa^+$, the restriction $S^\beta : \mathrm{dom}\, S^\beta \to F(\kappa)$ must be surjective, as well.
We will now lead this into a contradiction.

For the rest of this section, we fix some limit ordinal $\widetilde{\beta} < F(\kappa)$ *large enough* and let $\beta := \widetilde{\beta} + \kappa^+$. We want to capture S^β in an intermediate model $V[G^\beta \restriction (\kappa^+ + 1)]$, which will be a generic extension by a certain set forcing $\mathbb{P}^\beta \upharpoonright (\kappa^+ + 1)$. We will show that $V[G^\beta \upharpoonright (\kappa^+ + 1)]$ also contains an injection $\iota : \operatorname{dom} S^\beta \hookrightarrow \beta$, while $\mathbb{P}^\beta \upharpoonright (\kappa^+ + 1)$ preserves all cardinals $\geq F(\kappa)$ – a contradiction.

Roughly speaking, this forcing $\mathbb{P}^\beta \upharpoonright (\kappa^+ + 1)$ will be obtained from $\mathbb{P}$, by first cutting off at height $\kappa^+ + 1$, and then cutting off at width β. The latter procedure is rather clear for $\mathbb{P}_1$: For successor cardinals $\lambda^+ < \kappa$, $\lambda^+ \in Succ'$, we take for $(\mathbb{P}_1)^\beta(\lambda^+)$ the forcing $Fn([\lambda, \lambda^+) \times \beta, 2, \lambda^+)$ instead of $Fn([\lambda, \lambda^+) \times F(\lambda^+), 2, \lambda^+)$ in the case that $\beta < F(\lambda^+)$. However, the forcing notion $(\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1)$ requires a careful construction. One could try and restrict $\mathbb{P}_0$ to all those $p \in \mathbb{P}_0 \upharpoonright (\kappa^+ + 1)$ which have only maximal points (κ^+, i) with $i < \beta$. Nevertheless, their predecessors (λ, j) on lower levels $\lambda < \kappa^+$ might still have indices $j > \beta$, so our forcing would still be "too big".
Our idea will be to drop all indices at levels below κ^+ – then the domain $t(p)$ of the conditions $p \in (\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1)$ will be given by their maximal points (κ^+, i) and the structure of the tree below, i.e. for any two maximal points (κ^+, i) and (κ^+, i') we only need information about the level at which the branches below them meet.

We start with a "preliminary version" $(\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)$: Any condition $p \in (\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)$ will be of the form $p : t(p) \to V$ with a tree $t(p)$ given by its finitely many maximal points $(\kappa^+, \beta_0), \ldots, (\kappa^+, \beta_{k-1})$ and the tree structure below. We will now specify how this tree structure should be coded into the forcing conditions:

On the one hand, for any level $\alpha \leq \kappa^+$, the tree structure of $t(p)$ induces an equivalence relation $\sim_\alpha$ on the set $\{\beta_0, \ldots, \beta_{k-1}\}$ by setting $\beta_i \sim_\alpha \beta_j$ iff (κ^+, i) and (κ^+, j) have a common $t(p)$-predecessor on level α. This equivalence relation $\sim_\alpha$ induces a partition B_α on $\{\beta_0, \ldots, \beta_{k-1}\}$ such that for all $l, l' < k$, there exists $z \in B_\alpha$ with $\{\beta_l, \beta_{l'}\} \subseteq z$ iff the vertices (κ^+, β_l) and $(\kappa^+, \beta_{l'})$ have a common $t(p)$-predecessor on level α.

Conversely, the tree structure below $(\kappa^+, \beta_0), \ldots, (\kappa^+, \beta_{k-1})$ could be described by a sequence $(B_\alpha \mid \alpha \leq \kappa^+, \alpha \in \mathrm{Card})$ of partitions of the set $\{\beta_0, \ldots, \beta_{k-1}\}$ such that any B_{α^+} is finer than B_α, and $B_0 = \{\{\beta_0, \ldots, \beta_{k-1}\}\}$, $B_{\kappa^+} = \{\{\beta_0\}, \ldots, \{\beta_{k-1}\}\}$. Since for $F_{\lim}$-trees we do not allow splitting at limits, we have to require that for any limit cardinal $\alpha \leq \kappa$, there exists a cardinal $\overline{\alpha} < \alpha$ such that $B_\alpha = B_\beta$ for all β with $\overline{\alpha} \leq \beta \leq \alpha$.
We will give any $t(p)$-vertex on level $\alpha \leq \kappa^+$ a "name" (α, z), where $z \in B_\alpha$ is the collection of all $i < k$ with $(\alpha, z) \leq_{t(p)} (\kappa^+, \{\beta_i\})$. Then the vertices already determine the tree structure of $t(p)$.

Definition 3.4.3. Let $k < \omega$ and $\beta_0, \ldots, \beta_{k-1} < F_{\lim}(\kappa^+) = F(\kappa)$. We say that $(t, \leq_t)$ is a *tree below* $(\kappa^+, \beta_0), \ldots, (\kappa^+, \beta_{k-1})$ if there is a sequence $(B_\alpha \mid \alpha \leq \kappa^+, \alpha \in \mathrm{Card})$ of partitions of the set $\{\beta_0, \ldots, \beta_{k-1}\}$ such that

- for any cardinal $\alpha < \kappa^+$, it follows that B_{α^+} is finer than B_α, $B_0 = \{\{\beta_0, \ldots, \beta_{k-1}\}\}$, and $B_{\kappa^+} = \{\{\beta_0\}, \ldots, \{\beta_{k-1}\}\}$,
- for all limit cardinals α, there exists $\overline{\alpha} < \alpha$ with $B_\beta = B_\alpha$ for all $\overline{\alpha} \leq \beta \leq \alpha$,

such that

$$t := \bigcup_{\substack{\alpha \in \mathrm{Card} \\ \alpha \leq \kappa^+}} \{\alpha\} \times B_\alpha,$$

i.e. , the vertices of t are pairs (α, z) with $z \in B_\alpha$ a subset of $\{\beta_0, \dots, \beta_{k-1}\}$.

The order $\leq_t$ is defined as follows: $(\alpha, z) \leq_t (\beta, z')$ iff $\alpha \leq \beta$ and $z \supseteq z'$.

We call $\operatorname{supp} t = \{(\kappa^+, \beta_0), \dots, (\kappa^+, \beta_{k-1})\}$ the *support of* $(t, \leq_t)$.

For $\beta \leq F_{\lim}(\kappa^+)$, we denote by $T(\kappa^+, \beta)$ the collection of all $(t, \leq_t)$ such that $(t, \leq_t)$ is a tree below some $(\kappa^+, \beta_0), \dots, (\kappa^+, \beta_{k-1})$ with $k < \omega$ and $\beta_0, \dots, \beta_{k-1} < \beta$.

There is a canonical partial order $\leq_{T(\kappa^+,\beta)}$ on $T(\kappa^+, \beta)$: Set $(s, \leq_s) \leq_{T(\kappa^+,\beta)} (t, \leq_t)$ iff supp $\supseteq$ supp t, and the tree structures of s and t below supp t agree, i.e. for any $i, j \in \operatorname{supp} t$, the $(t, \leq_t)$-branches below (κ^+, i) and (κ^+, j) meet at the same level as they do in $(s, \leq_s)$.

Definition 3.4.4. Let $(t, \leq_t), (s, \leq_s) \in T(\kappa^+, \beta)$ with $\operatorname{supp} t = \{\beta_0, \dots, \beta_{k-1}\}$, $\operatorname{supp} s = \{\overline{\beta}_0, \dots, \overline{\beta}_{\overline{k}-1}\}$, and the according sequences of partitions $(B_\alpha \mid \alpha \in \mathrm{Card}, \alpha \leq \kappa^+)$ and $(\overline{B}_\alpha \mid \alpha \in \mathrm{Card}, \alpha \leq \kappa^+)$. Then $(s, \leq_s) \leq_{T(\kappa^+,\beta)} (t, \leq_t)$ iff the following hold:

- $\operatorname{supp} s = \{\overline{\beta}_0, \dots, \overline{\beta}_{\overline{k}-1}\} \supseteq \{\beta_0, \dots, \beta_{k-1}\} = \operatorname{supp} t$,
- for any $\alpha \leq \kappa^+$, the partition $\overline{B}_\alpha$ extends B_α, i.e. for any $\beta_l, \beta_{l'} \in \operatorname{supp} t$,

$$(\exists\, z \in B_\alpha \ \{\beta_l, \beta_{l'}\} \subseteq z) \ \Leftrightarrow \ (\exists\, \overline{z} \in \overline{A}_\alpha \ \{\beta_l, \beta_{l'}\} \subseteq \overline{z}).$$

One can check that $\leq_{T(\kappa^+,\beta)}$ is indeed a partial order.

For trees $(s, \leq_s)$ and $(t, \leq_t)$ in $T(\kappa^+, \beta)$ with $(s, \leq_s) \leq_{T(\kappa^+,\beta)} (t, \leq_t)$, we can define an embedding $\iota : (t, \leq_t) \hookrightarrow (s, \leq_s)$ as follows: $\iota(\alpha, z) := (\alpha, \overline{z})$, where $(\alpha, \overline{z}) \in s$ with $\overline{z} \supseteq z$ (then $z = \overline{z} \cap \operatorname{supp} t$). With $\leq_{\iota[t]} := \iota[\leq_t] = \{(\iota(\alpha, z), \iota(\beta, z')) \mid (\alpha, z) \leq_t (\beta, z')\}$, it follows that $\leq_{\iota[t]} = \leq_s \cap\, \iota[t]$, and $(\iota[t], \leq_{\iota[t]}) \subseteq (s, \leq_s)$ is a subtree.
Conversely, consider $s, t \in T(\kappa^+, \beta)$ with an embedding $\iota : (t, \leq_t) \hookrightarrow (s, \leq_s)$ such that for all $(\alpha, z) \in t$, we have $\iota(\alpha, z) = (\alpha, \overline{z})$ with $\overline{z} \supseteq z$. Then $(\iota[t], \iota[\leq_t]) \subseteq (s, \leq_s)$ is a subtree, and one can easily check that $(s, \leq_s) \leq_{T(\kappa^+,\beta)} (t, \leq_t)$.
Hence, the partial order $\leq_{T(\kappa^+,\beta)}$ can also be described via embeddings.

The maximal element of $T(\kappa^+, \beta)$ is the empty tree.

Now, we can define $(\widehat{\mathbb{P}}_0)^\beta \restriction (\kappa^+ + 1)$:

Definition 3.4.5. The forcing $(\widehat{\mathbb{P}}_0)^\beta \restriction (\kappa^+ + 1)$ consists of all $p : t(p) \to V$ such that

- $t(p) \in T(\kappa^+, \beta)$,
- $p(\alpha^+, z) \in Fn([\alpha, \alpha^+), 2, \alpha^+)$ for all $(\alpha^+, z) \in t(p)$ with α^+ a successor cardinal,
- $p(\aleph_0, z) \in Fn(\aleph_0, 2, \aleph_0)$ for all $(\aleph_0, z) \in t(p)$,
- $p(\alpha, z) = \varnothing$ for all $(\alpha, z) \in t(p)$ with α a limit cardinal, and
- $|p \restriction \alpha| < \alpha$ for all regular limit cardinals α.

For $\overline{p}, p \in (\widehat{\mathbb{P}}_0)^\beta \restriction (\kappa^+ + 1)$, set $\overline{p} \leq p$ iff

- $t(\overline{p}) \leq_{T(\kappa^+,\beta)} t(p)$,

- $\overline{p}(\alpha, \overline{z}) \supseteq p(\alpha, z)$ whenever $\overline{z} \supseteq z$.

The maximal element $\mathbb{1}$ in $(\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)$ is the empty condition with $t(\mathbb{1}) = \emptyset$.

Our argument for capturing S^β inside $V[G^\beta \upharpoonright (\kappa^+ + 1)]$ will roughly be as follows: We define a function $(S^\beta)'$ as the set of all $(\dot{X}^{G^\beta \upharpoonright (\kappa^+ + 1)}, \alpha)$ for an appropriate name $\dot{X}$, such that there exists $p \in \mathbb{P}$ with $p \Vdash (\dot{X}, \alpha) \in \dot{S}$ and $p^\beta \upharpoonright (\kappa^+ + 1) \in G^\beta \upharpoonright (\kappa^+ + 1)$. In order to show that $(S^\beta)' \subseteq S^\beta$, we use an isomorphism argument similarly as before: If there was $(\dot{X}^{G^\beta \upharpoonright (\kappa^+ + 1)}, \alpha) \in (S^\beta)' \setminus S^\beta$, one could take p and q in $\mathbb{P}$ with $p^\beta \upharpoonright (\kappa^+ + 1) \in G^\beta \upharpoonright (\kappa^+ + 1)$, $q \in G$ such that $p \Vdash (\dot{X}, \alpha) \in \dot{S}$ and $q \Vdash (\dot{X}, \alpha) \notin \dot{S}$. We construct an automorphism π with $\pi p \parallel q$ with $\pi \widetilde{\dot{X}}^{D_\pi} = \widetilde{\dot{X}}^{D_\pi}$ and $\pi \overline{S}^{D_\pi} = \overline{S}^{D_\pi}$, and obtain a contradiction.

Recall that prior to the proof of Proposition 3.4.2, we have fixed a condition $r \in G_0$ such that the maximal points of $t(r)$ are among $\{(\kappa_0, i_0), \ldots, (\kappa_{n-1}, i_{n-1})\} \subseteq t(r)$, and $\overline{r} \in G_0$, $\overline{r} \leq r$, such that all branches of $\overline{r}$ have height $\geq \kappa^+$. For $l < n$ with $\kappa_l \geq \kappa^+$, we denote by (κ^+, i_l') the $t(\overline{r})$-predecessor of (κ_l, i_l) on level κ^+; in the case that $\kappa_l < \kappa^+$, we have chosen for (κ^+, i_l') some $t(\overline{r})$-successor of (κ_l, i_l) on level κ^+.

Firstly, in order to make sure that $\pi p \parallel q$ is possible while at the same time $\pi \in Fix_0(\kappa_0, i_0) \cap \cdots \cap Fix_0(\kappa_{n-1}, i_{n-1})$, it will be necessary that from $(p_0)^\beta \upharpoonright (\kappa^+ + 1) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1)$, $q \in G$, it follows that p and q coincide on the tree $t(\overline{r})$. Thus, we will have to include $t(\overline{r})$ into our forcing $(\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)$: Namely, we will restrict $(\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)$ to those conditions that coincide with $t(\overline{r})$ below level κ^+.
Secondly, for $\pi \in Small_0(\lambda_0, [0, \alpha_0)) \cap \cdots \cap Small_0(\lambda_{m-1}, [0, \alpha_{m-1}))$, we will have to make sure that $(p_0)^\beta \upharpoonright (\kappa^+ + 1) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1)$, $q \in G$ implies that for all $l < m$, the indices (λ_l, i) at level λ_l agree for p and q for all $i < \alpha_l$. In order to achieve this, we enhance our forcing $(\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)$ and assign indices (λ, i) with $i < \alpha_l$ to some some vertices (λ_l, z).

We start with the second, defining a forcing $\big((\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)\big)_{(\lambda_0, \alpha_0), \ldots}$ that will be the collection of all $p \in (\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)$ equipped with an additional indexing function $N(p)$ on $\{(\lambda_l, z) \in t(p) \mid l < m, \lambda_l \leq \kappa\}$ such that

- $N(p)(\lambda_l, z) \in \{(\lambda_l, i) \mid i < \alpha_l\} \cup \{*\}$ for all $(\lambda_l, z) \in \operatorname{dom} N(p)$,
- any $(\lambda, i) \in rg\, N(p)$ has only one preimage:

$$\big(N(p)(\lambda_l, z) = N(p)(\lambda_l, z') \wedge z \neq z'\big) \Rightarrow N(p)(\lambda_l, z) = N(p)(\lambda_l, z') = *.$$

The idea about this indexing function $N(p)$ is that for a condition $p \in \big((\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)\big)_{(\lambda_0, \alpha_0), \ldots}$, any vertex $(\lambda_l, z) \in t(p)$ with $N(p)(\lambda_l, z) = (\lambda_l, i)$ for some $i < \alpha_l$ should correspond to the vertex (λ_l, i) for conditions in $\mathbb{P}_0$, while all vertices $(\lambda_l, z) \in t(p)$ with $N(p)(\lambda_l, z) = *$ should correspond to vertices (λ_l, i) with $i \geq \alpha_l$.

For $\overline{p}$, $p \in \big((\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)\big)_{(\lambda_0, \alpha_0), \ldots}$ with indexing functions $N(p)$ and $N(\overline{p})$, we set $\overline{p} \leq p$ iff $\overline{p} \leq p$ in $(\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)$, and $N(\overline{p})(\lambda_l, \overline{z}) = N(p)(\lambda_l, z)$ for all $\overline{z} \supseteq z$.

Now, we define our forcing $\big((\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)\big)^{\overline{r}}_{(\lambda_0,\alpha_0),\dots}$, which could be regarded the collection of all those conditions $p \in \big((\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)\big)_{(\lambda_0,\alpha_0),\dots}$ that coincide with $t(\overline{r})$ below (κ^+, i'_0), ... , (κ^+, i'_{n-1}), where the function $N(p)$ is now defined on

$$\{(\lambda_l, z) \in t(p) \mid l < m, \lambda_l \leq \kappa\} \cup \{(\alpha, z) \in t(\overline{r}) \mid \alpha \leq \kappa^+\}.$$

First, we define $T(\kappa^+, \beta)^{t(\overline{r})} \subseteq T(\kappa^+, \beta)$ as follows: The condition $t(\overline{r})$ induces on any level $\alpha \leq \kappa^+$ an equivalence relation $\sim_\alpha^{t(\overline{r})}$ on $\{i'_0, \dots, i'_{n-1}\}$ by setting $i'_l \sim_\alpha^{t(\overline{r})} i'_{\overline{l}}$ iff (κ^+, i'_l) and $(\kappa^+, i'_{\overline{l}})$ have a common $t(\overline{r})$-predecessor on level α.
Thus, let $(t, \leq_t) \in T(\kappa^+, \beta)^{t(\overline{r})}$ iff $(t, \leq_t) \in T(\kappa^+, \beta)$ with partitions $(B_\alpha \mid \alpha \in \mathrm{Card}, \alpha \leq \kappa^+)$ as in the definition of $T(\kappa^+, \beta)$, such that $\{(\kappa^+, i'_0), \dots, (\kappa^+, i'_{n-1})\} \subseteq \operatorname{supp} t$, and for any level $\alpha \leq \kappa^+$, the partition B_α coincides with $\sim_\alpha^{t(\overline{r})}$, i.e. for all $l, \overline{l} < n$, we have $i'_l \sim_\alpha^{t(\overline{r})} i'_{\overline{l}}$ iff there exists $z \in B_\alpha$ with $\{i'_l, i'_{\overline{l}}\} \subseteq z$.
In other words, we want the tree structure of t below $(\kappa^+, i'_0), \dots, (\kappa^+, i'_{n-1})$ coincide with the tree structure of $t(\overline{r})$.
The partial order $\leq_{T(\kappa^+,\beta)^{t(\overline{r})}}$ on $T(\kappa^+, \beta)^{t(\overline{r})}$ is inherited from $T(\kappa^+, \beta)$.

Now, any $p \in \big((\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)\big)^{\overline{r}}_{(\lambda_0,\alpha_0),\dots}$ will be of the form $p : t(p) \to V$ with $t(p) \in T(\kappa^+, \beta)^{t(\overline{r})}$ and the values $p(\alpha, z)$ as in Definition 3.4.5, equipped with an indexing function $N(p)$ defined on

$$\big\{\, (\lambda_l, z) \in t(p) \mid l < m, \lambda_l \leq \kappa \,\big\} \cup \big\{\, (\alpha, z) \mid \exists\, l < n \ \ (\alpha, z) \leq_{t(p)} (\kappa^+, \{i'_l\}) \,\big\}$$

with the following properties:

- For $(\alpha, z) \leq_{t(p)} (\kappa^+, \{i'_l\})$ with $N(p)(\alpha, z) = (\alpha, i)$, it follows that (α, i) is the $t(\overline{r})$-predecessor of (κ^+, i'_l) on level α.
- For all the (λ_l, z) remaining, $N(p)(\lambda_l, z) \in \{(\lambda_l, i) \mid i < \alpha_l\} \cup \{*\}$ as before with

$$\big(N(\lambda_l, z) = N(\lambda_l, z') \wedge z \neq z'\big) \Rightarrow N(\lambda, z) = N(\lambda, z') = *.$$

The idea about extending the domain of $N(p)$ is that any $(\alpha, z) \leq_{t(p)} (\kappa^+, \{i'_l\})$ with $N(p)(\alpha, z) = (\alpha, i)$ should correspond to the vertex $(\alpha, i) \in t(\overline{r})$.

The partial order "$\leq$" on $\big((\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)\big)^{\overline{r}}_{(\lambda_0,\alpha_0),\dots}$ is defined as follows: Set $\overline{p} \leq p$ iff $t(\overline{p}) \leq t(p)$ in $T(\kappa^+, \beta)^{t(\overline{r})}$, and for all $(\alpha, z) \in t(p)$, $(\alpha, \overline{z}) \in t(\overline{p})$ with $z \subseteq \overline{z}$, it follows that $\overline{p}(\alpha, \overline{z}) \supseteq p(\alpha, z)$, and $N(p)(\alpha, z) = N(\overline{p})(\alpha, \overline{z})$ in the case that $(\alpha, z) \in \operatorname{dom} N(p)$.

For the maximal element $\mathbb{1}$, we have for $t(\mathbb{1})$ a tree below $(\kappa^+, i'_0), \dots, (\kappa^+, i'_{n-1})$ with partitions $(B_\alpha \mid \alpha \in \mathrm{Card}, \alpha \leq \kappa^+)$ and the values $N(\mathbb{1})(\alpha, z)$ given by $t(\overline{r})$, and $\mathbb{1}(\alpha, z) = \emptyset$ for all $(\alpha, z) \in t(\mathbb{1})$.

This defines $(\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1) := \big((\widehat{\mathbb{P}}_0)^\beta \upharpoonright (\kappa^+ + 1)\big)^{\overline{r}}_{(\lambda_0,\alpha_0),\dots}$.

We will now see that there is a subforcing $(\widetilde{\mathbb{P}}_0)^{\overline{r}} \subseteq \mathbb{P}_0$ dense in $\mathbb{P}_0$ below $\overline{r}$ with a projection of forcing posets $\rho_0^\beta : (\widetilde{\mathbb{P}})^{\overline{r}} \to (\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1)$. Hence, G_0 induces a V-generic filter $(G_0)^\beta \upharpoonright (\kappa^+ + 1)$ on $(\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1)$.

Generally, for a condition $p \in \mathbb{P}_0$ with $t(p) \leq t(\overline{r})$ such that all the branches of $t(p)$ have height $\geq \kappa^+$, we can define $\rho_0^\beta(p) \in (\mathbb{P}_0)^\beta \restriction (\kappa^+ + 1)$ as follows: Roughly, we take all predecessors of the points $\{(\kappa^+, i) \in t(p) \mid i < \beta\}$ and drop the indices below level κ^+. We start with defining $t := t\big(\rho_0^\beta(p)\big)$. Let $\operatorname{supp} t := \{(\kappa^+, \beta_l) \mid l < k\} := \{(\kappa^+, i) \in t(p) \mid i < \beta\}$. For any level $\alpha \leq \kappa^+$, the condition p induces an equivalence relation $\sim_\alpha$ on $\{\beta_0, \dots, \beta_{k-1}\}$ by setting $\beta_l \sim_\alpha \beta_{\overline{l}}$ iff (κ^+, β_l) and $(\kappa^+, \beta_{\overline{l}})$ have a common $t(p)$-predecessor on level α. We take for t the sequence $(B_\alpha \mid \alpha \in \mathrm{Card}, \alpha \leq \kappa^+)$ of partitions such that any B_α corresponds to the equivalence relation $\sim_\alpha$: For any $\beta_l, \beta_{\overline{l}}$, we have $\beta_l \sim_\alpha \beta_{\overline{l}}$ iff there exists $z \in B_\alpha$ with $\{\beta_l, \beta_{\overline{l}}\} \subseteq z$. Together with the order relation $\leq_t$ given by $(\alpha, z) \leq_t (\beta, z')$ iff $\alpha \leq \beta$ and $z \supseteq z'$, this defines $t \in T(\kappa^+, \beta)$. From $t(p) \leq t(\overline{r})$ it follows that $t \in T(\kappa^+, \beta)^{t(\overline{r})}$.

The tree t can be embedded into $t(p)$: Namely, a canonical map $\iota_0^\beta(p) : t \hookrightarrow t(p)$ can be defined as follows. For $(\alpha, z) \in t$, consider $\beta_l \in z$. Let (α, i) denote the $t(p)$-predecessor of (κ^+, β_l) on level α. Then $(\alpha, z) \in t$ corresponds to the vertex $(\alpha, i) \in t(p)$, and we set $\iota_0^\beta(p)(\alpha, z) := (\alpha, i)$. This map is well-defined and injective, with $(\alpha, z) \leq_t (\beta, z')$ if and only if $\iota_0^\beta(p)(\alpha, z) \leq_{t(p)} \iota_0^\beta(p)(\beta, z')$.
Hence, $(\iota_0^\beta(p)[t], \iota_0^\beta(p)[\leq_t]) \subseteq (t(p), \leq_{t(p)})$ is a subtree.

For $(\alpha, z) \in t = t\big(\rho_0^\beta(p)\big)$, we set $\big(\rho_0^\beta(p)\big)(\alpha, z) := p\big(\iota_0^\beta(p)(\alpha, z)\big)$.

It remains to define the indexing function $N := N\big(\rho_0^\beta(p)\big)$: For $(\alpha, z) \in t$ with $(\alpha, z) \leq_t (\kappa^+, \{i'_l\})$ for some $l < n$, let $N(\alpha, z) := (\alpha, i) := \iota_0^\beta(p)(\alpha, z)$. For all $(\lambda_l, z) \in t$, $< m$, with $\iota_0^\beta(p)(\lambda_l, z) = (\lambda_l, i)$, let $N(\lambda_l, z) := \iota_0^\beta(p)(\lambda_l, z) = (\lambda_l, i)$ in the case that $i < \alpha_l$, and $N(\lambda_l, z) := *$, else.

This defines the projection $\rho_0^\beta(p)$.

Whenever $(\alpha, z) \in t$ with $(\alpha, z) \leq_t (\kappa^+, \{i'_l\})$ for some $l < n$, then $N(\alpha, z) = (\alpha, i)$ is the $t(p)$-predecessor of (κ^+, i'_l) on level α. Since $t(p) \leq t(\overline{r})$ it follows that (α, i) is also the $t(\overline{r})$-predecessor of (κ^+, i'_l) on level α. Hence, $\rho_0^\beta(p)$ is indeed a condition in $\big((\widehat{\mathbb{P}}_0)^\beta \restriction (\kappa^+ + 1)\big)^{\overline{r}}_{(\lambda_0, \alpha_0), \dots} = (\mathbb{P}_0)^\beta \restriction (\kappa^+ + 1)$.

Let now $(\widetilde{\mathbb{P}}_0)^{\overline{r}}$ denote the collection of all $p \in \mathbb{P}_0$ with $t(p) \leq t(\overline{r})$ such that all branches of p have height at least κ^+, and the following additional property holds:

(1) For $l < m$, every $(\lambda_l, k) \in t(p)$ with $k < \alpha_l$ has a $t(p)$-successor (κ^+, i) with $i < \beta$.

Then $\big((\widetilde{\mathbb{P}}_0)^{\overline{r}}, (\widetilde{\leq}_0)^{\overline{r}}\big)$ is a forcing with the partial order $(\widetilde{\leq}_0)^{\overline{r}}$ induced by $\leq_0$ and maximal element $\mathbb{1} : t(\overline{r}) \to V$ with $\mathbb{1}(\alpha, i) = \varnothing$ for all $(\alpha, i) \in t(\overline{r})$.

Since $(\widetilde{\mathbb{P}}_0)^{\overline{r}}$ is dense in $\mathbb{P}_0$ below $\overline{r}$, it follows that $(\widetilde{G}_0)^{\overline{r}} := \{p \in (\widetilde{\mathbb{P}}_0)^{\overline{r}} \mid p \in G_0\}$ is a V-generic filter on $(\widetilde{\mathbb{P}}_0)^{\overline{r}}$.

Proposition 3.4.6. *The map $\rho_0^\beta : (\widetilde{\mathbb{P}}_0)^{\overline{r}} \to (\mathbb{P}_0)^\beta \restriction (\kappa^+ + 1)$, $p \mapsto \rho_0^\beta(p)$ is a projection of forcing posets. In particular,*

$$(G_0)^\beta \restriction (\kappa^+ + 1) := \rho_0^\beta\big[(\widetilde{G}_0)^{\overline{r}}\big] = \big\{\rho_0^\beta(p) \mid p \in (\widetilde{\mathbb{P}}_0)^{\overline{r}} \cap G_0\big\}$$

is a V-generic filter on $(\mathbb{P}_0)^\beta \restriction (\kappa^+ + 1)$.

The latter will be important, since we want to work with models of the form $V[(G_0)^\beta \upharpoonright (\kappa^+ + 1)]$ as intermediate generic extensions to capture parts of the map S^β.

Proof. It is not difficult to see that ρ_0^β is order-preserving and surjective with $\rho_0^\beta(\mathbb{1}) = \mathbb{1}$. In order to show that ρ_0^β is a projection of forcing posets, it remains to verify the following property: *For any* $p \in (\widetilde{\mathbb{P}}_0)^{\overline{r}}$ *and* $q \in (\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1)$ *with* $q \leq \rho_0^\beta(p)$*, there exists* $s \in (\widetilde{\mathbb{P}}_0)^{\overline{r}}$*,* $s \leq p$ *with* $\rho_0^\beta(s) \leq q$*.*

Then it follows that $(G_0)^\beta \upharpoonright (\kappa^+ + 1)$ hits any open dense set $D \subseteq (\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1)$.

Let $p \in (\widetilde{\mathbb{P}}_0)^{\overline{r}}$ and $q \leq \rho_0^\beta(p)$ as above. First, we construct a condition $\overline{q} \in (\widetilde{\mathbb{P}}_0)^{\overline{r}}$ compatible with p such that $\rho_0^\beta(\overline{q}) = q$. We do not change the tree structure of q, but give any vertex $(\alpha, z) \in t(q)$ an index $\overline{N}(q)(\alpha, z) = (\alpha, i)$, where $\overline{N}(q)$ should extend the following indexing functions $N_{\kappa^+}(q)$, $N'(q)$ and $N_p(q)$:

- $N_{\kappa^+}(q)$ maps any $(\kappa^+, \{i\}) \in t(q)$ to the number (κ^+, i),
- $N'(q)$ is the restriction of $N(q)$ to the set of all $(\lambda_l, z) \in t(q)$, $\lambda_l \leq \kappa$, with $N(q)(\lambda_l, z) \neq *$,
- $N_p(q)$ maps any $(\alpha, \overline{z}) \in t(q)$ which corresponds to a vertex $(\alpha, z) \in t = t\big(\rho_0^\beta(p)\big)$ to the number (α, i) that (α, z) inherits from $t(p)$.
  More precisely: Since $q \leq \rho_0^\beta(p)$, there is an embedding $\iota : (t, \leq_t) \hookrightarrow (t(q), \leq_{t(q)})$ such that for all $(\alpha, z) \in t$, it follows that $\iota(\alpha, z) = (\alpha, \overline{z})$ for some $\overline{z} \supseteq z$. For any $(\alpha, \overline{z}) \in \operatorname{im} \iota$ with $(\alpha, \overline{z}) = \iota(\alpha, z)$, let $N_p(q)(\alpha, \overline{z})$ be the number (α, i) of the $t(p)$-vertex corresponding to (α, z): With our canonical map $\iota_0^\beta(p) : t \hookrightarrow t(p)$ with $\iota^\beta(p)(\alpha, z) = (\alpha, i)$, set $N_p(q)(\alpha, \overline{z}) := (\alpha, i) = \iota^\beta(p)\big(\iota^{-1}(\alpha, \overline{z})\big)$.

It is not difficult to see that $N_{\kappa^+}(q) \cup N'(q) \cup N_p(q)$ is well-defined and injective.
Since $t(p) \leq t(\overline{r})$, it follows that for any $(\alpha, \overline{z}) \in t(q)$ with $(\alpha, \overline{z}) \leq_{t(q)} (\kappa^+, \{i'_l\})$ for some $l < n$, we have $N_p(q)(\alpha, \overline{z}) = (\alpha, i)$, where (α, i) is the $t(\overline{r})$-predecessor of (κ^+, i'_l) on level α.

It remains to define $\overline{N}(q)(\alpha, z)$ for those $(\alpha, z) \in t(q)$ remaining with $(\alpha, z) \notin \operatorname{dom}\big(N_{\kappa^+}(q) \cup N'(q) \cup N_p(q)\big)$.

For $\alpha < \kappa^+$, $\alpha \notin \{\lambda_l \mid l < m\}$, let

$$Z_\alpha := \big\{\, (\alpha, i) \mid i < F_{\lim}(\alpha)\,,\, (\alpha, i) \notin t(p) \,\cup\, \operatorname{im}\big(N_{\kappa^+}(q) \cup N'(q) \cup N_p(q)\big) \big\}.$$

For $l < m$ with $\lambda_l \leq \kappa$, let

$$Z_{\lambda_l} := \big\{\, (\lambda_l, i) \mid i \in [\alpha_l, F_{\lim}(\lambda_l))\,,\, (\lambda_l, i) \notin t(p) \,\cup\, \operatorname{im}\big(N_{\kappa^+}(q) \cup N'(q) \cup N_p(q)\big) \big\}.$$

We take for $\overline{N}(q) : t(q) \to V$ an injective function with $\overline{N}(q) \supseteq N_{\kappa^+}(q) \cup N'(q) \cup N_p(q)$ such that $\overline{N}(q)(\alpha, z) \in Z_\alpha$ for all $(\alpha, z) \in t(q) \setminus \operatorname{dom}\big(N_{\kappa^+}(q) \cup N'(q) \cup N_p(q)\big)$.
The condition $\overline{q} \in (\widetilde{\mathbb{P}}_0)^{\overline{r}}$ is defined as follows: $t(\overline{q}) := \{\overline{N}(q)(\alpha, z) \mid (\alpha, z) \in t(q)\}$, with $\leq_{t(\overline{q})} := \{\big(\overline{N}(q)(\alpha, z), \overline{N}(q)(\beta, z')\big) \mid (\alpha, z) \leq_{t(q)} (\beta, z')\}$.
For any $(\alpha, i) = \overline{N}(q)(\alpha, z) \in t(\overline{q})$, let $\overline{q}(\alpha, i) := q(\alpha, z)$.
This finishes the construction of $\overline{q}$.

By construction, it follows that $\rho_0^\beta(\overline{q}) = q$. Also, $\overline{q} \parallel p$: Firstly, for any $(\kappa^+, j) \in t(p)$ with $j < \beta$, it follows by construction of $N_p(q)$ that the $t(p)$-branch below (κ^+, j) coincides with the $t(\overline{q})$-branch below (κ^+, j).
On the other hand, the set of all $(\alpha, i) \in t(p)$ which have no successor (κ^+, j) with $j < \beta$ is disjoint from $t(\overline{q})$: The sets Z_α and Z_{λ_l} are disjoint from $t(p)$ by construction, so $\overline{N}(q)(\alpha, \overline{z}) = (\alpha, i) \in t(p)$ would imply $(\alpha, i) \in \operatorname{im}\big(N_{\kappa^+}(q) \cup N'(q) \cup N_p(q)\big)$. But any $(\alpha, i) \in \operatorname{im} N_{\kappa^+}(q) \cup \operatorname{im} N_p(q)$ clearly *has* a $t(p)$-successor (κ^+, j) with $j < \beta$, so the only possibility remaining is that $(\alpha, i) = (\lambda_l, i) = N'(q)(\lambda_l, z) = N(q)(\lambda_l, z)$ for some $l < m$ with $i < \alpha_l$. But then it follows from property (1) for $(\widetilde{\mathbb{P}}_0)^{\overline{r}}$ that again, (λ_l, i) has a $t(p)$-successor (κ^+, j) with $j < \beta$ – contradiction.
For any $(\alpha, i) = \overline{N}(q)(\alpha, \overline{z}) \in t(\overline{q}) \cap t(p)$, we have $(\alpha, i) = N_p(q)(\alpha, \overline{z})$, and with the embedding $\iota : (t, \leq_t) \hookrightarrow (t(q), \leq_{t(q)})$ as in the definition of $N_p(q)$ with $\iota(\alpha, z) = (\alpha, \overline{z})$, it follows from $q \leq \rho_0^\beta(p)$ that $\overline{q}(\alpha, i) = q(\alpha, \overline{z}) \supseteq \rho_0^\beta(p)(\alpha, z) = p(\alpha, i)$.
Hence, $\overline{q} \parallel p$.

Setting $s := p \cup \overline{q}$, it follows that $s \leq p$ with $s \in (\widetilde{\mathbb{P}}_0)^{\overline{r}}$ and $\rho_0^\beta(s) \leq \rho_0^\beta(\overline{q}) = q$.
Hence, the condition s has all the desired properties, and it follows that ρ_0^β is indeed a projection of forcing posets.

□

For capturing S^β, we will consider the product forcing

$$(\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1) \times \big(\mathbb{P}_0 \upharpoonright t(\overline{r})\big) \upharpoonright [\kappa^+, \infty).$$

Then also the map $\overline{\rho}_0^\beta : (\widetilde{\mathbb{P}}_0)^{\overline{r}} \to (\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1) \times \big(\mathbb{P}_0 \upharpoonright t(\overline{r})\big) \upharpoonright [\kappa^+, \infty)$, which maps a condition $p \in (\widetilde{\mathbb{P}}_0)^{\overline{r}}$ to $\big(\rho_0^\beta(p), (p \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty)\big)$ is a projection of forcing posets; hence, $(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times \big(G_0 \upharpoonright t(\overline{r})\big) \upharpoonright [\kappa^+, \infty)$ is a V-generic filter on $(\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1) \times \big(\mathbb{P}_0 \upharpoonright t(\overline{r})\big) \upharpoonright [\kappa^+, \infty)$.

Now, we turn to $(\mathbb{P}_1)^\beta \upharpoonright (\kappa + 1)$. As already mentioned, we take for any $\lambda^+ \in Succ' \cap \kappa$ at stage λ^+ the forcing $\operatorname{Fn}\big([\lambda, \lambda^+) \times \min\{\beta, F(\lambda^+)\}, 2, \lambda^+\big)$ instead of $\operatorname{Fn}\big([\lambda, \lambda^+) \times F(\lambda^+), 2, \lambda^+\big)$.
More precisely, $(\mathbb{P}_1)^\beta \upharpoonright (\kappa + 1)$ consists of all conditions $p : Succ' \cap (\kappa + 1) \to V$ with $\operatorname{supp} p := \{\lambda^+ < \kappa \mid p(\lambda^+) \neq \varnothing\}$ finite such that for all $\lambda^+ \in \operatorname{supp} p$,

$$p(\lambda^+) \in \operatorname{Fn}\big([\lambda, \lambda^+) \times \min\{F(\beta, \lambda^+)\}, 2, \lambda^+\big)$$

with $\operatorname{dom} p$ rectangular, i.e.

$$\operatorname{dom} p(\lambda^+) = \operatorname{dom}_x p(\lambda^+) \times \operatorname{dom}_y p(\lambda^+)$$

for some $\operatorname{dom}_x p(\lambda^+) \subseteq [\lambda, \lambda^+)$ and $\operatorname{dom}_y p(\lambda^+) \subseteq \min\{\beta, F(\lambda^+)\}$. The partial order "$\leq$" is reverse inclusion, and the maximal element $\mathbb{1}$ is the empty condition.

For $p \in \mathbb{P}_1$, we can define a projection $\rho_1^\beta(p)$ as follows: $\operatorname{supp} \rho_1^\beta(p) := \operatorname{supp} p \cap (\kappa + 1)$, and for any $\lambda^+ < \kappa$ with $\lambda^+ \in \operatorname{supp} p$,

$$\operatorname{dom}\big(\rho_1^\beta(p)\big)(\lambda^+) := \operatorname{dom}_x p(\lambda^+) \times (\operatorname{dom}_y p(\lambda^+) \cap \beta),$$

with $\big(\rho_1^\beta(p)\big)(\lambda^+)(\zeta,i) = p(\lambda^+)(\zeta,i)$ for all $(\zeta,i) \in \mathrm{dom}\big(\rho_1^\beta(p)\big)(\lambda^+)$.

It is not difficult to check that ρ_1^β is indeed a projection from $\mathbb{P}_1$ onto $(\mathbb{P}_1)^\beta \upharpoonright (\kappa+1)$. Hence,

$$(G_1)^\beta \upharpoonright (\kappa+1) := \{\rho_1^\beta(p) \mid p \in G_1\}$$

is a V-generic filter on $(\mathbb{P}_1)^\beta \upharpoonright (\kappa+1)$.

For capturing S^β, we will work with the forcing

$$\big((\mathbb{P}_1)^\beta \upharpoonright (\kappa+1)\big) \times \mathbb{P}_1(\kappa^+) \times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n},\, \overline{\kappa}_l > \kappa^+\}.$$

The map $\overline{\rho}_1^\beta : \mathbb{P}_1 \to \big((\mathbb{P}_1)^\beta \upharpoonright (\kappa+1)\big) \times \mathbb{P}_1(\kappa^+) \times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n},\, \overline{\kappa}_l > \kappa^+\}$ that maps a condition $p \in \mathbb{P}_1$ to $\big(\rho_1^\beta(p), p_1(\kappa^+), p_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}\big)$ is a projection of forcing posets, as well. Hence, it follows that

$$(G_1)^\beta \upharpoonright (\kappa+1) \times G_1(\kappa^+) \times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n},\, \overline{\kappa}_l > \kappa^+\}$$

is a $V[G_0]$-generic filter on $\big((\mathbb{P}_1)^\beta \upharpoonright (\kappa+1)\big) \times \mathbb{P}_1(\kappa^+) \times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n},\, \overline{\kappa}_l > \kappa^+\}$.
In particular,

$$V\big[(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times \big(G_0 \upharpoonright t(\overline{r})\big) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa+1) \times G_1(\kappa^+) \times$$

$$\times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n},\, \overline{\kappa}_l > \kappa^+\}\big]$$

is a well-defined generic extension by the forcing

$$(\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1) \times \big(\mathbb{P}_0 \upharpoonright t(\overline{r})\big) \upharpoonright [\kappa^+, \infty) \times (\mathbb{P}_1)^\beta \upharpoonright (\kappa+1) \times \mathbb{P}_1(\kappa^+) \times$$

$$\times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n},\, \overline{\kappa}_l > \kappa^+\}.$$

Lemma 3.4.7.

$$(\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1) \times \big(\mathbb{P}_0 \upharpoonright t(\overline{r})\big) \upharpoonright [\kappa^+, \infty) \times (\mathbb{P}_1)^\beta \upharpoonright (\kappa+1) \times \mathbb{P}_1(\kappa^+) \times$$

$$\times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n},\, \overline{\kappa}_l > \kappa^+\}$$

preserves cardinals $\geq F(\kappa)$.

Proof. First, it is not difficult to see that the forcing $(\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1)$ has cardinality $\leq |\beta| < F(\kappa)$ (one has to use that β is *large enough*, which implies that $\beta > \alpha_l$ for all $l < m$ with $\lambda_l \leq \kappa^+$).

Concerning $(\mathbb{P}_1)^\beta \upharpoonright (\kappa+1)$, we have several cases to distinguish: If $|\beta|^+ < F(\kappa)$, then $|(\mathbb{P}_1)^\beta \upharpoonright (\kappa+1)| \leq |\beta|^+ < F(\kappa)$. For the rest of the proof, assume $|\beta|^+ = F(\kappa)$.

- If the class $Succ'$ has no maximal element below κ, it follows that $F(\lambda^+) < |\beta|$ for all $\lambda^+ < \kappa$ with $\lambda^+ \in Succ'$, since $F(\lambda^+) < F(\mu^+)$ for all $\lambda^+, \mu^+ \in Succ'$ with $\lambda^+ < \mu^+$. Hence, all the blocks $Fn\big([\lambda, \lambda^+) \times F(\lambda^+), 2, \lambda^+\big)$ in $(\mathbb{P}_1)^\beta \upharpoonright (\kappa+1)$ have cardinality $\leq |\beta|$; so $|(\mathbb{P}_1)^\beta \upharpoonright (\kappa+1)| < F(\kappa)$.

It remains to consider the case that $Succ'$ has a maximal element μ^+ below κ.
Now, we have to treat the block $(\mathbb{P}_1)^\beta(\mu^+) = Fn\big([\mu,\mu^+)\times\min\{F(\mu^+),\beta\},2,\mu^+\big)$ separately and consider the forcing $(\mathbb{P}_1)^\beta\restriction(\mu+1)$.

- In the case that $F(\mu^+)\leq|\beta|$ or "$F(\mu^+)=F(\kappa)=|\beta|^+$ *and the class* $Succ'$ *has no maximal element below* μ^+", it follows that $|(\mathbb{P}_1)^\beta\restriction(\mu+1)|<F(\kappa)$ similarly as before.
- Finally, if $F(\mu^+)=F(\kappa)=|\beta|^+$ and $Succ'$ *has* a maximal element ν^+ below μ^+, we have to treat the product $(\mathbb{P}_1)^\beta(\nu^+)\times(\mathbb{P}_1)^\beta(\mu^+)$ separately. Since $F(\nu^+)\leq|\beta|$, it follows that $F(\lambda^+)<|\beta|$ for all $\lambda^+\in Succ'$ with $\lambda^+<\nu^+$; hence, $|(\mathbb{P}_1)^\beta\restriction(\nu+1)|\leq|\beta|<F(\kappa)$.

For the rest of the proof, we restrict to the latter case with $(\mathbb{P}_1)^\beta\restriction(\kappa+1)\cong((\mathbb{P}_1)^\beta\restriction(\nu+1))\times(\mathbb{P}_1)^\beta(\nu^+)\times(\mathbb{P}_1)^\beta(\mu^+)$ and $|(\mathbb{P}_1)^\beta\restriction(\nu+1)|<F(\kappa)$ – the other cases can be treated similarly.

Consider the product forcing

$$(\mathbb{P}_0)^\beta\restriction(\kappa^++1)\times\big(\mathbb{P}_0\restriction t(\overline{r})\big)\restriction[\kappa^+,\infty)\times(\mathbb{P}_1)^\beta\restriction(\kappa+1)\times\mathbb{P}_1(\kappa^+)\times$$
$$\times\,\mathbb{P}_1\restriction\{(\overline{\kappa}_l,\overline{\imath}_l)\mid l<\overline{n},\,\overline{\kappa}_l>\kappa^+\}.$$

Similarly as in Proposition 3.1.9, it follows that the "upper part"

$$\big(\mathbb{P}_0\restriction t(\overline{r})\big)\restriction[\kappa^+,\infty)\times\mathbb{P}_1\restriction\{(\overline{\kappa}_l,\overline{\imath}_l)\mid l<\overline{n},\,\overline{\kappa}_l>\kappa^+\}$$

preserves cardinals. Since this forcing is also $\leq\kappa^+$-closed, it follows that the "lower part", namely,

$$(\mathbb{P}_0)^\beta\restriction(\kappa^++1)\times(\mathbb{P}_1)^\beta\restriction(\kappa+1)\times\mathbb{P}_1(\kappa^+),$$

is the same forcing in a $\big(\mathbb{P}_0\restriction t(\overline{r})\big)\restriction[\kappa^+,\infty)\times\mathbb{P}_1\restriction\{(\overline{\kappa}_l,\overline{\imath}_l)\mid l<\overline{n},\,\overline{\kappa}_l>\kappa^+\}$-generic extension as it is in V.

Thus, it suffices to show that $(\mathbb{P}_0)^\beta\restriction(\kappa^++1)\times(\mathbb{P}_1)^\beta\restriction(\kappa+1)\times\mathbb{P}_1(\kappa^+)$ preserves cardinals $\geq F(\kappa)$. We factor

$$(\mathbb{P}_0)^\beta\restriction(\kappa^++1)\times(\mathbb{P}_1)^\beta\restriction(\kappa+1)\times\mathbb{P}_1(\kappa^+)\ \cong$$
$$\cong\Big(\,(\mathbb{P}_0)^\beta\restriction(\kappa^++1)\times(\mathbb{P}_1)^\beta\restriction(\nu+1)\Big)\times\Big((\mathbb{P}_1)^\beta(\nu^+)\times(\mathbb{P}_1)^\beta(\mu^+)\times\mathbb{P}_1(\kappa^+)\Big).$$

The product $(\mathbb{P}_1)^\beta(\nu^+)\times(\mathbb{P}_1)^\beta(\mu^+)\times\mathbb{P}_1(\kappa^+)$ preserves all cardinals. Secondly, as we have argued before, the forcing $(\mathbb{P}_0)^\beta\restriction(\kappa^++1)\times(\mathbb{P}_1)^\beta\restriction(\nu+1)$ has cardinality $<F(\kappa)$ (in V and hence, also in any $(\mathbb{P}_1)^\beta(\nu^+)\times(\mathbb{P}_1)^\beta(\mu^+)\times\mathbb{P}_1(\kappa^+)$-generic extension). Hence, the product forcing $(\mathbb{P}_0)^\beta\restriction(\kappa^++1)\times(\mathbb{P}_1)^\beta\restriction(\kappa+1)\times\mathbb{P}_1(\kappa^+)$ preserves cardinals $\geq F(\kappa)$, which finishes the proof. □

We want to show by an isomorphism argument that our surjection $S^\beta:\operatorname{dom}S^\beta\to F(\kappa)$ is contained in

$$V[(G_0)^\beta\restriction(\kappa^++1)\times\big(G_0\restriction t(\overline{r})\big)\restriction[\kappa^+,\infty)\times(G_1)^\beta\restriction(\kappa+1)\times G_1(\kappa^+)\times$$

$$\times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}].$$

Also, we will see that in $V[(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times \big(G_0 \upharpoonright t(\overline{r})\big) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa + 1) \times G_1(\kappa^+) \times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}]$, there is also an injection $\iota : \operatorname{dom} S^\beta \hookrightarrow \beta$. Together with Lemma 3.4.7 this gives the desired contradiction.

Recall that any X in the domain of S^β is of the form

$$X = \dot{X}^{G_0 \upharpoonright t(s) \times G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times G_1(\kappa^+)}$$

where s is a condition in $G_0 \upharpoonright (\kappa^+ + 1)$ and $(s, (\overline{\mu}_0, \overline{\jmath}_0), \dots, (\overline{\mu}_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in M_\beta$, i.e. s has finitely many maximal points $(\kappa^+, j_0), \dots, (\kappa^+, j_{k-1})$ with $j_0 < \beta, \dots, j_{k-1} < \beta$, and $\overline{k} < \omega$, $\mu_0, \dots, \mu_{\overline{k}-1} \in \kappa \cap Succ'$, $\overline{\jmath}_0 < \min\{F(\mu_0), \beta\}, \dots, \overline{\jmath}_{\overline{k}-1} < \min\{F(\mu_{\overline{k}-1}), \beta\}$. For any such s, let $\overline{s} = s \cup \overline{r}$.

Since do not want to use $G_0 \upharpoonright (\kappa^+ + 1)$ for capturing S^β, but only $(G_0)^\beta \upharpoonright (\kappa^+ + 1)$, we would like to replace the filter

$$G_0 \upharpoonright t(s) = \{p \upharpoonright t(s) \mid p \in G_0, t(p) \leq t(s)\},$$

by something like

$$\text{“}\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(s) := \big\{\, p \upharpoonright t(s) \;\big|\; \rho_0^\beta(p) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1),\, t\big(\rho_0^\beta(p)\big) \leq t\big(\rho_0^\beta(\overline{s})\big) \big\}\text{”}$$

but we have to specify what we mean by $p \upharpoonright t(s)$ if not necessarily $t(p) \leq t(s)$, but we only know that $t\big(\rho_0^\beta(p)\big) \leq t\big(\rho_0^\beta(\overline{s})\big)$, i.e. merely the tree structures of $t(p)$ and $t(s)$ agree below the vertices $(\kappa^+, j) \in t(s)$.

We will have $t\big(p \upharpoonright t(s)\big) := t(s)$. For a vertex $(\alpha, m) \in t(s)$ with $t(s)$-successor (κ^+, j), let (α, m') denote the $t(p)$-predecessor of (κ^+, j) on level α. We will set $(p \upharpoonright t(s))(\alpha, m) := p(\alpha, m')$. From $t\big(\rho_0^\beta(p)\big) \leq t\big(\rho_0^\beta(\overline{s})\big)$ it follows that this is well-defined: If $(\kappa^+, j), (\kappa^+, j')$ are both $t(s)$-successors of (α, m), then also in the tree $t(p)$, the vertices (κ^+, j) and (κ^+, j') have a common predecessor (α, m') on level α.
In other words: The condition $p \upharpoonright t(s)$ is constructed from $p \upharpoonright \{(\kappa^+, j) \mid (\kappa^+, j) \in t(s)\}$ by exchanging any index (α, m') such that $(\alpha, m') \leq_{t(p)} (\kappa^+, j)$, with (α, m) such that $(\alpha, m) \leq_{t(s)} (\kappa^+, j)$.

Definition/Lemma 3.4.8. Let q denote a condition in $\mathbb{P}_0 \upharpoonright (\kappa^+ + 1)$ with maximal points $(\kappa^+, j_0), \dots, (\kappa^+, j_{\overline{k}-1})$ such that $j_0, \dots, j_{\overline{k}-1} < \beta$, and $q \parallel \overline{r}$. With $\overline{q} = q \cup \overline{r}$, assume $\rho_0^\beta(\overline{q}) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1)$. We define

$$\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(q)$$

as the set of all $p \upharpoonright t(q)$ with $p \in \mathbb{P}_0 \upharpoonright (\kappa^+ + 1)$ such that

$$\rho_0^\beta(p) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1)$$

and

$$t\big(\rho_0^\beta(p)\big) \leq t\big(\rho_0^\beta(\overline{q})\big),$$

with $p \upharpoonright t(q)$ as defined before.
This is a V-generic filter on $\mathbb{P}_0 \upharpoonright t(q)$, with $\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(q) = G_0 \upharpoonright t(q)$ in the case that $q \in G$.

Proof. More generally, for conditions $q_0, q_1 \in \mathbb{P}_0$ with maximal points in $\{(\kappa^+, j) \mid j < \beta\}$ and $q_0 \parallel \overline{r}$, $q_1 \parallel \overline{r}$, let $\overline{q}_0 = q_0 \cup \overline{r}$ and $\overline{q}_1 = q_1 \cup \overline{r}$ as before. If

$$t\big(\rho_0^\beta(\overline{q}_0)\big) = t\big(\rho_0^\beta(\overline{q}_1)\big),$$

there is the following canonical isomorphism $T(q_0, q_1) : \mathbb{P}_0 \upharpoonright t(q_0) \to \mathbb{P}_0 \upharpoonright t(q_1)$: For a condition $p \in \mathbb{P}_0 \upharpoonright t(q_0)$ and some vertex $(\alpha, m) \in t(q_1)$, consider a $t(q_1)$-successor (κ^+, j). Let (α, m') denote the according $t(q_0)$-predecessor of (κ^+, j) on level α. Set $\big(T(q_0, q_1)(p)\big)(\alpha, m) := p(\alpha, m')$. As argued before, it follows from $t\big(\rho_0^\beta(\overline{q}_0)\big) = t\big(\rho_0^\beta(\overline{q}_1)\big)$ that this is well-defined.
This isomorphism $T(q_0, q_1)$ extends to an isomorphism $\overline{T}(q_0, q_1) : \mathrm{Name}(\mathbb{P}_0 \upharpoonright t(q_0)) \to \mathrm{Name}(\mathbb{P}_0 \upharpoonright t(q_1))$ on the name space as usual: For $\dot{Y} \in \mathrm{Name}(\mathbb{P}_0 \upharpoonright t(q_0))$, define recursively:

$$\overline{T}(q_0, q_1)(\dot{Y}) := \big\{ \big(\overline{T}(q_0, q_1)(\dot{Z}), T(q_0, q_1)(p) \big) \mid (\dot{Z}, p) \in \dot{Y} \big\}.$$

In the case that $t\big(\rho_0^\beta(\overline{q}_0)\big) = t\big(\rho_0^\beta(\overline{q}_1)\big)$ agrees with the generic filter $(G_0)^\beta \upharpoonright (\kappa^+ + 1)$, it is not difficult to check that

$$\dot{Y}^{\left((G_0)^\beta \upharpoonright (\kappa^+ + 1)\right) \upharpoonright t(q_0)} = \big(\overline{T}(q_0, q_1)\dot{Y}\big)^{\left((G_0)^\beta \upharpoonright (\kappa^+ + 1)\right) \upharpoonright t(q_1)}.$$

Hence, using canonical names for the generic filter, it follows that

$$\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(q_1) = T(q_0, q_1)\big[\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(q_0)\big].$$

Now, let $q \in \mathbb{P}_0 \upharpoonright (\kappa^+ + 1)$ as in the statement of this lemma, with maximal points $(\kappa^+, j_0), \ldots, (\kappa^+, j_{\overline{k}-1})$ with $j_0, \ldots, j_{\overline{k}-1} < \beta$ such that $q \parallel r$, and $\rho_0^\beta(\overline{q}) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1)$ for $\overline{q} := q \cup r$.
Let $s \in G_0$ with the same maximal points $(\kappa^+, j_0), \ldots, (\kappa^+, j_{\overline{k}-1})$ and $\rho_0^\beta(\overline{s}) = \rho_0^\beta(\overline{q})$, where $\overline{s} := s \cup \overline{r}$ as before. Since $\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(s) = G_0 \upharpoonright t(s)$ is a V-generic filter on $\mathbb{P}_0 \upharpoonright t(s)$ and $T(s, q) : \mathbb{P}_0 \upharpoonright t(s) \to \mathbb{P}_0 \upharpoonright t(q)$ is an isomorphism of forcings, it follows from

$$\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(q) = T(s, q)[G_0 \upharpoonright t(s)]$$

that $\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(q)$ is a V-generic filter on $\mathbb{P}_0 \upharpoonright t(q)$ as desired. □

Now, we turn to $\mathbb{P}_1$: For finitely many $(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})$ with $\mu_0, \ldots, \mu_{\overline{k}-1} < \kappa$, $\overline{\jmath}_0 < \min\{F(\mu_0), \beta\}$, $\ldots$, $\overline{\jmath}_{\overline{k}-1} < \min\{F(\mu_{\overline{k}-1}), \beta\}$, let

$$\big((G_1)^\beta \upharpoonright (\kappa + 1)\big) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}$$

denote the collection of all $p_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}$ with $p_1 \in \mathbb{P}_1 \upharpoonright (\kappa + 1)$ such that

$$(p_1)^\beta \upharpoonright (\kappa + 1) \in (G_1)^\beta \upharpoonright (\kappa + 1).$$

Then $\big((G_1)^\beta \upharpoonright (\kappa + 1)\big) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} = G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}$.
Thus, for any $X \in \operatorname{dom} S^\beta$, $X = \dot{X}^{G_0 \upharpoonright t(s) \times G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times G_1(\kappa^+)}$, it follows that

$$X = \dot{X}^{\left((G_0)^\beta \upharpoonright (\kappa^+ + 1)\right) \upharpoonright t(s) \times \left((G_1)^\beta \upharpoonright (\kappa + 1)\right) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times G_1(\kappa^+)}.$$

This will help us prove the following proposition:

Proposition 3.4.9. *The restriction S^β is contained in*

$$V\big[\,(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times (G_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa + 1) \times G_1(\kappa^+) \times$$

$$\times\, G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}\,\big].$$

Proof. As in the proof of Proposition 3.4.2, fix a cardinal λ with $\lambda > \max\{\kappa^+, \kappa_0, \ldots, \kappa_{n-1}, \lambda_0, \ldots, \lambda_{m-1}, \overline{\kappa}_0, \ldots, \overline{\kappa}_{\overline{n}-1}, \overline{\lambda}_0, \ldots, \overline{\lambda}_{\overline{m}-1}\}$ such that $\dot{S} \in \mathrm{Name}(\mathbb{P} \upharpoonright (\lambda + 1))$. Then also $\dot{S}^\beta \in \mathrm{Name}(\mathbb{P} \upharpoonright (\lambda + 1))$.
Let $(S^\beta)'$ denote the collection of all

$$\Big(\,\dot{X}^{\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(q) \,\times\, \big((G_1)^\beta \upharpoonright (\kappa + 1)\big) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \,\times\, G_1(\kappa^+)}, \alpha\Big)$$

such that

(i) q is a condition in $\mathbb{P}_0 \upharpoonright (\kappa^+ + 1)$ where $t(q)$ has maximal points $(\kappa^+, j_0), \ldots, (\kappa^+, j_{k-1})$ with $j_0, \ldots, j_{k-1} < \beta$; moreover, $q \parallel \overline{r}$, and for $\overline{q} := q \cup \overline{r}$, it follows that $\rho_0^\beta(\overline{q}) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1)$,

(ii) $\overline{k} < \omega$, $\mu_0, \ldots, \mu_{\overline{k}-1} \in \kappa \cap Succ'$ and $\overline{\jmath}_0 < \min\{F(\mu_0), \beta\}, \ldots, \overline{\jmath}_{\overline{k}-1} < \min\{F(\mu_{\overline{k}-1}), \beta\}$,

(iii) $\dot{X}$ is a name for the forcing $\mathbb{P}_0 \upharpoonright t(q) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+)$,

(iv) there is a condition $p \in \mathbb{P} \upharpoonright (\lambda + 1)$ with $p_0 \in (\widetilde{\mathbb{P}}_0)^{\overline{r}}$, $p_0 \leq \overline{q}$ and

- $\rho_0^\beta(p) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1)$
- $(p_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \in (G_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty)$
- $(p_1)^\beta \upharpoonright (\kappa + 1) \in (G_1)^\beta \upharpoonright (\kappa + 1)$
- $p_1(\kappa^+) \in G_1(\kappa^+)$
- $p_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\} \in G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}$

such that $p \Vdash_{\mathbb{P} \upharpoonright (\lambda+1)} (\widetilde{\dot{X}}, \alpha) \in \dot{S}$.

It suffices to show that $S^\beta = (S^\beta)'$.

"$\supseteq$": For $(X, \alpha) \in S^\beta$, we have $X = \dot{X}^{G_0 \upharpoonright t(s) \times G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times G_1(\kappa^+)}$ for some $(s, (\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in M_\beta$ with $s \in G_0 \upharpoonright (\kappa^+ + 1)$, where $\dot{X}$ is a name for the forcing $\mathbb{P}_0 \upharpoonright t(s) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots\} \times \mathbb{P}_1(\kappa^+)$.
Then $(X, \alpha) = (\widetilde{\dot{X}}^{G \upharpoonright (\lambda+1)}, \alpha) \in \dot{S}^{G \upharpoonright (\lambda+1)}$, so there must be $p \in G \upharpoonright (\lambda+1)$, $p_0 \leq \overline{s} := s \cup r$, with $p \Vdash_{\mathbb{P} \upharpoonright (\lambda+1)} (\widetilde{\dot{X}}, \alpha) \in \dot{S}$.
Setting $q := s$, it follows that

$$(X, \alpha) = \Big(\,\dot{X}^{\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(q) \,\times\, \big((G_1)^\beta \upharpoonright (\kappa + 1)\big) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \,\times\, G_1(\kappa^+)}, \alpha\Big)$$

is contained in $(S^\beta)'$ as desired.

"$\subseteq$": Assume towards a contradiction, there was $(X,\alpha)\in (S^\beta)'\setminus S^\beta$. Let

$$X=\dot X^{\left((G_0)^\beta\restriction(\kappa^++1)\right)\restriction t(q)\times\left((G_1)^\beta\restriction(\kappa+1)\right)\restriction\{(\mu_0,\overline{\jmath}_0),\dots,(\mu_{\overline{k}-1},\overline{\jmath}_{\overline{k}-1})\}\times G_1(\kappa^+)}$$

as in the definition of $(S^\beta)'$ with $p\in\mathbb{P}\restriction(\lambda+1)$ as in (iv) such that

$$p\Vdash_{\mathbb{P}\restriction(\lambda+1)}(\widetilde{\dot X},\alpha)\in\dot S\qquad\qquad(\times).$$

Since $\rho_0^\beta(\overline{q})\in(G_0)^\beta\restriction(\kappa^++1)$, we can take a condition $\overline{s}\in G_0\restriction(\lambda+1)$, $\overline{s}\in(\widetilde{\mathbb{P}}_0)^{\overline{r}}$ with $\overline{s}\le\overline{r}$ and $\rho_0^\beta(\overline{s})=\rho_0^\beta(\overline{q})$. W.l.o.g. we can assume that $\overline{s}=s\cup\overline{r}$ for some $s\in G_0\restriction(\kappa^++1)$ which has the same maximal points $(\kappa^+,j_0),\dots,(\kappa^+,j_{k-1})$ as q. The isomorphism $T(q,s):\mathbb{P}_0\restriction q\to\mathbb{P}_0\restriction s$ from the proof of Definition / Lemma 3.4.8 can be extended to an isomorphism from $\mathbb{P}_0\restriction t(q)\times\mathbb{P}_1\restriction\{(\mu_0,\overline{\jmath}_0),\dots,(\mu_{\overline{k}-1},\overline{\jmath}_{\overline{k}-1})\}\times\mathbb{P}_1(\kappa^+)$ onto $\mathbb{P}_0\restriction t(s)\times\mathbb{P}_1\restriction\{(\mu_0,\overline{\jmath}_0),\dots,(\mu_{\overline{k}-1},\overline{\jmath}_{\overline{k}-1})\}\times\mathbb{P}_1(\kappa^+)$ that is the identity on the second and third coordinate. We will denote this extension by $T(q,s)$ as well, and consider the according isomorphism on the name space $\overline{T}(q,s):\mathrm{Name}\big(\mathbb{P}_0\restriction t(q)\times\mathbb{P}_1\restriction\{(\mu_0,\overline{\jmath}_0),\dots,(\mu_{\overline{k}-1},\overline{\jmath}_{\overline{k}-1})\}\times\mathbb{P}_1(\kappa^+)\big)\to\mathrm{Name}\big(\mathbb{P}_0\restriction t(s)\times\mathbb{P}_1\restriction\{(\mu_0,\overline{\jmath}_0),\dots,(\mu_{\overline{k}-1},\overline{\jmath}_{\overline{k}-1})\}\times\mathbb{P}_1(\kappa^+)\big)$.

Let $\ddot X:=\overline{T}(q,s)\dot X$. Then

$$X=\dot X^{\left((G_0)^\beta\restriction(\kappa^++1)\right)\restriction t(q)\times\left((G_1)^\beta\restriction(\kappa+1)\right)\restriction\{(\mu_0,\overline{\jmath}_0),\dots,(\mu_{\overline{k}-1},\overline{\jmath}_{\overline{k}-1})\}\times G_1(\kappa^+)}=$$
$$=\ddot X^{\left((G_0)^\beta\restriction(\kappa^++1)\right)\restriction t(s)\times\left((G_1)^\beta\restriction(\kappa+1)\right)\restriction\{(\mu_0,\overline{\jmath}_0),\dots,(\mu_{\overline{k}-1},\overline{\jmath}_{\overline{k}-1})\}\times G_1(\kappa^+)}=$$
$$=\ddot X^{G_0\restriction t(s)\times G_1\restriction\{(\mu_0,\overline{\jmath}_0),\dots,(\mu_{\overline{k}-1},\overline{\jmath}_{\overline{k}-1})\times G_1(\kappa^+)}=\widetilde{\ddot X}^G,$$

where as before, $\widetilde{\ddot X}$ denotes the canonical extension of $\ddot X$ to a $\mathbb{P}$-name.

Since $(X,\alpha)\notin S$, there exists $p'\in G\restriction(\lambda+1)$, $p'_0\in(\widetilde{\mathbb{P}}_0)^{\overline{r}}$, with

$$p'\Vdash_{\mathbb{P}\restriction(\lambda+1)}(\widetilde{\ddot X},\alpha)\notin\dot S\qquad\qquad(\times\times).$$

W.l.o.g. we can take $p'_0\le\overline{s}$, and assure by a density argument, that $\rho_0^\beta(p'_0)\le\rho_0^\beta(p_0)$.

We want to construct an isomorphism $\pi:\mathbb{P}\to\mathbb{P}$ with the following properties:

- $\pi p\parallel p'$
- $\pi\widetilde{\dot X}^{D_\pi}=\widetilde{\ddot X}^{D_\pi}$
- $\pi\overline{S}^{D_\pi}=\overline{S}^{D_\pi}$.

Together with $(\times)$ and $(\times\times)$, this gives the desired contradiction.

The third condition is satisfied if we make sure that π is contained in the intersection $Fix_0(\kappa_0,i_0)\cap\dots\cap Small_0(\lambda_0,[0,\alpha_0))\cap\dots\cap Fix_1(\overline{\kappa}_0,\overline{\imath}_0)\cap\dots\cap Small_1(\overline{\lambda}_0,[0,\overline{\alpha}_0))\cap\dots$.

We start with the construction of π_0. From $\rho_0^\beta(p_0') \leq \rho_0^\beta(p_0)$, it follows that the tree structures of $t(p_0)$ and $t(p_0')$ coincide below the vertices $(\kappa^+, i) \in t(p_0)$ with $i < \beta$. Hence, we can achieve $\pi_0 p_0 \parallel p_0'$ by changing any index (α, m) with $(\alpha, m) \leq_{t(p_0)} (\kappa^+, i)$ for some $i < \beta$, to (α, m'), where $(\alpha, m') \leq_{t(p_0')} (\kappa^+, i)$, i.e. (α, m') is the corresponding index in the tree structure of $t(p_0')$; and outside the branches below $\{(\kappa^+, i) \in t(p_0) \mid i < \beta\}$, we make $t(\pi_0 p_0)$ and $t(p_0')$ disjoint.

Let $\operatorname{ht} \pi_0 := \lambda + 1$. For a cardinal $\alpha < \operatorname{ht} \pi_0$ with $\alpha \notin \{\lambda_0, \ldots, \lambda_{m-1}\}$, take for $\pi_0(\alpha)$ a bijection on $\{(\alpha, j) \mid j < F_{\lim}(\alpha)\}$ with finite support such that the following hold:

- If $(\alpha, j) \in t(\overline{r})$, then $\pi_0(\alpha)(\alpha, j) := (\alpha, j)$.
- If (α, j) has a $t(p_0)$-successor (κ^+, i) with $i < \beta$, it follows from $\rho_0^\beta(p_0') \leq \rho_0^\beta(p_0)$ that also $(\kappa^+, i) \in t(p_0')$. Let $\pi_0(\alpha)(\alpha, j) := (\alpha, j')$ be the $t(p_0')$-predecessor of (κ^+, i) on level α.
- For all the $(\alpha, j) \in t(p_0)$ remaining, $j \in [\gamma(j), \gamma(j) + \omega)$ for $\gamma(j)$ a limit ordinal, let $\pi_0(\alpha)(\alpha, j) = (\alpha, j')$ for some $j' \in [\gamma(j), \gamma(j) + \omega)$ with $(\alpha, j') \notin t(p_0) \cup t(p_0')$.

This is well-defined: If (α, j) has two $t(p_0)$-successors (κ^+, i) and (κ^+, i') with $i, i' < \beta$, then it follows from $\rho_0^\beta(p_0') \leq \rho_0^\beta(p_0)$ that (κ^+, i) and (κ^+, i') also have the same $t(p_0')$-predecessor on level α. Also, if $(\alpha, j) \in t(\overline{r})$ has a $t(p_0)$-successor (κ^+, i) with $i < \beta$, it follows that in $t(p_0')$, the vertex (κ^+, i) has predecessor (α, j) as well, since $t(p_0)$ and $t(p_0')$ both extend $t(\overline{r})$. Thus, $\pi_0(\alpha)(\alpha, j) = (\alpha, j)$.

In the case that $\alpha = \lambda_l$ for some $l < m$, we have to be careful, since we want $\pi \in Small_0(\lambda_l, [0, \alpha_l))$. Thus, for any interval $[\gamma, \gamma + \omega) \subseteq \alpha_l$ with γ a limit ordinal and $j \in [\gamma, \gamma + \omega)$, we have to make sure that $\pi_0(\lambda_l)(\lambda_l, j) = (\lambda_l, j')$ such that also $j' \in [\gamma, \gamma + \omega)$:
Consider $(\lambda_l, j) \in t(p_0)$ with $t(p_0)$-successor (κ^+, i) for some $i < \beta$. Let $(\lambda_l, z) \in t\big(\rho_0^\beta(p_0)\big)$ with $i \in z$, and $(\lambda_l, \overline{z}) \in t\big(\rho_0^\beta(p_0')\big)$ with $i \in \overline{z}$.
Since $\rho_0^\beta(p_0') \leq \rho_0^\beta(p_0)$, it follows that $\overline{z} \supseteq z$, and in the case that $j < \alpha_l$, we have $N\big(\rho_0^\beta(p_0')\big)(\lambda_l, \overline{z}) = N\big(\rho_0^\beta(p_0)\big)(\lambda_l, z) = (\lambda_l, j)$. Hence, (λ_l, j) is also the $t(p_0')$-predecessor of (κ^+, i) on level λ_l, which gives $\pi_0(\lambda_l)(\lambda_l, j) = (\lambda_l, j)$.
In the case that $j \geq \alpha_l$, it follows from

$$N\big(\rho_0^\beta(p')\big)\,(\lambda_l, \overline{z}) = N\big(\rho_0^\beta(p)\big)(\lambda_l, z) = *$$

that for (λ_l, j') denoting the $t(p_0')$-predecessor of (κ^+, i) on level λ_l, i.e. $\pi_0(\lambda_l)(\lambda_l, j) = (\lambda_l, j')$, we have $j' \geq \alpha_l$, as well.

Thus, we can make sure that for any $l < m$, the following additional property holds for $\pi_0(\lambda_l)$:

- For any (λ_l, j) with γ a limit ordinal such that $j \in [\gamma(j), \gamma(j) + \omega) \subseteq \alpha_l$, we have $\pi_0(\lambda_l)(\lambda_l, j) = (\lambda_l, j')$ such that j' is contained in the interval $[\gamma(j), \gamma(j) + \omega)$, as well.

Then $\pi_0 \in Small_0(\lambda_0, [0, \alpha_0)) \cap \cdots \cap Small_0(\lambda_{m-1}, [0, \alpha_{m-1}))$, and $\pi_0 \in Fix_0(\kappa_0, i_0) \cap \cdots \cap Fix_0(\kappa_{n-1}, i_{n-1})$, since $(\kappa_l, i_l) \in t(\overline{r})$ for all $l < n$.

We now have to verify that $\pi_0 p_0 \parallel p_0'$. Firstly, on the tree $t(\overline{r})$, the conditions p_0 and p_0' coincide, and π_0 is the identity. Secondly, from $\rho_0^\beta(p_0') \leq \rho_0^\beta(p_0)$ and by construction of the map π_0, it follows that $\pi_0 p_0$ and p_0' agree on the branches below $\{(\kappa^+, i) \in t(\pi_0 p_0) \mid i < \beta\}$. All the remaining $t(\pi_0 p_0)$- and $t(p_0')$-branches are disjoint, i.e. whenever $(\alpha, j) \in t(p_0') \setminus t(\overline{r})$, and (α, j) has no $t(p_0')$-successor (κ^+, i) with $i < \beta$, then $(\alpha, j) \notin t(\pi_0 p_0)$. Hence, $\pi_0 p_0 \parallel p_0'$.

The map π_1 with $\pi_1 p_1 \parallel p_1'$ can be constructed as in Proposition 3.3.2, and since $p_1' \in G_1 \upharpoonright (\lambda + 1)$ and p satisfies (iv), it follows that $\pi_1 \in Fix_1(\overline{\kappa}_0, \overline{\imath}_0) \cap \cdots \cap Fix_1(\overline{\kappa}_{\overline{n}-1}, \overline{\imath}_{\overline{n}-1}) \cap Small_1(\overline{\lambda_0}, [0, \overline{\alpha}_0)) \cap \cdots \cap Small_1(\overline{\lambda}_{\overline{m}-1}, [0, \overline{\alpha}_{\overline{m}-1}))$ as desired.

It remains to check that $\pi \widetilde{\dot{X}}^{D_\pi} = \widetilde{\ddot{X}}^{D_\pi}$, where $\ddot{X} := \overline{T}(q, s)\dot{X}$.
Firstly, π_1 is the identity on $\mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}$, since $\mu_l < \kappa$, $\overline{\jmath}_l < \beta$ for all $l < k$; so from $(p_1)^\beta \upharpoonright (\kappa + 1) \in (G_1)^\beta \upharpoonright (\kappa^+ + 1)$, $p_1' \in G_1$, it follows that p_1 and p_1' coincide on $\mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}$. Similarly, π_1 is the identity on $\mathbb{P}_1(\kappa^+)$.
Now, consider π_0. Recall that any $(\alpha, j) \in t(p_0)$ with $(\alpha, j) \leq_{t(p_0)} (\kappa^+, i)$ for some $i < \beta$ is mapped to (α, j') such that (α, j') is the $t(p_0')$-predecessor of (κ^+, i) on level α. Since $p_0 \leq \overline{q} = q \cup \overline{r}$, $p_0' \leq \overline{s} = s \cup \overline{r}$ with $\rho_0^\beta(\overline{s}) = \rho_0^\beta(\overline{q})$, it follows that any $(\alpha, j) \in t(q)$ with $(\alpha, j) \leq_{t(q)} (\kappa^+, i)$ for some $i < \beta$ is mapped to the corresponding $t(s)$-predecessor of (κ^+, i) on level α: $\pi_0(\alpha)(\alpha, j) = (\alpha, j')$ with $(\alpha, j') \leq_{t(s)} (\kappa^+, i)$.
Hence, it follows for any condition $\widetilde{q} \in \mathbb{P}_0 \upharpoonright t(q)$ that $\pi_0 \widetilde{q} = T(q, s)(\widetilde{q}) \in \mathbb{P}_0 \upharpoonright t(s)$.
Inductively, this implies $\pi \widetilde{\dot{x}}^{D_\pi} = \widetilde{\ddot{x}}^{D_\pi}$ whenever $\dot{x}$ is a name for $\mathbb{P}_0 \upharpoonright t(q) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+)$ and $\ddot{x} := \overline{T}(q, s)\dot{x}$.

In particular, $\pi \widetilde{\dot{X}}^{D_\pi} = \widetilde{\ddot{X}}^{D_\pi}$, which finishes the proof.

□

Thus, we have shown that the surjection $S^\beta : \operatorname{dom} S^\beta \to F(\kappa)$ is contained in $V[(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times (G_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa + 1) \times G_1(\kappa^+) \times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}]$. We will now see that in this model, there is also an injection $\iota^\beta : \operatorname{dom} S^\beta \hookrightarrow \beta$. Together with Lemma 3.4.7 this gives the desired contradiction.

Proposition 3.4.10. *In* $V[(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times (G_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa + 1) \times G_1(\kappa^+) \times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}]$*, there is an injection* $\iota^\beta : \operatorname{dom} S^\beta \to \beta$.

Proof. We work inside $V[(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times (G_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa + 1) \times G_1(\kappa^+) \times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}] \vDash \mathsf{ZFC}$.

Let $\widetilde{M_\beta}$ denote the collection of all tuples $(q, (\mu_0, j_0), \ldots, (\mu_{\overline{k}-1}, j_{\overline{k}-1})) \in M_\beta$ with the property that $q \parallel \overline{r}$, and for $\overline{q} = q \cup \overline{r}$ as before, $\rho_0^\beta(\overline{q}) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1)$.

Fix some $(q, (\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in \widetilde{M_\beta}$. Then $\big((G_0)^\beta \upharpoonright (\kappa^+ + 1)\big) \upharpoonright t(q) \times \big((G_1)^\beta \upharpoonright (\kappa + 1)\big) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times G_1(\kappa^+)$ is a V-generic filter on $\mathbb{P}_0 \upharpoonright t(q) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \ldots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+)$.

From Proposition 3.1.9 we know that the forcing $\mathbb{P}_0 \upharpoonright t(q) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots\}$ preserves cardinals and the GCH. By the same proof, one can show that $\mathbb{P}_0 \upharpoonright t(q) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots\} \times \mathbb{P}_1(\kappa^+)$ preserves cardinals and the GCH below κ^+ (since $\mathbb{P}_1(\kappa^+)$ is $\leq \kappa$-closed):
For every $\alpha \leq \kappa$,

$$\left(2^\alpha\right)^{V[((G_0)^\beta \upharpoonright (\kappa^+ + 1)) \upharpoonright t(q) \times ((G_1)^\beta \upharpoonright (\kappa+1)) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots\} \times G_1(\kappa^+)]} = (\alpha^+)^V.$$

Hence, in $V[\left((G_0)^\beta \upharpoonright (\kappa^+ + 1)\right) \upharpoonright t(q) \times \left((G_1)^\beta \upharpoonright (\kappa + 1)\right) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots\} \times G_1(\kappa^+)]$, there is an injection $\iota : \wp(\kappa) \hookrightarrow (\kappa^+)^V$.

Now, we can use AC (in $V[(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times (G_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa + 1) \times G_1(\kappa^+) \times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}]$) to obtain a collection of injections $\iota_{(q, (\mu_0, \overline{\jmath}_0), \dots, (\mu_{k-1}, \overline{\jmath}_{\overline{k}-1}))} : \wp(\kappa) \cap V[\left((G_0)^\beta \upharpoonright (\kappa^+ + 1)\right) \upharpoonright t(q) \times \left((G_1)^\beta \upharpoonright (\kappa + 1)\right) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots\} \times G_1(\kappa^+)] \hookrightarrow (\kappa^+)^V$ for $(q, (\mu_0, \overline{\jmath}_0), \dots, (\mu_{k-1}, \overline{\jmath}_{k-1})) \in \widetilde{M}_\beta$.

Let $\widetilde{\widetilde{M}}_\beta$ denote the set of all tuples $((\kappa^+, j_0), \dots, (\kappa^+, j_{k-1}), (\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1}))$ with $k, \overline{k} < \omega$ and $j_0, \dots, j_{k-1} < \beta$, $\mu_0, \dots, \mu_{\overline{k}-1} \in \kappa \cap Succ'$, $\overline{\jmath}_0 < \min\{F(\mu_0), \beta\}, \dots, \overline{\jmath}_{\overline{k}-1} < \min\{F(\mu_{\overline{k}-1}), \beta\}$.

Let τ denote an injection that maps any tuple $((\kappa^+, j_0), \dots, (\kappa^+, j_{k-1}))$ with $j_0, \dots, j_{k-1} < \beta$ as above to some condition $q \in \mathbb{P}_0$ such that $t(q)$ has maximal points $(\kappa^+, j_0), \dots, (\kappa^+, j_{k-1})$, $q \parallel \overline{r}$, and for $\overline{q} := q \cup \overline{r}$ as before, $\rho_0^\beta(\overline{q}) \in (G_0)^\beta \upharpoonright (\kappa^+ + 1)$.

For any $((\kappa^+, j_0), \dots, (\kappa^+, j_{k-1}), (\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in \widetilde{\widetilde{M}}_\beta$, let

$$\iota_{((\kappa^+, j_0), \dots, (\mu_0, \overline{\jmath}_0), \dots)} := \iota_{(q, (\mu_0, \overline{\jmath}_0), \dots)},$$

where $q := \tau((\kappa^+, j_0), \dots, (\kappa^+, j_{k-1}))$.

Any $X \in \operatorname{dom} S^\beta$ is of the form

$$X = \dot{X}^{\left((G_0)^\beta \upharpoonright (\kappa^+ + 1)\right) \upharpoonright t(q) \times \left((G_1)^\beta \upharpoonright (\kappa+1)\right) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots\} \times G_1(\kappa^+)}$$

for some $\dot{X} \in \mathrm{Name}\left(\mathbb{P}_0 \upharpoonright t(q) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}(\kappa^+)\right)$ with $(q, (\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in \widetilde{M}_\beta$. Denote by $(\kappa^+, j_0), \dots, (\kappa^+, j_{k-1})$ the maximal points of $t(q)$ with $\tau((\kappa^+, j_0), \dots, (\kappa^+, j_{k-1})) =: q'$. Then $\rho_0^\beta(\overline{q}), \rho_0^\beta(\overline{q}') \in G_0^\beta \upharpoonright (\kappa^+ + 1)$ with the same maximal points; hence, $\rho_0^\beta(\overline{q}) = \rho_0^\beta(\overline{q}')$. With the isomorphism $T(q, q') : \mathbb{P}_0 \upharpoonright t(q) \to \mathbb{P}_0 \upharpoonright t(q')$ from Definition / Lemma 3.4.8 and its extension $\overline{T}(q, q') : \mathrm{Name}(\mathbb{P}_0 \upharpoonright t(q) \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+)) \to \mathrm{Name}(\mathbb{P}_0 \upharpoonright t(q') \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+))$, it follows that

$$X = \left(\overline{T}(q, q')\dot{X}\right)^{\left((G_0)^\beta \upharpoonright (\kappa^+ + 1)\right) \upharpoonright t(q') \times \left((G_1)^\beta \upharpoonright (\kappa+1)\right) \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots\} \times G_1(\kappa^+)}$$

where $\left(\overline{T}(q, q')\dot{X}\right) \in \mathrm{Name}(\mathbb{P}_0 \upharpoonright t(q') \times \mathbb{P}_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\} \times \mathbb{P}_1(\kappa^+))$.

Hence,

$$X \in \operatorname{dom} \iota_{(q', (\mu_0, \overline{\jmath}_0), \dots)} = \operatorname{dom} \iota_{((\kappa^+, j_0), \dots, (\mu_0, \overline{\jmath}_0), \dots)}.$$

There is a canonical bijection $b : \widetilde{\widetilde{\widetilde{M}}}_\beta \to \beta$. Hence, the injections $\iota_{((\kappa^+, j_0), \dots, (\mu_0, \overline{\jmath}_0), \dots)}$ for $((\kappa^+, j_0), \dots, (\mu_0, \overline{\jmath}_0), \dots) \in \widetilde{\widetilde{\widetilde{M}}}_\beta$ can be "glued together" to an injection $\overline{\iota} : \operatorname{dom} S^\beta \to (\kappa^+)^V \times \beta$ as follows: For $X \in \operatorname{dom} S^\beta$, take $((\kappa^+, j_0), \dots, (\mu_0, \overline{\jmath}_0), \dots) \in \widetilde{\widetilde{\widetilde{M}}}_\beta$ with $\delta := b((\kappa^+, j_0), \dots, (\mu_0, \overline{\jmath}_0), \dots) < \beta$ least such that $X \in \operatorname{dom} \iota_{((\kappa^+, j_0), \dots, (\mu_0, \overline{\jmath}_0), \dots)}$ and set

$$\overline{\iota}(X) := \big(\iota_{((\kappa^+, j_0), \dots, (\mu_0, \overline{\jmath}_0), \dots)}(X), \delta \big).$$

This gives an injection $\iota : \operatorname{dom} S^\beta \to \beta$ in $V[(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times (G_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa + 1) \times G_1(\kappa^+) \times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}]$ as desired.

□

Thus, we have shown that our assumption of a surjective function $S : \wp(\kappa) \to F(\kappa)$ in N leads to a contradiction.
Hence, $\theta^N(\kappa) \leq F(\kappa)$ for any limit cardinal κ.

It remains to show that $\theta^N(\kappa^+) \leq F(\kappa^+)$ for all

successor cardinals $\boldsymbol{\kappa^+}$,

which can be done by the same argument:

Like before, we assume towards a contradiction there was a surjective function $S : \wp(\kappa^+) \to F(\kappa^+)$ in N, $S = \dot{S}^G$ with $\pi \overline{\dot{S}}^{D_\pi} = \overline{\dot{S}}^{D_\pi}$ for all π that are contained in an intersection like $(I_{\dot{S}})$. Again, fix a condition $r \in G_0$ such that $\{(\kappa_0, i_0), \dots, (\kappa_{n-1}, i_{n-1})\} \subseteq t(r)$ contains all maximal points of $t(r)$, and an extension $\overline{r} \leq_0 r$, $\overline{r} \in G_0$ such that all $t(\overline{r})$-branches have height $\geq \kappa^+$.
From Corollary 3.3.8, it follows that any $X \in N$, $X \subseteq \kappa^+$, is contained in a model of the form

$$V[G_0 \upharpoonright \{(\kappa^+, j_0), \dots, (\kappa^+, j_{k-1})\} \times G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}],$$

where $j_0, \dots, j_{k-1} < F_{\lim}(\kappa^+) = F(\kappa)$ and $\mu_0, \dots, \mu_{\overline{k}-1} \in Succ' \cap (\kappa^+ + 1)$ with $\overline{\jmath}_0 < F(\mu_0), \dots, \overline{\jmath}_{\overline{k}-1} < F(\mu_{\overline{k}-1})$.

For a limit ordnal $\widetilde{\beta} < F(\kappa^+)$, our definition of *large enough for* $(I_{\dot{S}})$ has to be slightly modified: This time, we require that $\widetilde{\beta} > \overline{\imath}_l$ for all $l < \overline{n}$ with $\overline{\kappa}_l \leq \kappa^+$ (instead of just $\overline{\kappa}_l < \kappa$), and $\widetilde{\beta} > \overline{\alpha}_l$ for all $l < \overline{m}$ with $\overline{\lambda}_l \leq \kappa^+$ (instead of just $\overline{\lambda}_l < \kappa$).

Fix $\widetilde{\beta} < F(\kappa^+)$ *large enough for* $(I_{\dot{S}})$ and $\beta := \widetilde{\beta} + \kappa^+$ (addition of ordinals). We define the restriction S^β similarly as before: Let M' denote the collection of all tuples $(s, (\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1}))$ with $\overline{k} < \omega$, $\mu_0, \dots, \mu_{\overline{k}-1} \leq \kappa^+$, $\overline{\jmath}_0 < F(\mu_0), \dots, \overline{\jmath}_{\overline{k}-1} < F(\mu_{\overline{k}-1})$, and s a condition in $\mathbb{P}_0$ with maximal points $(\kappa^+, j_0), \dots, (\kappa^+, j_{k-1})$ where $j_0 < F_{\lim}(\kappa^+), \dots, j_{k-1} < F_{\lim}(\kappa^+)$. Moreover, we denote by M'_β the collection of all tuples $(s, (\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in M'$ with the additional property that $\overline{\jmath}_0 < \beta, \dots, \overline{\jmath}_{\overline{k}-1} < \beta$, and s has maximal points $(\kappa^+, j_0), \dots, (\kappa^+, j_{k-1})$ with $j_0 < \beta, \dots, j_{k-1} < \beta$.
Let

$$S^\beta := S \upharpoonright \big\{ X \subseteq \kappa \;\big|\; \exists (s, (\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})) \in M'_\beta : \; s \in G_0 \upharpoonright (\kappa^+ + 1), \\ X \in V[G_0 \upharpoonright t(s) \times G_1 \upharpoonright \{(\mu_0, \overline{\jmath}_0), \dots, (\mu_{\overline{k}-1}, \overline{\jmath}_{\overline{k}-1})\}] \big\}.$$

The same proof as for Proposition 3.4.2 shows that the surjectivity of S implies that S^β must be surjective, as well.

Now, with the same construction as before, one can capture S^β in an intermediate model $V[(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times (G_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa^+ + 1) \times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}]$, and like in Lemma 3.4.7, one can show that the according forcing $(\mathbb{P}_0)^\beta \upharpoonright (\kappa^+ + 1) \times (\mathbb{P}_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \times (\mathbb{P}_1)^\beta \upharpoonright (\kappa^+ + 1) \times \mathbb{P}_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}$ preserves cardinals $\geq F(\kappa^+)$.
Finally, one can show like in Proposition 3.4.10 that in this model $V[(G_0)^\beta \upharpoonright (\kappa^+ + 1) \times (G_0 \upharpoonright t(\overline{r})) \upharpoonright [\kappa^+, \infty) \times (G_1)^\beta \upharpoonright (\kappa^+ + 1) \times G_1 \upharpoonright \{(\overline{\kappa}_l, \overline{\imath}_l) \mid l < \overline{n}, \overline{\kappa}_l > \kappa^+\}]$, there is also an injection $\iota^\beta : \operatorname{dom} S^\beta \hookrightarrow \beta$. This gives the desired contradiction.

Hence, it follows that $\theta^N(\kappa^+) \leq F(\kappa^+)$ for all successor cardinals κ^+.

Thus, our model N has all the desired properties.

3.5 Discussion and Remarks

Our result generalizes Easton's Theorem to regular **and** singular cardinals: In the theory ZF, the θ-function can take almost arbitrary values. This extends the results from Chapter 2 to a proper class of cardinals, with the constraint that this time, we do not retain DC in the symmetric extension N.

One could ask whether it is possible to do a similar construction and obtain a ZF-model N where additionally DC holds. For this, we would need a countably closed forcing notion (and a symmetric system generated by countable intersections). A straightforward generalization of $\mathbb{P}_0$ would be a forcing with trees $(t, \leq_t)$ where countably many maximal points are allowed, instead of just finitely many.
However, this gives rise to the following appearance that we call an *open branch*: There might be a $\leq_t$-increasing chain of vertices $((\alpha, i_\alpha) \mid \alpha < \lambda)$ for some cardinal λ of countable cofinality such that there exists no $(\lambda, i) \in t$ with $(\alpha, i_\alpha) \leq_t (\lambda, i)$ for all $\alpha < \lambda$. The number of open branches might be $2^{\aleph_0} = \aleph_1$, so we can not always "close" all of them and retain a condition in the forcing.

Let us shortly discuss the following technical problem that comes along with these open branches: If conditions p and q in $\mathbb{P}_0$ agree on a subtree $t(r) \geq t(p)$, $t(q)$, it might not be possible to achieve $\pi p \parallel q$ by a small $\mathbb{P}_0$-automorphism π that is the identity on $t(r)$: Consider the case that the tree $t(r)$ has an open branch $((\alpha, i_\alpha) \mid \alpha < \lambda)$ such that in $t(p)$, there is a vertex (λ, i) with $(\lambda, i) \geq_{t(p)} (\alpha, i_\alpha)$ for all $\alpha < \lambda$, but in $t(q)$, there is a different vertex (λ, i') with $i' \neq i$ and $(\lambda, i') \geq_{t(q)} (\alpha, i_\alpha)$ for all $\alpha < \lambda$. An automorphism π with $\pi p \parallel q$ such that π is the identity on this branch $((\alpha, i_\alpha) \mid \alpha < \lambda)$, has to satisfy $\pi(\lambda)(\lambda, i) = (\lambda, i')$, since the tree $t(\pi p) \cup t(q)$ must not have a "splitting" at level λ. But then, there is no way to guarantee that π is small, since in general, i and i' will not be close to each other.

Thus, generalizing $\mathbb{P}_0$ to trees with countably many maximal points makes us lose an essential homogeneity property, and several crucial arguments in the original proof do not work any more.

One could try and allow trees with $< \mu$-many maximal points, where μ is an inaccessible cardinal. Then our conditions in the forcing have $< \mu$-many open branches, and we can now "close" all of them and still remain inside $\mathbb{P}_0$. Hence, our forcing will be $< \mu$-closed.
In this setting, we call a $\mathbb{P}_0$-automorphism *small*, if for any level κ and $\pi(\kappa)(\kappa, i) = (\kappa, i')$, it follows that there is an ordinal γ divisible by μ with $i, i' \in [\gamma, \gamma + \mu)$.
Concerning $\mathbb{P}_1$, we can use $< \mu$-support instead of finite support, and then take intersections of $< \mu$-many $Fix(\kappa, i)$- and $Small(\lambda, [0, \alpha))$-subgroups for generating our symmetric system. Then $N \vDash \mathsf{DC}_{<\mu}$ (cf. [Kar14, Lemma 1]).

Of course, $\mathsf{DC}_{<\mu}$ imposes further restrictions on the θ-function, and one cannot use this modified forcing for setting θ-values $\theta^N(\kappa)$ for cardinals $\kappa < \mu$. However, it might be conceivable to combine this $< \mu$-closed tree forcing with the set-sized forcing notion from Chapter 2, which could treat the cardinals below μ, while the $< \mu$-closed tree forcing could provide the "upper part", setting the θ-values $\theta^N(\kappa)$ of cardinals $\kappa \geq \mu$.

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