Moduli spaces of branched coverings of the plane

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Summary of the thesis

In this thesis I build a strong connection between Hurwitz spaces of branched coverings of the complex plane, and moduli spaces $\mathfrak{M}_{g,n}$ of Riemann surfaces of genus $g \geq 0$ with $n \geq 1$ boundary components.
I refine a construction by Bödigheimer, which gives a combinatorial model for $\mathfrak{M}_{g,n}$ based on slit pictures on the complex plane. For an integer $d \geq n$ and a splitting $d = d_1 + \cdots + d_n$ with $d_i \geq 1$, I consider a certain moduli space $\bar{\mathcal{O}}_{g,n}[d_1, \ldots, d_n]$ of branched coverings of Riemann surfaces $\Sigma_{g,n} \to \mathbb{C}P^1$ of degree d, with some prescribed behaviour near the n marked points of $\Sigma_{g,n}$.
The main results of the thesis are the following.

- The space $\bar{\mathcal{O}}_{g,n}[d_1, \ldots, d_n]$ is a complex manifold of complex dimension $d + 2g + n - 2$ and has a combinatorial cell structure analogue to Bödigheimer's model.
- For $d \geq 2g + n - 1$ the space $\bar{\mathcal{O}}_{g,n}[d_1, \ldots, d_n]$ is homotopy equivalent to $\mathfrak{M}_{g,n}$.
- The space $\bar{\mathcal{O}}_{g,n}[d_1, \ldots, d_n]$ has a natural filtration whose strata are the classical Hurwitz spaces of d-fold coverings of $\mathbb{C}$ branched over k points, where k depends on the stratum. Hence the construction of $\bar{\mathcal{O}}_{g,n}[d_1, \ldots, d_n]$ creates a bridge between the theory of configuration spaces and braids on one side, and the theory of moduli spaces of Riemann surfaces on the other side.
- The cellular chain complex of $\bar{\mathcal{O}}_{g,n}[d_1, \ldots, d_n]$ can be simplified using a technique due to Balázs Visy, thus obtaining a direct summand of the reduced cobar complex of $\mathcal{V}(d)$. The latter is a certain bialgebra in the category of $\mathfrak{S}_d$-Yetter-Drinfeld modules: this roughly means that $\mathcal{V}(d)$ is a graded and $\mathfrak{S}_d$-graded abelian group with an action of $\mathfrak{S}_d$, a unit, a counit, a multiplication and a comultiplication, and all these structures are interrelated. Therefore also the homology of $\bar{\mathcal{O}}_{g,n}[d_1, \ldots, d_n]$ is a direct summand of the cohomology of the reduced cobar complex of $\mathcal{V}(d)$.

Contents

1 Introduction

The purpose of this thesis is to give a connection between the following mathematical objects:

- moduli spaces $\mathfrak{M}_{g,n}$ of Riemann surfaces $\Sigma_{g,n}$ of genus $g \geq 0$ with $n \geq 1$ poles (marked points, each endowed with a chosen tangent vector);
- configuration spaces of points in the plane with various decorations;
- bialgebras.

The starting point is a classical construction by Hurwitz [26]: one can obtain Riemann surfaces of type $\Sigma_{g,n}$ as branched coverings of the projective line $f\colon \bar{\mathcal{F}} \to \mathbb{C}P^1$ of some degree d, by choosing a set $P = \{z_1, \dots, z_k\} \subset \mathbb{C}$ of ramification points, and by specifying a *monodromy* for the regular part of the covering. The monodromy can be regarded as a homomorphism $\varphi\colon \pi_1(\mathbb{C} \setminus P, *_P) \to \mathfrak{S}_d$, after having suitably fixed a basepoint $*_P \in \mathbb{C} \setminus P$ and having trivialised its fibre $f^{-1}(*_P) \cong \{1, \dots, d\}$.
If d, k, P and φ are chosen conveniently, the total space of the branched covering is a Riemann surface $\bar{\mathcal{F}}$ of type $\Sigma_{g,n}$; the poles $Q_1, \dots, Q_n$ are precisely the points in the fibre of $\infty \in \mathbb{C}P^1$ along the map f, and each pole can be endowed with a tangent vector in a canonical way.
If we perturb the configuration P to another configuration of k points, and adjust consequently the homomorphism φ, we obtain a new pair (P', φ') yielding a new Riemann surface $\bar{\mathcal{F}}'$. The surface $\bar{\mathcal{F}}'$ is also of type $\Sigma_{g,n}$, but in general it represents a distinct point in the moduli space $\mathfrak{M}_{g,n}$, i.e. $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}'$ are in general not biholomorphic to each other.
Using the previous recipe, we can even construct an entire *family* of Riemann surfaces, parametrised by the space $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}]$ of pairs (P, φ) yielding a surface of type $\Sigma_{g,n}$, where P is a configuration of k points in $\mathbb{C}$ and $\varphi\colon \pi_1(\mathbb{C} \setminus P, *_P) \to \mathfrak{S}_d$. Here the word *family* is informal: more formally we have constructed a bundle over the space $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}]$, with fibres Riemann surfaces of type $\Sigma_{g,n}$; there is a corresponding classifying map $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}] \to \mathfrak{M}_{g,n}$.
The space $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}]$ is a finite covering of the unordered configuration space $C(\mathbb{C}; k)$ of k points in $\mathbb{C}$, and thus we have obtained a diagram in which the spaces $C(\mathbb{C}; k)$ and $\mathfrak{M}_{g,n}$ are related to each other, although indirectly, i.e. through the space $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}]$:

$$\begin{array}{ccc} \mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}] & \longrightarrow & \mathfrak{M}_{g,n} \\ \downarrow & & \\ C(\mathbb{C}; k). & & \end{array}$$

In the case $d = 2$ we note that for every configuration $P = \{z_1, \dots, z_k\} \in C(\mathbb{C}; k)$ there is a particularly interesting choice for the map $\varphi\colon \pi_1(\mathbb{C} \setminus P) \to \mathfrak{S}_2$, namely the homomorphism whose *local monodromy* around each point z_i is the generator $(1, 2) \in \mathfrak{S}_d$. Thus the space $\mathrm{hur}(k, \mathfrak{S}_2)$ contains a copy of the space $C(\mathbb{C}; k)$, and we obtain, for all $g \geq 1$, maps

$$\varphi\colon C(\mathbb{C}; 2g-1) \to \mathfrak{M}_{g-1,1} \qquad \varphi\colon C(\mathbb{C}; 2g) \to \mathfrak{M}_{g-1,2}.$$

These maps have been first introduced by Birman and Hilden [7, 8], and their behaviour in homology has been investigated by Song and Tillmann [38], Segal and Tillmann [37], Callegaro and Salvetti [12], and the author [5, 6].

One can ask whether the previous construction yields a *good approximation* of the moduli space $\mathfrak{M}_{g,n}$, e.g. whether the map $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}] \to \mathfrak{M}_{g,n}$ is highly connected, for a suitable choice of k and maybe restricting to a suitable connected component of the space $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}]$: note indeed that $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}]$ is in general disconnected, whereas $\mathfrak{M}_{g,n}$ is connected. The following simple example suggests that this conjecture is too optimistic, and that one has to improve the construction considered above in order to obtain a good approximation of $\mathfrak{M}_{g,n}$.
Let $\varepsilon > 0$, let $P_\varepsilon := \{z_1 = 0, z_2 = \varepsilon\} \subset \mathbb{C}$ be a configuration of two points, and let $\varphi_\varepsilon\colon \pi_1(\mathbb{C} \setminus P_\varepsilon, *) \to \mathfrak{S}_3$ be defined by $\varphi_\varepsilon(\mathfrak{f}_1) := (1, 2) \in \mathfrak{S}_3$ and $\varphi_\varepsilon(\mathfrak{f}_2) := (2, 3) \in \mathfrak{S}_3$. Here we use use the fact that $\pi_1(\mathbb{C} \setminus P_\varepsilon)$ is a free group on two generators $\mathfrak{f}_1$ and $\mathfrak{f}_2$, represented by two loops spinning once, clockwise, around z_1 and z_2 respectively; $(1, 2)$ and $(2, 3)$ are transpositions in $\mathfrak{S}_3$ (see Figure 1.1, left). We denote by $\mathcal{L}$ the element $\mathfrak{f}_1 \cdot \mathfrak{f}_2 \in \pi_1(\mathbb{C} \setminus P_\varepsilon)$; then $\varphi(\mathcal{L})$ is the 3-cycle $(1, 2, 3) \in \mathfrak{S}_3$.

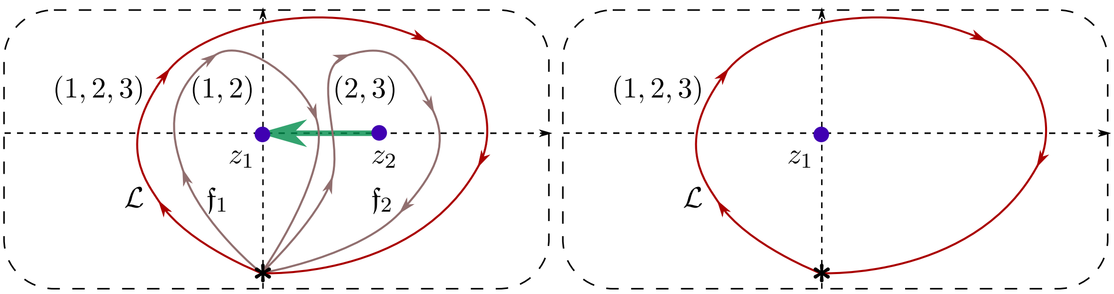

Figure 1.1. A point in $(P_0, \varphi_0) \in \mathrm{hur}(1, \mathfrak{S}_3)[\Sigma_{0,1}]$, on right, is the natural limit of a sequence of points $(P_\varepsilon, \varphi_\varepsilon) \in (\mathrm{hur}(2, \mathfrak{S}_3)[\Sigma_{0,1}]$, on left, for $\varepsilon \to 0$.

Using the previous recipe with the data $(P_\varepsilon, \varphi_\varepsilon)$ we obtain a surface $\bar{\mathcal{F}}_\varepsilon$ of type $\Sigma_{0,1}$, i.e. of genus 0 with one pole. The surface $\bar{\mathcal{F}}_\varepsilon$ is biholomorphic to $\mathbb{C}P^1$. More precisely, let $\lambda_\varepsilon := \sqrt[3]{\frac{27}{4}\varepsilon}$; then we can choose a biholomorphism $\bar{\mathcal{F}}_\varepsilon \cong \mathbb{C}P^1$ such that the map $f_\varepsilon\colon \bar{\mathcal{F}}_\varepsilon \to \mathbb{C}P^1$ corresponds to the map $\mathbb{C}P^1 \to \mathbb{C}P^1$ induced by the polynomial $p_\varepsilon(z) = z^3 + \lambda_\varepsilon z^2$.
To see this, note that the two roots of $p'_\varepsilon(z)$ (the two branch points of p_ε) are $w_1 = 0$ and $w_2 = -\frac{2}{3}\lambda_\varepsilon$, and their images under p_ε (the two branch values of p_ε) are precisely $p_\varepsilon(w_1) = 0 = z_1$ and $p_\varepsilon(w_2) = \varepsilon = z_2$.

For $\varepsilon \to 0$, the polynomial $p_\varepsilon(z)$ has an obvious limit, namely the polynomial $p_0(z) = z^3$. The induced map $\mathbb{C}P^1 \to \mathbb{C}P^1$ corresponds to the couple (P_0, φ_0); here $P_0 = \{z_1 = 0\}$ consists of only one point (the only branch value of p_0), the group $\pi_1(\mathbb{C} \setminus P_0, *)$ is free of rank *one*, generated by the loop $\mathcal{L}$, and $\varphi_0(\mathcal{L})$ is the 3-cycle $(1, 2, 3) \in \mathfrak{S}_3$ (see Figure 1.1, right).
According to our previous definition of the spaces $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}]$, we would then have that for $\varepsilon > 0$ the configuration $(P_\varepsilon, \varphi_\varepsilon)$ belongs to the space $\mathrm{hur}(2, \mathfrak{S}_3)[\Sigma_{0,1}]$, but for $\varepsilon = 0$ the limit configuration (P_0, φ_0) belongs to *another* space, namely $\mathrm{hur}(1, \mathfrak{S}_3)[\Sigma_{0,1}]$. It is then natural, for a fixed surface type $\Sigma_{g,n}$, to *amalgamate* the spaces $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}]$, for varying k, into a unique space $\mathrm{hur}(\mathfrak{S}_d)[\Sigma_{g,n}]$.

This amalgamation is the main construction of Chapters 3 and 4 of the thesis. Let $h = 2g + n + d - 2$; then the spaces $\mathrm{hur}(k, \mathfrak{S}_d)[\Sigma_{g,n}]$, for varying k, amalgamate into a disjoint union of connected spaces $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$, one for each permutation $\sigma \in \mathfrak{S}_d$ having a cycle decomposition consisting of precisely n cycles.
Each space $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$ is endowed with a natural map $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma) \to \mathfrak{M}_{g,n}$, and in Chapter 7 we will prove that, assuming $d \geq 2g + n - 1$, then each map $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma) \to \mathfrak{M}_{g,n}$ is a homotopy equivalence (see Theorem 7.3.2). This is the first, main result of the thesis, and it is the bulk of Chapter 7.
The preparation for this result is done in Chapter 6,where we recall the definition of the space $\mathfrak{Par}_{g,n}[\underline{d}]$, introduced by Bödigheimer [10, 1] and generalised by Boes and Hermann [11]. The space $\mathfrak{Par}_{g,n}[\underline{d}]$ is a combinatorial model for the moduli space $\mathfrak{M}_{g,n}$, and its construction depends on a chosen partition $\underline{d} = (d_1, \ldots, d_n)$ of d into n positive integers. It turns out that $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$ is a closed subspace of $\mathfrak{Par}_{g,n}[\underline{d}]$, if we choose $\underline{d}$ as the sequence of lengths of the cycles of σ (see Theorem 7.4.1).
The space $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$ arises as a particular connected component of a larger space $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$. In Chapters 3 and 4 we endow all spaces $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$ with a finite, combinatorial cell structure; our construction is similar in flavour to the stratification of the configuration spaces $C(\mathbb{C}; k)$ considered by Fox and Neuwirth [20] and Fuchs [22], and was independently developed by Kapranov and Schechtman in [27].
The connection between configuration spaces and bialgebras, and in particular *shuffle algebras*, is well-known: a good account on the subject can be found in [16]. In Chapter 8, elaborating on work of Visy [39], we will introduce for every $d \geq 2$ an algebra $\mathcal{V}(d)$. We will prove that $\mathcal{V}(d)$ is the Koszul dual of an algebra $\mathrm{grad}^N \mathbb{Z}[\mathfrak{S}_d]$: the latter arises as associated graded of the group ring $\mathbb{Z}[\mathfrak{S}_d]$, on which one can define a convenient filtration, called *norm filtration*. The first algebraic properties of $\mathcal{V}(d)$ are studied; in particular $\mathcal{V}(d)$ has a bialgebra structure in a suitable category, the category of $\mathfrak{S}_d$-Yetter-Drinfeld modules.
In Chapter 9 we apply the algebraic results from Chapter 8 and we describe in detail the cellular cochain complex associated with the one-point compactifications $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)^\infty$ of the spaces $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$. The second, main result of the thesis is Theorem 9.3.1, which connects the homology of the spaces $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$ with the cohomology of the reduced cobar complex $\bar{F}_{\bullet,\bullet}(\mathbb{Z}, \mathcal{V}(d), \mathbb{Z})$ of $\mathcal{V}(d)$ with coefficients in the coaugmentation module

$\mathbb{Z}$.
The construction of the reduced cobar complex only involves the structure of *coaugmented coalgebra* on $\mathcal{V}(d)$, i.e. the product, the counit and the unit. The additional product structure will correspond to the geometric structure that the spaces $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$ have when considered all together. In Chapter 5 we will define a homotopy associative H-space structure on the disjoint union

$$\mathrm{Hur}(\mathfrak{S}_d) := \coprod_{h \geq 0, \sigma \in \mathfrak{S}_d} \mathrm{Hur}(h, \mathfrak{S}_d, \sigma),$$

giving a Pontryagin product in homology: this is the same product induced by the product of $\mathcal{V}(d)$ on the cohomology $H^*\left(\bar{F}_{\bullet,\bullet}(\mathbb{Z}, \mathcal{V}(d), \mathbb{Z})\right)$, as we will see at the end of Chapter 9.
In Chapter 5 we will also consider the disjoint union $\mathrm{Hur}(\mathfrak{S}_d, \mathbb{1}) := \coprod_{h \geq 0} \mathrm{Hur}(h, \mathfrak{S}_d, \mathbb{1})$, where $\mathbb{1} \in \mathfrak{S}_d$ denotes the identity. The space $\mathrm{Hur}(\mathfrak{S}_d, \mathbb{1})$ is a sub-H-space of $\mathrm{Hur}(\mathfrak{S}_d)$ and it carries an action of the operad $\mathcal{C}_2$ of little squares [9, 29].
In Chapter 10 we will briefly discuss some open questions related to the topics of the thesis, and we will outline some possible future research.

This thesis is the main fruit of my PhD research at the University of Bonn. I am indebted to Professor Carl-Friedrich Bödigheimer for his patient supervision and his continuous encouragement. I would like to thank him and Felix Boes for many useful discussions, in particular for the detailed explanation of the content of Chapter 6. I would like to thank Oscar Randal-Williams for a useful discussion which led, in particular, to the development of Subsection 8.1.3, and for pointing the reference [17]. I would like to thank Florian Kranhold for pointing the reference [32], and Annika Kiefner, for some computer-aided computations which let me understand how complicated the homology of the little bundle operad may be. I would like to thank Professor Carl-Friedrich Bödigheimer, Alexander Thomas, Florian Kranhold and Bastiaan Cnossen for many useful comments on a first draft of the thesis. Finally, I would like to thank all my colleagues, my family and my friends for their moral support during these three years of research.

2 Preliminaries

In this chapter we recall some basic facts and set some conventions.

2.1 Simplices and multisimplices

2.1.1 Simplicial and multisimplicial notation

We fix some notation concerning simplices and multisimplices.

Definition 2.1.1. Let $[-\infty, +\infty]$ be the closed real line. For $n \geq 0$, the standard simplex Δ^n is defined as

$$\Delta^n = \{(x_1, \dots, x_n) \in [-\infty, +\infty]^n \,|\, x_1 \leq \dots \leq x_n\}.$$

The numbers x_r are called the *local coordinates* of Δ^n. The simplex Δ^n has a canonical CW-structure with $\binom{n+1}{i+1}$ cells in dimension i, for all $1 \leq i \leq n$; the *faces* of Δ^n are the closures of the cells of dimension $n-1$: these are called $\partial_r \Delta^n$ for $0 \leq r \leq n$, and each face $\partial_r \Delta^n$ is naturally identified with Δ^{n-1}.

Consider the cellular chain complex of Δ^n, denoted by $C_*(\Delta^n)$: then the cellular boundary $\partial \Delta^n$ takes the usual form

$$\partial \Delta^n = \sum_{r=0}^{n} (-1)^r \partial_r \Delta^n.$$

Definition 2.1.2. Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ be a sequence of numbers $\lambda_r \geq 0$. The multisimplex $\Delta^{\underline{\lambda}}$ is the space

$$\Delta^{\underline{\lambda}} = \Delta^{\lambda_1} \times \dots \times \Delta^{\lambda_k}.$$

For each $1 \leq r \leq k$ we denote by $x_{r,1} \leq \dots \leq x_{r,\lambda_r}$ the local coordinates of the r^{th}factor; all these coordinates are referred to as the *local coordinates* of $\Delta^{\underline{\lambda}}$. Whenever it is not explicitly stated otherwise, $\Delta^{\underline{\lambda}}$ carries the CW-structure of product of CW-complexes. Denote $\lambda = \sum_{r=1}^{k} \lambda_r$, and note that $\Delta^{\underline{\lambda}}$ has one maximal cell in dimension λ. The *faces* of $\Delta^{\underline{\lambda}}$ are the closures of the cells of dimension $\lambda - 1$: these are called $\partial_{r,s} \Delta^{\underline{\lambda}}$, for $1 \leq r \leq k$ and $0 \leq s \leq \lambda_r$, and each face $\partial_{r,s} \Delta^{\underline{\lambda}}$ is naturally identified with the multisimplex $\Delta^{\lambda_1} \times \dots \times \partial_s \Delta^{\lambda_r} \times \dots \times \Delta^{\lambda_k}$.

Consider the cellular chain complex of $\Delta^{\underline{\lambda}}$, denoted by $C_*(\Delta^{\underline{\lambda}})$: then the cellular boundary $\partial \Delta^{\underline{\lambda}}$ takes the form

$$\partial \Delta^{\underline{\lambda}} = \sum_{r=1}^{k} \sum_{s=0}^{\lambda_r} (-1)^{\left(\sum_{i=1}^{r-1} \lambda_i\right) + s} \partial_{r,s} \Delta^{\underline{\lambda}}.$$

2.1.2 Shuffle products

We recall the Eilenberg-Zilber shuffle product decomposition of a product of two simplices $\Delta^\lambda \times \Delta^{\lambda'}$; see also [15, p.68] or [24, p.286].

Definition 2.1.3. For all $\lambda \geq 1$ we denote by $[\lambda]$ the finite, ordered set $\{1 < \cdots < \lambda\}$. Let $\lambda, \lambda' \geq 1$. A *shuffle* η of the sets $[\lambda]$ and $[\lambda']$ is a couple of increasing injective maps $\eta_1 \colon [\lambda] \hookrightarrow [\lambda + \lambda']$ and $\eta_2 \colon [\lambda'] \hookrightarrow [\lambda + \lambda']$ with disjoint images. The set of all shuffles of $[\lambda]$ and $[\lambda']$ is denoted by $\mathfrak{Shuf}(\lambda, \lambda')$.
Equivalently, let $A = \{a_1 \prec_A \cdots \prec_A a_\lambda\}$ and $B = \{b_1 \prec_B \cdots \prec_B b_{\lambda'}\}$ be two disjoint, non-empty, finite ordered sets. A *shuffle* η of A and B is a total order $\prec = \prec_\eta$ on the set $C = A \sqcup B$ restricting to $\prec_A$ on A and to $\prec_B$ on B. The set of all shuffles of A and B is denote by $\mathfrak{Shuf}(A, B)$; this set is in natural bijection with $\mathfrak{Shuf}(\lambda, \lambda')$, and from now on we will pass from one notion to the other without explicitly stating the equivalence.
The *parity* of $\eta \in \mathfrak{Shuf}(A, B)$, denoted $\pi(\eta)$, is the parity (even or odd) of the cardinality of the set

$$\{(a_i, b_j) \,|\, a_i \in A\,,\, b_j \in B\,,\, b_j \prec_\eta a_i\}\,.$$

The product $\Delta^\lambda \times \Delta^{\lambda'}$ can be dissected into $\binom{\lambda+\lambda'}{\lambda}$ simplices of dimension $\lambda + \lambda'$. Let $(x_i)_{1 \leq i \leq \lambda}$ and $(x'_i)_{1 \leq i \leq \lambda'}$ be the local coordinates of the two simplices and consider the ordered sets $A = \{x_i\}_{1 \leq i \leq \lambda} \cong [\lambda]$ and $B = \{x'_i\}_{1 \leq i \leq \lambda'} \cong [\lambda']$, containing the coordinates of the two simplices.
Then for every shuffle $\eta \in \mathfrak{Shuf}(A, B)$ we can consider the subspace

$$\triangle_\eta \subset \Delta^n \times \Delta^{n'},$$

of points $\mathbb{P} \in \Delta^\lambda \times \Delta^{\lambda'}$ where for all pairs of coordinates $x_i \prec_\eta x'_j$ in $A \sqcup B$, the inequality $x_i|_\mathbb{P} \leq x'_j|_\mathbb{P}$ holds ($x_i|_P$ and $x'_j|_P$ are numbers in $[-\infty, +\infty]$). The space $\triangle_\eta$ is a $(\lambda+\lambda')$-simplex in a natural way: its local coordinates are just the coordinates in $A \sqcup B$, and they are ordered according to the order $\prec_\eta$. The simplices $\triangle_\eta$ for varying shuffle η give a simplicial decomposition of $\Delta^\lambda \times \Delta^{\lambda'}$.
Therefore we have two CW-structures on $\Delta^\lambda \times \Delta^{\lambda'}$: in the *product* CW-structure, cells are modelled on products of two simplices; in the *Eilenberg-Zilber* CW-structure, cells are modelled on single simplices. The identity of $\Delta^n \times \Delta^{n'}$ is a cellular map from the product CW-structure to the Eilenberg-Zilber CW-structure; the induced map between cellular $(\lambda+\lambda')$-chains $C^{prod}_{\lambda+\lambda'}(\Delta^\lambda \times \Delta^{\lambda'}) \to C^{EZ}_{\lambda+\lambda'}(\Delta^\lambda \times \Delta^{\lambda'})$ sends the generator $\Delta^\lambda \times \Delta^{\lambda'}$ of the first group to the following chain

$$\Delta^\lambda \times \Delta^{\lambda'} \mapsto \sum_{\eta \in \mathfrak{Shuf}(A,B)} (-1)^{\pi(\eta)} \triangle_\eta.$$

2.2 Finite groups and norms

In the whole thesis G represents a finite group; for many constructions we will also need to assume that G is endowed with a norm N. The motivating case that one has to keep in mind is, for some $d \geq 2$, $G = \mathfrak{S}_d$, i.e. a symmetric group, endowed with the word length norm with respect to all transpositions. The identity of G is usually denoted by $\mathbb{1} = \mathbb{1}_G \in G$.

Definition 2.2.1. A norm N on a group G is a function $N\colon G \to \mathbb{Z}_{\geq 0}$ such that

- $N(\gamma) = 0$ if and only if $\gamma = \mathbb{1}$;
- N is invariant under conjugation;
- $N(\gamma \cdot \gamma') \leq N(\gamma) + N(\gamma')$ for all $\gamma, \gamma' \in G$.

Definition 2.2.2. We denote by $\mathfrak{S}_d$ the symmetric group on d elements. Permutations are bijective functions from the set $[d] = \{1, \dots, d\}$ to itself and the product corresponds to composition of functions: in particular functions are composed from right to left.
For $i, j \in [d]$ with $i \neq j$ we denote by $(i, j) = (j, i)$ the transposition that exchanges i and j and fixes all other elements of $[d]$. More generally, for distinct elements $\iota_1, \dots, \iota_r \in [d]$, we denote by $(\iota_1, \dots, \iota_r) \in \mathfrak{S}_d$ the cycle mapping $\iota_i \mapsto \iota_{i+1}$, where indices are understood modulo r, and fixes all other elements in $[d]$.
By $\mathfrak{lc}_d \in \mathfrak{S}_d$ we denote the *long cycle*, i.e. the permutation

$$1 \mapsto 2 \mapsto \cdots \mapsto d \mapsto 1.$$

We consider on $\mathfrak{S}_d$ the word-length norm with respect to the generating set of all transpositions: for $\sigma \in \mathfrak{S}_d$ we denote by $N(\sigma)$ the smallest $s \geq 0$ such that there exist transpositions $\mathbf{t}_1, \dots, \mathbf{t}_s \in \mathfrak{S}_d$ with $\sigma = \mathbf{t}_1 \dots \mathbf{t}_s$.

Note that if the cycle decomposition of σ is $c_1 \dots c_l$, where fixed points of σ are treated as cycles of length 1, then $N(\sigma) = d - l$.
Note also that the norm of a permutation is invariant under conjugation, and that the norm of a permutation is even (respectively odd) if and only if the permutation is even (respectively odd).

3 Symmetric products

We fix $h \geq 1$ throughout the chapter. The case $h = 0$ is of little interest and we will add a few remarks for completeness.

Definition 3.0.1. The h-fold symmetric product of the plane, denoted by $SP^h(\mathbb{C})$, is the quotient of $\mathbb{C}^h$ by the action of the symmetric group $\mathfrak{S}_h$ permuting the coordinates. An element of $SP^h(\mathbb{C})$ can be regarded as a configuration of h points in the plane, where each point carries a multiplicity and points are not ordered; the sum of all multiplicities is equal to h. We will usually write $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\}$ for a configuration in $SP^h(\mathbb{C})$, with $z_1, \dots, z_k$ distinct points in $\mathbb{C}$, $\sum_{i=1}^k m_i = h$ and all $m_i \geq 1$. We define k to be the *absolute value* of P, and write $|P| = k$.
For an open set $U \subset \mathbb{C}$ we let $SP^h(U) \subset SP^h(\mathbb{C})$ be the subspace containing configurations $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\}$ with all points z_i in U.
The number h is also called the *weight* of the configuration P, and is denoted by $\mathfrak{w}(P)$.

The space $SP^h(\mathbb{C})$ is homeomorphic to $\mathbb{C}^h$: indeed $\mathbb{C}^h$ is the space of monic polynomials of degree h with coefficients in $\mathbb{C}$, and to every such polynomial we can associate its set of roots, which is precisely a configuration in $SP^h(\mathbb{C})$. Thus $SP^h(\mathbb{C})$ has a natural structure of complex manifold of dimension h. For $h = 0$, the space $SP^0(\mathbb{C})$ consists of only one configuration, i.e. the empty configuration in $\mathbb{C}$.
In this chapter we will study the geometry of the space $SP^h(\mathbb{C})$. In Section 3.1 we will construct a sheaf $\mathfrak{G}$ of free groups over $SP^h(\mathbb{C})$. In Section 3.2 we will define a cell decomposition for the one-point compactification $SP^h(\mathbb{C})^\infty$ of $SP^h(\mathbb{C})$. In Section 3.3 we will define a standard generating set for the free groups that arise as stalks of $\mathfrak{G}$.

3.1 A sheaf of free groups

In this section we introduce a sheaf of free groups $\mathfrak{G}$ over the space $SP^h(\mathbb{C})$ and study its main properties.

3.1.1 Definition of the sheaf $\mathfrak{G}$

Definition 3.1.1. Let $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$ be a configuration. The space $\mathbb{C} \setminus P$ is defined as the space $\mathbb{C} \setminus \{z_1, \dots, z_k\}$.

Note that for all $P \in SP^h(\mathbb{C})$ the space $\mathbb{C} \setminus P$ is homotopy equivalent to a wedge of circles, but both the number of wedge summands, and hence the homotopy type of $\mathbb{C} \setminus P$, depend on the absolute value $|P|$ (see Definition 3.0.1).

Definition 3.1.2. For every $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$ we endow the space $\mathbb{C} \setminus P$ with a basepoint $* = *_P$ *on the bottom*, which is the complex number

$$-\sqrt{-1}\left(1 + \max_{i=1}^{k} |z_i|\right) \in \mathbb{C} \setminus P.$$

This defines a continuous map $*\colon SP^h(\mathbb{C}) \to \mathbb{C}$, which assigns to every configuration $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$ a point $* = *_P \in \mathbb{C}$ distinct from all z_i's, lying on the negative imaginary axis underneath all z_i's (see Figure 3.1).

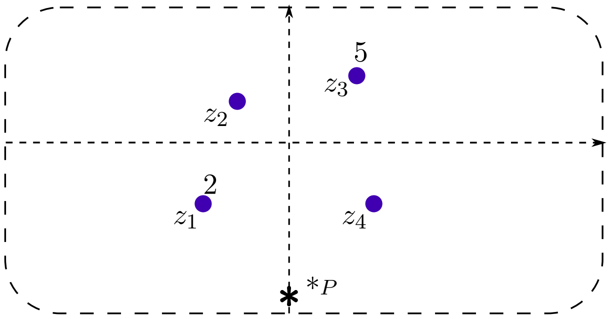

Figure 3.1. A configuration $P = \{2 \cdot z_1, z_2, 5 \cdot z_3, z_4\} \in SP^9(\mathbb{C})$ and the basepoint $*_P \in \mathbb{C} \setminus P$.

Definition 3.1.3. Let $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$. For each $1 \leq i \leq k$ let $U_i \subset \mathbb{C}$ be be a small open disc centered at z_i, and suppose that the discs $(U_i)_{1 \leq i \leq k}$ are disjoint. Then

$$\mathcal{U} = SP^{m_1}(U_1) \times \cdots \times SP^{m_k}(U_k) \subset SP^h(\mathbb{C})$$

is a neighbourhood of P in $SP^h(\mathbb{C})$; we call *normal neighbourhood* a neighbourhood obtained in this way (see Figure 3.2).
We denote by $\mathbb{C} \setminus \mathcal{U}$ the space $\mathbb{C} \setminus \left(\coprod_{i=1}^{k} U_i\right)$.

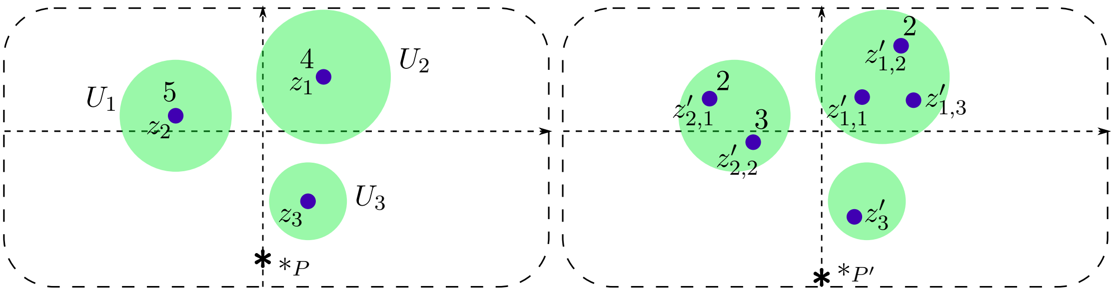

Figure 3.2. On left, a configuration $P = \{4 \cdot z_1, 5 \cdot z_2, z_3\} \in SP^{10}(\mathbb{C})$, and a choice of discs U_1, U_2, U_3. On right a configuration $P' = \left\{z'_{1,1}, 2 \cdot z'_{1,2}, z'_{1,3}, 2 \cdot z'_{2,1}, 3 \cdot z'_{2,2}, z'_3\right\}$ in the corresponding normal neighbourhood $\mathcal{U}$ of P. Note that $*_P \neq *_{P'}$.

Normal neighbourhoods form a fundamental system of neighbourhoods of a configuration in $SP^h(\mathbb{C})$. If we choose all radii of the discs U_i small enough (e.g., all smaller than $\frac{1}{2}$), then for all $P' \in \mathcal{U}$ the basepoint $*_{P'}$ of $\mathbb{C} \setminus P'$ does not belong to any U_i, and also the whole vertical segment joining $*_P$ and $*_{P'}$ is disjoint from the discs U_i. This condition will be very convenient later, so we will always assume it when considering normal neighbourhoods without further mention.

Lemma 3.1.4. *The collection of groups $(\pi_1(\mathbb{C} \setminus P, *_P))_{P \in SP^h(\mathbb{C})}$ forms a sheaf in groups $\mathfrak{G} = \mathfrak{G}_h$ over $SP^h(\mathbb{C})$; for each $P \in SP^h(\mathbb{C})$ and for each normal neighbourhood $P \in \mathcal{U} \subset SP^h(\mathbb{C})$ the restriction map $\mathfrak{G}(\mathcal{U}) \to \mathfrak{G}(P)$ is an isomorphism (we say that all germs of $\mathfrak{G}$ at P have a unique extension over $\mathcal{U}$).*

Proof. Let $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$ and let $\mathcal{U}$ be a normal neighbourhood of P, associated with the open discs $U_1, \dots, U_k \subset \mathbb{C}$. We can identify $\pi_1(\mathbb{C} \setminus P, *_P)$ with $\pi_1(\mathbb{C} \setminus \mathcal{U}, *_P)$ because the inclusion $\mathbb{C} \setminus \mathcal{U} \subset \mathbb{C} \setminus P$ is a homotopy equivalence.
For all $P' \in \mathcal{U}$ we consider the injective map $\pi_1(\mathbb{C} \setminus \mathcal{U}, *_P) \to \pi_1(\mathbb{C} \setminus P', *_P)$ induced by the inclusion, followed by the isomorphism $\pi_1(\mathbb{C} \setminus P', *_P) \simeq \pi_1(\mathbb{C} \setminus P', *_{P'})$ induced by translating the basepoint $P \to P'$ along the negative imaginary axis. Considering all $P' \in \mathcal{U}$ together, we obtain an injective homomorphism of groups

$$\varepsilon^P_{\mathcal{U}} \colon \pi_1(\mathbb{C} \setminus P, *_P) \to \prod_{P' \in \mathcal{U}} \pi_1\left(\mathbb{C} \setminus P', *_{P'}\right).$$

For all $P \in SP^h(\mathbb{C})$ we declare $\pi_1(\mathbb{C} \setminus P, *_P)$ to be the stalk $\mathfrak{G}(P)$ over P, and for all normal neighbourhoods $\mathcal{U}$ of a configuration P we declare the group $\mathfrak{G}(\mathcal{U})$ of sections of $\mathfrak{G}$ over $\mathcal{U}$ to be the image of $\varepsilon^P_{\mathcal{U}}$. Finally, for all open sets $\mathcal{V} \subset SP^h(\mathbb{C})$ we declare

$$\mathfrak{G}(\mathcal{V}) \subset \prod_{P \in \mathcal{V}} \pi_1(\mathbb{C} \setminus P, *_P)$$

to be the subgroup containing sections that on any normal neighbourhood $\mathcal{U} \subset \mathcal{V}$ of any configuration $P \in \mathcal{V}$ restrict to a section in $\mathfrak{G}(\mathcal{U})$.
Note that for $\mathcal{V} = \mathcal{U}$ a normal neighbourhood of a configuration P, the latter definition recovers the former: if $P' \in \mathcal{U}$ and $\mathcal{U}'$ is a normal neighbourhood of P' with $\mathcal{U}' \subset \mathcal{U}$, then the restriction of a section over $\mathcal{U}$ (one in the image of the map $\varepsilon^P_{\mathcal{U}}$) is a section over $\mathcal{U}'$ (lies in the image of $\varepsilon^{P'}_{\mathcal{U}'}$). To see this, note that for all $P'' \in \mathcal{U}'$ the inclusion $\mathbb{C} \setminus \mathcal{U} \subset \mathbb{C} \setminus P''$ can be written as a composition $\mathbb{C} \setminus \mathcal{U} \subset \mathbb{C} \setminus \mathcal{U}' \subset \mathbb{C} \setminus P''$. On the other hand, two functions in $\mathfrak{G}(\mathcal{U})$ agree on an open and closed subspace of $\mathcal{U}$.
By construction $\mathfrak{G}$ is a sheaf with the following property: for every configuration P and every normal neighbourhood $\mathcal{U}$ the restriction $\mathfrak{G}(\mathcal{U}) \to \mathfrak{G}(P)$ is an isomorphism. Therefore germs of $\mathfrak{G}$ have locally unique extensions to sections of $\mathfrak{G}$. □

For $h = 0$, the sheaf $\mathfrak{G}$ has only one stalk over the unique configuration $\emptyset$ of $SP^0(\mathbb{C})$, and this stalk is the trivial group.

3.1.2 Admissible generating sets

In the following, for $P = \{m_1 \cdot z_1, \ldots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$, we construct a set of generators of the free group $\pi_1(\mathbb{C} \setminus P; *)$.
We first choose small disjoint circles $\beta_1, \ldots, \beta_k$ around the points $z_1, \ldots, z_k$, and disjoint arcs $\zeta_1, \ldots, \zeta_k$ joining the basepoint $*$ with the points z_i's; we assume that ζ_i intersects once, transversely, the circle β_i in its *bottom point*, i.e. the point of β_i of minimal imaginary part; we also assume that ζ_i is disjoint from all other circles β_j for $j \neq i$ (see Figure 3.3, left).
For $1 \leq i \leq k$ consider the following loop $f_i \in \pi_1(\mathbb{C} \setminus P; *)$: it begins at $*$, runs along ζ_i until it reaches the intersection with β_i, spins clockwise around β_i and runs back to $*$ along ζ_i (see Figure 3.3, right).

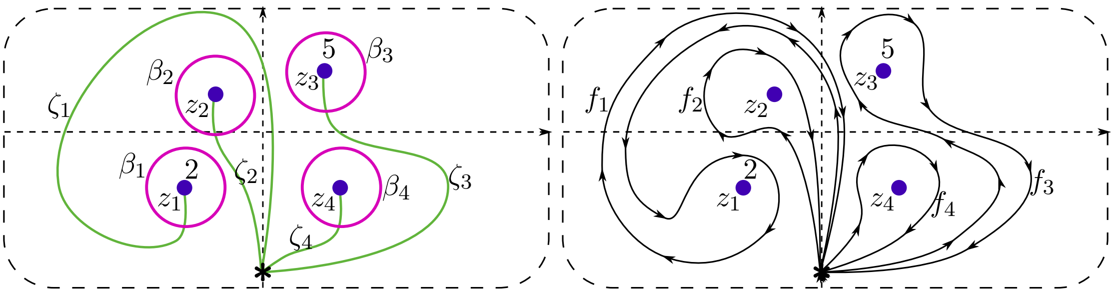

Figure 3.3. On left, for the configuration $P \in SP^9(\mathbb{C})$ from Figure 3.1, we choose circles $\beta_1, \beta_2, \beta_3, \beta_4$ and arcs $\zeta_1, \zeta_2, \zeta_3, \zeta_4$. On right, we obtain the corresponding admissible generating set f_1, f_2, f_3, f_4 for $\pi_1(\mathbb{C} \setminus P, *_P)$.

Then $f_1, \ldots, f_k$ form a free generating set for $\pi_1(\mathbb{C} \setminus P, *)$. This set of generators depends on the choice of the arcs ζ_i, but the conjugacy classes $[f_1]_{conj}, \ldots, [f_k]_{conj}$ in $\pi_1(\mathbb{C} \setminus P; *)$ do not depend on these arcs.

Definition 3.1.5. Let $P \in SP^h(\mathbb{C})$ be as above. A set of generators $f_1, \ldots, f_k$ for $\pi_1(\mathbb{C} \setminus P; *)$ obtained as described above is called an *admissible generating set*. The conjugacy classes of the elements in an admissible generating set, counted with multiplicity, do not depend on the admissible generating set.

Since the number $k = |P|$ depends on $P \in SP^h(\mathbb{C})$, we cannot determine in a continuous way a generating set of the stalk $\mathfrak{G}(P)$ for all $P \in SP^h(\mathbb{C})$. We will do it on certain subspaces $e^{\mathfrak{a}} \subset SP^h(\mathbb{C})$, see Definition 3.3.1.

3.2 Cell decomposition

In this section we describe a cell decomposition for the one-point compactification of $SP^h(\mathbb{C})$.

Definition 3.2.1. For a topological space X we denote by X^∞ its one-point compactification. The compactifying point is called *point at infinity*, serves as basepoint of X^∞

and is denoted by ∞ for all spaces X^∞. If X consists of one point (for instance $SP^0(\mathbb{C})$), then X^∞ is defined as $X \sqcup \{\infty\}$.

In [20] Fox and Neuwirth consider a stratification of the space $C(\mathbb{C};h)^\infty$ by subspaces homeomorphic to euclidean spaces of dimensions between $h+1$ and $2h$; in [22] uses this stratification to define a cell structure on the space $C(\mathbb{C};h)^\infty$, the one-point compactification of the unordered configuration space of h points in $\mathbb{C}$. We will mimic this construction for the space $SP^h(\mathbb{C})^\infty$; the same cell decomposition of $SP^h(\mathbb{C})$ was considered independently by Kapranov and Schechtman in [27].
It might seem cumbersome to define a complicated cell structure on the space $SP^h(\mathbb{C})^\infty$, which is homeomorphic to the sphere $\mathbb{S}^{2h}$; however we will need this construction in Chapter 4, in order to define a cell structure on the spaces $\widetilde{\mathrm{Hur}}(h,G)^\infty$ and $\mathrm{Hur}(h,G)^\infty$.

3.2.1 Columns and arrays

Here and in the following we identify $\mathbb{C}$ with $\mathbb{R}^2$ through real and imaginary part. We denote by $[-\infty,+\infty]$ the closed real line; hence $\mathbb{C} \subset [-\infty,+\infty]^2$.

Definition 3.2.2. A *column* $\mathfrak{col}$ of length $\lambda \geq 1$ is given by the following:

- a finite ordered set $A = \{a_1 \prec \cdots \prec a_\lambda\}$ of cardinality λ;
- a function $m\colon A \to \mathbb{Z}_{>0}$.

We write $\mathfrak{col} = (A, m)$. Two columns (A,m) and (A',m') are *equivalent* if $|A| = |A'|$ and $m \equiv m'$ after identifying linearly A and A'. We will consider columns up to equivalence. If $A = [\lambda] = \{1,\dots,\lambda\}$, we also write $\mathfrak{col} = (\lambda, \underline{m})$, with $\underline{m} = (m_1,\dots,m_\lambda) \in (\mathbb{Z}_{>0})^\lambda$; note that every equivalence class has a unique representative of this form, which we will use as preferred representative.
The *weight* of a column $\mathfrak{col} = (\lambda,\underline{m})$ is the number $\mathfrak{w}(\mathfrak{col}) = \sum_{i=1}^{\lambda} m_i$. The *absolute value* of $\mathfrak{col}$, denoted by $|\mathfrak{col}|$, coincides with its length. The *dimension* of a column is $\dim(\mathfrak{col}) = 1 + |\mathfrak{col}|$.

Definition 3.2.3. Let $\mathfrak{col} = (\lambda,\underline{m})$ and $\mathfrak{col}' = (\lambda',\underline{m}')$ be two columns and let $\eta \in \mathfrak{Shuf}(\lambda,\lambda')$ (see Definition 2.1.3). We define the *amalgamation* of $\mathfrak{col}$ and $\mathfrak{col}'$ along η as the following column

$$\mathfrak{col}'' = (\lambda+\lambda', \underline{m}'').$$

We choose representatives $\mathfrak{col} = (A,m)$ and $\mathfrak{col}' = (B,m')$, with A and B disjoint; then $\mathfrak{col}' = (A \cup_\eta B, m \cup m')$, where $m \cup m'$ is the function $A \cup_\eta B \to \mathbb{Z}_{>0}$ which restricts to m on A and to m' on B.

Definition 3.2.4. An *array* $\mathfrak{a}$ of length $l \geq 0$ is a sequence of l columns $\mathfrak{col}_1,\dots,\mathfrak{col}_l$. We will usually write $\mathfrak{a} = (l;\mathfrak{col}_1,\dots,\mathfrak{col}_l) = (l,\underline{\mathfrak{col}})$. Each column $\mathfrak{col}_i$ is usually expanded as $(\lambda_i,\underline{m}_i)$, and $\underline{m}_i$ is usually expanded as $(m_{i,1},\dots,m_{i,\lambda_i})$.
The *weight* of $\mathfrak{a}$ is defined as $\mathfrak{w}(\mathfrak{a}) = \sum_{i=1}^{l} \mathfrak{w}(\mathfrak{col}_i)$. The *absolute value* of $\mathfrak{a}$ is defined as $|\mathfrak{a}| = \sum_{i=1}^{l} |\mathfrak{col}_i|$. The *dimension* of $\mathfrak{a}$ is $\dim(\mathfrak{a}) = l + |\mathfrak{a}| = \sum_{i=1}^{l} \dim(\mathfrak{col}_i)$.

For an array $\mathfrak{a}$ of weight h, let $e^{\mathfrak{a}}$ be the subspace of $SP^h(\mathbb{C})$ of configurations satisfying the following conditions:

- there are exactly l distinct vertical lines in $\mathbb{C}$, of the form $\Re(z) = x_i$ for some real numbers $-\infty < x_1 < \cdots < x_l < +\infty$, containing at least one point of the configuration;
- from left to right, these lines contain exactly $\lambda_1, \ldots, \lambda_l$ distinct points;
- for each $1 \leq i \leq l$, the multiplicities of the λ_i points lying on the line $\Re(z) = x_i$ are, from bottom to top, exactly $m_{i,1}, \ldots, m_{i,\lambda_i}$ (see Figure 3.4).

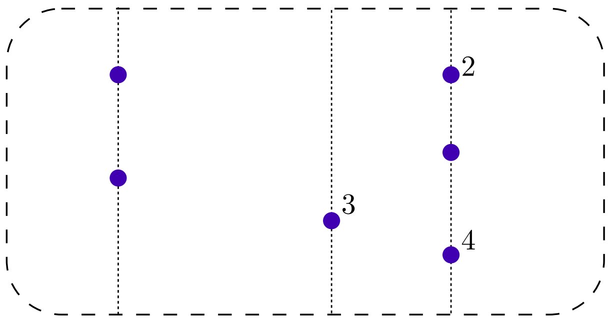

Figure 3.4. A configuration $P \in SP^{12}(\mathbb{C})$ in the space $e^{\mathfrak{a}}$, where $\mathfrak{a} = (\mathfrak{col}_1, \mathfrak{col}_2, \mathfrak{col}_3)$, $\mathfrak{col}_1 = (2; (1,1))$, $\mathfrak{col}_2 = (1; (3))$ and $\mathfrak{col}_3 = (3; (4,1,2))$.

There is a unique array $\mathfrak{a}$ of length 0, namely the *empty array* $\emptyset$. The corresponding space $e^{\emptyset}$ consists of the unique configuration in $SP^0(\mathbb{C})$. From now on, unless explicitly stated otherwise, we will only consider arrays of length $l \geq 1$.

3.2.2 Characteristic maps of cells

The space $e^{\mathfrak{a}}$ is homeomorphic to *the interior* of the following multisimplex (see Definition 2.1.2):

$$\Delta^{\mathfrak{a}} = \Delta^l \times \prod_{i=1}^{l} \Delta^{\lambda_i}.$$

The homeomorphism is given as follows:

- the local coordinates of the first factor Δ^l are the positions $x_1^{hor}, \ldots, x_l^{hor}$ of the vertical lines in $\mathbb{C}$ containing points of the configuration; these are also called *horizontal* coordinates;
- for $1 \leq i \leq l$, the local coordinates of the factor Δ^{λ_i} are the positions $x_{i,1}^{ver}, \ldots, x_{i,\lambda_i}^{ver}$ of the λ_i points lying on the vertical line $\Re(z) = x_i^{hor}$; these are also called *vertical* coordinates.

Note that the geometric dimension of $\Delta^{\mathfrak{a}}$ agrees with the dimension of $\mathfrak{a}$. The embedding $\mathring{\Delta}^{\mathfrak{a}} \cong e^{\mathfrak{a}} \hookrightarrow SP^h(\mathbb{C})^\infty$ extends to a continuous map $\Phi^{\mathfrak{a}} \colon \Delta^{\mathfrak{a}} \to SP^h(\mathbb{C})^\infty$, so that the image of $\partial\Delta^{\mathfrak{a}}$ is contained in the union of all subspaces $e^{\mathfrak{a}'}$ for arrays $\mathfrak{a}'$ with lower dimension than $\mathfrak{a}$, together with the point ∞.
The construction of the map $\Phi^{\mathfrak{a}}$ is as follows:

1. we regard the one-point compactification $\mathbb{C}^\infty$ of $\mathbb{C}$ as the quotient of $[-\infty, +\infty]^2$ that collapses $[-\infty, +\infty]^2 \setminus \mathbb{C}$ to the point ∞;

2. the homeomorphism $\mathring{\Delta}^{\mathfrak{a}} \to e^{\mathfrak{a}} \subset SP^h(\mathbb{C})$ extends now to a map $\tilde{\Phi}^{\mathfrak{a}} \colon \Delta^{\mathfrak{a}} \to SP^h\left([-\infty, +\infty]^2\right)$, using the same interpretation of local coordinates as above; if points collide, multiplicities are summed;

3. we can then further project $SP^h\left([-\infty, +\infty]^2\right)$ to $SP^h(\mathbb{C})^\infty$: the latter space is regarded as the quotient of $SP^h([-\infty, +\infty]^2)$ which collapses to ∞ the subspace $SP^h([-\infty, +\infty]^2) \setminus SP^h(\mathbb{C})$; the composition is the desired map $\Phi^{\mathfrak{a}} \colon \Delta^{\mathfrak{a}} \to SP^h(\mathbb{C})^\infty$.

Therefore the collection of subspaces $e^{\mathfrak{a}}$, for $\mathfrak{a}$ varying among arrays of weight h, together with the 0-cell ∞, gives a cell decomposition of $SP^h(\mathbb{C})^\infty$. The characteristic maps of the cells are the maps $\Phi^{\mathfrak{a}}$.
We remark that there is a natural quotient map $SP^h(\mathbb{C})^\infty \to C(\mathbb{C}; h)^\infty$, which collapses the closed subcomplex $SP^h(\mathbb{C})^\infty \setminus C(\mathbb{C}; h)$ to a point. Fuchs' cell structure on $C(\mathbb{C}; h)^\infty$ is precisely the one induced by this quotient map.

3.2.3 Boundary restrictions of characteristic maps

In this subsection we analyse more carefully the restriction of a characteristic map $\Phi^{\mathfrak{a}}$ to the boundary $\partial\Delta^{\mathfrak{a}}$. We restrict $\Phi^{\mathfrak{a}}$ to all faces of the multisimplex $\Delta^{\mathfrak{a}}$.

Definition 3.2.5. We introduce a new notation for the faces of the multisimplex $\Delta^{\mathfrak{a}}$ associated with an array $\mathfrak{a}$; compare it with Definition 2.1.2.
For all $0 \leq r \leq l$ we call $\partial^{hor}_r \Delta^{\mathfrak{a}}$ the face

$$\left(\partial_r \Delta^l\right) \times \Delta^{\lambda_1} \times \cdots \times \Delta^{\lambda_l} \subset \Delta^{\mathfrak{a}}.$$

This is also called a *horizontal face* of $\Delta^{\mathfrak{a}}$; for $r = 0$ or l we say that it is an *outer* face; for $1 \leq r \leq l-1$ it is an *inner* face.
For all $1 \leq r \leq l$ and $0 \leq s \leq \lambda_i$ we call $\partial^{ver}_{r,s} \Delta^{\mathfrak{a}}$ the face

$$\Delta^l \times \Delta^{\lambda_1} \times \cdots \times \partial_s \Delta^{\lambda_r} \times \cdots \times \Delta^{\lambda_l} \subset \Delta^{\mathfrak{a}}.$$

This is also called a *vertical face* of $\Delta^{\mathfrak{a}}$; for $s = 0$ or λ_r we say that it is an *outer* face; for $1 \leq s \leq \lambda_r - 1$ it is an *inner* face.

We note that $\Phi^{\mathfrak{a}}$ restricts to the constant map to $\infty \in SP^h(\mathbb{C})^\infty$ on every outer face, both horizontal and vertical. To see this, consider the map $\tilde{\Phi}^{\mathfrak{a}} \colon \Delta^{\mathfrak{a}} \to SP^h\left([-\infty,+\infty]^2\right)$ used to define $\Phi^{\mathfrak{a}}$; note that every outer face is mapped to the subspace of $SP^h\left([-\infty,+\infty]^2\right)$ of configurations in which at least one point lies on the boundary of the square $[-\infty,+\infty]^2$. These configurations are then mapped to $\infty \in SP^h(\mathbb{C})^\infty$ under the quotient map

$$SP^h\left([-\infty,+\infty]^2\right) \to SP^h(\mathbb{C})^\infty.$$

Lemma 3.2.6. *For all $1 \le r \le l$ and $1 \le s \le \lambda_r - 1$, the restriction of $\Phi^{\mathfrak{a}}$ to the inner, vertical face $\partial^{ver}_{r,s}\Delta^{\mathfrak{a}}$ is the characteristic map $\Phi^{\mathfrak{a}'}$ associated with the following array $\mathfrak{a}' = (l, \underline{\mathfrak{col}}')$:*

- $\mathfrak{col}'_i = \mathfrak{col}_i$ *for all $1 \le i \le l$ with $i \ne r$;*
- $\lambda'_r = \lambda_r - 1$*;*
- $\underline{m}'_r$ *is defined by $m'_{r,j} = m_{r,j}$ for $1 \le j \le s-1$, $m'_{r,s} = m_{r,s} + m_{r,s+1}$, $m'_{r,j} = m_{r,j+1}$ for all $s+1 \le j \le \lambda_r - 1$. .*

Proof. Recall the definition of the maps $\Phi^{\mathfrak{a}}$ and $\Phi^{\mathfrak{a}'}$. The statement follows from the observation that the restriction of $\tilde{\Phi}^{\mathfrak{a}}$ to $\partial^{ver}_{r,s}\Delta^{\mathfrak{a}}$ coincides with the map $\tilde{\Phi}^{\mathfrak{a}'}$: the coordinates $x^{ver}_{r,s}$ and $x^{ver}_{r,s+1}$ of $\Delta^{\mathfrak{a}}$ are equal on $\partial^{ver}_{r,s}\Delta^{\mathfrak{a}}$, therefore the $s^{\text{th}}$and the $(s+1)^{\text{th}}$points on the r^{th}vertical line, carrying multiplicities $m_{r,s}$ and $m_{r,s+1}$ respectively, collapse to a single point of multiplicity $m'_{r,s} = m_{r,s} + m_{r,s+1}$. □

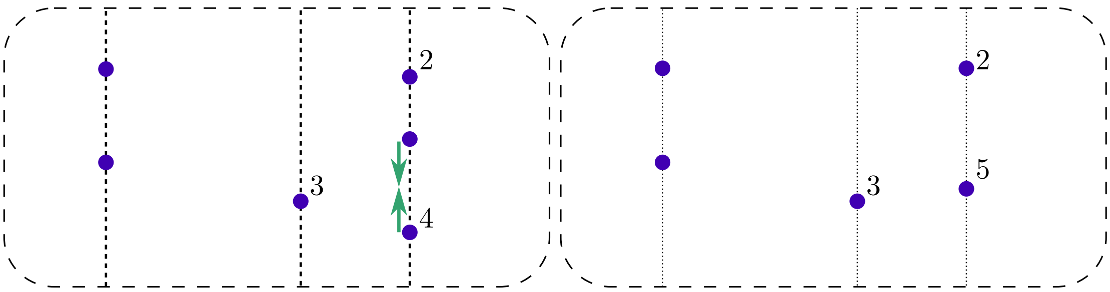

Figure 3.5. The face map $\partial^{ver}_{3,1}$ applied to the array $\mathfrak{a}$ from Figure 3.4 yields the array $\mathfrak{a}' = (\mathfrak{col}'_1, \mathfrak{col}'_2, \mathfrak{col}'_3)$, with $\mathfrak{col}'_1 = (2;(1,1))$, $\mathfrak{col}'_2 = (1;(3))$ and $\mathfrak{col}'_3 = (2;(5,2))$.

For $1 \le r \le l-1$ the restriction of $\Phi^{\mathfrak{a}}$ to the inner, horizontal face $\partial^{hor}_r\Delta^{\mathfrak{a}}$ is a more complicated map that we describe in the following.
Recall the Eilenberg-Zilber shuffle product decomposition discussed in Subsection 2.1.2, and apply it to the product of simplices $\Delta^{\lambda_r} \times \Delta^{\lambda_{r+1}}$. Denote by $\triangle_\eta$ the $(\lambda_r + \lambda_{r+1})$-subsimplex of $\Delta^{\lambda_r} \times \Delta^{\lambda_{r+1}}$ corresponding to the shuffle η of the sets $A = \{(r,s)\}_{1 \le s \le \lambda_r}$ and $B = \{(r+1,s)\}_{1 \le s \le \lambda_{r+1}}$.
We can find a corresponding decomposition of the face $\partial^{hor}_r\Delta^{\mathfrak{a}}$ in $\binom{\lambda_r+\lambda_{r+1}}{\lambda_r}$ multisimplices

$$\partial^{hor}_{r,\eta}\Delta^{\mathfrak{a}} = \partial^{hor}_r\Delta^l \times \Delta^{\lambda_1} \times \cdots \times \Delta^{\lambda_{r-1}} \times \triangle_\eta \times \Delta^{\lambda_{r+1}} \times \cdots \times \Delta^{\lambda_l}.$$

Lemma 3.2.7. *The restriction of $\Phi^{\mathfrak{a}}$ to the multisimplex $\partial^{hor}_{r,\eta}\Delta^{\mathfrak{a}} \subset \partial^{hor}_{r}\Delta^{\mathfrak{a}}$ is the characteristic map $\Phi^{\mathfrak{a}}$ associated with the following array $\mathfrak{a}' = (l-1, \underline{\mathfrak{col}'})$.*

- $\mathfrak{col}'_i = \mathfrak{col}_i$ *for all* $1 \leq i \leq r-1$*, and* $\mathfrak{col}'_i = \mathfrak{col}_{i+1}$ *for all* $r+1 \leq i \leq l-1$*;*
- $\mathfrak{col}'_r$ *is the amalgamation of the columns* $\mathfrak{col}_r$ *and* $\mathfrak{col}_{r+1}$ *along* η *(see Definition 3.2.3).*

Proof. As in the proof of Lemma 3.2.6, we will argue that already the restriction of $\tilde{\Phi}^{\mathfrak{a}}$ to $\partial^{hor}_{r,\eta}\Delta^{\mathfrak{a}}$ coincides with $\tilde{\Phi}^{\mathfrak{a}'}$. The coordinates x^{hor}_r and x^{hor}_{r+1} are equal, therefore the $r^{\text{th}}$and the $(r+1)$-st vertical lines of the configurations in $\tilde{\Phi}^{\mathfrak{a}}(\partial^{hor}_r\Delta^{\mathfrak{a}})$, and the λ_r points of the configuration parametrised by the coordinates $(x^{hor}_r, x^{ver}_{r,j})_{1\leq j\leq\lambda_r}$ are shuffled in some order on this vertical line with the λ_{r+1} points with coordinates $(x^{hor}_{r+1}, x^{ver}_{r+1,j})_{1\leq j\leq\lambda_{r+1}}$. In particular, if we further restrict $\tilde{\Phi}^{\mathfrak{a}}$ to the multisimplex $\partial^{hor}_{r,\eta}\Delta^{\mathfrak{a}}$, the multiplicities $(m_{r,j})_{1\leq j\leq\lambda_r}$ and $(m_{r+1,j})_{1\leq j\leq\lambda_{r+1}}$ are sorted on the r^{th}vertical line just as the coordinates $(x^{ver}_{r,j})_{1\leq j\leq\lambda_r}$ and $(x^{ver}_{r+1,j})_{1\leq j\leq\lambda_{r+1}}$ are sorted on $[-\infty,+\infty]$, and this information is precisely given by the shuffle η (see Figure 3.6). □

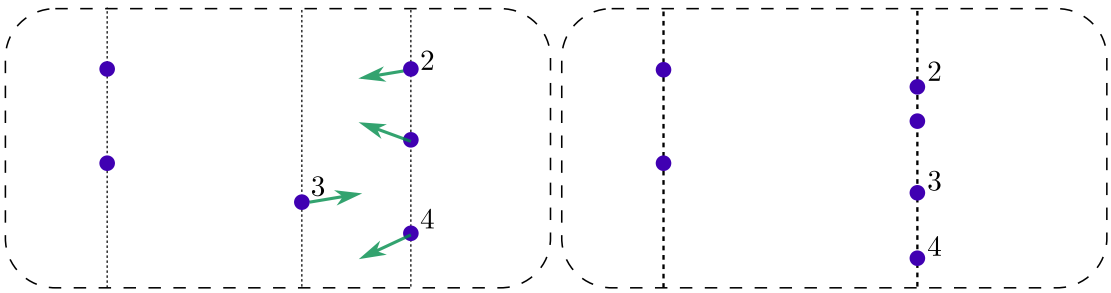

Figure 3.6. Let $\eta \in \mathfrak{Shuf}(1,3)$ assemble two ordered sets $A = \{a_1\}$ and $B = \{b_1 \prec_B b_2 \prec_B b_3\}$ into an ordered set $C = \{b_1 \prec a_1 \prec b_2 \prec b_3\}$. Then the face map $\partial^{hor}_{2,\eta}$ applied to the array $\mathfrak{a}$ from Figure 3.4 yields the array $\mathfrak{a}' = (\mathfrak{col}'_1, \mathfrak{col}'_2)$, with $\mathfrak{col}'_1 = (2;(1,1))$ and $\mathfrak{col}'_2 = (4;(4,3,1,2))$.

3.2.4 Absolute value filtration

Recall Definition 3.0.1; we introduce a filtration on the space $SP^h(\mathbb{C})^\infty$.

Definition 3.2.8. We set the absolute value of $\infty \in SP^h(\mathbb{C})^\infty$ to be $|\infty| = 0$. The absolute value gives a filtration in closed subcomplexes on the space $SP^h(\mathbb{C})^\infty$.
For $0 \leq k \leq h$ we define the k^{th}*filtration level* $F^{|\cdot|}_k SP^h(\mathbb{C})^\infty$ as the subspace of configurations $P \in SP^h(\mathbb{C})^\infty$ satisfying $|P| \leq k$: it is the union of ∞ and of all cells $e^{\mathfrak{a}}$ corresponding to arrays $\mathfrak{a}$ with $|\mathfrak{a}| \leq k$. We set $F^{|\cdot|}_{-1}SP^h(\mathbb{C})^\infty = \emptyset$. For $0 \leq k \leq h$, the k^{th}*filtration quotient* of $SP^h(\mathbb{C})^\infty$ is

$$F^{|\cdot|}_k/F^{|\cdot|}_{k-1}SP^h(\mathbb{C})^\infty = \left(F^{|\cdot|}_k SP^h(\mathbb{C})^\infty\right) / \left(F^{|\cdot|}_{k-1}SP^h(\mathbb{C})^\infty\right),$$

and the k^{th} *filtration stratum* of $SP^h(\mathbb{C})^\infty$ is the difference

$$\mathfrak{F}_k^{|\cdot|} SP^h(\mathbb{C})^\infty = \left(F_k^{|\cdot|} SP^h(\mathbb{C})^\infty\right) \setminus \left(F_{k-1}^{|\cdot|} SP^h(\mathbb{C})^\infty\right).$$

For $k \geq 1$ note that the stratum $\mathfrak{F}_k^{|\cdot|} SP^h(\mathbb{C})^\infty$ is contained in $SP^h(\mathbb{C})$, i.e. it avoids the point ∞: we will also denote it as $\mathfrak{F}_k^{|\cdot|} SP^h(\mathbb{C})$.

Definition 3.2.9. Let $\underline{\alpha} = (\alpha_j)_{j\geq 1}$ be a sequence of numbers $\alpha_j \geq 0$, satisfying the following conditions:

- all but finitely many α_j are equal to 0;
- $\sum_{j=1}^{\infty} \alpha_j = k$;
- $\sum_{j=1}^{\infty} j\alpha_j = h$.

We associate with $\underline{\alpha}$ a connected component of $\mathfrak{F}_k^{|\cdot|} SP^h(\mathbb{C})$, denoted $C(\mathbb{C}; \underline{\alpha}) \subset SP^h(\mathbb{C})$: it contains configurations $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$ such that, for all $j \geq 1$, there exist exactly α_j indices $1 \leq i \leq k$ with $m_i = j$.

The space $C(\mathbb{C}; \underline{\alpha})$ is a finite connected covering of the configuration space $C(\mathbb{C}; k)$ of k unordered points in the plane. It was proved in [18] that $C(\mathbb{C}; k)$ is a classifying space for Artin's braid group on k strands, which was introduced in [4]. The space $C(\mathbb{C}; \underline{\alpha})$ can be regarded as a coloured version of $C(\mathbb{C}; k)$, in which for every $j \geq 0$ there are α_j points of *colour* j; points of the same colour are indistinguishable. Note that, in the particular case $\alpha_1 = h$ and $\alpha_j = 0$ for all $j \geq 2$, we recover the space $C(\mathbb{C}; h) \subset SP^h(\mathbb{C})$ which coincides with the open, dense stratum $\mathfrak{F}_h^{|\cdot|} SP^h(\mathbb{C})$.

Lemma 3.2.10. *The restriction of the sheaf $\mathfrak{G}$ over $C(\mathbb{C}; \underline{\alpha})$ is a covering space.*

Proof. Let $P \in C(\mathbb{C}; \underline{\alpha}) \subset SP^h(\mathbb{C})$ and let $\mathcal{U}$ be a normal neighbourhood of P in $SP^h(\mathbb{C})$; we use the notation from Definition 3.1.3. Then the intersection $\mathcal{U} \cap C(\mathbb{C}; \underline{\alpha})$ is canonically identified with the product $U_1 \times \cdots \times U_k$: any configuration $P' \in \mathcal{U} \cap C(\mathbb{C}; \underline{\alpha})$ has the form $P = \{m_1 \cdot z'_1, \dots, m_k \cdot z'_k\}$ with $z'_i \in U_i$. In particular there is an isomorphism $\mathfrak{G}(P) \simeq \mathfrak{G}(\mathcal{U}) \xrightarrow{\simeq} \mathfrak{G}(P')$, and this gives a trivialisation of the sheaf $\mathfrak{G}$ over the open set $\mathcal{U} \cap C(\mathbb{C}; \underline{\alpha})$. □

3.3 Standard generating sets

In this section we construct in a continuous way a generating set for the free group $\mathfrak{G}(P)$, with P ranging in an open cell $e^{\mathfrak{a}}$.

3.3.1 Definition of the standard generating set

Definition 3.3.1. Use the notation from Definition 3.2.4, recall Definition 3.1.5 and fix a cell $e^{\mathfrak{a}} \subset SP^h(\mathbb{C})$ corresponding to an array $\mathfrak{a} = (l, \underline{\mathfrak{col}})$. Let $P \in e^{\mathfrak{a}}$, and represent P as

$$P = \{m_{i,j} \cdot z_{i,j}\}_{1 \le i \le l, 1 \le j \le \lambda_i},$$

where $z_{i,j}$ is the j^{th}point from the bottom on the i^{th}vertical line from left, among all points in P with a positive multiplicity, and $m_{i,j} > 0$ is the multiplicity of $z_{i,j}$ in P.
Let $\beta_{i,j}$ be a small circle in $\mathbb{C} \setminus P$ spinning clockwise around $z_{i,j}$, and assume that the circles $\beta_{i,j}$ are disjoint. We describe arcs $\bar{\zeta}_{i,j}$ in $\mathbb{C} \setminus P$ as follows: the arc $\bar{\zeta}_{i,j}$ begins at $*$, moves horizontally to a position just *on right* of the vertical line on which $z_{i,j}$ lies, then runs upwards until it reaches the height of $z_{i,j}$, and finally runs towards left and reaches the point $z_{i,j}$. Up to perturbing a little the arcs $\bar{\zeta}_{i,j}$ in $\mathbb{C} \setminus P$ we can assume that all requirements from Definition 3.1.5 are fulfilled.

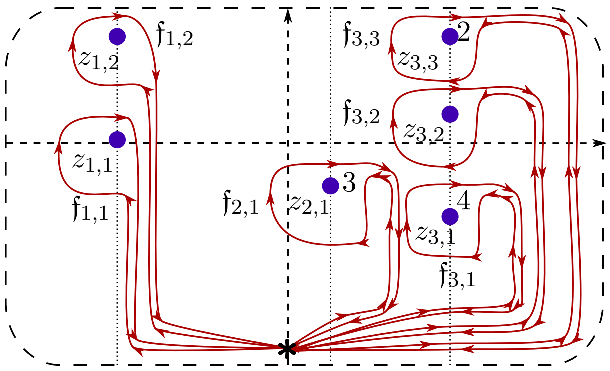

Figure 3.7. The standard generating set for $\pi_1(\mathbb{C} \setminus P, *_P)$, where P is the configuration from Figure 3.4

The associated admissible generating set $(\mathfrak{f}^P_{i,j})_{1 \le i \le l, 1 \le j \le \lambda_i}$ is called the *standard generating set* of $\pi_1(\mathbb{C} \setminus P, *)$. We abbreviate

$$\underline{\mathfrak{f}} = \underline{\mathfrak{f}}^P = (\mathfrak{f}^P_{i,j})_{1 \le i \le l, 1 \le j \le \lambda_i}.$$

We regard every $\mathfrak{f}_{i,j}$ as a section of the sheaf $\mathfrak{G}$ over the space $e^{\mathfrak{a}}$ (see Figure 3.7).

Let $P \in e^{\mathfrak{a}}$ and consider, for all $1 \le i \le l$ and $1 \le j \le \lambda_i$, the element $\mathfrak{f}^P_{i,j} \in \mathfrak{G}(P)$. By Lemma 3.1.4, in some neighbourhood $\mathcal{U} \subset SP^h(\mathbb{C})$ of P there is a unique extension of $\mathfrak{f}^P_{i,j}$ to a section $\mathfrak{f}^{P,\mathcal{U}}_{i,j} : \mathcal{U} \to \mathfrak{G}$.
Note however that if $P' \in \mathcal{U}$ is a configuration lying outside $e^{\mathfrak{a}}$, with $|P'| > |P|$, then the elements

$$\left(\mathfrak{f}^{P,\mathcal{U}}_{i,j}(P')\right)_{1 \le i \le l, 1 \le j \le \lambda_i} \in \mathfrak{G}(P')$$

do not form a generating set of the free group $\mathfrak{G}(P')$, because this free group has higher rank than $\mathfrak{G}(P)$. Even assuming $|P'| = |P|$, the elements $\mathfrak{f}^{P,\mathcal{U}}_{i,j}(P')$ give in general an *admissible* generating set for $\mathfrak{G}(P')$, but not the standard generating set of $\mathfrak{G}(P')$.

3.3.2 Boundary maps and standard generating sets

In the following we investigate how the standard generating set changes when passing from the interior of a cell $e^{\mathfrak{a}}$ to its boundary. Use the notation from Definition 3.2.4.
We denote by $\mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{a}} \subset \Delta^{\mathfrak{a}}$ the interior of the face $\partial^{ver}_{r,s}\Delta^{\mathfrak{a}} \subset \Delta^{\mathfrak{a}}$; similarly $\mathring{\partial}^{hor}_{r,\eta}\Delta^{\mathfrak{a}}$ denotes the interior of $\partial^{hor}_{r,\eta}\Delta^{\mathfrak{a}} \subset \Delta^{\mathfrak{a}}$.
We extend the sheaf in groups $\mathfrak{G}$, defined over $SP^h(\mathbb{C})$, to the space $SP^h(\mathbb{C})^\infty$, by declaring the stalk $\mathfrak{G}(\infty)$ to be the trivial group: for an open, connected set $\infty \in \mathcal{U} \subset SP^h(\mathbb{C})^\infty$, the only section in $\mathfrak{G}(\mathcal{U})$ is the trivial section. This extension makes the following discussion a little easier.
Consider the map $\Phi^{\mathfrak{a}}\colon \Delta^{\mathfrak{a}} \to SP^h(\mathbb{C})^\infty$ and denote by $\mathfrak{G}^{\mathfrak{a}} = (\Phi^{\mathfrak{a}})^*\,\mathfrak{G}$ the pullback sheaf in groups over $\Delta^{\mathfrak{a}}$. For $1 \le i \le l$ and $1 \le j \le \lambda_i$ there is a section $\mathfrak{f}^{\mathfrak{a}}_{i,j}\colon \mathring{\Delta}^{\mathfrak{a}} \to \mathfrak{G}^{\mathfrak{a}}$ defined on *the interior* of the multisimplex $\Delta^{\mathfrak{a}}$: $\mathfrak{f}^{\mathfrak{a}}_{i,j}$ is the pullback of the section $\mathfrak{f}_{i,j}$ of $\mathfrak{G}$ over $e^{\mathfrak{a}}$.
Consider an inner, vertical face $\partial^{ver}_{r,s}\Delta^{\mathfrak{a}}$, for some $1 \le r \le l$ and $1 \le s \le \lambda_r - 1$. Recall from Lemma 3.2.6 that the restriction of $\Phi^{\mathfrak{a}}$ to $\partial^{ver}_{r,s}\Delta^{\mathfrak{a}}$ can be identified with the characteristic map $\Phi^{\mathfrak{a}'}$ of another cell $e^{\mathfrak{a}'}$. We use the notation from Lemma 3.2.6, and in the following we consider $\Delta^{\mathfrak{a}'} \cong \partial^{ver}_{r,s}\Delta^{\mathfrak{a}}$ as a subspace of $\Delta^{\mathfrak{a}}$: in particular we identify $\mathfrak{G}^{\mathfrak{a}'}$ with the restriction of the sheaf $\mathfrak{G}^{\mathfrak{a}}$ to the space $\Delta^{\mathfrak{a}'}$.

Lemma 3.3.2. *The following sections $\mathfrak{s}\colon \mathring{\Delta}^{\mathfrak{a}} \to \mathfrak{G}^{\mathfrak{a}}$ extend to continuous sections of $\mathfrak{G}^{\mathfrak{a}}$ over $\mathring{\Delta}^{\mathfrak{a}} \cup \mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{a}}$ by adjoining the corresponding sections $\mathfrak{s}'\colon \mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{a}} \to \mathfrak{G}^{\mathfrak{a}}$.*

- *For $1 \le i \le l$, $i \ne r$ and $1 \le j \le \lambda_i$, the section $\mathfrak{s} = \mathfrak{f}^{\mathfrak{a}}_{i,j}$ extends with the section $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{i,j}$.*
- *For $1 \le j \le s-1$, the section $\mathfrak{s} = \mathfrak{f}^{\mathfrak{a}}_{r,j}$ extends with the section $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{r,j}$; for $s+2 \le j \le \lambda_r$, the section $\mathfrak{s} = \mathfrak{f}^{\mathfrak{a}}_{r,j}$ extends with the section $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{r,j-1}$.*
- *The section $\mathfrak{s} = \mathfrak{f}^{\mathfrak{a}}_{r,s} \cdot \mathfrak{f}^{\mathfrak{a}}_{r,s+1}$ extends with the section $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{r,s}$.*

Proof. In all cases we have to prove continuity of $\mathfrak{s} \cup \mathfrak{s}'$ at points in $\mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{a}}$. Consider a point $\mathbb{P}' \in \mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{a}}$, let $\mathcal{U}$ be a normal neighbourhood of $P' := \Phi^{\mathfrak{a}}(\mathbb{P}')$ in $SP^h(\mathbb{C})$; denote by $\mathcal{U}^{\mathfrak{a}}$ the connected component of $(\Phi^{\mathfrak{a}})^{-1}(\mathcal{U}) \subset \Delta^{\mathfrak{a}}$ containing $\mathbb{P}'$.
By Lemma 3.1.4, $\mathfrak{s}'(\mathbb{P}')$ admits a unique extension $\mathfrak{s}'_{\mathcal{U}^{\mathfrak{a}}}\colon \mathcal{U}^{\mathfrak{a}} \to \mathfrak{G}^{\mathfrak{a}}$. If we prove that $\mathfrak{s}'_{\mathcal{U}^{\mathfrak{a}}}$ coincides with $\mathfrak{s}$ over $\mathcal{U}^{\mathfrak{a}} \cap \mathring{\Delta}^{\mathfrak{a}}$, we have proved continuity of $\mathfrak{s} \cup \mathfrak{s}'$ on $\mathcal{U}^{\mathfrak{a}}$, and hence continuity at $\mathbb{P}'$.
Note now that the restriction of $\mathfrak{G}^{\mathfrak{a}}$ over $\mathring{\Delta}^{\mathfrak{a}}$ is a (trivial) covering space, see also Lemma 3.2.10. Since $\mathcal{U}^{\mathfrak{a}} \cap \mathring{\Delta}^{\mathfrak{a}}$ is path-connected, the two sections $\mathfrak{s}'_{\mathcal{U}^{\mathfrak{a}}}$ and $\mathfrak{s}$ agree on the entire $\mathcal{U}^{\mathfrak{a}} \cap \mathring{\Delta}^{\mathfrak{a}}$ if and only if they agree on any chosen point $\mathbb{P} \in \mathcal{U}^{\mathfrak{a}} \cap \mathring{\Delta}^{\mathfrak{a}}$.
We choose $\mathbb{P}$ as follows. We express $\mathbb{P}'$ in the coordinates of $\Delta^{\mathfrak{a}}$

$$\mathbb{P}' = \left(\left(x^{hor}_i|_{\mathbb{P}'} \right)_{1 \le i \le l} ; \left(x^{ver}_{i,j}|_{\mathbb{P}'} \right)_{1 \le i \le l, 1 \le j \le \lambda_i} \right),$$

where $x^{ver}_{r,s}|_{\mathbb{P}'} = x^{ver}_{r,s+1}|_{\mathbb{P}'}$ because $\mathbb{P}' \in \mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{a}}$. The coordinates of $\mathbb{P}$ are then equal to those of $\mathbb{P}'$, except $x^{ver}_{r,s}(\mathbb{P})$, which is changed to a slightly *lower* value than $x^{ver}_{r,s}(\mathbb{P}')$ (see Figure 3.8).

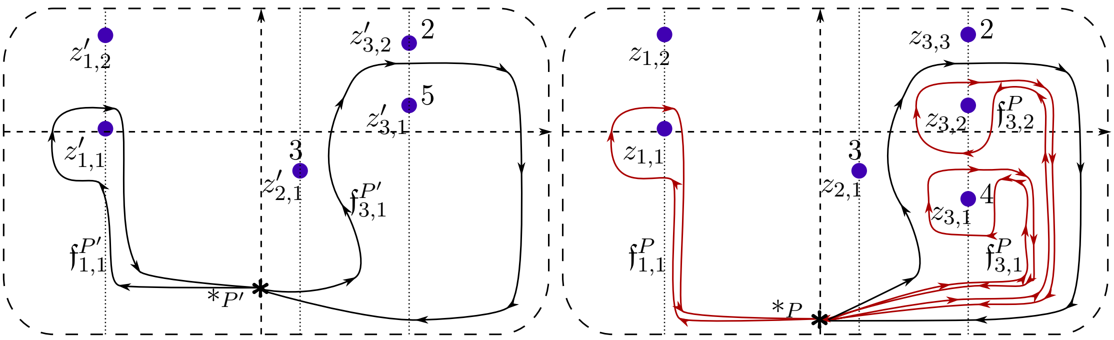

Figure 3.8. Let $\mathfrak{a}$ be the array from Figure 3.4. On left, a configuration $P' = \Phi^{\mathfrak{a}}(\mathbb{P}')$, for $\mathbb{P}' \in \mathring{\partial}^{ver}_{3,1}\Delta^{\mathfrak{a}}$. On right, its perturbation $P = \Phi^{\mathfrak{a}}(\mathbb{P})$, for $\mathbb{P} \in \mathring{\Delta}^{\mathfrak{a}}$. The element $\mathfrak{f}^{P'}_{1,1} \in \mathfrak{G}(P')$ is perturbed to the element $\mathfrak{f}^{P}_{1,1} \in \mathfrak{G}(P)$. The element $\mathfrak{f}^{P'}_{3,1}$ is perturbed to the element $\mathfrak{f}^{P}_{3,1} \cdot \mathfrak{f}^{P}_{3,2}$.

Then the configurations $P' = \Phi^{\mathfrak{a}}(\mathbb{P}')$ and $P := \Phi^{\mathfrak{a}}(\mathbb{P})$ in $SP^h(\mathbb{C})$ are equal, except for the following fact: the summand $m'_{r,s} \cdot z'_{r,s}$ in P' is split as $m_{r,s} \cdot z_{r,s}$ and $m_{r,s+1} \cdot z_{r,s+1}$ in P, using the notation from Definition 3.3.1. Moreover $z_{r,s+1} \equiv z'_{r,s} \in \mathbb{C}$ and $z_{r,s}$ lies on the same vertical line as $z_{r,s+1}$, slightly below it.
In particular the little circle $\beta'_{r,s}$ spinning around $z'_{r,s}$ becomes, after passing from P' to P, a little circle spinning around $z_{r,s}$ and $z_{r,s+1}$ (see Definition 3.1.5).
The result follows in the case $\mathfrak{s} = \mathfrak{f}^{\mathfrak{a}}_{r,s} \cdot \mathfrak{f}^{\mathfrak{a}}_{r,s+1}$ and $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{r,s}$; in all other cases it is a straightforward consequence of Definition 3.3.1. □

Consider now an inner, horizontal face $\partial^{hor}_r\Delta^{\mathfrak{a}}$, for some $1 \leq r \leq l-1$, and let $\partial^{hor}_{r,\eta}\Delta^{\mathfrak{a}} \subset \Delta^{\mathfrak{a}}$ be as in the discussion before Lemma 3.2.7, for some shuffle η of the sets $A = \{(r,s)\,|\,1 \leq s \leq \lambda_r\}$ and $B = \{(r+1,s)\,|\,1 \leq s \leq \lambda_{r+1}\}$. By Lemma 3.2.7 we can identify the restriction of $\Phi^{\mathfrak{a}}$ on $\partial^{hor}_{r,\eta}\Delta^{\mathfrak{a}}$ with the characteristic map $\Phi^{\mathfrak{a}'}$ of another cell $e^{\mathfrak{a}'}$. As before we consider $\Delta^{\mathfrak{a}'} \cong \partial^{hor}_{r,\eta}\Delta^{\mathfrak{a}}$ as a subspace of $\Delta^{\mathfrak{a}}$: in particular we identify $\mathfrak{G}^{\mathfrak{a}'}$ with the restriction of $\mathfrak{G}^{\mathfrak{a}}$ to $\Delta^{\mathfrak{a}'}$.

Lemma 3.3.3. *Let η be a shuffle of the ordered sets $A = \{(r,j)\}_{1\leq j\leq\lambda_r}$ and $B = \{(r+1,j)\}_{1\leq j\leq\lambda_{r+1}}$ into the ordered set $C = \{(r,j)\}_{1\leq j\leq\lambda'_r}$, where $\lambda'_r = \lambda_r + \lambda_{r+1}$. In particular we have two injective, increasing maps $\eta_1\colon A \hookrightarrow C$ and $\eta_2\colon B \hookrightarrow C$ with disjoint images.*
The following sections $\mathfrak{s}\colon \mathring{\Delta}^{\mathfrak{a}} \to \mathfrak{G}^{\mathfrak{a}}$ extend to continuous sections of $\mathfrak{G}^{\mathfrak{a}}$ over the union $\mathring{\Delta}^{\mathfrak{a}} \cup \mathring{\partial}^{hor}_{r,\eta}\Delta^{\mathfrak{a}}$ by adjoining the corresponding sections $\mathfrak{s}'\colon \mathring{\partial}^{hor}_{r,\eta}\Delta^{\mathfrak{a}} \to \mathfrak{G}^{\mathfrak{a}}$.

- *For $i \leq r-1$ and $1 \leq j \leq \lambda_i$, the section $\mathfrak{s} = \mathfrak{f}^{\mathfrak{a}}_{i,j}$ extends with the section $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{i,j}$; for $i \geq r+2$ and $1 \leq j \leq \lambda_i$, the section $\mathfrak{s} = \mathfrak{f}^{\mathfrak{a}}_{i,j}$ extends with the section $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{i-1,j}$.*

- *For $1 \le j \le \lambda_r$, let $(r+1,1) \prec_B \cdots \prec_B (r+1,\nu_j)$ be the list of elements of the form $(r+1,s) \in A$ such that $\eta_2(r+1,s) \prec_\eta \eta_1(r,j)$, for a suitable $0 \le \nu_j \le \lambda_{r+1}$, and define the section*
$$\xi_j = \mathfrak{f}^{\mathfrak{a}}_{r+1,1} \cdot \dots \cdot \mathfrak{f}^{\mathfrak{a}}_{r+1,\nu_j} \colon \mathring{\Delta}^{\mathfrak{a}} \to \mathfrak{G}^{\mathfrak{a}};$$
then the section $\mathfrak{s} = \xi_j^{-1} \mathfrak{f}^{\mathfrak{a}}_{r,j} \xi_j$ extends with the section $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{\eta_1(r,j)}$.

- *For $1 \le j \le \lambda_{r+1}$, the section $\mathfrak{s} = \mathfrak{f}^{\mathfrak{a}}_{r+1,j}$ extends with the section $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{\eta_2(r+1,j)}$.*

Proof. We use a similar strategy as in the proof of Lemma 3.3.2. This time the sheaf $\mathfrak{G}^{\mathfrak{a}}$ restricts to a trivial covering over the contractible space $\mathring{\Delta}^{\mathfrak{a}} \cup \mathring{\partial}^{hor}_{r,\eta} \Delta^{\mathfrak{a}}$, see also Lemma 3.2.10. Therefore it suffices to prove that the sections $\mathfrak{s}$ and $\mathfrak{s}'$ take values in the same connected component of this covering: to this purpose it suffices to check continuity of $\mathfrak{s} \cup \mathfrak{s}'$ at a single point $\mathbb{P}' \in \mathring{\partial}^{hor}_{r,\eta} \Delta^{\mathfrak{a}}$.
Fix $\mathbb{P}' \in \mathring{\partial}^{hor}_{r,\eta} \Delta^{\mathfrak{a}}$, and express $\mathbb{P}'$ in the coordinates of $\Delta^{\mathfrak{a}}$ as follows:

$$\mathbb{P}' = \left(\left(x_i^{hor}|_{\mathbb{P}'} \right)_{1 \le i \le l} ; \left(x_{i,j}^{ver}|_{\mathbb{P}'} \right)_{1 \le i \le l, 1 \le j \le \lambda_i} \right),$$

where $x_r^{hor}|_{\mathbb{P}'} = x_{r+1}^{hor}|\mathbb{P}'$ because $\mathbb{P}' \in \mathring{\partial}^{hor}_r \Delta^{\mathfrak{a}}$. Choose $\mathbb{P} \in \mathring{\Delta}^{\mathfrak{a}}$ near $\mathbb{P}'$ in a similar way as in the proof of Lemma 3.3.2: the coordinates of $\mathbb{P}$ are equal to those of $\mathbb{P}'$, except $x_{r+1}^{hor}|_{\mathbb{P}}$, which is changed to a slightly *higher* value than $x_{r+1}^{hor}|_{\mathbb{P}'}$ (see Figure 3.9).

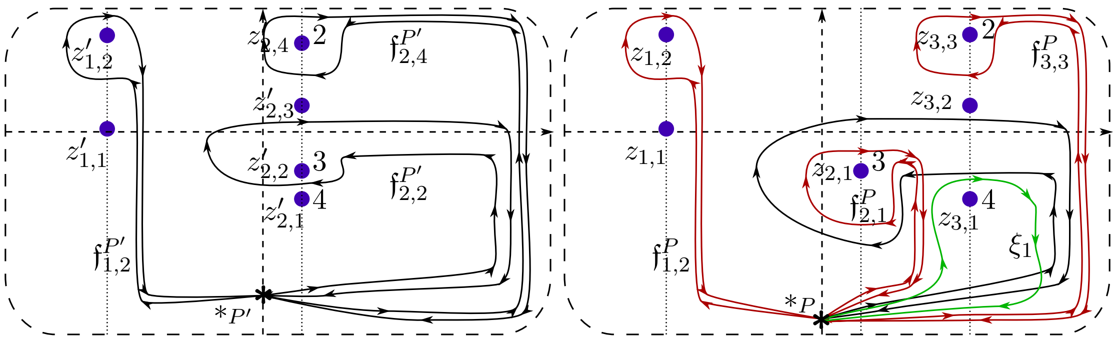

Figure 3.9. Let $\mathfrak{a}$ be the array from Figure 3.4, and let η be the shuffle from Figure 3.6. On left, a configuration $P' = \Phi^{\mathfrak{a}}(\mathbb{P}')$, for $\mathbb{P}' \in \mathring{\partial}^{hor}_{2,\eta} \Delta^{\mathfrak{a}}$. On right, its perturbation $P = \Phi^{\mathfrak{a}}(\mathbb{P})$, for $\mathbb{P} \in \mathring{\Delta}^{\mathfrak{a}}$. The element $\mathfrak{f}^{P'}_{1,2} \in \mathfrak{G}(P')$ is perturbed to the element $\mathfrak{f}^{P}_{1,2} \in \mathfrak{G}(P)$. The element $\mathfrak{f}^{P'}_{2,4}$ is perturbed to $\mathfrak{f}^{P}_{3,3}$. The element $\mathfrak{f}^{P'}_{2,2}$ is perturbed to $\xi_1^{-1} \cdot \mathfrak{f}^{P}_{2,1} \cdot \xi_1$, where $\xi_1 = \mathfrak{f}^{P}_{3,1}$ in this case.

Then the configurations $P' := \Phi^{\mathfrak{a}}(\mathbb{P}')$ and $P := \Phi^{\mathfrak{a}}(P)$ in $SP^h(\mathbb{C})$ are equal, except for the following fact: the points $\left\{ z'_{\eta_2(r+1,j)} \right\}_{1 \le j \le \lambda_{r+1}}$ of P' are slightly pushed to the right when passing to P, where they become the points $\{z_{r+1,j}\}_{1 \le j \le \lambda_{r+1}}$ lying on the $(r+1)$-st vertical line of P.
For all $1 \le j \le \lambda_r$, consider the loop $\xi_j \subset \mathbb{C} \setminus P$ described as follows: ξ_j begins at $*_P$, runs horizontally until it reaches a position just on left of the $(r+1)$-st vertical line of P,

then it runs upwards until it reaches a position just above z_{r+1,ν_j}, then it runs to right until it reaches a position just on right of the $(r+1)$-st vertical line of P, then it runs downwards until it reaches the height of $*_P$, and finally it runs back to $*_P$ horizontally. Note that ξ_j can be perturbed to a simple closed curve in $\mathbb{C} \setminus P$ which bounds a disc in $\mathbb{C}$ that contains, among all points in the configuration P, only the points $z_{r,1}, \dots, z_{r,\nu_j}$; if we regard ξ_j as an element of $\mathfrak{G}^{\mathfrak{a}}(\mathbb{P})$, then we have $\xi_j = \mathfrak{f}^{\mathfrak{a}}_{r+1,1}(\mathbb{P}) \cdot \dots \cdot \mathfrak{f}^{\mathfrak{a}}_{r+1,\nu_j}(\mathbb{P})$.
Note also that for $1 \le j \le \lambda_r$ the perturbation of P' into P transforms the arc $\bar{\zeta}'_{\eta 1(r,j)} \subset \mathbb{C} \setminus P'$, used to define $\mathfrak{f}^{\mathfrak{a}'}_{\eta 1(r,j)}(\mathbb{P}')$, to an arc $\zeta_j \subset \mathbb{C} \setminus P$ which is isotopic to the concatenation $\xi_j^{-1} \star \bar{\zeta}_{r,j}$, where $\bar{\zeta}_{r,j} \subset \mathbb{C} \setminus P$ is the arc used to define $\mathfrak{f}^{\mathfrak{a}}_{r,j}(\mathbb{P})$ (see Definition 3.3.1).
The result follows in the case $\mathfrak{s} = \xi_j^{-1} \mathfrak{f}^{\mathfrak{a}}_{r,j} \xi_j$ and $\mathfrak{s}' = \mathfrak{f}^{\mathfrak{a}'}_{\eta 1(r,j)}$; in all other cases it is a straightforward consequence of Definition 3.3.1. □

The proof of Lemma 3.3.3 justifies the following definition.

Definition 3.3.4. Let $\mathbb{G}$ be a group and let η be a shuffle of the two ordered sets $A = \{a_1 \prec_A \dots \prec_A a_n\}$ and $B = \{b_1 \prec_B \dots \prec_B b_{n'}\}$; let $\alpha \colon A \to \mathbb{G}$ and $\beta \colon B \to \mathbb{G}$ be functions of sets. The *twisted amalgamation* of α and β along η is the following function

$$\gamma = \alpha \cup_\eta \beta \colon (A \sqcup B) \to \mathbb{G}.$$

For all $b_j \in B$ we set $\gamma(b_j) = \beta(b_j)$. For $a_i \in A$ let $\{b_1 \prec_B \dots \prec_B b_{\nu_i}\} \subset B$ be the list of all elements $b_j \in B$ such that $b_j \prec_\eta a_i$, and define

$$\varepsilon_i = \beta(b_1) \cdot \dots \cdot \beta(b_{\nu_i}) \in \mathbb{G};$$

then $\gamma(a_i)$ is defined as $\varepsilon_i^{-1} \alpha(a_i) \varepsilon_i \in \mathbb{G}$.

We note that, for a fixed shuffle η, the twisted amalgamation along η gives a bijection between the sets $\mathbb{G}^A \times \mathbb{G}^B$ and $\mathbb{G}^{A \cup_\eta B}$.
We can rephrase as follows the last two statements of Lemma 3.3.3. Consider η as a shuffle of the ordered sets A and B into the set C, as in the statement of the lemma, and let $\mathbb{G}$ be the group of sections $\mathfrak{G}^{\mathfrak{a}}(\mathring{\Delta}^{\mathfrak{a}})$.
Define $\alpha \colon A \to \mathbb{G}$ by $(r,j) \mapsto \mathfrak{f}^{\mathfrak{a}}_{r,j}$ and $\beta \colon B \to \mathbb{G}$ by $(r+1,j) \mapsto \mathfrak{f}^{\mathfrak{a}}_{r+1,j}$; let $\gamma = \alpha \cup_\eta \beta \colon C \to \mathbb{G}$ be the twisted amalgamation of α and β along η. Then for all $1 \le j \le \lambda_r + \lambda_{r+1}$ the section $\gamma(r,j) \in \mathbb{G} = \mathfrak{G}^{\mathfrak{a}}(\mathring{\Delta}^{\mathfrak{a}})$ extends with the section $\mathfrak{f}^{\mathfrak{a}'}_{r,j}$ over $\mathring{\partial}^{hor}_{r,\eta} \Delta^{\mathfrak{a}}$.

3.3.3 The large loop

In this subsection we introduce an important section of the sheaf $\mathfrak{G}$, defined over the entire space $SP^h(\mathbb{C})$.

Definition 3.3.5. For all $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$ we define an element $\mathcal{L} = \mathcal{L}^P \in \mathfrak{G}(P) = \pi_1(\mathbb{C} \setminus P, *_P)$, called the *large loop*. It is represented by a simple closed curve based at $*$ that spins clockwise around all points $z_1, \dots, z_k$, i.e. this curve bounds a disc in $\mathbb{C}$ containing $z_1, \dots, z_k$ (see Figure 3.10).
The assigment $P \mapsto \mathcal{L}^P$ defines a global section $\mathcal{L}$ of the sheaf $\mathfrak{G}$.

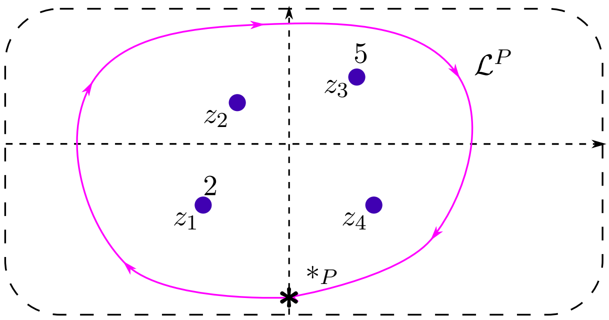

Figure 3.10. The large loop $\mathcal{L}^P \in \mathfrak{G}(P)$ for the configuration P from Figure 3.1.

We remark that for $P \in e^{\mathfrak{a}}$, using the notation from Definition 3.3.1, we can write $\mathcal{L}^P = \mathcal{L}_1^P \cdot \dots \cdot \mathcal{L}_l^P \in \mathfrak{G}(P)$, where for all $1 \le i \le l$ we define $\mathcal{L}_i^P = \mathfrak{f}_{i,1}^P \cdot \dots \cdot \mathfrak{f}_{i,\lambda_i}^P$.
Note also that any global section of the sheaf $\mathfrak{G} \to SP^h(\mathbb{C})$ must be in particular defined over configurations $P \in SP^h(\mathbb{C})$ of the form $P = \{h \cdot z\}$; in this case $\pi_1(\mathbb{C} \setminus P, *) \cong \mathbb{Z}$, generated by $\mathcal{L}^P$. Using that $SP^h(\mathbb{C})$ is connected and that germs of $\mathfrak{G}$ have locally unique extensions to sections of $\mathfrak{G}$, we obtain that all global sections of $\mathfrak{G}$ over $SP^h(\mathbb{C})$ are (possibly negative) powers of the section $\mathcal{L}\colon SP^h(\mathbb{C}) \to \mathfrak{G}$.
This characterisation of the section $\mathcal{L}$ will not be used in the rest of the thesis, but we still find it remarkable.

4 General and special Hurwitz spaces

Let G be a finite group. In this section we convert the sheaf $\mathfrak{G} \to SP^h(\mathbb{C})$ introduced in Lemma 3.1.4 into a space $\widetilde{\mathrm{Hur}}(h,G)$, called the *general Hurwitz space*. As we shall see, the space $\widetilde{\mathrm{Hur}}(h,G)$ inherits from $SP^h(\mathbb{C})$ a filtration by absolute value (see Definition 3.0.1), and the filtration strata are closely related to the classical Hurwitz spaces, studied by Hurwitz in [26].

If G is endowed with a norm N, the space $\widetilde{\mathrm{Hur}}(h,G)$ is also endowed with a filtration by norm and we can define the *special Hurwitz space* $\mathrm{Hur}(h,G)$.

We fix $h \geq 1$ throughout the section. Again the case $h = 0$ is of little interest and we will add a few remarks for completeness.

4.1 General Hurwitz spaces

Definition 4.1.1. Consider the contravariant functor $\mathrm{Hom}(\,\cdot\,;G)$ from the category of finitely generated free groups to the category of *finite* sets; if we apply this functor to stalks of the sheaf $\mathfrak{G}$ over $SP^h(\mathbb{C})$, we obtain a topological space $\widetilde{\mathrm{Hur}}(h,G)$, called the *general* Hurwitz space and described as follows.

There is a natural projection $p\colon \widetilde{\mathrm{Hur}}(h,G) \to SP^h(\mathbb{C})$; the fibre over a configuration $P \in SP^h(\mathbb{C})$ is given by

$$p^{-1}(P) = \mathrm{Hom}(\mathfrak{G}(P);G) = \mathrm{Hom}\left(\pi_1\left(\mathbb{C} \setminus P, *_P\right);G\right).$$

Hence a configuration in $\widetilde{\mathrm{Hur}}(h,G)$ can be described as a couple (P,φ) with $P \in SP^h(\mathbb{C})$ and $\varphi\colon \pi_1\left(\mathbb{C} \setminus P, *_P\right) \to G$ a homomorphism.

Given a normal neighbourhood $P \in \mathcal{U} \subset SP^h(\mathbb{C})$, the corresponding *normal neighbourhood* $\mathfrak{U}$ of (P,φ) in $\widetilde{\mathrm{Hur}}(h,G)$ is by definition the set of all configurations (P',φ') with $P' \in \mathcal{U}$ and such that the following composition is equal to φ:

$$\mathfrak{G}(P) \xrightarrow{\simeq} \mathfrak{G}(\mathcal{U}) \xrightarrow{res^{\mathcal{U}}_{P'}} \mathfrak{G}(P') = \pi_1\left(\mathbb{C} \setminus P', *_{P'}\right) \xrightarrow{\varphi'} G.$$

The map p restricts to a surjection $p\colon \mathfrak{U} \to \mathcal{U}$, because for P,$\mathcal{U}$ and P' as above the map $\mathfrak{G}(P) \to \mathfrak{G}(P')$ is a split injection of free groups (see also the proof of Lemma 3.1.4). Normal neighbourhoods define a Hausdorff topology on $\widetilde{\mathrm{Hur}}(h,G)$, such that the projection $p\colon \widetilde{\mathrm{Hur}}(h,G) \to SP^h(\mathbb{C})$ is continuous and has finite fibres.

The number h is also called the *weight* of the configuration (P,φ), and is denoted by $\mathfrak{w}(P,\varphi)$. The absolute value of (P,φ) is defined as $|P|$ (see Definition 3.0.1).

For $h = 0$ the general Hurwitz space $\widetilde{\mathrm{Hur}}(0, G)$ consists of one configuration $(\emptyset, \eta)$, where $\emptyset \in SP^0(\mathbb{C})$ is the empty configuration and $\eta\colon \{\mathbb{1}\} \to G$ is the inclusion of the trivial group. From now on we assume $h \geq 1$.

Definition 4.1.2. Let $(P, \varphi) \in \widetilde{\mathrm{Hur}}(h, G)$, with $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$, and let $f_1, \dots, f_k$ be an admissible generating set for $\pi_1(\mathbb{C} \setminus P, *)$ (see Definition 3.1.5). For each $1 \leq i \leq k$ we call $\varphi(f_i)$ the *local monodromy* of φ around z_i: as an element of G it depends on the chosen admissible generating set, but its conjugacy class in G is independent of this choice.

Definition 4.1.3. Recall Definitions 3.2.8 and 3.2.9. Let $C(\mathbb{C}; \underline{\alpha})$ be a connected component of $\mathfrak{F}_k^{|\cdot|} SP^h(\mathbb{C})$, associated with a sequence $\underline{\alpha}$ as in Definition 3.2.9. We denote by $\widetilde{\mathrm{hur}}(\underline{\alpha}, G) \subset \widetilde{\mathrm{Hur}}(h, G)$ the space $p^{-1}(C(\mathbb{C}; \underline{\alpha}))$, and call it the *weighted, classical Hurwitz space.*

By Lemma 3.2.10, the restriction

$$p\colon \widetilde{\mathrm{hur}}(\underline{\alpha}, G) \to C(\mathbb{C}; \underline{\alpha})$$

is a covering space of degree $|G|^k$, where $k = \sum_{j=1}^{\infty} \alpha_j$. The space $\widetilde{\mathrm{hur}}(\underline{\alpha}, G)$ is a coloured version of the spaces introduced by Hurwitz in [26] to parametrise branched coverings of the plane with a prescribed number k of branching values and with monodromies in G: here the colours are the multiplicities of the points z_i, similarly as in the discussion following Definition 3.2.9. This motivates the notations $\widetilde{\mathrm{Hur}}(h, G)$ and $\widetilde{\mathrm{hur}}(h, G)$ in Definitions 4.1.1 and 4.1.3.
Similarly, let $e^{\mathfrak{a}} \subset SP^h(\mathbb{C})$ be a cell associated with an array $\mathfrak{a} = (l, \underline{\mathfrak{col}})$, using the notation from Definition 3.2.4; then $e^{\mathfrak{a}}$ is contained in some connected component $C(\mathbb{C}; \underline{\alpha})$ of some stratum $\mathfrak{F}_k^{|\cdot|} SP^h(\mathbb{C})$, hence p restricts to a finite covering over $e^{\mathfrak{a}}$.
We note also that the map p is proper, so that it extends to a map

$$p^{\infty}\colon \widetilde{\mathrm{Hur}}(h, G)^{\infty} \to SP^h(\mathbb{C})^{\infty}.$$

We can therefore lift the CW structure on $SP^h(\mathbb{C})^{\infty}$ from Definition 3.2.4 to a CW structure on $\widetilde{\mathrm{Hur}}(h, G)^{\infty}$. We describe this CW structure in detail in the next section.

4.2 Cell decomposition of $\widetilde{\mathrm{Hur}}(h, G)^{\infty}$

Definition 4.2.1. Recall Definition 3.2.2, and use the same notation. A *G-column* $\mathfrak{Col}$ of length $\lambda \geq 1$ is given by a column $\mathfrak{col} = (A, m)$ of length λ together with a function $\gamma\colon A \to G$. We consider G-columns up to equivalence analogously as in Definition 3.2.2; if $A = [\lambda]$, we also write $\mathfrak{Col} = (\mathfrak{col}, \underline{\gamma}) = (\lambda, \underline{m}, \underline{\gamma})$, with $\underline{\gamma} = (\gamma_1, \dots, \gamma_\lambda) \in G^{\lambda}$.
The weight, the absolute value and the dimension of $\mathfrak{Col} = (\lambda, \underline{m}, \underline{\gamma})$ are defined as the weight, the absolute value and the dimension of the underlying column $\mathfrak{col}$.

Definition 4.2.2. Recall Definition 3.2.4, and use the same notation. A *G-array* $\mathfrak{A}$ of length $l \geq 0$ is a sequence of l G-columns $\mathfrak{Col}_1, \dots, \mathfrak{Col}_l$. We will usually write $\mathfrak{A} = (l; \mathfrak{Col}_1, \dots, \mathfrak{Col}_l) = (l, \underline{\mathfrak{Col}})$. Each G-column $\mathfrak{Col}_i$ is usually expanded as $(\mathfrak{col}_i, \underline{\gamma}_i) = (\lambda_i, \underline{m}_i, \underline{\gamma}_i)$, and $\underline{\gamma}_i$ is usually expanded as $(\gamma_{i,1}, \dots, \gamma_{i,\lambda_i})$.
The weight $\mathfrak{w}(\mathfrak{A})$, the absolute value $|\mathfrak{A}|$ and the dimension $\dim(\mathfrak{A})$ of the G-array $\mathfrak{A}$ are defined as the weight, the absolute value and the dimension of the underlying array.
We define a subspace $e^{\mathfrak{A}} \subset \widetilde{\mathrm{Hur}}(h, G)$. Denote by $\mathfrak{a}$ the array underlying the G-array $\mathfrak{A}$; a configuration $(P, \varphi) \in \widetilde{\mathrm{Hur}}(h, G)$ belongs to $e^{\mathfrak{A}}$ if the following conditions are satisfied.

- $P \in e^{\mathfrak{a}} \subset SP^h(\mathbb{C})$; let $P = \{m_{i,j} \cdot z_{i,j}\}$ using the notation from Definition 3.3.1;
- $\varphi(\mathfrak{f}^P_{i,j}) = \gamma_{i,j}$, where the elements $\mathfrak{f}^P_{i,j}$ constitute the standard generating set $\underline{\mathfrak{f}}^P$ of $\mathfrak{G}(P)$ (see Definition 3.3.1 and Figure 4.1).

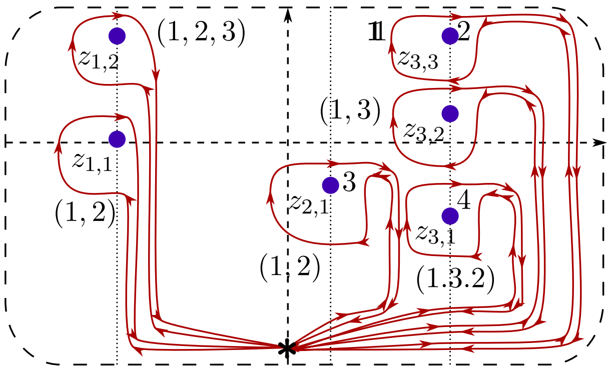

Figure 4.1. A configuration (P, φ) in the cell $e^{\mathfrak{A}} \subset \widetilde{\mathrm{Hur}}(12, \mathfrak{S}_3)$, where $P \in SP^{12}(\mathbb{C})$ is the configuration from Figure 3.4. Here $\mathfrak{A} = (\mathfrak{Col}_1, \mathfrak{Col}_2, \mathfrak{Col}_3)$, where for $1 \leq i \leq 3$ we set $\mathfrak{Col}_i = (\mathfrak{col}_i, \underline{\gamma}_i)$ using the column $\mathfrak{col}_i$ from Figure 3.4; moreover we set $\underline{\gamma}_1 = ((1,2), (1,2,3))$, $\underline{\gamma}_2 = ((1,2))$ and $\underline{\gamma}_3 = ((1,3,2), (1,3), \mathbb{1})$.

The space $e^{\mathfrak{A}} \subset \widetilde{\mathrm{Hur}}(h, G)^\infty$ is modelled on the interior of the product of simplices $\Delta^{\mathfrak{a}}$ (see the discussion after Definition 3.2.4), and the characteristic map $\Phi^{\mathfrak{A}} \colon \Delta^{\mathfrak{a}} \to \widetilde{\mathrm{Hur}}(h, G)^\infty$ covers the characteristic map $\Phi^{\mathfrak{a}} \colon \Delta^{\mathfrak{a}} \to SP^h(\mathbb{C})^\infty$ along the projection p. We define the multisimplex $\Delta^{\mathfrak{A}}$ as $\Delta^{\mathfrak{a}}$.

The previous discussion gives the following theorem.

Theorem 4.2.3. *The space $\widetilde{\mathrm{Hur}}(h, G)^\infty$ has a cell structure with only one 0-cell ∞; the other cells are in bijection with G-arrays of weight h.*
The map $p^\infty \colon \widetilde{\mathrm{Hur}}(h, G)^\infty \to SP^h(\mathbb{C})^\infty$ is cellular and restricts to a finite trivial covering over each cell $e^{\mathfrak{a}} \subset SP^h(\mathbb{C})^\infty$.

In the case $h = 0$ there is only one G-array of length 0 and weight 0, namely the empty G-array $\emptyset$: the corresponding 0-cell $e^{\emptyset}$ is contained in $\widetilde{\mathrm{Hur}}(0, G)^\infty$, giving the second point of the space $\widetilde{\mathrm{Hur}}(0, G)^\infty$ besides ∞.

Definition 4.2.4. Recall Definition 3.2.8. The absolute value filtration on $\widetilde{\mathrm{Hur}}(h,G)^\infty$ is the pullback along p of the absolute value filtration on $SP^h(\mathbb{C})^\infty$: for $-1 \le k \le h$ we set

$$F_k^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G) = (p^\infty)^{-1}\left(F_k^{|\cdot|}SP^h(\mathbb{C})^\infty\right).$$

Note that the k^{th}filtration level $F_k^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G)$ is a closed subcomplex of $\widetilde{\mathrm{Hur}}(h,G)^\infty$. For $0 \le k \le h$, the k^{th}filtration quotient of $\widetilde{\mathrm{Hur}}(h,G)^\infty$ is

$$F_k^{|\cdot|}/F_{k-1}^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G)^\infty = \left(F_k^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G)^\infty\right)/\left(F_{k-1}^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G)^\infty\right),$$

and the k^{th}filtration stratum of $\widetilde{\mathrm{Hur}}(h,G)^\infty$ is the difference

$$\mathfrak{F}_k^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G)^\infty = \left(F_k^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G)^\infty\right)\setminus\left(F_{k-1}^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G)^\infty\right).$$

For $k \ge 1$ note that the stratum $\mathfrak{F}_k^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G)^\infty$ is contained in $\widetilde{\mathrm{Hur}}(h,G)$: we will also denote it as $\mathfrak{F}_k^{|\cdot|}\widetilde{\mathrm{Hur}}(h,G)$.

4.3 Norm filtration

From now on assume now that G is endowed with a norm N (see Definition 2.2.1).

Definition 4.3.1. We use the notation from Definition 4.1.1; recall also Definition 3.1.5. Let $(P,\varphi) \in \widetilde{\mathrm{Hur}}(h,G)$ be a configuration and let $f_1,\dots,f_k$ be an admissible generating set for $\mathfrak{G}(P)$. The norm of (P,φ) is defined as

$$N(P,\varphi) = \sum_{i=1}^{k} N(\varphi(f_i)).$$

We set the norm $N(\infty)$ of the point at infinity $\infty \in \widetilde{\mathrm{Hur}}(h,G)^\infty$ to be 0.

Since the norm on G is conjugation invariant and the conjugacy classes of the elements of an admissible generating set are the same for all admissible generating sets, the norm is well-defined on configurations of $\widetilde{\mathrm{Hur}}(h,G)$. We define combinatorially a norm also for G-columns $\mathfrak{Col}$ and G-arrays $\mathfrak{A}$.

Definition 4.3.2. Using the notation from Definition 4.2.1, the *norm* of $\mathfrak{Col}$ is

$$N(\mathfrak{Col}) = \sum_{i=1}^{\lambda} N(\gamma_i).$$

Using the notation from Definition 4.2.2, the *norm* of $\mathfrak{A}$ is

$$N(\mathfrak{A}) = \sum_{i=1}^{l} N(\mathfrak{Col}_i).$$

Note that for $(P,\varphi) \in e^{\mathfrak{A}} \subset \widetilde{\mathrm{Hur}}(h,G)$ we have the equality $N(P,\varphi) = N(\mathfrak{A})$; in particular the norm is constant on each cell $e^{\mathfrak{A}}$.

Lemma 4.3.3. *The norm* $N \colon \widetilde{\mathrm{Hur}}(h,G)^\infty \to \mathbb{Z}_{\geq 0}$ *is lower semi-continuous.*

Proof. Obviously N is lower semi-continuous at ∞, where it attains a minimum. Let $(P,\varphi) \in \widetilde{\mathrm{Hur}}(h,G)$, with $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\}$, and let $\mathfrak{U}$ be a normal neighbourhood of (P,φ), associated with the discs $U_1, \dots, U_k \subset \mathbb{C}$ (see Definition 4.1.1); let $\mathcal{U} = p(\mathfrak{U})$ the corresponding normal neighbourhood of $P \in SP^h(\mathbb{C})$.
Choose disjoint arcs $\zeta_1, \dots, \zeta_k$ as in Definition 3.1.5, and let β_i be the circle ∂U_i, for $1 \leq i \leq k$; denote by $f_1, \dots, f_k$ be the associated admissible generating set for $\mathfrak{G}(P)$ (see Figure 4.2, left) Let $(P',\varphi') \in \mathfrak{U}$; then for all $1 \leq i \leq k$ we can determine

- a number $\kappa_i \geq 1$;
- numbers $(\mu_{i,j} \geq 1)_{1 \leq j \leq \kappa_i}$ satisfying $\sum_{j=1}^{\kappa_i} \mu_{i,j} = m_i$;
- distinct points $(z'_{i,j} \in U_i)_{1 \leq j \leq \kappa_i}$,

such that $P' = \left\{\mu_{i,j} \cdot z'_{i,j}\right\}_{1 \leq i \leq k, 1 \leq j \leq \kappa_i}$.

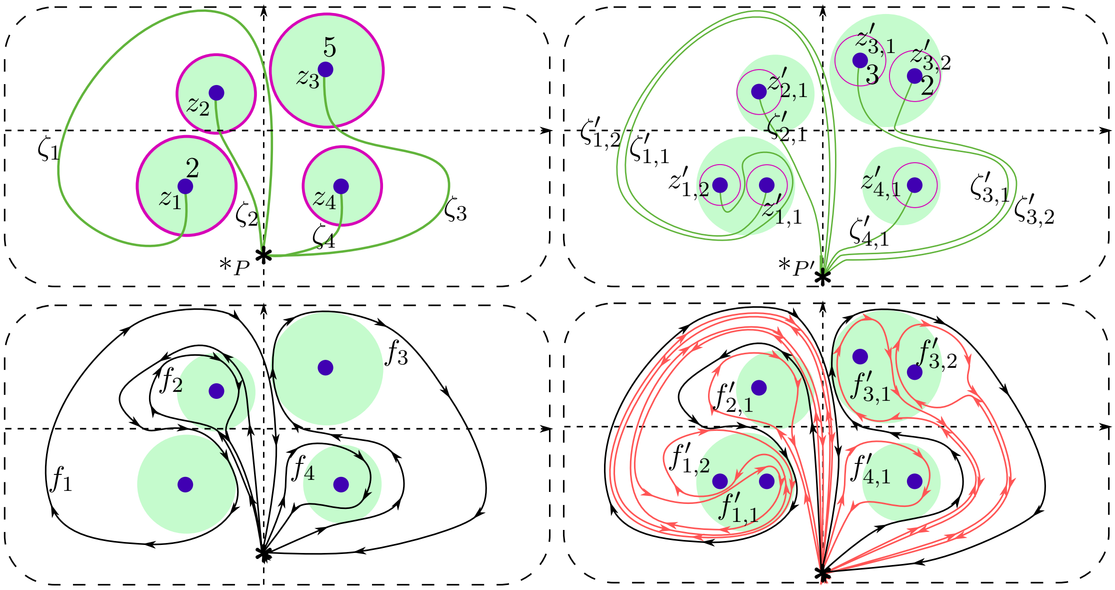

Figure 4.2. The configuration P from Figure 3.1 is perturbed to a new configuration $P' = \left\{z'_{1,1}, z'_{1,2}, z'_{2,1}, 3 \cdot z'_{3,1}, 2 \cdot z'_{3,2}, z'_{4,1}\right\}$. The arcs ζ_i (top left) are extended to arcs $\zeta'_{i,j}$ (top right). Correspondingly the elements f_1, f_2, f_3, f_4 in $\pi_1(\mathbb{C} \setminus P, *_P)$ (bottom left) are perturbed to the elements $f'_{1,1} \cdot f'_{1,2}, f'_{2,1}, f'_{3,1} \cdot f'_{3,2}, f'_{4,1}$ in $\pi_1(\mathbb{C} \setminus P', *_{P'})$ respectively (bottom right).

For all $1 \leq i \leq k$ and $1 \leq j \leq \kappa_i$, we extend the arc $\zeta_i \setminus U_i$ on both sides to obtain an arc $\zeta'_{i,j} \subset \mathbb{C} \setminus P'$ connecting $*_{P'}$ with $z'_{i,j}$: on one side we connect $*_P$ vertically to the

new basepoint $*_{P'}$, on the other side we connect $\zeta_i \cap \beta_i$ with $z'_{i,j}$ in $U_i \setminus \left\{z'_{i,1}, \dots, z'_{i,\kappa_i}\right\}$. We can choose all these extensions in such a way that, after a small perturbation, the arcs $(\zeta'_{i,j})_{1\leq i\leq k, 1\leq j\leq\kappa_i}$ are all disjoint.
Let $\left\{f'_{i,j}\right\}$ be the associated admissible generating set for $\mathfrak{G}(P')$; then for all $1 \leq i \leq k$ the image of $f_i \in \mathfrak{G}(P)$ under the map $\mathfrak{G}(P) \simeq \mathfrak{G}(\mathcal{U}) \to \mathfrak{G}(P')$ can be written as the product of the elements $(f'_{i,j})_{1\leq j\leq\kappa_i}$ taken in some order, depending on the disposition of the ends of the arcs $(\zeta'_{i,j})_{1\leq j\leq\kappa_i}$ near $*_{P'}$. Without loss of generality we assume that $f_i \mapsto f'_{i,1} \cdot \dots \cdot f'_{i,\kappa_i}$ (see Figure 4.2, right). We obtain

$$N\left(\varphi\left(f_i\right)\right) = N\left(\varphi'\left(\prod_{j=1}^{\kappa_i} f'_{i,j}\right)\right) \leq \sum_{j=1}^{\kappa_i} N\left(\varphi'\left(f'_{i,j}\right)\right),$$

and taking the sum over i we obtain the inequality $N(P, \varphi) \leq N(P', \varphi')$. □

We can then filter the space $\widetilde{\mathrm{Hur}}(h, G)^\infty$ according to the norm.

Definition 4.3.4. We define a filtration $F^N_\bullet$ on $\widetilde{\mathrm{Hur}}(h, G)^\infty$ into closed subcomplexes. For all $\nu \geq 0$ we define the ν^{th}filtration level $F^N_\nu\widetilde{\mathrm{Hur}}(h, G)^\infty \subset \widetilde{\mathrm{Hur}}(h, G)^\infty$ as the subspace of configurations of norm $\leq \nu$. By Lemma 4.3.3 this is a closed subspace. We set $F^N_{-1}\widetilde{\mathrm{Hur}}(h, G)^\infty = \emptyset$.
For $\nu \geq 0$ let the ν^{th}filtration quotient of $\widetilde{\mathrm{Hur}}(h, G)^\infty$ be

$$F^N_\nu/F^N_{\nu-1}\widetilde{\mathrm{Hur}}(h, G)^\infty = \left(F^N_\nu\widetilde{\mathrm{Hur}}(h, G)^\infty\right) / \left(F^N_{\nu-1}\widetilde{\mathrm{Hur}}(h, G)^\infty\right).$$

For $\nu \geq 0$ let the ν^{th}filtration stratum of $\widetilde{\mathrm{Hur}}(h, G)^\infty$ be the difference

$$\mathfrak{F}^N_\nu\widetilde{\mathrm{Hur}}(h, G)^\infty = \left(F^N_\nu\widetilde{\mathrm{Hur}}(h, G)^\infty\right) \setminus \left(F^N_{\nu-1}\widetilde{\mathrm{Hur}}(h, G)^\infty\right).$$

For $\nu \geq 1$ note that the stratum $\mathfrak{F}^N_\nu\widetilde{\mathrm{Hur}}(h, G)^\infty$ is contained in $\widetilde{\mathrm{Hur}}(h, G)$: we will also denote it as $\mathfrak{F}^N_\nu\widetilde{\mathrm{Hur}}(h, G)$.

Definition 4.3.5. Let $(P, \varphi) \in \widetilde{\mathrm{Hur}}(h, G)$, with $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\}$, and let $f_1, \dots, f_k$ be an admissible generating set for $\mathfrak{G}(P)$ (see Definition 3.1.5). We say that (P, φ) is *admissible* if for all $1 \leq i \leq k$ the inequality $N(\varphi(f_i)) \leq m_i$ holds (see Figure 4.3). The definition is well-posed because the norm on G is conjugation invariant and for all z_i's the conjugacy class $[f_i]_{conj}$ is well-defined. We declare $\infty \in \widetilde{\mathrm{Hur}}(h, G)^\infty$ also admissible. We call $\widetilde{\mathrm{Hur}}(h, G)^\infty_{adm} \subset \widetilde{\mathrm{Hur}}(h, G)^\infty$ the subspace of admissible configurations.
A G-array $\mathfrak{A}$ is admissible if, using the notation from Definition 4.2.2, for all $1 \leq i \leq l$ and $1 \leq j \leq \lambda_i$ the inequality $N(\gamma_{i,j}) \leq m_{i,j}$ holds.

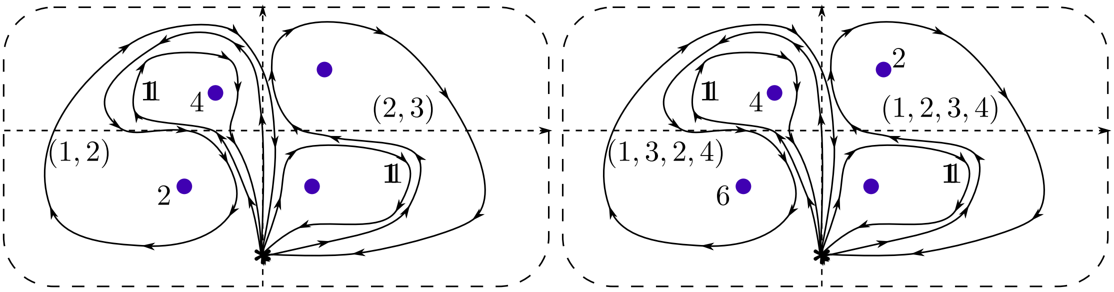

Figure 4.3. On left, an admissible configuration in $\widetilde{\mathrm{Hur}}(8, \mathfrak{S}_3)$. On right, a non-admissible configuration in $\widetilde{\mathrm{Hur}}(13, \mathfrak{S}_4)$: the 4-cycle $(1,2,3,4)$ has norm 3, which is strictly greater than 2.

Similarly as in the proof of Lemma 4.3.3, one can show that $\widetilde{\mathrm{Hur}}(h,G)^{\infty}_{adm}$ is a closed subcomplex of $\widetilde{\mathrm{Hur}}(h,G)^{\infty}$.

The filtrations on $\widetilde{\mathrm{Hur}}(h,G)^{\infty}$ from Definitions 4.2.4 and 4.3.4 give filtrations $F^{|\cdot|}$ and F^N on $\widetilde{\mathrm{Hur}}(h,G)^{\infty}_{adm}$.

Definition 4.3.6. The *special* Hurwitz space $\mathrm{Hur}(h,G)$ is defined as

$$\mathrm{Hur}(h,G) = \mathfrak{F}^N_h \widetilde{\mathrm{Hur}}(h,G)_{adm} \subset \widetilde{\mathrm{Hur}}(h,G).$$

For $(P,\varphi) \in \mathrm{Hur}(h,G)$, using the notation from Definition 4.3.5, one has equalities $N(\varphi(f_i)) = m_i$ for all $1 \le i \le k$ (see Figure 4.4).

In other words, $\mathrm{Hur}(h,G)$ is the union of all cells $e^{\mathfrak{A}}$ corresponding to G-arrays $\mathfrak{A}$ such that, using the notation from Definitions 3.2.4 and 4.2.2, for all $1 \le i \le l$ and $1 \le j \le \lambda_i$ the equality $m_{i,j} = N(\gamma_{i,j})$ holds.

We say that a G-column $\mathfrak{Col} = (\lambda, \underline{m}, \underline{\gamma})$ is *special* if, for all $1 \le j \le \lambda$ the equality $m_j = N(\gamma_j)$ holds, using the notation from Definition 4.2.1; then the previous condition on a G-array $\mathfrak{A}$ is that it consists of special G-columns: in this case we say that $\mathfrak{A}$ is *special*.

The one-point compactification $\mathrm{Hur}(h,G)^{\infty}$ has a cell decomposition given by one 0-cell, the point ∞, and cells $e^{\mathfrak{A}} \subset \widetilde{\mathrm{Hur}}(h,G)$ corresponding to special G-arrays.

The space $\mathrm{Hur}(h,G)$ inherits a filtration $F^{|\cdot|}$ from the space $\widetilde{\mathrm{Hur}}(h,G)$; we can extend this filtration to $\mathrm{Hur}(h,G)^{\infty}$ by declaring $|\infty| = 0$.

If $h = 0$ the special Hurwitz space $\mathrm{Hur}(0,G)$ coincides with the general Hurwitz space $\widetilde{\mathrm{Hur}}(0,G)$, and both spaces consist of one point.

In the following we analyse the strata of the absolute value filtration $F^{|\cdot|}_{\bullet}\mathrm{Hur}(h,G)$.

Definition 4.3.7. Let $\underline{\alpha} = (\alpha_j)_{j \ge 0}$ be as in Definitions 3.2.9 and 4.1.3, and use the same notation. We define the space $\mathrm{hur}(\underline{\alpha}, G) \subset \mathrm{Hur}(h,G)$ as the intersection

$$\mathrm{hur}(\underline{\alpha}, G) = \widetilde{\mathrm{hur}}(\underline{\alpha}, G) \cap \mathrm{Hur}(h,G),$$

and call it the *classical Hurwitz space*.

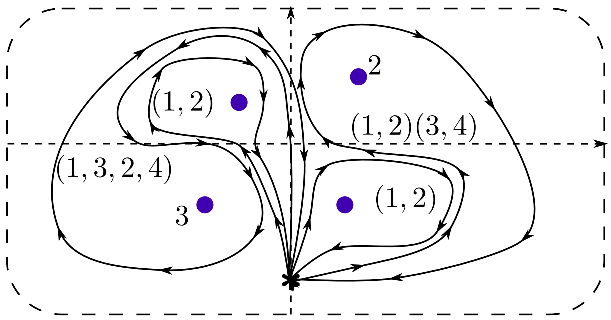

Figure 4.4. A configuration in the special Hurwitz space $\mathrm{Hur}(7, \mathfrak{S}_4)$.

It follows directly from Definitions 4.1.3 and 4.3.7 that for $k \geq 0$ we have a decomposition

$$\mathfrak{F}_k^{|\cdot|}\mathrm{Hur}(h, G) = \coprod_{\underline{\alpha}} \mathrm{hur}(\underline{\alpha}, G),$$

where $\underline{\alpha}$ ranges over all sequences $(\alpha_j)_{j\geq 0}$ with $\sum_{j=1}^{\infty} \alpha_j = k$ and $\sum_{j=1}^{\infty} j\alpha_j = h$.

For fixed k, note that the disjoint union $\coprod_{h\geq 0} \mathfrak{F}_k^{|\cdot|}\mathrm{Hur}(h, G)$ is precisely the space introduced by Hurwitz in [26] to parametrise branched coverings of $\mathbb{C}$ with k branching points and (non-trivial) local monodromies in G.

To see this, let $(P, \varphi) \in \mathfrak{F}_k^{|\cdot|}\mathrm{Hur}(h, G)$, with P of the form $\{m_1 \cdot z_1, \dots, m_k \cdot z_k\}$. Then one can recover the multiplicities $m_1, \dots, m_k$ just by knowing the set $\{z_1, \dots, z_k\} \subset \mathbb{C}$ and the homomorphism $\varphi\colon \pi_1(\mathbb{C} \setminus \{z_1, \dots, z_k\}; *_P) \to G$ (note also that the definition of the basepoint $*_P$ only depends on the set $\{z_1, \dots, z_k\}$ and not on the multiplicities $m_1, \dots, m_k$, see Definition 3.1.2). Indeed the multiplicity m_i of z_i is equal to the norm of the local monodromy of φ at z_i.

The disjoint union $\coprod_{h\geq 0} \mathfrak{F}_k^{|\cdot|}\mathrm{Hur}(h, G)$ splits into the subspaces $\mathfrak{F}_k^{|\cdot|}\mathrm{Hur}(h, G)$ according to the value of the total norm $h \geq 0$. Each space $\mathfrak{F}_k^{|\cdot|}\mathrm{Hur}(h, G)$ splits further into several subspaces $\mathrm{hur}(\underline{\alpha}, G)$ according to *which* positive integers (counted with multiplicity) occur as norms of the k local monodromies. The spaces $\mathrm{hur}(\underline{\alpha}, G)$ are in general still disconnected; different connected components can be for example distinguished considering the following:

- which conjugacy classes of G (counted with multiplicity) occur as local monodromies of φ at the k points of P;
- what the *total monodromy* of φ, as an element of G, is (see in Section 4.4);
- what the image of φ, as subgroup of G, is.

The previous list is in general not a complete set of invariants for connected components of the classical Hurwitz spaces, and classifying such connected components is a hard problem.

4.4 Total monodromy

In this section we introduce the notion of *total monodromy* for configurations of $\widetilde{\mathrm{Hur}}(h,G)$ and, combinatorially, for G-columns and G-arrays.

Definition 4.4.1. For a configuration $P \in \widetilde{\mathrm{Hur}}(h,G)$, the *total monodromy* $\omega(P,\varphi)$ is

$$\omega(P,\varphi) = \varphi(\mathcal{L}_P) \in G,$$

where the large loop $\mathcal{L}_P$ was introduced in Definition 3.3.5 (see Figure 4.5).
The *total monodromy* of a G-column $\mathfrak{Col}$ is

$$\omega(\mathfrak{Col}) = \gamma_1 \cdot \ldots \cdot \gamma_\lambda \in G,$$

where we use the notation from Definition 4.2.1.
The *total monodromy* of a G-array $\mathfrak{A}$ is

$$\omega(\mathfrak{A}) = \omega(\mathfrak{Col}_1) \cdot \ldots \cdot \omega(\mathfrak{Col}_l) \in G,$$

where we use the notation from Definition 4.2.2.

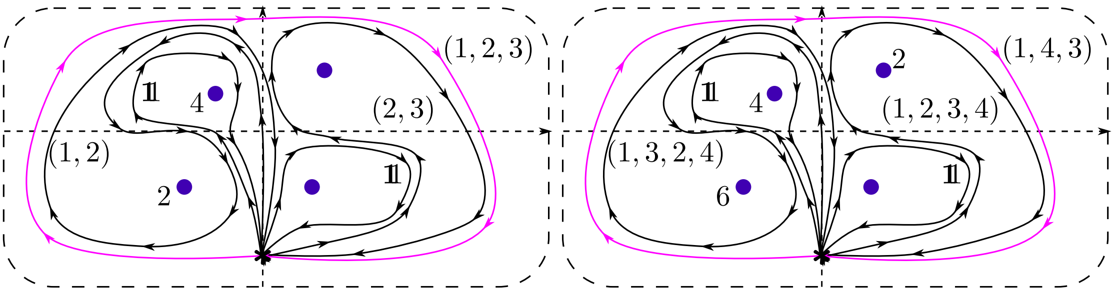

Figure 4.5. The total monodromies of the configurations in Figure 4.3 are $\mathbb{1} \cdot (1,2) \cdot \mathbb{1} \cdot (2,3) = (1,2,3) \in \mathfrak{S}_3$ and $\mathbb{1} \cdot (1,3,2,4) \cdot \mathbb{1} \cdot (1,2,3,4) = (1,4,3) \in \mathfrak{S}_4$ respectively.

The total monodromy is a continuous function $\omega\colon \widetilde{\mathrm{Hur}}(h,G) \to G$, so it is a discrete invariant that can distinguish connected components of $\widetilde{\mathrm{Hur}}(h,G)$ and, by restriction, of $\mathrm{Hur}(h,G)$.
Note that for $(P,\varphi) \in e^{\mathfrak{A}} \subset \widetilde{\mathrm{Hur}}(h,G)$ we have the equality $\omega(P,\varphi) = \omega(\mathfrak{A})$.

Definition 4.4.2. Let $\gamma \in G$ be an element. We define $\widetilde{\mathrm{Hur}}(h,G,\gamma) \subset \widetilde{\mathrm{Hur}}(h,G)$ as the subspace of configurations having total monodromy equal to γ.
Similarly we define $\mathrm{Hur}(h,G,\gamma) \subset \mathrm{Hur}(h,G)$ as the subspace of configurations having total monodromy equal to γ.

We can write both spaces $\widetilde{\mathrm{Hur}}(h,G)$ and $\mathrm{Hur}(h,G)$ as disjoint unions:

$$\widetilde{\mathrm{Hur}}(h,G) = \coprod_{\gamma \in G} \widetilde{\mathrm{Hur}}(h,G,\gamma); \qquad \mathrm{Hur}(h,G) = \coprod_{\gamma \in G} \mathrm{Hur}(h,G,\gamma).$$

The following lemma shows that, besides having interesting geometric properties (e.g. two filtrations, one by absolute value and one by norm), the space $\widetilde{\mathrm{Hur}}(h,G)$ is homotopically rather simple.

Lemma 4.4.3. *For $h \geq 1$, the total monodromy gives a homotopy equivalence*

$$\omega\colon \widetilde{\mathrm{Hur}}(h,G) \simeq G$$

Proof. We need to prove that for all $\gamma \in G$ the space $\widetilde{\mathrm{Hur}}(h,G,\gamma)$ is contractible. Let $P_0 = \{h \cdot 0\} \in SP^h(\mathbb{C})$ be the configuration in which we take $0 \in \mathbb{C}$ with multiplicity h, and let $\varphi_0\colon \pi_1(\mathbb{C} \setminus P_0; *) \simeq \mathbb{Z} \to G$ be given by mapping the standard generator, which is the large loop in this case, to γ. Then $(P_0, \varphi_0) \in \widetilde{\mathrm{Hur}}(h,G,\gamma)$, which is therefore non-empty.
If $(P,\varphi) \in \widetilde{\mathrm{Hur}}(h,G,\gamma)$ is a configuration, with $P = \{m_1 \cdot z_1, \ldots, m_k \cdot z_k\}$, then for all $0 < t \leq 1$ we can define $t(P,\varphi) = (tP, t\varphi)$ as the following configuration:

- $tP = \{m_1 \cdot tz_1, \ldots, m_k \cdot tz_k\}$, where for all $1 \leq i \leq k$ the point $tz_i \in \mathbb{C}$ is obtained multiplying $t \in \mathbb{R}$ and $z_i \in \mathbb{C}$;

- the homeomorphism $t\colon \mathbb{C} \setminus P \cong \mathbb{C} \setminus tP$ given by multiplication by t maps $*_P$ to the complex number $t(*_P) = (*_{(tP)} + \sqrt{-1}(1-t)) \in \mathbb{C} \setminus tP$, lying above $*_{(tP)}$ but below all points in the configuration tP (see Definition 3.1.2). The vertical segment joining $*_{(tP)}$ and $t(*_P)$ is disjoint from $\mathbb{C} \setminus tP$ and can be used to translate the basepoint of $\mathbb{C} \setminus tP$; we define $t\varphi$ as the composite

$$\pi_1(\mathbb{C} \setminus tP, *_{(tP)}) \xrightarrow{\cong} \pi_1(\mathbb{C} \setminus tP, t(*_P)) \xrightarrow{t_*^{-1}} \pi_1(\mathbb{C} \setminus P, *_P) \xrightarrow{\varphi} G.$$

We can then define a homotopy $H\colon \widetilde{\mathrm{Hur}}(h,G,\gamma) \times [0,1] \to \widetilde{\mathrm{Hur}}(h,G,\gamma)$ by setting $H(P,\varphi;t) = t(P,\varphi)$ for $0 < t \leq 1$, and $H(P,\varphi;0) = (P_0,\varphi_0)$. The map H is a retraction of $\widetilde{\mathrm{Hur}}(h,G,\gamma)$ onto the configuration (P_0,φ_0) (see Figure 4.6). □

The hypothesis $h \geq 1$ is only used to define the configuration (P_0,φ_0) having total monodromy $\gamma \in G$. On the other hand we know that $\widetilde{\mathrm{Hur}}(0,G)$ consists of only one configuration with total monodromy $\mathbb{1} \in G$.

4.5 Orbit partition

In this section we assume $G = \mathfrak{S}_d$ for some $d \geq 2$ and we introduce the notion of *orbit partition* for configurations of $\mathrm{Hur}(h,\mathfrak{S}_d)$.

Definition 4.5.1. Let $H \subset \mathfrak{S}_d$ be a subgroup. The action of H on the set $[d] = \{1,\ldots,d\}$ induces an orbit partition $\underline{\mathfrak{P}}(H) = \{\mathfrak{P}_1,\ldots,\mathfrak{P}_r\}$ on $[d]$, called the *orbit partition* of H. The number r is called the *orbit number* of H.

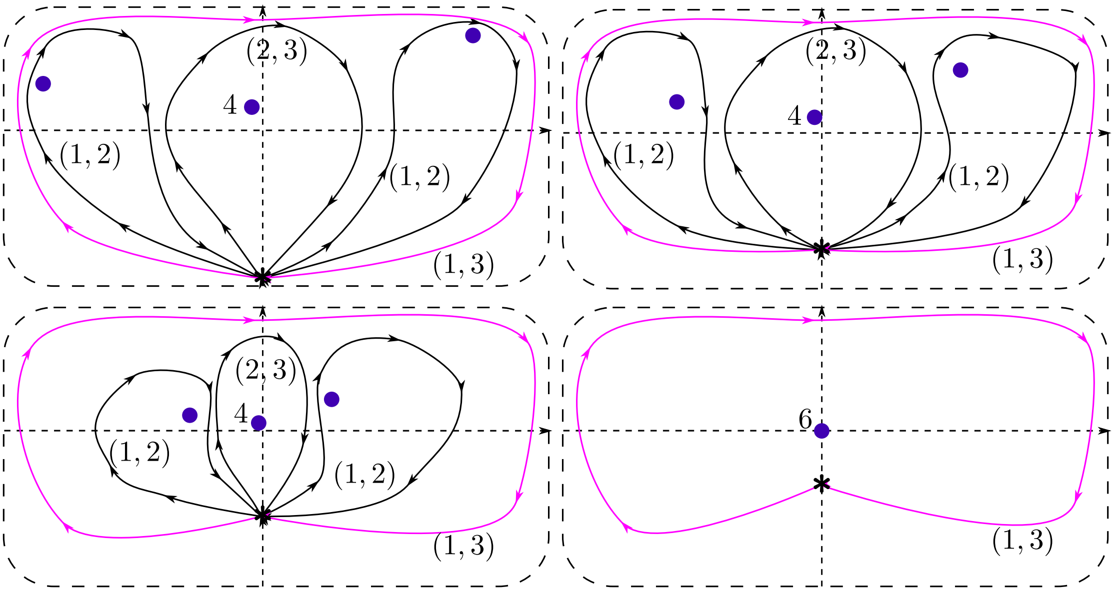

Figure 4.6. The homotopy H applied to a configuration in $\widetilde{\mathrm{Hur}}(6, \mathfrak{S}_3, (1,3))$ (top left) at times 1, $\frac{2}{3}$, $\frac{1}{3}$ and 0.

Let $(P, \varphi) \in \mathrm{Hur}(h, \mathfrak{S}_d)$; the *orbit partition* $\underline{\mathfrak{P}}(P, \varphi)$ is the orbit partition associated with the image of the homomorphism φ, i.e.

$$\underline{\mathfrak{P}}(P, \varphi) = \underline{\mathfrak{P}}(\mathrm{Im}(\varphi)),$$

and the orbit number of (P, φ) is the orbit number of $\mathrm{Im}(\varphi)$ (see Figure 4.7).

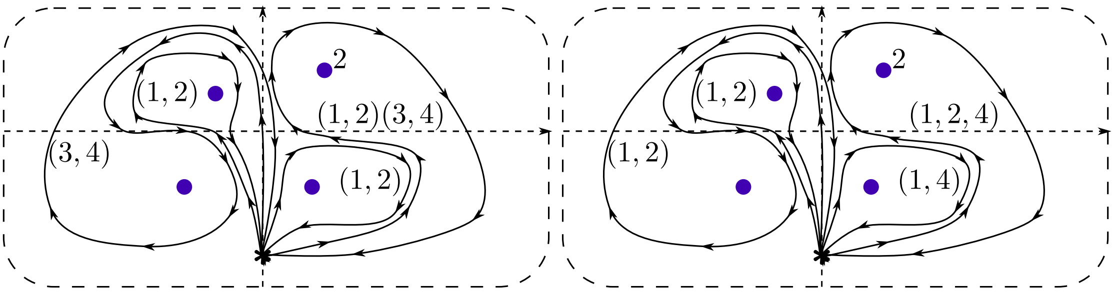

Figure 4.7. Two configurations in $\mathrm{Hur}(5, \mathfrak{S}_4, (1,2))$, having orbit partitions equal to $\{\{1,2\},\{3,4\}\}$ and $\{\{1,2,4\},\{3\}\}$ respectively.

The previous definition makes sense also for generic configurations in $\widetilde{\mathrm{Hur}}(h, \mathfrak{S}_d)$, but the following lemma, which motivates the definition, only holds for the subspace $\mathrm{Hur}(h, \mathfrak{S}_d)$.

Lemma 4.5.2. *Let $\mathfrak{Part}_d$ be the set of all partitions of $[d]$. Then the map*

$$\underline{\mathfrak{P}} \colon \mathrm{Hur}(h, \mathfrak{S}_d) \to \mathfrak{Part}_d$$

is continuous.

To prove this lemma we need the following definition.

Definition 4.5.3. Recall Definition 2.2.2, and let $H \subset \mathfrak{S}_d$ be a subgroup. We define the *norm* of H, denoted by $N(H)$, as the least number ν such that there exist transpositions $\mathbf{t}_1, \dots, \mathbf{t}_\nu \in \mathfrak{S}_d$ with H contained in the subgroup $\langle \mathbf{t}_1, \dots, \mathbf{t}_\nu \rangle \subseteq \mathfrak{S}_d$.

We make a list of remarks on the previous definition.

- For an element $\gamma \in \mathfrak{S}_d$ we have an equality $N(\gamma) = N(\langle\gamma\rangle)$. Note that for $H, H' \subset \mathfrak{S}_d$ there is an inequality
$$N(\langle H, H' \rangle) \leq N(H) + N(H').$$
- If $H \subseteq H' \subseteq \mathfrak{S}_d$, then $N(H) \leq N(H')$.
- The orbit number of H is equal to $d - N(H)$. To see this, let $\underline{\mathfrak{P}}(H) = \{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$; then H is contained in the product of the groups of permutations of the orbits
$$H \subseteq \mathfrak{S}_{|\mathfrak{P}_1|} \times \cdots \times \mathfrak{S}_{|\mathfrak{P}_r|} \subseteq \mathfrak{S}_d;$$
hence it suffices to take $(|\mathfrak{P}_1| - 1) + \cdots + (|\mathfrak{P}_r| - 1) = d - r$ transpositions to generate all elements of H; this shows the inequality $N(H) \leq d - r$. On the other hand let $\mathbf{t}_1, \dots, \mathbf{t}_\nu \in \mathfrak{S}_d$ be transpositions that suffice to generate all elements of H; we consider the graph $\mathcal{G}$ with set of vertices $[d]$ and one edge for each transposition $\mathbf{t}_i$. Each connected components of $\mathcal{G}$ is a union of some of the orbits $\mathfrak{P}_1, \dots, \mathfrak{P}_r$, therefore $|\pi_0(\mathcal{G})| \leq r$. Since $\mathcal{G}$ has ν edges, the inequality $\nu \geq d - |\pi_0(\mathcal{G})|$ holds; we obtain the inequality $\nu \geq d - r$ and hence $N(H) \geq d - r$.
- If $\underline{\mathfrak{P}}(H) = \underline{\mathfrak{P}}(H')$, then for all $H'' \subseteq \mathfrak{S}_d$ we have an equality
$$\underline{\mathfrak{P}}(\langle H, H'' \rangle) = \underline{\mathfrak{P}}(\langle H', H'' \rangle),$$
and in particular $N(\langle H, H'' \rangle) = N(\langle H', H'' \rangle)$.

We are now ready to prove Lemma 4.5.2.

Proof of Lemma 4.5.2. We refer to the proof of Lemma 4.3.3 for the notation. Let $(P, \varphi) \in \mathrm{Hur}(h, \mathfrak{S}_d)$, let $\mathfrak{U}$ be a normal neighbourhood of (P, φ) in $\widetilde{\mathrm{Hur}}(h, \mathfrak{S}_d)$ and let (P', φ') be a configuration in $\mathfrak{U} \cap \mathrm{Hur}(h, \mathfrak{S}_d)$. We want to show that $\underline{\mathfrak{P}}(P, \varphi) = \underline{\mathfrak{P}}(P', \varphi')$. Since $N(P, \varphi) = N(P', \varphi') = h$ we have, for all $1 \leq i \leq k$, the equality
$$N\left(\varphi\left(f_i\right)\right) = N\left(\varphi'\left(\prod_{j=1}^{\kappa_i} f'_{i,j}\right)\right) = \sum_{j=1}^{\kappa_i} N\left(\varphi'\left(f'_{i,j}\right)\right).$$

Note that $\mathrm{Im}(\varphi)$ is generated by the elements $\varphi(f_i)$, whereas $\mathrm{Im}(\varphi')$ is generated by the elements $\varphi'(f'_{i,j})$. Since $\varphi(f_i) = \prod_{j=1}^{\kappa_i} \varphi'\left(f'_{i,j}\right)$, we have an inclusion $\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\varphi')$, and therefore $\underline{\mathfrak{P}}(P, \varphi)$ is *finer or equal* to $\underline{\mathfrak{P}}(P', \varphi')$.

On the other hand, for all $1 \leq i \leq k$ let

$$H_i = \langle \varphi (f_i) \rangle$$

and let

$$H'_i = \left\langle \varphi' \left(f'_{i,1} \right), \ldots, \varphi' \left(f'_{i,\kappa_i} \right) \right\rangle .$$

Using the equality above we obtain

$$N (H_i) \leq N(H'_i) \leq \sum_{j=1}^{\kappa_i} N \left(\varphi' \left(f'_{i,j} \right) \right) = N (\varphi (f_i)) = N (H_i) ,$$

and in particular all inequalities must be equalities. This shows that $\underline{\mathfrak{P}}(H_i) = \underline{\mathfrak{P}}(H'_i)$: a priori $\underline{\mathfrak{P}}(H_i)$ is finer or equal to $\underline{\mathfrak{P}}(H'_i)$, but H_i and H'_i have the same orbit number. We can combine all the groups H_i and H'_i together and obtain

$$\underline{\mathfrak{P}}(\mathrm{Im}(\varphi)) = \underline{\mathfrak{P}}(\langle H_1, \ldots, H_k \rangle) = \underline{\mathfrak{P}}(\left\langle H'_1, \ldots, H'_k \right\rangle) = \underline{\mathfrak{P}}(\mathrm{Im}(\varphi')).$$

□

We conclude this section with the following definition.

Definition 4.5.4. We denote by $\mathrm{Hur}^*(h, \mathfrak{S}_d) \subset \mathrm{Hur}(h, \mathfrak{S}_d)$ the subspace containing configurations (P, φ) such that $\mathrm{Im}(\varphi)$ acts *transitively* on $[d]$: we call it the *transitive, special Hurwitz space*. It is a union of connected components of $\mathrm{Hur}(h, \mathfrak{S}_d)$ (see Figure 4.8).
Using the total monodromy (see Definitions 4.4.1 and 4.4.2) we obtain a splitting

$$\mathrm{Hur}^*(h, \mathfrak{S}_d) = \coprod_{\sigma \in \mathfrak{S}_d} \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma).$$

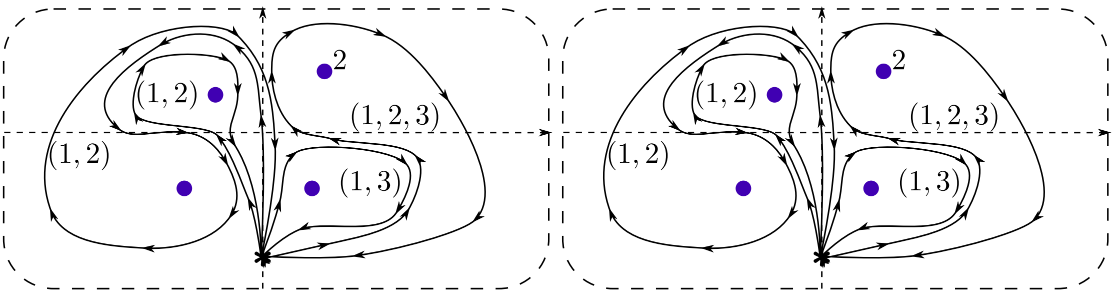

Figure 4.8. On left, a transitive configuration in $\mathrm{Hur}(5, \mathfrak{S}_3)$; on right, a non-transitive configuration in $\mathrm{Hur}(5, \mathfrak{S}_4)$. Note that the two configurations have formally the same description, but belong to different spaces.

We note that the special Hurwitz space $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$ is empty unless $h \geq N(\sigma)$ and the two numbers have the same parity. Similarly $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$ is empty unless $h \geq$

$2(d-1)-N(\sigma)$ and the two numbers have the same parity: note in particular that $2(d-1)-N(\sigma) \geq N(\sigma)$ for all $\sigma \in \mathfrak{S}_d$, because $N(\sigma) \leq d-1$.
The space $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$ is path connected. To see this, let $\underline{\alpha}$ be defined by $\alpha_1 = h$ and $\alpha_j = 0$ for all $j \geq 2$, and denote

$$\mathrm{hur}^*_{transp}(h, \mathfrak{S}_d) = \mathrm{hur}(\underline{\alpha}, \mathfrak{S}_d) \cap \mathrm{Hur}^*(h, \mathfrak{S}_d),$$

where the space $\mathrm{hur}(\underline{\alpha}, \mathfrak{S}_d)$ was introduced in Definition 4.3.7. Then $\mathrm{hur}^*_{transp}(h, \mathfrak{S}_d)$ is the classical Hurwitz space of connected d-fold branched coverings of $\mathbb{C}P^1$ with h distinct branching values in $\mathbb{C}$ and all corresponding local monodromies equal to transpositions (the monodromy around ∞, i.e. the total monodromy, can be arbitrary). It was proved by Hurwitz [26] that the total monodromy is a complete invariant for connected components of $\mathrm{hur}^*_{transp}(h, \mathfrak{S}_d)$, i.e. for each $\sigma \in \mathfrak{S}_d$ the following space is path connected, unless it is empty:

$$\mathrm{hur}^*_{transp}(h, \mathfrak{S}_d, \sigma) = \mathrm{hur}^*_{transp}(h, \mathfrak{S}_d) \cap \mathrm{Hur}(h, \mathfrak{S}_d, \sigma).$$

Note now that $\mathrm{hur}^*_{transp}(h, \mathfrak{S}_d, \sigma)$ is the open, dense stratum of $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$ with respect to the absolute value filtration $F^{|\cdot|}$ (see Definition 4.2.4): hence $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$ is connected, unless it is empty.
If $\underline{\mathfrak{P}} = \{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$ is a partition of $[d]$, we can denote by

$$\mathrm{Hur}(h, \mathfrak{S}_d)_{\underline{\mathfrak{P}}} \subset \mathrm{Hur}(h, \mathfrak{S}_d)$$

the union of connected components exhibiting and orbit partition $\underline{\mathfrak{P}}$. Note that for $(P, \varphi) \in \mathrm{Hur}(h, \mathfrak{S}_d)_{\underline{\mathfrak{P}}}$ the total monodromy $\omega(P, \varphi)$ belongs to the subgroup

$$\mathfrak{S}_{\underline{\mathfrak{P}}} = \mathfrak{S}_{\mathfrak{P}_1} \times \cdots \times \mathfrak{S}_{\mathfrak{P}_r} \subset \mathfrak{S}_d,$$

i.e. the subgroup of $\mathfrak{S}_d$ preserving the partition $\underline{\mathfrak{P}}$.
There is a natural homeomorphism

$$\mathrm{Hur}(h, \mathfrak{S}_d)_{\underline{\mathfrak{P}}} \cong \coprod_{\sigma \in \mathfrak{S}_{\underline{\mathfrak{P}}}} \left(\coprod_{h_1 + \cdots + h_r = h} \left(\prod_{i=1}^{r} \mathrm{Hur}^*(h_i, \mathfrak{S}_{\mathfrak{P}_i}, \sigma|_{\mathfrak{P}_i}) \right) \right),$$

where $\sigma|_{\mathfrak{P}_i} \in \mathfrak{S}_{\mathfrak{P}_i}$ is the restriction of $\sigma \in \mathfrak{S}_{\underline{\mathfrak{P}}}$ to the subset $\mathfrak{P}_i$, and the second disjoint union is extended over all splittings of h into r numbers $h_i \geq 0$. Here we use the convention for which $\mathrm{Hur}^*(0, \mathfrak{S}_1, \mathbb{1})$ contains only one configuration, and $\mathrm{Hur}^*(h, \mathfrak{S}_1, \mathbb{1})$ is empty for all $h \geq 1$.
To see this, note that if $(P, \varphi) \in \mathrm{Hur}(h, \mathfrak{S}_d)_{\underline{\mathfrak{P}}}$, then φ is a product of r homomorphisms of groups $\varphi_i \colon \pi_1(\mathbb{C} \setminus P, *) \to \mathfrak{S}_{\mathfrak{P}_i}$. If $f_1, \dots, f_k$ is an admissible generating set for $\pi_1(\mathbb{C} \setminus P, *)$ (see Definition 3.1.5), then for all $1 \leq j \leq k$ we have a natural splitting

$$N(\varphi(f_j)) = N(\varphi_1(f_j)) + \cdots + N(\varphi_r(f_j)),$$

and summing over j we obtain a splitting

$$h = h_1 + \cdots + h_r.$$

Each product space in the right-hand side of the decomposition of $\mathrm{Hur}(h, \mathfrak{S}_d)_{\underline{\mathfrak{P}}}$ is either empty or a path-connected space. We obtain the following theorem.

Theorem 4.5.5. *A connected component of* $\mathrm{Hur}(h, \mathfrak{S}_d)$ *is uniquely determined by the following:*

- *an orbit partition* $\underline{\mathfrak{P}} = (\mathfrak{P}_1, \dots, \mathfrak{P}_r)$*;*
- *a splitting* $h = h_1 + \cdots + h_r$ *of* h *into numbers* $h_i \geq 0$*;*
- *a total monodromy* $\sigma \in \mathfrak{S}_{\underline{\mathfrak{P}}} \subset \mathfrak{S}_d$*,*

subject to the following conditions:

- $h_i \geq 2(|\mathfrak{P}_i| - 1) - N(\sigma|_{\mathfrak{P}_i})$*;*
- h_i *and* $N(\sigma|_{\mathfrak{P}_i})$ *have the same parity;*
- $h_i = 0$ *whenever* $|\mathfrak{P}_i| = 1$*.*

4.6 The case of an infinite group G

The construction of the space $\widetilde{\mathrm{Hur}}(h, G)$ can be repeated for an infinite group G, but one has to be careful. The topological space $\widetilde{\mathrm{Hur}}(h, G)$, endowed with the topology introduced in Definition 4.1.1, is not any more locally compact, even if it is still covered by subspaces $e^{\mathfrak{A}}$ which are modelled on interiors of multisimplices.
The one-point compactification $\widetilde{\mathrm{Hur}}(h, G)^\infty$ can be written as a union of a 0-cell, the point ∞, and open cells $e^{\mathfrak{A}}$; there are continuous maps $\Phi^{\mathfrak{A}} \colon \Delta^{\mathfrak{A}} \to \widetilde{\mathrm{Hur}}(h, G)^\infty$, mapping $\mathring{\Delta}^{\mathfrak{A}}$ homeomorphically onto $e^{\mathfrak{A}}$ and mapping $\partial\Delta^{\mathfrak{A}}$ to the union of all cells of lower dimension; however one has to change the topology on $\widetilde{\mathrm{Hur}}(h, G)^\infty$ in order to get a CW-complex, namely one has to consider the weak topology induced by the described cell structure.
If the infinite group G is normed, one can define the subspace $\mathrm{Hur}(h, G) \subset \widetilde{\mathrm{Hur}}(h, G)$ in the same way. Our main application will be however with $G = \mathfrak{S}_d$, hence with a finite group: this is the reason why we have preferred to focus on the case G finite in the entire chapter.

5 Global properties of the Hurwitz spaces

Let G be a fixed finite, normed group throughout the chapter. We want to study the interaction between different special Hurwitz spaces $\mathrm{Hur}(h, G)$, for different values of h.
In Section 5.1 we will consider a natural action of G on each space $\mathrm{Hur}(h, G)$, making it into a *G-crossed space* (see Definition 5.1.1).
In Section 5.2 will endow the disjoint union $\mathrm{Hur}(G)$ of all spaces $\mathrm{Hur}(h, G)$, for $h \geq 0$, with an associative H-space structure.
In Section 5.3 we will restrict our attention to the spaces $\mathrm{Hur}(h, G, \mathbb{1})$ (see Definition 4.4.2), and construct an algebra over the operad $\mathcal{C}_2$ of little squares.

5.1 Action of G on the spaces $\mathrm{Hur}(h, G)$.

The action of G on itself by conjugation induces an action of G on the space $\widetilde{\mathrm{Hur}}(h, G)$, which is defined as follows: for $\gamma \in G$ and $(P, \varphi) \in \widetilde{\mathrm{Hur}}(h, G)$ we let

$$\gamma \cdot (P, \varphi) = \left(P, \gamma\varphi\gamma^{-1}\right),$$

where $\gamma\varphi\gamma^{-1} \colon \pi_1(\mathbb{C} \setminus P, *) \to G$ is the composition of φ and conjugation by γ (see Figure 5.1).

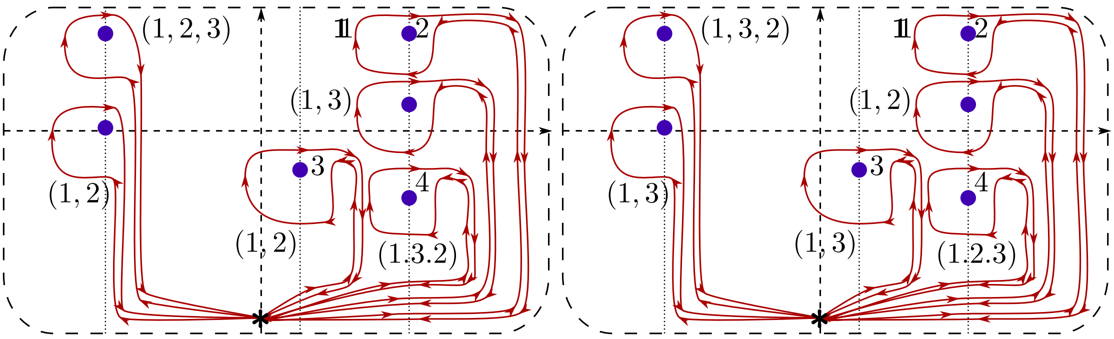

Figure 5.1. On left, the configuration $(P, \varphi) \in \widetilde{\mathrm{Hur}}(12, \mathfrak{S}_3)$ from Figure 4.1. The action of the element $(2, 3) \in \mathfrak{S}_3$ yields the configuration $(P', \varphi') \in \widetilde{\mathrm{Hur}}(12, \mathfrak{S}_3)$ on right.

We can extend this to an action on the one-point compactification $\widetilde{\mathrm{Hur}}(h, G)^{\infty}$ by letting ∞ be a G-fixed point. The following properties are straightforward:

- the action is cellular with respect to the cell structure on $\widetilde{\mathrm{Hur}}(h, G)^{\infty}$ from Section 4.2;

- the action preserves the absolute value filtration, and restricts to an action on the spaces $\widetilde{\mathrm{hur}}(\underline{\alpha}, G)$ from Definition 4.1.3;
- if G is normed, the action preserves the norm filtration (see Definition 4.3.4), and restricts to an action on the special Hurwitz space $\mathrm{Hur}(h, G)$ from Definition 4.3.6;
- consequently the action preserves the spaces $\mathrm{hur}(\underline{\alpha}, G)$ from Definition 4.3.7;
- the action induces conjugation on the total monodromy: $\omega(\gamma \cdot (P, \varphi)) = \gamma\omega(P, \varphi)\gamma^{-1}$;
- if $G = \mathfrak{S}_d$ for some $d \geq 2$, the action induces the standard action on the orbit partition: if $\underline{\mathfrak{P}}(P, \varphi) = \{\mathfrak{P}_1, \dots, \mathfrak{P}_r\}$, then $\underline{\mathfrak{P}}(\gamma \cdot (P, \varphi)) = \{\gamma(\mathfrak{P}_1), \dots, \gamma(\mathfrak{P}_r)\}$; in particular the orbit number is preserved.

We recall from [21] the notion of G-crossed space

Definition 5.1.1. A G-crossed space is a topological space X endowed with a (left) action of G and with a decomposition

$$X = \coprod_{\gamma \in G} X_\gamma,$$

such that for all $\alpha, \beta \in G$ the equality $\alpha \cdot X_\beta = X_{\alpha\beta\alpha^{-1}}$ holds.
The product of two G-crossed spaces X and Y is the G-crossed space $X \times Y$ with diagonal G-action and with

$$(X \times Y)_\gamma = \coprod_{\alpha \cdot \beta = \gamma} X_\alpha \times Y_\beta.$$

We can also define a braiding $\mathfrak{b}\colon X \times Y \to Y \times X$ by setting, for $x \in X_\alpha$ and $y \in Y_\beta$,

$$\mathfrak{b}(x, y) = ((\alpha \cdot y), x) \in Y_{\alpha\beta\alpha^{-1}} \times X_\alpha.$$

The category of G-crossed spaces becomes a braided monoidal category.

We can make $\mathrm{Hur}(h, G)$ into a G-crossed space: Definition 4.4.2 provides a decomposition

$$\mathrm{Hur}(h, G) = \coprod_{\gamma \in G} \mathrm{Hur}(h, G, \gamma).$$

5.2 H-space structure

Definition 5.2.1. We denote by $\widetilde{\mathrm{Hur}}(G)$ and $\mathrm{Hur}(G)$ the spaces

$$\widetilde{\mathrm{Hur}}(G) = \coprod_{h=0}^{\infty} \widetilde{\mathrm{Hur}}(h, G); \qquad \mathrm{Hur}(G) = \coprod_{h=0}^{\infty} \mathrm{Hur}(h, G).$$

In the following we make $\widetilde{\mathrm{Hur}}(G)$ and $\mathrm{Hur}(G)$ into homotopy associative H-spaces. The associativity can be strictified by replacing the spaces $\widetilde{\mathrm{Hur}}(h, G)$ and $\mathrm{Hur}(h, G)$ with other homotopy equivalent spaces, in the same spirit as the Moore construction gives a strictly associative version $\Omega^{\mathrm{Moore}}X$ of the loop space ΩX of a topological space X.

5.2.1 Replacing $\mathbb{C}$ with the unit square

Definition 5.2.2. Let $Q =]0,1[^2 \subset \mathbb{C}$ denote the open unit square, and $\bar{Q}$ its closure $[0,1]^2 \subset \mathbb{C}$. The basepoint of $\bar{Q}$ is fixed once and for all to be $*_{\bar{Q}} = (0,0) = 0 \in \mathbb{C}$.

For $h \geq 0$ we consider the subspace $\widetilde{\mathrm{Hur}}_Q(h,G) \subset \widetilde{\mathrm{Hur}}(h,G)$ of configurations (P,φ) with $P \in SP^h(Q) \subset SP^h(\mathbb{C})$ (see Definition 3.0.1). For $P \in SP^h(Q)$, the straight segment in $\mathbb{C}$ joining $*_P$ and $*_{\bar{Q}}$ is disjoint from P (see Definition 3.1.2), and we use it to identify the groups $\pi_1(\mathbb{C} \setminus P, *_P) \cong \pi_1(\mathbb{C} \setminus P, *_{\bar{Q}})$. The latter group is then isomorphic to $\pi_1(\bar{Q} \setminus P, *_{\bar{Q}})$ through the inclusion $\bar{Q} \setminus P \subset \mathbb{C} \setminus P$, and from now on we will regard φ as a homomorphism of groups $\pi_1(\bar{Q} \setminus P, *_{\bar{Q}}) \to G$.

The space $\mathrm{Hur}_Q(h,G)$ is defined as $\mathrm{Hur}(h,G) \cap \widetilde{\mathrm{Hur}}_Q(h,G)$ (see Figure 5.2).

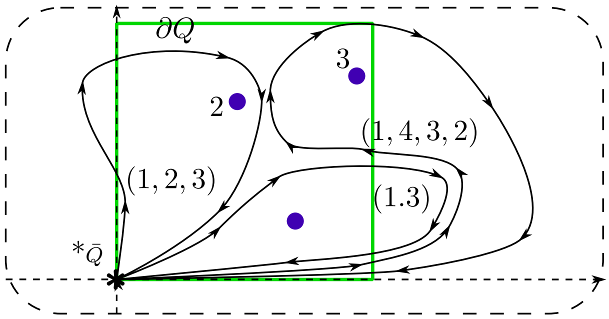

Figure 5.2. A configuration (P,φ) in $\mathrm{Hur}_Q(6,\mathfrak{S}_4)$.

Consider the homeomorphism $\mathbb{C} \cong Q$ given by shrinking both real coordinates of $\mathbb{C}$ along the function $\frac{e^t}{e^t+1} \colon \mathbb{R} \xrightarrow{\cong}]0,1[$; there is an induced homeomorphism $SP^h(\mathbb{C}) \cong SP^h(Q)$ and consequently homeomorphisms

$$\widetilde{\mathrm{Hur}}(h,G) \cong \widetilde{\mathrm{Hur}}_Q(h,G); \qquad \mathrm{Hur}(h,G) \cong \mathrm{Hur}_Q(h,G).$$

Using these homeomorphisms we can endow the one-point compactifications $\widetilde{\mathrm{Hur}}_Q(h,G)^\infty$ and $\mathrm{Hur}_Q(h,G)^\infty$ with the CW structures on $\mathrm{Hur}(h,G)$ and $\mathrm{Hur}(h,G)$ from Sections 4.2 and 4.3.

We will define homotopy-associative H-space structures on the following spaces:

$$\widetilde{\mathrm{Hur}}_Q(G) := \coprod_{h\geq 0} \widetilde{\mathrm{Hur}}_Q(h,G); \qquad \mathrm{Hur}_Q(G) := \coprod_{h\geq 0} \mathrm{Hur}_Q(h,G).$$

Definition 5.2.3. Let the embeddings $\iota_1, \iota_2 \colon \bar{Q} \to \bar{Q}$ be given by the following formulas (see also Figure 5.3):

$$\iota_1(x,y) = \left(\frac{1}{3}x + \frac{1}{12}, \frac{1}{3}y + \frac{1}{3}\right) \qquad \iota_2(x,y) = \left(\frac{1}{3}x + \frac{7}{12}, \frac{1}{3}y + \frac{1}{3}\right).$$

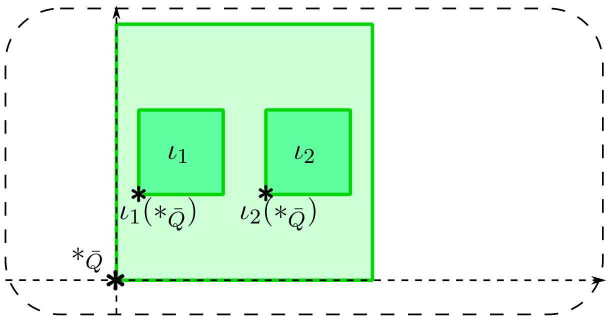

Figure 5.3. The embeddings ι_1 and ι_2.

For all $h_1, h_2 \geq 0$ we define a map

$$\mu\colon \widetilde{\mathrm{Hur}}_Q(h_1, G) \times \widetilde{\mathrm{Hur}}_Q(h_2, G) \to \widetilde{\mathrm{Hur}}_Q(h_1 + h_2, G).$$

For $i = 1, 2$, let $(P_i, \varphi_i) \in \mathrm{Hur}_Q(h_i, G)$, and define $P = \iota_1(P_1) \sqcup \iota_2(P_2) \in SP^{h_1+h_2}(Q)$. The fundamental group $\pi_1(\bar{Q} \setminus P, *_{\bar{Q}})$ can be identified as the free product of the groups $\pi_1(\bar{Q} \setminus P_1, *_{\bar{Q}})$ and $\pi_1(\bar{Q} \setminus P_2, *_{\bar{Q}})$ by including, for $i = 1, 2$, the i^{th}free factor along the map

$$\pi_1(Q \setminus P_i, *_{\bar{Q}}) \xrightarrow{(\iota_1)_*} \pi_1(Q \setminus P, \iota_i(*_{\bar{Q}})) \xrightarrow{\simeq} \pi_1(Q \setminus P, *_{\bar{Q}})$$

where the last isomorphism is obtained by translating the point $\iota_i(*_{\bar{Q}})$ to $*_{\bar{Q}}$, along a straight segment in $\bar{Q}$.
Let $\varphi\colon \pi_1(\bar{Q} \setminus P, *_{\bar{Q}}) \to G$ correspond to the couple of homomorphisms φ_1 and φ_2. Then we define

$$\mu\left((P_1, \varphi_1), (P_2, \varphi_2)\right) = (P, \varphi).$$

See Figure 5.4

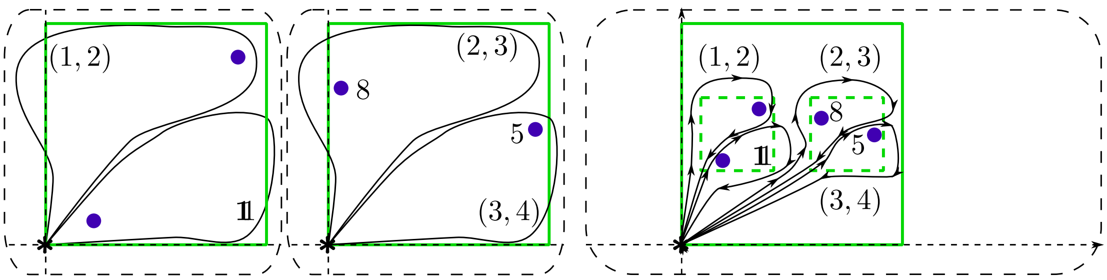

Figure 5.4. On left, two configurations in $\widetilde{\mathrm{Hur}}_Q(2, \mathfrak{S}_4)$ and $\widetilde{\mathrm{Hur}}_Q(13, \mathfrak{S}_4)$ respectively. On right, their product in $\widetilde{\mathrm{Hur}}(15, \mathfrak{S}_4)$.

The following properties of μ are straightforward:

- μ is homotopy associative;

- μ is equivariant with respect to the action of G from Subsection 5.1;
- μ is additive with respect to the weight, the absolute value and the norm (see Definitions 4.1.1, 4.3.1);
- μ restricts to a product map $\mu\colon : \mathrm{Hur}_Q(h_1, G) \times \mathrm{Hur}_Q(h_2, G) \to \mathrm{Hur}_Q(h_1 + h_2, G)$ for all $h_1, h_2 \geq 0$;
- μ is multiplicative with respect to the total monodromy (see Definition 4.4.1);

In particular μ is a map in the category of G-crossed spaces.

5.2.2 Homotopy equivalent topological monoids

We discuss briefly how to replace $\widetilde{\mathrm{Hur}}_Q(G)$ and $\mathrm{Hur}_Q(G)$ with strictly associative topological monoids.

Definition 5.2.4. Recall Definition 5.2.2. We define $\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h, G) \subset \widetilde{\mathrm{Hur}}(h, G) \times \mathbb{R}_{\geq 0}$ as the subspace of configurations (P, φ, t) with

$$P \in SP^h\left(]0, t[\times]0, 1[\right).$$

Note that when $t = 0$ the configuration P must be empty, hence in $\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h, G)$ there is only one configuration of the form $(P, \varphi, 0)$, belonging to $\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(0, G)$, namely the configuration $(\emptyset, \eta, 0)$, where $\eta\colon \{\mathbb{1}\} \to G$ is the inclusion of the trivial group.
We will let $(0, 0) = *_{\bar{Q}}$ serve as basepoint of $\bar{Q}_t := [0, t] \times [0, 1]$, and we will regard φ as a homomorphism $\varphi\colon \pi_1\left(\bar{Q}_t \setminus P, *_{\bar{Q}}\right) \to G$, with the same convention as in Definition 5.2.2.
The space $\mathrm{Hur}_Q^{\mathrm{Moore}}(h, G)$ is defined as the intersection $\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h, G) \cap \mathrm{Hur}(h, G)$ (see Figure 5.5).

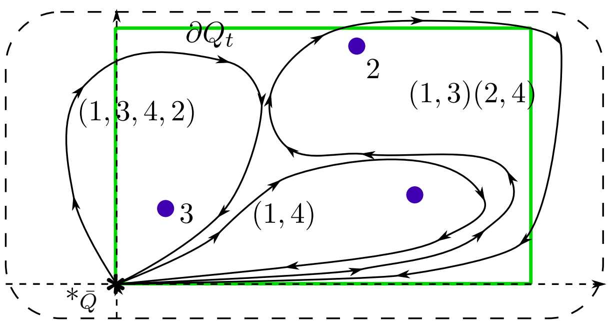

Figure 5.5. A configuration (P, φ, t) in $\mathrm{Hur}_Q^{\mathrm{Moore}}(6, \mathfrak{S}_4)$, with $t = \frac{\sqrt{5}+1}{2}$.

For all $h_1, h_2 \geq 0$ we define a map

$$\mu\colon \widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h_1, G) \times \widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h_2, G) \to \widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h_1 + h_2, G).$$

For $i=1,2$ let $(P_i,\varphi_i,t_i)\in\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h_i,G)$; let $t=t_1+t_2$, and let $P_2+t_1\in SP^{h_2}(Q_t)$ be defined by translating the points of P_2 (with their multiplicities) towards right of a distance t_1. Note that P_1 and P_2+t_1 are configurations of *distinct* points (counted with multiplicity) in Q_t; define $P=P_1\sqcup(P_2+t_1)\in SP^{h_1+h_2}(Q_t)$.
The space $\bar{Q}_t\setminus P$ is the union of $\bar{Q}_{t_1}\setminus P_1$ and $\left(\bar{Q}_{t_2}\setminus P_2\right)+t_1$ along the segment $\{t_1\}\times[0,1]$. Correspondingly we can write a free product decomposition

$$\pi_1\left(\bar{Q}_t\setminus P,*_{\bar{Q}}\right)\cong\pi_1\left(\bar{Q}_{t_1}\setminus P_1,*_{\bar{Q}}\right)\vee\pi_1\left(\left(\bar{Q}_{t_2}\setminus P_2\right)+t_1,*_{\bar{Q}}+t_1\right),$$

where for the second factor we use the horizontal segment $[0,t_1]\times\{0\}$ to translate $*_{\bar{Q}}+t_1=(t_1,0)$ to $*_{\bar{Q}}=(0,0)$.
Let $\varphi\colon\pi_1\left(\bar{Q}_t\setminus P,*_{\bar{Q}}\right)\to G$ correspond to the couple of homomorphisms φ_1 and φ_2. Then we define

$$\mu\left((P_1,\varphi_1,t_1),(P_2,\varphi_2,t_2)\right)=(P,\varphi,t).$$

See Figure 5.6.
The map μ makes

$$\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(G):=\coprod_{h\geq 0}\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h,G)$$

into a strictly associative topological monoid. The neutral element is the configuration $(\emptyset,\eta,0)\in\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(0,G)$ considered above.
For all $h_1,h_2\geq 0$ we can restrict the map $\mu\colon\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h_1,G)\times\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h_2,G)\to\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h_1+h_2,G)$ to a map

$$\mu\colon\mathrm{Hur}_Q^{\mathrm{Moore}}(h_1,G)\times\mathrm{Hur}_Q^{\mathrm{Moore}}(h_2,G)\to\mathrm{Hur}_Q^{\mathrm{Moore}}(h_1+h_2,G).$$

Thus also

$$\mathrm{Hur}_Q^{\mathrm{Moore}}(G):=\coprod_{h\geq 0}\mathrm{Hur}_Q^{\mathrm{Moore}}(h,G)$$

becomes a strictly associative topological monoid, with the same neutral element.

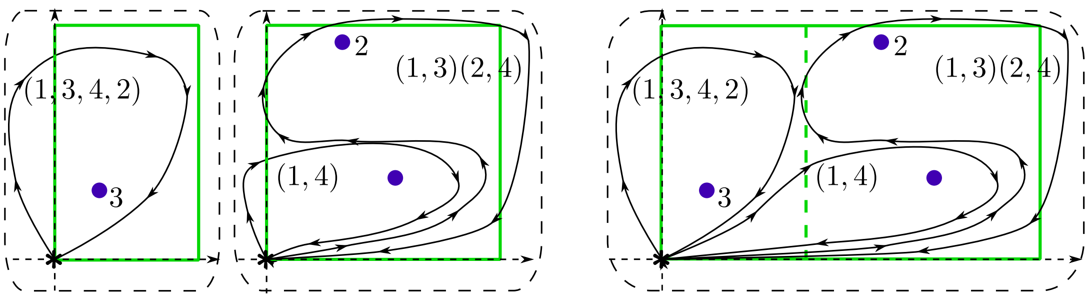

Figure 5.6. The configuration from Figure 5.5 (on right) can be obtained as a product of two configurations in $\mathrm{Hur}_Q^{\mathrm{Moore}}(3,\mathfrak{S}_4)$ (on left).

The forgetful map $\widetilde{\mathrm{Hur}}_Q^{\mathrm{Moore}}(h,G) \to \widetilde{\mathrm{Hur}}(h,G)$ sending (P,φ,t) to (P,φ) is a homotopy equivalence; similarly the spaces $\mathrm{Hur}_Q^{\mathrm{Moore}}(h,G)$ and $\mathrm{Hur}(h,G)$ are homotopy equivalent. In particular $H_*(\mathrm{Hur}(G)) \simeq H_*(Hur_Q(G)) \simeq H_*\left(\mathrm{Hur}_Q^{\mathrm{Moore}}(G)\right)$ is an associative Pontryagin ring. In Chapter 9 we will recover the Pontryagin product on $H_*(\mathrm{Hur}(\mathfrak{S}_d))$ in a different, more algebraic way.

5.3 Action of the operad of little squares

Definition 5.3.1. Recall Definition 4.4.2 and recall from Subsection 5.2.1 the homeomorphisms $\widetilde{\mathrm{Hur}}(h,G) \cong \widetilde{\mathrm{Hur}}_Q(h,G)$ and $\mathrm{Hur}(h,G) \cong \mathrm{Hur}_Q(h,G)$. Using the total monodromy we obtain decompositions

$$\widetilde{\mathrm{Hur}}_Q(h,G) = \coprod_{\gamma\in G} \widetilde{\mathrm{Hur}}_Q(h,G,\gamma); \qquad \mathrm{Hur}_Q(h,G) = \coprod_{\gamma\in G} \mathrm{Hur}_Q(h,G,\gamma).$$

For $\gamma \in G$ we define the following disjoint unions:

$$\widetilde{\mathrm{Hur}}_Q(G,\gamma) = \coprod_{h\geq 0} \widetilde{\mathrm{Hur}}_Q(h,G,\gamma); \qquad \mathrm{Hur}_Q(G,\gamma) = \coprod_{h\geq 0} \mathrm{Hur}_Q(h,G,\gamma).$$

Note in particular that $\widetilde{\mathrm{Hur}}_Q(G,\mathbb{1}) \subset \widetilde{\mathrm{Hur}}_Q(G)$ and $\mathrm{Hur}_Q(G,\mathbb{1}) \subset \mathrm{Hur}_Q(G)$ are sub-H-spaces, because the total monodromy is multiplicative with respect to the map μ from Subsection 5.2.1.

In this section we will prove that $\widetilde{\mathrm{Hur}}(G,\mathbb{1})$ and $\mathrm{Hur}(G,\mathbb{1})$ are algebras over the operad $\mathcal{C} = \mathcal{C}_2$ of little squares [9, 29].

5.3.1 Definition of the operad $\mathcal{C}$

We define $\mathcal{C}(k)$ as the space of configurations of disjoint embeddings $\iota_1,\dots,\iota_k$ of $\bar{Q}$ into Q which are a composition of a conformal dilation and a translation. Every $\iota_i(\bar{Q}) \subset Q$ is called a *little square*. As an example, the embeddings $\iota_1,\iota_2\colon \bar{Q} \to Q \subset \bar{Q}$ from Definition 5.2.3 give a configuration in $\mathcal{C}(2)$.

Note that this definition is slightly different from the standard one, in which one considers configurations of embeddings $\iota_1,\dots,\iota_k\colon \bar{Q} \to \bar{Q}$ and requires only the *interiors* of the little squares, i.e. $\iota_1(Q),\dots,\iota_k(Q)$, to be disjoint. In particular with our definition the operad $\mathcal{C}$ lacks a strict unit (which would be the identity of $\bar{Q}$, as a configuration in $\mathcal{C}(1)$). Nevertheless the two definitions yield homotopy equivalent spaces $\mathcal{C}(k)$.

For a configuration $\underline{\iota} = (\iota_1,\dots,\iota_k)$ in $\mathcal{C}(k)$, we denote by $\bar{Q}\setminus\underline{\iota}$ the closed, connected subspace

$$\bar{Q}\setminus\underline{\iota} = \bar{Q}\setminus\left(\coprod_{i=1}^{k}\iota_i(Q)\right) \subset \bar{Q}.$$

By our definition $\bar{Q}\setminus\underline{\iota}$ is a surface of genus 0 with $k+1$ disjoint boundary components: this property will be convenient later and justifies our slightly different definition of $\mathcal{C}$ from the standard one.

For all $k \geq 0$ and $l_1, \ldots, l_k \geq 0$ there are structure maps

$$\mu\colon \mathcal{C}(k) \times (\mathcal{C}(l_1) \times \cdots \times \ldots \mathcal{C}(l_k)) \to \mathcal{C}(l_1 + \cdots + l_k),$$

given by $\mu\colon (\underline{\iota}; \underline{\iota}'_1, \ldots, \underline{\iota}'_k) \mapsto \underline{\iota}''$, where for all $1 \leq r \leq k$ and $1 \leq s \leq l_r$ we define

$$\iota''_{\sum_{i=1}^{r-1} l_i + s} = \iota_r \circ \iota'_{r,s}.$$

5.3.2 Hurwitz spaces as algebras over $\mathcal{C}$

For all $k \geq 0$ and all $h_1, \ldots, h_k \geq 0$ we define a map

$$\mu\colon \mathcal{C}(k) \times \left(\widetilde{\mathrm{Hur}}_Q(h_1, G, \mathbb{1}) \times \cdots \times \widetilde{\mathrm{Hur}}_Q(h_k, G, \mathbb{1})\right) \to \widetilde{\mathrm{Hur}}_Q(h, G, \mathbb{1}),$$

where $h = h_1 + \cdots + h_k$.
Let $\underline{\iota} = (\iota_1, \ldots, \iota_k) \in \mathcal{C}(k)$ and for $1 \leq i \leq k$ let $(P_i, \varphi_i) \in \widetilde{\mathrm{Hur}}_Q(h_i, G, \mathbb{1})$, with $P_i = \{m_{i,j} \cdot z_{i,j}\}_{1 \leq j \leq \kappa_i} \in SP^{h_i}(Q)$ and $\varphi_i\colon \pi_1\left(\bar{Q} \setminus P_i, *_{\bar{Q}}\right) \to G$. Define P as the following disjoint union of configurations:

$$P = \iota_1(P_1) \sqcup \cdots \sqcup \iota_k(P_k) \in SP^h(Q).$$

Compare the following construction with the one in the proof of Lemma 4.3.3. Choose arcs $\zeta_1, \ldots, \zeta_k \subset \bar{Q} \setminus \underline{\iota}$ satisfying the following properties:

- for $1 \leq i \leq k$ the arc ζ_i connects $*_{\bar{Q}}$ with $\iota_i(*_{\bar{Q}})$;
- each arc ζ_i is contained in the interior of $\bar{Q} \setminus \underline{\iota}$, except its endpoints;
- the arcs $\zeta_1, \ldots, \zeta_k$ are disjoint except at their common endpoint $*_{\bar{Q}}$.

For $1 \leq i \leq k$ set $\check{Q}_i := \zeta_i \cup \iota_i(\bar{Q} \setminus P_i) \subset \bar{Q}$. Note that $\check{Q}_i$ deformation retracts onto $\iota_i(\bar{Q} \setminus P_i)$. We have a sequence of isomorphisms of groups

$$\pi_1\left(\bar{Q} \setminus P_i, *_{\bar{Q}}\right) \cong \pi_1\left(\iota_i(\bar{Q} \setminus P_i), \iota_i(*_{\bar{Q}})\right) \cong \pi_1\left(\check{Q}_i, \iota_i(*_{\bar{Q}})\right) \cong \pi_1\left(\check{Q}_i, *_{\bar{Q}}\right),$$

where the last isomorphism is obtained translating the basepoint along the arc ζ_i. Moreover the space $\bar{Q} \setminus P$ deformation retracts onto the union $\check{Q}_1 \cup \cdots \cup \check{Q}_k$, which is really a wedge sum based at $*_{\bar{Q}}$. Consequently we can write $\pi_1\left(\bar{Q} \setminus P, *_{\bar{Q}}\right)$ as a free product

$$\pi_1\left(\bar{Q} \setminus P, *_{\bar{Q}}\right) \cong \pi_1\left(\check{Q}_1, *_{\bar{Q}}\right) \vee \cdots \vee \pi_1\left(\check{Q}_k, *_{\bar{Q}}\right) \cong \pi_1\left(\bar{Q} \setminus P_1, *_{\bar{Q}}\right) \vee \cdots \vee \pi_1\left(\bar{Q} \setminus P_k, *_{\bar{Q}}\right).$$

We define $\varphi\colon \pi_1\left(\bar{Q} \setminus P, *_{\bar{Q}}\right) \to G$ as the homomorphism corresponding to the sequence of homomorphisms $\varphi_1, \ldots, \varphi_k$ under the identification above. Thus we obtain a configuration $(P, \varphi) \in \widetilde{\mathrm{Hur}}_Q(h, G)$ which a priori depends on the choice of the arcs ζ_i.
Note that the large loop $\mathcal{L}^P \in \pi_1\left(\bar{Q} \setminus P, *_{\bar{Q}}\right)$ corresponds, under the free product decomposition above, to the product of the large loops $\mathcal{L}^{P_i} \in \pi_1\left(\bar{Q} \setminus P_i, *_{\bar{Q}}\right)$, taken in some order. Correspondingly the total monodromy $\omega(P, \varphi)$ is the product of the total

monodromies $\omega(P_i,\varphi_i)$ taken in some order, and since $\omega(P_i,\varphi_i)=\mathbb{1}$ for all $1\le i\le k$, we also have $\omega(P,\varphi)=\mathbb{1}$: this shows that $(P,\varphi)\in\widetilde{\mathrm{Hur}}_Q(h,G,\mathbb{1})$.
More generally the following composition is the trivial map, using the fact that the subgroup $\pi_1\left(\bar{Q}\setminus\underline{\iota},*_{\bar{Q}}\right)\subset\pi_1(\bar{Q}\setminus P,*_{\bar{Q}})$ is generated by the large loops $\mathcal{L}^{P_i}$ under the identification above:

$$\pi_1\left(\bar{Q}\setminus\underline{\iota},*_{\bar{Q}}\right)\to\pi_1\left(\bar{Q}\setminus P,*_{\bar{Q}}\right)\to G.$$

Let now $\zeta'_1,\dots,\zeta'_i$ be another choice of arcs satisfying the conditions above, and for all $1\le i\le k$ denote by $\check{Q}'_i=\zeta'_i\cup\iota_i(\bar{Q}\setminus P_i)\subset\bar{Q}$; denote by $\varphi'\colon\pi_1\left(\bar{Q}\setminus P,*_{\bar{Q}}\right)\to G$ the homomorphism of groups obtained using the system of arcs $\zeta'_1,\dots,\zeta'_k$ instead of $\zeta_1,\dots,\zeta_k$ in the construction above.
Then we can embed the group $\pi_1\left(\bar{Q}\setminus P_i,*_{\bar{Q}}\right)\cong\pi_1\left(\iota_i\left(\bar{Q}\setminus P_i\right),\iota_i(*_{\bar{Q}})\right)$ in two different ways into $\pi_1\left(\bar{Q}\setminus P,*_{\bar{Q}}\right)$, namely as $\pi_1\left(\check{Q}_i,*_{\bar{Q}}\right)$, using the translation of the basepoint along ζ_i, and as $\pi_1\left(\check{Q}'_i,*_{\bar{Q}}\right)$, translating the basepoint along ζ'_i. The two subgroups of $\pi_1\left(\bar{Q}\setminus P,*_{\bar{Q}}\right)$ that we obtain are conjugate under the element of $\pi_1\left(\bar{Q}\setminus P,*_{\bar{Q}}\right)$ represented by the loop $\zeta_i\star(\zeta'_i)^{-1}$.
This is a loop in $\bar{Q}\setminus\underline{\iota}$, and therefore $\varphi(\zeta_i\star(\zeta'_i)^{-1})=\mathbb{1}$. Therefore the following two compositions are the same homomorphism of groups, up to conjugation by $\mathbb{1}$ (i.e. they are really the same homomorphism!):

$$\pi_1\left(\bar{Q}\setminus P_i,*_{\bar{Q}}\right)\cong\pi_1\left(\check{Q}_i,*_{\bar{Q}}\right)\subset\pi_1\left(\bar{Q}\setminus P_i,*_{\bar{Q}}\right)\xrightarrow{\varphi}G;$$

$$\pi_1\left(\bar{Q}\setminus P_i,*_{\bar{Q}}\right)\cong\pi_1\left(\check{Q}'_i,*_{\bar{Q}}\right)\subset\pi_1\left(\bar{Q}\setminus P_i,*_{\bar{Q}}\right)\xrightarrow{\varphi}G.$$

This shows that φ and φ' are the same homomorphism $\pi_1\left(\bar{Q}\setminus P,*_{\bar{Q}}\right)\to G$, and therefore the configuration (P,φ) constructed above does not depend on the choice of the arcs ζ_i.
We define

$$\mu\left(\underline{\iota};(P_1,\varphi_1),\dots,(P_k,\varphi_k)\right)=(P,\varphi).$$

See Figure 5.7.
This construction makes $\widetilde{\mathrm{Hur}}(G,\mathbb{1})$ into an algebra over the operad $\mathcal{C}$. The sssociativity of the product can be proved by an argument similar to the one of the proof of Lemmas 4.3.3 and Theorem 7.5.1, see in particular Subsection 7.5.3. We omit the details.
The maps μ defined above are additive with respect to the absolute value, the weight and the norm, and in particular they restrict to maps

$$\mu\colon\mathcal{C}(k)\times\left(\mathrm{Hur}_Q(h_1,G,\mathbb{1})\times\cdots\times\mathrm{Hur}_Q(h_k,G,\mathbb{1})\right)\to\mathrm{Hur}_Q(h_1+\cdots+h_k,G,\mathbb{1}).$$

Hence also $\mathrm{Hur}(G,\mathbb{1})$ is an algebra over the operad $\mathcal{C}$.
If G is not a commutative group it is not possible to enhance the above construction to a structure of algebra over $\mathcal{C}$ for the entire space $\mathrm{Hur}(G)$; one can however replace the operad $\mathcal{C}_2$ of little squares with a certain *coloured* operad, called the operad of *G-little bundles* and denoted $\mathcal{C}_2^G$: the operad $\mathcal{C}_2^G$ has colours in the set G, i.e. the spaces $\mathcal{C}_2^G(k)$ decompose as disjoint unions

$$\mathcal{C}_2^G(k)=\coprod_{\gamma_1,\dots,\gamma_k,\gamma\in G}\mathcal{C}_2^G(\gamma_1,\dots,\gamma_k;\gamma)$$

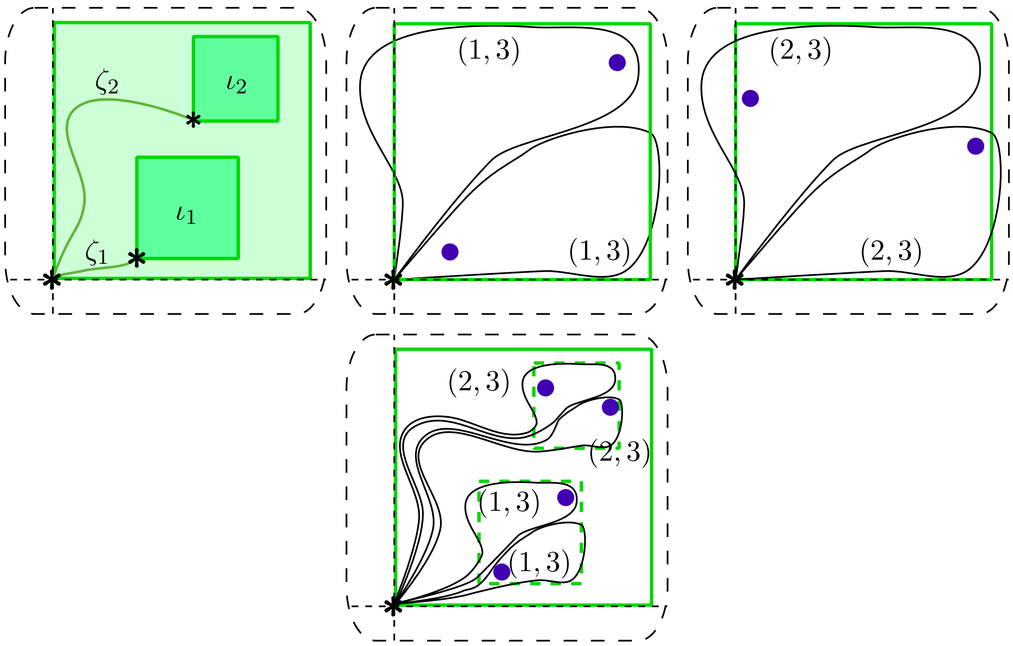

Figure 5.7. On top, a configuration $\underline{\iota} = (\iota_1, \iota_2) \in \mathcal{C}(2)$, with a choice of arcs ζ_1, ζ_2, and two configurations (P_1, φ_1) and (P_2, φ_2) in $\mathrm{Hur}_Q(2, \mathfrak{S}_3, \mathbb{1}) \subset \widetilde{\mathrm{Hur}}_Q(2, \mathfrak{S}_3, \mathbb{1})$. On bottom, the product $\mu(\underline{\iota}; (P_1, \varphi_1), (P_, \varphi_2)) \in \mathrm{Hur}_Q(4, \mathfrak{S}_3, \mathbb{1}) \subset \widetilde{\mathrm{Hur}}_Q(4, \mathfrak{S}_3, \mathbb{1})$.

and the composition maps are only defined for compatible sequences of inputs and outputs, i.e. they take the form

$$\mu\colon \mathcal{C}_2^G\left(\underline{\beta}; \gamma\right) \times \left(\mathcal{C}_2^G\left(\underline{\alpha}_1; \beta_1\right) \times \cdots \times \mathcal{C}_2^G\left(\underline{\alpha}_l; \beta_l\right)\right) \to \mathcal{C}_2^G\left(\underline{\alpha}; \gamma\right),$$

where $\underline{\beta} = (\beta_1, \ldots, \beta_l)$, $\underline{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,\lambda_i})$ for all $1 \le i \le l$, and $\underline{\alpha}$ is the sequence of $\lambda_1 + \cdots + \lambda_l$ elements of G obtained by concatenating the sequences $\underline{\alpha}_1, \ldots, \underline{\alpha}_l$.
The operad $\mathcal{C}_2^G$ was introduced in an equivalent way in [32], as an operad with colours in the loop space ΩBG; this space is however homotopy equivalent to G, and in fact one can replace the construction from [32] with one for which the spaces $\mathcal{C}_2^G(k)$ are finite dimensional manifolds, just as for the operad $\mathcal{C}_2$.
For $d \ge 3$ and $G = \mathfrak{S}_d$ the operad $\mathcal{C}_2^{\mathfrak{S}_d}$ seems to have a rather complicated homology, and understanding the homology of $\mathrm{Hur}(\mathfrak{S}_d)$ as an algebra over the homology of $\mathcal{C}_2^{\mathfrak{S}_d}$ seems a very difficult problem.

6 Classical models for the moduli space of Riemann surfaces

In this chapter we introduce the moduli space $\mathfrak{M}_{g,n}$ of Riemann surfaces of genus $g \geq 0$ with $n \geq 1$ marked points, called *poles*. Each pole is endowed with a fixed tangent vector.

Understanding the homotopy type of the spaces $\mathfrak{M}_{g,n}$ for different values of g and n, and in particular their homology, is the main goal of the thesis. In this chapter we will recall some classical properties and constructions with the moduli spaces $\mathfrak{M}_{g,n}$; this is done in preparation to Chapter 7, where we will establish a connection between the moduli spaces $\mathfrak{M}_{g,n}$ and the Hurwitz spaces discussed in Chapter 4.

In Section 6.1 we recall the main properties of the moduli space $\mathfrak{M}_{g,n}$.

In Section 6.2 we recall the construction of the space $\mathfrak{Par}_{g,n}[\underline{d}] \simeq \mathfrak{M}_{g,n}$ and the cell decomposition of its one-point compactification $\mathfrak{Par}_{g,n}[\underline{d}]^{\infty}$ (see [10, 1, 11]).

In Section 6.3 we discuss a refinement of the cell decomposition from Section 6.2, which will be useful in Chapter 7.

6.1 A brief introduction to moduli spaces

Let $g \geq 0$ and $n \geq 1$ be fixed throughout the chapter; assume for simplicity that $g \neq 0$ or $n \neq 1$: the discussion in the case $g = 0$ and $n = 1$ is slightly different and we will add some remarks when needed.

We denote by Σ_g a closed, smooth, connected, orientable surface of genus g. We denote by $\Sigma_{g,n}$ the surface Σ_g with a choice of n distinct points $Q_1, \dots, Q_n$, called *poles*; each Q_i is endowed with a fixed non-zero tangent vector $X_i \in T_{Q_i}\Sigma_g$ (see Figure 6.1.

We abbreviate $\underline{Q} = (Q_1, \dots, Q_n)$ and $\underline{X} = (X_1, \dots, X_n)$. The open surface $\Sigma_{g,n} \setminus \{Q_1, \dots, Q_n\}$ is usually denoted by $\Sigma_{g,n} \setminus \underline{Q}$.

A sequence $\underline{d} = (d_1, \dots, d_n)$ of numbers $d_i \geq 1$ is fixed throughout the chapter, and we denote by d the sum $\sum_{i=1}^{n} d_i$. It is convenient to assume already in this chapter that $d \geq 2$, although we will only see in Chapter 7, after Theorem 7.1.2, why the case $d = n = 1$ is of little interest for our purposes.

Finally, we set $h := 2g + n + d - 2$ throughout the chapter.

6.1.1 Definition of the moduli space $\mathfrak{M}_{g,n}$

A Riemann structure $\mathfrak{r}$ on $\Sigma_{g,n}$ is a Riemann structure $\mathfrak{r}$ on the smooth surface Σ_g, which can be regarded either as a maximal atlas with holomorphic transition maps, or as a conformal class of Riemannian metrics on Σ_g.

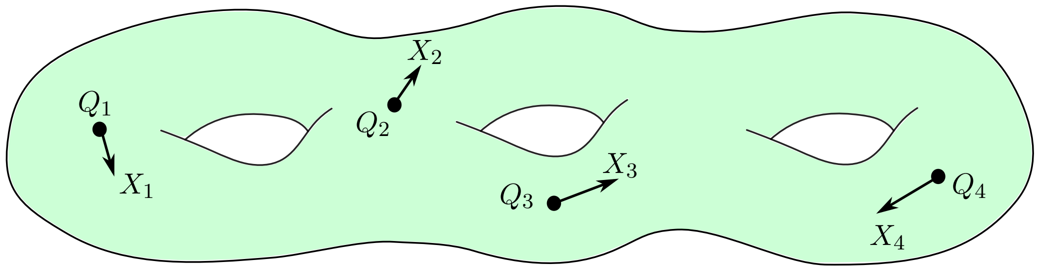

Figure 6.1. A surface of genus 3 with 4 poles.

Definition 6.1.1. Fix a Riemann structure $\mathfrak{r}$ on $\Sigma_{g,n}$. A *normal chart* at (Q_i, X_i) is a holomorphic chart $w_i \colon U_i \to \mathbb{C}$ defined on a neighbourhood $Q_i \in U_i \subset \Sigma_{g,n}$ with the following properties:

- $w_i \colon Q_i \mapsto 0 \in \mathbb{C}$;
- $Dw_i \colon X_i \mapsto \partial/\partial x$, where $x = \Re(z)$ and $y = \Im(z)$ are the standard coordinates on $\mathbb{C} \simeq \mathbb{R}^2$, and $\partial/\partial x$ is the dual vector of dx (informally, it is the horizontal unit vector pointing to right).

The composition of a normal chart with the inverse of another normal chart is a holomorphic function $f = f(z)$ defined on some neighbourhood of $0 \in \mathbb{C}$; its Taylor expansion has the form

$$f(z) = z + l.o.t.,$$

where the lower order terms correspond to higher powers of z. In particular the differential of a normal chart $Dw_i \colon T_{Q_i}\Sigma_{g,n} \to T_0\mathbb{C}$ does not depend on the normal chart.
Note also that X_i can be recovered from a normal chart w as $Dw_i^{-1}(\partial/\partial x)$, so given a Riemann surface $(\Sigma_g, \mathfrak{r})$ with a choice of poles $Q_1, \dots, Q_n$, we can define the vectors X_i by imposing that certain charts w_i around the Q_i's are normal.

Definition 6.1.2. We denote by $\mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right)$ the topological group of diffeomorphisms of $\Sigma_{g,n}$ that preserve the orientation and fix all points Q_i and all vectors X_i. The *mapping class group* $\Gamma_{g,n}$ is the group of isotopy classes of such diffeomorphisms, i.e.

$$\Gamma_{g,n} = \pi_0\left(\mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right)\right).$$

Definition 6.1.3. We denote by $\mathfrak{M}_{g,n}$ the moduli space of equivalence classes of Riemann structures $\mathfrak{r}$ on $\Sigma_{g,n}$. Two Riemann structures $\mathfrak{r}$ and $\mathfrak{r}'$ on $\Sigma_{g,n}$ are *equivalent* if there exists a diffeomorphism $\psi \in \mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right)$ which is a biholomorphism from $(\Sigma_{g,n}, \mathfrak{r})$ to $(\Sigma_{g,n}, \mathfrak{r}')$.
More formally, we consider the action of the topological group $\mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right)$ on the space $\mathfrak{Riem}(\Sigma_{g,n})$ of Riemann structures on $\Sigma_{g,n}$; the moduli space $\mathfrak{M}_{g,n}$ is the orbit space

$$\mathfrak{M}_{g,n} = \mathfrak{Riem}(\Sigma_{g,n}) / \mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right).$$

We will usually denote by $\mathfrak{m} = [\mathfrak{r}] \in \mathfrak{M}_{g,n}$ a *modulus* on $\Sigma_{g,n}$, i.e. an equivalence class of Riemann structures on $\Sigma_{g,n}$; here $\mathfrak{r}$ is a Riemann structure representing $\mathfrak{m}$.

6.1.2 Basic properties of the moduli spaces

If $g = 0$ and $n = 1$ the space $\mathfrak{M}_{g,n}$ consists of only one point, that is, all Riemann structures on $\Sigma_{0,1}$ are equivalent. Our preferred model of Riemann surface $(\Sigma_{0,1}, \mathfrak{r})$ is $\mathbb{C}P^1$: the unique pole Q_1 is the point at infinity $\infty = [1 : 0]$; the vector $X_1 \in T_\infty \mathbb{C}P^1$ is the unique vector such that the chart $[a : b] \mapsto b/a \in \mathbb{C}$, defined on a neighbourhood of ∞, is normal (see Definition 6.1.1). The difference $\mathbb{C}P^1 \setminus \{Q_1\}$ is identified with $\mathbb{C}$ through the map $[a : b] \mapsto a/b$.
The action of $\mathrm{Diff}^+(\Sigma_{1,0})$ on $\mathfrak{Riem}(\Sigma_{0,1})$ is transitive but not free. The isotropy group is the group of biholomorphisms $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ that fix ∞ and $T_\infty\mathbb{C}P^1$: this group is isomorphic to the additive group $\mathbb{C}$, where $\lambda \in \mathbb{C}$ corresponds to the map $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ mapping $z \mapsto z + \lambda$ for $z \in \mathbb{C}$, and $\infty \mapsto \infty$.
Assuming $g \neq 0$ or $n \neq 1$, then for all Riemann structures $\mathfrak{r}$ and $\mathfrak{r}'$ on $\Sigma_{g,n}$ there is at most one biholomorphism $\psi \in \mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right)$ from $(\Sigma_{g,n}, \mathfrak{r})$ to $(\Sigma_{g,n}, \mathfrak{r}')$: that is, the action of $\mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right)$ on $\mathfrak{Riem}(\Sigma_{g,n})$ is free. Moreover the space $\mathfrak{M}_{g,n}$ is a connected, orientable, open manifold of dimension $6g - 6 + 4n$, and it is an Eilenberg-Maclane space of type $K(\Gamma_{g,n}, 1)$, i.e. a classifying space for the mapping class group $\Gamma_{g,n}$. See [13] and [23] for these classical results in Teichmüller theory.

6.1.3 Combinatorial models

In order to study the homotopy type of $\mathfrak{M}_{g,n}$, and in particular to compute its homology, it is convenient to have a *combinatorial model* for this space. Here by *combinatorial model* we mean in great generality a pair of finite CW-complexes $(\mathcal{X}, \mathcal{X}')$ whose definition (cells and attaching maps) is combinatorial in flavour, and such that the difference $\mathcal{X} \setminus \mathcal{X}'$ is homotopy equivalent to $\mathfrak{M}_{g,n}$.
We will present two such models: in both of them the difference $\mathcal{X} \setminus \mathcal{X}'$ will also be a connected, orientable, open manifold, of higher dimension than $\mathfrak{M}_{g,n}$; moreover the quotient $\mathcal{X}/\mathcal{X}'$ will be homeomorphic to the one-point compactification of $\mathcal{X} \setminus \mathcal{X}'$.
In Section 6.2 we will recall the space $\mathfrak{Par}_{g,n}[\underline{d}] \simeq \mathfrak{M}_{g,n}$ introduced by Bödigheimer [10] in the case $n = d = 1$ (see also [1]), and generalised to the case $d \geq 2$ by Boes and Hermann [11]; the space $\mathfrak{Par}_{g,n}[\underline{d}]$ arises as difference of finite CW-complexes $P_{g,n}[\underline{d}] \setminus P'_{g,n}[\underline{d}]$.
In Section 6.3 we will slightly refine the cell decomposition of $P_{g,n}[\underline{d}]$, and describe in detail the attaching maps.
In Chapter 7 we will then consider a particular subspace $\bar{\mathcal{O}}_{g,n}[\underline{d}] \subset \mathfrak{Par}_{g,n}[\underline{d}]$. We will prove in particular that if $d = \sum_{i=1}^n d_i \geq 2g - 1 + n$, then the latter inclusion is a homotopy equivalence, and hence $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ provides a new combinatorial model for $\mathfrak{M}_{g,n}$.

6.2 The parallel slit complex

The model $\mathfrak{Par}_{g,n}[\underline{d}]$ is constructed by considering Riemann surfaces $(\Sigma_{g,n}, \mathfrak{r})$ equipped with a harmonic function $u\colon \Sigma_{g,n} \setminus \underline{Q} \to \mathbb{R}$ having some prescribed behaviour near the poles.

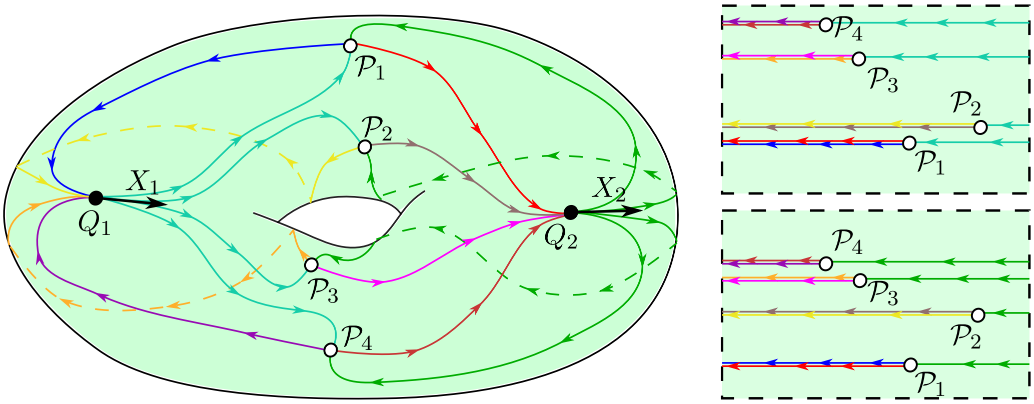

Figure 6.2. On left, a Riemann surface $\mathcal{S}$ of type $\Sigma_{1,2}$, together with a harmonic function $u\colon \mathcal{S} \setminus \{Q_1, Q_2\} \to \mathbb{R}$ having directed poles of order 1 at Q_1 and Q_2: the critical points of u are $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$, and the flow lines of $-\nabla u$ entering or exiting from the critical points are shown. On right, the associated slit picture. Here $d = 2$ and $d_1 = d_2 = 1$.

One can cut the surface along the critical lines of the flow associated with $-\nabla u$; the cut surface consists of d connected components, each of which can be embedded in $\mathbb{C}$ as the complement of a finite collection of *slits*. A slit is a horizontal halfline in $\mathbb{C}$ running towards left, i.e. of the form $\{z_0 - t \,|\, t \geq 0\}$ for some $z_0 \in \mathbb{C}$ (see Figure 6.2).
The process of dissecting a *single* Riemann surface as described above, in order to represent its parts as nice open sets of the plane, is called *Hilbert uniformisation* [25].
Bödigheimer [10] considers *all Riemann surfaces* at the same time and describes the model $\mathfrak{Par}_{1,g}[1]$ for the moduli space $\mathfrak{M}_{g,1}$; the same technique works for surfaces with more than one pole and with higher values for the numbers d_i: a detailed description of this generalisation can be found in [11]; a brief description of the construction in the case $n = d = 1$ can also be found in [1].

6.2.1 The auxiliary moduli space $\mathfrak{H}_{g,n}[\underline{d}]$

Definition 6.2.1. We introduce a space $\tilde{\mathfrak{H}}_{g,n}[\underline{d}]$. A point in $\tilde{\mathfrak{H}}_{g,n}[\underline{d}]$ is given by a choice of the following:

- a Riemann structure $\mathfrak{r}$ on $\Sigma_{g,n}$;
- a harmonic function $u\colon \Sigma_{g,n} \setminus \underline{Q} \to \mathbb{R}$;
- a harmonic function $v\colon \Sigma_{g,n} \setminus \mathcal{K}_0 \to \mathbb{R}$, where $\mathcal{K}_0 = \mathcal{K}_0(u) \subset \Sigma_{g,n}$ is the critical graph of u,

satisfying some conditions that we describe in the following.
First, we understand that the functions u and v are harmonic with respect to $\mathfrak{r}$. To define $\mathcal{K}_0 = \mathcal{K}_0(u)$ we choose a Riemannian metric $\mathfrak{g}$ on $\Sigma_{g,n}$ whose conformal class is $\mathfrak{r}$, and consider the flow associated with $-\nabla u$, the opposite of the gradient of u, on $\Sigma_{g,n} \setminus \underline{Q}$.
The graph $\mathcal{K}_0$ contains the following points of $\Sigma_{g,n}$:

- all critical points of u, i.e. points where $\nabla u = 0$;
- all flow lines having a critical point of u as negative limit, and having either a pole Q_i or another critical point of u as positive limit;
- all poles Q_i (see Figure 6.3).

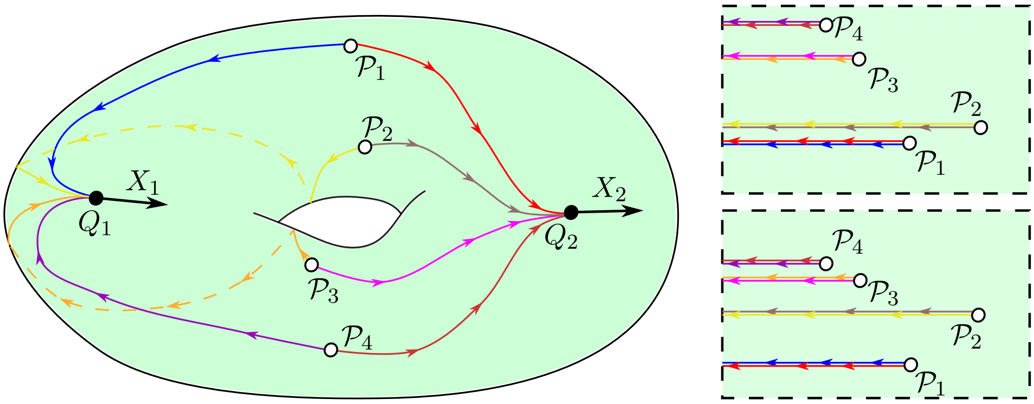

Figure 6.3. On left, the critical graph $\mathcal{K}_0 \subset \mathcal{S}$ in the example of Figure 6.2. It is an oriented graph with 6 vertices and 8 edges embedded in the surface $\mathcal{S}$. The complement $\mathcal{S} \setminus \mathcal{K}_0$ is endowed with a holomorphic embedding into $\mathbb{C}_1 \sqcup \mathbb{C}_2$; the image of this embedding is the complement of the infinite halflines, called *slits*, shown on right.

Note that the subspace $\mathcal{K}_0 \subset \Sigma_{g,n}$ only depends on $\mathfrak{r}$ and not on $\mathfrak{g}$. A triple $(\mathfrak{r}, u, v)$ in $\tilde{\mathfrak{H}}_{g,n}[\underline{d}]$ must fulfil the following properties.

- For all $1 \le i \le n$, in a normal chart $w_i \colon U_i \to \mathbb{C}$ around Q_i we can write u as

$$u(w_i) = \Re\left(1/w_i^{d_i} + l.o.t.\right) - B_i \log|w_i|,$$

  for some real number B_i; the lower order terms correspond to powers of w_i with exponent higher than $-d_i$. We say that u has a *directed* pole of order d_i at Q_i.

- The function $u + \sqrt{-1}v$ is holomorphic on $\Sigma_{g,n} \setminus \mathcal{K}_0$, i.e. v is a harmonic conjugate of u on $\Sigma_{g,n} \setminus \mathcal{K}_0$.

Note that the first condition does not depend on the normal chart. We also say that u is a $\underline{d}$-directed harmonic function on $\Sigma_{g,n} \setminus \underline{Q}$.

There is a bundle map

$$\tilde{\mathcal{H}} \colon \tilde{\mathfrak{H}}_{g,n}[\underline{d}] \to \mathfrak{Riem}(\Sigma_{g,n}),$$

mapping $(\mathfrak{r}, u, v)$ to $\mathfrak{r}$; the fibre, denoted by $\tilde{\mathfrak{H}}(\mathfrak{r}, \underline{d})$, has the structure of a real affine space of dimension $3h - \dim(\mathfrak{M}_{g,n})$.
To see that $\tilde{\mathfrak{H}}(\mathfrak{r}, \underline{d})$ has a real affine structure, choose for every $1 \le i \le n$ and $1 \le j \le d_i$, a germ of a smooth path $L_{i,j} \colon [0, \varepsilon) \to \Sigma_{g,n}$ with

- $L_{i,j}(0) = Q_i$;
- $DL_{i,j}|_{t=0}(\partial/\partial t) = e^{2\pi\sqrt{-1}(j-1)/d_i} X_i$; here $\partial/\partial t \in T_0[0,\varepsilon)$ is the standard generator, and the rotation by $e^{2\pi\sqrt{-1}(j-1)/d_i}$ is taken with respect to $\mathfrak{r}$ (see Figure 6.4).

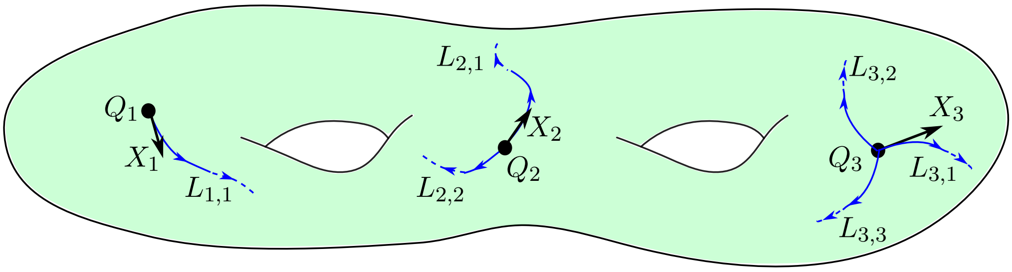

Figure 6.4. An example of choices of the germs $L_{i,j}$ for $n = 3$ and $\underline{d} = (1, 2, 3)$.

Then for all $\underline{d}$-directed harmonic functions u on $(\Sigma_{g,n}, \mathfrak{r})$ the space $\Sigma_{g,n} \setminus \mathcal{K}_0(u)$ has d connected components, containing each one of the germs of paths $L_{i,j}$. The harmonic conjugate v is uniquely determined by the germs of its restrictions on the paths $L_{i,j}$. Given (u, v) and $(u', v') \in \tilde{\mathfrak{H}}(\mathfrak{r}, \underline{d})$ and $\lambda \in \mathbb{R}$, we can define $\lambda(u, v) + (1 - \lambda)(u', v')$ as the couple (u'', v'') with:

- $u'' = \lambda u + (1 - \lambda)u' \colon \Sigma_{g,n} \setminus \underline{Q} \to \mathbb{R}$;
- v'' is the unique harmonic conjugate of u'' on $\Sigma_{g,n} \setminus \mathcal{K}_0(u'')$ whose germs of restrictions on the paths $L_{i,j}$ are equal to the corresponding germs of the function $\lambda v + (1 - \lambda)v'$.

The group $\mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right)$ acts on $\tilde{\mathfrak{H}}_{g,n}[\underline{d}]$ by pulling back both the Riemann structure $\mathfrak{r}$ and the functions u and v on their respective domains.

Definition 6.2.2. We define the space $\mathfrak{H}_{g,n}[\underline{d}]$ as the quotient space

$$\mathfrak{H}_{g,n}[\underline{d}] := \tilde{\mathfrak{H}}_{g,n}[\underline{d}] / \mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right).$$

The bundle map $\tilde{\mathcal{H}}$ yields a bundle map

$$\mathcal{H} \colon \mathfrak{H}_{g,n}[\underline{d}] \to \mathfrak{M}_{g,n}.$$

If $g \neq 0$ or $n \neq 1$, the fibre of $\mathcal{H}$ over $\mathfrak{m} \in \mathfrak{M}_{g,n}$ is canonically identified with $\tilde{\mathfrak{H}}(\mathfrak{r}, \underline{d})$, for any $\mathfrak{r}$ representing $\mathfrak{m}$.
Otherwise $\mathfrak{M}_{0,1}$ is a point and $\mathfrak{H}_{0,1}[d]$ can be identified with the affine space $\mathfrak{NMonPol}_d$ of normalised monic polynomials of degree d, see Subsection 7.5.1: in particular $\mathfrak{H}_{0,1}[d]$ has a natural structure of real affine space of real dimension $2d - 2$.
The fact that $\tilde{\mathcal{H}}$ is a bundle map, and in particular that it is surjective, relies ultimately on Theorem 7.1.2: we refer to [10] for this fact and for the computation of the real dimension of $\tilde{\mathfrak{H}}(\mathfrak{r}, \underline{d})$

Since the fibres of $\mathcal{H}$ are contractible, $\mathcal{H}$ is a homotopy equivalence. The space $\mathfrak{H}_{g,n}[\underline{d}]$ is an orientable manifold (actually a real analytic manifold, see [14]) of dimension $3h$. Moreover one can recover combinatorially $\mathfrak{H}_{g,n}[\underline{d}]$ as follows. There exists a finite multisimplicial complex $P = P_{g,n}(\underline{d})$ with a subcomplex $P' = P'_{g,n}(\underline{d})$, and one defines $\mathfrak{Par}_{g,n}[\underline{d}]$ as the difference between their geometric realisations

$$\mathfrak{Par}_{g,n}[\underline{d}] = |P| \setminus |P'| .$$

Then the space $\mathfrak{Par}_{g,n}[\underline{d}]$ is homeomorphic to $\mathfrak{H}_{g,n}[\underline{d}]$; $\mathfrak{Par}_{g,n}[\underline{d}]$ is a combinatorial model for $\mathfrak{M}_{g,n}$. Moreover the quotient $P/P' \cong \mathfrak{Par}_{g,n}[\underline{d}]^{\infty}$ is a CW-complex homeomorphic to the one-point compactification $\mathfrak{H}_{g,n}[\underline{d}]^{\infty}$.

6.2.2 Cell decomposition of P/P'

We recall from [11] the cell decomposition of P/P'. The point at infinity ∞ yields a 0-cell; the other cells are described as follows.

Definition 6.2.3. Let $\underline{p} = (p_1, \dots, p_d)$ be a sequence of integers $p_i \geq 1$. We denote by $\mathfrak{S}^0_{\underline{p}}$ the group of permutations of the set

$$S^0_{\underline{p}} = \{(i,j) \,|\, 1 \leq i \leq d\,,\, 0 \leq j \leq p_i\} .$$

The group $\mathfrak{S}^0_{\underline{p}}$ is isomorphic to $\mathfrak{S}_{p+d}$, where $p = \sum_{i=1}^n p_i$.
We denote by $\mathfrak{lc}^0_{\underline{p}} \in \mathfrak{S}^0_{\underline{p}}$ the permutation that, for every $1 \leq i \leq d$, permutes cyclically the elements $(i,0) \mapsto (i,1) \mapsto \cdots \mapsto (i,p_i) \mapsto (i,0)$ (compare with Definition 2.2.2).
Let $q \geq 1$ and consider a sequence $[\underline{\sigma}] = [\sigma_0 \colon \dots : \sigma_q]$ of permutations $\sigma_i \in \mathfrak{S}^0_{\underline{p}}$ satisfying the following combinatorial conditions:

(1) for all $0 \leq k \leq q$ and all $1 \leq i \leq d$ the permutation σ_k maps $(i,p_i) \mapsto (i,0)$;

(2) $\sigma_0 = \mathfrak{lc}^0_{\underline{p}}$;

(3) for all $1 \leq i \leq d$ and $0 \leq j \leq p_i - 1$ there exists an index $1 \leq k \leq q$ such that $\sigma_k(i,j) \neq (i,j+1)$;

(4) σ_q has a cycle decomposition consisting of precisely n cycles; for each $1 \leq l \leq n$ denote $\bar{d}_l = \sum_{\nu=1}^{l-1} d_l$: then there is a cycle in σ_q in which the elements

$$(\bar{d}_l + 1, 0), (\bar{d}_l + 2, 0), \dots, (\bar{d}_l + d_l, 0)$$

appear in this cyclic order, and no other element of the form $(i,0)$ appears;

(5) for $1 \leq k \leq q$ the permutation $\tau_k = \sigma_k \sigma_{k-1}^{-1} \in \mathfrak{S}^0_{\underline{p}}$ is not the identity;

(6) using the previous notation, we have $\sum_{k=1}^q N(\tau_k) = h$ (see Definition 2.2.2);

(7) the subgroup $H \subseteq \mathfrak{S}^0_{\underline{p}}$ generated by $\sigma_0, \dots, \sigma_q$ acts transitively on the set $S^0_{\underline{p}}$.

For every such choice of $\underline{p}$, q and $[\underline{\sigma}]$ there is a cell $e^{[\underline{\sigma}]} \subset P/P'$, which is modelled on the interior of the multisimplex

$$\Delta^{[\underline{\sigma}]} = \Delta^q \times \Delta^{p_1} \times \cdots \times \Delta^{p_d}.$$

We denote by $(x_k^{hor})_{1 \le k \le q}$ the coordinates of $\Delta^{[\underline{\sigma}]}$ coming from the first factor Δ^q and by $(x_{i,j}^{ver})_{1 \le i \le d, 1 \le j \le p_i}$ the coordinates coming from the other factors.
Similarly, for $0 \le k \le q$ we denote by $\partial_k^{hor} \Delta^{[\underline{\sigma}]}$ the *horizontal* faces of $\Delta^{[\underline{\sigma}]}$, coming from the factor Δ^q, and for $1 \le i \le d$, $0 \le j \le p_i$ we denote by $\partial_{i,j}^{ver} \Delta^{[\underline{\sigma}]}$ the *vertical* faces of $\Delta^{[\underline{\sigma}]}$, coming from the other factors Δ^{p_i}. The *outer* faces of $\Delta^{[\underline{\sigma}]}$ are $\partial_k^{hor} \Delta^{[\underline{\sigma}]}$ for $k = 0, q$, and $\partial_{i,j}^{ver} \Delta^{[\underline{\sigma}]}$ for $j = 0, p_i$; all other faces are *inner* faces.

6.2.3 A first recipe to construct Riemann surfaces

We recall briefly how to use the combinatorial data of $[\underline{\sigma}]$ to construct a point in $\mathfrak{H}_{g,n}[\underline{d}]$ from a point

$$\mathbb{P} = \left(\left(x_k^{hor}|_{\mathbb{P}} \right)_{1 \le k \le q}, \left(x_{i,j}^{ver}|_{\mathbb{P}} \right)_{1 \le i \le d, 1 \le j \le p_i} \right) \in \mathring{\Delta}^{[\underline{\sigma}]}.$$

See also Figure 6.5.
Let $\mathbb{C}_1, \dots, \mathbb{C}_d$ be d distinct copies of the complex plane $\mathbb{C}$. For all $1 \le i \le d$, and $1 \le j \le p_i$ we cut the plane $\mathbb{C}_i$ along the p_i horizontal lines of equations $\Im(z) = x_{i,j}^{ver}|_{\mathbb{P}}$: the plane $\mathbb{C}_i$ is thus dissected into $p_i + 1$ horizontal strips called, from bottom to top, $\mathbb{B}_0^{(i)}, \dots, \mathbb{B}_{p_i}^{(i)}$. Similarly we cut each plane $\mathbb{C}_i$ along the q vertical lines of equations $\Re(z) = x_k^{hor}|_{\mathbb{P}}$ for $1 \le k \le q$, obtaining vertical strips called $\mathbb{A}_0^{(i)}, \dots, \mathbb{A}_q^{(i)}$ from left to right.
Let $R_{k,j}^{(i)} \subset \mathbb{C}_i$ the region obtained as the intersection between the strips $\mathbb{B}_j^{(i)}$ and $\mathbb{A}_k^{(i)}$, where $0 \le k \le q$ and $0 \le j \le p_i$: it is a rectangle for $k \ne 0, q$ and $j \ne 0, p_i$, and it is an unbounded triangle or an unbounded bigon in the other cases.
The combinatorial information of $[\underline{\sigma}]$ is used to reassemble the regions $R_{k,j}^{(i)}$ in order to obtain a new surface:

- for $k \ne q$ we glue the right side of $R_{k,j}^{(i)}$ to the left side of $R_{k+1,j}^{(i)}$;
- for $j \ne p_i$ we glue the upper side of $R_{k,j}^{(i)}$ to the lower side of $R_{k,j'}^{(i')}$, where $(i', j') = \sigma_{q-k}(i, j)$.

Note that $\sigma_0 = \mathfrak{lc}_{\underline{p}}^0$ is used to reglue the regions $R_{q,j}^{(i)}$ coming from the rightmost vertical strips $\mathbb{A}_q^{(i)}$: in particular these vertical strips are reassembled after glueing exactly as they were before cutting. Similarly σ_q provides data to reglue the regions coming from the leftmost vertical strips $\mathbb{A}_0^{(i)}$.
We obtain a non-compact, orientable surface $\mathcal{F}$; the Euclidean metric on $\mathbb{C}$ induces a metric on the regions $R_{k,j}^{(i)}$ and, after glueing, a flat metric with conical singularities on $\mathcal{F}$: this metric determines a Riemann structure on $\mathcal{F}$.

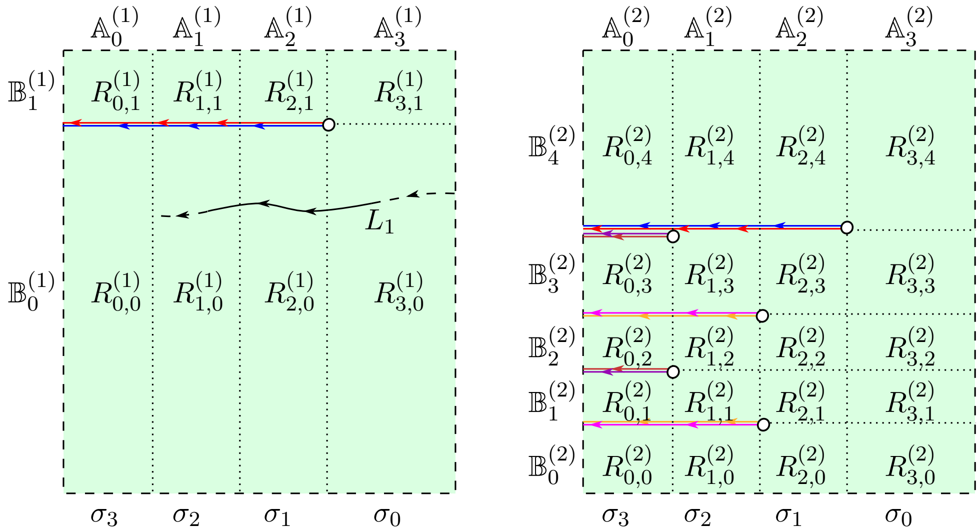

Figure 6.5. A slit configuration in $e^{[\underline{\sigma}]} \subset \mathfrak{Par}_{1,1}[2]$. Here $q = 3$, $p_1 = 1$ and $p_2 = 4$; the horizontal and vertical positions of the dotted lines are the coordinates of the bisimplex $\Delta^{[\underline{\sigma}]} = \Delta^3 \times (\Delta^1 \times \Delta^4)$; the permutations in $[\underline{\sigma}]$ have the following cycle decompositions:
$\sigma_0 = \;((1,0),(1,1)) \quad ((2,0),(2,1),(2,2),(2,3),(2,4));$
$\sigma_1 = \;((1,0),(2,4),(2,0),(2,1),(2,2),(2,3),(1,1));$
$\sigma_2 = \;((1,0),(2,4),(2,0),(2,3),(1,1)) \quad ((2,1),(2,2));$
$\sigma_3 = \;((1,0),(2,4),(2,0),(2,3),(2,2),(2,1),(1,1)).$
The two copies of $\mathbb{C}$ are cut into regions that, using the combinatorial information of $[\underline{\sigma}]$, can be reassembled to give a surface $\mathcal{F} \cong \Sigma_{1,1} \setminus \{Q_1\}$, with a harmonic function $u\colon \mathcal{F} \to \mathbb{R}$ having a pole of order 2 near Q_1. The germ of a path $L_1\colon [0,\varepsilon) \to \bar{\mathcal{F}}$ exiting from Q_1 with velocity X_1 lives for small positive times in the strip $\mathbb{A}_q^{(1)} = \mathbb{A}_3^{(1)}$.

Property (1) in Definition 6.2.3 ensures that, when glueing $R_{k,j}^{(i)}$ to the lower side of $R_{k,j'}^{(i')}$ according to the second rule, the region $R_{k,j'}^{(i')}$ has indeed a lower side and the glueing is well-defined.

Properties (2) and (4) ensure that the surface $\mathcal{F}$ can be compactified by adding n points $Q_1, \dots, Q_n$, in order to obtain a closed Riemann surface $\bar{\mathcal{F}}$; the points Q_i will be the poles of $\bar{\mathcal{F}}$, and we will see soon how to determine the tangent vectors X_i in a canonical way. For $1 \le l \le n$ the point Q_l is adherent to the strips $\mathbb{A}_q^{\bar{d}_l+1}, \dots, \mathbb{A}_q^{\bar{d}_l+d_l}$.

Property (7) ensures that $\bar{\mathcal{F}}$ is connected, and property (6) ensures that it has the right Euler characteristic, hence the right genus g.

Property (3) ensures that for each horizontal line $\Im(z) = x_{i,j}^{ver}$ along which we cut the plane $\mathbb{C}_i$ there is at least a couple of regions $R_{k,j}^{(i)}$ and $R_{k,j+1}^{(i)}$, for some $0 \le k \le q$, that are separated by that line and afterwards are not glued together again along the same side: in other words, no horizontal line is neglectable. Property (5) ensures in a similar way that no vertical line is neglectable.

On $\mathcal{F} = \bar{\mathcal{F}} \setminus \underline{Q}$ there is a harmonic function $u = \Re(z)$, given by glueing the restrictions of the function $\Re\colon \mathbb{C} \to \mathbb{R}$ on the regions $R_{k,j}^{(i)}$. For $1 \le l \le n$ the tangent vectors $X_l \in T_{Q_l}\bar{\mathcal{F}}$

can then be defined by imposing the following conditions:

- u is a $\underline{d}$-directed harmonic function on $\mathcal{F}$;
- the germ of a path $L_l\colon [0,\varepsilon) \to \bar{\mathcal{F}}$ exiting from Q_l with velocity X_l lies for small positive times in the image in $\mathcal{F}$ of the vertical strip $\mathbb{A}_q^{(\bar{d}_l+1)} \subset \mathbb{C}_{\bar{d}_i+1}$.

The germ L_l plays here a similar role as the germ $L_{l,1}$ in the discussion after Definition 6.2.2.
The restrictions of the function $\Im\colon \mathbb{C} \to \mathbb{R}$ on the regions $R_{k,j}^{(i)}$ do not determine a continuous function on the surface $\mathcal{F}$ after glueing. Nevertheless one can define a continuous function v in this way on the complement of the critical graph $\mathcal{K}_0$ of u.
Note that $\mathcal{K}_0$ consists of the points $Q_1, \dots, Q_n$ together with some of the horizontal sides of the regions $R_{k,j}^{(i)}$. Moreover the following property holds: if the upper side of $R_{k,j}^{(i)}$ does not belong to $\mathcal{K}_0(u)$, then it is glued to the lower side of $R_{k,j+1}^{(i)}$, so that the function $\Im$ is continuous along the glueing. We obtain a harmonic function $v\colon \mathcal{F} \setminus \mathcal{K}_0 \to \mathbb{R}$ which is harmonic conjugate to u on $\mathcal{F} \setminus \mathcal{K}_0$.
Property (4) and the definition of $\underline{X}$ ensure that u has a directed pole of order d_l at Q_l, for all $1 \le l \le n$. Thus we have determined a point in $\mathfrak{H}_{g,n}[\underline{d}]$.

6.2.4 Boundary operators for P/P'

We describe the attaching maps of the cells $e^{[\underline{\sigma}]} \subset P/P'$; the results of this subsection are contained in greater detail in [11], we will recall them briefly for completeness.
We use the notation from Subsection 6.2.2. A cell $e^{[\underline{\sigma}]}$ is modelled on the interior of the multisimplex $\Delta^{[\underline{\sigma}]}$; the restriction of its characteristic map

$$\Phi^{[\underline{\sigma}]}\colon \Delta^{[\underline{\sigma}]} \to P/P'$$

to each face $\partial_k^{hor}\Delta^{[\underline{\sigma}]} \subset \partial\Delta^{[\underline{\sigma}]}$ or $\partial_{i,j}^{ver}\Delta^{[\underline{\sigma}]} \subset \partial\Delta^{[\underline{\sigma}]}$ is either the constant map to the point ∞ or the characteristic map of another cell $e^{[\underline{\sigma}']}$. In case of an outer face, both horizontal or vertical, we always obtain the constant map to ∞, so in the following we focus on inner faces of $\Delta^{[\underline{\sigma}]}$.
Let $1 \le k \le q-1$ and consider the inner, horizontal face $\partial_k^{hor}\Delta^{[\underline{\sigma}]}$. Let $[\underline{\sigma}']$ be the sequence of $q-1$ permutations in $\mathfrak{S}_{\underline{p}}^0$ obtained from $[\underline{\sigma}]$ by omitting σ_k; then the restriction of $\Phi^{[\underline{\sigma}]}$ to $\partial_k^{hor}\Delta^{[\underline{\sigma}]}$ is

- the characteristic map of $e^{[\underline{\sigma}']}$, if $[\underline{\sigma}']$ satisfies Properties (1)-(7) from Definition 6.2.3;
- the constant map to ∞ otherwise (see Figure 6.6.

The description of vertical faces is more complicated.
Fix $1 \le r \le d$ and $1 \le s \le p_r - 1$ and consider the inner, vertical face $\partial_{r,s}^{ver}\Delta^{[\underline{\sigma}]}$. Let $\underline{p}' = (p_1', \dots, p_d')$ be defined by $p_i' = p_i$ for all $i \ne r$, and $p_r' = p_r - 1$; denote also $p' = p - 1 = \sum_{i=1}^d p_i'$.

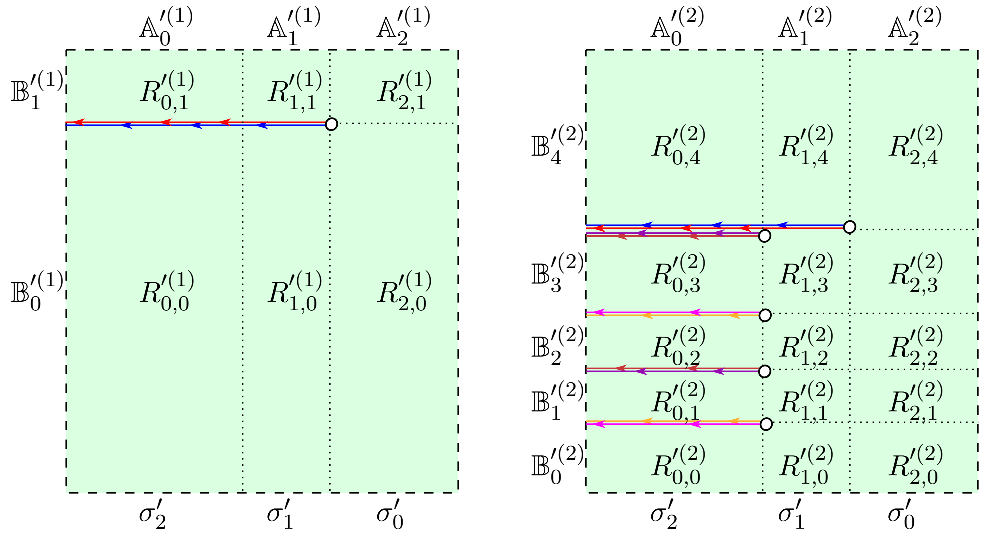

Figure 6.6. Applying the face map ∂_1^{hor} to the cell $e^{[\underline{\sigma}]}$ from Figure 6.5, we obtain a cell $e^{[\underline{\sigma}']} \subset \mathfrak{Par}_{1,1}[2]$, modelled on the multisimplex $\Delta^{[\underline{\sigma}']} = \Delta^2 \times (\Delta^1 \times \Delta^4)$; the permutations in $[\underline{\sigma}']$ have the following cycle decompositions:
$\sigma_0' = \;((1,0),(1,1)) \quad ((2,0),(2,1),(2,2),(2,3),(2,4));$
$\sigma_1' = \;((1,0),(2,4),(2,0),(2,1),(2,2),(2,3),(1,1));$
$\sigma_2' = \;((1,0),(2,4),(2,0),(2,3),(2,2),(2,1),(1,1)).$

Definition 6.2.4. Let S be a finite set, $\mathfrak{s} \in S$ an element and $T = S \setminus \{\mathfrak{s}\}$. Let $\mathfrak{S}_S$ and $\mathfrak{S}_T$ be the groups of permutations of the sets S and T respectively. We define a map *of sets* $D_{\mathfrak{s}} \colon \mathfrak{S}_S \to \mathfrak{S}_T$: a permutation $\sigma \colon S \to S$ is mapped to the following permutation $\tau \colon T \to T$:

- $\tau(t) = \sigma(t)$ if $t \in T$ and $\sigma(t) \neq \mathfrak{s}$;
- $\tau(t) = \sigma(\sigma(t))$ if $t \in T$ and $\sigma(t) = \mathfrak{s}$.

In few words, $D_{\mathfrak{s}}$ transforms a permutation of S into one of T in which the preimage of $\mathfrak{s}$ is sent to the image of $\mathfrak{s}$.
Note that $D_{\mathfrak{s}}$ is a right inverse to the inclusion of groups $\mathfrak{S}_T \subset \mathfrak{S}_S$ identifying $\mathfrak{S}_T$ as the group of permutations of $\mathfrak{S}_S$ fixing $\mathfrak{s}$.
Note also that if $\mathfrak{s}$ and $\mathfrak{s}'$ are two distinct elements of S, then for $T' = S \setminus \{\mathfrak{s}, \mathfrak{s}'\}$ the two maps of sets

$$D_{\mathfrak{s}} \circ D_{\mathfrak{s}'} \text{ and } D_{\mathfrak{s}'} \circ D_{\mathfrak{s}} \colon \mathfrak{S}_S \to \mathfrak{S}_{T'}$$

are equal: in this sense the operations $D_{\mathfrak{s}}$ for different values of $\mathfrak{s}$ commute with each other.

Definition 6.2.5. Let $1 \leq r \leq d$ and $1 \leq s \leq p_r - 1$. We define a map *of sets* $D_{r,s} \colon \mathfrak{S}^0_{\underline{p}} \to \mathfrak{S}^0_{\underline{p}'}$.
Let T denote the set $S^0_{\underline{p}} \setminus \{(r,s)\}$, and let $\mathfrak{S}_T$ be the group of permutations of T.
The following assignment gives a bijection $T \cong S^0_{\underline{p}'}$:

- $(i, j) \mapsto (i, j)$ for $i \neq r$ and $0 \leq j \leq p_i$, or for $i = r$ and $0 \leq j \leq s - 1$;
- $(r, j) \mapsto (r, j - 1)$ for $s + 1 \leq j \leq p_r$.

Under this bijection the groups $\mathfrak{S}_T$ and $\mathfrak{S}^0_{\underline{p}'}$ are identified. We define then $D_{r,s}$ as the composition

$$D_{r,s} \colon \mathfrak{S}^0_{\underline{p}} \xrightarrow{D_{(r,s)}} \mathfrak{S}_T \xrightarrow{\simeq} \mathfrak{S}^0_{\underline{p}'}$$

using the map $D_{(r,s)}$ from Definition 6.2.4 with $\mathfrak{s} = (r, s)$.

Let $[\underline{\sigma}'] = [\sigma'_1, \ldots, \sigma'_q]$, where for all $0 \leq k \leq q$ we set $\sigma'_k := D_{r,s}(\sigma_k)$; then the restriction of $\Phi^{[\underline{\sigma}]}$ to $\partial^{ver}_{r,s} \Delta^{[\underline{\sigma}]}$ is

- the characteristic map of $e^{[\underline{\sigma}']}$, if $[\underline{\sigma}']$ satisfies Properties (1)-(7) from Definition 6.2.3;
- the constant map to ∞ otherwise (see Figure 6.7).

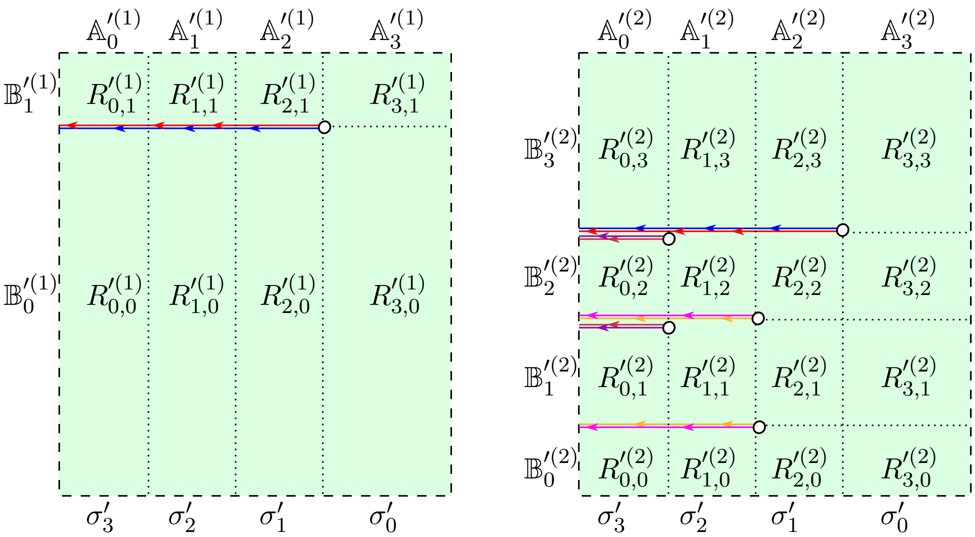

Figure 6.7. Applying the face map $\partial^{ver}_{2,2}$ to the cell $e^{[\underline{\sigma}]}$ from Figure 6.5, we obtain a cell $e^{[\underline{\sigma}']} \subset \mathfrak{Par}_{1,1}[2]$, modelled on the multisimplex $\Delta^{[\underline{\sigma}']} = \Delta^3 \times (\Delta^1 \times \Delta^3)$; the permutations in $[\underline{\sigma}']$ have the following cycle decompositions:
$\sigma'_0 = ((1,0),(1,1)) \quad ((2,0),(2,1),(2,2),(2,3))$
$\sigma'_1 = ((1,0),(2,3),(2,0),(2,1),(2,2),(1,1))$
$\sigma'_2 = ((1,0),(2,3),(2,0),(2,2),(1,1)) \quad ((2,1))$
$\sigma'_3 = ((1,0),(2,3),(2,0),(2,2),(2,1),(1,1))$.

This describes P/P' as a cell complex with a 0-cell ∞ and other cells modelled on multisimplices $\Delta^{[\underline{\sigma}]}$; the attaching map of $\Delta^{[\underline{\sigma}]}$ is either simplicial or constant ∞ on every face of the multisimplex $\Delta^{[\underline{\sigma}]}$.

6.3 A refined cell decomposition on P/P'

For our purposes it will be convenient to refine the cell decomposition of the complex P/P' introduced in Section 6.2. Using the notation from Definition 6.2.3, we consider the Eilenberg-Zilber shuffle product decomposition on the factor $\Delta^{p_1} \times \cdots \times \Delta^{p_d}$ of $\Delta^{[\underline{\sigma}]}$, for every cell $e^{[\underline{\sigma}]} \subset P/P'$: this decomposition is obtained by iterating $d-1$ times the decomposition discussed in Subsection 2.1.2 for a product of two simplices.
Thus we dissect $\Delta^{[\underline{\sigma}]}$ into several bisimplices of the form $\Delta^q \times \Delta^p$. We perform this decomposition on all multisimplices of P/P' at the same time, obtaining a new, refined cell decomposition.

6.3.1 Combinatorial definition of the new cells

The refined cell decomposition of P/P' consists of the 0-cell ∞ and of other cells that we describe briefly in the following; compare with Subsection 6.2.2.

Definition 6.3.1. For $p \geq 1$ let $\mathfrak{S}^0_{d\times p}$ be the group of permutations of the set

$$S^0_{d\times p} = \{(i,j) \mid 1 \leq i \leq d\,,\, 0 \leq j \leq p\}\,.$$

We denote by $\mathfrak{lc}^0_{d\times p} \in \mathfrak{S}^0_{d\times p}$ the permutation that, for every $1 \leq i \leq d$, permutes cyclically the elements $(i,0) \mapsto (i,1) \mapsto \cdots \mapsto (i,p) \mapsto (i,0)$ (compare with Definition 2.2.2).
Let $q \geq 1$ and consider a sequence $\underline{\tilde{\sigma}} = [\tilde{\sigma}_0 \colon \dots \colon \tilde{\sigma}_q]$ of permutations $\tilde{\sigma}_i \in \mathfrak{S}^0_{d\times p}$ satisfying the following combinatorial properties (compare with the list in Definition 6.2.3)

(1) for all $1 \leq k \leq q$ and all $1 \leq i \leq d$ the permutation $\tilde{\sigma}_k$ maps $(i,p) \mapsto (i,0)$;

(2) $\tilde{\sigma}_0 = \mathfrak{lc}^0_{d\times p}$;

(3) for all $0 \leq j \leq p-1$ there exist $1 \leq k \leq q$ and all $1 \leq i \leq d$ such that $\tilde{\sigma}_k(i,j) \neq (i,j+1)$;

(4) $\tilde{\sigma}_q$ has a cycle decomposition consisting of precisely n cycles; for each $1 \leq l \leq n$ denote again $\bar{d}_l = \sum_{\nu=1}^{i-1} d_l$: then there is a cycle in $\tilde{\sigma}_q$ in which the elements

$$(\bar{d}_l+1,0), (\bar{d}_l+2,0), \dots, (\bar{d}_l+d_l,0)$$

appear in this cyclic order, and no other element of the form $(i,0)$ appears;

(5) for $1 \leq k \leq q$ the permutation $\tilde{\tau}_k = \tilde{\sigma}_k\tilde{\sigma}_{k-1}^{-1} \in \mathfrak{S}^0_{d\times p}$ is not the identity;

(6) using the previous notation, we have $\sum_{k=1}^q N(\tilde{\tau}_k) = h$;

(7) the subgroup $\tilde{H} \subseteq \mathfrak{S}^0_{d\times p}$ generated by $\tilde{\sigma}_0, \dots, \tilde{\sigma}_q$ acts transitively on the set $S^0_{d\times p}$.

For every such choice of $[\underline{\tilde{\sigma}}]$ there is a cell $e^{[\underline{\tilde{\sigma}}]} \subset P/P'$, modelled on the bisimplex

$$\Delta^{[\underline{\tilde{\sigma}}]} = \Delta^q \times \Delta^p.$$

For $0 \leq k \leq q$ we denote by $\partial_k^{hor}\Delta^{[\underline{\tilde{\sigma}}]}$ the *horizontal* faces of $\Delta^{[\underline{\tilde{\sigma}}]}$, i.e. the faces coming from the factor Δ^q, and for $0 \leq j \leq p$ we denote by $\partial_j^{ver}\Delta^{[\underline{\tilde{\sigma}}]}$ the *vertical* faces of $\Delta^{[\underline{\tilde{\sigma}}]}$, coming from the factor Δ^p. The *outer* faces are the horizontal faces with $k = 0, q$ and the vertical faces with $j = 0, p$; all other faces are *inner* faces.

6.3.2 Mild variation of the recipe

The construction of the Riemann surface $\mathcal{F}$ is analogue to the one in Subsection 6.2.3. Let $\mathbb{P} \in \mathring{\Delta}^{[\underline{\sigma}]}$ as in Subsection 6.2.3, and assume for simplicity that the vertical coordinates $x_{i,j}^{ver}|_{\mathbb{P}} \in (-\infty, +\infty)$ are all different, for $1 \leq i \leq d$ and $1 \leq j \leq p_i$. Then $\mathbb{P}$ belongs to the interior of a top-dimensional cell of the decomposition of $\Delta^{[\underline{\sigma}]}$. This cell is modelled on the interior of the bisimplex $\Delta^q \times \Delta^p$ and corresponds to a suitable sequence $[\underline{\tilde{\sigma}}]$.
This time we cut all planes $\mathbb{C}_i$ along all horizontal lines of equations $\Im(z) = x_{i',j}^{ver}|_{\mathbb{P}}$, even for $i' \neq i$; each old region $R_{k,j}^{(i)}$ is thus possibly dissected into several smaller regions. We call $\tilde{R}_{k,j}^{(i)} \subset \mathbb{C}_i$ the new regions, for all $1 \leq i \leq d$, $0 \leq k \leq q$ and $0 \leq j \leq p$; these regions are glued as follows:

- for $k \neq q$ we glue the right side of $\tilde{R}_{k,j}^{(i)}$ to the left side of $\tilde{R}_{k+1,j}^{(i)}$;
- for $j \neq p$ we glue the upper side of $\tilde{R}_{k,j}^{(i)}$ to the lower side of $\tilde{R}_{k,j'}^{(i')}$, where $(i', j') = \tilde{\sigma}_{q-k}(i, j)$.

Note in particular that the glueing reassembles each old region $R_{k,j}^{(i)}$ from its subregions $\tilde{R}_{k,j'}^{(i)}$ exactly as it was originally; note also that the obtained surface $\tilde{\mathcal{F}}$ is canonically identified with the old $\mathcal{F}$: the rest of the construction follows in the same way.
Figure 6.8 shows the same point of $\mathfrak{Par}_{1,1}[2]$ represented in Figure 6.5, which now belongs to a new, smaller cell $e^{[\underline{\tilde{\sigma}}]} \subset e^{[\underline{\sigma}]}$. The cell $e^{[\underline{\tilde{\sigma}}]}$ is modelled on the interior of the bisimplex $\Delta^3 \times \Delta^5$. The permutations in $[\underline{\tilde{\sigma}}]$ have the following cycle decompositions:
$\tilde{\sigma}_0 = \; ((1,0),(1,1),(1,2),(1,3),(1,4),(1,5)) \quad ((2,0),(2,1),(2,2),(2,3),(2,4),(2,5))$;
$\tilde{\sigma}_1 = \; ((1,0),(1,1),(1,2),(1,3),(1,4),(2,4),(2,5),(2,0),(2,1),(2,2),(2,3),(1,5))$;
$\tilde{\sigma}_2 = \; ((1,0),(1,1),(1,2),(1,3),(1,4),(2,4),(2,5),(2,0),(2,3),(1,5)) \quad ((2,1),(2,2))$;
$\tilde{\sigma}_3 = \; ((1,0),(1,1),(1,2),(1,3),(1,4),(2,4),(2,5),(2,0),(2,3),(2,2),(2,1),(1,5))$.

6.3.3 Refined boundary operators

We briefly describe the attaching maps for the refined cell decomposition; compare with Subsection 6.2.4.
We use the notation from Subsection 6.2.2. A cell $e^{[\underline{\tilde{\sigma}}]}$ is modelled on the interior of the bisimplex $\Delta^{[\underline{\tilde{\sigma}}]} = \Delta^q \times \Delta^p$; the restriction of its characteristic map

$$\Phi^{[\underline{\tilde{\sigma}}]} \colon \Delta^{[\underline{\tilde{\sigma}}]} \to P/P'$$

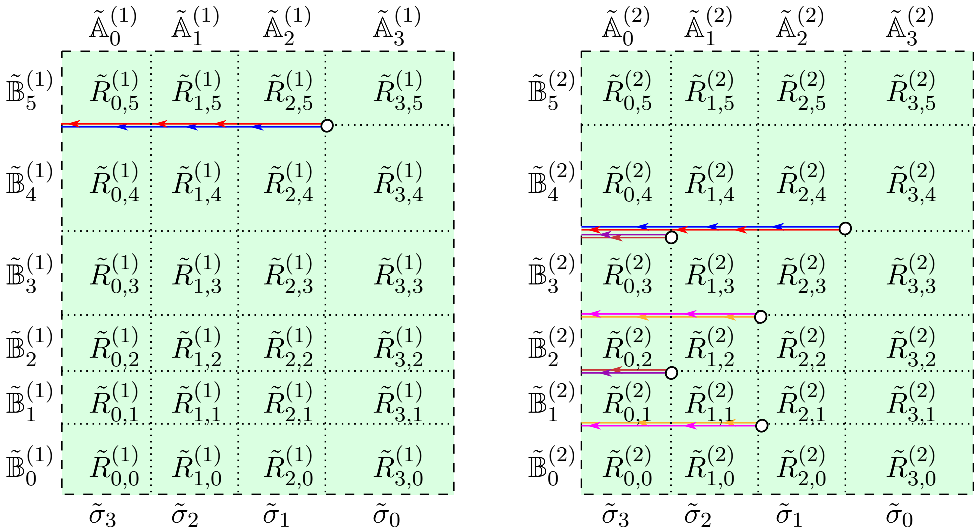

Figure 6.8. The new, smaller regions $\tilde{R}^{(i)}_{k,j}$ are reassembled to give the same surface $\mathcal{F}$ as in Figure 6.5.

to each face $\partial^{hor}_k\Delta^{[\tilde{\sigma}]}$ or $\partial^{ver}_j\Delta^{[\tilde{\sigma}]}$ is either the constant map to ∞ or the characteristic map of another cell $e^{[\tilde{\sigma}']}$. In the case of an outer face we obtain the constant map to ∞, so in the following we focus on inner faces of $\Delta^{[\tilde{\sigma}]}$.
Let $1 \le k \le q-1$ and consider the inner, horizontal face $\partial^{hor}_k\Delta^{[\tilde{\sigma}]}$. Let $[\underline{\tilde{\sigma}}']$ be the sequence of $q-1$ permutations in $\mathfrak{S}^0_{d\times p}$ obtained from $[\underline{\tilde{\sigma}}]$ by omitting σ_k; then the restriction of $\Phi^{[\tilde{\sigma}]}$ to $\partial^{hor}_k\Delta^{[\tilde{\sigma}]}$ is

- the characteristic map of $e^{[\underline{\tilde{\sigma}}']}$, if $[\underline{\tilde{\sigma}}']$ satisfies Properties (1)-(7) from Definition 6.3.1;
- the constant map to ∞ otherwise.

Fix now $1 \le s \le p-1$ and consider the inner, vertical face $\partial^{ver}_s\Delta^{[\tilde{\sigma}]}$.

Definition 6.3.2. Recall Definitions 6.2.4 and 6.2.5, and let $1 \le s \le p-1$. Denote $p' = p-1$; we define a map *of sets* $\tilde{D}_s\colon \mathfrak{S}^0_{d\times p} \to \mathfrak{S}^0_{d\times p'}$.
Let T denote the set

$$S^0_{d\times p} \setminus \{(i,s)\,|\,1 \le i \le d\} = S^0_{d\times p} \setminus ([d]\times\{s\}),$$

and let $\mathfrak{S}_T$ be the group of permutations of T.
We define a bijection $\Xi\colon T \cong S^0_{d\times p'}$:

- $\Xi(i,j) = (i,j)$ for $0 \le j \le s-1$;
- $\Xi(i,j) = (i,j-1)$ for $s+1 \le j \le p$.

Under this bijection the groups $\mathfrak{S}_T$ and $\mathfrak{S}^0_{d\times p'}$ are identified. We define then $\tilde{D}_s$ as the composition

$$\tilde{D}_s\colon \mathfrak{S}^0_{d\times p} \xrightarrow{D_{(1,s)}\circ\cdots\circ D_{(d,s)}} \mathfrak{S}_T \xrightarrow{\quad\cong\quad} \mathfrak{S}^0_{d\times p'}$$

where we use the composition of the maps $D_{(i,s)}$ for $1 \leq i \leq d$; note that these d maps commute, as observed in the discussion after Definition 6.2.4.

Let $[\underline{\tilde{\sigma}}'] = [\tilde{\sigma}'_1, \dots, \tilde{\sigma}'_q]$ with $\tilde{\sigma}'_k = \tilde{D}_s(\tilde{\sigma}_k)$; then the restriction of $\Phi^{[\underline{\tilde{\sigma}}]}$ to $\partial_s^{ver}\Delta^{[\underline{\tilde{\sigma}}]}$ is

- the characteristic map of $e^{[\underline{\tilde{\sigma}}']}$, if $[\underline{\tilde{\sigma}}']$ satisfies Properties (1)-(7) from Definition 6.3.1;
- the constant map to ∞ otherwise.

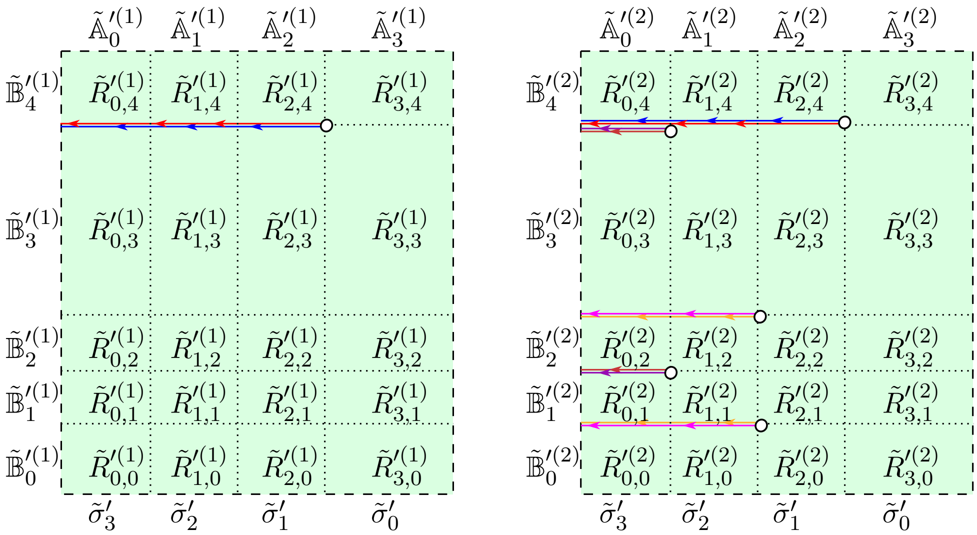

Figure 6.9. Applying the face map ∂_4^{ver} to the cell $e^{[\underline{\tilde{\sigma}}]}$ from Figure 6.8, we obtain a cell $e^{[\underline{\tilde{\sigma}}']} \subset \mathfrak{Par}_{1,1}[2]$, modelled on the bisimplex $\Delta^{[\underline{\tilde{\sigma}}']} = \Delta^3 \times \Delta^4$; the permutations in $[\underline{\sigma}']$ have the following cycle decompositions:
$\tilde{\sigma}'_0 = ((1,0),(1,1),(1,2),(1,3),(1,4)) \quad ((2,0),(2,1),(2,2),(2,3),(2,4))$;
$\tilde{\sigma}'_1 = ((1,0),(1,1),(1,2),(1,3),(2,4),(2,0),(2,1),(2,2),(2,3),(1,4))$;
$\tilde{\sigma}'_2 = ((1,0),(1,1),(1,2),(1,3),(2,4),(2,0),(2,3),(1,4)) \quad ((2,1),(2,2))$;
$\tilde{\sigma}'_3 = ((1,0),(1,1),(1,2),(1,3),(2,4),(2,0),(2,3),(2,2),(2,1),(1,4))$.

This construction describes P/P' as a bisimplicial complex, whose (non-degenerate) bisimplices are the $(0,0)$-bisimplex ∞ and the bisimplices $e^{[\underline{\tilde{\sigma}}]}$. Note that a face of a non-degenerate (p,q)-bisimplex $e^{[\underline{\tilde{\sigma}}]}$ may be a degenerate $(p,q-1)$ or $(p-1,q)$-bisimplex, namely one obtained from the $(0,0)$-bisimplex ∞ by applying a suitable degeneracy map.

7 The moduli space of branched coverings of $\mathbb{C}P^1$

In this chapter we will establish a connection between the Hurwitz spaces introduced in Chapter 4 and the moduli spaces $\mathfrak{M}_{g,n}$ introduced in Chapter 6. We will in particular prove that the transitive, special Hurwitz spaces introduced in Definition 4.5.4 approximate in a good way the homotopy type of the moduli spaces $\mathfrak{M}_{g,n}$.
This is the main reason for our interest in the constructions from Chapter 4, besides the fact that for all normed groups G the spaces $\widetilde{\mathrm{Hur}}(h,G)$ and $\mathrm{Hur}(h,G)$ give an interesting amalgamation of the classical Hurwitz spaces $\mathrm{hur}(\underline{\alpha},G)$ from Definition 4.3.7.
In Section 7.1 we will recall the theorem of Riemann-Roch.
In Section 7.2 we will consider a certain moduli space $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ of Riemann surfaces $(\Sigma_{g,n},\mathfrak{r})$ endowed with a meromorphic function $f\colon \Sigma_{g,n}\to\mathbb{C}P^1$ having some prescribed behaviour near the poles Q_i; the definition of $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ depends on a sequence $\underline{d}=(d_1,\dots,d_n)$ of numbers $d_i\geq 1$.
We will interpret $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ as a closed, proper subspace of $\mathfrak{H}_{g,n}[\underline{d}]$ (see Definition 6.2.2), and prove that the corresponding inclusion

$$\bar{\mathcal{O}}_{g,n}[\underline{d}]^{\infty}\subset\mathfrak{H}_{g,n}[\underline{d}]^{\infty}\cong P/P'$$

is the inclusion of a subcomplex if we consider the refined cell structure on P/P' from Section 6.3.
In Section 7.3 we will prove that if $d:=\sum_{i=1}^n d_i\geq 2g-1+n$, then the inclusion $\bar{\mathcal{O}}_{g,n}[\underline{d}]\subset\mathfrak{H}_{g,n}[\underline{d}]$ is a homotopy equivalence; hence in this case $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ is homotopy equivalent to $\mathfrak{M}_{g,n}$.
In Section 7.4 we will prove that $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ is homeomorphic to the transitive, special Hurwitz space $\mathrm{Hur}^*(h,\mathfrak{S}_d,\sigma)$ (see Definition 4.5.4), where $h=2g+n+d-2$ as in Chapter 6, and σ is any permutation in $\mathfrak{S}_d$ whose cycle decomposition consists of n cycles of lengths $d_1,\dots,d_n$.
In Section 7.5 we will prove that the special Hurwitz space $\mathrm{Hur}(h,\mathfrak{S}_d)$ is a complex, open manifold of complex dimension h.
We fix $g\geq 0$, $n\geq 1$ and $\underline{d}=(d_1,\dots,d_n)$ throughout the chapter; we will denote $d=\sum_{i=1}^n d_i$ and $h=2g+n+d-2$ throughout the chapter.

7.1 The theorem of Riemann-Roch

Definition 7.1.1. A divisor $\mathcal{D}$ on $\Sigma_{g,n}$ is an element of the free abelian group generated by points of $\Sigma_{g,n}$. The degree of a divisor $\mathcal{D}=\sum_{i=1}^k\lambda_iP_i$ is defined as $\sum_{i=1}^k\lambda_i$.

We say that $\mathcal{D}$ is *normal* if it is of the form $\sum_{i=1}^{n} \lambda_i Q_i$, with coefficients $\lambda_i \geq 1$.
We denote by $D = D(\underline{d}) = \sum_{i=1}^{n} d_i \cdot Q_i$ the normal divisor of $\Sigma_{g,n}$ associated with the sequence $\underline{d}$; D has degree $d = \sum_{i=1}^{n} d_i$.

Let $\mathfrak{r}$ be a Riemann structure on $\Sigma_{g,n}$, let $\mathcal{D} = \sum_{i=1}^{k} \lambda_i P_i$ be a divisor and consider the complex vector space $\mathcal{O}(\mathfrak{r}, \mathcal{D})$ containing all meromorphic functions $f\colon (\Sigma_g, \mathfrak{r}) \to \mathbb{C}P^1$ such that f is holomorphic away from $\{P_1, \ldots, P_k\}$ and, for all $1 \leq i \leq k$, f has a pole of order at most λ_i at P_i. This means that if $\lambda_i < 0$, then f must have a zero of order at least $-\lambda_i$ at P_i.
The complex dimension of $\mathcal{O}(\mathfrak{r}, \mathcal{D})$ depends in general on $\mathfrak{r}$ and $\mathcal{D}$. Recall that we can associate with every Riemann surface $(\Sigma_{g,n}, \mathfrak{r})$ a *canonical divisor* $K(\mathfrak{r})$, whose corresponding holomorphic line bundle is isomorphic to the cotangent bundle $T^*\Sigma_{g,n}$. In general $K(\mathfrak{r})$ is not a normal divisor.
We state the theorem of Riemann-Roch in the form that we will need later; we refer to [19] for the full statement and for the proof of this classical theorem.

Theorem 7.1.2 (Riemann-Roch)**.** *Let $\mathfrak{r}$ be a Riemann structure on $\Sigma_{g,n}$ and denote by $K(\mathfrak{r})$ the canonical divisor associated with the Riemann surface $(\Sigma_{g,n}, \mathfrak{r})$. Recall from Definition 7.1.1 that D denotes the normal divisor associated with $\underline{d}$. Then*

$$\dim_{\mathbb{C}} \mathcal{O}(\mathfrak{r}, D) - \dim_{\mathbb{C}} \mathcal{O}(\mathfrak{r}, K(\mathfrak{r}) - D) = d - g + 1.$$

Note in particular that the inequality $\dim_{\mathbb{C}} \mathcal{O}(\mathfrak{r}, D) \geq d-g+1$ always holds. If moreover $d \geq 2g-1$, then the divisor $K(\mathfrak{r})-D$ has degree ≤ -1, because $K(\mathfrak{r})$ has degree $2g-2$ for all $\mathfrak{r}$. In this case $\mathcal{O}(K(\mathfrak{r}) - D) = 0$ and we obtain the equality $\dim_{\mathbb{C}} \mathcal{O}(\mathfrak{r}, D) = d-g+1$.
The particular case $d = n = 1$ is of little interest: if $f\colon (\Sigma_{g,1}, \mathfrak{r}) \to \mathbb{C}P^1$ has at most one pole at Q_1 of order at most 1, then either f is constant, and therefore it captures no information about the Riemann structure $\mathfrak{r}$, or f is a biholomorphism, which means that $g = 0$. Since we are mainly interested in the moduli space of Riemann surfaces of positive genus, from now on we assume $d \geq 2$.

Definition 7.1.3. Let $\mathfrak{r}$ be a Riemann structure on $\Sigma_{g,n}$. Define

$$\bar{\mathcal{O}}(\mathfrak{r}, \underline{d}) \subset \mathcal{O}(\mathfrak{r}, D)$$

as the affine subspace containing all meromorphic functions $f\colon (\Sigma_{g,n}, \mathfrak{r}) \to \mathbb{C}P^1$ such that for all $1 \leq i \leq n$ and for any normal chart w_i around Q_i, the Laurent expansion of f in the coordinate w_i has the form

$$f(w_i) = \frac{1}{w_i^{d_i}} + l.o.t.,$$

where the lower order terms correspond powers of w_i with exponent higher than $-d_i$: we say that f has a *directed pole* of order d_i at Q_i, and that f is a $\underline{d}$-directed meromorphic function on $(\Sigma_{g,n}, \mathfrak{r})$. Note that this property does not depend on the chosen normal charts.

Note that $\bar{\mathcal{O}}(\mathfrak{r},\underline{d}) \subset \mathcal{O}(\mathfrak{r})$ is closed under convex combination with complex coefficients, i.e. if $f, g \in \bar{\mathcal{O}}(\mathfrak{r},\underline{d})$ and $\lambda \in \mathbb{C}$, then also $\lambda f + (1-\lambda) g \in \bar{\mathcal{O}}(\mathfrak{r},\underline{d})$; therefore $\bar{\mathcal{O}}(\mathfrak{r},\underline{d})$ is a *complex affine* subspace of $\mathcal{O}(\mathfrak{r}, D)$.
Recall Definition 6.2.2. Given $f \in \bar{\mathcal{O}}(\mathfrak{r},\underline{d})$, we can define a harmonic function $u = \Re(f)$ on $\Sigma_{g,n} \setminus \underline{Q}$, which admits a conjugate harmonic function $v = \Im(f)$ on the entire subspace $\Sigma_{g,n} \setminus \underline{Q}$, and in particular on $\Sigma_{g,n} \setminus \mathcal{K}_0(u)$: thus we determine a point $(\mathfrak{r}, u, v) \in \tilde{\mathfrak{H}}(\mathfrak{r},\underline{d})$. We obtain an inclusion

$$\bar{\mathcal{O}}(\mathfrak{r},\underline{d}) \subset \tilde{\mathfrak{H}}(\mathfrak{r},\underline{d});$$

the image of this inclusion consists of all points $(\mathfrak{r}, u, v) \in \tilde{\mathfrak{H}}(\mathfrak{r},\underline{d})$ such that v can be continuously extended on $\mathcal{K}_0(u) \setminus \underline{Q}$.

7.2 The moduli space $\bar{\mathcal{O}}_{g,n}[\underline{d}] \subset \mathfrak{H}_{g,n}[\underline{d}]$

7.2.1 Definition of $\bar{\mathcal{O}}_{g,n}[\underline{d}]$

We elaborate on Definition 7.1.3 by considering all Riemann structures $\mathfrak{r}$ on $\Sigma_{g,n}$ at the same time and by quotienting out the action of $\mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right)$ (see Definition 6.1.2).

Definition 7.2.1. Recall definition 7.1.3. We define $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ as the moduli space of Riemann surfaces $(\Sigma_{g,n}, \mathfrak{r})$ endowed with a $\underline{d}$-directed meromorphic function $f \in \bar{\mathcal{O}}(\mathfrak{r},\underline{d})$. Two couples $(\mathfrak{r}, f)$ and $(\mathfrak{r}', f')$ are equivalent if there is $\psi \in \mathrm{Diff}^+\left(\Sigma_{g,n}; \underline{Q}, \underline{X}\right)$ pulling back $\mathfrak{r}'$ to $\mathfrak{r}$ and f' to f.
We define the topology on $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ to be that of closed subspace of $\mathfrak{H}_{g,n}[\underline{d}]$. There is a map $\rho\colon \bar{\mathcal{O}}_{g,n}[\underline{d}] \to \mathfrak{M}_{g,n}$ which forgets the function f; the map ρ is the restriction of the map $\mathcal{H}\colon \mathfrak{H}_{g,n}[\underline{d}] \to \mathfrak{M}_{g,n}$ on the subspace $\bar{\mathcal{O}}_{g,n}[\underline{d}]$.

Assuming $g \neq 0$ or $n \neq 1$, for $\mathfrak{m} \in \mathfrak{M}_{g,n}$, represented by some $\mathfrak{r}$, there is a natural identification between $\rho^{-1}(\mathfrak{m})$ and $\bar{\mathcal{O}}(\mathfrak{r},\underline{d})$. In the case $g = 0$ and $n = 1$, the space $\mathfrak{M}_{g,n}$ consists of a single point and the space $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ coincides with the entir space $\mathfrak{H}_{g,n}[\underline{d}]$, or equivalently with the affine space $\mathfrak{NMonPol}_d$ of normalised monic polynomials of degree d, see Subsections 6.2.1 and 7.5.1.
The inclusion $\bar{\mathcal{O}}_{g,n}[\underline{d}] \subset \mathfrak{H}_{g,n}[\underline{d}]$ is closed, hence proper. We obtain an inclusion

$$\bar{\mathcal{O}}_{g,n}[\underline{d}]^{\infty} \hookrightarrow \mathfrak{H}_{g,n}[\underline{d}]^{\infty} \cong \mathfrak{Par}_{g,n}[\underline{d}]^{\infty} = P_{g,n}(\underline{d})/P'_{g,n}(\underline{d});$$

with image a closed *subspace* of the complex $P/P' = \mathfrak{Par}_{g,n}[\underline{d}]^{\infty}$.

7.2.2 Cell decomposition of $\bar{\mathcal{O}}_{g,n}[\underline{d}]^{\infty}$

We consider the bisimplicial structure on P/P' from Section 6.3; then the image of the previous inclusion is a subcomplex of P/P', spanned by the 0-cell ∞ and by all cells $e^{[\underline{\tilde{\sigma}}]}$ satisfying the following additional property, where we use the notation and the conventions from Section 6.3.

Definition 7.2.2. Let $[\underline{\tilde{\sigma}}] = [\tilde{\sigma}_0 \colon \dots \colon \tilde{\sigma}_q]$ be a sequence of permutations in $\mathfrak{S}^0_{d\times p}$ as in Definition 6.3.1. We say that $\underline{\tilde{\sigma}}$ has the Property $\bullet$ if:

- for all $1 \le i \le d$, $0 \le k \le q$ and $0 \le j \le p-1$, the permutation $\tilde{\sigma}_k$ maps (i, j) to some element of the form $(i', j+1)$, for some $1 \le i' \le d$ (see Figures 7.1 and 7.2).

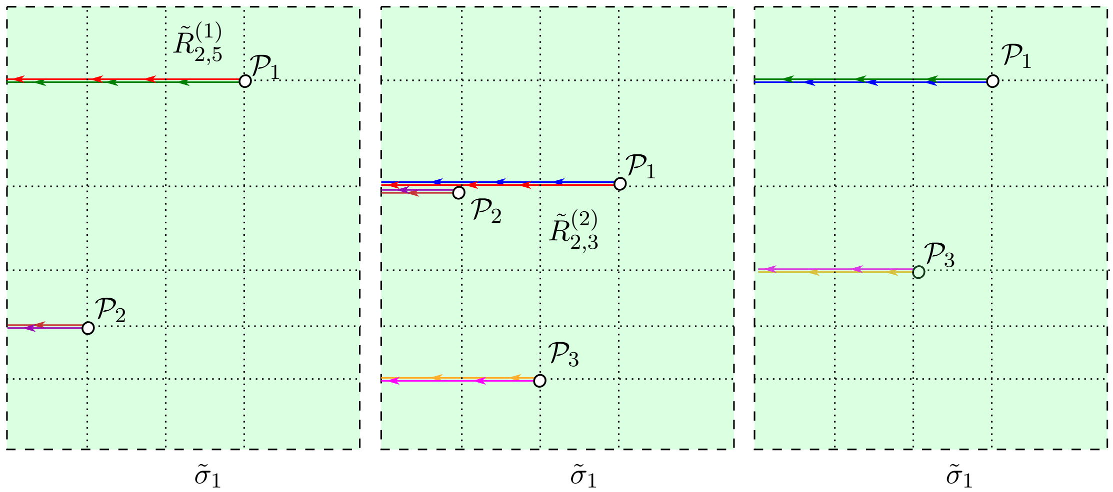

Figure 7.1. A configuration in $\mathfrak{Par}_{0,3}[1,1,1]$ which does not belong to the subspace $\bar{\mathcal{O}}_{0,3}[1,1,1]$: this can be read off from the picture by noticing that there are slits of the same colour (corresponding to the same critical line in $\mathcal{K}_0$) but at different heights. Combinatorially, note for example that $\tilde{\sigma}_1$ maps $(2,3) \mapsto (1,5)$, hence Property • is not fulfilled. Note that $\mathcal{P}_1$ is a critical point for the harmonic function u of index 2, whereas the other critical points have index 1. The red critical line exits from $\mathcal{P}_1$ and enters $\mathcal{P}_2$.

Property • is the combinatorial reformulation of the following geometric property, which is necessary and sufficient for a cell $e^{[\underline{\tilde{\sigma}}]} \subset P/P'$ to lie in $\bar{\mathcal{O}}_{g,n}[\underline{d}]^\infty$.
Let $\mathbb{P}$ be a point in $\mathring{\Delta}^{[\underline{\tilde{\sigma}}]}$ and perform the construction from Subsection 6.3.2; the conjugate harmonic function v extends continuously on the surface $\mathcal{F}$ if and only if the following property holds: for all $0 \le k \le q$, $1 \le i \le d$ and $0 \le j \le p-1$, the upper side of $\tilde{R}^{(i)}_{k,j}$ and the lower side of $\tilde{R}^{(i')}_{k,j'}$, with $(i', j') = \tilde{\sigma}_{q-k}(i, j)$, project to the same real number under the map $\Im$. By the assumption that $\mathbb{P}$ is in the interior of $\Delta^{[\underline{\tilde{\sigma}}]}$, this means that $j' = j+1$. We obtain the following lemma.

Lemma 7.2.3. *The inclusion $\bar{\mathcal{O}}_{g,n}[\underline{d}]^\infty \hookrightarrow \mathfrak{H}_{g,n}[\underline{d}]^\infty$ exhibits $\bar{\mathcal{O}}_{g,n}[\underline{d}]^\infty$ as a subcomplex of P/P', where the latter is endowed with the finer cell structure from Section 6.3: this subcomplex contains ∞ and all cells $e^{[\underline{\tilde{\sigma}}]}$ satisfying Property •.*

We introduce now a slight variation of our notation $e^{[\underline{\tilde{\sigma}}]}$ for cells in P/P' and in particular in $\bar{\mathcal{O}}_{g,n}[\underline{d}]^\infty$.

Definition 7.2.4. Let $[\underline{\tilde{\sigma}}]$ be as in Definition 6.3.1; for all $1 \le k \le q$ define

$$\check{\sigma}_k = \tilde{\sigma}_k^{-1} \cdot \tilde{\sigma}_0 \in \mathfrak{S}^0_{d \times p};$$

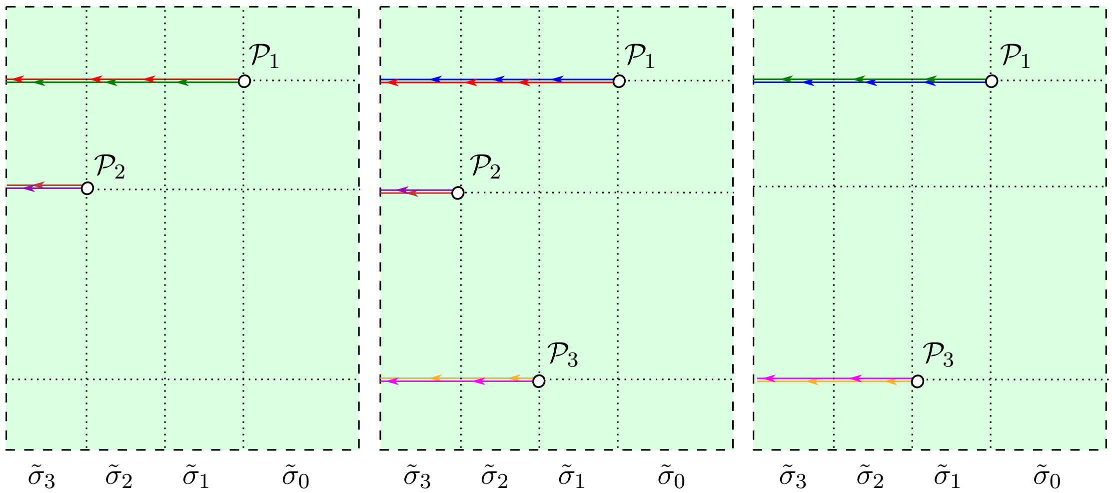

Figure 7.2. A configuration in $\bar{\mathcal{O}}_{0,3}[1,1,1] \subset \mathfrak{Par}_{0,3}[1,1,1]$. Note that $\mathcal{P}_1$ is a critical point for the harmonic function u of index 2, whereas the other critical points have index 1.

then the sequence $[\underline{\tilde\sigma}]$ can be recovered from the sequence $\underline{\check\sigma} = (\check\sigma_1, \dots, \check\sigma_q)$ by the evident inversion formula

$$\tilde\sigma_k = \mathfrak{lc}^0_{d\times p} \cdot \check\sigma_k^{-1},$$

where $\mathfrak{lc}^0_{d\times p} \in \mathfrak{S}^0_{d\times p}$ was introduced in Definition 6.3.1.

We will denote the *same* cell in P/P' as $e^{[\underline{\tilde\sigma}]}$ or as $e^{\underline{\check\sigma}}$: the first is called the *homogeneous* notation, whereas the second is the *reduced* notation.

The homogeneous notation is taken from [1, 11], whereas the reduced notation is a variation of the *inhomogeneous notation* from [1, 11].

Consider the configuration in Figure 7.2: it belongs to a cell $e^{[\underline{\tilde\sigma}]} \subset \bar{\mathcal{O}}_{0,3}[1,1,1]$ which is modelled on the interior of the bisimplex $\Delta^3 \times \Delta^3$. The permutations for the homogeneous notation $[\underline{\tilde\sigma}]$ have the following cycle decompositions:

$\tilde\sigma_0 = \ ((1,0),(1,1),(1,2),(1,3)) \quad ((2,0),(2,1),(2,2),(2,3)) \quad ((3,0),(3,1),(3,2),(3,3));$

$\tilde\sigma_1 = \ ((1,0),(1,1),(1,2),(3,3),(3,0),(3,1),(3,2),(2,3),(2,0),(2,1),(2,2),(1,3));$

$\tilde\sigma_2 = \ ((1,0),(1,1),(1,2),(3,3),(3,0),(2,1),(2,2),(1,3)) \quad ((2,0),(3,1),(3,2),(2,3));$

$\tilde\sigma_3 = \ ((1,0),(1,1),(2,2),(1,3)) \quad ((2,0),(3,1),(3,2),(2,3)) \quad ((3,0),(2,1),(1,2),(3,3)).$

The same cell can be expressed as $e^{\underline{\check\sigma}}$ in the reduced notation, where the permutations in $\underline{\check\sigma}$ belong to $\mathfrak{S}^0_{d\times p}$ and have the following cycle decompositions (we omit all fixpoints and just write the non-trivial cycles):

$\underline{\check\sigma}_1 = \ ((1,2),(2,2),(3,2));$

$\underline{\check\sigma}_2 = \ ((2,0),(3,0)) \quad ((1,2),(2,2),(3,2));$

$\underline{\check\sigma}_3 = \ ((2,0),(3,0)) \quad ((1,1),(2,1)) \quad ((1,2),(2,2),(3,2)).$

By Property (1) in Definition 6.3.1, the permutations $\check\sigma_k \in \mathfrak{S}^0_{d\times p}$ fix all elements of the form (i,p) for $1 \le i \le d$: therefore they can be regarded as permutations in the group

$\mathfrak{S}^0_{d\times(p-1)}$ of permutations of the set

$$S^0_{d\times(p-1)} = \{(i,j) \mid 1 \le i \le d\,,\, 0 \le j \le p-1\}\,.$$

From now on we will only consider sequences $[\check{\underline{\sigma}}]$ yielding cells in $\bar{\mathcal{O}}_{g,n}[\underline{d}]^\infty$. Property $\bullet$ from Definition 7.2.2 can now be rephrased as follows, using the reduced notation:

- for all $1 \le i \le d$, $1 \le k \le q$ and $0 \le j \le p-1$, the permutation $\check{\sigma}_k$ maps (i,j) to some element of the form (i',j), for some $1 \le i' \le d$.

Let $(\mathfrak{S}_d)^p \subset \mathfrak{S}^0_{d\times(p-1)}$ denote the subgroup of permutations that, for each $0 \le j \le p-1$, leave the set $[d]\times\{j\} = \{(i,j) \mid 1 \le i \le d\} \subset S^0_{d\times(p-1)}$ invariant; then Property $\bullet$ can be rephrased by saying that the permutations $\check{\sigma}_k$ are in $(\mathfrak{S}_d)^p$.
We can therefore represent $\check{\sigma}_k$, for $1 \le k \le q$, as a sequence of p permutations in $\mathfrak{S}_d$, using for all $0 \le j \le p-1$ the canonical identification $[d] \cong [d]\times\{j\}$. We call these permutations $(\check{\sigma}_{k,1},\dots,\check{\sigma}_{k,p})$, where $\check{\sigma}_{k,j}$ corresponds to the action of $\check{\sigma}_k$ on the set $[d]\times\{j-1\}$; we regard $\check{\underline{\sigma}}$ as a bi-indexed collection of elements of $\mathfrak{S}_d$:

$$\check{\underline{\sigma}} = (\check{\sigma}_{k,j})_{1\le k\le q, 1\le j\le p}.$$

The advantage of the reduced notation is the following: a cell $e^{\check{\underline{\sigma}}} \subset \bar{\mathcal{O}}_{g,n}[\underline{d}]$ is described by a collection of permutations $\check{\sigma}_{k,j}$ belonging a group, namely $\mathfrak{S}_d$, which *does not depend on the chosen cell*, i.e. it only depends on the space $\bar{\mathcal{O}}_{g,n}[\underline{d}]$.
We list the properties that the permutations $\check{\sigma}_{k,j} \in \mathfrak{S}_d$ must fulfil in order to give a cell of $\bar{\mathcal{O}}_{g,n}[\underline{d}]$; the enumeration is given in a way that invites a comparison with the list from Definition 6.3.1.
For all $1 \le k \le q$ and $1 \le j \le p$ let $\check{\tau}_{k,j} = \check{\sigma}_{k,j}\check{\sigma}^{-1}_{k-1,j}$, where by convention we set $\check{\sigma}_{0,j} = \mathbb{1} \in \mathfrak{S}_d$ for all $1 \le j \le p$:

(3) for all $1 \le j \le p$ there exists some $1 \le k \le q$ such that $\check{\tau}_{k,j} \ne \mathbb{1} \in \mathfrak{S}_d$;

(4) the product
$$\check{\sigma}_{q,p} \cdot \check{\sigma}_{q,p-1} \cdot \dots \cdot \check{\sigma}_{q,1} \in \mathfrak{S}_d$$
is the following permutation with n cycles: for all $1 \le l \le n$ there is a cycle
$$\left(\bar{d}_l + 1\right) \mapsto \left(\bar{d}_l + d_l\right) \mapsto \left(\bar{d}_l + d_l - 1\right) \mapsto \dots \mapsto \left(\bar{d}_l + 2\right) \mapsto \left(\bar{d}_l + 1\right),$$
where we denote again $\bar{d}_l = \sum_{\nu=1}^{i-1} d_l$ for all $1 \le l \le n$;

(5) for all $1 \le k \le q$ there exists $1 \le j \le p$ such that $\check{\tau}_{k,j} \ne \mathbb{1} \in \mathfrak{S}_d$;

(6) $\sum_{k=1}^{q}\sum_{j=1}^{p} N(\check{\tau}_{k,j}) = h$;

(7) the subgroup $\tilde{H} \subseteq \mathfrak{S}_d$ generated by the elements $\check{\sigma}_{k,j}$ acts transitively on the set $[d] = \{1,\dots,d\}$.

For all $1 \leq k \leq q$ the sequence of permutations

$$(\check{\tau}_{k,1}, \dots, \check{\tau}_{k,p}) \in (\mathfrak{S}_d)^p \subset \mathfrak{S}^0_{d\times(p-1)} \subset \mathfrak{S}^0_{d\times p}$$

corresponds to the permutation

$$\check{\tau}_k := \check{\sigma}_k \check{\sigma}_{k-1}^{-1} = \tilde{\sigma}_k^{-1} \tilde{\sigma}_{k-1} = \tilde{\sigma}_{k-1}^{-1} (\tilde{\tau}_k)^{-1} \tilde{\sigma}_{k-1} \in \mathfrak{S}^0_{d\times p},$$

which is a conjugate of the inverse of the element $\tilde{\tau}_k = \tilde{\sigma}_k \tilde{\sigma}_{k-1}^{-1}$ considered in Properties (5) and (6) from Definition 6.3.1; in particular $N(\tilde{\tau}_k) = N(\check{\tau}_k) = \sum_{j=1}^{p} N(\check{\tau}_{k,j})$.
Properties (1) and (2) from Definition 6.3.1 are missing because we have already used them to restrict the ranges of the indices $0 \leq j \leq p$ and $0 \leq k \leq q$ to $1 \leq j \leq p$ and $1 \leq k \leq q$ respectively; Property • is also missing because it is entailed in the new notation with permutations $\check{\sigma}_{k,j} \in \mathfrak{S}_d$.

7.2.3 Boundary operators in reduced notation

In this subsection we adapt the description of the boundary operators from Subsection 6.3.3 to the reduced notation, restricting our attention to the subcomplex $\bar{\mathcal{O}}_{g,n}[\underline{d}] \subset P/P'$.
Recall Definition 7.2.4 and use the notation from Subsection 7.2.2. A cell $e^{\underline{\check{\sigma}}}$ is modelled on the interior of the bisimplex $\Delta^{\underline{\check{\sigma}}} = \Delta^q \times \Delta^p$; the restriction of its characteristic map

$$\Phi^{\underline{\check{\sigma}}} \colon \Delta^{\underline{\check{\sigma}}} \to P/P'$$

to each face $\partial_k^{hor} \Delta^{\underline{\check{\sigma}}}$ or $\partial_j^{ver} \Delta^{\underline{\check{\sigma}}}$ is either the constant map to ∞ or the characteristic map of another cell $e^{\underline{\check{\sigma}}'}$. In the case of an outer face we obtain the constant map to ∞, so in the following we focus on inner faces of $\Delta^{\underline{\check{\sigma}}}$.
Let $1 \leq r \leq q-1$ and consider the inner, horizontal face $\partial_r^{hor} \Delta^{\underline{\check{\sigma}}}$. Let

$$\underline{\check{\sigma}}' = (\check{\sigma}'_{k,j})_{1\leq k\leq q-1, 1\leq j\leq p}$$

be the collection of permutations in $\mathfrak{S}_d$ obtained from $\underline{\check{\sigma}}$ by omitting the p permutations $\check{\sigma}_{r,1}, \dots, \check{\sigma}_{r,p}$ corresponding to $\check{\sigma}_r$, and renormalising the indices in a convenient way. Then the restriction of $\Phi^{[\underline{\check{\sigma}}]}$ to $\partial_k^{hor} \Delta^{[\underline{\check{\sigma}}]}$ is

- the characteristic map of $e^{\underline{\tilde{\sigma}}'}$, if $\underline{\tilde{\sigma}}'$ satisfies Properties (3)-(7) from the list in Subsection 7.2.2;
- the constant map to ∞ otherwise (see Figure 7.3).

Let now $1 \leq s \leq p-1$ and consider the inner, vertical face $\partial_s^{ver} \Delta^{\underline{\check{\sigma}}}$. We have to understand the behaviour of the map $\tilde{D}_s$ from Definition 6.3.2 in the reduced notation.

Lemma 7.2.5. *Recall Definition 6.3.2. Let $\rho \in (\mathfrak{S}_d)^p \subset \mathfrak{S}^0_{d\times(p-1)} \subset \mathfrak{S}^0_{d\times p}$, and split $\rho = (\rho_1, \dots, \rho_p)$, where $\rho_j \in \mathfrak{S}_d$ describes the action of ρ on the set $[d] \times \{j-1\}$, for $0 \leq j \leq p$.*

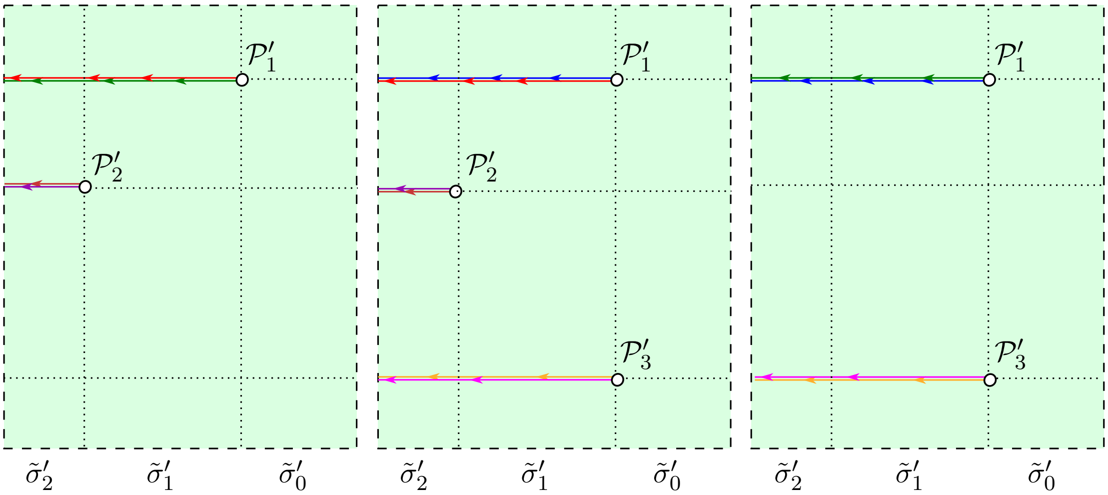

Figure 7.3. The face map ∂_2^{hor} applied to the cell $e^{[\underline{\tilde{\sigma}}]}$ from Figure 7.2 yields a cell $e^{[\underline{\tilde{\sigma}}']}$, modelled on the interior of the bisimplex $\Delta^2 \times \Delta^3$. We have $[\underline{\tilde{\sigma}}'] = [\tilde{\sigma}'_0 : \tilde{\sigma}'_1 : \tilde{\sigma}'_2] = [\tilde{\sigma}_0 : \tilde{\sigma}_1 : \tilde{\sigma}_3]$, and passing to the reduced notation we have $\underline{\check{\sigma}}' = (\check{\sigma}'_1, \check{\sigma}'_2) = (\check{\sigma}_1, \check{\sigma}_3)$.

Let $1 \leq s \leq p-1$*, denote* $p' = p-1$*, and define*

$$\rho' = (\rho'_1, \dots, \rho'_{p'}) = (\rho_1\,,\,\dots\,,\,\rho_{s-1}\,,\,\rho_s\rho_{s+1}\,,\,\rho_{s+2}\,,\,\dots\,,\,\rho_p);$$

regard ρ' *as an element in* $(\mathfrak{S}_d)^{p'} \subset \mathfrak{S}^0_{d\times(p'-1)} \subset \mathfrak{S}^0_{d\times p'}$.
For $l = p, p'$ *let* $\mathfrak{lc}^0_{d\times l} \in \mathfrak{S}^0_{d\times p}$ *as introduced in Definition 6.3.1. Then*

$$\tilde{D}_s\left(\mathfrak{lc}^0_{d\times p} \cdot \rho^{-1}\right) = \mathfrak{lc}^0_{d\times p'} \cdot (\rho')^{-1} \in \mathfrak{S}^0_{d\times p'}.$$

Proof. Let T be the set

$$T = S^0_{d\times p} \setminus ([d] \times \{s\})\,.$$

By Definition 6.3.2, to compute $\tilde{D}_s\left(\mathfrak{lc}^0_{d\times p} \cdot \rho^{-1}\right) \in \mathfrak{S}^0_{d\times p'}$ we first have to compute the permutation

$$\upsilon := D_{(1,s)} \circ \dots \circ D_{(d,s)}\left(\mathfrak{lc}^0_{d\times p} \cdot \rho^{-1}\right) \in \mathfrak{S}_T,$$

and then to use the bijection $\Xi\colon T \cong S^0_{d\times p'}$ from Definition 6.3.2 to identify the groups $\mathfrak{S}_T$ and $\mathfrak{S}^0_{d\times p'}$. For $(i,j) \in T$ we have the following equalities.

- For all $j \neq s-1$ and $j \neq p$, we have

  $$\upsilon(i,j) = \mathfrak{lc}_{d\times p} \cdot \rho^{-1}(i,j) = (\rho^{-1}_{j+1}(i), j+1),$$

  because $\rho^{-1}(i,j) = \left(\rho^{-1}_{j+1}(i), j\right)$ and $\mathfrak{lc}_{d\times p}\left(\rho^{-1}_{j+1}(i), j\right) = \left(\rho^{-1}_{j+1}(i), j+1\right)$.

- For $j = p$ we have

  $$\upsilon(i,p) = \mathfrak{lc}_{d\times p} \cdot \rho^{-1}(i,p) = (i,0),$$

  because (i,p) is a fixpoint for ρ.

- For $j = s-1$ we have $\left(\mathfrak{lc}^0_{d\times p}\cdot\rho^{-1}\right)(i,s-1) = (\rho_s^{-1}(i),s)\notin T$, hence

$$\upsilon(i,s-1) = \left(\mathfrak{lc}_{d\times p}\cdot\rho^{-1}\right)^2(i,s-1) = \left(\rho_{s+1}^{-1}\rho_s^{-1}(i),s+1\right) = \Big((\rho_s\rho_{s+1})^{-1}(i),s+1\Big).$$

Using now the bijection Ξ, we can rewrite the previous equalities as follows.

- For all $0\le j\le p'-1$ with $j\ne s-1$, let $\Xi^{-1}(i,j) = (i,\bar{j})$, with $0\le\bar{j}\le p-1$ and $\bar{j}\ne s-1,s$; then
$$\tilde{D}_s\left(\mathfrak{lc}^0_{d\times p}\cdot\rho^{-1}\right)(i,\bar{j}) = \Big((\rho')^{-1}_{\bar{j}+1},\bar{j}+1\Big).$$

- $\tilde{D}_s\left(\mathfrak{lc}^0_{d\times p}\cdot\rho^{-1}\right)(i,p') = (i,0)$.

- $\tilde{D}_s\left(\mathfrak{lc}^0_{d\times p}\cdot\rho^{-1}\right)(i,s-1) = \Big((\rho_s\rho_{s+1})^{-1}(i),s\Big) = \Big((\rho'_s)^{-1}(i),s\Big)$.

The latter is precisely the description of the permutation $\mathfrak{lc}_{d\times p'}\cdot(\rho')^{-1}\in\mathfrak{S}^0_{d\times p'}$. □

Let $p' = p-1$ and define
$$\underline{\check{\sigma}}' = \left(\check{\sigma}'_{k,j}\right)_{1\le k\le q,1\le j\le p'},$$
where for all $1\le k\le q$ we set

$$\left(\check{\sigma}'_{k,1},\dots,\check{\sigma}'_{k,p'}\right) = \left(\check{\sigma}_{k,1},\dots,\check{\sigma}_{k,s-1},\check{\sigma}_{k,s}\cdot\check{\sigma}_{k,s+1},\check{\sigma}_{k,s+2},\dots,\check{\sigma}_{k,p}\right).$$

Then the restriction of $\Phi^{\underline{\check{\sigma}}}$ to $\partial_s^{ver}\Delta^{\underline{\check{\sigma}}}$ is

- the characteristic map of $e^{\underline{\check{\sigma}}'}$, if $\underline{\check{\sigma}}'$ satisfies Properties (3)-(7) from the list in Subsection 7.2.2;

- the constant map to ∞ otherwise (see Figure 7.4).

7.3 A new combinatorial model for $\mathfrak{M}_{g,n}$

In this section we prove that if $d\ge 2g+n-1$, then the space $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ is homotopy equivalent to $\mathfrak{M}_{g,n}$: the condition $d\ge 2g+n-1$ is needed in the following lemma, in which we apply Theorem 7.1.2 to prove that $\bar{\mathcal{O}}(\mathfrak{r},\underline{d})$ is non-empty, for all Riemann structures $\mathfrak{r}$ on $\Sigma_{g,n}$.

Lemma 7.3.1. *In the notation used above, suppose that $d\ge 2g+n-1$. Then for every Riemann structure $\mathfrak{r}$ on $\Sigma_{g,n}$, $\bar{\mathcal{O}}(\mathfrak{r},\underline{d})$ is an affine space of complex dimension $d-g+1-n$; in particular it is non-empty and contractible.*

Proof. We consider the divisor $\mathcal{D} := D-\sum_{i=1}^n Q_i$, whose degree is $\ge 2g-1$: by Theorem 7.1.2 we have
$$\dim_{\mathbb{C}}\mathcal{O}(\mathfrak{r},\mathcal{D}) = d-g+1-n.$$

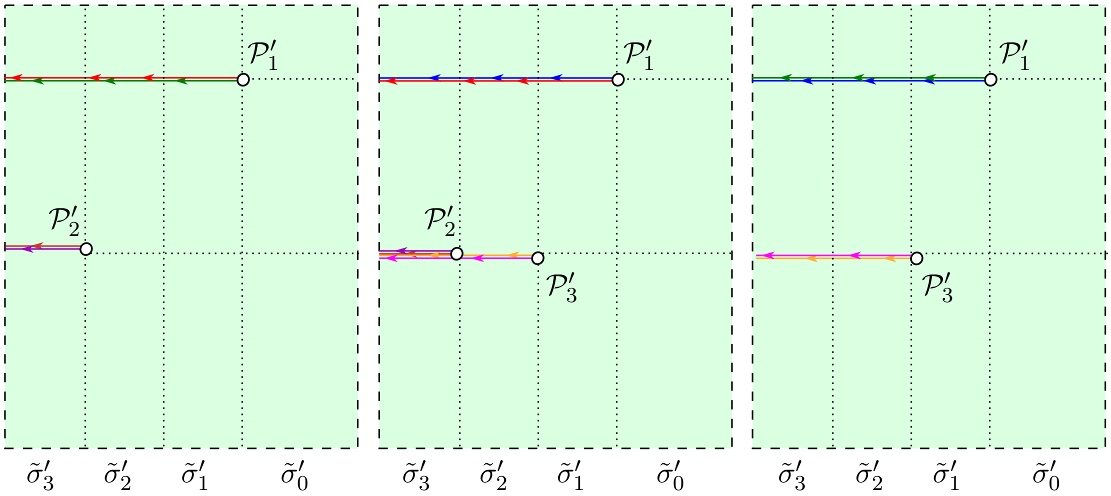

Figure 7.4. The face map ∂_1^{ver} applied to the cell $e^{[\tilde{\underline{\sigma}}]}$ from Figure 7.2 yields a cell $e^{[\tilde{\underline{\sigma}}']}$, modelled on the interior of the bisimplex $\Delta^3 \times \Delta^2$. The permutations in $[\tilde{\underline{\sigma}}']$ have the following cycle decompositions:
$\tilde{\sigma}'_0 = \ ((1,0),(1,1),(1,2)) \quad ((2,0),(2,1),(2,2)) \quad ((3,0),(3,1),(3,2));$
$\tilde{\sigma}'_1 = \ ((1,0),(1,1),(3,2),(3,0),(3,1),(2,2),(2,0),(2,1),(1,2));$
$\tilde{\sigma}'_2 = \ ((1,0),(1,1),(3,2),(3,0),(2,1),(1,2)) \quad ((2,0),(3,1),(2,2));$
$\tilde{\sigma}'_3 = \ ((1,0),(2,1),(1,2)) \quad ((2,0),(3,1),(2,2)) \quad ((3,0),(1,1),(3,2)).$
The same cell can be expressed as $e^{\check{\underline{\sigma}}'}$ in the reduced notation, where the permutations in $\check{\underline{\sigma}}'$ have the following cycle decompositions (we omit all fixpoints and just write the non-trivial cycles):
$\check{\sigma}'_1 = \ ((1,1),(2,1),(3,1));$
$\check{\sigma}'_2 = \ ((2,0),(3,0)) \quad ((1,1),(2,1),(3,1));$
$\check{\sigma}'_3 = \ ((1,0),(3,0),(2,0)) \quad ((1,1),(2,1),(3,1)).$

The previous is a sub-vector space of $\mathcal{O}(\mathfrak{r}, D)$, and we note that $\bar{\mathcal{O}}(\mathfrak{r}, \underline{d})$ is either empty or a translate of $\mathcal{O}(\mathfrak{r}, \mathcal{D})$ in $\mathcal{O}(\mathfrak{r}, D)$.
To prove that $\bar{\mathcal{O}}(\mathfrak{r}, \underline{d})$ is non-empty, consider for all $1 \leq j \leq n$ the divisor $\mathcal{D} + Q_j$, whose degree is $\geq 2g$: again by Theorem 7.1.2 we have

$$\dim_{\mathbb{C}} \mathcal{O}(\mathfrak{r}, \mathcal{D} + Q_j) = d - g + 2 - n,$$

so that, by comparing dimensions, we can find a function

$$f_j \in \mathcal{O}(\mathfrak{r}, \mathcal{D} + Q_j) \setminus \mathcal{O}(\mathfrak{r}, \mathcal{D}).$$

Note that f_j has a pole of order exactly d_j at Q_j; up to multiplying by a suitable constant in $\mathbb{C}^*$ we may assume that the Laurent expansion of f_j around Q_j, read in a normal chart w_j, has the form $1/w_j^{d_j} + l.o.t.$.
We can now consider the sum $f = \sum_{j=1}^{n} f_j$, which belongs to $\bar{\mathcal{O}}(\mathfrak{r}, \underline{d})$ and witnesses that the latter is non-empty. □

Assuming $g \neq 0$ or $n \neq 1$, Lemma 7.3.1 has the following implication: if $d \geq 2g+n-1$ the map $\rho\colon \bar{\mathcal{O}}_{g,n}[\underline{d}] \to \mathfrak{M}_{g,n}$ is a *fibre bundle* with fibre a complex affine space of dimension $d-g+1-n$. More precisely, $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ is a subbundle of the bundle $\mathcal{H}\colon \mathfrak{H}_{g,n}[\underline{d}] \to \mathfrak{M}_{g,n}$. In particular ρ has contractible fibres and we obtain the following theorem.

Theorem 7.3.2. *If $d \geq 2g+n-1$ the following map is a homotopy equivalence.*

$$\rho\colon \bar{\mathcal{O}}_{g,n}[\underline{d}] \xrightarrow{\simeq} \mathfrak{M}_{g,n}.$$

In the case $g=0$ and $n=1$, the map $\bar{\mathcal{O}}_{g,n}[\underline{d}] \to \mathfrak{M}_{g,n}$ is also a homotopy equivalence (it is the surjection of the space $\mathfrak{NMonPol}_d$ onto a point), but the complex dimension of the (unique) fibre is $d-1$.
In all cases we obtain a diagram of fibre bundles; when $d \geq 2g+n-1$ all vertical maps are homotopy equivalences

$$\begin{array}{ccc} \bar{\mathcal{O}}(\mathfrak{r}, \underline{d}) & \hookrightarrow & \bar{\mathcal{O}}_{g,n}[\underline{d}] \\ \downarrow \subseteq & & \downarrow \subseteq \\ \tilde{\mathfrak{H}}(\mathfrak{r}, D) & \hookrightarrow & \mathfrak{H}_{g,n}[\underline{d}] \\ & & \downarrow \simeq \\ & & \mathfrak{M}_{g,n} \end{array}$$

7.4 Connection with Hurwitz spaces

In this section we establish a connection between the space $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ and the transitive, special Hurwitz space $\mathrm{Hur}^*(h, \mathfrak{S}_d)$ from Definition 4.5.4. Recall that $h := 2g+n+d-2$ was set at the beginning of the chapter.

Theorem 7.4.1. *Let $\sigma \in \mathfrak{S}_d$ be a permutation whose cycle decomposition has the form*

$$\sigma = c_1 \cdot \ldots \cdot c_n,$$

where c_i is a cycle on d_i elements. Then there is a homeomorphism

$$\mathfrak{O}\colon \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma) \cong \bar{\mathcal{O}}_{g,n}[\underline{d}],$$

depending on a choice of elements $\iota_1, \ldots, \iota_n \in [d] = \{1, \ldots, d\}$ with $\iota_i \in c_i$.

The proof of Theorem 7.4.1 relies on the cell structures introduced in Definition 4.1.1 and Lemma 7.2.3 respectively, and will be given in Subsection 7.4.2.

7.4.1 A second recipe to construct Riemann surfaces

Before proving Theorem 7.4.1, we will describe a recipe to convert a configuration in $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$ into a Riemann surface $\bar{\mathcal{F}}$ of type $\Sigma_{g,n}$ with a $\underline{d}$-directed meromorphic

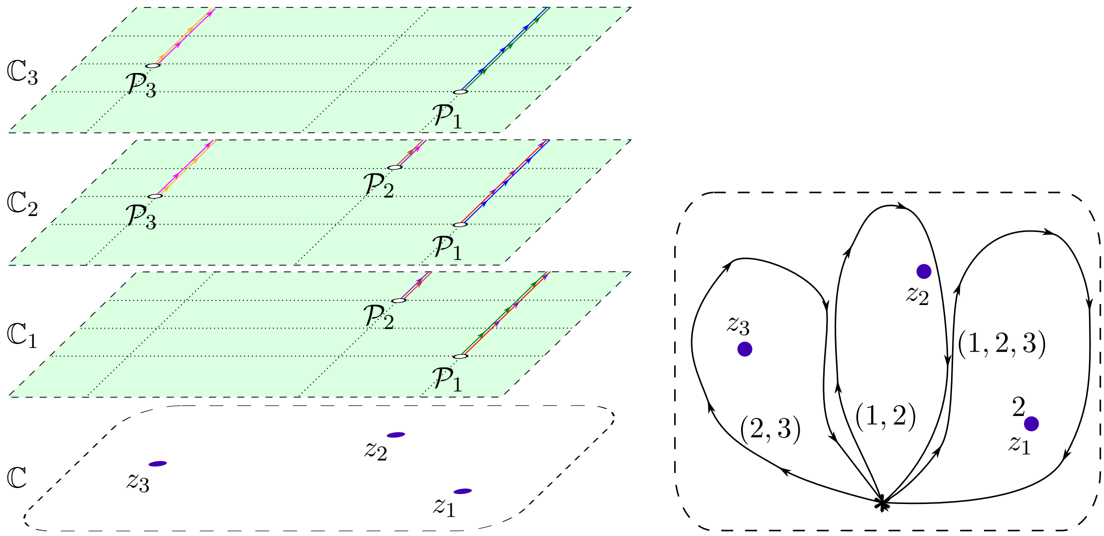

Figure 7.5. On left, the configuration in $\bar{\mathcal{O}}_{0,3}[1,1,1]$ from Figure 7.2 gives rise to a triple branched covering of the plane. The data of this covering corresponds to the configuration in $\mathrm{Hur}^*(5, \mathfrak{S}_3, \mathbb{1})$ on right.

function $f\colon \bar{\mathcal{F}} \to \mathbb{C}P^1$, that is, a point in $\bar{\mathcal{O}}_{g,n}[\underline{d}]$. Compare this recipe with the one from Subsection 6.2.3, and see also Figure 7.5.

First recall that if $\mathcal{X}$ is a locally contractible topological space and $\varphi\colon \pi_1(\mathcal{X}, *) \to \mathfrak{S}_d$ is a group homomorphism, there is a corresponding d-fold covering $\mathcal{Y} \to \mathcal{X}$ of $\mathcal{X}$, given by:

$$\pi\colon \mathcal{Y} := \tilde{\mathcal{X}} \times_{\pi_1(\mathcal{X},*)} [d] \to \tilde{\mathcal{X}}/\pi_1(\mathcal{X}, *) = \mathcal{X}.$$

Here $\tilde{\mathcal{X}}$ denotes the universal covering of $\mathcal{X}$; the group $\pi_1(\mathcal{X}, *)$ acts on $\tilde{\mathcal{X}}$ by deck transformations; the action of $\pi_1(\mathcal{X}, *)$ on $[d]$ is induced by φ.

The covering π is characterised by the following property: the fibre over the basepoint $\pi^{-1}(*)$ is canonically identified with the set $\{1, \dots, d\}$ and the monodromy of $\pi_1(\mathcal{X}, *)$ on this fibre is precisely given by φ. The identification $\pi^{-1}(*) \cong [d]$ is induced by the following natural map, where $*_{\tilde{\mathcal{X}}}$ denotes the basepoint of $\tilde{\mathcal{X}}$:

$$[d] \cong \left\{*_{\tilde{\mathcal{X}}}\right\} \times [d] \subset \tilde{\mathcal{X}} \times [d] \to \mathcal{Y}.$$

Let now $(P, \varphi) \in \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$, where $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\} \in SP^h(\mathbb{C})$ and φ is a homomorphism $\pi_1(\mathbb{C} \setminus P, *) \to \mathfrak{S}_d$ such that the norm of the local monodromy of φ at z_i is m_i, for all $1 \le i \le k$.

Consider the covering space $\check{f}\colon \mathcal{F} \to \mathbb{C} \setminus P$ associated with the homomorphism φ: note that $\mathcal{F}$ has a natural structure of (non-compact) Riemann surface; note also that $\mathcal{F}$ is connected, because the action of φ on the fibre $\check{f}^{-1}(*_P) \cong [d]$ is assumed transitive.

Let $(U_i)_{1 \le i \le k}$ be open neighbourhoods in $\mathbb{C}$ of the points $(z_i)_{1 \le i \le k}$ as in Definition 3.1.3. Choose also a neighbourhood U_∞ of $\infty \in \mathbb{C}P^1$ of the form

$$U_\infty = \{z \in \mathbb{C} \mid |z| > N\} \cup \{\infty\},$$

and assume that $U_1, \dots, U_k$ and $*_P$ are contained in the closed ball $\{z \in \mathbb{C} \mid |z| \leq N\} = \mathbb{C}P^1 \setminus U_\infty$. For $1 \leq i \leq k$ let $U_i^* = U_i \setminus \{z_i\}$; moreover let $U_\infty^* = U_\infty \setminus \{\infty\}$.
Let $f_1, \dots, f_k$ be an admissible generating set for $\pi_1(\mathbb{C} \setminus P, *)$ (see Definition 3.1.5), and for $1 \leq i \leq k$ let $\gamma_i = \varphi(f_i)$. Moreover denote $\gamma_\infty = \omega(P, \varphi) = \sigma$ (see Definition 4.4.1).
Then for each $i = 1, \dots, k, \infty$ the restriction of the covering $\check{f}$ over U_i^* is a finite covering of a space homotopy equivalent to $\mathbb{S}^1$, hence it is a disjoint union of finite cyclic coverings.
Let γ_i have a cycle decomposition

$$\gamma_i = c_{i,1} \cdot \dots \cdot c_{i,\nu_i};$$

note in particular that $\nu_\infty = n$, and up to relabeling we may assume that $c_{\infty,j} = c_j$, using the notation from the statement of the Theorem 7.4.1.
Then $\check{f}|_{U_i^*}$ is a disjoint union of ν_i cyclic coverings; more precisely, $\check{f}|_{U_i^*}$ contains a cyclic covering $V_{i,j}^* \to U_i^*$ of degree $|c_{i,j}|$ for every $1 \leq j \leq \nu_i$.
The standard model for such a covering is the map $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^{|c_{i,j}|}$, restricted to a neighbourhood of $0 \in \mathbb{C}$ intersected with $\mathbb{C}^*$. This covering map $\mathbb{C}^* \to \mathbb{C}^*$ extends to a map $\mathbb{C} \to \mathbb{C}$ by the same formula $z \mapsto z^{|c_{i,j}|}$: in particular $0 \mapsto 0$. The new map is not a genuine covering map unless $|c_{i,j}| = 1$, because the fibre over 0 consists of only one point, whereas the other fibres consist of $|c_{i,j}|$ points. We say that there is a *branching* at 0, and in the next Definition we elaborate on this idea.

Definition 7.4.2. A map of (non-compact) Riemann surfaces $\pi \colon \mathcal{S} \to \mathcal{T}$ is called a *finite branched covering* if it is proper and if for every $s \in \mathcal{S}$ one can find the following:

- neighbourhoods $s \in U_s \subset \mathcal{S}$ and $t = \pi(s) \in U_t \subset \mathcal{T}$;
- charts $w_s \colon U_s \to \mathbb{C}$ and $w_t \colon U_t \to \mathbb{C}$ with $w_s(s) = w_t(t) = 0$;
- an integer $r \geq 0$,

such that the map π restricts to a map $\pi \colon U_s \to U_t$ and this restriction, read in the charts, has the form $z \mapsto z^r$.
Points $s \in \mathcal{S}$ for which $r \geq 2$ are called *branching points* of π; their images $t = \pi(s) \in \mathcal{T}$ are called *branching values*. Note that the set of branching points is discrete.

The open Riemann surface $\mathcal{F}$, constructed as d-fold covering $f \colon \mathcal{F} \to \mathbb{C} \setminus P$, can now be compactified to a closed Riemann surface $\bar{\mathcal{F}}$ by adding some points; we can moreover extend $\check{f}$ to a branched covering $\check{f} \colon \bar{\mathcal{F}} \to \mathbb{C}P^1$:

- for $1 \leq i \leq k$ and $1 \leq j \leq \nu_i$ we add a point $C_{i,j}$ to close the end of $\mathcal{F}$ lying in $V_{i,j}^*$; we set $\check{f}(C_{i,j}) = z_i$;
- for $1 \leq j \leq \nu_\infty = n$ we add a point Q_j to close the end of $\mathcal{F}$ lying in $V_{\infty,j}^*$; we set $\check{f}(Q_j) = \infty$.

The map $\check{f}\colon \bar{\mathcal{F}} \to \mathbb{C}P^1$ can be equivalently regarded as a meromorphic function on $\bar{\mathcal{F}}$, holomorphic on $\mathcal{F}$ and having a pole of order exactly $d_j = |c_{\infty,j}|$ at Q_j, for all $1 \le j \le n$. The Euler characteristic $\chi\left(\bar{\mathcal{F}}\right)$ can be computed as follows. First observe that $\chi(\mathcal{F}) = d \cdot \chi(\mathbb{C} \setminus P) = d(1-k)$, because $\check{f}$ is a d-fold covering over $\mathbb{C} \setminus P$. Then note that each point $C_{i,j}$ or Q_j raises the Euler characteristic by 1; therefore

$$\chi(\bar{\mathcal{F}}) = d(1-k) + n + \sum_{i=1}^{k} \nu_i = d(1-k) + n + \sum_{i=1}^{k} (d - N(\gamma_i)) = d + n - h = 2 - 2g.$$

Thus we have shown that $\bar{\mathcal{F}}$ has genus g; in the following we will define tangent vectors $X_j \in T_{Q_j}\bar{\mathcal{F}}$ and replace $\check{f}$ with another map $f\colon \bar{\mathcal{F}} \to \mathbb{C}P^1$, such that f is a $\underline{d}$-directed meromorphic function (see Definition 7.1.3): the whole construction then yields a point in $\bar{\mathcal{O}}_{g,n}[\underline{d}]$. We will explain at the end of the subsection the technical reasone for which we prefer to replace $\check{f}$ with a new map f.
Let $L \subset \mathbb{C}P^1$ be the segment which joins $*$ to ∞ along the vertical halfline in $\mathbb{C}$ which goes in the negative imaginary direction; then $\check{f}^{-1}(L) \subset \bar{\mathcal{F}}$ consists of d segments: each segment starts from one of the d points in $\check{f}^{-1}(*)$ and approaches one of the points Q_j. The canonical identification $\check{f}^{-1}(*) \cong [d]$ allows us to label these d segments as $L_1, \dots, L_d$.
For all $1 \le i \le d$ and $1 \le j \le n$, the segment L_i has an endpoint in Q_j if and only if i is contained in the cycle $c_j = c_{\infty,j}$. We define $f = \sqrt{-1} \cdot \check{f}\colon \bar{\mathcal{F}} \to \mathbb{C}P^1$; note that the branching values of f in $\mathbb{C}$ are not the points $z_i \in \mathbb{C}$, but rather the points $\sqrt{-1} \cdot z_i$.
For all $1 \le j \le n$ we let $X_j \in T_{Q_j}\bar{\mathcal{F}}$ be the unique vector having the following two properties:

1. X_j is tangent to L_{ι_j} at Q_j and points in the direction of L_{ι_j};
2. f has a *directed* pole of order d_j at Q_j, i.e. for every chart w_j defined on a neighbourhood of Q_j such that $w_j(Q_j) = 0 \in \mathbb{C}$ and $Dw_j(X_j) = \partial/\partial x \in T_0\mathbb{C}$, the function f has the form
$$f(w_j) = \frac{1}{w_j^{d_i}} + l.o.t..$$

Note that the first property determines X_j up to real positive multiples, whereas the second determines it up a to rotation by a multiple of $2\pi/d_j$.
To check that the two properties are compatible, note that $f(L_{\iota_j}) = \sqrt{-1} \cdot L$ is a horizontal halfline in $\mathbb{C}$ going to $\infty \in \mathbb{C}P^1$ in the positive, real direction. More formally, $f(L_{\iota_j})$ exits from $\infty \in \mathbb{C}P^1$ along a positive multiple of the vector $\partial/\partial\Re(w) \in T_\infty\mathbb{C}P^1$; here w is the canonical chart near ∞ mapping $[a\colon b] \mapsto b/a$, and $\partial/\partial\Re(w)$ is the vector dual to $d\Re(w) \in T^*_\infty\mathbb{C}P^1$ with respect to the basis $d\Re(w), d\Im(w)$.
Therefore if X_j satisfies property 1. and if w_j is adapted to X_j in the sense of property 2., then the expression of f near Q_j has automatically the form

$$f(w_j) = \frac{N}{w_j^{d_i}} + l.o.t..$$

for some *positive real* constant $N > 0$. This shows that the properties 1. and 2. are compatible.
The proof of Theorem 7.4.1 will give an explicit homeomorphism $\mathfrak{O}\colon \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma) \cong \bar{\mathcal{O}}_{g,n}[\underline{d}]$, so that one can compare the construction from this subsection and the other construction from Subsection 6.2.3 as follows.
Let $(P, \varphi) \in \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$, and consider the point $\mathfrak{O}(P, \varphi) \in \bar{\mathcal{O}}_{g,n}[\underline{d}] \subset \mathfrak{H}_{g,n}[\underline{d}]$, where $\mathfrak{O}$ is the homeomorphism described in Theorem 7.4.1.
On the one hand, using the recipe above, we can construct directly from (P, φ) a Riemann surface $\bar{\mathcal{F}}$ of type $\Sigma_{g,n}$ with a $\underline{d}$-direct meromorphic function $f\colon \bar{\mathcal{F}} \to \mathbb{C}P^1$.
On the other hand, by Definition 7.2.1, the point $\mathfrak{O}(P, \varphi)$ belongs to the subspace $\bar{\mathcal{O}}_{g,n}[\underline{d}] \subset \mathfrak{Par}_{g,n}[\underline{d}]$, hence the recipe from Subsection 6.2.3 yields a Riemann surface $\bar{\mathcal{F}}'$ of type $\Sigma_{g,n}$ endowed with a $\underline{d}$-directed meromorphic function $f'\colon \bar{\mathcal{F}}' \to \mathbb{C}P^1$.
It can be checked that $(\bar{\mathcal{F}}, f)$ and $(\bar{\mathcal{F}}', f')$ represent the same point in $\bar{\mathcal{O}}_{g,n}[\underline{d}]$: this follows from a careful analysis of all definitions and conventions: in particular the replacement of $\check{f}$ with $f = \sqrt{-1} \cdot \check{f}$ is needed for the two recipes to agree on the nose. We will omit the details.
In other words, the recipe in this subsection provides an explicit, geometric description of the homeomorphism $\mathfrak{O}$, which instead will be defined in an abstract way, by identifying two bisimplicial complexes.

7.4.2 Proof of Theorem 7.4.1

Let $\mathfrak{S}_{\underline{d}}$ be the group of permutations of the set

$$S_{\underline{d}} = \{(l, \nu) \,|\, 1 \le l \le n, 1 \le \nu \le d_l\},$$

and denote by $\mathfrak{lc}_{\underline{d}} \in \mathfrak{S}_{\underline{d}}$ the permutation that, for all $1 \le l \le n$, permutes cyclically the elements

$$(l, 1) \mapsto (l, d_l) \mapsto (l, d_l - 1) \mapsto \cdots \mapsto (l, 1) \mapsto (l, d_l).$$

Then there is a unique bijection $\Xi\colon [d] \cong S_{\underline{d}}$ satisfying the following properties:

- $\Xi(\iota_l) = (l, 1)$ for all $1 \le l \le n$, where $\iota_l \in c_l$ is the chosen element in the l^{th}cycle of σ (see statement of Theorem 7.4.1);
- along the isomorphism of groups $\mathfrak{S}_d \cong \mathfrak{S}_{\underline{d}}$ induced by Ξ, the permutations σ and $\mathfrak{lc}_{\underline{d}}$ correspond to each other.

From now on we will identify the sets $[d] \cong S_{\underline{d}}$ and the groups $\mathfrak{S}_d \cong \mathfrak{S}_{\underline{d}}$ as explained above, without further mention.
We will prove that there is a homeomorphism

$$\mathfrak{O}^{\infty}\colon \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^{\infty} \cong \bar{\mathcal{O}}_{g,n}[\underline{d}]^{\infty}$$

mapping $\infty \mapsto \infty$; the desired homeomorphism $\mathfrak{O}\colon \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma) \cong \bar{\mathcal{O}}_{g,n}[\underline{d}]$ will then be obtained by restriction.

To define $\mathfrak{O}^\infty$ we will first refine the cell structure on $\mathrm{Hur}^*(h,\mathfrak{S}_d,\sigma)^\infty$ from Theorem 4.2.3, again by applying a suitable shuffle product decomposition. The space $\mathrm{Hur}^*(h,\mathfrak{S}_d,\sigma)$ with the new cell structure will be a bisimplicial complex, and we will prove that $\mathrm{Hur}^*(h,\mathfrak{S}_d,\sigma)^\infty$ and $\bar{O}_{g,n}[\underline{d}]^\infty$ are isomorphic as bisimplicial complexes (for the bisimplicial structure on the latter see Sections 6.3 and 7.2).

Recall that a cell $e^{\mathfrak{A}} \subset \mathrm{Hur}^*(h,\mathfrak{S}_d,\sigma)^\infty$ is modelled on the multisimplex $\Delta^{\mathfrak{A}} = \Delta^l \times \Delta^{\lambda_1} \times \cdots \times \Delta^{\lambda_l}$, using the notation from Definition 4.2.2. The $\mathfrak{S}_d$-array $\mathfrak{A}$ must fulfil the following properties in order to give a cell in $\mathrm{Hur}^*(h,\mathfrak{S}_d,\sigma)^\infty$, where we use the notation from Definition 4.2.2:

(a) $m_{i,j} \geq 1$ for all $1 \leq i \leq l$ and $1 \leq j \leq \lambda_i$;

(b) $\sum_{i=1}^{l}\sum_{j=1}^{\lambda_i} m_{i,j} = h$ (these two properties are needed for the general Hurwitz space $\widetilde{\mathrm{Hur}}(h,\mathfrak{S}_d)$, see Definition 4.1.1);

(c) $N(\gamma_{i,j}) = m_{i,j}$ for all $1 \leq i \leq l$ and $1 \leq j \leq \lambda_i$, i.e. $\mathfrak{A}$ is a *special* $\mathfrak{S}_d$-array (property needed for the special Hurwitz space $\mathrm{Hur}(h,\mathfrak{S}_d)$, see Definition 4.3.6);

(d) the total monodromy $\omega(\mathfrak{A})$ is $\sigma \in \mathfrak{S}_d$ (property needed for the subspace $\mathrm{Hur}(h,\mathfrak{S}_d,\sigma)$, see Definition 4.4.2);

(e) the subgroup $\tilde{H} \subset \mathfrak{S}_d$ generated by all elements $\gamma_{i,j}$ acts transitively on $[d]$ (property needed for the subspace $\mathrm{Hur}^*(h,\mathfrak{S}_d,\sigma)$, see Definition 4.5.4).

We can now get rid of the numbers $m_{i,j}$ from our notation, by using Property (c) as a *definition* of the $m_{i,j}$'s in terms of the $\gamma_{i,j}$'s. The condition $m_{i,j} \geq 1$ can be rephrased as $\gamma_{i,j} \neq \mathbb{1} \in \mathfrak{S}_d$ for all i and j. Thus we can regard a special $\mathfrak{S}_d$-array $\mathfrak{A}$ as a collection $(\gamma_{i,j})_{1\leq i\leq l, 1\leq j\leq\lambda_i}$ of elements of $\mathfrak{S}_d$.

We introduce a new notation for cells in $\mathrm{Hur}(h,\mathfrak{S}_d)$. For $1 \leq i \leq l$ and $1 \leq j \leq \lambda_i$ let

$$\theta_{i,j} = \gamma_{i,1} \cdot \ldots \cdot \gamma_{i,j} \in \mathfrak{S}_d.$$

Then the elements $\gamma_{i,j}$ can be recovered from the elements $\gamma_{i,j}$ by the inversion formula

$$\gamma_{i,j} = \theta_{i,j-1}^{-1}\theta_{i,j},$$

where by convention we set $\theta_{i,0} = \mathbb{1} \in \mathfrak{S}_d$. The collection

$$[\underline{\theta}] = [\theta_{i,j}]_{1\leq i\leq l, 1\leq j\leq\lambda_i},$$

gives an alternative notation to represent the cell $e^{\mathfrak{A}} \subset \mathrm{Hur}(h,\mathfrak{S}_d)$: we will write $e^{[\underline{\theta}]}$ for this cell, which is modelled on the multisimplex $\Delta^{[\underline{\theta}]} \cong \Delta^{\mathfrak{A}}$, and refer to $[\underline{\theta}]$ as the *integral* notation; the old notation $\mathfrak{A}$ is instead called the *differential* notation.

The element $\theta_{i,j} \in \mathfrak{S}_d$ is the image along φ of the element $\xi_{i,j} \in \pi_1(\mathbb{C} \setminus P, *)$ represented by the following loop in $\mathbb{C} \setminus P$ (we use the notation from Definition 3.3.1): the loop $\xi_{i,j}$ begins at $*$, runs horizontally to a position just on left of the i^{th}vertical line of P, runs upwards to a position just above $z_{i,j}$, runs to right to a position just on right of $z_{i,j}$, runs

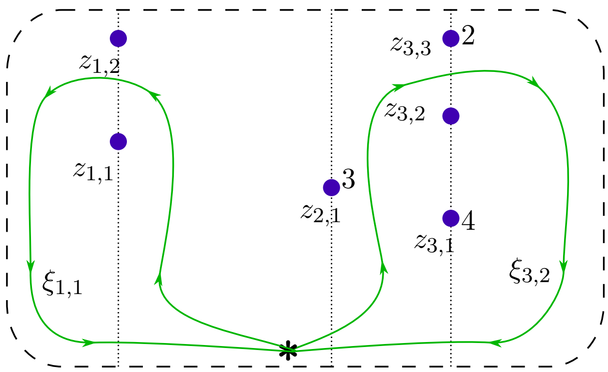

Figure 7.6. The loops $\xi_{1,1}$ and $\xi_{3,2}$ for the configuration $P \in SP^{12}(\mathbb{C})$ from Figure 3.4. Note that $\xi_{1,1} = \mathfrak{f}^P_{1,1}$ and $\xi_{3,2} = \mathfrak{f}^P_{3,1} \cdot \mathfrak{f}^P_{3,2}$, using the notation from Definition 3.3.1.

downwards to a position on the same height as $*$, runs back to $*$ horizontally. Compare with the elements ξ_j introduced in the proof of Lemma 3.3.3, and see Figure 7.6.
We can now rephrase the properties listed above in the integral notation. A collection $[\underline{\theta}]$ yields a cell $e^{[\underline{\theta}]} \subset \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^\infty$ if the following properties are fulfilled:

(a) for all $1 \le i \le l$ and $1 \le j \le \lambda_i$, the permutation $\gamma_{i,j} := \theta^{-1}_{i,j-1}\theta_{i,j}$ is different from $\mathbb{1} \in \mathfrak{S}_d$, where by convention $\theta_{i,0} = \mathbb{1}$;

(b) using the previous notation, $\sum_{i=1}^{l} \sum_{j=1}^{\lambda_i} N(\gamma_{i,j}) = h$;

(d) the product $\theta_{1,\lambda_1} \cdot \dots \cdot \theta_{l,\lambda_l}$ is equal to $\sigma \in \mathfrak{S}_d$;

(e) the subgroup $\tilde{H} \subset \mathfrak{S}_d$ generated by all elements $\theta_{i,j}$ acts transitively on $[d]$.

We now operate an Eilenberg-Zilber decomposition on the factor $\Delta^{\lambda_1} \times \dots \times \Delta^{\lambda_l}$ of every multisimplex $\Delta^{[\underline{\theta}]}$, similarly as how we did in Section 6.3: thus we dissect $\Delta^{[\underline{\theta}]}$ into several bisimplices of the form $\Delta^l \times \Delta^\lambda$, where $\lambda = \sum_{i=1}^{l} \lambda_i$ is the absolute value of $\mathfrak{A}$. We obtain a finer cell decomposition on the space $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^\infty$, whose cells are described in the following. Let

$$[\underline{\tilde{\theta}}] = [\tilde{\theta}_{i,j}]_{1 \le i \le l, 1 \le j \le \lambda}$$

be a collection of elements of $\mathfrak{S}_d$, for some $l \ge 1$ and $\lambda \ge 1$; for $1 \le i \le l$ and $1 \le j \le \lambda$ define $\tilde{\gamma}_{i,j} = \tilde{\theta}^{-1}_{i,j}\tilde{\theta}_{i,j}$, where by convention $\tilde{\theta}_{i,0} = \mathbb{1} \in \mathfrak{S}_d$. Then there is a cell $e^{[\underline{\tilde{\theta}}]} \subset \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^\infty$, if $[\underline{\tilde{\theta}}]$ fulfils the following list of properties; the enumeration invites a comparison with the list from the end of Subsection 7.2.2:

(3) for all $1 \le i \le l$ there exist $1 \le j \le \lambda$ such that $\tilde{\gamma}_{i,j} \ne \mathbb{1} \in \mathfrak{S}_d$;

(4) the product

$$\tilde{\theta}_{1,\lambda} \cdot \dots \cdot \tilde{\theta}_{l,\lambda} \in \mathfrak{S}_d$$

is the permutation σ;

(5) for all $1 \leq j \leq \lambda$ there exists $1 \leq i \leq l$ such that $\tilde{\gamma}_{i,j} \neq \mathbb{1} \in \mathfrak{S}_d$;

(6) $\sum_{i=1}^{l} \sum_{j=1}^{\lambda} N(\tilde{\gamma}_{i,j}) = h$;

(7) the subgroup $\tilde{H} \subseteq \mathfrak{S}_d$ generated by all elements $\tilde{\theta}_{i,j}$ acts transitively on $[d]$.

The subspace $e^{[\tilde{\theta}]} \subset \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma) \subset \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^\infty$ contains configurations (P, φ) satisfying the following properties (compare with Definitions 3.2.4,4.2.2):

- there are exactly l distinct vertical lines in $\mathbb{C}$, of the form $\Re(z) = x_i$ for some real numbers $-\infty < x_1 < \cdots < x_l < +\infty$, containing at least one point of the configuration P;
- there are exactly λ distinct horizontal lines in $\mathbb{C}$, of the form $\Im(z) = y_i$ for some real numbers $-\infty < y_1 < \cdots < y_\lambda < +\infty$, containing at least one point of the configuration P;
- let $\tilde{z}_{i,j} = x_i + \sqrt{-1} y_j$, let $P = \{\tilde{m}_{i,j} \cdot \tilde{z}_{i,j}\}$, for some numbers $\tilde{m}_{i,j} \geq 0$, and let $\tilde{f}_{i,j} \in \pi_1(\mathbb{C} \setminus P, *)$ be represented by a loop that winds only around $\tilde{z}_{i,j}$, once and clockwise (compare with Definition 3.1.5); then $N(\varphi(\tilde{f}_{i,j})) = \tilde{m}_{i,j}$; this condition is vacuous when $\tilde{m}_{i,j} = 0$, because then $\tilde{z}_{i,j}$ is not a point of the configuration P and thus $\tilde{f}_{i,j}$ represents $\mathbb{1} \in \pi_1(\mathbb{C} \setminus P, *)$;
- $\omega(P, \varphi) = \sigma$ (see Definition 4.4.1);
- the orbit partition $\underline{\mathfrak{P}}(P, \varphi)$ consists of the only set $[d]$ (see Definition 4.5.1).

The element $\tilde{\theta}_{i,j} \in \mathfrak{S}_d$ is the image along φ of the element $\tilde{\xi}_{i,j} \in \pi_1(\mathbb{C} \setminus P, *)$ represented by the following loop in $\mathbb{C} \setminus P$: the loop $\tilde{\xi}_{i,j}$ starts at $*$, runs horizontally to a position just on left of x_i, runs upwards to a position just above y_j, runs to right to a position just on right of x_i, runs downwards to a position on the same height as $*$, runs back to $*$ horizontally (see Figure 7.7).
The cell $e^{[\tilde{\theta}]}$ is modelled on the bisimplex $\Delta^{[\tilde{\theta}]} = \Delta^l \times \Delta^\lambda$; the coordinates of the factor Δ^l are the numbers $x_1, \ldots, x_l$, and the coordinates of the factor Δ^λ are the numbers $y_1, \ldots, y_\lambda$.
Properties (3) and (5) in the list for $[\tilde{\theta}]$ correspond to the fact that every vertical line $\Re(z) = x_i$ and every horizontal line $\Im(z) = y_i$ must contain at least one point of the configuration P with multiplicity ≥ 1; they are the reformulation of Property (a) for $[\underline{\theta}]$ after the Eilenberg-Zilber decomposition.
Property (4) is the reformulation of Property (d), and corresponds to the requirement $\omega(P, \varphi) = \sigma$.
Property (6) is the reformulation of Property (b).
Property (7) is the reformulation of Property (e), and corresponds to the requirement $\underline{\mathfrak{P}}(P, \varphi) = ([d])$.

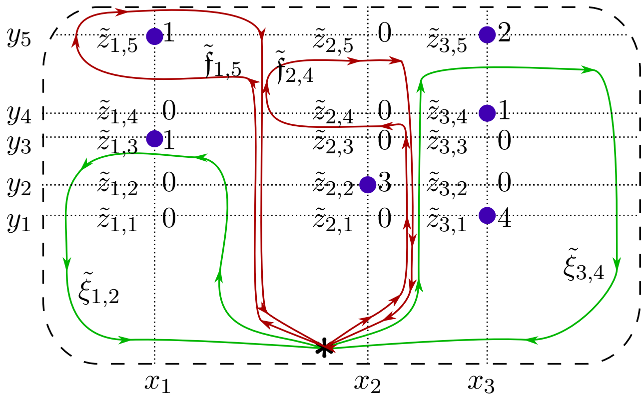

Figure 7.7. The configuration $P \in SP^{12}(\mathbb{C})$ from Figure 3.4 can be expressed as $P = \{\tilde{m}_{i,j} \cdot \tilde{z}_{i,j}\}_{1 \le i \le 3, 1 \le j \le 5}$, where nine of the coefficients $\tilde{m}_{i,j}$ are equal to zero. The loops $\tilde{\xi}_{1,2}$ and $\tilde{\mathfrak{f}}_{2,4}$ are trivial in $\pi_1(\mathbb{C} \setminus P, *)$, because the space $\mathbb{C} \setminus P$ is by definition $\mathbb{C} \setminus \{\tilde{z}_{1,3}, \tilde{z}_{1,5}, \tilde{z}_{2,2}, \tilde{z}_{3,1}, \tilde{z}_{3,4}, \tilde{z}_{3,5}\}$; the loop $\tilde{\mathfrak{f}}_{1,5}$ is homotopic to the loop $\mathfrak{f}_{1,2}$; the loop $\tilde{\xi}_{3,4}$ is homotopic to the loop $\xi_{3,2}$ and represents the product $\mathfrak{f}_{3,1} \cdot \mathfrak{f}_{3,2} \in \pi_1(\mathbb{C} \setminus P, *)$.

It is now evident that there is a bijection between cells $e^{[\underline{\tilde{\theta}}]} \subset \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^\infty$ and cells $e^{[\underline{\check{\sigma}}]} \subset \check{\mathcal{O}}_{g,n}[\underline{d}]^\infty$: corresponding cells satisfy $q = \lambda$, $p = l$ and $\check{\sigma}_{k,j} = \tilde{\theta}_{j,k}$, and the 0-cells ∞ also correspond to each other.

We study now the boundary operators in the refined cell structure on $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^\infty$. Let $e^{[\underline{\tilde{\theta}}]}$ be a cell; using the notation above, we write $\Delta^{[\underline{\tilde{\theta}}]} = \Delta^l \times \Delta^\lambda$. For $0 \le i \le l$ let $\partial_i^{hor} \Delta^{[\underline{\tilde{\theta}}]} = \partial_i \Delta^l \times \Delta^\lambda$ be the i^{th}horizontal face, and for $0 \le j \le \lambda$ let $\partial_j^{ver} \Delta^{[\underline{\tilde{\theta}}]} = \Delta^l \times \partial_j \Delta^\lambda$ be the j^{th}vertical face. When $i = 0, l$ or $j = 0, \lambda$ we say that $\partial_i^{hor} \Delta^{[\underline{\tilde{\theta}}]}$ and $\partial_j^{ver} \Delta^{[\underline{\tilde{\theta}}]}$ are outer faces, otherwise they are inner faces.

The restriction of the characteristic map $\Phi^{[\underline{\tilde{\theta}}]} \colon \Delta^{[\underline{\tilde{\theta}}]} \to \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^\infty$ to a face is computed by letting two consecutive coordinates x_i, or two consecutive y_j's, become equal (inner face), or by letting x_1 respectively x_l, or y_1 respectively y_λ, go to $-\infty$ respectively $+\infty$ (outer face). In case of an outer face, at least one point of the configuration P goes to $\infty \in (\mathbb{C})^\infty \cong \mathbb{C}P^1$; therefore the restriction of $\Phi^{[\underline{\tilde{\theta}}]}$ to any outer face of $\Delta^{[\underline{\tilde{\theta}}]}$ is the constant map to $\infty \in \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^\infty$.

To understand the restriction of $\Phi^{[\underline{\tilde{\theta}}]}$ to an inner, horizontal face $\partial_r^{hor} \Delta^{[\underline{\tilde{\theta}}]}$, with $1 \le r \le l-1$, denote $l' = l - 1$ and let

$$[\underline{\tilde{\theta}}'] = [\tilde{\theta}'_{i,j}]_{1 \le i \le l', 1 \le j \le \lambda}$$

be defined by letting, for all $1 \le j \le \lambda$,

$$[\tilde{\theta}'_{1,j}, \dots, \tilde{\theta}'_{l',j}] = [\tilde{\theta}_{1,j}, \dots, \tilde{\theta}_{i-1,j}, \tilde{\theta}_{i,j} \cdot \tilde{\theta}_{i+1,j}, \tilde{\theta}_{i+2,j}, \dots, \tilde{\theta}_{l,j}];$$

then the restriction of $\Phi^{[\underline{\tilde{\theta}}]}$ to $\partial_r^{hor} \Delta^{[\underline{\tilde{\theta}}]}$ is:

- the characteristic map of $e^{[\underline{\tilde{\theta}}']}$, if $[\underline{\tilde{\theta}}']$ statisfies Properties (3)-(7) in the list above;

- the constant map to ∞ otherwise.

Similarly, to understand the restriction of $\Phi^{[\tilde{\underline{\theta}}]}$ to an inner, vertical face $\partial_s^{ver}\Delta^{[\tilde{\underline{\theta}}]}$, with $1 \leq s \leq \lambda - 1$, denote $\lambda' = \lambda - 1$ and let

$$[\tilde{\underline{\theta}}'] = [\tilde{\theta}'_{i,j}]_{1\leq i\leq l, 1\leq j\leq \lambda'}$$

be defined by omitting all elements $\tilde{\theta}_{i,s}$, for $1 \leq i \leq l$, and renormalising the indices; then the restriction of $\Phi^{[\tilde{\underline{\theta}}]}$ to $\partial_s^{ver}\Delta^{[\tilde{\underline{\theta}}]}$ is:

- the characteristic map of $e^{[\tilde{\underline{\theta}}']}$, if $[\tilde{\underline{\theta}}']$ statisfies Properties (3)-(7) in the list above;
- the constant map to ∞ otherwise.

The proof of these facts is similar to the proof of Lemmas 3.3.2, 3.3.3 and, in Chapter 9, of Lemmas 9.1.1 and 9.1.2. We omit the details.
We have recovered precisely the description of the boundary operators for the refined cell structure on $\bar{\mathcal{O}}_{g,n}[\underline{d}]^{\infty}$, see Subsection 7.2.3. The main difference is in the fact that a bisimplex $\Delta^l \times \Delta^{\lambda}$ representing a cell in $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^{\infty}$ corresponds to a bisimplex $\Delta^p \times \Delta^q$ with $q = l$ and $p = \lambda$, hence the factors of the simplices are swapped.
We have therefore obtained an isomorphism $\mathfrak{O}\colon \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^{\infty} \cong \bar{\mathcal{O}}_{g,n}[\underline{d}]^{\infty}$ of bisimplicial complexes, up to changing the bisimplicial structure of either of the two by swapping the order of the two factors in each bisimplex.

7.5 Complex manifolds

In this section we prove the following theorem.

Theorem 7.5.1. *The special Hurwitz space* $\mathrm{Hur}(h, \mathfrak{S}_d)$ *has a natural structure of complex manifold of complex dimension* h.

The proof is split in three parts. We first prove the theorem in a very special case, namely we will set $h = d-1$ and consider the space $\mathrm{Hur}^*(d-1, \mathfrak{S}_d, \mathfrak{lc}_d)$, where $\mathfrak{lc}_d \in \mathfrak{S}_d$ is the long cycle (see Definition 2.2.2). We then prove that a normal neighbourhood $\mathfrak{U}$ of a configuration $(P, \varphi) \in \mathrm{Hur}(h, \mathfrak{S}_d)$ has a canonical complex structure in the special case in which $h = N(\sigma)$. Finally, we treat the general case.

7.5.1 The space of monic polynomials

By Theorem 7.4.1 there is a homeomorphism $\mathrm{Hur}^*(d-1, \mathfrak{S}_d, \mathfrak{lc}_d) \cong \bar{\mathcal{O}}_{0,1}[d]$; the latter is by Definition 7.2.1 the space of d-directed meromorphic functions $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$, up to precomposition with a translation of the form $z \mapsto z + \lambda$ (see the discussion after Definition 6.1.3).
Note that a d-directed meromorphic function $f\colon \mathbb{C}P^1 \to \mathbb{C}P^1$ is precisely a monic polynomial of degree d, i.e. having leading monomial equal to z^d. We denote by $\mathfrak{MonPol}_d$

the space of monic polynomials of degree d: it is an affine subspace of the $\mathbb{C}$-vector space $\mathbb{C}[z]$; we have $\dim_{\mathbb{C}} \mathfrak{MonPol}_d = d$. The space $\bar{\mathcal{O}}_{0,1}[d]$ is therefore the orbit space

$$\bar{\mathcal{O}}_{0,1}[d] = \mathfrak{MonPol}_d/\mathbb{C},$$

where $\mathbb{C}$ acts by precomposition with a translation. Note that each orbit has a unique *normalised* representative of the form $z^d + a_{d-2}z^{d-2} + \cdots + a_0$, i.e. with coefficient of the monomial z^{d-1} equal to 0. These normalised polynomials form an affine subspace $\mathfrak{NMonPol}_d \subset \mathfrak{MonPol}_d$ of complex dimension $d-1$.
We have a sequence of homeomorphisms

$$\mathrm{Hur}^*(d-1, \mathfrak{S}_d, \mathfrak{lc}_d) \cong \bar{\mathcal{O}}_{0,1}[d] \cong \mathfrak{NMonPol}_d,$$

where the latter space is in particular a complex manifold of complex dimension $d-1$. We use the above sequence of homeomorphisms to define the complex structure on $\mathrm{Hur}^*(d-1, \mathfrak{S}_d, \mathfrak{lc}_d)$.
If $\sigma \in \mathfrak{S}_d$ is a permutation consisting of a unique cycle, then σ and $\mathfrak{lc}_d$ are conjugate by some element $\gamma \in \mathfrak{S}_d$. Using the action of $\mathfrak{S}_d$ on $\mathrm{Hur}^*(d-1, \mathfrak{S}_d)$ from Section 5.1, we obtain a homeomorphism

$$\gamma\colon \mathrm{Hur}^*(d-1, \mathfrak{S}_d, \mathfrak{lc}_d) \to \mathrm{Hur}^*(d-1, \mathfrak{S}_d, \sigma).$$

Therefore also $\mathrm{Hur}^*(d-1, \mathfrak{S}_d, \sigma)$ is a complex manifold of complex dimension $d-1$.

7.5.2 The case $h = N(\sigma)$

If $h = N(\sigma) \geq 1$, then we can fix a configuration $(P, \varphi) \in \mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$ with $P = \{h \cdot z\}$ for some $z \in \mathbb{C}$.
Decompose $\sigma = c_1 \cdot \ldots \cdot c_n$ into cycles; since σ is also a generator of the group $\mathrm{Im}(\varphi) \subset \mathfrak{S}_d$, the orbit partion of (P, φ) (see Definition 4.5.1) is

$$\underline{\mathfrak{P}}(P, \varphi) = \{c_1, \ldots, c_n\},$$

where we regard each c_i as a subset of $[d]$. By Definition 4.3.6 we have

$$h = N(\sigma) = N(c_1) + \cdots + N(c_n) = d - n.$$

We can use Theorem 4.5.5 and find a canonical homeomorphism between $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$ and the product

$$\prod_{i=1}^{n} \mathrm{Hur}^*(N(c_i), \mathfrak{S}_{c_i}, c_i),$$

where $\mathfrak{S}_{c_i}$ denotes the group of permutations of the set $c_i \subset [d]$. Note that $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$ is connected because there is only one way to split h into k numbers $h_1 + \cdots + h_n$ with $h_i \geq N(c_i)$, namely one must set $h_i = N(c_i) = |c_i| - 1$.
The factor $\mathrm{Hur}^*(h_i, \mathfrak{S}_{c_i}, c_i)$ can be identified with $\mathbb{C}^{h_i}$ by Subsection 7.5.1. Therefore we have found a homeomorphism between $\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)$ and $\mathbb{C}^h$, and thus we obtain a complex structure on the former space.

7.5.3 The general case

Let now $(P,\varphi) \in \mathrm{Hur}(h,\mathfrak{S}_d)$ be any configuration, and denote $P = \{m_1 \cdot z_1, \dots, m_k \cdot z_k\}$. Choose disjoint open disks $z_i \in U_i \subset \mathbb{C}$ giving a normal neighbourhood $\mathfrak{U}$ for (P,φ) in $\mathrm{Hur}(h,\mathfrak{S}_d)$ (see Definition 4.1.1, and recall by Definition 4.3.6 that $\mathrm{Hur}(h,\mathfrak{S}_d) \subseteq \widetilde{\mathrm{Hur}}(h,\mathfrak{S}_d)$ is open).
Choose arcs $\zeta_1,\dots,\zeta_k$ as in Definition 3.1.5 and Lemma 4.3.3: ζ_i connects $*_P$ with z_i, and these arcs have only one point in common, namely their endpoint $*_P$. For $1 \le i \le k$ we denote by $*_i \in \partial U_i$ the intersection between ζ_i and ∂U_i, which is assumed to be the bottom point of ∂U_i.
Let $\mathcal{U} \subset SP^h(\mathbb{C})$ be the normal neighbourhood of $P \in SP^h(\mathbb{C})$ associated with the discs U_i (see Definition 3.1.3); let $P' \in \mathcal{U}$ and write $P'_i \in SP^{m_i}(U_i)$ for the *intersection* between P' and U_i. We can then identify the groups

$$\pi_1\left(\mathbb{C}\setminus P', *_{P'}\right) \cong \pi_1(\bar{U}_1 \setminus P'_i; *_1) \vee \cdots \vee \pi_1(\bar{U}_k \setminus P'_k; *_k),$$

where $\vee$ denotes the free product of groups. The isomorphism is essentially constructed by translating all basepoints to $*_P$ using the vertical segment joining $*_P$ and $*_{P'}$ on the left-hand side, and the arcs ζ_i on the right-hand side. More precisely, for every $1 \le i \le k$ there is an isomorphism

$$\pi_1(\bar{U}_i \setminus P'_i; *_i) \cong \pi_1(\left(\bar{U}_i \cup \zeta_i\right) \setminus P'_i; *_P),$$

and the space $\mathbb{C}\setminus P$ deformation retracts onto the union (which is topologically a wedge) of the spaces $\left(\bar{U}_i \cup \zeta_i\right) \setminus P'_i$.
Let $f_1,\dots,f_k$ be the admissible generating set for $\pi_1(\mathbb{C}\setminus P, *)$ associated with the arcs ζ_i. Then a configuration (P',φ') in $\mathfrak{U}$ can be equivalently represented as a sequence $(P'_1,\varphi'_1),\dots,(P'_k,\varphi'_k)$, where for all $1 \le i \le k$ the following are satisfied:

- $P'_i \in SP^{m_i}(U_i)$;
- $\varphi'_i \colon \pi_1(\bar{U}_i \setminus P'_i; *_i) \to \mathfrak{S}_d$ is a homomorphism of groups with $\varphi'_i(\partial U_i) = \varphi(f_i)$.

Here $\partial U_i \in \pi_1(\bar{U}_i \setminus P'_i; *_i)$ is the loop running clockwise on the circle ∂U_i, and plays a similar role as the large loop (see Definition 3.3.5).
We can regard each (P'_i,φ'_i) as a configuration in $\mathrm{Hur}(m_i,\mathfrak{S}_d,\varphi(f_i))$: we can translate the basepoint $*_i$ to $*_{P'_i}$ by a straight segment, because $*_i$ is the bottom point of ∂U_i and $*_{P'_i}$ lies below P'_i.
There is therefore a canonical, open inclusion

$$\mathfrak{U} \subset \prod_{i=1}^{k} \mathrm{Hur}(m_i,\mathfrak{S}_d,\varphi(f_i)).$$

Note now that $N(\varphi(f_i)) = m_i$, hence the factor on the right-hand side cab be identified with $\mathbb{C}^h$ by Subsection 7.5.2. Therefore also $\mathfrak{U}$ can be identified with an open subset of $\mathbb{C}^h$, and this defines a complex structure on $\mathfrak{U}$.

7.5.4 Consequences of Theorem 7.5.1

For $d \geq 2g+n-1$ we recall that, by Theorem 7.3.2, the space $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ is a bundle over $\mathfrak{M}_{g,n}$ with fibre an affine complex space: since the base space and the fibre are manifolds, also the total space must be a manifold. Theorem 7.5.1 enhances this remark by proving that $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ is a complex manifold of complex dimension $h = 2g+n+d-2$ even in the case $d < 2g+n-1$, in which the map $\rho\colon \bar{\mathcal{O}}_{g,n}[\underline{d}] \to \mathfrak{M}_{g,n}$ need not even be surjective. Moreover, by Poincaré-Lefschetz duality we obtain an isomorphism

$$H_*(\bar{\mathcal{O}}_{g,n}[\underline{d}]) \simeq \tilde{H}^{2h-*}(\bar{\mathcal{O}}_{g,n}[\underline{d}]^\infty);$$

the right-hand side can be computed as $\tilde{H}^{2h-*}(\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)^\infty)$, where $\sigma \in \mathfrak{S}_d$ is a permutation as in the statement of Theorem 7.4.1. This is a first justification of our interest for the reduced cochain complex $\tilde{C}^*\left(\mathrm{Hur}(h, \mathfrak{S}_d)^\infty\right)$ that we will consider in Chapter 9, because $\tilde{C}^*\left(\mathrm{Hur}(h, \mathfrak{S}_d, \sigma)^\infty\right)$ is naturally a direct summand of $\tilde{C}^*\left(\mathrm{Hur}(h, \mathfrak{S}_d)^\infty\right)$.

8 The algebra $\mathcal{V}(d)$

Let $d \geq 2$ be fixed throughout the chapter. We elaborate on the work of Visy [39] and define a particular bialgebra $\mathcal{V}(d)$, which plays a central role in the computation of the homology of the spaces $\mathrm{Hur}(h, \mathfrak{S}_d)$ introduced in Chapter 4. The precise connection will be established in Chapter 9, whereas in this chapter we focus on the algebraic properties of $\mathcal{V}(d)$. We hope that a systematic study of the algebraic properties of $\mathcal{V}(d)$ may shed some light on the homology of the spaces $\mathrm{Hur}(h, \mathfrak{S}_d)$.

8.1 Definition of $\mathcal{V}(d)$

We will give two alternative definitions of $\mathcal{V}(d)$. The first is a reformulation of Theorem 8.1.3 and uses a certain norm filtration on the *reduced bar complex* $\bar{B}_\bullet\mathfrak{S}_d$ of $\mathfrak{S}_d$. The second exhibits $\mathcal{V}(d)$ as the quadratic dual of the quadratic algebra $\mathrm{grad}^N\mathbb{Z}[\mathfrak{S}_d]$, the associated graded of the group ring $\mathbb{Z}[\mathfrak{S}_d]$ filtered by norm: as we will see, the latter is actually a Koszul duality.

8.1.1 Norm filtration on $\bar{B}_\bullet\mathfrak{S}_d$

We consider the *normalised bar complex* $\bar{B}_\bullet\mathfrak{S}_d$ of the symmetric group $\mathfrak{S}_d$: it is a chain complex generated by *simplices* $(\sigma_1, \dots, \sigma_q)$, where the σ_i's are permutations in $\mathfrak{S}_d \setminus \{\mathbb{1}\}$. The boundary operator on $\bar{B}_\bullet\mathfrak{S}_d$ takes the usual form

$$\partial(\sigma_1, \dots, \sigma_q) = (\sigma_2, \dots, \sigma_q) + \sum_{i=1}^{q-1} (-1)^i (\sigma_1, \dots, \sigma_i \cdot \sigma_{i+1}, \dots, \sigma_q) + (-1)^q (\sigma_1, \dots, \sigma_{q-1}),$$

where all terms in which a permutation $\mathbb{1}$ appears are omitted.

Definition 8.1.1. The norm of a q-simplex $(\sigma_1, \dots, \sigma_q) \in \bar{B}_\bullet\mathfrak{S}_d$ is

$$N(\sigma_1, \dots, \sigma_q) = N(\sigma_1) + \dots + N(\sigma_q),$$

where the norm of a permutation was introduced in Definition 2.2.2. Note that the norm is weakly decreasing along boundaries in $\bar{B}_\bullet\mathfrak{S}_d$, since for all $\sigma, \sigma' \in \mathfrak{S}_d$ we have $N(\sigma\sigma') \leq N(\sigma)N(\sigma')$. This gives a filtration F^N on $\bar{B}_\bullet\mathfrak{S}_d$ by sub-chain complexes, called *norm filtration*.
We denote by $F^N_p = F^N_p\bar{B}_\bullet\mathfrak{S}_d$ the $p-$th filtration level, generated by all simplices of norm $\leq p$.
We denote by $\mathfrak{F}^N_p = \mathfrak{F}^N_p\bar{B}_\bullet\mathfrak{S}_d = F^N_p/F^N_{p-1}\bar{B}_\bullet\mathfrak{S}_d$ the p^{th}filtration stratum.

Note that each level F_p^N and each stratum $\mathfrak{F}_p^N$ is a finitely generated chain complex. Indeed $\mathfrak{S}_d$ is a finite group, hence in each degree there are finitely many simplices in the reduced bar complex. Moreover if a q-simplex $(\sigma_1, \dots, \sigma_q)$ has norm $\leq p$, then $q \leq p$ because each σ_i is different from $\mathbb{1} \in \mathfrak{S}_d$, hence $N(\sigma_i) \geq 1$.
In particular simplices of $\mathfrak{F}_p^N$ have the form $(\sigma_1, \dots, \sigma_q)$, with $\sigma_i \neq \mathbb{1}$ and $\sum_{i=1}^q N(\sigma_i) = p$. The boundary operator of $\mathfrak{F}_p^N$ is induced from the boundary operator of $\bar{B}_\bullet \mathfrak{S}_d$:

$$\partial_{\mathfrak{F}_p}(\sigma_1, \dots, \sigma_q) = \sum_{i=1}^{q-1} (-1)^i (\sigma_1, \dots, \sigma_i \cdot \sigma_{i+1}, \dots, \sigma_q),$$

where in the sum we omit all terms lying in the level F_{p-1}^N: in particular the summands $(\sigma_2, \dots, \sigma_q)$ and $(-1)^q(\sigma_1, \dots, \sigma_{q-1})$ are always omitted.

Definition 8.1.2. The *height* of a permutation $\sigma \in \mathfrak{S}_d$, denoted by $ht(\sigma)$, is the greatest index $i \in [d] = \{1, \dots, d\}$ such that $\sigma(i) \neq i$.
A p-tuple $(\sigma_1, \dots, \sigma_p)$ of permutations in $\mathfrak{S}_d$ is *monotone* if $ht(\sigma_1) \geq \dots \geq ht(\sigma_p)$.

The homology of $\mathfrak{F}_p^N$ was computed by Visy [39].

Theorem 8.1.3 (Visy)**.** *The homology of $\mathfrak{F}_p^N \bar{B}_\bullet \mathfrak{S}_d$ with coefficients in $\mathbb{Z}$ is concentrated in degree p; $H_p(\mathfrak{F}_p^N \bar{B}_\bullet \mathfrak{S}_d)$ is a free abelian group with generators in bijection with monotone p-tuples of transpositions* $(\mathbf{t}_1, \dots, \mathbf{t}_p)$.

The rank of $H_p(\mathfrak{F}_p^N)$ can be computed as the coefficient of t^p in the formal power series

$$\prod_{i=2}^{d} \frac{1}{1-(i-1)t} = \prod_{i=2}^{d} (1 + (i-1)t + (i-1)^2 t^2 + \dots).$$

To see this, note that in order to determine a monotone p-tuple $(\mathbf{t}_1, \dots, \mathbf{t}_p)$ one has to choose a splitting $p = p_2 + \dots + p_d$ with $p_i \geq 0$, and then choose for each $2 \leq i \leq d$ a list of p_i transpositions of height i.
We describe now a dual version of Theorem 8.1.3, which will be convenient later.
Let $\left(\mathfrak{F}_p^N\right)^\bullet$ denote the dual of the chain complex $\mathfrak{F}_p^N = \left(\mathfrak{F}_p^N\right)_\bullet$. Note that this cochain complex is trivial in degree strictly greater than p, hence the entire $\left(\mathfrak{F}_p^N\right)^p$ consists of cocycles.
Note also that $\left(\mathfrak{F}_p^N\right)^p$ is free abelian with basis consisting of elements $\underline{\mathbf{t}}^*$ dual to the basis elements $\underline{\mathbf{t}} = (\mathbf{t}_1, \dots, \mathbf{t}_p) \in \bar{B}_p \mathfrak{S}_d$, where all $\mathbf{t}_i$'s are transpositions: a priori each $\mathbf{t}_i$ has norm ≥ 1, but if the simplex $\underline{\mathbf{t}}$ lies exactly in the stratum $\mathfrak{F}_p^N$, then we must have $N(\mathbf{t}_i) = 1$ for all $1 \leq i \leq p$.
Similarly note that $\left(\mathfrak{F}_p^N\right)^{p-1}$ is a free abelian with generators of the form

$$(\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \sigma, \mathbf{t}_{r+1}, \dots, \mathbf{t}_{p-1})^*,$$

where $1 \leq r \leq p-1$, all $\mathbf{t}_i$'s are transpositions and σ is a permutation of norm 2. In particular σ can be either a 3-cycle or a product of two disjoint transpositions.

Consider again the chain complex $\mathfrak{F}_p^N = \left(\mathfrak{F}_p^N\right)_\bullet$ and in particular its boundary operator $\partial\colon \left(\mathfrak{F}_p^N\right)_p \to \left(\mathfrak{F}_p^N\right)_{p-1}$. Given simplices

$$(\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \sigma, \mathbf{t}_{r+1}, \dots, \mathbf{t}_{p-1}) \in \left(\mathfrak{F}_p^N\right)_{p-1}$$

and

$$(\mathbf{t}'_1, \dots, \mathbf{t}'_{r-1}, \mathbf{t}'_r, \mathbf{t}''_r, \mathbf{t}'_{r+1}, \dots, \mathbf{t}'_{p-1}) \in \left(\mathfrak{F}_p^N\right)_p,$$

we want to determine when the first simplex appears in the boundary of the second, and with which coefficient. The only possibility is that the first simplex is the r^{th}face of the second, i.e. that $\mathbf{t}_i = \mathbf{t}'_i$ for all $1 \le i \le p-1$ with $i \ne r$, and $\sigma = \mathbf{t}'_r \cdot \mathbf{t}''_r$. In this case the boundary coefficient is $(-1)^r$.
Dualising, the coboundary of $(\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \sigma, \mathbf{t}_{r+1}, \dots, \mathbf{t}_{p-1})^*$ in $\left(\mathfrak{F}_p^N\right)^\bullet$, is $(-1)^r$ times the sum of all duals $(\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \mathbf{t}'_r, \mathbf{t}''_r, \mathbf{t}_{r+1}, \dots, \mathbf{t}_{p-1})^*$, one for each decomposition $\mathbf{t}'_r \cdot \mathbf{t}''_r$ of σ.
It suffices now to determine which factorisations as a product of two transpositions σ has.

- If σ is a 3-cycle (xyz) for some $x, y, z \in [d]$, then σ can be decomposed in three ways as a product of transpositions $\mathbf{t}'_r \cdot \mathbf{t}''_r$, namely $\sigma = (xy) \cdot (yz) = (yz) \cdot (zx) = (zx) \cdot (xy)$. Therefore $(\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \sigma, \mathbf{t}_{r+1}, \dots, \mathbf{t}_{p-1})$ appears in the boundary of exactly three simplices in $\left(\mathfrak{F}_p^N\right)_p$, always with coefficient $(-1)^r$.

- If σ is a product of two transpositions $(xy)(zw)$ for some $x, y, z, w \in [d]$, then σ can be decomposed in two ways as an *ordered* product of transpositions $\mathbf{t}'_r \cdot \mathbf{t}''_r$, namely $\sigma = (xy) \cdot (zw) = (zw) \cdot (xy)$. Therefore $(\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \sigma, \mathbf{t}_{r+1}, \dots, \mathbf{t}_{p-1})$ appears in the boundary of exactly two simplices in $\left(\mathfrak{F}_p^N\right)_p$, always with coefficient $(-1)^r$.

We obtain a formula for the coboundary operator $\delta\colon \left(\mathfrak{F}_p^N\right)^{p-1} \to \left(\mathfrak{F}_p^N\right)^p$:

$$\delta(\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \sigma, \mathbf{t}_{r+1}, \dots, \mathbf{t}_{p-1})^* = (-1)^r \sum_{\mathbf{t}_i \cdot \mathbf{t}'_i = \sigma} (\mathbf{t}_1, \dots, \mathbf{t}_{r-1}, \mathbf{t}_r, \mathbf{t}'_r, \mathbf{t}_{r+1}, \dots, \mathbf{t}_{p-1})^*,$$

where the sum contains either two or three summands.

Definition 8.1.4. The algebra $\mathcal{V}(d)$ is the free associative algebra over $\mathbb{Z}$ with the following generators and relations:

Generators	For all $x < y \in [d]$ there is a generator $[xy] = [yx]$.
Relations	$[xy][yz] + [yz][zx] + [zx][xy] = 0$ for all distinct elements $x, y, z \in [d]$.
	$[xy][zw] + [zw][xy] = 0$ for all distinct elements $x, y, z, w \in [d]$.

For a transposition $\mathbf{t} = (xy) \in \mathfrak{S}_d$ we denote by $[\mathbf{t}] = [xy]$ the corresponding generator of $\mathcal{V}(d)$.
We denote by $\mathbb{V} = \mathbb{V}_d$ the free abelian group generated by the elements $[xy]$ for $1 \le x < y \le d$, and we regard $\mathcal{V}(d)$ as a quotient of the tensor algebra $T_\bullet(\mathbb{V})$; the quotient map is denoted by $\pi\colon T_\bullet(\mathbb{V}) \to \mathcal{V}(d)$.

By definition $\mathcal{V}(d)$ is a quadratic algebra: it is the quotient of $T_\bullet(\mathbb{V})$ by a two-sided ideal generated by some elements in $T_2(\mathbb{V}) = \mathbb{V}^{\otimes 2}$. In particular $\mathcal{V}(d)$ inherits a grading from $T_\bullet(\mathbb{V})$, such that every generator $[xy] \in \mathcal{V}(d)$ has degree 1. The previous discussion proves the following reformulation of Visy's theorem:

Theorem 8.1.5. *The cohomology of $\left(\mathfrak{F}_p^N\right)^\bullet$ with coefficients in $\mathbb{Z}$ is concentrated in degree p; $H^p\left(\left(\mathfrak{F}_p^N\right)^\bullet\right)$ is isomorphic to $\mathcal{V}(d)_p$, the degree p summand of $\mathcal{V}(d)$.*

Proof. Note that $\mathcal{V}(d)_p$ is the quotient of $\mathbb{V}^{\otimes p}$ by the degree-p part of the two-sided ideal generated by the relations in Definition 8.1.4. Similarly $H_p\left(\left(\mathfrak{F}_p^N\right)^\bullet\right)$ is the quotient of $\left(\mathfrak{F}_p^N\right)^p \cong \mathbb{V}^{\otimes p}$ by the coboundaries of the elements in $\left(\mathfrak{F}_p^N\right)^{p-1}$. Using the formula given before Definition 8.1.4, it is evident that the image of the coboundary map $\delta\colon \left(\mathfrak{F}_p^N\right)^{p-1} \to \left(\mathfrak{F}_p^N\right)^p$ is precisely the intersection between the aforementioned two-sided ideal and $\mathbb{V}^{\otimes p}$. □

Together with Visy's original statement, we have proved that $\mathcal{V}(d)_p$ is a free abelian group with basis given by monomials of the form $[\mathbf{t}_1] \cdot \dots \cdot [\mathbf{t}_p]$, where $(\mathbf{t}_1, \dots, \mathbf{t}_p)$ is a monotone p-tuple of transpositions: such monomials are also said to be *monotone*.

Definition 8.1.6. The *standard representation* of $a \in \mathcal{V}(d)_p$ is the representation of a as a linear combination of monotone monomials of degree p.

8.1.2 Yetter-Drinfeld modules and bialgebras

The algebra $\mathcal{V}(d)$ carries a richer structure that we describe in this subsection.

Definition 8.1.7. Let G be a finite group. A G-Yetter Drinfeld module M is an abelian group M endowed with a left action of G and a decomposition

$$M = \bigoplus_{\gamma \in G} \bigoplus_{h \geq 0} M_h^\gamma,$$

such that for all $\sigma \in G$ we have $\sigma \cdot M_h^\gamma = M_h^{\sigma\gamma\sigma^{-1}}$. Hence M has a grading and a G-grading, and the action of G conjugates the G-grading.
A map of G-Yetter-Drinfeld modules $f\colon M \to M'$ is a G-equivariant map respecting the grading and the G-grading. Thus G-Yetter-Drinfeld modules form a category, denoted by $\mathcal{YD}(G)$.
The tensor product of two G-Yetter-Drinfeld modules M and M' is the abelian group $M \otimes M'$ endowed with the diagonal G-action and with grading and G-grading given by the following formula

$$\left(M \otimes M'\right)_h^\gamma = \bigoplus_{i=0}^{h} \bigoplus_{\sigma \in G} M_i^\sigma \otimes M_{h-i}^{\sigma^{-1}\gamma}.$$

The tensor product makes $\mathcal{YD}(G)$ into a monoidal category; the neutral element for the tensor product is $\mathbb{Z}$, concentrated in grading 0 and G-grading $1 \in G$.

For $M, M' \in \mathcal{YD}(G)$ define a braiding $\mathfrak{b} = \mathfrak{b}_{M,M'} \colon M \otimes M' \to M' \otimes M$ as follows: let $m \in M$ have grading h and G-grading γ, and let $m' \in M'$ have grading h'; then

$$\mathfrak{b}_{M,M'}(m \otimes m') = (-1)^{h \cdot h'} (\gamma \cdot m') \otimes m.$$

This makes $\mathcal{YD}(G)$ into a braided monoidal category.

We will now show that $\mathcal{V}(d)$ is an algebra (monoid object) in the category $\mathcal{YD}(\mathfrak{S}_d)$. Generators $[xy]$ of $\mathcal{V}(d)$ are in bijection with transpositions $(xy) \in \mathfrak{S}_d$, and $\mathfrak{S}_d$ acts on the set of its transpositions by conjugation. We let $[xy] \in \mathcal{V}(d)_1^{(xy)}$, and extend the grading and the $\mathfrak{S}_d$-grading multiplicatively. More precisely, we first extend the $\mathfrak{S}_d$-grading multiplicatively on $T_\bullet(\mathbb{V})$, and then note that the relations in Definition 8.1.4 are homogeneous also with respect to the $\mathfrak{S}_d$-grading, so that there is an induced $\mathfrak{S}_d$-grading on $\mathcal{V}(d)$.
The action of $\mathfrak{S}_d$ on $\mathcal{V}(d)$ is defined on generators by permuting them according to the action by conjugation on transpositions, and extending multiplicatively; again note that the relations in Definition 8.1.4 are invariant under conjugation.

Definition 8.1.8. Let A and B be two algebras in $\mathcal{YD}(G)$; then the tensor product $A \otimes B$ carries also a structure of algebra in $\mathcal{YD}(G)$, where the product

$$\mu_{A\otimes B} \colon (A \otimes B) \otimes (A \otimes B) \to A \otimes B$$

is the concatenation

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{Id}_A \otimes \mathfrak{b}_{B,A} \otimes \mathrm{Id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\quad \mu_A \otimes \mu_B \quad} A \otimes B.$$

Definition 8.1.9. A $\mathfrak{S}_d$-Yetter-Drinfeld bialgebra is a bialgebra object in $\mathcal{YD}(G)$. Concretely, it is an object $A \in \mathcal{YD}(G)$ with the following structure maps:

- a multiplcation $\mu \colon A \otimes A \to A$;
- a unit $\eta \colon \mathbb{Z} \to A$;
- a comultiplication $\Delta \colon A \to A \otimes A$;
- a counit (or augmentation) $\varepsilon \colon A \to \mathbb{Z}$,

such that (A, μ, η) is an algebra in $\mathcal{YD}(G)$, (A, Δ, ε) is a coalgebra (comonoid object) in $\mathcal{YD}(G)$, and the maps Δ and ε are maps of *algebras*, where $A \otimes A$ is given the algebra structure from Definition 8.1.8. Here $\mathbb{Z}$ is the trivial $\mathfrak{S}_d$-Yetter-Drinfeld algebra concentrated in grading 0 and $\mathfrak{S}_d$-grading $\mathbb{1}$.

We can now enhance the $\mathfrak{S}_d$-Yetter-Drinfeld algebra structure on $\mathcal{V}(d)$ to a $\mathfrak{S}_d$-Yetter-Drinfeld bialgebra structure.
First define $\tilde{\Delta} \colon T_\bullet(\mathbb{V}) \to T_\bullet(\mathbb{V})^{\otimes 2}$ by setting, for all transpositions (xy),

$$\tilde{\Delta}([xy]) = [xy] \otimes 1 + 1 \otimes [xy],$$

and extending multiplicatively; similarly define $\tilde{\varepsilon}\colon T_\bullet(\mathbb{V}) \to \mathbb{Z}$ by setting $\tilde{\varepsilon}([xy]) = 0$ and extending multiplicatively. This defines a $\mathfrak{S}_d$-Yetter-Drinfeld bialgebra structure on the tensor algebra $T_\bullet(\mathbb{V})$: the coproduct $\tilde{\Delta}$ is coassociative on generators $[xy]$, because they are primitive, and one can retrieve coassociativity on the entire $T_\bullet(\mathbb{V})$ multiplicatively, using the fact that $\tilde{\Delta}$ is a map of algebras; the other conditions are straightforward.
To induce a $\mathfrak{S}_d$-Yetter-Drinfeld bialgebra on $\mathcal{V}(d)$ it suffices to prove that the kernel of π is contained in the kernels of the following two compositions:

$$T_\bullet(\mathbb{V}) \xrightarrow{\tilde{\Delta}} T_\bullet(\mathbb{V})^{\otimes 2} \xrightarrow{\pi\otimes\pi} \mathcal{V}(d)^{\otimes 2};$$

$$T_\bullet(\mathbb{V}) \xrightarrow{\tilde{\varepsilon}} \mathbb{Z}.$$

Note that both $(\pi \otimes \pi) \circ \tilde{\Delta}$ and $\tilde{\varepsilon}$ are morphisms of algebras, so it suffices to check that their kernels contain the generators of the kernel of π, i.e. all elements in $T_2(\mathbb{V})$ of the forms

- $[xy][yz] + [yz][zx] + [zx][xy]$ for all distinct elements $x, y, z \in [d]$;
- $[xy][zw] + [zw][xy]$ for all distinct elements $x, y, z, w \in [d]$.

This is immediate for the kernel of $\tilde{\varepsilon}$, which contains all elements of $T_\bullet(\mathbb{V})$ of positive degree; note also that

$$\begin{aligned}
&\tilde{\Delta}([xy][yz] + [yz][zx] + [zx][xy]) = \textstyle\sum_{cyc} \tilde{\Delta}[xy]\tilde{\Delta}[yz] = \\
&= \textstyle\sum_{cyc}([xy][yz]) \otimes 1 + \sum_{cyc}[xy] \otimes [yz] + \\
&- \textstyle\sum_{cyc}[yz] \otimes ((yz) \cdot [xy]) + \sum_{cyc} 1 \otimes ([xy][yz]) = \\
&= \textstyle\sum_{cyc}([xy][yz]) \otimes 1 + \sum_{cyc}[xy] \otimes [yz] - \sum_{cyc}[yz] \otimes [zx] + \sum_{cyc} 1 \otimes ([xy][yz]) = \\
&= \left(\textstyle\sum_{cyc}[xy][yz]\right) \otimes 1 + 1 \otimes \left(\textstyle\sum_{cyc}[xy][yz]\right) \in \ker(\pi \otimes \pi),
\end{aligned}$$

where we have used the equality $\mathfrak{b}([xy] \otimes [yz]) = -[yz] \otimes ((yz) \cdot [xy]) = -[yz] \otimes [zx]$, because both $[xy]$ and $[yz]$ have degree 1 and $(yz)(xy)(yz)^{-1} = (zx) \in \mathfrak{S}_d$. Similarly

$$\begin{aligned}
&\tilde{\Delta}([xy][zw] + [zw][xy]) = \tilde{\Delta}[xy]\tilde{\Delta}[zw] + \tilde{\Delta}[zw]\tilde{\Delta}[xy] = \\
&= ([xy][zw] + [zw][xy]) \otimes 1 + ([xy] \otimes [zw] + [zw] \otimes [xy]) + \\
&- \left([zw] \otimes \left((zw)^{-1} \cdot [xy]\right) + [xy] \otimes ((xy) \cdot [zw])\right) + 1 \otimes ([xy][zw] + [zw][xy]) = \\
&= ([xy][zw] + [zw][xy]) \otimes 1 + ([xy] \otimes [zw] + [zw] \otimes [xy]) + \\
&- ([zw] \otimes [xy] + [xy] \otimes [zw]) + 1 \otimes ([xy][zw] + [zw][xy]) = \\
&= ([xy][zw] + [zw][xy]) \otimes 1 + 1 \otimes ([xy][zw] + [zw][xy]) \in \ker(\pi \otimes \pi),
\end{aligned}$$

where we used that $[xy]$ and $[zw]$ have degree 1 and that the transpositions (xy) and (zw) commute. Thus $\mathcal{V}(d)$ becomes a $\mathfrak{S}_d$-Yetter-Drinfeld bialgebra.

8.1.3 Koszul duality

In this subsection we prove that $\mathcal{V}(d)$ is a Koszul quadratic algebra and we give a new, short proof of Theorem 8.1.5 which relies on the general theory of Koszul algebras; our main reference is [34].

Let $\mathbb{Z}[\mathfrak{S}_d]$ be the group ring of $\mathfrak{S}_d$. We define a filtration $F^N_\bullet\mathbb{Z}[\mathfrak{S}_d]$: for $k \geq 0$ let $F^N_k = F^N_k\mathbb{Z}[\mathfrak{S}_d]$ be the sub-abelian group generated by permutations $\sigma \in \mathfrak{S}_d$ with $N(\sigma) \leq k$. Note that $\mathbb{Z}[\mathfrak{S}_d]$ becomes a filtered ring, i.e. for all $k, k' \geq 0$ there is an inclusion $F^N_k \cdot F^N_{k'} \subset F^N_{k+k'}$. Note also that $F_{d-1} = \mathbb{Z}[\mathfrak{S}_d]$.

Set $F^N_{-1} = \{0\}$. For $k \geq 0$ let

$$\mathfrak{F}^N_k = \mathfrak{F}^N_k\mathbb{Z}[\mathfrak{S}_d] = F^N_k\mathbb{Z}[\mathfrak{S}_d]/F^N_{k-1}\mathbb{Z}[\mathfrak{S}_d]$$

be the k^{th}filtration stratum, and let

$$\check{\mathcal{V}} := \mathrm{grad}^N\mathbb{Z}[\mathfrak{S}_d] = \bigoplus_{k=0}^{\infty}\mathfrak{F}^N_k\mathbb{Z}[\mathfrak{S}_d] = \bigoplus_{k=0}^{d-1}\mathfrak{F}^N_k\mathbb{Z}[\mathfrak{S}_d]$$

be the associated graded. Note that $\check{\mathcal{V}}(d)$ inherits from the filtered ring $\mathbb{Z}[\mathfrak{S}_d]$ a structure of *graded* ring.

One can present $\check{\mathcal{V}}(d)$ as the free associative $\mathbb{Z}$-algebra with the following generators and relations (compare with Definition 8.1.4):

Generators For all $x < y \in [d]$ there is a generator $[\![xy]\!] = [\![yx]\!]$.

Relations $[\![xy]\!]^2 = 0$ for all distinct elements $x, y \in [d]$.
$[\![xy]\!][\![yz]\!] = [\![yz]\!][\![zx]\!] = [\![zx]\!][\![xy]\!]$ for all distinct elements $x, y, z \in [d]$.
$[\![xy]\!][\![zw]\!] = [\![zw]\!][\![xy]\!]$ for all distinct elements $x, y, z, w \in [d]$.

Recall Definition 8.1.4, and note that $\mathcal{V}(d)$ and $\check{\mathcal{V}}(d)$ are dual quadratic algebras. Both algebras are quadratic, i.e. generated in degree 1 with relations generated in degree 2. There is a duality (of finitely generated, free $\mathbb{Z}$-modules) $\left(\check{\mathcal{V}}(d)\right)_1 \simeq (\mathcal{V}(d)_1)^* = \mathbb{V}^*$ given by letting the basis of generators $[\![xy]\!]$ be dual to the basis of generators $[xy]$. The quadratic relations of $\check{\mathcal{V}}(d)$ are then precisely the orthogonal complement of the quadratic relations of $\mathcal{V}(d)$. Here we note that, although we are working over $\mathbb{Z}$ and not over a field, the relations of $\mathcal{V}(d)$ and $\check{\mathcal{V}}(d)$, considered as submodules of the free abelian groups $\mathbb{V}^{\otimes 2}$ and $(\mathbb{V}^*)^{\otimes 2}$ respectively, admit a complementary free abelian group.

Note that $\check{\mathcal{V}}(d)$ is a graded, quadratic algebra over $\mathbb{Z}$, endowed with an augmentation $\check{\mathcal{V}}(d) \to \mathbb{Z}$ given by projection on the 0-component.

Note also that the normalised bar complex $\bar{B}(\mathbb{Z}, \check{\mathcal{V}}(d), \mathbb{Z})$ is isomorphic to the direct sum $\bigoplus_{p\geq 0}\mathfrak{F}^N_p\bar{B}_\bullet\mathfrak{S}_d$: this is an isomorphism of *graded* chain complexes, where the grading is in both cases induced by the norm $N\colon \mathfrak{S}_d \to \mathbb{Z}_{\geq 0}$.

Therefore we obtain a bigraded isomorphism

$$\mathrm{Ext}^*_{\check{\mathcal{V}}(d)}(\mathbb{Z}, \mathbb{Z}) \simeq \bigoplus_{p\geq 0} H^*(\mathfrak{F}^N_p\bar{B}_\bullet\mathfrak{S}_d),$$

and Theorem 8.1.5 can be rephrased to say that $\mathrm{Ext}^*_{\check{\mathcal{V}}(d)}(\mathbb{Z},\mathbb{Z})$ is concentrated *on the diagonal*, i.e. in bigradings for which the cohomological grading is equal to the norm grading. This is one of the equivalent definitions of being Koszul for the algebra $\check{\mathcal{V}}(d)$.
An equivalent reformulation of Theorem 8.1.5 is therefore that $\check{\mathcal{V}}(d)$ is a Koszul quadratic algebra; its dual quadratic algebra, which is

$$\mathcal{V}(d) \cong \mathrm{Ext}^*_{\check{\mathcal{V}}(d)}(\mathbb{Z},\mathbb{Z}),$$

is then also Koszul. In the following we prove, by a simple argument, that $\check{\mathcal{V}}(d)$ is Koszul: this gives a new proof of Theorem 8.1.5, and therefore of Theorem 8.1.3.

Alternative proof of Theorem 8.1.5. We will prove that $\check{\mathcal{V}}(d)$ admits a Poincaré-Birkhoff-Witt (PBW) basis: a result by Priddy [35] then ensures that $\check{\mathcal{V}}(d)$ is Koszul; see also Theorem 3.1 in [34, Chapter 4].
We give a total order on the generators of $\check{\mathcal{V}}(d)$: let $[\![xy]\!]$ and $[\![x'y']\!]$ be two different generators, with $x < y$ and $x' < y'$; then $[\![xy]\!] \prec [\![x'y']\!]$ if and only if $y < y'$, or $y = y'$ and $x < x'$. The set $\{([\![xy]\!], [\![x'y']\!])\}$ of couples of generators, is given the lexicographic order, also denoted by $\prec$.
For a transposition $\mathbf{t} = (xy) \in \mathfrak{S}_d$ we denote by $[\![xy]\!]$ the corresponding generator of $\check{\mathcal{V}}(d)$.
More generally, for $\sigma \in \mathfrak{S}_d$ we denote by $[\![\sigma]\!] \in \mathfrak{F}^N_p\mathbb{Z}[\mathfrak{S}_d]$ the image of $\sigma \in F^N_p\mathbb{Z}[\mathfrak{S}_d]$, where $p = N(\sigma)$. The elements $[\![\sigma]\!]$ for varying $\sigma \in \mathfrak{S}_d$ form a basis for $\check{\mathcal{V}}(d)$.
Define $S \subset \{([\![xy]\!], [\![x'y']\!])\}$ as the subset containing couples $([\![\mathbf{t}]\!], [\![\mathbf{t}']\!])$ such that the product $[\![\mathbf{t}]\!][\![\mathbf{t}']\!] \in \check{\mathcal{V}}(d)$ cannot be expressed as a linear combination $\sum_{i=1}^k \lambda_1 [\![\mathbf{t}_i]\!][\![\mathbf{t}'_i]\!]$, with $([\![\mathbf{t}_i]\!], [\![\mathbf{t}'_i]\!]) \prec ([\![\mathbf{t}]\!], [\![\mathbf{t}']\!])$ for all i. A PBW-monomial in the generators $[\![xy]\!]$ is then a monomial $[\![\mathbf{t}_1]\!] \cdot \ldots \cdot [\![\mathbf{t}_p]\!]$ with $([\![\mathbf{t}_i]\!], [\![\mathbf{t}_{i+1}]\!]) \in S$ for all $1 \le i \le p-1$, and our aim is to prove that PBW-monomials form a basis of $\check{\mathcal{V}}(d)$, called PBW-basis.
By the relations in the presentation of $\check{\mathcal{V}}(d)$ it is straightforward to see that $([\![\mathbf{t}]\!], [\![\mathbf{t}']\!]) \in S$ if and only if $ht(\mathbf{t}) < ht(\mathbf{t}')$.
Moreover note that every permutation $\sigma \in \mathfrak{S}_d$ can be written in a unique way as a product of transpositions $\mathbf{t}_1 \cdot \ldots \mathbf{t}_p$, with $p = N(\sigma)$ and $ht(\mathbf{t}_1) < \cdots < ht(\mathbf{t}_p)$; viceversa every product $\mathbf{t}_1 \cdot \ldots \mathbf{t}_p$, with $ht(\mathbf{t}_1) < \cdots < ht(\mathbf{t}_p)$ gives a permutation $\sigma \in \mathfrak{S}_d$ of norm p.
We see therefore that PBW-monomials form precisely the standard basis of elements $[\![\sigma]\!]$ of $\check{\mathcal{V}}(d)$. □

For the sake of completeness, we note that $\check{\mathcal{V}}(d)$ has a natural structure of algebra in $\mathcal{YD}(\mathfrak{S}_d)$; in particular $\mathrm{Ext}^*_{\check{\mathcal{V}}(d)}(\mathbb{Z},\mathbb{Z})$ inherits a $\mathfrak{S}_d$-grading and an action of $\mathfrak{S}_d$, and it becomes automatically an algebra in $\mathcal{YD}(\mathfrak{S}_d)$. It is no surprise that the above is an isomorphism of of algebras in $\mathcal{YD}(\mathfrak{S}_d)$ (see Subsection 8.1.2).

8.2 Algebraic properties of $\mathcal{V}(d)$

In this section we investigate some properties of the algebra $\mathcal{V}(d)$.

8.2.1 Zero-divisors

The aim of this subsection is to prove the following theorem.

Theorem 8.2.1. *The algebra $\mathcal{V}(d)$ has no zero-divisors.*

Definition 8.2.2. Let $\underline{\mathbf{t}} = (\mathbf{t}_1, \dots, \mathbf{t}_p)$ be a p-tuple of transpositions. We associate to $\mathbf{t}$ the monomial $m(\underline{\mathbf{t}}) = [\mathbf{t}_1] \dots [\mathbf{t}_p] \in \mathcal{V}(d)_p$.
For $2 \leq r \leq d$, we define the r^{th}weight $\varepsilon_r(\underline{\mathbf{t}})$ of $\underline{\mathbf{t}}$ as the number of transpositions $\mathbf{t}_i$ such that $ht(\mathbf{t}_i) = r$. The multiweight $\varepsilon(\underline{\mathbf{t}})$ is the vector $(\varepsilon_2(\underline{\mathbf{t}}), \dots, \varepsilon_d(\underline{\mathbf{t}}))$.
Multiweights of different p-tuples are compared lexicographically: we write

$$(\varepsilon_2, \dots, \varepsilon_d) \prec (\varepsilon'_2, \dots, \varepsilon'_d)$$

if, for the smallest index $r \geq 2$ satisfying $\varepsilon_r \neq \varepsilon'_r$, the inequality $\varepsilon_r < \varepsilon'_r$ holds.
For an element $0 \neq a \in \mathcal{V}(d)_p$, the multiweight is defined as follows. Let $a = \sum_{i=1}^k \lambda_i \alpha_i$ be the standard representation of a (see Definition 8.1.6; then the multiweight $\varepsilon(a)$ of a is the maximum among the multiweights associated with the monotone p-tuples corresponding to the monomials α_i.
We also decompose $0 \neq a \in \mathcal{V}(d)_p$ as a sum $\mathcal{P}(a) + (a - \mathcal{P}(a))$, where $\mathcal{P}(a)$, called the *principal part* of a, is generated by monotone monomials of multiweight precisely $\varepsilon(a)$, whereas $(a - \mathcal{P}(a))$ is zero or has stricly lower multiweight.

In the following lemma we compute multiweights for all monomials.

Lemma 8.2.3. *Let $m(\underline{\mathbf{t}}) = [\mathbf{t}_1] \cdot \ldots \cdot [\mathbf{t}_p] \in \mathcal{V}(d)_p$ be the monomial associated with the p-tuple $\underline{\mathbf{t}} = (\mathbf{t}_1, \dots, \mathbf{t}_p)$. Then $\varepsilon(m(\underline{\mathbf{t}})) = \varepsilon(\underline{\mathbf{t}})$.*

Proof. For a p-tuple $\underline{\mathbf{t}}$, denote by $\iota(\underline{\mathbf{t}})$ the number of couples of indices $1 \leq i < j \leq p$ such that $ht(\mathbf{t}_i) < ht(\mathbf{t}_j)$. Clearly $\iota(\underline{\mathbf{t}}) = 0$ if and only if $\underline{\mathbf{t}}$ is monotone. We will prove the statement for a generic p-tuple $\underline{\mathbf{t}}$ by lexicographic induction on the couple $(\varepsilon(\underline{\mathbf{t}}), \iota(\underline{\mathbf{t}}))$.
As basis for the induction we take the case in which $\underline{\mathbf{t}}$ is monotone. In this case the standard representation of $m(\underline{\mathbf{t}})$ is $m(\underline{\mathbf{t}})$ itself, and by definition $\varepsilon(m(\underline{\mathbf{t}})) = \varepsilon(\underline{\mathbf{t}})$.
Suppose now that $\underline{\mathbf{t}}$ is not monotone, and let $1 \leq i \leq p-1$ be some index such that $ht(\mathbf{t}_i) < ht(\mathbf{t}_{i+1})$. Denote $\mathbf{t}_i = (xy)$ with $x < y = ht(\mathbf{t}_i)$. There are two cases.

- $\mathbf{t}_i$ and $\mathbf{t}_{i+1}$ commute. Define the p-tuple $\underline{\mathbf{t}}' = (\mathbf{t}_1, \dots, \mathbf{t}_{i-1}, \mathbf{t}_{i+1}, \mathbf{t}_i, \mathbf{t}_{i+2}, \dots, \mathbf{t}_p)$; then $m(\underline{\mathbf{t}}) = -m(\underline{\mathbf{t}}') \in \mathcal{V}(d)_p$, and therefore also the standard representations of $m(\underline{\mathbf{t}})$ and $m(\underline{\mathbf{t}}')$ are equal up to a sign; in particular $\varepsilon(m(\underline{\mathbf{t}})) = \varepsilon(m(\underline{\mathbf{t}}'))$. Moreover $\varepsilon(\underline{\mathbf{t}}') = \varepsilon(\underline{\mathbf{t}})$ and $\iota(\underline{\mathbf{t}}') < \iota(\underline{\mathbf{t}})$. We apply the inductive hypothesis to $\underline{\mathbf{t}}'$.

- $\mathbf{t}_i$ and $\mathbf{t}_{i+1}$ are braided. Let $z = ht(\mathbf{t}_{i+1}) > y > x$; there are two subcases:
  - if $\mathbf{t}_{i+1} = (yz)$ define the p-tuples $\underline{\mathbf{t}}' = (\mathbf{t}_1, \dots, \mathbf{t}_{i-1}, (yz), (zx), \mathbf{t}_{i+2}, \dots, \mathbf{t}_p)$ and $\underline{\mathbf{t}}'' = (\mathbf{t}_1, \dots, \mathbf{t}_{i-1}, (zx), (xy), \mathbf{t}_{i+2}, \dots, \mathbf{t}_p)$;
  - if $\mathbf{t}_{i+1} = (xz)$ define the p-tuples $\underline{\mathbf{t}}' = (\mathbf{t}_1, \dots, \mathbf{t}_{i-1}, (xz), (yz), \mathbf{t}_{i+2}, \dots, \mathbf{t}_p)$ and $\underline{\mathbf{t}}'' = (\mathbf{t}_1, \dots, \mathbf{t}_{i-1}, (yz), (xy), \mathbf{t}_{i+2}, \dots, \mathbf{t}_p)$.

The following argument holds for both cases. First, note that $m(\underline{\mathbf{t}}) = -m(\underline{\mathbf{t}}') - m(\underline{\mathbf{t}}'') \in \mathcal{V}(d)$. Note also that $\varepsilon(\underline{\mathbf{t}}') \prec \varepsilon(\underline{\mathbf{t}})$: indeed the equality $\varepsilon_y(\underline{\mathbf{t}}') = \varepsilon_y(\underline{\mathbf{t}}) - 1$ holds; moreover we have $\varepsilon(\underline{\mathbf{t}}'') = \varepsilon(\underline{\mathbf{t}})$ and $\iota(\underline{\mathbf{t}}'') < \iota(\underline{\mathbf{t}})$. We apply the inductive hypothesis to $\underline{\mathbf{t}}'$ and $\underline{\mathbf{t}}''$, obtaining $\varepsilon(m(\underline{\mathbf{t}}')) = \varepsilon(\underline{\mathbf{t}}')) \prec \varepsilon(\underline{\mathbf{t}})$ and $\varepsilon(m(\underline{\mathbf{t}}'')) = \varepsilon(\underline{\mathbf{t}}'') = \varepsilon(\underline{\mathbf{t}})$. Note that the standard representation of $m(\underline{\mathbf{t}})$ is minus the sum of the standard representations of $m(\underline{\mathbf{t}}')$ and $m(\underline{\mathbf{t}}'')$, and the second contains some monotone monomials of multiweight $\varepsilon(\underline{\mathbf{t}})$, whereas the first only contains monotone monomials of multiweight strictly $\prec$ $\varepsilon(\underline{\mathbf{t}})$. This shows that $\varepsilon(m(\underline{\mathbf{t}})) = \varepsilon(\underline{\mathbf{t}})$.

□

Lemma 8.2.4. *For a p-tuple $\underline{\mathbf{t}} = (\mathbf{t}_1, \ldots, \mathbf{t}_p)$, define $ht(\underline{\mathbf{t}}) = (ht(\mathbf{t}_1), \ldots, ht(\mathbf{t}_p))$, which is a sequence in $\{2, \ldots, d\}^p$. Note that both functions $\varepsilon(\underline{\mathbf{t}})$ and $\iota(\underline{\mathbf{t}})$ from the proof of Lemma 8.2.3 can be expressed as functions of $ht(\underline{\mathbf{t}})$.*
Fix a sequence $\underline{\mathfrak{h}} \in \{2, \ldots, d\}^p$ and let S be the set of p-tuples $\underline{\mathbf{t}}$ satisfying $ht(\underline{\mathbf{t}}) = \underline{\mathfrak{h}}$. Then the monomials associated with all p-tuples in S are linearly independent in $\mathcal{V}(d)_p$, even modulo all monomials of strictly lower multiweight. More precisely, if

$$a = \sum_{\underline{\mathbf{t}} \in S} \lambda_{\underline{\mathbf{t}}} m(\underline{\mathbf{t}})$$

and if at least one coefficient $\lambda_{\underline{\mathbf{t}}} \neq 0$, then $a \neq 0$ and $\varepsilon(a) = \varepsilon(\underline{\mathfrak{h}})$.

Proof. We note that the functions $\varepsilon(\underline{\mathbf{t}})$ and $\iota(\underline{\mathbf{t}})$ are equal to $\varepsilon(\underline{\mathfrak{h}})$ and $\iota(\underline{\mathfrak{h}})$ respectively, for all $\underline{\mathbf{t}} \in S$; we denote briefly their values by $\varepsilon = \varepsilon(\underline{\mathfrak{h}})$ and $\iota = \iota(\underline{\mathfrak{h}})$.
If all p-tuples of S are monotone (which means $\iota = 0$), then their corresponding monomials are linearly independent by Theorem 8.1.5; moreover if $a \in \mathcal{V}(d)_p$ is a non-zero linear combination of these monomials, then by definition $\varepsilon(a) = \varepsilon(\underline{\mathfrak{h}})$.
Suppose now $\iota \geq 1$ and let $1 \leq i \leq p-1$ be a position such that $\mathfrak{h}_i < \mathfrak{h}_{i+1}$. Denote by $\underline{\mathfrak{h}}'$ the vector obtained by swapping the values $\mathfrak{h}_i$ and $\mathfrak{h}_{i+1}$ in $\underline{\mathfrak{h}}$, and let S' be the set of p-tuples $\underline{\mathbf{t}}$ satisfying $ht(\underline{\mathbf{t}}) = \underline{\mathfrak{h}}'$. Clearly $\iota(\underline{\mathfrak{h}}') = \iota - 1 < \iota$; we abbreviate $\iota' = \iota(\underline{\mathfrak{h}}')$. Note on the other hand that $\varepsilon(\underline{\mathfrak{h}}') = \varepsilon$.
We establish a bijection between S and S'. A p-tuple $\underline{\mathbf{t}} = (\mathbf{t}_1, \ldots, \mathbf{t}_p)$ is associated with the following p-tuple in S':

$$\underline{\mathbf{t}}' = (\mathbf{t}_1, \ldots, \mathbf{t}_{i-1}, (\mathbf{t}_i \mathbf{t}_{i+1} \mathbf{t}_i^{-1}), \mathbf{t}_i, \mathbf{t}_{i+2}, \ldots, \mathbf{t}_p).$$

Note that conjugating a transposition $\mathbf{t}_{i+1}$ by a transposition $\mathbf{t}_i$ of strictly lower height yields a transposition $\mathbf{t}_i \mathbf{t}_{i+1} \mathbf{t}_i^{-1}$ of the same height as $\mathbf{t}_{i+1}$. Note that by the argument of the proof of Lemma 8.2.3 the equality $m(\underline{\mathbf{t}}) = -m(\underline{\mathbf{t}}')$ holds if $\mathbf{t}_i$ and $\mathbf{t}_{i+1}$ commute, and holds up to an error of stricly lower multiweight if $\mathbf{t}_i$ and $\mathbf{t}_{i+1}$ are braided.
By inductive hypothesis we can assume that the monomials $m(\underline{\mathbf{t}}')$, with $\underline{\mathbf{t}}'$ varying in S', are linearly independent up to errors of multiweight stricly lower than ε; therefore the same statement holds for the monomials $m(\underline{\mathbf{t}})$, with $\underline{\mathbf{t}}$ varying in S. □

We can now prove Theorem 8.2.1

Proof of Theorem 8.2.1. Let $a, b \in \mathcal{V}(d)$ and assume that both a and b are $\neq 0$. Since $\mathcal{V}(d)$ is a $\mathbb{Z}_{\geq 0}$-graded ring, it suffices to prove that $ab \neq 0$ under the additional hypothesis that both a and b are homogeneous of some degrees p and q respectively.
We claim that $\mathcal{P}\left(()\,a\right)\mathcal{P}\left(()\,b\right) \neq 0$; then a straightforward consequence of Lemma 8.2.3 would be that $\mathcal{P}\left(()\,ab\right) = \mathcal{P}\left(()\,\mathcal{P}\left(()\,a\right)\mathcal{P}\left(()\,b\right)\right) \neq 0$, so that $ab \neq 0$ and even $\varepsilon(ab) = \varepsilon(a) + \varepsilon(b)$ (the last equality is beyond the statement of the Lemma).
From now on we assume that both a and b are equal to their principal part. Let the standard representations of a and b be $a = \sum_{i=1}^{k} \lambda_i \alpha_i$ and $b = \sum_{i=1}^{l} \mu_i \beta_i$. Let $\underline{\mathfrak{h}}^a$ be the common height of the monotone p-tuples associated with all α_i's; similarly define $\underline{\mathfrak{h}}^b$ by looking at the β_i's. Denote by $\underline{\mathfrak{h}} \in \{2, \dots, d\}^{p+q}$ the juxtaposition of $\underline{\mathfrak{h}}^a$ and $\underline{\mathfrak{h}}^b$.
Then the product

$$ab = \sum_{i=1}^{k} \sum_{j=1}^{l} (\lambda_i \mu_j) \cdot \alpha_i \beta_j$$

is a non-trivial linear combination of monomials corresponding to different $(p+q)$-tuples of height $\underline{\mathfrak{h}}$. By Lemma 8.2.4 this linear combination does not vanish. □

8.2.2 Center of $\mathcal{V}(d)$

In this subsection we compute the center of the algebra $\mathcal{V}(d)$. For $d = 2$ the algebra $\mathcal{V}(d)$ is just the polynomial algebra $\mathbb{Z}[t]$, where $t = [12]$; in particular $\mathcal{V}(d)$ is commutative: this is a special case, corresponding to the fact that $\mathfrak{S}_2$ is an abelian group. From now on assume for simplicity $d \geq 3$.

Definition 8.2.5. We define an element $\mathbf{z} \in \mathcal{V}(d)$:

$$\mathbf{z} = \sum_{1 \leq x < y \leq d} [xy]^2.$$

Note that $\mathbf{z} \in \mathcal{V}(d)_2^{\mathbb{1}}$, where we use the structure of $\mathfrak{S}_d$-Yetter-Drinfeld module of $\mathcal{V}(d)$ from Subsection 8.1.2. Moreover $\mathbf{z}$ is invariant under the action of $\mathfrak{S}_d$. We will prove the following theorem

Theorem 8.2.6. *The center $\mathcal{Z}(\mathcal{V}(d))$ of the algebra $\mathcal{V}(d)$ is the polynomial ring $\mathbb{Z}[\mathbf{z}]$ generated by the element $\mathbf{z}$.*

The first step in the proof is the following lemma.

Lemma 8.2.7. *The element $\mathbf{z} \in \mathcal{V}(d)$ is central.*

Proof. It suffices to prove that $\mathbf{z}$ commutes with every generator $[xy] \in \mathcal{V}(d)$; as the group $\mathfrak{S}_d$ permutes transitively generators of $\mathcal{V}(d)$ and fixes $\mathbf{z}$, it suffices to prove that $\mathbf{z}$ commutes with $[12]$, and to this purpose we will split $\mathbf{z}$ into convenient summands.
Trivially $[12]^2$ commutes with $[12]$. For all distinct $x, y \geq 3$ we have that $[xy]^2$ commutes with $[12]$:

$$\begin{aligned}[xy]^2[12] &= -[xy][12][xy] \\ &= [12][xy]^2.\end{aligned}$$

For all $x \geq 3$ we have that $[1x]^2 + [2x]^2$ commutes with $[12]$:

$$\begin{aligned}([1x]^2 + [2x]^2)[12] &= [1x][1x][12] + [2x][2x][12] \\ &= -([1x][12][2x] + [1x][2x][1x]) - ([2x][12][1x] - [2x][1x][2x]) \\ &= -([1x][12] + [2x][1x])[2x] - ([1x][2x] + [2x][12])[1x] \\ &= [12][2x][2x] + [12][1x][1x] \\ &= [12]([1x]^2 + [2x]^2).\end{aligned}$$

□

By Theorem 8.2.1 there is an inclusion of rings $\mathbb{Z}[\mathbf{z}] \subseteq \mathcal{Z}(\mathcal{V}(d)) \subset \mathcal{V}(d)$; we still have to prove that $\mathcal{Z}(\mathcal{V}(d)) \subseteq \mathbb{Z}[\mathbf{z}]$.
We note that $\mathcal{Z}(\mathcal{V}(d))$ is a graded subring of $\mathcal{V}(d)$, i.e. if $a \in \mathcal{V}(d)$ is central and $a = \sum_{i=0}^{p} a_i$, with $a_i \in \mathcal{V}(d)_i$, then all $a_i \in \mathcal{Z}(\mathcal{V}(d))$: for each generator $[xy]$ the equality $[xy]a = a[xy]$ read in degree $i+1$ gives the equality $[xy]a_i = a_i[xy]$.

Definition 8.2.8. Recall Definition 8.2.2. For a p-tuple $\underline{\mathbf{t}} = (\mathbf{t}_1, \ldots, \mathbf{t}_p)$ we define the *maximal height* $mht(\underline{\mathbf{t}})$ as

$$mht(\underline{\mathbf{t}}) = \max_{i=1}^{p} mht(\mathbf{t}_i).$$

For all $2 \leq i \leq d$ denote by $\mathcal{V}(d)_{p,mht=i} \subset \mathcal{V}(d)_p$ the submodule generated by monomials $m(\underline{\mathbf{t}})$ associated with p-tuples $\underline{\mathbf{t}}$ of maximal height i.

Note that the relations defining $\mathcal{V}(d)$ only combine monomials in the tensor algebra $T_\bullet(\mathbb{V})$ corresponding to p-tuples of the same maximal height; therefore we have a splitting

$$\mathcal{V}(d)_p = \bigoplus_{i=2}^{d} \mathcal{V}(d)_{p,mht=i};$$

we can therefore write $0 \neq a \in \mathcal{V}(d)$ uniquely as

$$a = a_{mht=2} + \cdots + a_{mht=d},$$

with $a_{mht=y} \in \mathcal{V}(d)_{p,mht=y}$ for all $2 \leq y \leq d$; we call $mht(a)$ the maximal index y such that $a_{mht=y} \neq 0$. Note that if $0 \neq a \in \mathcal{V}(d)_{p,mht=y}$ and $b \in \mathcal{V}(d)_{q,mht=z}$, with $y \leq z$, then $ab \in \mathcal{V}(d)_{p+q}$ is generated by (non-monotone) monomials of length $p+q$ and maximal height z, hence $ab \in \mathcal{V}(d)_{p+q,mht=z}$.

Lemma 8.2.9. *Let $2 \leq y \leq d$ and let $a \in \mathcal{V}(d)_{p,mht=y}$; then there exist unique $b_1, \ldots, b_{y-1} \in \mathcal{V}(d)_{p-1}$ with $mht(b_x) \leq y$ for all $1 \leq x < y$ and*

$$a = \sum_{x=1}^{y-1} [xy] b_x.$$

Proof. Let $a = \sum_{i=1}^{k} \lambda_i m(\underline{\mathbf{t}}_i)$ be the standard representation of a, where, for $1 \leq i \leq k$, $\underline{\mathbf{t}}_i = (\mathbf{t}_{i,1}, \ldots, \mathbf{t}_{i,p})$ is a monotone p-tuple of maximal height y. Then for all $1 \leq i \leq k$ the transposition $\mathbf{t}_{i,1}$ has the form $(x_i y)$, for some $1 \leq x_i < y$.
For all $1 \leq x < y-1$ we put together all summands in the standard representation of a corresponding to monotone p-tuples starting with $[xy]$: the partial sum that we obtain has clearly the form $[xy]b_x$, for some $b_x \in \mathcal{V}(d)_{p-1}$ having height $\leq y$. This shows the existence of $b_1, \ldots, b_{y-1}$.
Conversely, let $b_1, \ldots, b_{y-1} \in \mathcal{V}(d)_{p-1}$ of height $\leq y$ with $\sum_{x=1}^{y-1}[xy]b_x = 0$. For all $1 \leq x < y$ let $b_x = \sum_{i=1}^{l_x} \mu_{x,i}\beta_{x,i}$ be the standard representation of b_x: then the standard representation of $0 = \sum_{x=1}^{y-1}[xy]b_x$ is precisely

$$\sum_{x=1}^{y-1}\sum_{i=1}^{l_x} \mu_{x,i}[xy]\beta_{x,i},$$

where every summand corresponds to a different monotone monomial $[xy]\beta_{x,i}$. Therefore all coefficients $\mu_{x,i}$ must vanish, hence $b_x = 0$ for all $1 \leq x \leq y-1$. This shows the uniqueness of $b_1, \ldots, b_{y-1}$. □

Lemma 8.2.10. *Let $a \in \mathcal{Z}(\mathcal{V}(d))_p$, and assume that $a_{mht=2} = 0$, i.e. the monomial $[12]^p$ does not appear in the standard representation $a = \sum_{i=1}^{k} \lambda_i \alpha_i$; then $a = 0$.*

Proof. Assume instead $a \neq 0$; it suffices to find a generator $[\bar{x}y]$ such that $a[\bar{x}y] \neq [\bar{x}y]a$.
For $1 \leq i \leq k$ let $\underline{\mathbf{t}}_i = (\mathbf{t}_{i,1}, \ldots, \mathbf{t}_{i,p})$ be the monotone p-tuple with $m(\underline{\mathbf{t}}_i) = \alpha_i$.
Let $y = \min_{i=1}^{k} mht(\underline{\mathbf{t}}_i)$, with $3 \leq y \leq d$ because $[12]^p$ does not appear among the α_i's.
Up to reordering the summands in the standard representation of a, we may assume that there is $1 \leq \nu \leq k$ with $a_{mht=y} = \sum_{i=1}^{\nu} \lambda_i \alpha_i$. Write

$$a_{mht=y} = \sum_{x=1}^{y-1}[xy]b_x$$

as according to Lemma 8.2.9.
Let $1 \leq \bar{x} < y$ be chosen with the following property: there exists some $x \neq \bar{x}$ with $b_x \neq 0$ (here it is essential that $y \geq 3$).
It suffices to prove that

$$(a[\bar{x}y])_{mht=y} \neq ([\bar{x}y]a)_{mht=y}\,.$$

Note that $(a[\bar{x}y])_{mht=y} = a_{mht=y} \cdot [\bar{x}y]$ and similarly $([\bar{x}y]a)_{mht=y} = [\bar{x}y] \cdot a_{mht=y}$. Hence from now on we assume $a = a_{mht=y}$.
There is a unique way of writing $[\bar{x}y] \cdot a = \sum_{x=1}^{y-1}[xy]b'_x$ as in Lemma 8.2.9, and this way is precisely $[\bar{x}y] \cdot a$, i.e. $b'_{\bar{x}} = a$ and $b'_x = 0$ for $x \neq \bar{x}$.
On the other hand we can set $b''_x := b_x[\bar{x}y]$, with $b''_x \neq 0$ for some $x \neq 0$ by our choice of $\bar{x}$ and by Theorem 8.2.1. Then

$$a \cdot [\bar{x}y] = \sum_{x=1}^{y-1}[xy]b''_x$$

is the unique way of representing $a \cdot [\bar{x}y]$, according to Lemma 8.2.9. The two representations are different, hence $a \cdot [\bar{x}y] \neq [\bar{x}y] \cdot a$. □

Proof of Theorem 8.2.6. Thanks to Lemma 8.2.7, we only have to prove that a homogeneous element $a \in \mathcal{Z}(\mathcal{V}(d))_p$ belongs to $\mathbb{Z}[\mathbf{z}]_p$.
Suppose first that p is even, and write $p = 2q$; assuming $a \neq 0$, the standard representation of a must contain by Lemma 8.2.10 a summand of the form $\lambda[12]^{2q}$. We consider then the element

$$a - \lambda \mathbf{z}^q \in \mathcal{V}(d)_p.$$

This element is also central in $\mathcal{V}(d)$ and the monomial $[12]^{2q}$ does not appear in its standard representation; hence this element is zero, and we have $a = \lambda[12]^q$.
Suppose now that p is odd. By Lemma 8.2.10 the standard representation of a must contain a summand of the form $\lambda[12]^p$, hence $\varepsilon(a) = (p, 0, 0, \ldots, 0)$ and the principal part of a (see Definition 8.2.2) is equal to $\lambda[12]^p$.
We will show that $a[13] \neq [13]a$. First note that by Lemma 8.2.3, as already remarked in the Proof of Theorem 8.2.1, we have $\varepsilon(a[13]) = \varepsilon([13]a) = (p, 1, 0, \ldots, 0)$, so it suffices to prove that $\mathcal{P}(a[13]) \neq \mathcal{P}([13]a)$. We have

$$\begin{aligned}\mathcal{P}([13]a) &= \mathcal{P}(\mathcal{P}([13])\,\mathcal{P}(a)) \\ &= \mathcal{P}([13] \cdot \lambda[12]^p) \\ &= [13] \cdot \lambda[12]^p,\end{aligned}$$

where in the last equality we use that $[13][12]^p$ is monotone. On the other hand

$$\begin{aligned}\mathcal{P}(a[13]) &= \mathcal{P}(\mathcal{P}(a)\,\mathcal{P}([13])) \\ &= \mathcal{P}(\lambda[12]^p[13]) \\ &= -[23] \cdot \lambda[12]^p.\end{aligned}$$

For the last equality we use several times the relations $[12][13] = -[23][12] - [13][23]$ and $[12][23] = -[13][12] - [23][13]$ to bring the factor $[13]$ at the beginning of the monomial. For $x = 1, 2$, every time we swap a factor $[x3]$ with a factor $[12]$ the following happen:

- we gain a sign -1;
- we replace the factor $[x3]$ with the factor $[y3]$, where $y = 1, 2$ and $y \neq x$;
- we create a new summand that, as long as we are interested in computing only the principal part of $\lambda[12]^p[13]$, we can neglect.

Since p is odd, after an odd number of swaps the factor $[13]$ at the end of the monomial becomes a factor $-[23]$ at the beginning. □

9 Chain complexes

In this chapter we will consider the *reduced* cellular chain complex $\tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^\infty\right)$. In Section 9.1 we will describe explicitly its generators and boundary maps; assuming that G is normed, we will then deduce a description of the chain complex $\tilde{C}_*\left(\mathrm{Hur}(h,G)^\infty\right)$.
In Section 9.2 we will consider the dual description of the reduced cochain complex $\tilde{C}^*\left(\mathrm{Hur}(h,G)^\infty\right)$, and we will identify the direct sum $\oplus_{h\geq 0}\tilde{C}^*\left(\mathrm{Hur}(h,G)^\infty\right)$ with the reduced cobar complex of a certain differential, coaugmented coalgebra $\mathbb{A}(G)$.
In Section 9.3 we will simplify the reduced cobar complex of $\mathbb{A}(\mathfrak{S}_d)$ using the results from Chapter 8.
Recall from Section 7.5 we have proved that $\mathrm{Hur}(h,\mathfrak{S}_d)$ is a non-compact, complex manifold of complex dimension h: by Poincaré-Lefschetz duality the cohomology of $\tilde{C}^*\left(\mathrm{Hur}(h,\mathfrak{S}_d)^\infty\right)$ will be isomorphic to the homology of $\mathrm{Hur}(h,\mathfrak{S}_d)$. This justifies our shift to the reduced *cochain* complexes of $\widetilde{\mathrm{Hur}}(h,G)^\infty$ and $\mathrm{Hur}(h,G)^\infty$.
We fix a finite group G throughout the chapter; from Section 9.2 onwards we assume that G is normed; in Section 9.3 we will assume $G=\mathfrak{S}_d$ for some $d\geq 2$.

9.1 Reduced chain complexes

9.1.1 The chain complex $\tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^\infty\right)$

We fix $h\geq 1$ throughout the section; all G-arrays are assumed of weight h. By Theorem 4.2.3 the chain complex $\tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^\infty\right)$ is concentrated in degrees $2\leq *\leq 2h$. For all $2\leq k\leq 2h$, $\tilde{C}_k\left(\widetilde{\mathrm{Hur}}(h,G)^\infty\right)$ is a free abelian group generated by cells $e^{\mathfrak{A}}$ of dimension k.
For each G-array $\mathfrak{A}$ of dimension k we denote by $\tilde{\mathfrak{A}}$ the generator of $\tilde{C}_k\left(\widetilde{\mathrm{Hur}}(h,G)^\infty\right)$ corresponding to the cell $e^{\mathfrak{A}}$. We want now to express $\partial(\tilde{\mathfrak{A}})$ as a linear combination of generators of $\tilde{C}_{k-1}\left(\widetilde{\mathrm{Hur}}(h,G)^\infty\right)$.
We use for the G-array $\mathfrak{A}$ the notation from Definitions 3.2.4 and 4.2.2; we can expand $\mathfrak{A}=(\mathfrak{a},\underline{\gamma})$, where $\mathfrak{a}$ is the array underlying $\mathfrak{A}$ and $\underline{\gamma}=(\gamma_{i,j})_{1\leq i\leq l,1\leq j\leq\lambda_i}$. Recall that the cell $e^{\mathfrak{A}}$ is modeled on the interior of the multisimplex

$$\Delta^{\mathfrak{A}}=\Delta^{\mathfrak{a}}=\Delta^l\times\Delta^{\lambda_1}\times\cdots\times\Delta^{\lambda_l};$$

the corresponding characteristic map is $\Phi^{\mathfrak{A}}\colon\Delta^{\mathfrak{A}}\to\widetilde{\mathrm{Hur}}(h,G)^\infty$.
Recall Definition 3.2.5 and the discussion before Lemma 3.2.7, and define $\partial^{ver}_{r,s}\Delta^{\mathfrak{A}}\subset\Delta^{\mathfrak{A}}$ and $\partial^{hor}_{r,\eta}\Delta^{\mathfrak{A}}\subset\Delta^{\mathfrak{A}}$ in the same way.

We first note that, as in the case of $SP^h(\mathbb{C})^\infty$, the restriction of $\Phi^{\mathfrak{A}}$ to any *outer* face of $\Delta^{\mathfrak{A}}$ is the constant map to ∞.

Lemma 9.1.1. *For all $1 \leq r \leq l$ and $1 \leq s \leq \lambda_r - 1$, the restriction of $\Phi^{\mathfrak{A}}$ to the inner, vertical face $\partial^{ver}_{r,s}\Delta^{\mathfrak{A}}$ is the characteristic map $\Phi^{\mathfrak{A}'}$ associated with the G-array $\mathfrak{A}' = (l, \underline{\mathfrak{Col}}') = (\mathfrak{a}', \underline{\gamma}')$ such that $\mathfrak{a}'$ is obtained from $\mathfrak{a}$ as in Lemma 3.2.6, and the following hold:*

- $\gamma'_{i,j} = \gamma_{i,j}$ *for $i \neq r$ and $1 \leq j \leq \lambda_i$.*
- $\gamma'_{r,j} = \gamma_{r,j}$ *for $1 \leq j \leq s-1$, and $\gamma'_{r,j} = \gamma_{r,j+1}$ for $s+1 \leq j \leq \lambda_r - 1$;*
- $\gamma'_{r,s} = \gamma_{r,s} \cdot \gamma_{r,s+1}$*;*

Proof. The cell structure on $\widetilde{\mathrm{Hur}}(h,G)^\infty$ is the pullback of the cell structure of $SP^h(\mathbb{C})^\infty$ along the map $p^\infty \colon \widetilde{\mathrm{Hur}}(h,G)^\infty \to SP^h(\mathbb{C})^\infty$ (see Section 4.2), therefore the restriction of $\Phi^{\mathfrak{A}}$ to $\partial^{ver}_{r,s}\Delta^{\mathfrak{A}}$ is the characteristic map $\Phi^{\mathfrak{A}'}$ associated with some G-array $\mathfrak{A}' = (\mathfrak{a}', \underline{\gamma}')$, where $\mathfrak{a}'$ is obtained as in Lemma 3.2.6.
We need only to check that $\underline{\gamma}'$ satisfies the required properties. Let f and f' be sections of the sheaf $\mathfrak{G}^{\mathfrak{a}}$ as in Lemma 3.3.2; then $f \cup f'$ is a continuous section of $\mathfrak{G}^{\mathfrak{a}}$ over $\mathring{\Delta}^{\mathfrak{a}} \cup \mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{a}} \cong \mathring{\Delta}^{\mathfrak{A}} \cup \mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{A}}$.
Consider the following map of sets $\mathring{\Delta}^{\mathfrak{A}} \cup \mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{A}} \to G$: a point $\mathbb{P} \in \mathring{\Delta}^{\mathfrak{A}} \cup \mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{A}}$ is mapped to $\varphi((f \cup f')(\mathbb{P}))$, where $\Phi^{\mathfrak{A}}(\mathbb{P})$ takes the form (P, φ). The topology of $\widetilde{\mathrm{Hur}}(h,G)$ from Definition 4.1.1 ensures that this map is continuous, hence locally constant, hence constant because $\mathring{\Delta}^{\mathfrak{A}} \cup \mathring{\partial}^{ver}_{r,s}\Delta^{\mathfrak{A}}$ is connected. The result follows in the three cases listed, corresponding to the three cases of Lemma 3.3.2. ☐

Lemma 9.1.2. *Let $1 \leq r \leq l-1$ and let $\eta \in \mathfrak{Shuf}(\lambda_r, \lambda_{r+1})$. The restriction of $\Phi^{\mathfrak{A}}$ to the multisimplex $\partial^{hor}_{r,\eta}\Delta^{\mathfrak{A}}$ is the characteristic map $\Phi^{\mathfrak{A}}$ associated with the G-array $\mathfrak{A}' = (l, \underline{\mathfrak{Col}}') = (\mathfrak{a}', \underline{\gamma}')$ such that $\mathfrak{a}'$ is obtained from $\mathfrak{a}$ as in Lemma 3.2.7, and the following hold:*

- $\gamma'_{i,j} = \gamma_{i,j}$ *for $1 \leq i \leq r-1$ and $1 \leq j \leq \lambda_i$, and $\gamma'_{i,j} = \gamma_{i+1,j}$ for $r+1 \leq i \leq l-1$ and $1 \leq j \leq \lambda_{i+1}$;*
- *the sequence $(\gamma'_{r,j})_{1 \leq j \leq \lambda_r + \lambda_{r+1}}$ of elements of G is the twisted amalgamation along η of the sequences $(\gamma_{r,j})_{1 \leq j \leq \lambda_r}$ and $(\gamma_{r+1,j})_{1 \leq j \leq \lambda_{r+1}}$;*

Proof. The proof is the equal to the one of Lemma 9.1.1, considering the space $\mathring{\Delta}^{\mathfrak{a}} \cup \mathring{\partial}^{hor}_{r,\eta}\Delta^{\mathfrak{a}} \cong \mathring{\Delta}^{\mathfrak{A}} \cup \mathring{\partial}^{hor}_{r,\eta}\Delta^{\mathfrak{A}}$ and referring to Lemma 3.3.3. ☐

We obtain the following theorem, which computes the differential of $\tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^\infty\right)$.

Theorem 9.1.3. *Let $\mathfrak{A}$ be a G-array of dimension k, where we use the notation from Definition 4.2.2. Then $\partial(\tilde{\mathfrak{A}}) \in \tilde{C}_{k-1}\left(\widetilde{\mathrm{Hur}}(h,G)^{\infty}\right)$ can be computed as*

$$\begin{aligned}\partial(\tilde{\mathfrak{A}}) &= \partial^{ver}(\tilde{\mathfrak{A}}) + \partial^{hor}(\tilde{\mathfrak{A}}) \\ &= \sum_{r=1}^{l} \partial_r^{ver}(\tilde{\mathfrak{A}}) + \sum_{r=1}^{l-1} \partial_r^{hor}(\tilde{\mathfrak{A}}) \\ &= \sum_{r=1}^{l}\left(\sum_{s=1}^{\lambda_r-1} \partial_{r,s}^{ver}(\tilde{\mathfrak{A}})\right) + \sum_{r=1}^{l-1}\left(\sum_{\eta\in\mathfrak{Shuf}(\lambda_r,\lambda_{r+1})} \partial_{r,\eta}^{hor}(\tilde{\mathfrak{A}})\right).\end{aligned}$$

The term $\partial_{r,s}^{ver}(\tilde{\mathfrak{A}})$ is defined as

$$\partial_{r,s}^{ver}(\tilde{\mathfrak{A}}) = (-1)^{\left(\sum_{i=1}^{r-1}\lambda_i\right)+s}\,\tilde{\mathfrak{A}}',$$

where $\mathfrak{A}'$ is obtained from $\mathfrak{A}$ as in Lemma 9.1.1.
The term $\partial_{r,\eta}^{hor}(\tilde{\mathfrak{A}})$ is defined as

$$\partial_{r,\eta}^{hor}(\tilde{\mathfrak{A}}) = (-1)^{r+\pi(\eta)}\,\tilde{\mathfrak{A}}',$$

where $\mathfrak{A}'$ is obtained from $\mathfrak{A}$ as in Lemma 9.1.2, and $\pi(\eta)$ was defined in Definition 2.1.3.

Proof. The statement is a straightforward consequence of the definition of cellular chain complex and Lemmas 9.1.1 and 9.1.2. The signs are chosen according to the standard orientation of the multisimplices $\Delta^{\mathfrak{A}}$, and follow the rules described in Section 2.1. □

We note in particular that $\tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^{\infty}\right)$ has a natural structure of double chain complex, i.e. the differential ∂ is the sum of two anti-commuting differentials $\partial^{ver}+\partial^{hor}$.

9.1.2 The chain complex $\tilde{C}_*\left(\mathrm{Hur}(h,G)^{\infty}\right)$

If G is normed, the chain complex $\tilde{C}_*\left(\mathrm{Hur}(h,G)^{\infty}\right)$ has a similar description: for every *special* G-array $\mathfrak{A}$ of weight h and dimension k we have a generator in $\tilde{C}_k(\mathrm{Hur}(h,G)^{\infty})$. For simplicity, we call this generator $\mathfrak{A}$ (see Definition 4.3.6). The chain complex $\tilde{C}_*\left(\mathrm{Hur}(h,G)^{\infty}\right)$ can be regarded as a subquotient of $\tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^{\infty}\right)$, as we describe in the following.
Recall Definition 4.3.5. We consider the subcomplex

$$\tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^{\infty}_{adm}\right) \subset \tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^{\infty}\right)$$

spanned by generators $\mathfrak{A}$ corresponding to admissible G-arrays (see Definition 4.3.5).

Consider the norm filtration $F_\bullet^N \tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^\infty\right)$ induced from the norm filtration on $\widetilde{\mathrm{Hur}}(h,G)^\infty$, and restrict this filtration to $\tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^\infty_{adm}\right)$. Then

$$\tilde{C}_*\left(\mathrm{Hur}(h,G)^\infty\right) \cong F_h^N/F_{h-1}^N\left(\tilde{C}_*\left(\mathrm{Hur}(h,G)^\infty_{adm}\right)\right).$$

The differential ∂ of $\tilde{C}_*\left(\mathrm{Hur}(h,G)^\infty\right)$ has the same description given in Theorem 9.1.3, but we leave out in the computation of $\partial(\mathfrak{A})$ all terms $\partial^{ver}_{r,s}(\mathfrak{A})$ yielding a G-array $\mathfrak{A}'$ of norm strictly lower than h: note that this cannot happen for the terms $\partial^{hor}_{r,\eta}(\mathfrak{A})$, i.e. the horizontal differentials $\partial^{hor}(\tilde{\mathfrak{A}})$ in $\tilde{C}_*\left(\widetilde{\mathrm{Hur}}(h,G)^\infty\right)$ and $\partial^{hor}(\mathfrak{A})$ in $\tilde{C}_*\left(\mathrm{Hur}(h,G)^\infty\right)$ have formally the same value, for all special G-arrays $\mathfrak{A}$.

9.2 The cochain complex $\tilde{C}^*\left(\mathrm{Hur}(h,G)^\infty\right)$

In this section we translate the results of the previous section into a description of the *reduced* cochain complex $\tilde{C}^*\left(\mathrm{Hur}(h,G)^\infty\right)$.

9.2.1 All values of h at the same time

It will be easier to give a unified description of all cochain complexes $\tilde{C}^*\left(\mathrm{Hur}(h,G)^\infty\right)$ at the same time, for varying $h \geq 0$, rather than to focus on a specific value of h.

Definition 9.2.1. We define the cochain complex $\mathcal{A} = \mathcal{A}(G)$ as

$$\mathcal{A} = \bigoplus_{h\geq 0} \tilde{C}^*\left(\mathrm{Hur}(h,G)^\infty\right).$$

It is generated by the duals of the generators of the chain complexes $\tilde{C}_*\left(\mathrm{Hur}(h,G)^\infty\right)$ for varying $h \geq 0$. Let $\mathfrak{A}$ be a special G-array of dimension k and weight h; then we call $\mathfrak{A}^*$ the dual of the generator $\mathfrak{A} \in \tilde{C}_k(\mathrm{Hur}(h,G))$.

Note that $\mathcal{A}(G)$ is a double cochain complex, i.e. its differential δ splits naturally as a sum of two anti-commuting differentials $\delta^{hor} + \delta^{ver}$, dual to the splitting $\partial = \partial^{hor} + \partial^{ver}$ discussed after Theorem 9.1.3.
Note also that in the case $G = \mathfrak{S}_d$ we can use the Poincaré-Lefschetz duality argument from Subsection 7.5.4 and find a homotopy equivalence of the cochain complex $\mathcal{A}$ with the (singular) chain complex of $\mathrm{Hur}(\mathfrak{S}_d)$ (see Definition 5.2.1):

$$\mathcal{A} \simeq C_*\left(\mathrm{Hur}(\mathfrak{S}_d)\right).$$

The change in degree in the previous homotopy equivalence is governed by a different formula for each summand of $\mathcal{A}$ and each corresponding subspace $\mathrm{Hur}(h,\mathfrak{S}_d)$ of $\mathrm{Hur}(\mathfrak{S}_d)$; we omit the details. The previous homotopy equivalence justifies our shift to the reduced *cochain* complexes of the spaces $\mathrm{Hur}(h,G)^\infty$.

In the following we will represent $\mathcal{A}$ as the reduced cobar complex of a certain differential coalgebra $\mathbb{A} = \mathbb{A}(G)$, which is coaugmented over $\mathbb{Z}$. Our construction is similar to the construction underlying Theorem 1.3 in [16], where the *cohomology* of a certain *quantum shuffle algebra* is compared with the homology of braid groups with coefficients in a braided vector space. The construction in [16] is applied to study Hurwitz spaces in the original sense [26], see also the discussion after Definition 4.3.7; we are instead interested in the *amalgamations* $\mathrm{Hur}(h, G)$, which contain the spaces $\mathrm{hur}(\underline{\alpha}, G)$ as strata of the absolute value filtration.

Our construction yields the reduced cobar complex of a differential, coaugmented coalgebra with coefficients in the left and right comodule $\mathbb{Z}$; instead the construction in [16] yields the bar complex of a certain augmented algebra, with coefficients in the left and right module $\mathbb{Z}$. The difference is only due to the fact that we prefer to focus on the reduced *cochain* complexes $\tilde{C}^*(\mathrm{Hur}(h, G)^\infty)$ rather than the reduced *chain* complexes $\tilde{C}_*(\mathrm{Hur}(h, G)^\infty)$.

The fact that our coalgebra $\mathbb{A}$ is differential (i.e. endowed with a non-trivial differential δ^{ver}) is due to the fact that we are considering the special Hurwitz spaces $\mathrm{Hur}(h, G)$, which have natural projections to the symmetric products $SP^h(\mathbb{C})$. If our focus were instead, similarly as in [16], on the classical Hurwitz spaces spaces $\mathrm{hur}(\underline{\alpha}, G)$, which project naturally to the coloured configuration spaces $C(\mathbb{C}; \underline{\alpha})$, then we would have a trivial vertical differential, and we would get the cobar complex of a coaugmented (non-differential) coalgebra.

9.2.2 The cochain complex $\mathbb{A}(G)$

Definition 9.2.2. For $\lambda \geq 1$, we denote by $\mathbb{A}_\lambda = \mathbb{A}_\lambda(G)$ the free abelian group generated by all special G-columns of absolute value λ (see Definitions 4.2.1 and 4.3.6).

A special G-column $\mathfrak{Col} = (\lambda, \underline{m}, \underline{\gamma})$ is uniquely determined by its sequence $\underline{\gamma}$ of elements in $G \setminus \{\mathbb{1}\}$: indeed λ is the length of $\underline{\gamma}$, and for all $1 \leq j \leq \lambda$ we can recover $m_j = N(\gamma_j) \geq 1$. In the following we will use this reduced notation, so we will write $\mathfrak{Col} = \underline{\gamma}$ for a special G-column.

We define $\mathbb{A}_0 = \mathbb{A}_0(G)$ to be $\mathbb{Z}$, generated by the *empty column*, which has absolute value 0 and is denoted by 1.

We denote by $\mathbb{A} = \mathbb{A}(G)$ the direct sum $\bigoplus_{\lambda \geq 0} \mathbb{A}_\lambda$: it is a graded abelian group, where the *degree* of $\mathfrak{Col} = \underline{\gamma}$ is given by the length λ of $\underline{\gamma}$; we denote $\mathbb{A}_{>0} = \bigoplus_{\lambda \geq 1} \mathbb{A}_\lambda$ the part of strictly positive degree.

To every special G-column $\mathfrak{Col} = \underline{\gamma}$ we can assign a weight $\mathfrak{w}(\mathfrak{Col}) = \sum_{j=1}^{\lambda} N(\gamma_j)$ (see Definitions 4.2.1 and 4.3.6) and a total monodromy $\omega(\mathfrak{Col}) = \gamma_1 \cdot \ldots \cdot \gamma_\lambda$ (see Definition 4.4.1: this gives a *grading* and a G-grading on $\mathbb{A}$. The group G acts on the set of special G-columns by conjugation; the action preserves degree and grading, and induces the action by conjugation on the G-grading. Hence $\mathbb{A}$ is a graded G-Yetter-Drinfeld module (see Definition 8.1.7).

The differential δ^{ver} makes $\mathbb{A}$ into a cochain complex: it is defined by considering special G-columns as special G-arrays of length 1. More precisely, for a special G-column

$\mathfrak{Col} = (\gamma_1, \dots, \gamma_\lambda)$ we set

$$\delta^{ver}(\mathfrak{Col}) = \sum_{j=1}^{\lambda} (-1)^j \sum_{\alpha,\beta} (\gamma_1, \dots, \gamma_{j-1}, \alpha, \beta, \gamma_{j+1}, \dots, \gamma_\lambda),$$

where the last sum is extended over all couples α, β of elements of $G \setminus \{\mathbb{1}\}$ with $\alpha \cdot \beta = \gamma_j$ and $N(\alpha) + N(\beta) = N(\gamma_j)$.
The differential δ^{ver} increases the degree by 1, preserves the grading and the G-grading and is G-equivariant; thus $\mathbb{A}$ is a G-Yetter-Drinfeld cochain complex.

Recall Definition 8.1.1: one can define in an analogous way a norm filtration $F^N_\bullet$ on the normalised bar complex $\bar{B}_\bullet G$ for any normed group G; considering the cochain complexes associated with the filtration strata, we obtain immediately by Definition 9.2.2 the following lemma.

Lemma 9.2.3. *There is an isomorphism of cochain complexes*

$$\mathbb{A}(G) \cong \bigoplus_{h=0}^{\infty} \left(\mathfrak{F}^N_h \bar{B}_\bullet G\right)^\bullet,$$

in which the grading on left corresponds to the filtration grading on right.

In the case of $G = \mathfrak{S}_d$ we can therefore compute explicitly the cohomology of $\mathbb{A}(G)$ by virtue of Theorem 8.1.3 and its reformulation 8.1.5.

9.2.3 Bialgebra structure on $\mathbb{A}(G)$

In the following we will make $\mathbb{A}(G)$ into a G-Yetter-Drinfeld differential bialgebra.

Definition 9.2.4. The unit $\eta\colon \mathbb{Z} \to \mathbb{A}$ is defined as the inclusion $\mathbb{Z} = \mathbb{A}_0 \subset \mathbb{A}$; viceversa the counit (or augmentation) $\varepsilon\colon \mathbb{A} \to \mathbb{Z}$ is the projection onto the direct summand $\mathbb{A}_0$: hence the kernel of ε is $\mathbb{A}_{>0}$.
The product $\mu\colon \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ is defined by juxtaposition of special G-columns: for $\mathfrak{Col} = \underline{\gamma} \in \mathbb{A}_\lambda$ and $\mathfrak{Col}' = \underline{\gamma}' \in \mathbb{A}_{\lambda'}$ we set

$$\mu\left((\gamma_1, \dots, \gamma_{\lambda_1}) \otimes (\gamma'_1, \dots, \gamma'_{\lambda'})\right) = (\gamma_1, \dots, \gamma_\lambda, \gamma'_1, \dots, \gamma'_{\lambda'}) \in \mathbb{A}_{\lambda+\lambda'}.$$

The product is associative and it is a map of G-Yetter-Drinfeld modules. Note that, as an algebra, $\mathbb{A}$ is isomorphic to the free tensor algebra over $\mathbb{A}_1$, i.e. the free tensor algebra generated by special G-columns of length 1; note also that $\mathbb{A}_1$ is canonically isomorphic to the free abelian group generated by $G \setminus \{\mathbb{1}\}$.
The coproduct $\Delta\colon \mathbb{A} \to \mathbb{A} \otimes \mathbb{A}$ is defined as follows:

- every special G-column $\mathfrak{Col} = \gamma \in G \setminus \{\mathbb{1}\}$ of length 1 is *primitive*, i.e. we set

$$\Delta(\gamma) = 1 \otimes (\gamma) + (\gamma) \otimes 1;$$

- the coproduct is extended multiplicatively on $\mathbb{A}$, using the algebra structure on $\mathbb{A}$ given by μ and regarding $\mathbb{A} \otimes \mathbb{A}$ as the tensor algebra in the category of G-Yetter-Drinfeld modules (see Definition 8.1.8).

The coproduct is coassociative on generators in $\mathbb{A}_1$, because they are primitive; hence it coassociative on the entire $\mathbb{A}$. Moreover the coproduct is by construction a map of G-Yetter-Drinfeld algebras, and it also preserves the degree, since generators in $\mathbb{A}_1$ are mapped to elements in degree 1 in $\mathbb{A} \otimes \mathbb{A}$, namely to $\mathbb{A}_0 \otimes \mathbb{A}_1 \oplus \mathbb{A}_1 \otimes \mathbb{A}_0$.
Thus $\mathbb{A}$ is a graded bialgebra in the category of G-Yetter-Drinfeld modules; in the following we check that the differential δ^{ver} is compatible with this bialgebra structure, i.e., that η, ε, μ and Δ are maps of cochain complexes.
For η and ε it is straightforward: $\mathbb{Z}$ is regarded as the trivial cochain complex sitting in degree 0.
For μ and Δ, recall that the differential δ^{ver} is extended on $\mathbb{A} \otimes \mathbb{A}$ by imposing the Leibniz rule: using the notation from Definition 9.2.4 we set

$$\delta^{ver}(\mathfrak{Col} \otimes \mathfrak{Col}') := \delta^{ver}(\mathfrak{Col}) \otimes \mathfrak{Col}' + (-1)^{\lambda}\mathfrak{Col} \otimes \delta^{ver}(\mathfrak{Col}');$$

it is then straightforward from the formula for δ^{ver} on $\mathbb{A}$ in Definition 9.2.2 that μ is a map of cochain complexes, i.e. the following equality holds:

$$\delta^{ver}(\mathfrak{Col} \cdot \mathfrak{Col}') = \delta^{ver}(\mathfrak{Col}) \cdot \mathfrak{Col}' + (-1)^{\lambda}\mathfrak{Col} \cdot \delta^{ver}(\mathfrak{Col}').$$

Note also that δ^{ver} makes $\mathbb{A} \otimes \mathbb{A}$ into a differential algebra, i.e. the following formula holds, where $\mathfrak{Col}_1 \in \mathbb{A}_{\lambda_1}$ and $\mathfrak{Col}'_1 \in \mathbb{A}_{\lambda'_1}$:

$$\begin{aligned}\delta^{ver}\left((\mathfrak{Col}_1 \otimes \mathfrak{Col}'_1) \cdot (\mathfrak{Col}_2 \otimes \mathfrak{Col}'_2)\right) = \left(\delta^{ver}(\mathfrak{Col}_1 \otimes \mathfrak{Col}'_1)\right) \cdot (\mathfrak{Col}_2 \otimes \mathfrak{Col}'_2) + \\ + (-1)^{\lambda_1+\lambda'_1}(\mathfrak{Col}_1 \otimes \mathfrak{Col}'_1) \cdot \left(\delta^{ver}(\mathfrak{Col}_2 \otimes \mathfrak{Col}'_2)\right).\end{aligned}$$

We still need to check that $\Delta(\delta^{ver}(\mathfrak{Col})) = \delta^{ver}(\Delta(\mathfrak{Col}))$ for all special G-columns $\mathfrak{Col}$; by the previous discussion it suffices to check this on generators in $\mathbb{A}_1$, i.e. on G-columns of length 1, having the form (γ) for some $\gamma \in G \setminus \{\mathbb{1}\}$:

$$\begin{aligned}\Delta\left(\delta^{ver}(\gamma)\right) &= \Delta\left(\sum_{\alpha,\beta}(\alpha,\beta)\right) \\ &= \sum_{\alpha,\beta}\Delta(\alpha,\beta) \\ &= \sum_{\alpha,\beta}(1 \otimes (\alpha) + (\alpha) \otimes 1) \cdot (1 \otimes (\beta) + (\beta) \otimes 1) \\ &= \sum_{\alpha,\beta}(1 \otimes (\alpha,\beta) + (\alpha,\beta) \otimes 1) + \sum_{\alpha,\beta}\left((\alpha) \otimes (\beta) - (\alpha\beta\alpha^{-1}) \otimes (\alpha)\right) \\ &= \delta^{ver}\left(1 \otimes (\gamma) + (\gamma) \otimes 1\right) = \delta^{ver}\left(\Delta(\gamma)\right),\end{aligned}$$

where all sums are extended over all couples $\alpha, \beta \in G \setminus \{\mathbb{1}\}$ with $\alpha\beta = \gamma$ and $N(\alpha) + N(\beta) = N(\gamma)$, see Definition 9.2.2. In the fourth equality we have used the product on $\mathbb{A} \otimes \mathbb{A}$ from Definition 8.1.8; the additional sign comes from the fact that both (α) and (β) have degree 1. In the fifth equality we have used that the sum $\sum_{\alpha,\beta} \big((\alpha) \otimes (\beta) - (\alpha\beta\alpha^{-1}) \otimes (\alpha)\big)$ vanishes.
We have proved the following lemma:

Lemma 9.2.5. *The operations introduced in Definitions 9.2.2 and 9.2.4 make $\mathbb{A}$ into a differential bialgebra in the category of G-Yetter-Drinfeld modules.*

9.2.4 The cobar complex of $\mathbb{A}(G)$

Let $C_\bullet$ be a differential, connected, coaugmented coalgebra in the category of abelian groups: $C_\bullet$ is in particular endowed with a coaugmentation (or unit) $\mathbb{Z} \to C_\bullet$ and an augmentation (or counit) $C_\bullet \to \mathbb{Z}$, and we can decompose $C_\bullet = \mathbb{Z} \oplus C_{>0}$, where $C_{>0}$ is both the positive part of $C_\bullet$ and the augmentation ideal.
Following [2], we can associate with $C_\bullet$ a cobar complex $F_{\bullet,\bullet} = F_{\bullet,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z})$, which is a double chain complex in the category of abelian groups. For all $l \geq 0$ we have

$$F_{l,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z}) = \mathbb{Z} \otimes (C_\bullet)^{\otimes l} \otimes \mathbb{Z};$$

for a generator $1 \otimes c_1 \otimes \cdots \otimes c_l \otimes 1$, the first degree, called *length*, is the number l, whereas the second degree, called *absolute value*, is the number $\lambda = \lambda_1 + \cdots + \lambda_l$, assuming that $c_i \in C_{\lambda_i}$ for all $1 \leq i \leq l$.
The *vertical* differential $\delta^{ver} \colon F_{l,\lambda} \to F_{l,\lambda+1}$ is defined by using the differential δ_C of $C_\bullet$ and imposing the Leibniz rule: using the notation above we have

$$\delta^{ver}(1 \otimes c_1 \otimes \ldots c_l \otimes 1) = \sum_{i=1}^{l} (-1)^{\sum_{j=1}^{i-1} \lambda_j} 1 \otimes c_1 \otimes \cdots \otimes c_{i-1} \otimes \delta_C(c_i) \otimes c_{i+1} \otimes \cdots \otimes c_l \otimes 1.$$

The *horizontal* differential $\delta^{hor} \colon F_{l,\lambda} \to F_{l+1,\lambda}$ is defined by using the coproduct Δ_C of $C_\bullet$ and the unit $\mathbb{Z} \subset C_\bullet$, making $\mathbb{Z}$ into a left and right comodule over $C_\bullet$:

$$\begin{aligned}\delta^{hor}(1 \otimes c_1 \otimes \ldots c_l \otimes 1) = {} & (1 \otimes 1) \otimes c_1 \otimes \ldots c_l \otimes 1 + \\ & + \sum_{i=1}^{l} (-1)^i \, 1 \otimes c_1 \otimes \cdots \otimes c_{i-1} \otimes \Delta_C(c_i) \otimes c_{i+1} \otimes \cdots \otimes c_l \otimes 1 + \\ & + (-1)^{l+1} \, 1 \otimes c_1 \otimes \ldots c_l \otimes (1 \otimes 1).\end{aligned}$$

There is a reduced version of the construction, called the *reduced* bar complex, and denoted by $\bar{F}_{\bullet,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z})$. For all $l \geq 0$ we have

$$\bar{F}_{l,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z}) = \mathbb{Z} \otimes (C_{>0})^{\otimes l} \otimes \mathbb{Z};$$

there are natural projections $\pi_l \colon F_{l,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z}) \to \bar{F}_{l,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z})$ induced by the projection $C_\bullet \to C_{>0}$, and the horizontal and vertical differentials δ^{hor} and δ^{ver} on $F_{\bullet,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z})$ induce on the quotient $\bar{F}_{\bullet,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z})$ differentials $\bar{\delta}^{hor}$ and $\bar{\delta}^{ver}$.

Concretely, we can define the *reduced* coproduct $\bar{\Delta}_C\colon C_{>0} \to C_{>0}$ by the formula $\bar{\Delta}_C(c) = \Delta_C(c) - 1 \otimes c - c \otimes 1$; then the vertical differential $\bar{\delta}^{ver}$ on $\bar{F}_{\bullet,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z})$ has precisely the same formula as the differential δ^{ver} on $F_{\bullet,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z})$, whereas the horizontal differential $\bar{\delta}^{hor}$ on $F_{\bullet,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z})$ takes the form

$$\bar{\delta}^{hor}(1 \otimes c_1 \otimes \dots c_l \otimes 1) = \sum_{i=1}^{l} (-1)^i 1 \otimes c_1 \otimes \cdots \otimes c_{i-1} \otimes \bar{\Delta}_C(c_i) \otimes c_{i+1} \otimes \cdots \otimes c_l \otimes 1$$

It is a standard result that the projection map $\pi\colon F_{\bullet,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z}) \to \bar{F}_{\bullet,\bullet}(\mathbb{Z}, C_\bullet, \mathbb{Z})$ is a homotopy equivalence of double cochain complexes; the construction of the homotopy equivalence uses the counit of $C_\bullet$.

Theorem 9.2.6. *Let G be a finite normed group. Recall Definitions 9.2.1, 9.2.2 and 9.2.4, and consider $\mathbb{A}(G)$ as a differential, connected, coaugmented coalgebra in the category of abelian groups. There is an isomorphism of double cochain complexes*

$$\mathcal{A}(G) \cong \bar{F}_{\bullet,\bullet}(\mathbb{Z}, \mathbb{A}(G), \mathbb{Z});$$

Proof. Given a special G-array $\mathfrak{A} = (l, \underline{\mathfrak{Col}})$ of length l and absolute value λ, we identify the generator $\mathfrak{A}^* \in \mathcal{A}$ with the tensor product

$$1 \otimes \mathfrak{Col}_1 \otimes \cdots \otimes \mathfrak{Col}_l \otimes 1 \in \bar{F}_{l,\lambda}(\mathbb{Z}, \mathbb{A}(G), \mathbb{Z}).$$

This assignment gives an isomorphism $\mathcal{A}(G) \cong \bar{F}_{\bullet,\bullet}(\mathbb{Z}, \mathbb{A}(G), \mathbb{Z})$ as bigraded abelian groups; we still have to check that the vertical and horizontal differentials of the two cochain complexes correspond to each other.
The correspondence between δ^{ver} on $\mathcal{A}$ and $\bar{\delta}^{ver}$ on $\bar{F}_{\bullet,\bullet}(\mathbb{Z}, \mathbb{A}(G), \mathbb{Z})$ follows immediately from Theorem 9.1.3 and Definition 9.2.2.
To check that δ^{hor} on $\mathcal{A}$ and $\bar{\delta}^{hor}$ on $\bar{F}_{\bullet,\bullet}(\mathbb{Z}, \mathbb{A}(G), \mathbb{Z})$ also correspond to each other, note first that for a special G-array $\mathfrak{A} = (l, \underline{\mathfrak{Col}})$ we have a natural splitting

$$\delta^{hor}(\mathfrak{A}^*) = \sum_{r=1}^{l} (-1)^r \delta_r^{hor}(\mathfrak{A}^*),$$

where $\delta_r^{hor}(\mathfrak{A}^*)$ is defined as follows. Let $(\mathfrak{Col}_r)$ denote the special G-array of length 1 consisting only of the column $\mathfrak{Col}_r$, and write $\delta^{hor}\left((\bar{\mathfrak{Col}}_r)^*\right) = \sum_{i=1}^{k} \mu_i (\mathfrak{A}_i)^*$, for suitable $k \geq 0$, $\mu_i \in \mathbb{Z}$ and for suitable special G-arrays of length 2 $\mathfrak{A}_i = (\mathfrak{Col}_{i,1}, \mathfrak{Col}_{i,2})$. For $1 \leq i \leq k$ define the special G-array $\mathfrak{A}'_i$ of length $l+1$ by

$$\mathfrak{A}'_i = (\mathfrak{Col}_1, \dots, \mathfrak{Col}_{r-1}, \mathfrak{Col}_{i,1}, \mathfrak{Col}_{i,2}, \mathfrak{Col}_{r+1}, \dots, \mathfrak{Col}_l).$$

Then $\delta_r^{hor}(\mathfrak{A}^*) = \sum_{i=1}^{k} \mu_i (\mathfrak{A}'_i)^*$; in few words, δ_r^{hor} is obtained by applying the horizontal differential δ^{hor} only to the $r^{\text{th}}$$G$-column of $\mathfrak{A}$.
In the same way we have a splitting

$$\bar{\delta}^{hor}(1 \otimes \mathfrak{Col}_1 \otimes \cdots \otimes \mathfrak{Col}_l \otimes 1) = \sum_{r=1}^{l} (-1)^r \bar{\delta}_r^{hor}(1 \otimes \mathfrak{Col}_1 \otimes \cdots \otimes \mathfrak{Col}_l \otimes 1),$$

where for $1 \le r \le l$ we have

$\bar{\delta}_r^{hor}(1 \otimes \mathfrak{Col}_1 \otimes \cdots \otimes \mathfrak{Col}_l \otimes 1) = 1 \otimes \mathfrak{Col}_1 \otimes \cdots \otimes \mathfrak{Col}_{r-1} \otimes \bar{\Delta}(\mathfrak{Col}_r) \otimes \mathfrak{Col}_{r+1} \otimes \cdots \otimes \mathfrak{Col}_l \otimes 1.$

Therefore we only need to check that for a special G-array consisting of a single G-column $\mathfrak{Col}$ of some length λ, the horizontal differential $\delta^{hor}\left((\bar{\mathfrak{Col}})^*\right) \in \mathcal{A}$ corresponds to $1 \otimes \bar{\Delta}(\mathfrak{Col}) \otimes 1 \in \bar{F}_{1,\lambda}(\mathbb{Z}, \mathbb{A}(G), \mathbb{Z})$.
Write $\mathfrak{Col} = \underline{\gamma} = (\gamma_1, \ldots, \gamma_\lambda)$; from Theorem 9.1.3, we obtain the formula

$$\delta^{hor}\left((\bar{\mathfrak{Col}})^*\right) = \sum_{i=1}^{\lambda-1} \sum_{\eta \in \mathfrak{Shuf}(i,\lambda-i)} (-1)^{\pi(\eta)}(\alpha_1, \ldots, \alpha_i) \otimes (\beta_1, \ldots, \beta_{\lambda-i}),$$

where for every shuffle $\eta \in \mathfrak{Shuf}(i, \lambda - i)$ we denote by $\underline{\alpha}$ and $\underline{\beta}$ the sequences of elements in G corresponding, under the bijection given by the twisted amalgamation along η, to $\underline{\gamma}$: see Definition 3.3.4 and the discussion after it. Recall also that δ^{hor}, just as ∂^{hor}, is formally the same in $\tilde{C}^*\left(\widetilde{\mathrm{Hur}}(h, G)\right)$ and in $\tilde{C}^*\left(\mathrm{Hur}(h, G)\right)$, for this see the discussion at the end of Subsection 9.1.2.
On the other hand we have

$$\bar{\Delta}(\mathfrak{Col}) = (1 \otimes \gamma_1 + \gamma_1 \otimes 1) \cdot \ldots \cdot (1 \otimes \gamma_\lambda + \gamma_\lambda \otimes 1) - 1 \otimes \mathfrak{Col} - \mathfrak{Col} \otimes 1.$$

If we expand the product of binomials, we obtain a total of 2^λ smaller products, that we call *terms*. The term $(1 \otimes \gamma_1) \cdot \ldots \cdot (1 \otimes \gamma_l)$ cancels out with $-1 \otimes \mathfrak{Col}$; similarly the term $(\gamma_1 \otimes 1) \cdot \ldots \cdot (\gamma_l \otimes 1)$ cancels out with $-\mathfrak{Col} \otimes 1$.
The other $2^\lambda - 2$ terms are in bijection with bipartitions of the set $[\lambda]$ into two non-empty sets A and B of some cardinalities i and $\lambda - i$: the set A contains the indices $j \in [\lambda]$ for which we choose the summand $\gamma_j \otimes 1$ from the j^{th}parenthesis, and the set B contains indices $j \in [\lambda]$ for which we choose the summand $1 \otimes \gamma_j$. The inclusions of A and B in $[\lambda]$ determine a shuffle $\eta \in \mathfrak{Shuf}(i, \lambda - i)$.
Recall Definition 8.1.8, and note that for $\alpha, \beta \in G$ the following equality holds in $\mathbb{A} \otimes \mathbb{A}$:

$$(1 \otimes \beta) \cdot (\alpha \otimes 1) = -(\beta\alpha\beta^{-1}) \otimes \beta = -((\beta\alpha\beta^{-1}) \otimes 1) \cdot (1 \otimes \beta).$$

We can therefore reduce every term to a normal form by pushing all factors of the form $1 \otimes \gamma_j$ for $j \in B$ to the right and all factors of the form $\gamma_j \otimes 1$ for $j \in A$ to the left. We obtain a product of the form

$$\pm(\alpha_1 \otimes 1) \cdot \ldots \cdot (\alpha_i \otimes 1) \cdot (1 \otimes \beta_1) \cdot \ldots \cdot (1 \otimes \beta_{\lambda-i}),$$

and it is straightforward to check that the sign is $(-1)^{\pi(\eta)}$ and that $\underline{\alpha}$ and $\underline{\beta}$ are precisely the functions $A \to G$ and $B \to G$ whose twisted amalgamation along η is $\underline{\gamma}$. This shows that δ^{hor} and $\bar{\delta}^{hor}$ agree, and completes the proof. □

9.3 Further simplification for symmetric groups

In this section we assume that $G = \mathfrak{S}_d$ for some $d \ge 2$, and write $\mathbb{A} = \mathbb{A}(\mathfrak{S}_d)$.

9.3.1 A spectral sequence argument

Filtering $\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathbb{A},\mathbb{Z})$ by *length* (see Subsection 9.2.4), we obtain a spectral sequence with first page $E_1^{\bullet,\bullet} = \bar{F}_{\bullet,\bullet}(\mathbb{Z}, H^*(\mathbb{A},\delta^{ver}),\mathbb{Z})$, converging to the cohomology of the double complex $\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathbb{A},\mathbb{Z})$.
Recall from Subsection 9.2.3 that $(\mathbb{A},\delta^{ver})$ carries the richer structure of bialgebra in the category of $\mathfrak{S}_d$-Yetter-Drinfeld modules.
In particular it has a grading as a $\mathfrak{S}_d$-Yetter-Drinfeld modules, which was defined using the weight of $\mathfrak{S}_d$-columns. The spectral sequence above respects this grading, essentially because both differentials $\bar{\delta}^{hor}$ and $\bar{\delta}^{ver}$ of $\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathbb{A},\mathbb{Z})$ respect the grading. So the spectral sequence E can be written as a direct sum of spectral sequences $\oplus_{h\geq 0}E(h)$.
On the other hand the cohomology $H^*(\mathbb{A},\delta^{ver})$ was computed in Theorems 8.1.3 and 8.1.5, and it is equal, as bialgebra in $\mathfrak{S}_d$-Yetter Drinfeld modules, to the algebra $\mathcal{V}(d)$ (see Subsection 8.1.2); in particular this algebra is non-trivial only in degree equal to the grading. This implies that the second page $E(h)_2$ has only one non-trivial row, namely the row $E(h)_2^{\bullet,h}$, where we have

$$E(h)_2^{l,h} = \oplus_{h_1,\dots,h_l}\mathbb{Z}\otimes\mathcal{V}(d)_{h_1}\otimes\cdots\otimes\mathcal{V}(d)_{h_l}\otimes\mathbb{Z},$$

where the direct sum is extended over all sequences $h_1,\dots,h_l$ with $h_i\geq 1$ for all $1\leq i\leq l$ and $\sum_{i=1}^l h_i = h$. In particular the spectral sequences $E(h)$ collapse on the second page, and therefore also E does.
The differential of the first page is the differential induced by $\bar{\delta}^{hor}$ on $\bar{F}_{\bullet,\bullet}(\mathbb{Z},H^*(\mathbb{A},\delta^{ver}),\mathbb{Z})$, and makes the first page $E_2^{\bullet,\bullet}$ isomorphic to the reduced cobar complex of the coaugmented coalgebra $\mathcal{V}(d)$. We obtain the following theorem.

Theorem 9.3.1. *There is an isomorphism of $\mathfrak{S}_d$-Yetter-Drinfeld modules*

$$H_*\left(\mathrm{Hur}(\mathfrak{S}_d)\right)\cong H^*\left(\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathcal{V}(d),\mathbb{Z})\right),$$

such that the part of degree $*$ and grading h on left corresponds to the part of degree $2h-*$ and grading h on right.

The previous theorem establishes a strong connection between the algebra $\mathcal{V}(d)$ and the special Hurwitz space $\mathrm{Hur}(\mathfrak{S}_d)$, which for $d\geq 2g+n-1$ contains a connected component homotopy equivalent to $\mathfrak{M}_{g,n}$ (see Theorems 7.3.2 and 7.4.1).

9.3.2 Comparison between two products

Recall from Subsection 5.2.1 the multiplication μ making $\mathrm{Hur}_Q(\mathfrak{S}_d)$ into an H-space. Note that $\mathrm{Hur}_Q(h,\mathfrak{S}_d)$ can be regarded as an open subset of $\mathrm{Hur}(h,\mathfrak{S}_d)$ (see Definition 5.2.2), hence by Theorem 7.5.1 also the space $\mathrm{Hur}_Q(h,\mathfrak{S}_d)$ is a complex manifold of complex dimension h. Note that the map μ gives an open embedding of complex manifolds

$$\mu\colon \mathrm{Hur}_Q(h_1,\mathfrak{S}_d)\times\mathrm{Hur}_Q(h_2,\mathfrak{S}_d)\hookrightarrow\mathrm{Hur}_Q(h_1+h_2,\mathfrak{S}_d);$$

Consider the corresponding collapse map

$$\mu^{\infty}\colon \mathrm{Hur}_Q(h_1+h_2,\mathfrak{S}_d)^{\infty} \to \mathrm{Hur}_Q(h_1,\mathfrak{S}_d)^{\infty} \wedge \mathrm{Hur}_Q(h_2,\mathfrak{S}_d)^{\infty}$$

and consider on the latter spaces the CW structure described in Subection 5.2.1. Then μ^{∞} is a cellular map, and it induces on reduced cochain complexes a map

$$(\mu^{\infty})^* : \tilde{C}^* \left(\mathrm{Hur}_Q(h_1,\mathfrak{S}_d)^{\infty}\right) \otimes \tilde{C}^* \left(\mathrm{Hur}_Q(h_2,\mathfrak{S}_d)^{\infty}\right) \to \tilde{C}^* \left(\mathrm{Hur}_Q(h_1+h_2,\mathfrak{S}_d)^{\infty}\right).$$

For two special $\mathfrak{S}_d$-arrays $\mathfrak{A}_1, \mathfrak{A}_2$ of weights h_1 and h_2 respectively, consider the product cell $e^{\mathfrak{A}_1} \times e^{\mathfrak{A}_2} \subset \mathrm{Hur}_Q(h_1,\mathfrak{S}_d)^{\infty} \wedge \mathrm{Hur}_Q(h_2,\mathfrak{S}_d)^{\infty}$.
There is a unique cell in $\mathrm{Hur}_Q(h_1+h_2,\mathfrak{S}_d)^{\infty}$ having dimension equal to $\dim(e^{\mathfrak{A}_1}) + \dim(e^{\mathfrak{A}_2})$ and intersecting the preimage of $e^{\mathfrak{A}_1} \times e^{\mathfrak{A}_2}$ along the map μ^{∞}: it is the cell $e^{\mathfrak{A}} \subset \mathrm{Hur}_Q(h_1+h_2,\mathfrak{S}_d)^{\infty}$, where the $\mathfrak{S}_d$-array $\mathfrak{A}$ is obtained by juxtaposition of the $\mathfrak{S}_d$-arrays $\mathfrak{A}_1$ and $\mathfrak{A}_2$. Hence we obtain, at the level of reduced cochain complexes, the equality

$$(\mu^{\infty})^* \left(\mathfrak{A}_1^* \otimes \mathfrak{A}_2^*\right) = \mathfrak{A}^*.$$

Considering all values of h at the same time we obtain a (strictly) associative product on $\mathcal{A}(\mathfrak{S}_d)$ (see Definition 9.2.1). This product clearly corresponds, under the isomorphism from Theorem 9.2.6, to the natural product on the reduced cobar complex $\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathbb{A},\mathbb{Z})$ coming from the product of $\mathbb{A}$, and essentially given by juxtaposition of tensor powers of $\mathbb{A}$.
Finally one can replace $\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathbb{A},\mathbb{Z})$ by $\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathcal{V}(d),\mathbb{Z})$ using Theorem 9.3.1. As a consequence the isomorphism from Theorem 9.3.1 is an isomorphism of rings, where on left we consider the Pontryagin product and on right the product induced on the cobar complex from the product of $\mathcal{V}(d)$.
In this way we have used the entire structure of $\mathcal{V}(d)$ as bialgebra: the coproduct and the unit are needed to define the cobar complex $F_{\bullet,\bullet}(\mathbb{Z},\mathcal{V}(d),\mathbb{Z})$; the counit allows the replacement with the reduced cobar complex $\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathcal{V}(d),\mathbb{Z})$; the product, which is not strictly necessary to state the isomorphism of Theorem 9.3.1 as $\mathfrak{S}_d$-Yetter-Drinfeld modules, gives a product on $\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathcal{V}(d),\mathbb{Z})$ and hence on $H^*\left(\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathcal{V}(d),\mathbb{Z})\right)$, recovering in a purely algebraic way the Pontryagin product from Subsection 5.2.1.

10 Outlook

In this chapter we consider some questions arising naturally from the work of the thesis. The cohomology of the reduced cobar complex $\bar{F}_{\bullet,\bullet}(\mathbb{Z},\mathcal{V}(d),\mathbb{Z})$, appearing in the statement of Theorem 9.3.1 seems very difficult to compute explicitly, even in the small case $d=3$. Nevertheless, to obtain some information about $H_*(\mathfrak{M}_{g,n})$, by Theorem 7.3.2 one should consider a high value of d, namely $d \geq 2g+n-1$.
Note that in the case $d=2$ the space $\mathrm{Hur}(h,\mathfrak{S}_2)$ is homeomorphic to the configuration space $C(\mathbb{C};h)$, whose homology is well-understood, see for example [3, 22, 40].

10.1 A Leray spectral sequence

Recall Definitions 4.2.4 and 4.3.7, and fix $h \geq 1$. The space $\mathrm{Hur}(h,\mathfrak{S}_d)^\infty$ has an absolute value filtration $F^{|\cdot|}_\bullet$; for all $k \geq 0$ the filtration level $F^{|\cdot|}_k\mathrm{Hur}(h,\mathfrak{S}_d)^\infty$ is a closed subcomplex of $\mathrm{Hur}(h,\mathfrak{S}_d)^\infty$.
There is therefore a Leray spectral sequence $\mathcal{E}(h)$ converging to the reduced cohomology of $\mathrm{Hur}(h,\mathfrak{S}_d)^\infty$, whose first page contains the reduced cohomology of the filtration quotients $F^{|\cdot|}_k/F^{|\cdot|}_{k-1}\mathrm{Hur}(h,\mathfrak{S}_d)^\infty$:

$$\mathcal{E}(h)_1^{p,q} = \tilde{H}^{p+q}\left(F^{|\cdot|}_p/F^{|\cdot|}_{p-1}\mathrm{Hur}(h,\mathfrak{S}_d)^\infty\right) \Rightarrow \tilde{H}^{p+q}\left(\mathrm{Hur}(h,\mathfrak{S}_d)^\infty\right).$$

By Theorem 7.5.1 and the remarks in Subsection 7.5.4, we have an isomorphism

$$\tilde{H}^{p+q}\left(\mathrm{Hur}(h,\mathfrak{S}_d)^\infty\right) \cong H_{2h-p-q}\left(\mathrm{Hur}(h,\mathfrak{S}_d)\right).$$

Moreover for $k \geq 1$ the stratum $\mathfrak{F}^{|\cdot|}_k\mathrm{Hur}(h,\mathfrak{S}_d)^\infty$ splits as

$$\mathfrak{F}^{|\cdot|}_k\mathrm{Hur}(h,\mathfrak{S}_d)^\infty = \coprod_{\underline{\alpha}} \mathrm{hur}(\underline{\alpha},\mathfrak{S}_d),$$

where the disjoint union is taken over all sequences $\underline{\alpha} = (\alpha_j)_{j\geq 1}$ with $\sum_{j=1}^\infty \alpha_j = k$ and $\sum_{j=1}^\infty j\alpha_j = h$. In particular $\mathfrak{F}^{|\cdot|}_k\mathrm{Hur}(h,\mathfrak{S}_d)^\infty$ is a complex manifold of complex dimension k, and we have an isomorphism

$$\tilde{H}^{p+q}\left(F^{|\cdot|}_p/F^{|\cdot|}_{p-1}\mathrm{Hur}(h,\mathfrak{S}_d)^\infty\right) \cong \tilde{H}^{p+q}\left(\left(\mathfrak{F}^{|\cdot|}_p\mathrm{Hur}(h,\mathfrak{S}_d)\right)^\infty\right) \cong H_{p-q}\left(\mathfrak{F}^{|\cdot|}_p\mathrm{Hur}(h,\mathfrak{S}_d)\right).$$

Hence the previous spectral sequence takes the form

$$\mathcal{E}(h)_1^{p,q} = H_{p-q}\left(\mathfrak{F}^{|\cdot|}_p\mathrm{Hur}(h,\mathfrak{S}_d)\right) \Rightarrow H_{2h-p-q}\left(\mathrm{Hur}(h,\mathfrak{S}_d)\right).$$

It would be interesting to understand whether the previous spectral sequence collapses at a finite stage. Motivated by some small computations, we state the following conjecture.

Conjecture 10.1.1. The spectral sequence $\mathcal{E}(h)$ collapses at the second page $\mathcal{E}(h)_2$, which is therefore isomorphic to the page $\mathcal{E}(h)_\infty$.

10.2 Algebraic varieties and Mumford-Miller-Morita classes

Fix $h \geq 1$. Theorem 7.5.1 ensures that $\mathrm{Hur}(h, \mathfrak{S}_d)$ is a smooth complex manifold of complex dimension h. One can enhance this statement to the following conjecture.

Conjecture 10.2.1. The complex manifold $\mathrm{Hur}(h, \mathfrak{S}_d)$ carries a structure of smooth, affine, algebraic variety, such that the natural map $p\colon \mathrm{Hur}(h, \mathfrak{S}_d) \to SP^h(\mathbb{C})$ is an algebraic map.

Let now S denote the set of non-trivial conjugacy classes of $\mathfrak{S}_d$ (i.e. we leave out the class of $\mathbb{1}$), and for all $\mathfrak{s} \in S$ let $N(\mathfrak{s}) \geq 1$ be the norm of any permutation in $\mathfrak{s}$.
Let $1 \leq k \leq h-1$, (the case $k = h$ makes the following discussion rather trivial, so we leave it out), and let $\beta\colon S \to \mathbb{Z}_{\geq 0}$ be a function satisfying $\sum_{\mathfrak{s}\in S} \beta(\mathfrak{s}) = k$ and $\sum_{\mathfrak{s}\in S} N(\mathfrak{s}) \cdot \beta(\mathfrak{s}) = h$; for $j \geq 1$ define

$$\alpha_j := \sum_{\mathfrak{s}\in S, N(\mathfrak{s})=j} \beta(\mathfrak{s}).$$

Then we can consider the subspace

$$\mathrm{hur}(\beta, \mathfrak{S}_d) \subset \mathrm{hur}(\underline{\alpha}, \mathfrak{S}_d) \subset \mathrm{Hur}(h, \mathfrak{S}_d)$$

containing configurations (P, φ) satisfying the following properties:

- $|P| = k$, i.e. P has the form $P = \{m_1 \cdot z_1, \ldots, m_k \cdot z_k\}$;
- for all $\mathfrak{s} \in S$, exactly $\beta(\mathfrak{s})$ of the local monodromies of φ around the points $z_1, \ldots, z_k$ belong to $\mathfrak{s}$ (see Definition 4.1.2).

Note that $\mathrm{hur}(\beta, \mathfrak{S}_d) \subset \mathrm{hur}(\underline{\alpha}, \mathfrak{S}_d)$ is a union of connected components; in particular $\mathrm{hur}(\beta, \mathfrak{S}_d)$ is naturally a (non-closed) complex sub-manifold of $\mathrm{Hur}(h, \mathfrak{S}_d)$ of complex dimension k. We denote by

$$\overline{\mathrm{hur}}(\beta, \mathfrak{S}_d) \subset \mathrm{Hur}(h, \mathfrak{S}_d)$$

the closure of $\mathrm{hur}(\beta, \mathfrak{S}_d)$ in $\mathrm{Hur}(h, \mathfrak{S}_d)$.

Conjecture 10.2.2. Assume Conjecture 10.2.1. The subspace $\overline{\mathrm{hur}}(\beta, \mathfrak{S}_d)$ is a closed, algebraic subvariety of $\mathrm{Hur}(h, \mathfrak{S}_d)$.

Fix now $\sigma \in \mathfrak{S}_d$ with h and $N(\sigma)$ of the same parity, assume $h \geq 2d - 2 - N(\sigma)$ and consider the connected component $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma) \subset \mathrm{Hur}(h, \mathfrak{S}_d)$. Define

$$\overline{\mathrm{hur}}^*(\beta, \mathfrak{S}_d, \sigma) := \overline{\mathrm{hur}}(\beta, \mathfrak{S}_d) \cap \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma).$$

Assuming Conjecture 10.2.2, the space $\overline{\mathrm{hur}}^*(\beta, \mathfrak{S}_d, \sigma)$ is an algebraic, closed subvariety of dimension k of the algebraic, smooth variety $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)$, hence by Poincaré-Lefschetz duality we can associate with it a cohomology class

$$w_\beta \in H^{2h-2k}\left(\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)\right).$$

One can also note that the space $\overline{\mathrm{hur}}^*(\beta, \mathfrak{S}_d, \sigma)$ is a smooth complex manifold of complex dimension k away from a subspace of complex dimension $k-1$, hence one can define a corresponding fundamental class in $H^{2h-2k}\left(\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)\right)$.
Let n be the number of cycles of σ and let $g \geq 0$ be such that $h = 2g + n + d - 2$; by Theorem 7.4.1 and Definition 7.2.1 we have a map

$$\rho\colon \mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma) \to \mathfrak{M}_{g,n},$$

inducing a map $\rho^*\colon H^{2h-2k}(\mathfrak{M}_{g,n}) \to H^{2h-2k}\left(\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma)\right)$. For $i \geq 1$ denote by $\kappa_i \in H^{2i}(\mathfrak{M}_{g,n})$ the i^{th}Mumford-Miller-Morita class [33, 30, 31].

Conjecture 10.2.3. Consider cohomology with rational coefficients; then the pull-back of the product

$$\prod_{\mathfrak{s} \in S} \kappa_{N(\mathfrak{s})-1}^{\beta(\mathfrak{s})} \in H^{2h-2k}(\mathfrak{M}_{g,n}; \mathbb{Q})$$

along the map ρ is proportional, inside $H^{2h-2k}\left(\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma); \mathbb{Q}\right)$, to the class w_β.

The previous conjecture can be proved in the following particular case by using an argument by Kranhold [28]. Fix $3 \leq r \leq d$, let $k = h + 2 - r$ and let $\beta_r\colon S \to \mathbb{Z}_{\geq 0}$ be defined by

- $\beta_r(cyc_r) = 1$, where cyc_r denotes the conjugacy class of permutations consisting of one r-cycle and $d - r$ fixpoints;

- $\beta_r(cyc_2) = k - 1$, where cyc_2 denotes the conjugacy class of transpositions;

- $\beta_r(\mathfrak{s}) = 0$ for all other conjugacy classes $\mathfrak{s} \in S$.

Then we claim that the class w_{β_r} is *equal* to the class $\rho^*(\kappa_{r-2})$ (the coefficient of proportionality is 1). In his work Kranhold proves a similar statement concerning the space $\mathfrak{Par}_{g,1}[1]$, which is a real analytic manifold of dimension $6g$ [14], and a certain subspace $W_r \subset \mathfrak{Par}_{g,1}[1]$, which is conjecturally a real analytic subvariety of $\mathfrak{Par}_{g,1}[1]$ of dimension $6g + 4 - 2r$.
One can associate with W_r a cohomology class $[W_r] \in H^{2r-4}(\mathfrak{Par}_{g,1}[1])$, and Kranhold proves that the class $[W_r]$ corresponds to κ_{r-2} under the homotopy equivalence $\mathfrak{Par}_{g,1}[1] \simeq \mathfrak{M}_{g,1}$. The construction can be generalised to any space $\mathfrak{Par}_{g,n}[\underline{d}]$, yielding a class $[W_r] \in H^{2r-2}\left(\mathfrak{Par}_{g,n}[\underline{d}]\right)$ corresponding to κ_{r-2} under the homotopy equivalence $\mathfrak{Par}_{g,n}[\underline{d}] \simeq \mathfrak{M}_{g,n}$; the class $[W_r]$ is supported on a subspace $W_r \subset \mathfrak{Par}_{g,n}[\underline{d}]$, which is conjecturally a real analytic subvariety.

One can also use a more direct argument: W_r is a smooth, real manifold of dimension $6g-2r+4$ away from a subspace of dimension $6g-2r+2$, hence we can associate with it a fundamental cohomology class in $H^{2r-4}(\mathfrak{Par}_{g,1}[1])$.
Kranhold argues that $(W_r)^\infty \subset \mathfrak{Par}_{g,n}[\underline{d}]^\infty$ is a closed subcomplex, and writes explicitly a fundamental cycle representing the class

$$[W_r^\infty] \in H_{6g-2r+4}\left(\mathfrak{Par}_{g,n}[\underline{d}]^\infty\right) \cong H^{2r-4}\left(\mathfrak{Par}_{g,n}[\underline{d}]\right);$$

he then uses this combinatorial description to prove that $[W_r]$ corresponds to κ_{r-2}.
Let now $\underline{d} = (d_1, \dots, d_n)$ be the sequence of lengths of cycles of σ; by Theorem 7.4.1 the space $\mathrm{Hur}^*(h, \mathfrak{S}_d, \sigma) \cong \bar{\mathcal{O}}_{g,n}[\underline{d}]$ is a closed, smooth submanifold of $\mathfrak{Par}_{g,n}[\underline{d}]$. The set-theoretic intersection between W_r and $\bar{\mathcal{O}}_{g,n}[\underline{d}]$ inside $\mathfrak{Par}_{g,n}[\underline{d}]$ is precisely the space $\overline{\mathrm{hur}}^*(\beta, \mathfrak{S}_d, \sigma)$. This intersection is transverse away from a subspace of $\overline{\mathrm{hur}}^*(\beta, \mathfrak{S}_d, \sigma)$ of real dimension $2k-2$: hence the fundamental cohomology class w_r is the pullback of $[W_r]$ along the inclusion $\bar{\mathcal{O}}_{g,n}[\underline{d}] \subset \mathfrak{Par}_{g,n}[\underline{d}]$. This shows the conjecture in the special case.

10.3 Homology stability

Recall the multiplication map μ from Definition 5.2.3, making $\mathrm{Hur}_Q(\mathfrak{S}_d) \cong \mathrm{Hur}(\mathfrak{S}_d)$ into an H-space. Fix a transposition $\mathbf{t} \in \mathfrak{S}_d$ and a configuration $(P, \varphi)_\mathbf{t} \in \mathrm{Hur}_Q(1, \mathfrak{S}_d, \mathbf{t})$, and for $h \geq 0$ and $\sigma \in \mathfrak{S}_d$, consider the map

$$s^\mathbf{t} = \mu\left((P, \varphi)_\mathbf{t}, \cdot\right) : \mathrm{Hur}(h, \mathfrak{S}_d) \to \mathrm{Hur}(h+1, \mathfrak{S}_d).$$

A naive conjecture could be that the sequence of spaces $\mathrm{Hur}(h, \mathfrak{S}_d)$ for $h \geq 0$ exhibits homology stability with respect to the maps $s^\mathbf{t}$, i.e. the map $s^\mathbf{t}$ induces an isomorphism in homology $s^\mathbf{t}_*\colon H_*\left(\mathrm{Hur}(h, \mathfrak{S}_d)\right) \cong H_*\left(\mathrm{Hur}(h+1, \mathfrak{S}_d)\right)$ in a range of degrees $* < \psi(h)$, for a diverging function $\psi\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$.
However it is relatively easy to see that, for $d \geq 4$, there can be no homology stability even in degree $* = 0$: indeed the number of connected components of $\mathrm{Hur}(h, \mathfrak{S}_d)$ becomes arbitrarily large when $h \to \infty$, as follows from Theorem 4.5.5.
Note however that $s^\mathbf{t}$ restricts to a map

$$s^\mathbf{t}\colon \mathrm{Hur}^*_Q(h, \mathfrak{S}_d) \to \mathrm{Hur}^*_Q(h+1, \mathfrak{S}_d),$$

where for $h \geq 0$ we set $\mathrm{Hur}^*_Q(h, \mathfrak{S}_d) = \mathrm{Hur}^*(h, \mathfrak{S}_d) \cap \mathrm{Hur}_Q(h, \mathfrak{S}_d)$ (see Definition 4.5.4).

Conjecture 10.3.1. The sequence of spaces $\mathrm{Hur}^*_Q(h, \mathfrak{S}_d)$ for $h \geq 0$ exhibits homology stability with respect to the maps $s^\mathbf{t}$, for some stability range $\psi\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$.

Note that for different transpositions $\mathbf{t}, \mathbf{t}' \in \mathfrak{S}_d$ the maps

$$s^\mathbf{t}, s^{\mathbf{t}'}\colon \mathrm{Hur}^*_Q(h, \mathfrak{S}_d) \to \mathrm{Hur}^*_Q(h+1, \mathfrak{S}_d)$$

behave differently on connected components: for all $\sigma \in \mathfrak{S}_d$ such that h and $N(\sigma)$ have the same parity and $h \geq 2d-2-N(\sigma)$, the connected component $\mathrm{Hur}^*_Q(h, \mathfrak{S}_d, \sigma)$ is

mapped to the component $\mathrm{Hur}^*_Q(h+1,\mathfrak{S}_d,\mathbf{t}\cdot\sigma)$ in the first case and to the component $\mathrm{Hur}^*_Q(h+1,\mathfrak{S}_d,\mathbf{t}'\cdot\sigma)$ in the second case.
However both double iterations $(s^{\mathbf{t}})^2$ and $(s^{\mathbf{t}'})^2$ restrict to maps

$$\mathrm{Hur}^*_Q(h,\mathfrak{S}_d,\sigma)\to\mathrm{Hur}^*_Q(h+2,\mathfrak{S}_d,\sigma),$$

so that $(s^{\mathbf{t}}_*)^2$ and $(s^{\mathbf{t}'}_*)^2$ induce the same isomorphism

$$H_0\left(\mathrm{Hur}^*_Q(h,\mathfrak{S}_d)\right)\to H_0\left(\mathrm{Hur}^*_Q(h+2,\mathfrak{S}_d)\right).$$

Conjecture 10.3.2. In the stable range $*\le\psi(h)$ and for all $\sigma\in\mathfrak{S}_d$, the two maps $(s^{\mathbf{t}})^2_*$ and $(s^{\mathbf{t}'})^2_*$ exhibit *the same* isomorphism

$$H_*\left(\mathrm{Hur}^*_Q(h,\mathfrak{S}_d)\right)\to H_*\left(\mathrm{Hur}^*_Q(h+2,\mathfrak{S}_d)\right).$$

More generally one could conjecture that for any two sequences of transpositions $\mathbf{t}_1,\dots,\mathbf{t}_k$ and $\mathbf{t}'_1,\dots,\mathbf{t}'_k$ in $\mathfrak{S}_d$ such that the products $\mathbf{t}_1\cdot\ldots\cdot\mathbf{t}_k$ and $\mathbf{t}'_1\cdot\ldots\cdot\mathbf{t}'_k$ are the same permutation in $\mathfrak{S}_d$, the compositions $s^{\mathbf{t}_1}_*\circ\dots\circ s^{\mathbf{t}_k}_*$ and $s^{\mathbf{t}'_1}_*\circ\dots\circ s^{\mathbf{t}'_k}_*$ exhibit the same isomorphism $H_*\left(\mathrm{Hur}^*_Q(h,\mathfrak{S}_d)\right)\to H_*\left(\mathrm{Hur}^*_Q(h+k,\mathfrak{S}_d)\right)$ in the stable range.
Homology stability for the spaces $\mathrm{Hur}^*_Q(h,\mathfrak{S}_d)\cong\mathrm{Hur}(h,\mathfrak{S}_d)$ could be derived from a suitable homology stability statement for the classical Hurwitz spaces $\mathrm{hur}(\underline{\alpha},\mathfrak{S}_d)$ (see Definition 4.3.7), by noting that the map $s^{\mathbf{t}}$ considered above shifts by 1 the absolute value filtration $F^{|\cdot|}_\bullet$ in a strong sense, i.e.

$$s^{\mathbf{t}}\colon\mathcal{F}^{|\cdot|}_k\mathrm{Hur}^*_Q(h,\mathfrak{S}_d)\to\mathcal{F}^{|\cdot|}_{k+1}\mathrm{Hur}^*_Q(h+1,\mathfrak{S}_d).$$

In particular $s^{\mathbf{t}}$ induces a map $s^{\mathbf{t}}_*\colon\mathcal{E}(h)\to\mathcal{E}(h+1)$ between the Leray spectral sequences considered in Section 10.1: these spectral sequences compute the homology of $\mathrm{Hur}^*_Q(h,\mathfrak{S}_d)$ and $\mathrm{Hur}^*_Q(h+1,\mathfrak{S}_d)$ respectively, and start from the homology of the strata of the absolute value filtration, i.e. the spaces $\mathrm{hur}(\underline{\alpha},\mathfrak{S}_d)$. If $s^{\mathbf{t}}$ induces an isomorphism $\mathcal{E}(h)_1\to\mathcal{E}(h+1)_1$ on the first page in a range of degrees $*\le\psi(h)$, the same happens on the infinity page.
To the best of my knowledge, very little is known about homology stability for classical Hurwitz spaces with coefficients in a non-abelian group G. An important result [17] is the rational homology stability for Hurwitz spaces with coefficients in a finite group G having a particular algebraic property, called the *non-splitting property*. Unfortunately, for $d\ge4$ the symmetric group $\mathfrak{S}_d$ does not fulfil this property, so that one can possibly adapt the argument in [17] only to show rational homology stability for the spaces $\mathrm{Hur}^*(h,\mathfrak{S}_3)$.

10.4 Stable homology

Even without assuming Conjecture 10.3.1, one can try to calculate the colimit

$$\mathrm{colim}_{h\to\infty}H_*\left(\mathrm{Hur}^*(h,\mathfrak{S}_d)\right),$$

where we use all stabilisation maps $s_*^{\mathbf{t}}$ for a all transposition $\mathbf{t} \in \mathfrak{S}_d$ at the same time. Assuming Conjecture 10.3.2, and noting that the previous diagram contains a cofinal subdiagram spanned by the groups $H_*\left(\mathrm{Hur}^*(h, \mathfrak{S}_d, \mathbb{1})\right)$, one can replace the previous colimit with

$$\mathrm{colim}_{h\to\infty} H_*\left(\mathrm{Hur}^*(h, \mathfrak{S}_d, \mathbb{1})\right),$$

where h ranges among even numbers and we use all stabilisation maps $(s_*^{\mathbf{t}})^2$ at the same time. One can now note that the previous colimit coincides also with

$$\mathrm{colim}_{h\to\infty} H_*\left(\mathrm{Hur}(h, \mathfrak{S}_d, \mathbb{1})\right),$$

again because the diagram spanned by the groups $H_*\left(\mathrm{Hur}^*(h, \mathfrak{S}_d, \mathbb{1})\right)$ is cofinal in the diagram spanned by the groups $H_*\left(\mathrm{Hur}(h, \mathfrak{S}_d, \mathbb{1})\right)$.
Recall now from Section 5.3 that $\mathrm{Hur}(\mathfrak{S}_d, \mathbb{1}) \cong \mathrm{Hur}_Q(\mathfrak{S}_d, \mathbb{1})$ is an algebra over the operad $\mathcal{C}_2$ of little squares. The group completion of the discrete monoid $\pi_0\left(\mathrm{Hur}(\mathfrak{S}_d, \mathbb{1})\right)$ is isomorphic to the group $2\mathbb{Z}$ of even integers, where the natural map $\pi_0\left(\mathrm{Hur}(\mathfrak{S}_d, \mathbb{1})\right)$ maps the entire $\mathrm{Hur}(h, \mathfrak{S}_d, \mathbb{1})$ to h. The group completion theorem [36] applies and one obtains an isomorphism

$$\bigoplus_{2\mathbb{Z}} \left(\mathrm{colim}_{h\to\infty} H_*\left(\mathrm{Hur}(h, \mathfrak{S}_d, \mathbb{1})\right)\right) \cong H_*\left(\Omega B\mathrm{Hur}(\mathfrak{S}_d, \mathbb{1})\right).$$

The latter homology is the homology of an Ω^2-space, namely the double loop space of the double bar construction $B^2\mathrm{Hur}(\mathfrak{S}_d, \mathbb{1})$. It would be interesting to find a convenient model for the latter space, compute its homology and deduce from it the stable homology of the special, transitive Hurwitz spaces $\mathrm{Hur}(h, \mathfrak{S}_d)$ for $h \to \infty$.

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