Probabilistic and differential geometric methods for relativistic and Euclidean Dirac and radiation fields

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Luigi Marcello Borasi

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1. Gutachter: Prof. Dr. Juan J. L. Velázquez
2. Gutachter: Prof. Dr. Sergio Albeverio

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A mia mamma, mio papà,

mio fratello e la sua famiglia,

con grande affetto.

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Summary

The main objective of this thesis is to study relativistic and Euclidean Fermionic quantum fields from a geometrical and probabilistic point of view as opposed to the standard treatment which is more algebraic in nature. The main motivation lays in the practical need of being able to apply to Fermionic systems a number of results from probability theory, stochastic analysis, calculus of variations, and infinite dimensional analysis which are readily available in the case of Bosonic quantum fields. A more general context for this work is the development of alternative models for Fermionic quantum fields.

This thesis comprises of six chapters including the first one which is the introduction. The chapters from the second to the fifth deal with Fermionic theories. In chapter two we present a model for a finite dimensional system of Fermions which is described in terms of a stochastic diffusion process on the Lie group $\mathbf{Spin}(2n + 1)$. In chapter three we review the notion of induced representations applied to the representation theory of the Poincaré group. Moreover, in that chapter, we introduce Wightman and Schwinger functions and we relate them to the representation theory of the Poincaré group. In chapter four we simplify the derivation of a known result in the literature which allows for a description of Euclidean Dirac Fermions in four dimensions in terms of complex Gaussian random fields. Our simplification should prove useful in future applications, particularly to the non Gaussian case. In chapter five we present a new model for relativistic Dirac fields. In this model we consider a complexified space time and the representation theory of the complexified Poincaré spin group. In this way we can treat both the real Poincaré group and the Euclidean group at once. We show that starting from this complexified spacetime one arrives naturally to a Bosonic description of free Dirac Fermions. We finally give a method to recover the usual Fermionic Fock space and the free Wightman functions.

In the last chapter we pass to the Bosonic case and in particular to gauge theories. There we give a rigorous formulation of a simple "naive" Faddeev-Popov quantization for (Bosonic) gauge theories. We apply such a formulation to the case of the Euclidean radiation field and prove a representation of the corresponding Euclidean probability measure as a limit of essentially finite dimensional natural probability measures inspired by our version of the Faddeev-Popov procedure.

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I

Introduction

Contents

1 A very incomplete slice of history.

The beginning of quantum field theory (QFT). Quantum field theory (QFT) originates as early as in the late 20s, soon after the first steps of quantum mechanics [114], with the seminal work of, among others[1], Dirac, Fermi, Heisenberg, Pauli, Fock, and Jordan and Wigner. One of the main motivations, in the early development of QFT, was the desire of getting a theory consistent both with the principles of special relativity and with those of quantum mechanics. The phenomenological motivation was the formulation of *quantum electrodynamics* (QED), that is of a theory describing the interaction of electrons with the electromagnetic field at a fundamental level. The requirement of the formulation of a (quantum mechanical) relativistic theory was natural, after the basic principles of dynamics based on special relativity had been accepted by the physics community at large.

Wigner notion of elementary particles. A cornerstone in the development of QFT was the idea, due to Wigner ([140], [14]), of giving a mathematical definition of the intuitive notion of elementary particle. Intuitively a particle is *elementary* when it cannot be broken further. Nevertheless, as it later turned out, one needs to allow fundamental particles to transform into one another. Hence, the problem of giving a good definition of such an abstract entity is certainly a nontrivial one.

Wigner's idea is, roughly speaking, as follows. He defines an elementary particle in terms of its transformation properties under the most fundamental symmetry transformations in nature. These fundamental symmetry transformations must include, if we are to produce a relativistic theory, rotations, translations, and pure relativistic transformations (boosts). This group of transformations is usually called the Poincaré group. More precisely he defines a fundamental particle as a state of a quantum mechanical system which has transformation properties, under the Poincaré group, which are in a sense "minimal". By "minimal" we mean that, such transformation properties, can be described by a minimal set of numbers. Mathematically, one says that an elementary particle is a vector which transforms under the Poincaré group according to a unitary, projective, *irreducible* representation on a Hilbert space. Different stateswhich have the same transformation properties correspond to possible configurations of the same elementary particle. Hence, if one thinks of an elementary particle as an abstract concept which can have different states, then an elementary particle is completely characterized by a given "minimal" set of numbers, i.e. an irreducible, projective, unitary representation. The set of numbers which characterize a given elementary particle

[1] A beautiful selection of the original papers is given in [121]. More on the history of quantum electrodynamics and quantum field theory can be found e.g. in [72, 118].

are historically called *quantum numbers*. In Wigner analysis the Poincaré group is considered as the fundamental symmetry group, but the definition of elementary particle can be generalized to more general groups. This generalization is necessary, for example, if one wants to have the electric charge among the quantum numbers describing the elementary particles. The irreducible unitary projective representations of the Poincaré group are characterized by two numbers (s, ρ) with $s = 0, 1/2, 1, 3/2, \dots$ and $\rho \in \mathbb{R}$. The number s is called *spin* quantum number whereas the number ρ denotes the *square* of the *mass* quantum number m.

Wigner idea, on one side, gives a definition of elementary particles which is completely independent from the phenomenology or any prior intuitive notion of particle. On the other side, it gives striking physical meaning to the representation theory of certain (Lie) groups. Thanks to Wigner, and later on, Yang and Mills in the 50s, and Gell-Mann and Ne'eman in the 60s, the theory of Lie groups became of prominent importance in high energy theoretical physics. Such a development is undoubtedly a consequence also of the far seeing result by Noether [103]

Bosons and Fermions. In quantum mechanics a state of a given system is usually described by a wave function. Such a wave function is a vector v in some infinite dimensional Hilbert space and can represent for example a system of n *identical* elementary particles. Roughly speaking, a wave function describing n identical elementary particles depends on n independent variables. Permuting the order of the n elementary particles correspond to permuting these variables. When these n particles are identical the state described by the wave function should not change under such a permutation. Now, the wave function gives more information than that necessary to fully describe the state of the system. If we think of the wave function as a vector v in an infinite dimensional Hilbert space, then the state of the system is determined only by the line on which the vector v lies. Mathematically speaking any vector of the form $v = \lambda v_0$, $\lambda \in \mathbb{C}$, corresponds to the same state of the system. In particular the vector v and the reversed vector $-v$ determine the same state of the system.

As a result, when we permute identical particles the vector (i.e. the wave function) v is allowed to reverse its orientation. We can now define the notion of Bosons and Fermions. We call n identical elementary particles *Bosons* if they are described by a vector which, when we permute the order of the elementary particles, remains the same. One also says that these n identical particles obey the *Bose-Einstein statistics*. Similarly, we call n identical elementary particles *Fermions* if, when we permute their order, the vector v describing them is reversed into $-v$. In this case one also says that these particles obey the *Fermi-Dirac statistics*. In general there are other possibilities. The vector v, describing n identical elementary particles, could be changed by a permutation by a unitary phase $v \mapsto e^{\mathrm{i}\theta} v$, $\theta \in \mathbb{R}$ (We want a unitary phase $e^{\mathrm{i}\theta}$, because symmetry transformations in quantum mechanics are assumed to preserve the length of the vector v describing any state and to be invertible). Particles that transform under permutation according to this more general rule are said to obey a *para statistics*.

The difficulties of QFT. QFT, in spite of its early history, which intertwines with the early stages of quantum mechanics, does not share with quantum mechanics the same role of a classical, well established theory. There are perhaps two reasons for this difference.

One cause could be that the original approach to QFT was strongly changed by the development of gauge theories in the 60s and the parallel evolution of statistical physics and critical phenomena.

The second, and perhaps biggest difference, is that quantum mechanics was, from almost the beginning ([114, 61, 32] cf. also [133]), within the grasp of the mathematics which was being developed around the same time (see e.g. [137, 102, 111] and reference therein). On the other hand, QFT underwent a development which lead to explore directions very far from the known realms of Mathematics. This is especially true if one restricts the attention to the *theory of relativistic quantum fields* (RQFT). Perhaps the distance between the theoretical physical aspects of QFT and the rigorous mathematical physical ones were not realized at first. In fact it was hoped that the combination of special relativity with quantum mechanics would have resolved some of the mathematical issues which afflicted classical relativistic theories (action at a distance cf. e.g. [138], etc…). It did not take long to realize that this was not the case, because of the persistence of divergences e.g. in QED. By this we mean that a superficial application of the theory of

QED would erroneously lead to the prediction that some physical observable quantities are "divergent", that is infinite. This problem, which already exists in classical (non quantum) physics, is not acceptable if we want to claim that QED is really a "fundamental" theory. Hence some reformulation of QED was necessary. That point in time was the beginning of renormalization theory[2]. Since then, many efforts have been made to try and give to RQFT a solid mathematical foundation which would resolve the appearance of such infinite quantities. Instead of focusing on a specific model, which could be a priori very hard to define and would inevitably have some ad hoc features, one could formulate a set of general axioms motivated from physical intuition and hindsight. These axiom would define the scope of a well defined theory of relativistic quantum fields. The problem then remains of finding models which satisfy those axioms. Such an approach is usually referred to as the **axiomatic approach** to QFT.

Wightman axiomatization. One of the early attempts at the axiomatization of RQFT was proposed by Wightman [139] (a recent review can be found in [128], moreover let us cite [124, 122]). This approach starts from a set of physically motivated axioms, called *Wightman axioms*, for a system of quantum fields. This set of axioms has been proved (cf. [139]) to be equivalent to a set of properties regarding a denumerable set of relativistically covariant, analytic functions in an increasing number of variables, called Wightman functions. As a result, the problem of finding a system of fields fulfilling the set of axioms can be converted into the problem of finding Wightman functions which satisfy the corresponding set of properties.

One basic axiom expresses the property (called positivity) of getting a suitable Hilbert space associated with the functions. The program of constructing Wightman functions satisfying all the axioms is sometimes called *nonlinear program*, the same leaving out the positivity condition is sometimes called *linear program*.

The Wightman axioms are satisfied by *free* (i.e. non interacting), relativistic quantum fields that are characterized by a strictly positive mass quantum number and an arbitrary (integer or half-integer) value of the intrinsic spin quantum number. The problem of finding non-free models satisfying these axioms remains open and it is at the core of the mathematical research regarding QFT.

Perhaps the most fruitful attempt at generalizing the Wightman axioms to quantum fields with *zero* mass has its origin in [127, 126] (for further reference and comments cf. [125]).

Wightman approach considers the analytic continuation of distributions, the Fourier transform of which is supported on some set contained in the solid light cone (meaning the interior of the light cone united to its boundary). In this way on obtains functions, called **Wightman functions** which are well defined an analytic in some complex domain.

Other directions have been investigated to axiomatize RQFT. We devote our attention to the Wightman approach because is tightly linked with Euclidean quantum field theory to be described below. Another axiomatization is the Haag-Kastler one, based on the theory of operator algebras. We limit ourselves to mentioning the references [58, 10, 12] since it is only marginally related to the approach we take in this work.

Euclidean quantum field theory (EQFT). Another direction in the development of QFT begun with the work of Schwinger [120, 119] and Nakano [94] who defined in physical terms the notion of Euclidean quantum field theory (and in particular Euclidean quantum electrodynamics). The main point proposed by those authors is to analytically "rotate" the relativistic time parameter which appears in the heuristic formulation of QFT to an imaginary time (the rotation is currently generally called "Wick rotation"). Schwinger shows that such a procedure is quite natural and, at least at the formal level, the original QFT and the "rotated" one are equivalent. Moreover the "rotated" theory is formally invariant (respectively covariant) under the Euclidean group in four dimensions whenever the original theory was invariant (respectively covariant) under the Poincaré group. For this reason the theory which originates from this approach is called *Euclidean quantum field theory* (EQFT). The point of view of going to this "Euclidean

[2] For an early perspective on the problem of renormalization see e.g. [135, 136]; a very informative short commentary about the early efforts of renormalization and the relation with the problems in the classical (non quantum) theory is given in [117, chapters 15-16]; a recent thorough historical discussion of renormalization theory is given by [22]. Finally, for a critical modern perspective on renormalization, from a phenomenological point of view, cf. [134, Section 12.3].

setting" seems to be particularly fruitful especially in the case of scalar fields. Indeed, Symanzik [130, 129] realized that the theory for Euclidean scalar fields is related to the theory of particular Euclidean random fields. This realization opened a new direction of investigation and attempts at axiomatization. Nelson, in his germinal work [98, 95], proposed a set of axioms for Euclidean, Markovian (in the sense of Nelson), random fields. These axioms, when satisfied, guarantee (via a procedure of analytic continuation) the existence of a well defined scalar relativistic quantum field (i.e. satisfying the Wightman axioms). A weaker set of axioms (where the character of random fields was abandoned) was later proposed by Osterwalder and Schrader in [105, 106] (see also [51]) who formulated axioms in terms of Euclidean analogues of the Wightman functions (these analogues are called Schwinger functions). From these Schwinger functions one gets the Wightman functions and vice versa, see also Zinoviev [142] for a complete proof (cf. [123] for a detailed review of the different set of axioms, moreover we cite [52, 31]).

New work in progress. At the moment, new mathematical methods are being developed which one one hand interpret in a new light the ideas from the old QFT (in particular but not restricted to EQFT) and on the other introduce innovative ideas which are leading to an overall renewed hope in succeeding in the old goad of giving QFT, RQFT, and EQFT a solid mathematical foundation. We give a very incomplete list of reference to delineate some of the present line of research which we find particularly stimulating [59, 60, 6, 4, 56, 54, 13, 55, 53].

2 Motivation and some of the existing results

The need of different approaches to quantum mechanics. Perhaps because of the novelty of the ideas which lead to quantum mechanics and quantum field theory, there has been in the physical and mathematical communities a strong desire to develop different approaches for describing nature at the microscopical level.

Schrödinger "wave mechanics" and Heisenberg "matrix mechanics" were proved by Schrödinger to be in fact not different alternatives. Nevertheless they do carry a different flavor, the former more analytical and the latter more algebraic. Among the "classical" alternative approaches to quantum mechanics, we cite the Feynamn path integral approach [38][3] and Bohmian mechanics[4] [30, 19, 20]. Another, perhaps less well known, alternative approach, is Nelson's stochastic mechanics [96] (cf. also below), where he proposes a model for quantum mechanics which takes probability theory, instead of Hilbert space techniques, as basic mathematical foundation of the theory.

One reason of wishing for alternative formulations resides perhaps in the intrinsic philosophical difficulty in accepting the novelty of the ideas which lead to quantum mechanics and quantum field theory.

Another, maybe stronger, justification lies in the fact that quantum mechanics in not felt as an independent theory, as perhaps one could say of general relativity. Quantum mechanics in a sense depends upon a classical picture that then is "quantized". This procedure of going from a classical intuition to a quantum model is perhaps, philosophically, quite unsatisfactory. Indeed, quantum mechanics should describe the world at a scale where classical mechanics is known to fail. Our, perhaps naive, opinion is that the tools from classical mechanics, rather than classical mechanics itself, are very malleable and capture and display some properties in a very efficient way. This is what makes, at least for us, the applicability of classical mechanical techniques to QFT particularly desirable.

The appearance of divergences in quantum field theory, both in relativistic quantum field theory and statistical quantum field theory, fuels once again the question whether the quantum theory is insufficient and incomplete. Henceforth one feels even strongly the need for alternative approaches.

The Bosonic case. We can describe Bosonic quantum field theories within different formulations each of them carrying different useful insights on the problem.

[3]There exists a large literature regarding mathematical, rigorous treatment of Feynman path integral. We restrict ourselves only to the following references: [5, 91, 3, 62, 76]. A more complete list can be found e.g. in [5] (cf. also [7]).

[4]We mention: [11] which includes an historical account about the role of de Broglie in the development of Bohmian mechanics; [36, 37] for a rigorous mathematical and physical perspective, and [66] for further Physical perspectives.

The parallel with classical mechanics leads to the idea that free Bosonic quantum fields are described by an infinite system of quantum harmonic oscillators.

The parallel between relativistic QFT and Euclidean QFT gives a new interpretation of free Bosonic quantum fields. Free Bosonic quantum fields, from the probabilistic EQFT perspective, are described by infinite random variables (random fields) which are independent and Gaussianly distributed. The description of free Bosonic quantum fields in terms of Gaussian random variables is, we believe, particularly satisfactory. Indeed, one could interpret this Gaussian distribution as resulting from very many effects at lower scales. These effect being unknown could, at first order, be described as giving random independent contributions. Summing all those contributions would lead, with a naive application of the central limit theorem, to a Gaussian distribution. Hence free Bosonic quantum fields could be considered as a first order approximation of a collectively phenomenon of effects at lower scales.

The Fermionic case. It is of course desirable to obtain, in the case of free Fermionic quantum fields, a similar level of understanding to the one present for free Bosonic quantum fields. We give three justifications.

Physically, Fermions are just as fundamental as Bosons. Indeed all known matter is composed by Fermions (quarks, electrons, etc...).

Mathematically, one believes that accounting for Fermions should improve the rate of summability of the formal asymptotic series the non convergence of which constitutes the basic obstruction to a mathematically rigorous definition of QFT.

A third, perhaps more technical, reason is found in the fact that Lagrangian and more general variational techniques, which are very powerful in the context of Bosonic field theories, are difficult to generalize to Fermionic fields. Indeed, because of the intrinsic anticommuting character of Fermions, attempts to generalize these techniques leads often to merely formal manipulations. On the problem of Lagrangian formulation and Fermion fields cf. e.g. [79]. At the heuristic level, the Lagrangian formulation is usually employed in combination with Feynman path integration, either in the Minkowski or Euclidean setting, to "quantize" a field theory. In case of certain Bosonic field theories this procedure of "quantization" can be made fully rigorous. There exists a parallel approach in the case of Fermions which also in some cases can be made fully rigorous. One has to employ in place of Feynman path integral the notion of **Berezin integral** (cf. [18, 17] and for an instructive application of these techniques in the context of constructive renormalization and supersymmetry cf. [33]. Regarding Fermionic integration and supersymmetry we would like to mention the recent result by Gñeysu and Ludewig [57]). The Berezin integration is a powerful tool when introduced rigorously. Nevertheless, in the context of Berezin integration the Lagrangian and variational techniques are lost to some extent. The reason is that the variables, e.g. of the Lagrangian, need now to *anticommute*. It is therefore difficult to recover in this anticommutative context the analysis which is familiar in the standard commutative setting.

Regardless of the personal motivation, mainly the physical but also the mathematical community has produced, in almost a century of history, a huge literature in the effort of obtaining such a level of understanding. In spite of such efforts, a completely satisfactory answer to such a problem can be considered to be still open.

Existing results. Beyond the standard, perhaps complete and satisfactory, algebraic description, many other partial interpretation of Fermions have been given.

One possible direction is to find a classical analog to Fermions in a similar way as one can look at Bosons, classically, as harmonic oscillators. Intuitively spin is associated to a classical system which rotates, for example a **spinning top**. This parallel is well known at least for a single *non-relativistic* Fermion, that is a non-relativistic $1/2$-spin particle[5]. Early work in this direction includes: [21, 113]. This parallel can be considered the starting point of subsequent research: [116] (cf. also [115]), [15, 16, 70]. A generalization to n-Fermion states is given in [41] and Chapter II of this thesis. We feel these techniques to be very promising because they give a simple intuitive classical picture to the algebraic notion of Fermions

[5] In non relativistic physics we cannot use Wigner's definition of "elementary particle". In the non relativistic context, *particle* will just mean the solution of a Schrödinger type equation.

as anticommuting objects. In particular, because they come from a classical picture, they allow for a Lagrangian treatment, a feature which is very desirable in the case of quantum fields. Unfortunately this description of Fermions is, for the time being and to our knowledge, limited to *non-relativistic* systems.

Another direction of research can be considered as originating from what is now known as **Feynman checkerboard model** (cf. [115, pp. 367-380]). The idea is particularly stimulating because it uses a stochastic jump process which in some sense would describe a random "switching" up and down of the spin of the particle. Feynman original model works in $1 + 1$-dimensions. There exists generalizations to higher dimensions. Perhaps the state of the art in this stochastic, **jump process**, description of Fermionic degrees of freedom is reached in [29, 27, 28] [8]. Another, related direction, is the use of **determinantal point-processes**. In this regard we would like to mention [89] and [131, 132]. Citing only these few references, we cannot give justice to the development of these stochastic ideas in the description of Fermions. We point out two limitations which, to our knowledge, are common among the approaches which use jump processes to describe Fermions. First, they do not directly provide an intuitive picture. The jump process does give a notion of "anticommutativity" but the process itself is in a sense artificial. Moreover, so far, these methods have been successful only in the *non-relativistic case*.

A third direction originates from the work of Jordan and Wigner [71]. They describe a way to embed the Fermionic Fock space[6] into the Bosonic one. This embedding is an isometry and preserves the grading, that is a state describing n Fermions is mapped to a state of n Bosons. This embedding procedure is now called the **Jordan-Wigner transformation** and has many applications. For example it is the tightly linked to what is usually called the **Boson-Fermion correspondence** (starting perhaps with the work by Coleman [23]) which has applications in PDE theory (e.g. Korteweg–de Vries equation), representation theory, low-dimensional conformal field theory and string theory (e.g. affine Lie algebras, Kac-Moody algebras) (cf. [93, 73, 109, 92, 67]) This Boson-Fermions correspondence gives an equivalence between a Fermionic model, for example the Thirring model and a Bosonic one, for example the sine-Gordon model. This correspondence is unfortunately mainly restricted to *models in* $1 + 1$*-dimensions*.

Jordan-Wigner transformation is defined for a discrete system, that is for a Fock space built from a 1-particle Hilbert space like $\ell^2(\mathbb{N})$. **Friedrichs** [39, 40] gave a generalization of this embedding to the case where the 1-particle space is a space like $L^2(\mathbb{R}^d)$. A remarkable fact, is that in this case the embedding becomes an *isomorphism*. Among the works in physics which employ Friedrichs' idea, we mention Klauder [75], many works by Garbaczewski among which we cite [44, 42, 50, 45, 46, 47, 48, 49, 43], related works by Dobaczewski [34, 35] (and Holland [65]), the works in Mathematics by Parthasarathy et al. [68, 69, 108, 88, 107], Accardi [1], Applebaum [9], Kopietz [77], and the work by Kupsch[7] [84, 83, 85, 81, 82] and Lehmann [87]. We discuss further these ideas in Chapter IV. Friedrichs' isomorphism has the advantage of being ab initio suited to the *relativistic* (as well as the *Euclidean*) case *in any dimension*. The drawback resides in the fact that such an isomorphism is to some extent artificial.

We mention the approach by Laidlaw and DeWitt in [86] and rediscovered by Nelson [99] within the context of his stochastic mechanics (cf. also [101] where a short list or errata for [99] is given and [97, 100] for reviews). This approach uses the idea that the configuration space of unordered n-tuples $\{x_1, \dots, x_n\}$, $x_j \in \mathbb{R}^d$, $j = 1, \dots, n$, is a *non-simply-connected* differentiable manifold. The property of being non-simply-connected would, in this approach, be the origin of the different statistics (Bose-Einstein, Fermi-Dirac, and para statistics). We remark that this formulation is related to the approach in terms of determinantal processes mentioned above.

A final approach we would like to mention is the one in [24, 25] This approach considers expectations with respect to a Poisson process on a topological groups. It connects the Feynman-Kac formula for the time evolution of a non-relativistic system to unitary projective representations of a topological group acting on a topological space equipped with a quasi-invariant measure. The perspective of relating the construction of (interacting) quantum fields and with the representation theory of Abelian topological groups is also investigated in [2] in connection with interacting Bosonic fields. In [24] the authors show

[6]For the definition of Bosonic and Fermionic Fock spaces cf. [26] and e.g. [110]

[7]In particular, for ease of reference, we point out [81] where Kupsch' approach is compared with the probabilistic approach of Meyer, Parthasaraty, et al.

that, by choosing a special topological group, one can describe with this method quantum spin systems[8] on a lattice as well as more general Fermi systems.

3 Main results obtained and structure of the thesis

The content of the other Chapters of this Thesis is as follows.

Chapter II. In Chapter II we discuss the relation between finite dimensional Fermion systems and stochastic diffusion processes on the spin group. Finite dimensional Fermion systems are described by elements of an exterior algebra over an n-dimensional complex space. Such Fermions are by definition spinless but possess the characterizing property of relativistic Fermions in as much as the states describing them belong to an exterior algebra.

These states can be embedded in the space $L^2(\mathbf{Spin}(2n+1))$. Under this embedding the creation and annihilation operators of the Fermions are lifted to left invariant vector fields on the Lie group $\mathbf{Spin}(2n+1)$.

We prove that the time evolution of the Fermionic system can be described in terms of a stochastic process with a well defined second order, positive, selfadjoint generator. In fact we describe the time evolution as a Feynman-Kac type formula with respect to that stochastic process, the perturbation being described by a first order complex valued operator.

We find a description which is quite similar to the case of the evolution generated by the Laplacian perturbed by a scalar potential in which the solution is described by a Feynman-Kac formula with respect to a Brownian motion.

The relation between the Fermionic Fock space and the Lie group $\mathbf{Spin}(2n+1)$ is standard (cf. [41]) even though, perhaps, not very well known (let us remark, apologetically, that we did not know such a relation at the time when we wrote the content in this chapter). In this regard we hope to have clarified that the correct relation is between the Fermionic Fock space and $\mathbf{Spin}(2n+1)$ and *not* $\mathbf{SO}(2n+1)$. The confusion between $\mathbf{Spin}(n)$ and $\mathbf{SO}(n)$ is unfortunately still present in the physical literature (a related problem is found in [141] where the term 3-sphere is employed to mean the 2-sphere).

The analysis of how and to what extent one can use this correspondence to give a stochastic description of Fermions is new. The main results are in Theorem §**4.13** and Theorem §**5.6**. Standard functional analytical and stochastic methods are employed in our analysis.

Chapter III. In Chapter III we study the Schwinger (reduced, two-point) function for the Euclidean Dirac field (in $3+1$ dimensions). We derive the Schwinger function from first principles. In doing so we give a detailed review of standard Wigner-Mackey theory of induced representations applied to the universal cover of the Poincaré group (which is sometimes also called the Poincaré spinor group).

Moreover, we discuss some technical points. In particular those related to the representation of reflections (in particular the parity transformation) within the *full* Poincaré group and its different universal covers.

In our analysis we derive the Schwinger function in a natural, canonical way up to the final step. The final step corresponds to the choice of embedding of the Schwinger function into the Clifford algebra $\mathbb{C}\ell(4,\mathbb{C})$. The final Euclidean 2-point function, as given by a bilinear form, depends upon this choice in a relevant way.

Most of the chapter is to be understood as a detailed review of standard material which will be needed in the rest of the thesis. There are numerous little glitches in the literature which we hope to have clarified in our presentation. To our knowledge, the way we introduce Wightman and Schwinger functions from the group theory fills a gap in the literature. The main results are the constructions summarized in Theorem §**5.13** and Theorem §**5.28**. The construction explained in section 4 will be used in Chapter V.

Chapter IV. In a paper by Kupsch the n-point function of Euclidean Dirac field is given as expectation of a certain function of complex Gaussian random fields. In Chapter IV we give a simplification of Kupsch' result in three different directions.

[8]Other approaches suitable for spin systems which could allow for a generalization to relativistic fields include [63, 64], [90]

1. We avoid the artificial doubling of spinor fields;
2. We use the chaos decomposition of complex Gaussian random fields;
3. We employ Friedrichs' idea to obtain an isomorphism between the Fermionic Fock space with the Bosonic Fock space over the same 1-particle space.

Our approach permits to treat the antisymmetric property of Fermions and the non-positivity of the 2-point function from a more unified perspective. Moreover, we believe, it is better suited for future generalizations where non-Gaussian random fields will be considered.

The main result is Theorem §**5.10** and its Corollary which describe the Euclidean n-point functions for the Euclidean Dirac field in term of the complex Gaussian random fields mentioned above. Theorem §**5.10** and the discussion leading to it constitute the simplified (in the sense mentioned above) version of the result by Kupsch mentioned above.

Chapter V. In Chapter 5. we give a novel Bosonic realization for the free Dirac fields in $3+1$ dimensions. We solve two problems that in general arise when attempting a Bosonic representation of Fermions:

1. The Fermionic 2-point Schwinger function is *not* positive definite (whereas the 2-point function for relativistic Bosons is positive definite);
2. The Fermionic Fock space consists of states which are antisymmetric (whereas the Bosonic Fock space consists of symmetric states).

The solution we propose solves both problems at once. The main point, in our construction, is to consider, in place of the real Poincaré group, its complexification. Such a complexification arises already in the context of Wightman functions independently of the aim of associating Euclidean functions to Wightman functions.

The representation theory of the complexified Poincaré group leads to 2-point functions which are positive definite. Moreover one can recover, as two limits, the standard Wightman, respectively Schwinger, 2-point functions.

The 1-particle space $\mathscr{H}$ for such half-spin representations of the complexified Poincaré spinor group splits as a tensor product of two (infinite dimensional) Hilbert spaces: $\mathscr{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We therefore, correspondingly, construct a Bosonic Fock space over this 1-particle space. Finally, because of the tensor product structure of this 1-particle space we obtain that the Bosonic Fock space over this space includes naturally the Fermionic Fock space for Dirac Fields.

The representation theory of the complexified Poincaré spin group and its application to Physical problems is not new (cf. [112]). Nor the complexified spacetime is a new concept (cf. the whole theory of Wightman functions and [74]). The proposal of employing a complexified spacetime and the representation of the complexified Poincaré group to give to Dirac Fermions an Euclidean (as well as relativistic) covariant 2-point function which is *positive definite* is new.

Similarly, the Bosonic Fock space over the space of "infinite matrices" (Hilbert-Schmidt operators) nor is its decomposition into irreducible representations of the symmetric group are new (cf. e.g. [104]). The application of this decomposition to obtain a *Bosonic* Fock space to describe *Fermions* is new. Moreover the fact that the positive definite 2-point described above leads naturally and canonically to this Bosonic Fock space for Fermions is new an perhaps the relevant part in our proposal.

These ideas should be understood in the context of our wider program which aims at bringing together standard results from infinite dimensional global analysis (cf. e.g. [80]) and perhaps less standard results from infinite dimensional abstract harmonic analysis (we have in mind e.g. [78]) to develop new models, for the description of Fermions, which are both natural and amenable from the perspective of standard probability theory on infinite dimensional spaces.

The main results of this chapter are contained in the propositions §**5.3** and §**5.8** (cf. also the remark following it) where we summarize the main points of the construction, and in §**5.10** where we describe the Bosonic Fock space realization and the procedure of recovering the standard Fermionic Fock space from it.

Chapter VI. In all the chapters before the last one we concentrated on problems related to the description of Fermion systems and Dirac fields, in terms of associated Bosonic and probabilistic methods. The last chapter of this Thesis is dedicated to a problem concerning spin-1 Euclidean gauge fields, hence Bosonic (Euclidean) particles.

In the final goal of describing fields interacting with matter, one would like to consider coupled systems of Bosonic spin-1 particles interacting with Dirac fields. (this is the case, e.g. in QED, or more generally in Yang-Mills theories). The goal in this chapter is to define a rigorous version of the Faddeev-Popov procedure which is a quantization procedure well suited to gauge theories.

The general formal idea of this procedure is to introduce in the Feynman type path integral a density which depends on the gauge degrees of freedom. This allows, at least formally, to develop a perturbative analysis of gauge theories.

The mathematical implementation of this technique presents problems that are wide open in the case of the Yang-Mills fields.

This chapter have three objectives.

First, we give a review of important (fairly) recent developments regarding analytic aspects related to the topology and geometry of the gauge orbit space.

Second, we describe a general procedure of quantization which we call "naive Faddeev-Popov quantization". This procedure is fully rigorous. It is "naive" in the sense that it applies to theories which do *not* require any renormalization (this nevertheless includes models with gauge invariant regularizations, before such regularizations are removed).

Third, we specialize to the Maxwell field. There we give two procedures of quantization. First we quantize the free Maxwell field (which we also call radiation field) by taking quotient of the phase space by the gauge action. This is possible to do for the Maxwell field because the gauge symmetry in this case is Abelian. Second we apply our "naive Faddeev-Popov quantization" which in principle would be applicable to more general gauge theories. In this case we analyze the situation thoroughly explaining how infrared and ultraviolet regularizations can be introduced and then removed.

The main results are contained in the Theorems §**7.15**, §**7.18**, and §**7.19**. These results employ somewhat technical results from Hodge-Friedrichs-Kodaira theory together with Hilbert-space methods, probabilistic methods, and PDE methods. Each of these techniques are standard (at the very least within its own community). We employ these techniques to show that it is possible to make the ideas by Faddeev and Popov perfectly rigorous when taken independently from the (huge) *separate* problem of renormalizing gauge theories (in four dimensions). We consider as the natural next step the application of our approach of a rigorous, non-perturbative, Faddeev-Popov quantization to models where renormalization theory and geometry play a non-trivial role. This would put our approach to the test to see whether it can lead to some simplification in the rigorous definition of such models.

Each chapter is written with the objective of being completely independent from the others, at the cost of some unavoidable redundancy. We include a separate abstract, introduction, and list of references in every chapter. We hope that this will improve the readability. For convenience we also include the complete list of references cited in this thesis at the very end.

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II

Finite dimensional Fermions and stochastic processes on the spin group

Abstract

In this chapter we consider a "finite system of Fermions" represented by an element of the exterior algebra of the n-dimensional complex space. The Fermions are spinless but possess the characterizing anticommutativity property. We associate invariant vector fields on the Lie group $\mathbf{Spin}(2n+1)$ to the Fermionic creation and annihilation operators. These vector fields implement the regular representation of the corresponding Lie algebra $\mathfrak{so}(2n+1)$. As such, they do not satisfy the canonical anti-commutation relations in general, however once they have been projected onto an appropriate subspace of $L^2(\mathbf{Spin}(2n+1))$ these relations are satisfied. We define a time evolution of the system of Fermions in terms of a symmetric positive-definite quadratic form in the creation-annihilation operators. The realization of Fermionic creation and annihilation operators brought by the (invariant) vector fields allows us to interpret this time evolution in terms of a positive selfadjoint operator which is the sum of a second order operator which generates a stochastic process and a first order, complex valued operator which strongly commutes with the second order operator. A probabilistic interpretation is give in terms of a stochastic process associated with the second order operator.

Contents

1 Introduction

Probabilistic methods in Quantum Field Theory have proved to be particularly fruitful (cf. e.g. [37], [14], [17]). These methods have been almost exclusively restricted to *Bosonic* Field Theories. Some ideas of the Bosonic probabilistic methods carry over, to an extent, to the Fermionic case using the beautiful algebraic technique of Berezin integration [4]. However, the Berezin integral, being defined in terms of Grassmann variables, does not lend itself easily to an interpretation in the context of probability theory or measure theory, but cf. [7] on this regard.

The results in this chapter are inspired in part by the work of Schulman [36] (cf. also [35, Chapters 22-24]) who gives a description of a single $\frac{1}{2}$-spin particle in terms of the Feynman path integral. The precedents for Schulman's idea can be found in early work on Quantum Mechanics connecting the Pauli $\frac{1}{2}$-spin formalism with the *quantum spinning top* [5, 33] (cf. also the more recent work [3]).

During the final drafting of this paper we have found a setting similar to ours in [12] and [9] (cf. also [8, 10, 11, 24, 23, 26, 27, 25]). The motivations and aims of these papers are however different from those in the present manuscript. The idea presented in the references quoted above is, however, similar the starting point in our construction. Given a system of $2n$ anticommuting creation-annihilation operators, the idea consists in establishing an association between each anticommuting creation-annihilation operator and a corresponding element in the complexified Lie algebra $\mathfrak{so}_{\mathbb{C}}(2n+1)$. In turn the elements of the complexified Lie algebra are interpreted as some differential operators in an L^2 space. As we detail in section 2, the notion of the $1/2$-spin representation of the Lie algebra $\mathfrak{so}_{\mathbb{C}}(2n+1)$ is by definition realized as a representation on the exterior algebra $\bigwedge \mathbb{C}^n$. The irreducibility of such a representation then implies that the Fermionic algebra of anticommuting creation-annihilation operators is generated by the elements of $\mathfrak{so}_{\mathbb{C}}(2n+1)$ represented, according to the $1/2$-spin representation, as matrices acting on $\bigwedge \mathbb{C}^n$. In section 3 we then describe the association to the elements of the Lie algebra to corresponding differential operators.

The difference of our result with respect to the one cited above is the following. Here we associate to the Fermionic Hamiltonian a new operator, which we call *quasi-Hamiltonian*. This quasi-Hamiltonian differs from the one proposed in the literature above by having also a second order part (in addition to the first order part which also appears in the above publications). This difference stems from the fact that our motivation is to study the (Wick rotated) time evolution of the original Fermionic system in terms of a stochastic process in some probability space. This second order part is therefore crucial to our analysis, since it constitutes the generator of our stochastic process of diffusion type. This difference implies that the operator which is the objective of our investigation is very different and needed a separate analysis. Finally, our presentation devotes as much attention as possible to the rigorous mathematical derivation of the results. It provides, in our opinion, deeper physical understanding of the problem and offers the possibility of further generalizations, especially to infinite dimensional settings related to quantum electrodynamics.

General motivation

Our motivation originates, in fact, from quantum electrodynamics (QED). In particular we wish to investigate the possible ways in which one can associate to a relativistic Dirac quantum field a stochastic process in some probability space.

For an interacting scalar Bosonic quantum fields this association can be obtained following, for example, the following procedure.

1. Consider first the free Bosonic system associated to the original interacting one. One can prove that the Bosonic Fock space is isomorphic to a probability space $L^2(\mathcal{N}, \mu_0)$, where $\mathcal{N}$ is the dual of a nuclear space and μ_0 is a probability measure.

2. Continue the free time evolution of the quantum states, given by a quantum Hamiltonian H, to imaginary (Euclidean) time (Wick rotation) and associate H to a generator $-\mathcal{L}$ of a stochastic process acting on the probability space $L^2(\mathcal{N}, \mu_0)$.

3. The description of the (Wick rotated) free time evolution in terms of a stochastic process, whenever possible, allows for the application of powerful stochastic techniques to the original interacting quantum system.

For example the Feynman path integral is sometimes difficult to rigorously define and apply to the original relativistic quantum theory (see, however [1, Chapter 9]). On the other hand, in the stochastic description the integral in the space of paths is often a well defined measure-theoretic object. Moreover, this stochastic description leads, in addition to powerful estimates, useful for constructing models of relativistic quantum fields, also to a more intuitive picture of the models under consideration because offers a bridge between the theory of relativistic quantum fields and quantum statistical physics (see, e.g., [37] [14][17]).

This procedure is however not easily generalizable to *Fermionic* relativistic quantum fields.

To the knowledge of the author a probabilistic description of Fermionic relativistic quantum fields, parallel to the one explained here for Bosonic fields, is still missing. We are here using the term “probabilistic” in a “strict sense”, that is not including, e.g., non commutative probability.

Motivation for considering a “finite system of Fermions without spin”

In describing the Dirac Fermions one usually (c.f. [40, Chapter 10]) introduces a Fock space

$$\mathcal{F}_{\text{Dirac}}(\mathfrak{H}_+,\mathfrak{H}_-) = \bigoplus_{n=0}^{\infty}(\mathfrak{H}_+)^{\wedge n} \otimes \bigoplus_{m=0}^{\infty}(C\mathfrak{H}_-)^{\wedge m}$$

where $\mathfrak{H}_+, \mathfrak{H}_-$ are respectively the one-particle Hilbert space and one-antiparticle Hilbert space and C denotes the operator of charge conjugation. Alongside this Fock space one usually introduces a *CAR*[1] *algebra* of creation-annihilation operators for particles and antiparticles respectively $a^*(f), a(f), b^*(g), b(g)$, $f \in \mathfrak{H}_+, g \in \mathfrak{H}_-$. The definition of these operators, can be found in [40, Chapter 10], cf. also e.g. [28].

Finally, the dynamics of the free Dirac fields is prescribed by the Hamiltonian (which we write in the familiar physicists notation cf. e.g. [41, Equation (7.5.52), Chapter 7, §7.5, p.325])

$$H = \int_{\mathbb{R}^3} \omega(\mathbf{p}) \sum_{\sigma=1,2} \left(a^\dagger_\sigma(\mathbf{p})a_\sigma(\mathbf{p}) + b^\dagger_\sigma(\mathbf{p})b_\sigma(\mathbf{p})\right) \,\mathrm{d}\mathbf{p},$$

where $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$.

We shall consider here a radical simplification of this model. We will consider a finite dimensional Fock space of particles without spin (and no antiparticles):

$$\mathcal{F}_{\text{anti-symmetric}}(\mathbb{C}^n) := \bigwedge \mathbb{C}^n := \bigoplus_{k=0}^{\infty}(\mathbb{C}^n)^{\wedge k},$$

that is the Fermionic Fock space in this context is taken to be just the exterior algebra of $\mathbb{C}^n$. This Fermionic Fock space can be thought of as describing a system of spinless Fermions, where each of the Fermions, once created can occupy one of n possible states.

We introduce a finite dimensional CAR algebra in terms of creation-annihilation operators $c^\dagger_k, c_k$, $k = 1, \dots, n$ and in analogy with the Hamiltonian associated with the Dirac field we consider as Hamiltonian an operator on $\bigwedge \mathbb{C}^n$

$$\widetilde{H}_{\underline{E}} = \sum_{k=1}^{n} E_k c^\dagger_k c_k\,, \tag{II.1}$$

parametrized by $\underline{E} = 0 \leq E_1 \leq \cdots \leq E_n$. The set $\underline{E}$ is thought of as the collection of energy eigenstates and takes the role, in this simplified model, of $\omega(\mathbf{p})$ in the theory of Dirac fields.

We can write down the Schrödinger equation for this finite dimensional Fermionic model as

$$i\partial_t\Psi_t = H\Psi_t\,, \quad \Psi_t \in \bigwedge \mathbb{C}^n, t \in \mathbb{R}.$$

We are interested in the "Euclidean" version of this Schrödinger equation, namely

$$\partial_\tau\Psi_\tau = -H\Psi_\tau\,, \quad \Psi_\tau \in \bigwedge \mathbb{C}^n, \tau \geq 0$$

which corresponds to a Wick rotation of the time to the imaginary axis.

[1] CAR stays for canonical anticommutation relations. About CAR algebras cf. e.g. [6] .

Statement of the results

The results obtained here can be concisely phrased as follows.

The constant $1 \in \bigwedge \mathbb{C}^n$ is called the vacuum state in the Fermionic Fock space $\bigwedge \mathbb{C}^n$. The Fermionic creation-annihilation operators $c_k, c_k^\dagger \in \mathrm{End}(\bigwedge \mathbb{C}^n)$ satisfy the canonical anticommutation relations (CAR):

$$\{c_k, c_\ell^\dagger\} = \delta_{nm}, \quad \{c_k, c_\ell\} = \{c_k^\dagger, c_\ell\} = 0, \quad k, \ell = 1, \ldots, n. \tag{CAR}$$

We denote by $\mathbf{Spin}(2n+1)$ the real spin group in $2n+1$ dimensions, that is the universal cover of the real special orthogonal Lie group $\mathbf{SO}(2n+1)$ in $2n+1$ dimensions. The real Lie algebra associated with both of these Lie groups will be denoted by $\mathfrak{so}(2n+1)$. The space $L^2(\mathbf{Spin}(2n+1))$ will denote the space of *complex valued* functions which square modulus integrable with respect to the normalized Haar measure on $\mathbf{Spin}(2n+1)$. The space $C^\infty(\mathbf{Spin}(2n+1))$ will denote the space of *complex valued* smooth functions on the real spin group $\mathbf{Spin}(2n+1)$.

We associate to $c_k, c_k^\dagger$ elements of the complex Lie algebra $\mathfrak{so}_{\mathbb{C}}(2n+1)$ which is the complexification of the real Lie algebra $\mathfrak{so}(2n+1)$. This association is explained in section 2.

Then we interpret these elements as differential operators $D_k, D_k^\dagger$ on $L^2(\mathbf{Spin}(2n+1))$ with common domain given by $C^\infty(\mathbf{Spin}(2n+1))$.

We embed the Fermionic Fock space $\bigwedge \mathbb{C}^n$ into $L^2(\mathbf{Spin}(2n+1))$ as Hilbert spaces, and denote the image of this embedding by F_{Ψ_0}, of $L^2(\mathbf{Spin}(2n+1))$. This embedding is not canonical but is uniquely characterized as follows:

(A) $D_k, D_k^\dagger$ restricted to F_{Ψ_0} satisfy (CAR).

(B) There is a $\Psi_0 \in L^2(\mathbf{Spin}(2n+1))$ which corresponds to the image of $1 \in \bigwedge \mathbb{C}^n$ under this embedding.

The choice of this vector is restricted by the requirement (A). Different vectors which satisfy (A) give rise to equivalent embeddings. Peter-Weyl theorem gives a characterization of the equivalent embeddings. This point will be explained in section 3.

Much as we have associated $c_k^\dagger, c_k$ to $D_k, D_k^\dagger$, given an n-tuple $\underline{E} = (E_1, \ldots, E_n)$ of *strictly positive* numbers in non-decreasing order, we associate to a Fermionic Hamiltonian

$$\widetilde{H}_{\underline{E}} \stackrel{\text{def}}{=} \sum_{k=1}^{n} E_k c_k^\dagger c_k \ \in \mathrm{End}(\bigwedge \mathbb{C}^n)$$

an unbounded operator $H_{\underline{E}}$ in $L^2(\mathbf{Spin}(2n+1))$ given by

$$H_{\underline{E}} \stackrel{\text{def}}{=} \sum_{k=1}^{n} E_k D_k^\dagger D_k\,, \quad \mathrm{Dom}(H_{\underline{E}}) \stackrel{\text{def}}{=} C^\infty(\mathbf{Spin}(2n+1)).$$

The specific choice of $\underline{E}$ is immaterial. We therefore leave it free as a parametrization and sometimes refer to $H_{\underline{E}}$ as a *family of operators* implying "parametrized by $\underline{E}$". Some other times we refer to $H_{\underline{E}}$ as *quasi-Hamiltonian*. This practice should not lead to any confusion.

The choice of the family of operators $H_{\underline{E}}$, which we associate to the original Fermionic Hamiltonian $\widetilde{H}_{\underline{E}}$, is *not* unique. We make the choice based on two facts: (1) The restriction of $H_{\underline{E}}$ to F_{Ψ_0} coincides, up to isomorphism, with $\widetilde{H}_{\underline{E}}$ on $\bigwedge \mathbb{C}^n$; (2) The quasi-Hamiltonian is required to be a second order differential operator. We impose requirement (2) because we are interested in giving a stochastic process interpretation to the model we are constructing.

We are now almost ready to state the two main results of our analysis.

For the sake of definiteness, fix $X_{k,\ell}$, $k < \ell = 1, \ldots, 2n+1$ to be the standard basis of the Lie algebra $\mathfrak{so}_{\mathbb{C}}(2n+1)$ where $X_{k,\ell}$ are the matrices with elements $(X_{k,\ell})_{ij} = \delta_{ki}\delta_{\ell j}\,\mathrm{sgn}(i-j)$, $i, j = 1, \ldots, 2n+1$. As explained in details in section 3, we can regard the elements $X_{k,\ell}$ of the standard basis of $\mathfrak{so}_{\mathbb{C}}(2n+1)$ as right-invariant differential operators on $C^\infty(\mathbf{Spin}(2n+1)) \subset L^2(\mathbf{Spin}(2n+1))$.

Then the operators $D_k^\dagger$, D_k will be defined in section 4 as follows

$$D_k^\dagger \stackrel{\text{def}}{=} X_{2k-1,2n+1} + \mathrm{i}X_{2k,2n+1}\,, \quad D_k \stackrel{\text{def}}{=} X_{2k-1,2n+1} - \mathrm{i}X_{2k,2n+1}\,, \qquad k = 1,\dots,n\,,$$

In section 4 we define the quasi-Hamiltonian operator $H_{\underline{E}}$ and study its properties proving a number of results, the most important of which is the following:

Result I: Quasi-Hamiltonian (§4.13) The family of unbounded operators $H_{\underline{E}}$ defined on the domain $C^\infty(\mathbf{Spin}(2n+1))$ in $L^2(\mathbf{Spin}(2n+1))$ is a family of *essentially selfadjoint* operators. Moreover the quasi-Hamiltonian can be decomposed on $C^\infty(\mathbf{Spin}(2n+1))$ as

$$H_{\underline{E}} = \sum_{k=1}^{n} E_k L_k + \mathrm{i}\sum_{k=1}^{n} E_k T_k,$$

where $T_k \stackrel{\text{def}}{=} X_{2k-1,2k}$, and the operators $L_k \stackrel{\text{def}}{=} (X_{2k-1,2n+1})^2 + (X_{2k,2n+1})^2$, for $k = 1,\dots,n$, are positive definite and essentially selfadjoint on $C^\infty(\mathbf{Spin}(2n+1))$. The operators T_k, $k = 1,\dots,n$, are essentially-selfadjoint on $C^\infty(\mathbf{Spin}(2n+1))$ and their closure $\overline{T}_k$, $k = 1,\dots,n$, defines a family of strongly commuting unbounded operators in $L^2(\mathbf{Spin}(2n+1))$. Moreover $\overline{T}_k$ strongly commutes with $\overline{L}_\ell$, for any $k,\ell = 1,\dots,n$.

In section 5 we study how to associate to $H_{\underline{E}}$ a stochastic process. In that section we will write the quasi-Hamiltonian as

$$H_{\underline{E}} = P_0 + \mathrm{i}B_0,$$

where

$$P_0 \stackrel{\text{def}}{=} \sum_{k=1}^{n} E_k L_k, \quad B_0 \stackrel{\text{def}}{=} \sum_{k=1}^{n} E_k T_k,$$

are unbounded operators on $L^2(\mathbf{Spin}(2n+1))$ with common domain given by $C^\infty(\mathbf{Spin}(2n+1))$.

The following can be considered the most important result in that section.

Result II: Stochastic evolution (§5.6) The operator P_0 is essentially selfadjoint on the domain $C^\infty(\mathbf{Spin}(2n+1)) \subset L^2(\mathbf{Spin}(2n+1))$ and its closure $\overline{P_0}$ is the generator of a stochastic diffusion process on $\mathbf{Spin}(2n+1)$. Moreover, we have the following representation of the semigroup generated by $H_{\underline{E}}$

$$(f, e^{-t\overline{H_{\underline{E}}}} g)_{L^2(\mathbf{Spin}(2n+1))} = \mathbb{E}_X\left[\overline{f(0)}\left(e^{\mathrm{i}\,t\,\overline{B_0}} g\right)(X(t))\right], \tag{II.2}$$

where $\mathbb{E}_X$ denotes the expectation with respect to the process generated by P_0, $\overline{H_{\underline{E}}}$ and $\overline{B_0}$ denote the closure of the operators, $\overline{f(0)}$ denotes complex conjugation, and $f, g \in C(\mathbf{Spin}(2n+1)) \subset L^2(\mathbf{Spin}(2n+1))$.

2 Clifford algebra, exterior algebra, and the orthogonal Lie algebra

In this section we provide some preliminary definitions. In particular we define the half-integer representation of the Lie group $\mathbf{Spin}(2n+1)$.

§ 2.1 Clifford algebra. Let V be a finite dimensional vector space over $\mathbb{R}$, Q a real valued quadratic form on V, and $T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ the algebraic tensor algebra defined by V. Let $\mathscr{I}$ be the (two sided) ideal in $T(V)$ generated by elements of the form

$$v \otimes v + Q(v)1\,, \quad v \in V.$$

The **Clifford algebra** $C\ell(V,Q)$ is by definition the algebra given by taking $T(V)$ modulo $\mathscr{I}$, i.e.

$$C\ell(V,Q) := T(V)/\mathscr{I}.$$

We denote by $C\ell(n)$ the Clifford algebra $C\ell(\mathbb{R}^n, \|\cdot\|)$, where $\|\cdot\|$ is the norm on $\mathbb{R}^n$ given by the Euclidean quadratic form on $\mathbb{R}^n$, $n \in \mathbb{N}$.

We define the **complex Clifford algebra** $\mathbb{C}\ell(V,Q)$ to be the complexification of $C\ell(V,Q)$:

$$\mathbb{C}\ell(V,Q) := \mathbb{C} \otimes_{\mathbb{R}} C\ell(V,Q),$$

where $\otimes_{\mathbb{R}}$ denotes the tensor product over $\mathbb{R}$. Finally we denote by $\mathbb{C}\ell(n)$ the complexification of the Clifford algebra $C\ell(n)$, that is $\mathbb{C}\ell(n) \overset{\text{def}}{=} \mathbb{C} \otimes_{\mathbb{R}} C\ell(\mathbb{R}^n, \|\cdot\|) \cong C\ell(\mathbb{C}^n, \|\cdot\|_{\mathbb{C}^n})$, where $\|\cdot\|_{\mathbb{C}^n}$ denotes the norm in $\mathbb{C}$.

§ 2.2 Even and odd parts Consider the automorphism $\alpha : C\ell(V,Q) \to C\ell(V,Q)$ which on elements $v \in V$ acts by $\alpha(v) = -v$. Since $\alpha^2 = \mathrm{Id}$ there is a decomposition

$$C\ell(V,Q) = C\ell^{\,\text{even}}(V,Q) \oplus C\ell^{\,\text{odd}}(V,Q)$$

where

$$\begin{aligned} C\ell^{\,\text{even}}(V,Q) &:= \operatorname{span}\{v_1 \cdots v_{2k}, k \in \mathbb{N}, v_j \in V\}, \\ C\ell^{\,\text{odd}}(V,Q) &:= \operatorname{span}\{v_1 \cdots v_{2k+1}, k \in \mathbb{N}, v_j \in V\}. \end{aligned}$$

We shall call the elements of $C\ell^{\,\text{even}}(V,Q)$ and $C\ell^{\,\text{odd}}(V,Q)$ **even** and **odd elements** in $C\ell(V,Q)$ respectively.

Note that $C\ell^{\,\text{even}}(V,Q)$ is a subalgebra of $C\ell(V,Q)$. On the other hand $C\ell^{\,\text{odd}}(V,Q)$ is not an algebra because the multiplication of an even number of odd elements yields an even element.

We will use the shorthands $C\ell^{\,\text{even}}(k)$, $C\ell^{\,\text{even}}(k)$ when $V = \mathbb{R}^k$, $Q(v) = \|v\|^2$.

We now turn to exterior algebras and their relation with Clifford algebras.

§ 2.3 Exterior algebra Let V be a real or complex , finite dimensional, vector space. Let $\mathscr{I}_0$ be the (two sided) ideal in $T(V)$ generated by elements of the form

$$v \otimes v, \quad v \in V.$$

Then we define the **exterior algebra** of the vector space V to be

$$\bigwedge V := T(V)/\mathscr{I}_0,$$

where $T(V)$ is now respectively the real or complex tensor algebra of V.

§ 2.4 Proposition For any quadratic form Q on a finite dimensional vector space V over a field $\mathbb{K}$, the Clifford algebra $C\ell(V,Q)$ is *naturally isomorphic* (as a graded algebra) to the exterior algebra $\bigwedge V$.

> PROOF. A standard reference is [21, Chapter 1, Proposition 1.2]. A somewhat more extended proof can be found e.g. in [39, Chapter 11, Proposition 1.1] □

§ 2.5 Proposition We have the *algebra isomorphism* $\mathbb{C}\ell(2k) \cong \operatorname{End} \bigwedge \mathbb{C}^k$.

> PROOF. See, e.g., [13, Lemma 20.9, p.304] or [38, Proposition 43.1, p.183]. □

§ 2.6 Explicit example We clarify the connection between Clifford algebra and exterior algebra with a simple example.

Let $\gamma_1, \ldots, \gamma_{2n}$ be *complex* $2^{2n} \times 2^{2n}$ matrices which satisfy

$$\{\gamma_i, \gamma_j\} = -2\delta_{ij}\mathrm{Id}, \qquad i, j = 1, \ldots, 2n,$$

where $\{\cdot,\cdot\}$ stands for the anticommutator (i.e. $\{A,B\} \stackrel{\text{def}}{=} AB + BA$ for any finite dimensional matrices A, B). The matrices γ_i, $i = 1, \ldots, n$ generate $\mathbb{C}\ell(2n)$. If we now define[2]

$$c_j := \frac{1}{2}\left(\gamma_{2j} + i\gamma_{2j-1}\right), \quad c_j^* := \frac{1}{2}\left(\gamma_{2j} - i\gamma_{2j-1}\right), \qquad j = 1, \ldots, n,$$

then the matrices b_j, b_j^* satisfy the **canonical anti-commutation relations**

$$\{c_j, c_k\} = \{c_j^*, c_k^*\} = 0, \quad \{c_j, c_k^*\} = \delta_{jk}1, \qquad j,k = 1, \ldots n.$$

Finally note that c_j, c_k^* generate $\mathrm{End}(\bigwedge\mathbb{C}^n)$ as an algebra.

§ 2.7 Lemma Let $\mathfrak{so}_{\mathbb{C}}(2n+1)$ denote the complexification of the real Lie algebra $\mathfrak{so}(2n+1)$. Then we have the following chain of relations

$$\mathfrak{so}_{\mathbb{C}}(2n+1) \hookrightarrow \mathbb{C}\ell^{\text{ even}}(2n+1) \cong \mathbb{C}\ell(2n) \cong \mathrm{End}(\bigwedge\mathbb{C}^n)$$

where the first $\hookrightarrow$ is an embedding and a Lie algebra homomorphism and is *not* surjective (but only injective).

PROOF. The first embedding is proved in [15, Lemma 6.2.2, p. 313].

The first isomorphism $\cong$ is proved in [15, Lemma 6.1.7, p. 310] or can be deduced from [13, Lemma 20.16 p.306 + Lemma 20.9 p.304].

The second embedding is the $\mathbb{R}$-linear algebra homomorphism which extends to $C\ell(2n)$ the map which embeds $\mathbb{R}^{2n}$ into $\mathbb{C}^{2n}$.

The last isomorphism follows from the Proposition in §**2.5** (compare also [13, formula (20.18), p. 306]). □

§ 2.8 Definition (Half-spin representation) The chain of relations of the Lemma in §**2.7** gives rise to a representation by endomorphisms

$$\pi^{(1/2)} : \mathfrak{so}_{\mathbb{C}}(2n+1) \to \mathrm{End}(\bigwedge\mathbb{C}^n)$$

of the Lie algebra $\mathfrak{so}_{\mathbb{C}}(2n+1)$ into the carrier space $\bigwedge\mathbb{C}^n$. We shall call this representation the **half-spin representation**[3] of $\mathfrak{so}_{\mathbb{C}}(2n+1)$.

The terminology *half-spin representation* is justified by the following proposition.

§ 2.9 Proposition The half-spin representation $\pi^{(1/2)}$ in §**2.8** of $\mathfrak{so}_{\mathbb{C}}(2n+1)$ is *the* irreducible representation of $\mathfrak{so}_{\mathbb{C}}(2n+1)$ with highest weight

$$\lambda^{(1/2)} = (\underbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}_{n \text{ times}}).$$

[2]Note the similarity with creation annihilation operators for the (Bosonic) harmonic oscillator

$$a_j = \frac{x_j + ip_j}{\sqrt{2}}, \quad a_j^* = \frac{x_j - ip_j}{\sqrt{2}},$$

where $[a_j, a_k^*] = \delta_{jk}$ and $[x_j, p_k] = i\delta_{jk}$. This suggests the striking correspondence

$$\gamma_{2j} \rightsquigarrow \sqrt{2}\,x_j, \quad \gamma_{2j-1} \rightsquigarrow \sqrt{2}\,p_j$$

together with the following correspondence anticommutators $\rightsquigarrow$ commutators

$$\{\cdot,\cdot\} \rightsquigarrow i[\cdot,\cdot].$$

[3]Sometimes this representation is simply called the *spin representation*. We prefer, in our context, to use the more self-explanatory term *half-spin representation*.

A general weight λ of the representation $\pi^{(1/2)}$ is given by

$$\lambda = \underbrace{(\pm\tfrac{1}{2}, \dots, \pm\tfrac{1}{2})}_{n \text{ times}},$$

with all possible distributions of $\pm$ in the components.

Moreover the lowest weight of this representation is given by $-\lambda^{(1/2)}$. Finally the (normalized) *highest weight vector* is the constant $1 \in \bigwedge^0 \mathbb{C}^n$ and the (normalized) *lowest weight vector* is $e_1 \wedge \cdots \wedge e_n \in \bigwedge^n \mathbb{C}^n$.

PROOF. Cf. [13, §20.1, Proposition 20.20 p.307] or [15, Section 6.2.2, Proposition 6.2.4 p.315]. The statement about the highest/lowest weight *vectors* follows easily from the proofs of the cited propositions. □

§ 2.10 Remark The image $\pi^{(1/2)}(\mathfrak{so}_{\mathbb{C}}(2n+1))$ of $\mathfrak{so}_{\mathbb{C}}(2n+1)$ in $\mathrm{End}(\bigwedge \mathbb{C}^n)$ generates $\mathrm{End}(\bigwedge \mathbb{C}^n)$ as an algebra.

PROOF. This is a direct consequence of the fact that $\pi^{(1/2)}$ is an *irreducible* representation. □

We (finally) bring Fermions into the picture.

§ 2.11 (Finite dimensional) Fermionic Fock space In general given a Hilbert space $\mathscr{H}$ one calls Fermionic Fock space (or antisymmetric Fock space) the Hilbert space

$$\mathscr{F}_{\text{anti-symmetric}}(\mathscr{H}) \;:=\; \dot{\bigwedge}\, \mathscr{H} \;:=\; \bigoplus_{k=0}^{\infty} \mathscr{H}^{\dot{\wedge} k},$$

where the $k = 0$ term in the direct sum is just $\mathbb{C}$, and where the dot over the wedges denotes completion of the exterior product with respect to the usual scalar product (cf. [30] and [31]).

Here we are interested in the finite dimensional case, i.e. we are interested in a system of at most $n \in \mathbb{N}$ Fermions. So in our context the Fock space will consist of the exterior algebra (defined in 2.3) of the vector space $\mathbb{C}^n$:

$$\mathscr{F}_{\text{anti-symmetric}}(\mathbb{C}^n) \;:=\; \bigwedge \mathbb{C}^n \;:=\; \bigoplus_{k=0}^{n} (\mathbb{C}^n)^{\wedge k}.$$

We call the element $1 \in \mathbb{C} \hookrightarrow \bigwedge \mathbb{C}^n$ the **vacuum (state)**. We define the **Fermionic creation**, respectively **annihilation operators** $c_j^\dagger$, respectively c_k , with $j, k = 1, \dots, n$, as the *linear operators* mapping $\bigwedge \mathbb{C}^n$ into $\bigwedge \mathbb{C}^n$ which satisfy for every $j, k = 1, \dots, n$, the following conditions

(i) $c_j\, 1 = 0$, i.e. the c_j's annihilate the vacuum.

(ii) $(\omega_1, c_j \omega_2)_{\mathbb{C}^n} = (c_j^\dagger \omega_1, \omega_2)_{\mathbb{C}^n}$, for any $\omega_1, \omega_2 \in \mathbb{C}^n$, i.e. $c_j, c_j^\dagger$ are adjoint to one-another

(iii) $\{c_j, c_k\} = \{c_j^\dagger, c_k^\dagger\} = 0$, $\{c_j, c_k^\dagger\} = \delta_{jk}$ i.e. the canonical anti-commutation relations are satisfied.

3 Fermions and $L^2(\mathrm{Spin}(2n+1))$

In this section we connect the half-spin representation of $\mathfrak{so}_{\mathbb{C}}(2n+1)$, constructed by algebraic means in the previous section, with a more geometrical and analytical representation, i.e. the *regular* representation (to be defined below).

§ 3.1 $\mathfrak{so}(2n+1)$ as a Lie subalgebra of $\mathfrak{gl}(2n+1)$ We take as the *standard basis* for the Lie algebra $\mathfrak{gl}(2n+1)$ the matrices E_{ij}, $i, j = 1, \dots, 2n+1$ with matrix elements $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, $k, l = 1, \dots, 2n+1$, i.e. all entries vanish except for the (i, j)-th entry, which is equal to one.

Considering $\mathfrak{so}(2n+1)$ as a Lie subalgebra of $\mathfrak{gl}(2n+1)$, we can take as generators of $\mathfrak{so}(2n+1)$ the matrices

$$X_{ij} = E_{ij} - E_{ji}. \tag{II.3}$$

We also consider the Cartan-Weyl basis (adopting the notation in [29])

$$\begin{aligned} H_k &:= X_{2k-1,2k}\,, & k &= 1,\ldots,n \\ Q_{\pm k} &:= X_{2k-1,2n+1} \pm iX_{2k,2n+1}\,, & k &= 1,\ldots,n \\ \rho_{kh} &:= [Q_h, Q_k]\,, & h &\neq -k, \quad h,k = \pm 1,\ldots,\pm n\,. \end{aligned}$$

Note: H_k are the elements of the Cartan subalgebra. The $Q_{\pm k}$ and the ρ_{hk} are lowering-raising *generators* [4]. The $Q_{\pm k}$ connect the Lie algebra $\mathfrak{so}(2n+1)$ with its subalgebras while the ρ_{hk} operate within the subalgebras of $\mathfrak{so}(2n+1)$ ([29]).

The matrices H_k, Q_h satisfy the following commutation relations

$$\begin{aligned} [H_h, H_k] &= 0\,, & h,k &= 1,\ldots,n\,, \\ [H_h, Q_k] &= (\delta_{hk} - \delta_{h,-k})Q_k\,, & k &= \pm 1,\ldots,\pm n\,, \quad h,k = 1,\ldots,n\,. \end{aligned}$$

Other useful relations can be found in [29].

§ 3.2 The manifold $\mathbf{Spin}(2n+1)$ as a $\mathbf{Spin}(2n+1)$-space We collect some well-known results about the Lie group $\mathbf{Spin}(k)$ seen as a $k(k-1)/2$ dimensional manifold.

1. We can think of $\mathbf{Spin}(k)$ as a **Lie group of transformations** on itself, looked upon as a manifold denoted again by $\mathbf{Spin}(k)$. In particular we regard the *manifold* $\mathbf{Spin}(k)$ as equipped with a left action of the *Lie group* $\mathbf{Spin}(k)$: we have the differentiable map $\mathbf{Spin}(k) \times \mathbf{Spin}(k) \to \mathbf{Spin}(k)$ which sends $(g,x) \mapsto gx$ such that $(g_1g_2)x = g_1(g_2x)$, where $g, g_1, g_2 \in \mathbf{Spin}(k)$ are thought of as transformations, and $x \in \mathbf{Spin}(k)$ as a point on the manifold. Similarly, we can define a *right action* of $\mathbf{Spin}(k)$ on itself by taking the differentiable map $\mathbf{Spin}(k) \times \mathbf{Spin}(k) \to \mathbf{Spin}(k)$, $(g,x) \mapsto xg$, $x(g_1g_2) = (xg_1)g_2$, where x, g, g_1, g_2 are as before.

2. We denote by $\mathrm{d}g$ the unique **Haar measure** on a compact Lie group $\mathbf{G}$ normalized by the condition $\int_{\mathbf{G}} \mathrm{d}g(\mathbf{G}) = 1$. In particular we denote by $\mathrm{d}g$ the Haar measure for $\mathbf{G} = \mathbf{Spin}(k)$.

§ 3.3 Definition Let $\mathbf{G}$ be a compact Lie group. Let us denote by $C^\infty(\mathbf{G})$ the space of smooth functions from $\mathbf{G}$ to $\mathbb{C}$. Note that for convenience we will always consider spaces of functions into $\mathbb{C}$ and not $\mathbb{R}$. We define a representation of $\mathbf{G}$ on $C^\infty(\mathbf{G})$ by the action of *left translation*, i.e.

$$\mathbf{G} \to C^\infty(\mathbf{G}), \qquad g \mapsto L_g\,,$$

where L_g is defined as the map

$$L_g : C^\infty(\mathbf{G}) \to C^\infty(\mathbf{G}), \qquad L_g f = f \circ g^{-1},\ f \in C^\infty(\mathbf{G})$$

i.e. $(L_g f)(x) := f(g^{-1}x)$, $x, g \in \mathbf{G}$. We call this representation the **left regular representation** of $\mathbf{G}$.

Similarly we define the **right regular representation** of $\mathbf{G}$ by the action of *right translation* $R_.$, i.e.

$$R_. : G \to C^\infty(\mathbf{G}), \qquad g \mapsto R_g\,,$$

where the map R_g is defined by

$$(R_g f)(x) := f(xg), \qquad f \in C^\infty(\mathbf{G}), \quad x, g \in \mathbf{G}.$$

[4] In [29] the difference between lowering and rising *generators* (here denoted by $Q_{\pm k}$, ρ_{kh}) and lowering and rising *operators* (in the terminology of Gelfand-Zeitlin, see e.g. [2, Chapter 10, §1]) is explained.

Let us denote by $L^2(\mathbf{G})$ the complex space of functions from the manifold $\mathbf{G}$ into $\mathbb{C}$, which are square-integrable with respect to the (normalized) Haar measure on $\mathbf{Spin}(k)$.

We extend by continuity and linearity the *left regular representation* of $\mathbf{G}$ from the space $C^\infty(\mathbf{G})$ to the space $L^2(\mathbf{G})$ because the operators $(L_g)f(x) = f(g^{-1}x)$ are *bounded* linear operators from $L^2(\mathbf{G})$ to itself. Note in particular that the left regular representation extended to $L^2(\mathbf{G})$ gives rise to a unitary (infinite dimensional) representation of $\mathbf{G}$. For this reason in the following we shall denote the left regular representation of $\mathbf{G}$ on $L^2(\mathbf{G})$ by

$$(U(g)f)(x) \stackrel{\text{def}}{=} f(g^{-1}x), \qquad f \in L^2(\mathbf{G}), \quad x, g \in \mathbf{G}.$$

§ 3.4 Invariant vector fields on a Lie group A smooth vector field on a Lie group $\mathbf{G}$ is called **left invariant** when it commutes with all left translations L_g (defined in §3.3). Similarly a vector field on a Lie group is called **right invariant** when it commutes with all the right translations R_g.

To describe the invariant vector fields on $\mathbf{Spin}(2n+1)$ we employ the fact that the exponential map from the Lie algebra of a Lie group to the Lie group itself is *natural*. That is, given a homomorphism of Lie groups $\psi : G \to H$ inducing a map $\mathrm{d}\psi : T_eG \to T_eH$ the following diagram

$$\begin{tikzcd} T_eG \arrow[r, "\mathrm{d}\psi"] \arrow[d, "\exp"'] & T_eH \arrow[d, "\exp"] \\ G \arrow[r, "\psi"] & H \end{tikzcd}$$

commutes. In more categorical language, we have a natural transformation

$$\begin{tikzcd} \mathbf{Lie} \arrow[rr, "G\mapsto G"] \arrow[dr, "G\mapsto A"'] & \Uparrow & \mathbf{Mfld} \\ & \mathbf{LAlg} \arrow[ur, "A\mapsto A"'] & \end{tikzcd}$$

between the forgetful functor $\mathbf{Lie} \to \mathbf{Mfld}$ and the functor $\mathbf{Lie} \to \mathbf{Mfld}$ which first computes the Lie algebra, and then passes through the forgetful functor $\mathbf{LAlg} \to \mathbf{Mfld}$.

This fact allows us to consider an invariant vector field on a matrix Lie group $\mathbb{G}$ as the restriction of a corresponding invariant vector field on $\mathbf{GL}(2n+1)$.

We now use this naturality property in the case of the Lie group $\mathbf{G} = \mathbf{Spin}(2n+1)$.

Consider the Lie group $\mathbf{GL}(n)$ which is an open, connected, submanifold of the manifold $M(n \times n) \cong \mathbb{R}^{n^2}$ of real $n \times n$ matrices. Consider $\mathbf{Spin}(2n+1)$ as a Lie subgroup of $GL(2n+1)$ and consider the Lie algebra $\mathfrak{so}(2n+1)$ of $\mathbf{G}$ as a Lie subalgebra of $\mathfrak{gl}(k) \cong M(n \times n)$ of $\mathbf{GL}(n)$ (where $\cong$ stands here for Lie algebra isomorphism).

A standard result[5] gives that any left-invariant vector field X_A on $\mathbf{G}$ can be obtained in the form

$$X_A f(g) = \frac{d}{dt} f\left(ge^{tA}\right)\Big|_{t=0}, \qquad f \in C^\infty(\mathbf{G}), \tag{II.4}$$

where $A \in \mathfrak{so}(2n+1) \subset M(2n+1)$, $g \in \mathbf{G} = \mathbf{Spin}(2n+1) \subset \mathbf{GL}(2n+1) \subset M(2n+1)$.

Consider $\mathbf{GL}(n)$ with a non-singular[6] parametrization given by coordinates $c : \mathbf{GL}(n) \to \mathbb{R}^{n^2}$, $c : g \to c(g)$, that is $c = (c_{kj}(\cdot))_{i,j=1,\ldots,n}$ sends an element $g \in \mathbf{GL}(n)$ to a matrix $c(g) = (c_{jk}(g))_{j,k=1,\ldots,n} \in \mathbb{R}^{n^2}$.

[5] Cf. e.g. [15, Proposition 1.3.17 p. 33]

[6] Note that for example the Euler angle parametrization of $\mathbf{Spin}(3)$ is *singular* in the sense that the coefficients of the invariant vector fields in this coordinate system are singular functions. Non singular local parametrizations exist in a connected Lie group $\mathbf{G}$ because a Lie group is a smooth manifold.

We get

$$\begin{aligned} X_M^{\mathbf{GL}} f(g) &= \frac{d}{dt} f\left(g e^{tM}\right)\bigg|_{t=0} \\ &= \sum_{ij} m_{ij} \sum_k c_{ki}(g) \frac{\partial}{\partial c_{kj}} f(g) \\ &= \operatorname{Tr}\{M\,T\}\, f(g), \qquad \text{with } T_{ij} = \sum_k c_{ki}(g) \frac{\partial}{\partial c_{kj}}, \end{aligned}$$

where $M \in M(k)$ (the real $k \times k$ matrices) with elements m_{ij}.

At the identity $g = 1 \in \mathbf{GL}(n)$ the vector field $X_M^{\mathbf{GL}}$ becomes

$$X_M^{\mathbf{GL}} f(g)|_{g=1} = \sum_{ij} m_{ij} \left.\frac{\partial f(g)}{\partial c_{ij}}\right|_{g=1}.$$

Noting from (II.3) that a basis for the Lie algebra $\mathfrak{so}(2n+1)$ is given in terms of elements in the Lie algebra of $\mathfrak{gl}(2n+1)$ by $X_{ij} = E_{ij} - E_{ji}$ we get for the invariant vector fields on $\mathbf{Spin}(2n+1)$

$$X_A^{\mathbf{Spin}(2n+1)} f(g) = \sum_{i,j=1}^{n} A_{ij} \sum_{k=1}^{2n+1} \left(c_{ki}(g) \frac{\partial}{\partial c_{kj}} - c_{kj}(g) \frac{\partial}{\partial c_{ki}} \right) \tilde{f}(g),$$

with $A \in \mathfrak{so}(2n+1)$, $g \in \mathbf{Spin}(2n+1) \subset \mathbf{GL}(2n+1)$ and $\tilde{f}$ is the smooth extension of f from $C^\infty(\mathbf{Spin}(2n+1))$ to $C^\infty(\mathbf{GL}(2n+1))$. At the identity $g = 1 \in \mathbf{Spin}(2n+1)$ we have thus

$$X_A^{\mathbf{Spin}(2n+1)} f(1) = \sum_{i,j=1}^{n} A_{ij} \left(\frac{\partial}{\partial c_{ij}} - \frac{\partial}{\partial c_{ji}} \right) \tilde{f}(1).$$

Similarly any right-invariant vector field X_A on $\mathbf{G}$ can be obtained by the formula

$$X_A f(g) = \frac{d}{dt} f\left(e^{-tA} g\right)\bigg|_{t=0}, \qquad f \in C^\infty(\mathbf{G}), \tag{II.5}$$

where $A \in \mathfrak{so}(2n+1) \subset M(2n+1)$, $g \in \mathbf{G} = \mathbf{Spin}(2n+1) \subset \mathbf{GL}(2n+1) \subset M(2n+1)$.

Notice the “intertwining” of the terms of left-translation and right-invariance, right-translation and left-invariance. That is, note that if we differentiate the action by right translation as in (II.4) we obtain a left-invariant vector field. On the other hand, if we differentiate the left-action as in (II.5) we obtain a right-invariant vector field.

In the sequel, as a matter of choice, we will be deal only with the *left-regular* representation (which comes from the left action). Hence the vector fields we will handle will be *right-invariant*.

§ 3.5 Universal enveloping algebra Let $\mathfrak{a}$ be a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. Let $T(\mathfrak{a})$ be the tensor algebra over $\mathfrak{a}$ as a vector space. Let $\mathscr{J}$ be the (two sided) ideal generated by elements of the form

$$X \otimes Y - Y \otimes X - [X, Y], \qquad X, Y \in \mathfrak{a}.$$

Then we define the **universal enveloping algebra** of $\mathfrak{a}$ to be

$$\mathfrak{U}(\mathfrak{a}) := T(\mathfrak{a})/\mathscr{J}$$

§ 3.6 Proposition Let $\mathbf{G}$ be a Lie group with Lie algebra $\mathfrak{g}$. Denote by $\mathfrak{D}(\mathbf{G})$ the algebra of differential operators on $C^\infty(\mathbf{G})$ generated by the right-invariant vector fields on $\mathbf{G}$ and the identity. Then the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is isomorphic (as a an algebra) to $\mathfrak{D}(\mathbf{G})$. Moreover, by this isomorphism, the Lie algebra $\mathfrak{g}$ is represented on $\mathfrak{U}(\mathfrak{g}) \cong \mathfrak{D}(\mathbf{G})$ by the representation[7] $dU : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g}) \cong \mathfrak{D}(\mathbf{G})$ which associates to each element in $\mathfrak{g}$ the corresponding right-invariant vector field.

[7] The notation dU will be justified in Section 4.

PROOF. Cf. [16, Ch. II, Proposition 1.9 and its proof, p. 108] □

§ 3.7 Remark Denote by φ the isomorphism in Proposition §3.6. The fact that the universal enveloping algebra is isomorphic (as a Lie algebra) to $\mathfrak{D}(\mathbf{G})$ means that the invariant vector fields $\varphi(X_1), \dots, \varphi(X_n)$ associated with the generators of the Lie algebra $\mathfrak{g}$ satisfy the Lie algebra *commutation* relations. That is they satisfy, on the domain $C^\infty(\mathbf{G})$, the following relation [8]

$$[\varphi(X), \varphi(Y)] = \varphi([XY]).$$

Now in the case of the Lie group $\mathbf{SO}(2n+1)$ we constructed in the previous section its Lie algebra as a *matrix algebra* starting from the generators of the Clifford algebra $C\ell(2n)$. So in the representation of the previous section we have elements $\gamma_1, \dots \gamma_{2n}$ which satisfy the Lie algebra *commutation* relations of $\mathfrak{so}(2n+1)$ and the *anti-commutation* relations of the Clifford algebra. Then we can associate to every element in this "Clifford-Lie" algebra an invariant vector field as a differential operator in $\mathfrak{D}(\mathbf{G})$. The so defined invariant vector fields will satisfy the *commutation* relations of the Lie algebra $\mathfrak{so}(2n+1)$ but will *not* satisfy the Clifford *anti-commutation* relations of the original elements $\gamma_1, \dots \gamma_{2n}$ in the starting "Clifford-Lie" matrix algebra.

§ 3.8 Example To clarify the previous remark we consider the example of $SO(3)$. Using Euler's angle parametrization of $\mathbf{Spin}(3)$ we have for the invariant vector fields [3, 19]:

$$\begin{aligned}
s_1 &= i\left(-\cos\phi\,\partial_\phi + \sin\phi\cot\theta\,\partial_\phi - \frac{\sin\phi}{\sin\theta}\partial_\chi\right),\\
s_2 &= i\left(\sin\phi\,\partial_\phi + \cos\phi\cot\theta\,\partial_\phi - \frac{\sin\phi}{\sin\theta}\partial_\chi\right),\\
s_3 &= -i\,\partial_\phi\,,
\end{aligned}$$

where $(\theta, \phi, \chi) \in [0,\pi]\times[0,2\pi]\times[0,4\pi]$. With straight forward computations (or abstractly applying the Proposition in §3.6) it can be shown that these generators satisfy the commutation relations

$$[s_j, s_k] = i\sum_{l=1}^{3} \epsilon_{jkl} s_l\,,$$

where ϵ_{ijk} is the Levi-Civita symbol in three dimensions. On the other hand they *cannot* in general satisfy the Fermionic anti-commutation relations, we have for example

$$\{s_1 + is_2, s_1 - is_2\} \neq 1.$$

Indeed the left hand side is a nontrivial second order differential operator which is not a constant operator on the space $C^\infty(\mathbf{Spin}(3))$, as one easily verifies.

§ 3.9 Peter-Weyl theorem Let $\mathbf{G}$ by a compact Lie group. Denote by $\widehat{\mathbf{G}}$ the set of all irreducible non-equivalent unitary (complex) representations α of $\mathbf{G}$. Denote by D^α_{ij} the matrix elements of α in the irreducible unitary representation given by α, and denote by d_α the dimension of such a representation.

Then:

1. The regular representation decomposes into a direct sum of irreducible unitary representations. Every irreducible unitary representation of the compact Lie group $\mathbf{G}$ appears in this decomposition with a multiplicity equal to its dimension.

[8] We denote by $[XY]$ (no comma) the commutation relations in the Lie algebra $\mathfrak{g}$ and by $[A, B] = AB - BA$ (with comma) the commutation relations in $\mathfrak{D}(\mathbf{G})$ or in $\mathfrak{U}(\mathfrak{g})$.

2. We have an orthonormal basis for $L^2(\mathbf{G})$ given by the following vectors

$$Y^{\alpha}_{(i)j}(x) = \sqrt{d_\alpha}\, D^{\alpha}_{ij}(x)\,, \quad \alpha \in \widehat{\mathbf{G}}\,, \quad i,j = 1,\ldots,d_\alpha \quad x \in \mathbf{G}.$$

gewhere d_α denotes the dimension of the representation labeled by α. Take α fixed. Then the set of functions $(Y^{\alpha}_{(i)j})_{j=1,\ldots,d_\alpha}$ spans for every $i = 1,\ldots,d_\alpha$ an invariant irreducible subspace of dimension d_α for the *right* regular representation, and this subspace realizes one of the d_α copies of irreducible representations for the right regular representation parametrized by α. This means that the following decomposition holds:

$$L^2(\mathbf{G}) = \bigoplus_{\alpha\in\widehat{\mathbf{G}}} \Big(\overbrace{V_\alpha \oplus \cdots \oplus V_\alpha}^{d_\alpha \text{ copies}}\Big),$$

where V_α is the carrier space of the irreducible representation α and, as above, $d_\alpha = \dim V_\alpha$.

3. We have another orthonormal basis given by the following vectors

$$\widetilde{Y}^{\alpha}_{(i)j}(x) = \sqrt{d_\alpha}\, \overline{D^{\alpha}_{ij}}(x) \quad \alpha \in \widehat{\mathbf{G}}\,, \quad i,j = 1,\ldots,d_\alpha\,,$$

where the over-line in $\overline{D^{\alpha}_{ij}}(x)$ denotes complex conjugation. These vectors play the same role for the *left*-regular representation as the previous vectors $Y^{\alpha}_{(i)j}$ did for the right regular representation.

PROOF. Cf. [2, Chapter 7 §2, Theorem 1 p.172 and Theorem 2 p.174] □

§ 3.10 Remark The previous theorem tells us nothing about the actual form of the matrix elements $D^{\alpha}_{ij}(g)$, $g \in \mathbf{G}$, which need to be computed by other means, for example with the help of Gelfand-Zeitlin construction (cf. [2, Chapter 11 §1], [15, Chapter 8], or [42, Chapter X, Chapter XVIII]). On the other hand the previous theorem does tell us among other things that, once we have for an irreducible representation the matrix elements $D^{\alpha}_{ij}(g)$, $g \in \mathbf{G}$, α, then

$$(Y^{\alpha}_{(i)j}, L_g Y^{\alpha'}_{(i')j'}) = \delta_{\alpha\alpha'}\, \delta_{i(i')}\, \delta_{j(j')}\, D^{\alpha}_{ij}(g).$$

We now embed the Fermionic Fock space $\bigwedge \mathbb{C}^n$ into $L^2(\mathbf{Spin}(2n+1))$.

§ 3.11 Lemma Let $(\cdot,\cdot)_{\bigwedge \mathbb{C}^n}$ denote the Hermitian scalar product in $\bigwedge \mathbb{C}^n$ obtained by extension of the standard Hermitian scalar product of $\mathbb{C}^n$. Let $\pi_{1/2}(g)$ denote the half-spin representation of an element $g \in \mathbf{Spin}(2n+1)$.

Then the map

$$\imath : \bigwedge \mathbb{C}^n \hookrightarrow L^2(\mathbf{Spin}(2n+1)), \quad \imath : \psi \mapsto \big(\psi, \pi_{1/2}(g)\,\psi\big)_{\bigwedge \mathbb{C}^n}.$$

defines an embedding of $\bigwedge \mathbb{C}^n$ into $L^2(\mathbf{Spin}(2n+1))$.

The restriction of the regular representation of $\mathbf{Spin}(2n+1)$ to the image of $\imath$ defines a representation which coincides with the half-spin representation of $\mathbf{Spin}(2n+1)$.

PROOF. Follows directly from the Peter-Weyl theorem in §3.9. □

§ 3.12 Remark We note that from the Proposition in §2.9 we have that the weights of the half spin representations are

$$\underbrace{(\pm\tfrac{1}{2},\ldots,\pm\tfrac{1}{2})}_{n \text{ times}}.$$

Each of the corresponding *weight vectors* are elements of the carrier space of the half-spin representation. Hence by the Peter-Weyl theorem they can be embedded into $L^2(\mathbf{Spin}(2n+1))$.

Consider a weight λ of the form $(\underbrace{\pm\frac{1}{2},\dots,\pm\frac{1}{2}}_{n\text{ times}})$ with a given number of "plus signs" and the complementary number of "minus signs". Then the corresponding *weight vector* Ψ_λ is interpreted physically as a vector in the Fermionic Fock space. The plus signs in the *weight* λ denote "filled states", that is to every "plus sign' corresponds a Fermionic particle in the respective state.

For later convenience we give the following definition.

§ 3.13 Definition We denote by Ψ_0 the image of the vacuum state $1 \in \bigwedge\mathbb{C}^n$ under the embedding ι. That is, we define

$$\Psi_0(g) \stackrel{\text{def}}{=} \big(1, \pi_{1/2}(g)\,1\big)_{\bigwedge\mathbb{C}^n},$$

We will denote by F_{Ψ_0} the image of the embedding $\iota : \bigwedge\mathbb{C}^n \hookrightarrow L^2(\mathbf{Spin}(2n+1))$ given in point 2. of the Lemma in §3.11.

§ 3.14 Remark The closure of the orbit of Ψ_0 under the regular representation of $\mathbf{Spin}(2n+1)$ coincides with F_{Ψ_0}.

PROOF. Denote by $G_{\Psi_0} \subset L^2(\mathbf{Spin}(2n+1))$ the span of the orbit of Ψ_0 under the left-regular representation of $\mathbf{Spin}(2n+1)$ and consider the embedding $\iota : \bigwedge\mathbb{C}^n \to F_{\Psi_0} \subset L^2(\mathbf{Spin}(2n+1))$. Then G_{Ψ_0} is a subset of F_{Ψ_0} because $U(g)\Psi_0 = U(g)\,\iota(1) = \iota(\pi^{(1/2)}1) \subset \iota(\bigwedge\mathbb{C}^n)$, where $U(g)$ denotes the right regular representation of an element $g \in \mathbf{Spin}(2n+1)$. Moreover the inverse image $\iota^{-1}(G_{\Psi_0})$ is a $\mathbf{Spin}(2n+1)$-invariant subspace of $\bigwedge\mathbb{C}^n$. But then $\iota^{-1}(G_{\Psi_0}) = \bigwedge\mathbb{C}^n$ because the representation of $\mathbf{Spin}(2n+1)$ on $\bigwedge\mathbb{C}^n$ is irreducible. □

4 Time evolution of a Fermionic state

In order to define the time evolution we need some function analytic concepts. In particular regarding the infinitesimal representation dU of a Lie $\mathfrak{g}$ of a Lie group $\mathbf{G}$ obtained as the differential of the right regular representation U of $\mathbf{G}$. For simplicity, we restrict the discussion to the case where the group G is compact.

§ 4.1 Let $\mathbf{G}$ be a connected, simply connected, compact Lie group. Let $\mathfrak{g}$ be its Lie algebra. Let U denote the left regular representation of $\mathbf{G}$ introduced in §3.3. Note that U is a unitary infinite dimensional representation. For $X \in \mathfrak{g}$ we define the operator $dU(X)$ with domain $C^\infty(\mathbf{G})$ by

$$dU(X)f = \frac{d}{dt}U(e^{tX})f|_{t=0}, \quad f \in C^\infty(\mathbf{G}).$$

Then by definition dU coincides with the representation of the Lie algebra $\mathfrak{g}$ by *right*-invariant vector fields given in §3.6 (for more details cf., e.g. [34, §10.1]).

By the universal property of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ the representation dU can be extended uniquely (up to isomorphism) to a representation of the full universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$. For convenience we shall still denote by dU this extension. In this way we end up with a representation of every element in $\mathfrak{U}(\mathfrak{g})$ by an algebra of differential operators on the common domain $C^\infty(\mathbf{G})$.

§ 4.2 Denote by $\mathfrak{g}_\mathbb{C}$ the complexification of the real Lie algebra $\mathfrak{g}$. We can equip $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ with an antilinear involution $*$ defined on elements in the real Lie algebra $\mathfrak{g}$ by

$$X^* = -X, \quad X \in \mathfrak{g},$$

and then extended (uniquely because of the universal property of $\mathfrak{U}(\mathfrak{g})$) to the full enveloping algebra $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$. The enveloping algebra together with this involution becomes a $*$-algebra. An element $X \in \mathfrak{U}(\mathfrak{g})$ is said to be *Hermitian* (as an element of the universal enveloping algebra) when $X = X^*$, where the $*$ denotes the antilinear involution defined above.

On the algebra $\mathfrak{D}(\mathbf{G})$ of right-invariant smooth differential operators in $L^2(\mathbf{G})$ with common invariant domain $C^\infty(\mathbf{G})$ we have an antilinear involution, which we also denote by $*$, which sends the unbounded operator $D \in \mathfrak{D}(\mathbf{G})$ to its Hilbert-adjoint D^*. This involution makes $\mathfrak{D}(\mathbf{G})$ into a $*$-algebra.

The representation dU is by definition an algebra isomorphism

$$dU : \mathfrak{U}(\mathfrak{g}_\mathbb{C}) \overset{\cong}{\to} \mathfrak{D}(\mathbf{G}).$$

Once we equip $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ and $\mathfrak{D}(\mathbf{G})$ with the involutions described above in principle one would like to extend dU to a $*$-isomorphism. But this is in general not possible, since in general we will not have $dU(X) = dU(X)^*$, for $X = X^* \in \mathfrak{U}(\mathfrak{g}_\mathbb{C})$, because the domain of the Hilbert adjoint $dU(X)^*$ can in general be bigger than the domain of $dU(X)$, that is the operator $dU(X)$ is Hermitian[9] but not selfadjoint. Now one could try to extend the operator $dU(X)$ to a selfadjoint operator by enlarging its domain. This might be possible for one operator $dU(X)$ for a fixed $X \in \mathfrak{U}(\mathfrak{g})$. But for different $X, Y \in \mathfrak{U}(\mathfrak{g})$ we need to have a *common* invariant domain of definition for $dU(X)$ and $dU(Y)$ because we want an *algebra* of operators. Hence in general one cannot expect to find an extension of dU to a $*$-isomorphism.

One could argue that being a $*$-isomorphism is too strong a property and not necessarily the most natural. The best situation that we can hope for is contained in the following proposition.

§ 4.3 Proposition Let $\mathbf{G}$ be a *compact* Lie group with Lie algebra $\mathfrak{g}$. Then

$$\overline{dU(X^*)} = dU(X)^*, \quad X \in \mathfrak{U}(\mathfrak{g}_\mathbb{C}), \tag{II.6}$$

where the overline on the right hand side denotes the operator closure.

PROOF. The proof can be found in e.g. [34, Corollary 10.2.10, p.270]. □

Remark First note that the Proposition above implies that any Hermitian element $D \in \mathfrak{D}(\mathbf{G})$ is automatically *essentially selfadjoint*.

For this reason we could call an algebra isomorphism with the property in (II.6) an *essentially $*$-isomorphism*.

We now turn to the notion of commuting unbounded operators. There are two natural notions of commuting unbounded operators, weakly commuting and strongly commuting. We give the precise definitions.

§ 4.4 Given two unbounded operators A, B with common domain $\mathscr{D}$ in a Hilbert space $\mathfrak{H}$, we say that A, B **weakly commute** on $\mathscr{D}$ when $ABv = BAv$ for all $v \in \mathscr{D}$. Given two *selfadjoint* unbounded operators A, B we say that A, B **strongly commute** when for all $s, t \in \mathbb{R}$, $e^{\mathrm{i}tA}e^{\mathrm{i}sB} = e^{\mathrm{i}sB}e^{\mathrm{i}tA}$, where $e^{\mathrm{i}tC}$ denotes the unitary group generated by a selfadjoint operator C (cf. [30, Theorem VIII.13]).

Regarding the relation between strong and weak commutativity of operators on a Hilbert space we have the following result due to Nelson.

§ 4.5 Lemma ([22, Corollary 9.2]) Let A, B be two Hermitian unbounded operators on a Hilbert space $\mathcal{H}$ and let Q be a dense linear subspace of $\mathcal{H}$ such that Q is contained in the domain of A, B, A^2, AB, BA, and B^2, and such that A, B *weakly commute* on Q. If the restriction of $A^2 + B^2$ to Q is essentially selfadjoint then A and B are essentially selfadjoint and their closures $\overline{A}$, $\overline{B}$ *strongly commute*. □

A direct consequence of this lemma to our situation is the following.

[9] In the context of unbounded operators in a Hilbert space, an operator T with domain $\mathrm{Dom}(T)$ is *Hermitian* when it satisfies $\mathrm{Dom}(T) \subset \mathrm{Dom}(T^*)$ and $T|_{\mathrm{Dom}(T)} = T^*_{\mathrm{Dom}(T)}$. The operator T is selfadjoint when in addition the stronger condition $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$ holds. In the algebraic context of universal enveloping algebras, an element $X \in \mathfrak{U}(\mathfrak{g}_\mathbb{C})$ is said to be Hermitian when $X = X^*$ in the sense of section §**4.2**. These two, in general different, concepts for an object to be Hermitian coincides when we identify the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ with the algebra of smooth right-invariant vector fields $\mathfrak{D}(\mathbf{G})$.

§ 4.6 Proposition. Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ of a compact group $\mathbf{G}$. Let $X, Y \in \mathfrak{U}(\mathfrak{g})$ be two *commuting* operators (in the algebraic sense of elements in the universal enveloping algebra). Then

(i) the closed operators $\overline{dU(X)}, \overline{dU(Y)} \in \mathfrak{D}(\mathbf{G})$ *strongly commute*;

(ii) if $dU(X)$ is positive (semi-)definite, and $dU(Y)$ is Hermitian, then $\exp(-\overline{dU(X)})\exp(\mathrm{i}\overline{dU(Y)}) = \exp(\mathrm{i}\overline{dU(Y)})\exp(-\overline{dU(X)})$, where we recall that $\overline{dU(X)}$ and $\overline{dU(Y)}$ are the *unique* closed extensions of $dU(x)$, respectively $dU(Y)$, and $\overline{dU(X)} > 0$.

PROOF. The statement in (i) follows from the Proposition in §4.3 and Nelson's Lemma in §4.5. Indeed, if X, Y commute in the universal enveloping algebra then $dU(X)$ and $dU(Y)$ weakly commute on $C^\infty(\mathbf{G})$ because dU is a representation of $\mathfrak{U}(\mathfrak{g})$ with domain $C^\infty(\mathbf{G})$.

Now for $\mathbf{G}$ a compact group the Proposition in §4.3 tells us that *any* Hermitian element in the $\mathfrak{D}(\mathbf{G})$ is essentially self adjoint on $C^\infty(\mathbf{G}) \subset L^2(\mathbf{G})$. Therefore in particular, for any $X, Y \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$, we have that the operators $dU(X)$, $dU(X)^2 = dU(X^2)$, $dU(X)dU(Y) = dU(XY)$, $dU(X) + dU(Y) = dU(X+Y)$ have the same domain $C^\infty(\mathbf{G})$, and are essentially selfadjoint there. Hence the hypothesis of the Lemma in §4.5 are satisfied with $A = dU(X)$ and $B = dU(Y)$ and statement (i) follows.

The statement in (ii) is a straight forward application of spectral calculus (cf [30, Section VIII.5]). □

§ 4.7 Remark Because of the above proposition we only need to check whether two operators commute as elements of the universal enveloping algebra. From the proposition in §4.6 it then follows automatically that their closures are *selfajdoint* and *strongly commuting*.

With this proposition we have completed the considerations from the general theory. We can now turn to the application that we have in mind.

§ 4.8 Quasi-Fermionic vector fields Let X_{ij} ,$i, j = 1, \dots, 2n+1$, be (as in §3.1) the invariant vector fields on $\mathbf{Spin}(2n+1)$ which form the standard basis[10] of the Lie algebra[11] $\mathrm{Lie}(\mathbf{Spin}(2n+1))$.

We define the following operators

$$D_k^+ \stackrel{\text{def}}{=} X_{2k-1,2n+1} + \mathrm{i}X_{2k,2n+1}\,,$$
$$D_k^- \stackrel{\text{def}}{=} X_{2k-1,2n+1} - \mathrm{i}X_{2k,2n+1}\,, \qquad k = 1, \dots, n\,,$$

as linear operators on $C^\infty(\mathbf{Spin}(2n+1), \mathbb{C}) \subset L^2(\mathbf{Spin}(2n+1))$.

We call these operators "quasi-Fermionic" because they satisfy the Canonical Anti-Commutation Relations *only* when projected onto the correct (in the sense of the Remark in §3.7) subspace of $L^2(\mathbf{Spin}(2n+1))$.

§ 4.9 Quasi-Hamiltonian operator Let us choose an n-tuple of *strictly positive* numbers $\underline{E} = (E_1, \dots, E_n)$ with $0 < E_1 \le \cdots \le E_n$. Using for $D_k^\pm$ the notation of the previous paragraph we call a **quasi-Hamiltonian** the operator

$$H_{\underline{E}} = \sum_{k=1}^{n} E_k D_k^+ D_k^-\,,$$

acting on $C^\infty(\mathbf{Spin}(2n+1))$. For different n-tuples $\underline{E}$ we have different quasi-Hamiltonians. In the special case where $\underline{E} = (1, \dots, 1)$ the quasi-Hamiltonians will be called quasi-number operator.

[10] That is, they satisfy the commutation relations [2, Chapter 9, (42), p.260]

$$[X_{ri}, X_{sj}] = \delta_{is}X_{rj} + \delta_{rj}X_{is} - \delta_{ij}X_{rs} - \delta_{rs}X_{ij}\,, \quad \text{where } r, i, s, j \in \{1, \dots, 2n+1\}.$$

[11] We denote by $\mathrm{Lie}(\mathbf{G})$ the Lie algebra of a Lie group $\mathbf{G}$ when we want to underline that we consider the Lie algebra as the vector space of right-invariant vector fields on $\mathbf{G}$.

This is well defined, since $D^{\pm}$ are linear combinations of smooth vector fields, in particular D^{-} maps $C^{\infty}(\mathbf{Spin}(2n+1))$ into $C^{\infty}(\mathbf{Spin}(2n+1))$ (indeed these differential operators are elements of an algebra: $D^{\pm} \in \mathfrak{D}(\mathbf{Spin}(2n+1))$). In fact, using the above definition of the operators D_k^+, D_k^- in terms of the operators X_{ij} in §4.8 we have, on $C^{\infty}(\mathbf{Spin}(2n+1))$:

$$
\begin{aligned}
H_{\underline{E}} &= \sum_{k=1}^{n} E_k (X_{2k-1,2n+1} + \mathrm{i} X_{2k,2n+1})(X_{2k-1,2n+1} - \mathrm{i} X_{2k,2n+1}) \\
&= \sum_{k=1}^{n} E_k (X_{2k-1,2n+1})^2 + \sum_{k=1}^{n} E_k (X_{2k,2n+1})^2 + \mathrm{i} \sum_{k=1}^{n} E_k \left[X_{2k-1,2n+1}, X_{2k,2n+1}\right] \\
&= \sum_{k=1}^{n} E_k ((X_{2k-1,2n+1})^2 + (X_{2k,2n+1})^2) + \mathrm{i} \sum_{k=1}^{n} E_k X_{2k-1,2k}.
\end{aligned}
$$

Therefore we obtain

$$
H_{\underline{E}} = \sum_{k=1}^{n} E_k L_k + \mathrm{i} B_0 \tag{II.7}
$$

where $B_0 := \sum_{k=1}^{n} E_k X_{2k-1,2k}$ and $L_k \overset{\text{def}}{=} X_{2k-1,2n+1}^2 + X_{2k,2n+1}^2$, $k = 1, \ldots, n$.

Remark The operators $D_k^{\pm}$ restricted to the subspace $F_{\Psi_0} \subset C^{\infty}(\mathbf{Spin}(2n+1))$, defined in §3.13, satisfy the canonical anticommutation relations.

Similarly the operator $H_{\underline{E}}$ restricted to the subspace $F_{\Psi_0} \cong \bigwedge \mathbb{C}^n$ coincides with the **Fermionic Hamiltonian operator** $\widetilde{H}_{\underline{E}} = \sum_k E_k c_k^{\dagger} c_k$ defined in (II.1).

We now prove some Lemmas which culminate in the theorem in §4.13 below.

§ 4.10 Lemma The operators $X_{2k-1,2k}$, $k = 1, \ldots, n$, defined in §4.8, form a commuting family of operators in the universal enveloping algebra of $\mathfrak{so}_{\mathbb{C}}(2n+1)$.

PROOF. The statement follows from the standard fact[12] that the maximal commutative subalgebra (Cartan subalgebra) of the Lie algebra $\mathfrak{so}(2n+1)$ is generated by the elements $X_{2k-1,2k}$, $k = 1, \ldots, n$. □

§ 4.11 Lemma. Consider

$$
L_\ell \overset{\text{def}}{=} (X_{2\ell-1,2n+1})^2 + (X_{2\ell,2n+1})^2, \quad \ell = 1, \ldots, n,
$$

as an element of the universal enveloping algebra of $\mathfrak{so}(2n+1)$. Then L_ℓ commutes with $X_{2k-1,2k}$, for all $\ell, k \in \{1, \ldots, n\}$.

PROOF. We first observe that, thanks to the Proposition in §4.6, it is enough to prove that

$$
[L_\ell, X_{2k-1,2k}] = 0, \quad \text{for all } \ell, k = 1, \ldots, n,
$$

as a relation in the universal enveloping algebra $\mathfrak{U}(\mathfrak{so}(2n+1)_{\mathbb{C}})$. Hence all relations until the end of the proof are meant to hold on this algebra.

Using the identity $[X^2, Y] = X[X,Y] + [X,Y]X$ for any $X, Y \in \mathfrak{U}(\mathfrak{so}(2n+1)_{\mathbb{C}})$ we get

$$
\begin{aligned}
[L_\ell, X_{2k-1,2k}] = X_{2\ell-1,2n+1}[X_{2\ell-1,2n+1}, X_{2k-1,2k}] + [X_{2\ell-1,2n+1}, X_{2k-1,2k}]X_{2\ell-1,2n+1} \\
+ X_{2\ell,2n+1}[X_{2\ell,2n+1}, X_{2k-1,2k}] + [X_{2\ell,2n+1}, X_{2k-1,2k}]X_{2\ell,2n+1}
\end{aligned} \tag{II.8}
$$

[12] Cf. e.g. [38, Chapter 42.]

Now using in this expression the commutation relations (cf. footnote 10)

$$[X_{ri}, X_{sj}] = \delta_{is}X_{rj} + \delta_{rj}X_{is} - \delta_{ij}X_{rs} - \delta_{rs}X_{ij}\,, \quad \text{where } r,i,s,j \in \{1,\dots,2n+1\}, \tag{II.9}$$

we obtain for $\ell, k = 1, \dots, n$

$$\begin{aligned}[L_\ell, X_{2k-1,2k}] = &-X_{2\ell-1,2n-1}\,\delta_{2\ell-1,2k-1}X_{2n+1,2k} - \delta_{2\ell-1,2k-1}X_{2n+1,2k}\,X_{2\ell-1,2n+1}\\ &+ X_{2\ell,2n+1}\,\delta_{2\ell,2k}X_{2n+1,2k-1} + \delta_{2\ell,2k}X_{2n+1,2k-1}\,X_{2\ell,2n+1}\end{aligned}$$

Now, using in this expression the fact that $X_{ij} = -X_{ji}$ for all $i, j = 1, \dots, 2n+1$, and collecting the Kronecker deltas into a unique Kronecker delta which multiplies everything, we get

$$\begin{aligned}[L_\ell, X_{2k-1,2k}] = \delta_{k,\ell}\,(&X_{2\ell-1,2n+1}\,X_{2k,2n+1} + X_{2k,2n+1}\,X_{2\ell-1,2n+1}\\ &- X_{2\ell,2n+1}\,X_{2k-1,2n+1} - X_{2k-1,2n+1}\,X_{2\ell,2n+1})\end{aligned}$$

Finally using the identity $\delta_{ij}f(i,j) = \delta_{ij}f(i,i)$ where $f(i,j)$ is any function of $i,j \in \mathbb{N}$ we get

$$\begin{aligned}[L_\ell, X_{2k-1,2k}] &= \delta_{k,\ell}\,(X_{2k-1,2n+1}\,X_{2k,2n+1} + X_{2k,2n+1}\,X_{2k-1,2n+1}\\ &\qquad - X_{2k,2n+1}\,X_{2k-1,2n+1} - X_{2k-1,2n+1}\,X_{2k,2n+1})\\ &= 0.\end{aligned}$$

□

From this lemma we have the following straightforward corollary.

§ 4.12 Corollary. $\sum_{k=1}^n E_k L_k$ commutes with B_0 (with B_0 as in (II.7)). □

We collect all the properties of the quasi-Hamiltonian proved so far in the following theorem.

§ 4.13 Theorem. The family of unbounded operators $H_{\underline{E}}$ defined on the domain $C^\infty(\mathbf{Spin}(2n+1))$ in $L^2(\mathbf{Spin}(2n+1))$ is a family of *essentially selfadjoint* operators. Moreover the quasi-Hamiltonian can be decomposed on $C^\infty(\mathbf{Spin}(2n+1))$ as

$$H_{\underline{E}} = \sum_{k=1}^n E_k L_k + \mathrm{i}\sum_{k=1}^n E_k T_k,$$

where $T_k \stackrel{\text{def}}{=} X_{2k-1,2k}$, and the operators L^k, $k = 1, \dots, n$ are positive definite and essentially selfadjoint on $C^\infty(\mathbf{Spin}(2n+1))$. The operators T_k, $k = 1, \dots, n$, are essentially-selfadjoint on $C^\infty(\mathbf{Spin}(2n+1))$ and their closure $\overline{T}_k$, $k = 1, \dots, n$, defines a family of strongly commuting unbounded operators in $L^2(\mathbf{Spin}(2n+1))$. Moreover $\overline{T}_k$ strongly commutes with $\overline{L}_\ell$, for any $k, \ell = 1, \dots, n$. . □

5 Stochastic process associated to the quasi-Hamiltonian

§ 5.1 Operators associated to $H_{\underline{E}}$. Let us write the quasi-Hamiltonian in §4.13 as the sum of two operators, that is let

$$H_{\underline{E}} = P_0 + \mathrm{i}B_0$$

where $P_0 \stackrel{\text{def}}{=} \sum_{k=1}^n E_k L_k$ and $B_0 \stackrel{\text{def}}{=} \sum_{k=1}^n E_k T_k$, where all the operators are defined on $C^\infty(\mathbf{Spin}(2n+1))$.

Since the operator B_0 appears in $H_{\underline{E}}$ multiplied by the imaginary unit i we cannot associate directly to the closure $\overline{H_{\underline{E}}}$ a (real) stochastic process. For this reason we consider, together with P_0, B_0, and $H_{\underline{E}}$ above, the following operator

$$P \stackrel{\text{def}}{=} P_0 + B_0, quad\, \mathrm{Dom}(P) \stackrel{\text{def}}{=} C^\infty(\mathbf{Spin}(2n+1)). \tag{II.10}$$

We show now that it is possible to associate a stochastic processes to both $\overline{P_0}$ and $\overline{P}$. First we see that both $\overline{P_0}$ and $\overline{P}$ generate a probability semigroup in the following sense.

§ 5.2 Lemma. The operators P, P_0 are essentially selfadjoint on $C^\infty(\mathbf{Spin}(2n+1))$ and their closure $\overline{P}, \overline{P_0}$ are infinitesimal generators of a strongly continuous semigroup which acts on $L^2(\mathbf{Spin}(2n+1))$ as a convolution semigroup of probability measures with support on $\mathbf{Spin}(2n+1)$.

PROOF. The statement follows from [20, Theorem 3.1]. □

Now we characterize the stochastic processes generated by $\overline{P_0}$ and $\overline{P}$ in terms of the SDEs these processes satisfy. Before doing so let us spend a paragraph introducing the notion of stochastic differential equations (SDEs) on a manifold and the notion of the generator of a diffusion process, basically following [18].

§ 5.3 SDE on a manifold Let $\mathcal{M}$ be a smooth, connected manifold of dimension d. Moreover for convenience let us assume $\mathcal{M}$ to be compact. This assumption simplifies somewhat the discussion and is sufficient for our purposes because we will in the sequel only deal with manifolds associated to *compact* Lie groups. In particular if $\mathcal{M}$ is a compact manifold then every C^∞-vector field on it is complete, that is the flow associated to the given vector field can be extended to all times. This allows us to define a stochastic process globally on the manifold $\mathcal{M}$ (without the need of the introduction of an explosion time).

Let us denote by $\mathcal{X}(\mathcal{M})$ the set of C^∞-vector fields on $\mathcal{M}$. Let $A_0, A_1, \ldots, A_r \in \mathcal{X}(\mathcal{X})$, $r \in \mathbb{N}$ be vector fields on $\mathcal{M}$.

Let $(\Omega, (\mathscr{F}_t)_{0\le t<\infty}, \mathbb{P})$ be a filtered probability space. Let $(W(t)) = (W^1(t), \ldots, W^r(t))$ be an r-dimensional $\mathscr{F}_t$-adapted Brownian motion with $B(0) = 0$. Finally, let ξ be an $\mathscr{F}_0$-measurable $\mathcal{M}$-valued random variable.

Consider now an $\mathscr{F}_t$-adapted stochastic process $X = X(t)$ on $\mathcal{M}$, that is an $\mathscr{F}_t$-adapted random variable $X = (X(t))$ with values in the continuous functions $C^0([0,\infty); \mathcal{M})$.

Suppose that for every $f \in C^\infty(\mathcal{M})$ the stochastic process $X = (X(t))$ satisfies $\mathbb{P}$-almost surely the following integral equation

$$f(X(t)) - f(\xi) = \int_0^t \sum_{k=1}^r (A_k f)(X(s)) \circ dW^k(s) + \int_0^t (A_0 f)(X(s))\,\mathrm{d}s, \tag{II.11}$$

for all[13] $t \in [0,\infty)$, where $\circ dB$ denotes integration in the Stratonovich sense (see, e.g. [18]). Then we will say that the $\mathcal{M}$-valued stochastic process $X = (X(t))$ is a **solution** to (II.11).

Let us spend few words on the notion of *strong solution* regardless whether we are on a manifold $\mathcal{M}$ or just in $\mathbb{R}^d$. Given a notion of solution it is natural to ask whether it satisfies some given *initial condition*. That is we would like to specify the random variable ξ to be actually equal to a fixed point $x \in \mathcal{M}$ without any randomness. A solution to (II.11) with non random initial conditions $\xi = x$, would be a stochastic process X_x starting at x for $t = 0$.

Actually what we are asking for is a function $F : \mathcal{M} \times C^0([0,\infty); \mathbb{R}^r) \to C^0([0,\infty); \mathcal{M})$ which maps the initial condition $x \in \mathcal{M}$ and the given realization of the Brownian motion $W = (W(t))$ to a realization of a process $X = (X(t))$ on the manifold $\mathcal{M}$ which is a solution $X_x = F(x, W)$ to (II.11) with initial condition $\xi = x$ with probability one and with given Brownian motion $W = (W(t))$. Since at some point we would like to integrate X_x both with respect to $x \in \mathcal{M}$ and with respect to $\mathbb{P}$ it is natural to ask that F be jointly measurable in x and $W = (W(t))$. It turns out that this is *not* always possible. When it is we will call $X_x = F(x, W)$ a **strong solution** to (II.11) with initial condition $\xi = x \in \mathcal{M}$ with probability one (cf. the discussion in [32, Section V.10] and [18, Chapter IV, section 1, esp. pp.162-163]).

In the context of smooth manifolds the situation is particularly good because we are considering SDE with smooth coefficients. Indeed one has a result (cf. [18, Chapter V, Section 1., Theorem 1.1, p.249])

[13] By saying that the equality holds $\mathbb{P}$-almost surely, for all t, we are saying that the right hand side and the left hand side define *indistinguishable* processes.

which states that given an initial condition $x \in \mathcal{M}$ and an r-dimensional Brownian motion $W = (W(t))$, then a strong solution to (II.11) always exists and is unique[14].

Once this important detail regarding the meaning of initial conditions is understood we give meaning to the following shorthand, which we shall refer to as a *Stratonovich SDE* on the (compact) manifold $\mathcal{M}$:

$$\begin{cases} dX(t) = \sum_{k=1}^{r} A_k(X(t))\circ dW^k(t) + A_0(X(t))dt\,, \\ X(0) = x\,, \qquad x \in \mathcal{M}. \end{cases} \tag{II.12}$$

The meaning associated to (II.12) is that we consider a strong solution X of (II.11) (with initial conditions $\xi = x$ with probability one) and then define a solution to (II.12) to be the random variable $X_x = F(x, W)$, where F is the map which defines our strong solution X.

Let us now discuss the notion of generator associated to the strong solution of (II.12).

First consider a more general case. For $x \in \mathcal{M}$, let X_x be a continuous stochastic process adapted to a filtration $\mathscr{F}_t$ in the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. For simplicity we consider a stochastic process defined for all $t \in [0, \infty)$ and with values in the space of continuous maps $[0, \infty) \to \mathcal{M}$ (where $\mathcal{M}$ is always assumed to be compact) such that $X(0) = x$ (where equality means $\mathbb{P}$-a.s.).

Let P_x be the probability law associated to the random variable $(X_x(t))$. This means that P_x is the image measure under the measurable mapping $X_x = (X_x(t))$ of the probability measure $\mathbb{P}$. Assume that $x \mapsto P_x$ is universally measurable[15] and that P_x is uniquely determined by $x \in \mathcal{M}$.[16] Moreover assume that there exists a linear operator $\mathfrak{L}$ with domain $\mathrm{Dom}(\mathfrak{L})$ in $C(\mathcal{M})$, such that for every $f \in \mathrm{Dom}(\mathfrak{L})$,

$$X_f(t) \overset{\text{def}}{=} f(X(t)) - f(X(0)) - \int_0^t (\mathfrak{L}f)(X(s))\,\mathrm{d}s$$

is a martingale with continuous sample paths and adapted to the filtration $\mathscr{F}_t$ associated to $X_x(t)$ (cf. [18, Chapter IV, Theorem 5.2, p.207]). Then the family of probability measures $(P_x)_{x\in\mathcal{M}}$ is called a **diffusion generated** by the operator $\mathfrak{L}$.

When, for every $x \in \mathcal{M}$, X_x is the stochastic process on the manifold $\mathcal{M}$ which is the strong solution to (II.12) with initial condition $X(0) = x$, then we have the following [18, Chapter V, Theorem 1.2, p.253].

The family of probability laws $(P_x)_{x\in\mathcal{M}}$, associated with the strong solutions X_x to (II.12) with initial conditions $x \in \mathcal{M}$, is a diffusion generated by the operator

$$\mathfrak{L} \overset{\text{def}}{=} \frac{1}{2}\sum_{j=1}^{r} A_k(A_k f) + A_0 f, \quad f \in C^\infty(\mathcal{M}),$$

(where, as before, the manifold $\mathcal{M}$ is assumed to be compact) and $A_0, A_1, \ldots, A_r \in \mathcal{X}(\mathcal{M})$ are interpreted as differential operators with common domain $C^\infty(\mathcal{M})$.

We now go back to our setting where the manifold $\mathcal{M} = \mathbf{Spin}(2n+1)$ and collect the specialized version of the standard results discussed in the previous paragraph. Doing so we give the characterization of the generators $\overline{P_0}$ and $\overline{P}$ (defined in §**5.1**), as promised above, in terms of stochastic processes.

§ 5.4 Notation We need to perform a slight change of notation. In this section we use the symbols $A_0, A_1, \ldots$ to denote vector fields. In the last section we used the symbols X_{ij}, $i, j = 1, \ldots, 2n+1$, to denote both elements of the Lie algebra $\mathfrak{so}(2n+1)$, and the associated vector fields. We now change the notation and denote by A_k what we denoted in the last section by $X_{2n+1,k}$, that is we define:

$$A_k \overset{\text{def}}{=} X_{2n+1,k}, \quad k = 1, \ldots, 2n.$$

[14] The idea behind this result is that the manifold $\mathcal{M}$ is locally diffeomorphic to $\mathbb{R}^d$ where d is the dimension of the manifold $\mathcal{M}$. This means that locally the SDE (II.12) (and hence (II.11)) can be written in coordinates as a standard SDE on $\mathbb{R}^d$. One can apply standard results about existence and uniqueness of solutions of SDEs to these local realizations. Finally one needs to patch together different local solutions into a global solution. Details can be found (as usual) in the above mentioned [18].

[15] Cf., e.g. [18, p.1].

[16] These conditions are actually automatically satisfied when X_x is the strong solution to (II.12).

As before, we will not use different notations when we consider A_k as a vector field or a differential operator. Similarly, the differential operator B_0 defined after (II.7) will be considered also as a vector field without changing notation.

§ 5.5 Lemma: Stochastic processes associated to P_0 and P. 1. The following Stratonovich SDEs on $\mathbf{Spin}(2n+1)$

$$(P) \quad \begin{cases} dY(t) = \sum_{k=1}^{2n} \sqrt{E'_k} A_k(Y(t)) \circ dW^k(t) + B_0(Y(t))dt\,, \\ Y(0) = x\,, \qquad x \in \mathbf{Spin}(2n+1) \end{cases}$$

$$(P_0) \quad \begin{cases} dX(t) = \sum_{k=1}^{2n} \sqrt{E'_k} A_k(X(t)) \circ dW^k(t) \\ X(0) = x\,, \qquad x \in \mathbf{Spin}(2n+1), \end{cases}$$

where $(W^k(t), k = 1, \dots, 2n)$, is a standard Brownian motion in $\mathbb{R}^{2n}$, are well defined and admit a *unique strong solution*.

2. The operators $\overline{P}$ and $\overline{P_0}$ are the generators of the *diffusion processes* given by the strong solutions of (P), (P_0) respectively.

PROOF. For the first statement see [18, Chapter 5, Theorem 1.1 p.249]. The second statement is proved in [18, Theorem 1.2, p.253]. □

The following result describes the time evolution given by the quasi-Hamiltonian $H_{\underline{E}}$ in term of a stochastic diffusion process generated by the second order part in $H_{\underline{E}}$.

§ 5.6 Theorem. We have the following representations of the semigroup generated by $H_{\underline{E}}$

$$(f, e^{-t\overline{H_{\underline{E}}}} g)_{L^2(\mathbf{Spin}(2n+1))} = \mathbb{E}_X \left[\overline{f(0)} \left(e^{\mathrm{i}\, t\, \overline{B_0}} g \right) \left(X(t) \right) \right], \tag{II.13}$$

where $\mathbb{E}_X$ denotes the expectation with respect to the process generated by P_0, $\overline{H_{\underline{E}}}$ and $\overline{B_0}$ denote the closure of the operators, $\overline{f(0)}$ denotes complex conjugation, and $f, g \in C(\mathbf{Spin}(2n+1)) \subset L^2(\mathbf{Spin}(2n+1))$.

PROOF. First note that $e^{-t\overline{H_{\underline{E}}}}$ is a bounded operator for all $t \in \mathbb{R}^+$. Hence f, g can be taken in $L^2(\mathbf{Spin}(2n+1))$. The equality follows directly from the representation of the Hamiltonian as $H_{\underline{E}} = P_0 + \mathrm{i} B_0$, the fact that $\overline{P_0}$ and B_0 strongly commute, and the strong Markov property of P_0 (which is a consequence of point 2. of §5.5):

$$\begin{aligned} (f, e^{-t\overline{H_{\underline{E}}}} g)_{L^2(\mathbf{Spin}(2n+1))} &= (f, e^{-t\,(\overline{P_0 + \mathrm{i} B_0})} g)_{L^2(\mathbf{Spin}(2n+1))} \\ &= (f, e^{-t\,\overline{P_0}} e^{\mathrm{i}\, t\, \overline{B_0}} g)_{L^2(\mathbf{Spin}(2n+1))} \\ &= \mathbb{E}_X \left[\overline{f(0)} \left(e^{\mathrm{i}\, t\, \overline{B_0}} g \right) \left(X(t) \right) \right]. \end{aligned}$$ □

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III

Schwinger functions for Euclidean Dirac Fermions and induced representations

Abstract

We give a detailed analysis on how the 2-point Schwinger function (distribution) for the Dirac field is obtained from first principles. In particular we show how the Schwinger function is uniquely determined from the Lorentz covariance of the theory, arriving in a natural way to exhibit its vector valued character. To this vector valued Schwinger function one can associate a bilinear form, however this procedure is not unique.

Contents

1 Introduction

The goal of this chapter is to introduce the Schwinger two-point function for the Dirac field in the most natural way.

To do so we start from the Wigner-Mackey analysis of induced representations as applied to the full Poincaré spin group. This topic is often discussed in the literature. We give here a full account clarifying some points which might result as obscure from the standard treatment given in the literature.

Then we introduce the notion of Wightman two-point function. We explain the relation of the Wightman two-point function for the free Dirac field with the Wigner-Mackey analysis of the representations of the Poincaré spin group mentioned above.

Finally we pass from the two-point Wightman function for Dirac field to the two-point Schwinger function employing the Bargmann-Hall-Wightman theorem. By this theorem, the two point Wightman function determines uniquely the two-point Schwinger function. Whereas the two-point Wightman function is seen, e.g. from our analysis, to originate from a scalar product in a Hilbert space, when we pass to the two-point Schwinger function such a connection is lost. In particular, if we want to associate to the two-point Schwinger function a Euclidean invariant bilinear form, we can only do so in a non canonical way.

The structure of this chapter is as follows. In **section 2** we give some remarks about the projective representations of a symmetry group in quantum mechanics and how they correspond to true representations of the universal cover of such a symmetry group. Then we specialize to the situation where the symmetry group is chosen to be the full Poincaré group. In the last part of the section we explain how the Euclidean rotation group in four dimensions $\mathbf{SO}(4)$ is obtained from the proper, orthochronous Lorentz group $\mathbf{SO}^0(1,3)$ via complexification.

In **section 3** we introduce some aspects of the standard theory of Wigner and Mackey about induced representations and its applications to a special class of Lie groups split into a semidirect product. In the last subsection, we specialize the analysis to the case of the universal cover of the proper orthochronous Lorentz group.

In **section 4** we discuss the two fundamental representations for the 1-particle Hilbert space describing the possible states of a single $1/2$-spin massive elementary (quantum) particle. These two fundamental representation are usually called Wigner representation and Dirac representations. The Dirac representation is constructed in the last subsection and is obtained from what we called the "covariant representation".

In **section 5** we describe the Wightman and Schwinger two-point functions. We introduce Wightman two-point functions in relation with the analysis, done in section 4, of the Dirac realization of the 1-particle Hilbert space. The Schwinger functions are discussed in the last subsection. There we also discuss the ambiguity which arises if one wants, as it is usually the case, interpret the two-point Schwinger function as a kernel in a Euclidean invariant, bilinear form.

2 Some preliminary definitions.

Conventions. We define the Minkowski pseudo metric g on $\mathbb{R}^4$ to be given, in standard form, by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We employ the convention by which an **Hermitian scalar product** is a positive definite form $(\cdot,\cdot)$, *antilinear* in the *left* argument and *linear* in the right one.

Symmetries in quantum mechanics

Comment. In the standard presentation of quantum mechanics one usually interprets the absolute value squared of Schrödinger wave functions as probability densities. This physical assumption leads mathematically, to associate physical states with *normalized* wave functions. This in turn leads to some complications because, instead of considering wave functions in a linear space, in particular a Hilbert space, we have to consider them as elements in the *projectivization* of some Hilbert space. Moreover, the symmetry properties of the theory are represented in this projective Hilbert space only up to a phase, meaning that we have to look, on the projective Hilbert space, for *projective representations* of any symmetry group of our physical system. This picture, with a projective Hilbert space and projective representations, is more complicated than a description just in terms of a Hilbert space and actual (linear, unitary) representations because the constraint of having *normalized* states is a non linear constraint. Luckily, in most cases, one can define a new (linear) Hilbert space and a new group of transformations, acting on the Hilbert space by a (quasilinear) representation (to be defined below). The original projective system can then be uniquely reconstructed by this linear model. This analysis is due mainly to Wigner and Bargmann. We give a here the skeleton of this analysis mainly for two reasons. First we want to introduce some notation and motivate the next subsection where we pass from the proper orthochronous Lorentz group $\mathbf{SO}^0(1,3)$ to its double cover $\mathbf{Spin}^0(1,3)$. Second, the very existence of Fermions is justified mathematically by the, above mentioned, projective nature of quantum mechanics. In particular, since the sign of a wave function has no meaning from the projective standpoint, we can allow a wave function, describing *indistinguishable* particles, to be also antisymmetric in its variables, not just symmetric (and in general we can allow for parastatistics).

§ 2.1 Let $\mathcal{H}$ be a (separable) complex Hilbert space. We denote by $\mathbb{P}(\mathcal{H})$ the **projective Hilbert space** of $\mathcal{H}$. That is, we let $\mathbb{P}(\mathcal{H}) = (\mathcal{H} \setminus 0)/\sim$, where $\sim$ denotes the equivalence relations which identifies two vectors, $v \sim w$, $v, w \in \mathcal{H}$, if there exists a complex number $\lambda \in \mathbb{C}$ such that $v = \lambda w$.

§ 2.2 Let $\mathbf{G}$ be a Lie group and $\mathcal{H}$ be a complex Hilbert space. We denote by $\mathbb{C}^\star$ the multiplicative group of non-zero complex scalars. We call **projective representation** of $\mathbf{G}$ a Lie group endomorphism $\pi : \mathbf{G} \to \mathrm{Aut}(\mathcal{H})/\mathbb{C}^\star$, where $\mathrm{Aut}(\mathcal{H})$ denotes the space of invertible, bounded, linear maps from $\mathcal{H}$ to itself.

§ 2.3 We call a map ϕ from a complex vector space V into itself **antilinear**, when it satisfies, for all $v, w \in V$, $\lambda \in \mathbb{C}$, $\phi(v+w) = \phi(v) + \phi(w)$, $\phi(\lambda v) = \overline{\lambda}\phi(v)$, where $\overline{\lambda}$ denotes complex conjugation. Let $\mathcal{H}$ be a complex Hilbert space with scalar product $(\cdot,\cdot)$. An **antiunitary** map is an antilinear map $T\mathcal{H} \to \mathcal{H}$ such that $(Tv, Tw) = (w, v)$, for all $v, w \in V$. Let us call **quasiunitary** a map which is either unitary or antiunitary. We note that quasilinear maps are *real-linear* in the sense that a quasilinear map ϕ satisfies $\phi(rv) = r\phi(v)$ for all $r \in \mathbb{R}$, $v \in V$.

§ 2.4 We define a **quasiunitary representation** of a Lie group $\mathbf{G}$ to be a continuous group-homomorphism ρ of $\mathbf{G}$ into the space of quasiunitary endomorphisms of a complex Hilbert space $\mathcal{H}$.

§ 2.5 We define the **space of symmetry transformations** of a complex Hilbert space $\mathcal{H}$ with scalar product $(\cdot,\cdot)$ to be the topological group of all real-linear (real-linear is needed because we want to allow for antilinear maps which are real-linear but *not* complex-linear) maps ϕ from $\mathcal{H}$ to $\mathcal{H}$ such that $|(\phi(v), \phi(w))| = |(v, w)|$, where group structure is given by composition and the topology is the operator norm topology. We define a **quasiunitary projective representation** to be a continuous group-homomorphism π into the space of symmetry transformations. We note that in some references (cf. e.g. [26]) what we call a "quasiunitary projective representation" is simply called a "projective representation".

§ 2.6 Remark. One could say that a projective representation of a group $\mathbf{G}$ adds more structure to just a representation of a group $\mathbf{G}$. In some sense this "extra structure" is at the core of the difference between quantum an classical mechanics. We have in mind the case of the Heisenberg group [25, 2]. It is often convenient to go from a *projective representation* of a group $\mathbf{G}$ to just a *representation* but of a larger group $\mathbf{G}'$. This possibility is very convenient because it dispenses us of the *non-linear* nature of a projective space. Moreover, often, the group $\mathbf{G}'$ is "nicer" than the original group $\mathbf{G}$. We briefly describe this process (of going from a *projective representation* of $\mathbf{G}$ to a *representation* of $\mathbf{G}'$) following [7, 6, Sections III.5 and VIII.4]

§ 2.7 Let $\mathbf{N}, \mathbf{G}$ be topological groups with $\mathbf{N}$ Abelian. A **central extension** of $\mathbf{N}$ by $\mathbf{G}$ is a triple $(\mathbf{G}', i, j)$ where $\mathbf{G}'$ is a topological group, i is a homeomorphic isomorphism of $\mathbf{N}$ onto a closed subgroup $\mathbf{N}'$ of the center of $\mathbf{G}'$, and j is an open homomorphism of $\mathbf{G}'$ onto $\mathbf{G}$ whose kernel coincides with $\mathbf{N}'$. The triple $(\mathbf{G}', i, j)$ is usually displayed as a short exact sequence $\mathbf{N} \to \mathbf{G}' \to \mathbf{G}$.

§ 2.8 Proposition. *Let π be a (non trivial) unitary projective representation of a topological group* $\mathbf{G}$*. Then π is constructible from a unique (up to isomorphisms) central extension* $\mathbf{U}(1) \to \mathbf{G}' \to \mathbf{G}$ *of the circle group* $\mathbf{U}(1)$ *by* $\mathbf{G}$*.*

PROOF. Cf. [6, p. 834]. □

Lorentz and Poincaré groups, and universal covers, complexification

Comment. In this subsection we collect basic facts about the Lorentz and Poincaré group, their universal covers, and complexification. We employ the, by now, standard notation of Clifford algebras and Spin and Pin groups (cf. e.g. [22, 12]). We mention the problem of describing all the inequivalent Lie groups which are universal cover of the *full* Poincaré group, meaning the Poincaré group including space-time reflections.

§ 2.9 We call **Lorentz group** $\mathcal{L} \stackrel{\text{def}}{=} \mathbf{O}(1,3)$ (also sometimes called **full Lorentz group**, or **homogeneous Lorentz group**) the group of linear maps $\mathbb{R}^4 \to \mathbb{R}^4$ which leave invariant the Minkowski metric. Under this definition the Lorentz group has four connected components and includes three dimensional rotations, boosts, as well as spacetime reflections. It is customary to denote the connected component of the identity of $\mathcal{L}$ by $\mathcal{L}_+^\uparrow$. Said differently we have $\mathcal{L}_+^\uparrow \stackrel{\text{def}}{=} \mathbf{SO}^0(1,3)$, where $\mathbf{SO}^0(1,3)$ is the connected component of the identity of $\mathbf{SO}(1,3)$.

§ 2.10 We call **Poincaré group** $\mathcal{P}$ (also called **inhomogeneous Lorentz group**) the semidirect product of the Lorentz group acting on the group of four-dimensional translations, that is $\mathcal{P} \stackrel{\text{def}}{=} \mathbb{R}^4 \rtimes \mathbf{O}(1,3)$, where $\mathbb{R}^4$ denotes here the additive, Abelian, group of four-dimensional translations. Its connected subgroup is $\mathcal{P}_+^\uparrow \stackrel{\text{def}}{=} \mathbb{R}^4 \rtimes \mathbf{SO}^0(1,3)$.

§ 2.11 As discussed above, when dealing with *projective representations* of these groups, is convenient to consider their universal covers. We denote by $\widetilde{\mathcal{L}}_+^\uparrow$, respectively $\widetilde{\mathcal{P}}_+^\uparrow$ the universal cover of $\mathcal{L}_+^\uparrow$, respectively $\mathcal{P}_+^\uparrow$:

$$\widetilde{\mathcal{L}}_+^\uparrow \stackrel{\text{def}}{=} \mathbf{Spin}^0(1,3), \qquad \widetilde{\mathcal{P}}_+^\uparrow \stackrel{\text{def}}{=} \mathbb{R}^4 \rtimes_\tau \mathbf{Spin}^0(1,3),$$

where we have employed the notation $\rtimes_\tau$ to point out that $\mathbf{Spin}^0(1,3)$ acts on $\mathbb{R}^4$ via the covering map $\tau : \mathbf{Spin}^0(1,3) \to \mathbf{SO}^0(1,3)$. Let us denote by $\mathbf{SL}(2;\mathbb{C})_\mathbb{R}$ the complex Lie group $\mathbf{SL}(2;\mathbb{C})$ seen as a *real* Lie group of twice its dimension. Then we have (cf. e.g. [12, p. 56])

$$\mathbf{Spin}^0(1,3) \cong \mathbf{SL}(2;\mathbb{C})_\mathbb{R}.$$

§ 2.12 Choice of universal cover. The groups $\mathcal{L}$ and $\mathcal{P}$ are *non–connected–Lie* groups hence more care is needed when passing to the universal covers: cf. [19, 5, 24, 27]. In particular the universal cover is not unique. Since our primary interest is toward Dirac Fermions we follow the common practice choosing for universal covers of respectively the Lorentz and Poincaré group the following disconnected Lie groups

$$\widetilde{\mathcal{L}} \stackrel{\text{def}}{=} \mathbf{Pin}(1,3), \qquad \widetilde{\mathcal{P}} \stackrel{\text{def}}{=} \mathbb{R}^4 \rtimes_\tau \mathbf{Pin}(1,3).$$

Note that we have the following short exact sequences (cf. [12, Thorem 2.10])

$$\mathbb{Z}_2 \to \mathbf{Spin}(1,3) \to \mathbf{SO}(1,3), \qquad \mathbb{Z}_2 \to \mathbf{Pin}(1,3) \to \mathbf{O}(1,3).$$

In particular these universal covers are “**double covers**”.

§ 2.13 A **complexification** of a Lie group $\mathbf{G}$ is a pair $(\mathbf{F}, i)$ of a complex analytic group $\mathbf{F}$ and a Lie group homomorphism $i : \mathbf{G} \to \mathbf{F}$ such that the following universal property is satisfied: Given another complexification $(\mathbf{F}', i')$ there exists a unique analytic group homomorphism $\phi : F \to F'$ such that the diagram

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[dr, "i'"] & F \arrow[d, "\phi"] \\ & F' \end{tikzcd}$$

commutes.

§ 2.14 The complexification of $\mathbf{Spin}^0(1,3)$ is isomorphic (as a complex Lie group, that is as an analytic group) to $\mathbf{Spin}(4;\mathbb{C}) \cong \mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})$. Note that $\mathbf{Spin}^0(1,3)$ is isomorphic (as a Lie group) to $\mathbf{SL}(2,\mathbb{C})_{\mathbb{R}}$, that is to $\mathbf{SL}(2,\mathbb{C})$ seen as a *real* six-dimensional Lie group. On the other hand we have that $\mathbf{Spin}(4;\mathbb{C})$ splits into the direct product $\mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})$. However this time $\mathbf{SL}(2,\mathbb{C})$ is regarded as a *complex*, three-dimensional, Lie group. Let $\mathbf{Spin}(4)$ be the universal cover of the real Lie group $\mathbf{SO}(4)$ (cf. e.g. [20, p. 151]). The Lie group $\mathbf{Spin}(4)$ is *not* simple (but only semisimple), indeed one has $\mathbf{Spin}(4) \cong \mathbf{SU}(2)\times\mathbf{SU}(2)$ (cf. [12, p. 50]), where $\mathbf{SU}(2) \cong \mathbf{Spin}(3)$ (cf. e.g. [20, p. 152]). By the universal property, the complexification preserves the Cartesian product structure, in particular the complexification of $\mathbf{Spin}(4) \cong \mathbf{SU}(2) \times \mathbf{SU}(2)$ is $\mathbf{Spin}(4,\mathbb{C}) \cong \mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})$. It will be important the fact that the groups $\mathbf{Spin}^0(1,3)$, $\mathbf{Spin}(4,\mathbb{C})$, $\mathbf{Spin}(4)$ can all be realized inside the same complex Clifford algebra $\mathbb{C}\ell(4)$ (we will say more about this below).

§ 2.15 Let us consider the Lie group $\mathbf{SO}(4)$ of Euclidean rotations in four dimensions and the proper, orthochronous, Lorentz group $\mathbf{SO}^0(1,3)$. The complexification of each Lie group is isomorphic to $\mathbf{SO}(4,\mathbb{C})$. To define and manipulate Wightman and Schwinger functions, we will need to fix embeddings of $\mathbf{SO}(4)$ and $\mathbf{SO}^0(1,3)$ into $\mathbf{SO}(4,\mathbb{C})$. In this section we explicitly construct such embeddings.

Consider $\mathbf{SO}(4)$ as the real Lie subgroup of $\mathbf{SO}(4,\mathbb{C})$ consisting of those matrices in $\mathbf{SO}(4,\mathbb{C})$ all of whose entries are real. In this way we have defined and embedding of $\mathbf{SO}(4)$ into $\mathbf{SO}(4,\mathbb{C})$. Let us parameterize $\mathbf{SO}(4)$ by generalized Euler angles (cf. e.g. [28, Section 9.1.5 (8), p. 11]), that is, local coordinates:

$$(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6), \quad \theta_1 \in [0,2\pi),\ \theta_j \in [0,\pi), 2 \le j \le 6.$$

Any element R in $\mathbf{SO}(4)$ is an analytic functions in these six real parameters. We analytically continue these functions to functions on a set of 6 complex parameters. These 6 complex parameters now parameterize $\mathbf{SO}(4,\mathbb{C})$ in such a way that if we restrict to the real part we obtain an element of $\mathbf{SO}(4)$ (as a Lie subgroup of $\mathbf{SO}(4,\mathbb{C})$).

Let us now consider the proper orthochronous Lorentz group $\mathbf{SO}^0(1,3)$. We want to find an embedding of $\mathbf{SO}^0(1,3)$ into $\mathbf{SO}(4,\mathbb{C})$. Let us introduce a set of generalized Euler angles for $\mathbf{SO}^0(1,3)$, that is a set of local coordinates:

$$(\boldsymbol{\psi},\boldsymbol{\theta}) = (\psi_1,\psi_2,\psi_3,\theta_1,\theta_2,\theta_3), \quad \psi_1,\psi_2,\psi_3 \in \mathbb{R},\ \theta_1,\theta_3 \in [0,2\pi), \theta_2 \in [0,\pi),$$

where ψ_1,ψ_2,ψ_3 parametrize Lorentz boosts and $\theta_1,\theta_2,\theta_3$ parametrize three dimensional rotations. Let $R \in \mathbf{SO}(4) \subset \mathbf{SO}(4,\mathbb{C})$. We denote by $\widetilde{R}$ the analytic continuation of R to the complex coordinates which parametrize $\mathbf{SO}(4,\mathbb{C})$. Then there exists a unique element $R_\Lambda \in \mathbf{SO}(4)$ such that its analytic continuation $\widetilde{R}_\Lambda$ satisfies

$$\Lambda(\boldsymbol{\theta},\boldsymbol{\psi}) = \begin{pmatrix} -\mathrm{i} & \mathbf{0} \\ \mathbf{0} & \mathbb{1}_3 \end{pmatrix} \widetilde{R}_\Lambda(\boldsymbol{\theta},\mathrm{i}\boldsymbol{\psi}) \begin{pmatrix} \mathrm{i} & \mathbf{0} \\ \mathbf{0} & \mathbb{1}_3 \end{pmatrix},$$

for all $\psi_1,\psi_2,\psi_3 \in \mathbb{R}$, $\theta_1,\theta_3 \in [0,2\pi), \theta_2 \in [0,\pi)$, where the angles $\boldsymbol{\theta},\boldsymbol{\psi}$ are as above and $\mathrm{i}\boldsymbol{\psi}$ denotes the set of three imaginary angles: $\mathrm{i}\psi_1, \mathrm{i}\psi_2, \mathrm{i}\psi_3$.

Let us denote by $(\theta_j)_{j=1}^6$ the subset of the subset of the parametrization of $\mathbf{SO}(4,\mathbb{C})$ such that $\theta_1 \in [0,2\pi),\ \theta_j \in [0,\pi), 2 \le j \le 6$. Similarly, let us denote by $(\psi_k, \mathrm{i}\theta_k)_{k=1}^3$ the subset of parameters of $\mathbf{SO}(4,\mathbb{C})$ such that $\psi_1,\psi_2,\psi_3 \in \mathbb{R},\ \theta_1,\theta_3 \in [0,2\pi), \theta_2 \in [0,\pi)$.

Then, we define the following embeddings:

$$\mathbf{SO}(4) \hookrightarrow \mathbf{SO}(4,\mathbb{C}), \quad R \mapsto \widetilde{R}\restriction_{(\theta_j)_{j=1}^6},$$

and

$$\mathbf{SO}^0(1,3) \hookrightarrow \mathbf{SO}(4,\mathbb{C}), \quad \Lambda \mapsto \widetilde{R}_\Lambda\restriction_{(\mathrm{i}\psi_k,\theta_k)_{k=1}^3}.$$

For later reference, we call these embeddings the **standard embeddings** of $\mathbf{SO}(4)$ and $\mathbf{SO}^0(1,3)$ into $\mathbf{SO}(4,\mathbb{C})$ respectively.

3 Remarks on induced representations

Comment. In this section we collect some basic facts about irreducible unitary representations of the Poincaré group (more precisely, its double cover). We will not go into the details of induced representations as discussed by Wigner and Mackey, but rather will restrict ourselves to a presentation of such results as will be necessary in the sequel. We refer the reader to [10, 1, 23, 22, 13, 8, 30].

Induced representations of locally compact groups

Comment. We give, in this subsection, a set of general definitions and results which will become important when we turn to the specific situation treated in this thesis, i.e. the case of the Poincaré group in connection with Wightman and Schwinger functions. There are multiple (equivalent) approaches to define induced representations. We select the one that we feel is best suited for the application that we have in mind.

§ 3.1 Let $\mathbf{G}$ be a locally compact topological group, $\mathbf{H}$ a closed subgroup, and π a unitary representation of $\mathbf{H}$ on a complex Hilbert space $\mathcal{H}(\pi)$. The quotient $\mathbf{G}/\mathbf{H} = \{g\mathbf{H} : g \in \mathbf{G}\}$ equipped with the quotient topology is a locally compact Hausdorff space. $\mathbf{G}$ acts on the quotient $\mathbf{G}/\mathbf{H}$ by left multiplication, that is we have a continuous map $\mathbf{G} \times \mathbf{G}/\mathbf{H} \to \mathbf{G}/\mathbf{H}$, $(g, \omega) \mapsto g \cdot \omega$, where, if $\omega = g'H \in \mathbf{G}/\mathbf{H}$ for some $g' \in \mathbf{G}$, then $g \cdot \omega = gg'H$. Given an element $g \in \mathbf{G}$ and Borel measure μ on $\mathbf{G}/\mathbf{H}$ we denote by μ^g the push-forward measure given by $\mu^g(B) \stackrel{\text{def}}{=} \mu(g \cdot B)$ for any Borel set $B \in \mathbf{G}/\mathbf{H}$. Finally, we assume that a quasi-invariant measure μ is given on $\mathbf{G}/\mathbf{H}$, where a quasi-invariant measure is a regular Borel measure μ such that μ^g is absolutely continuous with respect to μ for all $g \in \mathbf{G}$.

§ 3.2 Consider the special case where $\mathbf{G}$ is a finite group and $\mathbf{H}$ a subgroup of $\mathbf{G}$. In the case of induced representations for finite groups (cf. e.g.[18, Chapter 8]), it is often useful to fix a section, that is a way to identify cosets in $\mathbf{G}/\mathbf{H}$ with elements in $\mathbf{G}$. Going back to the case of $\mathbf{G}$ a topological group and $\mathbf{H}$ a topological subgroup of $\mathbf{G}$, we now impose on a section the extra condition of being measurable. That is, we define a **measurable cross section** of a topological group $\mathbf{G}$ with respect to a topological subgroup $\mathbf{H}$ to be a measurable map $s : \mathbf{G}/\mathbf{H} \to \mathbf{G}$ such that $q(s(x)) = x$, for all $x \in \mathbf{G}/\mathbf{H}$, where $q : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is the quotient map. In the following we will refer to a measurable cross section just as **section**. Such a section always exists at least if we assume $\mathbf{G}$ to be a second countable, locally compact, topological group (cf.[8, pp.167, 71]). We assume that we have fixed a choice of a measurable cross section. Such a choice is, in general, not natural.

§ 3.3 Let us set $\mathcal{M} \stackrel{\text{def}}{=} \operatorname{Ran} s$ to be the image of s. Note that $\mathcal{M}$ is then a Borel set in $\mathbf{G}$ (because the quotient map $q : \mathbf{G} \to \mathbf{G}/\mathbf{H}$ is continuous, hence (Borel-)measurable) and intersects every $\mathbf{H}$-orbit in $\mathbf{G}$ in precisely one point. (Conversely, if we are given a Borel set $\mathcal{M}' \subset \mathbf{G}$ which intersect each $\mathbf{H}$-orbit in one point, we can define a measurable section s' by setting $s' \stackrel{\text{def}}{=} q\restriction_{\mathcal{M}'}$.) By definition, the preimage of the section s coincides with the restriction $q\restriction_{\mathcal{M}}$ of the quotient map q to $\mathcal{M}$. Hence, if we denote by μ^s the pushforward of the measure μ under s, then μ^s is supported on $\mathcal{M} \subset \mathbf{G}$ and $\mu^s(B) = \mu(q\restriction_{\mathcal{M}}(B))$, for any Borel set $B \subset \mathcal{M}$.

§ 3.4 The define the **representation** of $\mathbf{G}$ **induced** by the representation $(\pi, \mathcal{H}(\pi))$ of $\mathbf{H}$ to be the pair $\left(U^{\pi}_{\mu,s}, L^2(\mathcal{M}, \mu^s; \mathcal{H}(\pi))\right)$ defined as follows. We denote by

$$U^{\pi}_{\mu,s} : \mathbf{G} \to \mathbf{U}\left(L^2(\mathcal{M}, \mu^s; \mathcal{H}(\pi))\right),$$

the continuous group-homomorphism from the locally compact group $\mathbf{G}$ to the space of unitary operators on the complex Hilbert space $L^2(\mathcal{M}, \mu^s; \mathcal{H}(\pi))$. defined by:

$$U^{\pi}_{\mu,s}(g) f(p) \stackrel{\text{def}}{=} \sqrt{\frac{\mathrm{d}\mu_{g^{-1}}}{\mathrm{d}\mu}(q(p))}\, \pi\left(s(q(p))^{-1}\, g\, s(g^{-1} \cdot q(p))\right) f(g^{-1} \cdot q(p)),$$
$$p \in \mathcal{M}, \quad f \in L^2(\mathcal{M}, \mu^s; \mathcal{H}(\pi)), \quad \text{(III.1)}$$

where $\mu_{g^{-1}}$ is the push forward of the measure μ under the left action of $\mathbf{G}$ on $\mathbf{G}/\mathbf{H}$, that is $\int_{\mathbf{G}/\mathbf{H}} f(\omega)\,\mathrm{d}\mu_{g^{-1}}(\omega) = \int_{\mathbf{G}/\mathbf{H}} f(g\cdot\omega)\,\mathrm{d}\mu(\omega)$; moreover $\frac{\mathrm{d}\mu_{g^{-1}}}{\mathrm{d}\mu}(\omega)$ represents the Radon–Nikodym derivative of $\mu_{g^{-1}}$ with respect to *mu* at $\omega \in \mathbf{G}/\mathbf{H}$.

§ 3.5 We have presented here one way to induce a representation. There are other (unitarily equivalent) realizations of an induced representations (cf. e.g. [10, 1, 6]). When it is irrelevant which of these equivalent models we use, we will employ the notation $\mathrm{Ind}_{\mathbf{H}}^{\mathbf{G}}(\pi)$ to denote the representation of $\mathbf{G}$ induced by the representation π of the subgroup $\mathbf{H}$.

Wigner-Mackey theory for semidirect products

Comment. The objective of this subsection is to arrive at theorem §**3.13** which sometimes goes by the name of "Mackey machine". This theorem is a cornerstone for characterizing all unitary irreducible representations (up to equivalence) for a special class of topological groups. This class includes finite dimensional Lie groups which are semidirect products of a topological group and an Abelian topological group, and which satisfy an important condition regarding the orbits of the elements in the Abelian (normal) subgroup.

§ 3.6 Let $\mathbf{G}$ be a locally compact group. The set of equivalence classes of (continuous) unitary representations of $\mathbf{G}$ is, with the appropriate topology (Fell topology, cf. e.g.[10, p. 38]), a topological space, called the **dual space** of $\mathbf{G}$ and is denoted by $\widehat{\mathbf{G}}$.

§ 3.7 Let $\mathbf{N}$ be an *Abelian*, locally compact group. Then the dual space $\widehat{\mathbf{N}}$ can be given the structure of an Abelian topological group called the **dual group** (or **Pontryagin dual**) of $\mathbf{N}$. An element $\chi \in \widehat{\mathbf{N}}$ is (by definition) an irreducible unitary representation of $\mathbf{N}$ and is called a **character** of the Abelian group $\mathbf{N}$. By Schur's lemma, any character of $\mathbf{N}$ is a *one dimensional* unitary representation. Hence a character of $\mathbf{N}$ is a continuous homomorphism from $\mathbf{N}$ into the one dimensional torus $\mathbb{T}$ of complex numbers of modulus one. Therefore we can make $\widehat{\mathbf{N}}$ into a locally compact Abelian group by taking as our product the pointwise multiplication of complex valued functions, and as topology the compact-open topology (giving uniform convergence on compact sets).

§ 3.8 In this subsection we consider the case where we are given a topological group $\mathbf{G}$ with a normal, Abelian, closed subgroup $\mathbf{N}$. In particular we have in mind the "semidirect product case" where $\mathbf{G} = \mathbf{N} \rtimes \mathbf{L}$ is a semidirect product of an Abelian topological group $\mathbf{N}$ with a topological group $\mathbf{L}$.

§ 3.9 We define an action $\chi \mapsto g\cdot\chi$ of $\mathbf{G}$ on the dual group $\widehat{\mathbf{N}}$ of $\mathbf{N}$, by letting $(g\cdot\chi)(n) = \chi(g^{-1}ng)$, for $g \in \mathbf{G}$, $\chi \in \widehat{\mathbf{N}}$, $n \in \mathbf{N}$. This gives a jointly continuous map $\mathbf{G} \times \widehat{\mathbf{N}} \to \widehat{\mathbf{N}}$, $(g,\chi) \mapsto g\cdot\chi$. Note that $\mathbf{N}$, as a subgroup of $\mathbf{G}$, acts trivially on the dual space $\widehat{\mathbf{G}}$ of $\mathbf{G}$. If we specialize to the case of the semidirect product $\mathbf{G} = \mathbf{N} \rtimes \mathbf{L}$, then the action of $\mathbf{G}$ on $\widehat{\mathbf{N}}$ naturally descends to an action of $\mathbf{L}$ on $\widehat{\mathbf{N}}$.

§ 3.10 We let $\mathbf{G}_\chi \overset{\text{def}}{=} \{g \in \mathbf{G} : g\cdot\chi = \chi\}$, $\mathbf{G}(\chi) \overset{\text{def}}{=} \{g\cdot\chi : g \in G\}$ be, respectively, the **stabilizer** and the $\mathbf{G}$-**orbit** of a character[1] $\chi \in \widehat{\mathbf{N}}$. The stabilizer $\mathbf{G}_\chi$, $\chi \in \widehat{\mathbf{N}}$, is a closed subgroup of $\mathbf{G}$. If, as above, we specialize to the semidirect product case $\mathbf{G} = \mathbf{N} \rtimes \mathbf{L}$, then we define $\mathbf{L}_\chi$, as a subgroup of $\mathbf{L}$, to be the stabilizer of $\chi \in \widehat{\mathbf{N}}$ under the action of $\mathbf{L}$ on $\widehat{\mathbf{N}}$ as remarked in §**3.9**.

§ 3.11 Note that two $\mathbf{G}$-orbits never intersect unless they coincide. Therefore the set of $\mathbf{G}$-orbits of elements of $\widehat{\mathbf{N}}$ is the quotient space $\widehat{\mathbf{N}}/(\mathbf{G}\restriction_{\widehat{\mathbf{N}}}) = \{\mathbf{G}(\chi) : \chi \in \widehat{\mathbf{N}}\}$, where $\mathbf{G}\restriction_{\widehat{\mathbf{N}}}$ denotes the equivalence relation which makes of $x, y \in \widehat{\mathbf{N}}$ equivalent if and only if they lie on the same orbit under $\mathbf{G}$.

§ 3.12 A **cross-section** of the $\mathbf{G}$-orbits in $\widehat{\mathbf{N}}$, is defined to be a subset $\widehat{X}$ of $\widehat{\mathbf{N}}$ such that $X \cap O$ is a singleton for each $O \in \widehat{\mathbf{N}}/\mathbf{G}\restriction_{\widehat{\mathbf{N}}}$.

[1]We introduce here the stabilizer and the orbit for the action of $\mathbf{G}$ on the dual group $\widehat{\mathbf{N}}$ because we will only need these concepts in this context. The notion of a **stabilizer** (also called an **isotropy group** or a **little group**) and of a **G-orbit** are, in general, defined for a generic action of the group $\mathbf{G}$ on a generic set.

We are now ready to state the fundamental theorem in this subsection which is a version of what is often referred to as "Mackey machine" for semidirect products. The statement we present holds under weaker assumptions (cf. [10, p. 160-161]). The assumption we impose are more restrictive but are more intuitive, and wholly sufficient for our purposes.

§ 3.13 Theorem (Mackey). Suppose $\mathbf{G} = \mathbf{N} \rtimes \mathbf{L}$ is a semidirect product of a *second countable* topological group $\mathbf{L}$ acting on a *Abelian*, *second countable* topological group. Moreover, suppose there exists a *Borel set* $\widehat{X}$ in $\widehat{\mathbf{N}}$ which is a cross-section of the $\mathbf{G}$-orbits in $\widehat{\mathbf{N}}$. Then the dual space $\widehat{\mathbf{G}}$ is given by

$$\widehat{\mathbf{G}} = \left\{ \mathrm{ind}_{\mathbf{G}_\chi}^{\mathbf{G}} (\chi \times \rho) : \rho \in \widehat{\mathbf{L}_\chi}, \chi \in X \right\},$$

where $\chi \times \rho$ denotes the unitary representation $(\chi \times \rho, \mathcal{H})$, defined by taking as carrier space $\mathcal{H}$ the carrier space of the representation ρ, and letting $(\chi \times \rho)(n, l) = \chi(n)\rho(l)$, for all $(n, l) \in \mathbf{G} = \mathbf{N} \rtimes \mathbf{L}$.

The concrete case of $\mathbf{ISpin}^0(1,3) = \mathbb{R}^4 \rtimes \mathbf{Spin}^0(1,3)$

Comment. We now specialize Mackey's theorem in §3.13 to the specific case which interests us, i.e. the proper orthochronous Poincaré group.

§ 3.14 We specialize the discussion of the previous subsection to the case

$$\mathbf{ISpin}^0(1,3) = \mathbb{R}^4 \rtimes \mathbf{Spin}^0(1,3),$$

where $\mathbb{R}^4$ denotes the additive group of translations in the Euclidean four dimensional space $\mathbb{R}^4$, and $\mathbf{Spin}^0(1,3)$ is the universal cover of the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow = \mathbf{SO}^0(1,3)$. The semidirect product is induced by the covering map $\mathbf{Spin}^0(1,3) \to \mathbf{SO}^0(1,3)$ and the standard action of $\mathbf{SO}^0(1,3) \subset \mathbf{SO}(1,3)$ on $\mathbb{R}^4$ which defines $\mathbf{SO}(1,3)$. Let us remark that $\mathbf{ISpin}^0(1,3)$ is the universal cover of the proper orthochronous Poincaré group.

We translate the theorem in §3.13 to the present case.

§ 3.15 The action of $\mathbf{Spin}^0(1,3)$ on $\mathbb{R}^4$, in the semidirect product, coincides by hypothesis with the action of $\mathbf{SO}^0(1,3)$. The dual group $\widehat{\mathbb{R}}^4$ of $\mathbb{R}^4$ can be again identified with $\mathbb{R}^4$. Moreover, the action of $\mathbf{SO}^0(1,3)$ on the dual group of $\mathbb{R}^4$ can be identified with the action of $\mathbf{SO}^0(1,3)$ on $\mathbb{R}^4$ itself if we replace every element of $\mathbf{SO}^0(1,3)$ with its inverse. Hence the $\mathbf{Spin}^0(1,3)$-orbits in the dual group of $\mathbb{R}^4$ can be identified with the orbits of $\mathbb{R}^4$ itself under the standard action of $\mathbf{SO}^0(1,3)$. Each orbit (cf. e.g. [1, p. 517]) is one of the following subsets of $\mathbb{R}^4 \cong \widehat{\mathbb{R}}^4$, for some $m \geq 0$,

$$\begin{aligned}
\widehat{O}_m^+ &= \left\{\widehat{n} \in \widehat{\mathbb{R}}^4 : \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 = m^2,\ n_0 > 0\right\} \\
\widehat{O}_m^- &= \left\{\widehat{n} \in \widehat{\mathbb{R}}^4 : \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 = m^2,\ n_0 < 0\right\} \\
\widehat{O}_{\mathrm{i}m}^+ &= \left\{\widehat{n} \in \widehat{\mathbb{R}}^4 : \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 = -m^2, \right\} \\
\widehat{O}_0^+ &= \left\{\widehat{n} \in \widehat{\mathbb{R}}^4 : \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 = m^2,\ n_0 > 0\right\} \\
\widehat{O}_0^- &= \left\{\widehat{n} \in \widehat{\mathbb{R}}^4 : \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 + \widehat{n}_1^2 = m^2,\ n_0 < 0\right\} \\
\widehat{O}_0^0 &= \left\{\widehat{n} = (0,0,0,0)\right\}.
\end{aligned} \tag{III.2}$$

A cross-section $\widehat{X}$ must be a set in $\widehat{\mathbb{R}}^4$ which intersects every orbit in precisely one point. The list of sets of orbits in (III.2) implies that, if we let

$$\widehat{X}_> = \{(m,0,0,0) : m > 0\},\ \widehat{X}_< = \{(-m,0,0,0) : m > 0\},\ \widehat{X}_\mathrm{i} = \{(0,0,0,N) : N > 0\},$$
$$\widehat{X}_+ = \{(1/2,0,0,1/2)\},\ \widehat{X}_- = \{(-1/2,0,0,1/2)\},\ \widehat{X}_0 = \{(0,0,0,0)\},$$

then a possible choice of cross-section $\widehat{X}$ is the union

$$\widehat{X} = \widehat{X}_> \cup X_< \cup X_\mathrm{i} \cup X_+ \cup X_- \cup X_0. \tag{III.3}$$

For every $\widehat{n} \in \widehat{X}$ we have a corresponding $\mathbf{L}_{\widehat{n}}$. In the various cases, we have the following isomorphisms (for details cf. e.g. [1, Section 17.2.C])

$$\mathbf{L}_{\widehat{n}} \cong \begin{cases} \mathbf{Spin}(3) \cong \mathbf{SU}(2) & \widehat{n} \in \widehat{X}_> \cup \widehat{X}_< \\ \mathbf{SL}(2, \mathbb{R}) & \widehat{n} \in \widehat{X}_\mathrm{i} \\ \mathbf{ISO}(2) & \widehat{n} \in \widehat{X}_+ \cup X_- \\ \mathbf{Spin}(1,3) & \widehat{n} \in \widehat{X}_0 \end{cases} .$$

Now the theorem in §**3.13** says that every unitary irreducible representation (up to isomorphism) of $\mathbf{G} = \mathbf{Spin}^0(1,3)$ is uniquely determined by a character $\chi = \widehat{n} \in \widehat{X}$ and one (up to isomorphism) unitary irreducible representation $(\rho, \mathcal{H})$ of the corresponding group $\mathbf{L}_{\widehat{n}}$. Moreover the theorem asserts that each such a representation (up to isomorphism) of $\mathbf{ISpin}^0(1,3)$ is induced from the representation $(\widehat{n} \times \rho, \mathcal{H})$ of $\mathbb{R}^4 \rtimes \mathbf{L}_{\widehat{n}}$ via the induced representation method.

4 One particle states

Comment. We present here the description of elementary particles with positive mass and spin $1/2$ as one (up to equivalence) irreducible unitary representation of the (double cover) of the Poincaré group. We give three equivalent unitary irreducible representations of the (double cover) of the Poincaré group, each in a separate subsection. Each of these representations is important. The contents of the subsections is the following.

1. In the first subsection, we discuss Wigner original construction. This construction has the advantage of being the most natural. Unfortunately it has the disadvantage that it is non-trivial to extend this unitary representation to a (non unitary) representation of the complexified Poincaré group. The representation of the complexified Poincaré group will be a pivotal point in the next section, where we will deal with Wightman theory (for relativistic spin 1/2 positive mass quantum fields).

2. In the second subsection, we discuss a "covariant representation". This representation is unitarily equivalent to the Wigner representation of point 1. but has an important new feature. In this representation, the spin operator is finite dimensional (that is, a matrix) and commutes (strongly) with the angular momentum operator. Moreover, this representation can be easily extended to a (non unitary) representation of the complexified Poincaré group. Nevertheless, this representation has a problem. The parity transformation (which reverses the orientation of three dimensional space) *cannot* be realized in this representation.

3. This leads to the approach in the third subsection. To be able to represent parity and keep the representation amenable to an extension to the complexified Poincaré group, we need to add nonphysical degrees of freedom.

   We stress that, from the perspective taken here, these extra degrees of freedom are *not* directly linked with antiparticles.

   Antiparticles are described, in Wigner representation, by another copy of the same Wigner representation (up to equivalence) which is used for particles. The only difference is that particles and antiparticles have opposite charge. We do *not* discuss anti-particles because, for our discussion, we do not need the notion of charge.

   Explicitly, we consider in the last subsection the covariant representations (in the sense of point 2.) together with a representation obtained from that by applying (a lift of) the parity transformation (to be defined there).

   Then, we consider the direct sum of these representations, hence obtaining a *reducible* representation. Finally, we project onto the *even parity* subspace of the representation space obtaining an irreducible representation, which we call *covariant 1-particle representation*. This representation will be the starting point in our presentation of Wightman theory.

This section concerns material discussed in the rich literature that has appeared since the paper by Wigner and Barmann. Nevertheless we believe that our presentation clarifies some points where the existing literature is somewhat "cryptic".

Wigner states with positive mass and spin one half

§ 4.1 In the following we will be interested only in the irreducible unitary representation of $\mathbf{ISpin}^0(1,3)$ which is uniquely (up to isomorphism) given by a fixed character $\widehat{n} = (m, 0, 0, 0) \in \widehat{X}_>$, for a fixed

$m > 0$, and the defining representation of $\mathbf{SU}(2)$, that is the representation of $\mathbf{SU}(2)$ given by *complex* two-by-two matrices with determinant one acting on $\mathbb{C}^2$. In the framework of quantum mechanics, the parameter m is associated to the physical observable of *rest mass*, and the choice of representation of $\mathbf{SU}(2)$ corresponds to the physical observable called *total angular momentum quantum number* and denoted by J. We have made the choice of a fixed $m > 0$ and the defining representation of $\mathbf{SU}(2)$ which corresponds to $J = 1/2$ (the numerical value $1/2$ corresponds to the highest weight (with appropriate normalization) of the representation of the Lie algebra $\mathfrak{so}(3)$ which corresponds to the defining representation of the Lie group $SU(2)$). For this reason we call the induced representation in this case the "positive mass and spin one half" representation of $\mathbf{ISpin}^0(1,3)$.

We now explicitly construct the induced representation in this "positive mass and spin one half" case.

§ 4.2 In the definition of induced representation we gave in §3.4, we assumed fixed a choice of measurable section $s : \mathbf{G}/\mathbf{H} \to \mathbf{G}$. In this case this translates to a measurable section $s : \mathbf{ISpin}^0(1,3)/(\mathbb{R}^4 \rtimes \mathbf{L}_{\hat{n}}) \to \mathbf{ISpin}^0(1,3)$, that is we have to find a Borel set $\mathcal{M} \subset \mathbf{ISpin}^0(1,3)$ which intersects every $(\mathbb{R}^4 \rtimes \mathbf{L}_{\hat{n}})$-orbit in exactly one point. We proceed as follow. Under the identification $\widehat{\mathbb{R}}^4 \cong \mathbb{R}^4$ the character $\hat{n}$ is identified to some point $n \cong \hat{n}$ in $\mathbb{R}^4$. Then we choose as $\mathcal{M}$ the $\mathbf{ISpin}^0(1,3)$-orbit $\mathbf{ISpin}^0(1,3)(n)$ of n under $\mathbf{ISpin}^0(1,3)$. This procedure does give the desired result because by hypothesis the group $\mathbb{R}^4 \rtimes \mathbf{L}_{\hat{n}}$ is the stabilizer $\mathbf{ISpin}^0(1,3)_{\hat{n}}$ of $\hat{n} \cong n$, which means that the $\mathbf{ISpin}^0(1,3)$-orbit of $\hat{n} \cong n$ is naturally identified with $\mathbf{ISpin}^0(1,3)/\mathbf{ISpin}^0(1,3)_{\hat{n}}$. Finally, note that, under the identification $n \cong \hat{n}$, the $\mathbf{ISpin}^0(1,3)$-orbit $\mathbf{ISpin}^0(1,3)(n)$ is identified with one of the orbits in the sets of orbits we have listed in (III.2). To summarize, in our case, we have picked a character $\hat{n} = (m,0,0,0)$, for some fixed $m > 0$. This gives a natural choice of section $s : \mathbf{ISpin}^0(1,3)/\mathbf{ISpin}^0(1,3)_{\hat{n}} \to \mathbf{ISpin}^0(1,3)$ which sends $\mathbf{ISpin}^0(1,3)/\mathbf{ISpin}^0(1,3)_{(m,0,0,0)}$ to the $\mathbf{ISpin}^0(1,3)$-orbit of $(m,0,0,0)$ which we denote by O_m^+. Moreover the invariant measure μ on $\mathbf{Spin}(1,3)/\mathbf{Spin}(3) \cong \mathbf{SO}^0(1,3)/\mathbf{SO}(3)$ is pushed forward by s to a measure on O_m^+ which we denote by μ_m. It is well known (cf. e.g. [16, IX.8 p. 70 and Theorem IX.37 p. 84]) that the measure μ_m is unique up to multiplication by a scalar. We can explicitly parametrize the manifold O_m^+ with a single chart diffeomorphic to $\mathbb{R}^3$. In this parametrization we set

$$\mu_m(\mathbf{p}) \overset{\text{def}}{=} \frac{1}{2\sqrt{\mathbf{p}^2 + m^2}}\,\mathrm{d}\mathbf{p}, \quad \mathbf{p} \in \mathbb{R}^3.$$

where $\mathrm{d}\mathbf{p}$ denotes the Lebesgue measure on $\mathbb{R}^3$.

§ 4.3 The induced representation $(U^{\pi}_{\mu,s}, L^2(\mathcal{M}, \mu^s; \mathcal{H}(\pi)))$ in the general definition we gave in §3.4 specialize in this "positive mass, spin one half" case to the irreducible unitary representation $\left(U^{m,1/2}, \mathcal{W}^{m,1/2}\right)$ of $\mathbf{ISpin}^0(1,3)$, given as follows. The representation $(\pi, \mathcal{H}(\pi))$ of $\mathbb{R}^4 \rtimes \mathbf{SU}(2)$ is given, in our case, by setting $\mathcal{H}(\pi) = \mathbb{C}^2$ and, for all $(n,u) \in \mathbb{R}^4 \rtimes \mathbf{SU}(3)$,

$$\pi(x,u) = e^{\mathrm{i}mx_0}\,u,$$

where x_0 is the first component of a vector $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$ representing a translation in four dimensions, and $u \in \mathbf{SU}(2)$ is seen as a 2-by-2 complex matrix acting on $\mathbb{C}^2$. Moreover in our case, since μ is *invariant*, we have

$$\sqrt{\frac{\mathrm{d}\mu_{g^{-1}}}{\mathrm{d}\mu}(q(p))} = 1, \quad p \in O_m^+.$$

The only non trivial step in our "translation" is to give an explicit form to the term

$$\pi\left(s(q(p))^{-1}\, g\, s(g^{-1} \cdot q(p))\right)$$

in (III.1) of §3.4. Let us identify $\mathbf{Spin}^0(1,3) \cong \mathbf{SL}(2,\mathbb{C})_{\mathbb{R}}$. We define the matrix

$$\widetilde{p} \overset{\text{def}}{=} \begin{pmatrix} p_0 - p_3 & p_2 + \mathrm{i}p_1 \\ p_2 - \mathrm{i}p_1 & p_0 + p_3 \end{pmatrix}, \quad p \in O_m^+. \tag{III.4}$$

Note that $\widetilde{p} \in \mathbf{SL}(2,\mathbb{C})$, that is $\det \widetilde{p} = 1$, and is positive definite (indeed, $\widetilde{p}$ has eigenvalues $p_0 \pm \sqrt{\mathbf{p}^2}$, where $\mathbf{p}^2 \overset{\text{def}}{=} p_1^2 + p_2^2 + p_3^2$. Now, since we assume $p \in O_m^+$, $p^0 = \sqrt{\mathbf{p}^2 + m^2}$. Hence both eigenvalues are positive). Moreover, let us define

$$V(A,p) \overset{\text{def}}{=} \sqrt{\widetilde{p}^{-1}}\, A \sqrt{A^{-1}\widetilde{p}A^{*-1}}, \quad A \in \mathbf{SL}(2,C),\ p \in O_m^+. \tag{III.5}$$

We note that $V(A,p) \in \mathbf{SU}(2)$, for all $A \in \mathbf{SL}(2,\mathbb{C})$, $p \in O_m^+$. We state the following result, referring to, e.g. [4, Section 7.2.C] and [1, Section 17.2.D] for the explicit computation. For any $g = (x, A) \in \mathbf{ISpin}^0(1,3) = \mathbb{R}^4 \rtimes \mathbf{Spin}^0(1,3)$,

$$\pi\left(s(q(p))^{-1}\, g\, s(g^{-1} \cdot q(p))\right) = e^{ix_\mu p^\mu} V(A,p), \quad p \in O_m^+,$$

where $x_\mu p^\mu = t\sqrt{\mathbf{p}^2 + m^2} - \mathbf{x} \cdot \mathbf{p}$ denotes the Minkowski scalar product of $x = (t, \mathbf{x}) \in \mathbb{R}^4$ with $p = (p_0, \mathbf{p}) \in O_m^+$ and identifying O_m^+ with the set $\{p \in \mathbb{R}^4 : p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}),\ \mathbf{p} \in \mathbb{R}^3\} \subset \mathbb{R}^4$. We collect the main results discussed in §4.2, §4.3 in the following paragraph.

§ 4.4 The induced representation $(U^{m,1/2}, \mathcal{W}^{m,1/2})$ is given by

$$\mathcal{W}^{m,1/2} = L^2(O_m^+, \mu_m; \mathbb{C}^2), \qquad U^{m,1/2}(x,A)f(p) = e^{ix_\mu p^\mu} V(A,p) f(\Lambda(A)p), \tag{III.6}$$

where $f \in L^2(O_m^+, \mu_m; \mathbb{C}^2)$, $x_\mu p^\mu = x_0 p_0 - x_1 p_1 - x_2 p_2 - x_3 p_3$ denotes the Minkowski pseudo metric, $p = (p_\mu)_{\mu=0,1,2,3} \in O_m^+$, $(x, A) \in \mathbf{ISpin}^0(1,3)$ with $x = (x_\mu)_{\mu=0,1,2,3} \in \mathbb{R}^4$, $A \in \mathbf{SL}(2,\mathbb{C})$, and $\Lambda : \mathbf{Spin}^0(1,3) \to \mathbf{SO}^0(1,3)$ denotes the covering map. We call $(U^{m,1/2}, \mathcal{W}^{m,1/2})$ the **Wigner representation** of $\mathbf{ISpin}^0(1,3) = \mathbb{R}^4 \rtimes \mathbf{Spin}^0(1,3)$. Note that, because of the Mackey machine, the Wigner representation is an *irreducible*, *unitary representation* of $\mathbf{ISpin}^0(1,3)$.

§ 4.5 Remarks.

1. The matrix $\widetilde{p}$ defined in (III.4) can be expressed in terms of the three Pauli matrices σ_j, $j = 1,2,3$, as follows

   $$\widetilde{p} = p_0\, \mathbb{1}_2 - \sum_{j=1,2,3} p_j \sigma_j, \quad p \in O_m^+.$$

   If we let $A \in \mathbf{SL}(2,\mathbb{C})$ the matrix $\widetilde{p}$ satisfies (cf. [21, (1-17) p. 12])

   $$\widetilde{p}_{A^{-1}} = A^* \widetilde{p} A, \quad p_{A^{-1}} \overset{\text{def}}{=} \Lambda(A^{-1})p, \tag{III.7}$$

   where A^* denotes the complex conjugate transpose of the matrix A.

2. Alongside $\widetilde{p}$ we also introduce the notation

   $$\underset{\sim}{p} \overset{\text{def}}{=} p_0\, \mathbb{1}_2 + \sum_{j=1,2,3} p_j \sigma_j, \quad p \in O_m^+.$$

   We have a similar relation to the remark above: for $A \in \mathbf{SL}(2,\mathbb{C})$

   $$\underset{\sim}{p}_A = A \underset{\sim}{p} A^*, \quad p_A = \Lambda(A)p,$$

   in fact, this relation is the usual way to *define* the covering map $\Lambda : SL(2,\mathbb{C})_\mathbb{R} \to \mathbf{SO}^0(1,3)$ (cf. [21, p. 12]).

3. A straight forward computation using the properties of the Pauli matrices shows the following relation between the matrices $\widetilde{p}$ and $\underset{\sim}{p}$

   $$\widetilde{p}\underset{\sim}{p} = m^2, \quad p \in O_m^+.$$

4. The square root $\sqrt{\widetilde{p}/m}$ of the positive definite matrix $\widetilde{p}/m$, $p \in O_m^+$, is explicitly given by (cf. [4, p. 283])
$$\sqrt{\widetilde{p}/m} = \frac{m\,\mathbb{I} + \widetilde{p}}{\sqrt{2m(p_0 + m)}}.$$

5. Let $C\ell(1,3)$ denote the real Clifford algebra over $\mathbb{R}^4$ with the Minkowski metric. We denote by $\gamma : \mathbb{R}^4 \to C\ell(1,3)$ the natural embedding of $\mathbb{R}^4$ in $C\ell(1,3)$. Let us define the following shorthand notations. If $(e_k)_{k=0,1,2,3}$ denotes a basis for $\mathbb{R}^4$, then let us set $\gamma_k \stackrel{\text{def}}{=} \gamma(e_k)$. Moreover, we set
$$\widehat{p} = \gamma(p), \quad p \in O_m^+.$$
Now, we note that the matrix $\gamma_0\widehat{p}$, $p \in O_m^+$, is positive definite. Hence, we can take its square-root. In particular we have (cf. [23, (3.55) p. 91])
$$\sqrt{\widehat{p}\gamma_0/m} = \frac{m\,\mathbb{I}_4 + p_0\,\mathbb{I}_4 + \mathbf{p}\cdot\boldsymbol{\alpha}}{\sqrt{2m(p_0 + m)}}, \quad p \in O_m^+, \tag{III.8}$$
where we have denoted by $\boldsymbol{\alpha}$ the vector of matrices with components $[\boldsymbol{\alpha}]_j = \gamma_0\gamma_j$, $j = 1,2,3$, and $\mathbf{p}\cdot\boldsymbol{\alpha} = \sum_{j=1,2,3} p_j\alpha_j$.

6. In terms of a basis $(e_k)_{k=0,1,2,3}$ we have $\widehat{p} = \sum_{k=0}^{3} p_k\gamma_k$. Similarly, we define
$$\check{p} \stackrel{\text{def}}{=} \sum_{k=0}^{3} p_k\gamma_k^{-1} = p_0\gamma^0 - \sum_{j=1,2,3} p_j\gamma_j,$$
where γ_k^{-1} denotes the inverse of $\gamma_k = \gamma(e_k)$, $k = 0,1,2,3$. We employ the convention of defining the Clifford algebra $C\ell(1,3)$ such that
$$\{\gamma(v), \gamma(w)\} = g(v,w),$$
Where $v, w \in \mathbb{R}^4$, $\{\cdot,\cdot\}$ denotes the anticommutator, and $g(\cdot,\cdot)$ denotes the Minkowski pseudo metric with signature $(1,-1,-1,-1)$. In the usual convention for the gamma matrices $\gamma_k^{-1} = \gamma_k^*$, where the $*$ denotes complex-conjugate transpose. Similarly to the remark 3 we have
$$\widehat{p}\check{p} = m^2, \quad p \in O_m^+.$$

Covariant realization

§ 4.6 The Wigner representation of $\mathbf{ISpin}^0(1,3)$ depends in a non trivial way on $A \in \mathbf{SL}(2,\mathbb{C})_{\mathbb{R}} \cong \mathbf{Spin}^0(1,3)$ through the "twist" $V(p,A)$ which "mixes" $p \in O_m^+$ with $A \in \mathbf{Spin}^0(1,3)$. On the other hand the Hilbert space $\mathcal{W}^{m,1/2}$ is "not twisted" in the following sense. The Hilbert space $\mathcal{W}^{m,1/2} = L^2(O_m^+, \mu_m; \mathbb{C}^2) = L^2(O_m^+, \mu_m)\widehat{\otimes}\mathbb{C}^2$, where $\widehat{\otimes}$ denotes the completion of the tensor product with respect to the scalar product $(\cdot,\cdot)_{\mathcal{W}^{m,1/2}}$ of $\mathcal{W}^{m,1/2}$. This scalar product is given by
$$(h_1 \otimes v_1, h_2 \otimes v_2)_{\mathcal{W}^{m,1/2}} = (h_1, h_2)_{L^2(O_m^+,\mu_m)}(v_1, v_2)_{\mathbb{C}^2},$$
for $h_1, h_2 \in L^2(O_m^+, \mu_m)$, $v_1, v_2 \in \mathbb{C}^2$, and where $(v_1, v_2)_{\mathbb{C}^2}$ denotes the standard Hermitian scalar product in $\mathbb{C}^2$. Hence we say that the Hilbert space $\mathcal{W}^{m,1/2}$ is not twisted because its scalar product does not "mix" $L^2(O_m^+, \mu_m)$ with $\mathbb{C}^2$.

§ 4.7 We define a different representation, unitarily equivalent to the Wigner representation, by "removing" part of the twist in the definition of the homomorphism $U^{m,1/2}$ and "adding it" to the scalar product the Hilbert space. We define
$$\mathcal{W}_{(1,0)} \stackrel{\text{def}}{=} W\mathcal{W}^{m,1/2}, \quad U_{(1,0)} \stackrel{\text{def}}{=} WU^{m,1/2}W^{-1} \tag{III.9}$$

where

$$W f(p) \stackrel{\text{def}}{=} \sqrt{\widetilde{p}/m}\, f(p), \quad f \in \mathcal{W}^{m,1/2},\ p \in O_m^+, \tag{III.10}$$

and $\widehat{p}$ is given in §**4.3**. Note that this new representation $(\mathcal{W}_{(1,0)}, U_{(1,0)})$ is, by construction, unitarily equivalent to the original Wigner representation $(W^{m,1/2}, U^{m,1/2})$. Indeed, by (III.9), the map W is an isometric isomorphism $\mathcal{W}^{m,1/2} \to \mathcal{W}_{(1,0)}$ and intertwines $U^{m,1/2}$ with $U_{(1,0)}$. The explicit form of $U_{(1,0)}$ is directly computed from (III.10), (III.6), and (III.7) (also, compare with [1, Formula (41) p. 523]). We obtain

$$U_{(1,0)}(x, A)\psi(p) = e^{\mathrm{i} x_\mu p^\mu} A\psi(\Lambda(A)p),$$

where $\psi \in \mathcal{W}_{(1,0)}$, $p \in O_m^+$, $(x, A) \in \mathbf{ISpin}^0(1,3)$. It is clear that, in this formula, the dependence of $U_{(1,0)}$ on A is simpler than the A dependence of $U^{m,1/2}$. Moreover $U_{(1,0)}$ does *not* "mix" $p \in O_m^+$ with $A \in \mathbf{Spin}^0(1,3)$. The scalar product in $\mathcal{W}_{(1,0)}$, which makes the map $W : \mathcal{W}^{m,1/2} \to \mathcal{W}_{(1,0)}$ an isometric isomorphism, is given by

$$\big(\psi_1, \psi_2\big)_{\mathcal{W}_{(1,0)}} = \int_{O_m^+} \big(\psi_1(p), (\widetilde{p}/m)\,\psi_2(p)\big)_{\mathbb{C}^2}\, \mathrm{d}\mu_m, \quad \psi_1, \psi_2 \in \mathcal{W}_{(1,0)}.$$

We see that this scalar product is indeed "twisted", that is, if we take, for $i = 1, 2$, $\psi_i = \phi_i(p) \otimes v_i$ with ϕ_i a smooth, compactly supported, complex valued function, and $v_i \in \mathbb{C}^2$, then the scalar product in $\mathcal{W}_{(1,0)}$ does *not* decompose into the product of a scalar product, of just ϕ_1 with ϕ_2, *times* a scalar product, of just v_1 with v_2. This irreducible unitary representation $(\mathcal{W}_{(1,0)}, U_{(1,0)})$ of $\mathbf{ISpin}^0(1,3)$ is sometimes called **covariant representation**. We call it the **(1,0)-spinor representation**. This representation is, in fact, the one chosen in [21, in particular, see Section 1–4] as fundamental representation from which a multispinor representation is assembled.

§ 4.8 The $(1,0)$-representation, besides having a simpler transformation property under $\mathbf{SL}(2,\mathbb{C})_\mathbb{R}$, it has a *new* important feature. The generators of the rotations (that is, the elements of the Lie algebra of $\mathbf{SL}(2,\mathbb{C})$ seen as unbounded selfadjoint operators on the appropriate Gårding domain) split into the sum of two selfadjoint operators which *strongly commute* [3, p. 189]. One of these two strongly commuting operators is unbounded and it is interpreted physically as angular-momentum operator. The other is a bounded matrix operator which is interpreted as spin operator. The original Wigner representation does not have this property. This means that, with the *non-local* transformation which we used to go from Wigner representation to the $(1,0)$-representation, we have "decoupled" angular-momentum and spin variables.

Parity is a troublemaker

§ 4.9 It is well known (cf. e.g. [1, Section 17.3]) that it is impossible to represent the "parity" symmetry by a (complex-)linear representation in the carrier space of the (1,0)-spinor representation. The "solution" to this issue, which we will present in the following sections, is to define a dual representation, the $(\mathbf{0},\mathbf{1})$-spinor representation, and form a new, *reducible*, unitary representation, the $(\mathbf{1},\mathbf{0}) \oplus (\mathbf{0},\mathbf{1})$-spinor representation or, for short, **bispinor representation**. To recover an irreducible representation we will need to project onto a subspace.

§ 4.10 Remark. Consider the full Lorentz group $\mathbf{O}(1,3)$. In the defining representation of $\mathbf{O}(1,3)$ parity is represented by the matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The **parity transformation** acts on the Lie subgroup $\mathbf{SO}^0(1,3)$ by conjugation with the matrix P: $g \mapsto PgP^{-1}$, $g \in \mathbf{SO}^0(1,3)$. By taking the Jacobian of this map $g \mapsto PgP^{-1}$, we induce an action on the

Lie algebra $\mathfrak{so}(1,3)$ of $\mathbf{SO}^0(1,3)$. We briefly describe this action. Let $\mathfrak{so}(1,3) = K + N$ be a Cartan decomposition of $\mathfrak{so}(1,3)$. The generators in K are associated with Lorentz boosts and the generators in K with three-dimensional rotations. The action of the parity transformation on the Lie algebra $\mathfrak{so}(1,3)$ has the effect of what is sometimes called **Cartan involution** (cf. e.g. [11, p. 1]). That is, it reverses the sign of the elements in N (the generators of the boosts) while leaving unchanged the elements in K (the generators of three-dimensional rotations). Now, being this transformation defined on the Lie algebra, it can be uniquely lifted to the universal cover $\mathbf{Spin}^0(1,3)$ of $\mathbf{SO}^0(1,3)$. Hence, no matter which double cover we choose for the disconnected Lie group $\mathbf{O}(1,3)$, we will have the same action of the parity transformation on the generators of the Lie algebra. Since the parity transformation is, by definition, the Cartan involution at the Lie algebra level, when we pass to the Lie group, the parity transformation will correspond to the Cartan involution for the Lie group. If we regard a Lie group as a subgroup of an algebra of complex matrices, then Cartan involution is the map $g \mapsto g^{*-1}$, where the $*$ denotes complex-conjugate-transpose, in the sense of complex matrices..

Going back to the particular case of $\mathbf{SL}(2,\mathbb{C})_{\mathbb{R}} \cong \mathbf{Spin}^0(1,3)$, seen as a group of 2-by-2 complex matrices, the parity transformation induces the transformation $A \mapsto A^{*-1}$, $A \in \mathbf{SL}(2,\mathbb{C})$.

This discussion shows that the parity transformation intertwines the $(1,0)$-spinor representation with a representation, to be defined in the following paragraph, where the matrix A is "replaced" by A^{*-1}.

§ 4.11 By the discussion in the previous paragraph the parity transformation has a natural action on the positive mass hyperboloid O_m^+. Let us parametrize O_m^+ by identifying it with $\mathbb{R}^3$. Then the parity transformation P, discussed in the paragraph above, acts by sending $\mathbf{p} \in \mathbb{R}^3$ into $-\mathbf{p}$. This means that taken a point $p \in O_m^+$ which is parametrized by a vector $\mathbf{p}$, under the isomorphism $O_m^+ \cong \mathbb{R}^3$, is sent by the parity transformation to a point $p^P \in O_m^+$ which is parametrized by the vector $\mathbf{p}$, under the isomorphism $O_m^+ \cong \mathbb{R}^3$. Let us now lift this action from $O_m^+ \cong \mathbb{R}^3$ to the spaces $\mathcal{W}^{m,1/2}$ defined in §4.4 and the space $\mathcal{W}_{(1,0)}$ defined in §4.7. Explicitly, let us denote by $P^{(1,0)}$ the lift of P to $\mathcal{W}_{(1,0)}$. This lift acts on a function $\psi \in \mathcal{W}_{(1,0)}$ $\psi : O_m^+ \cong \mathbb{R}^3$ to $\mathbb{C}^2$ by $\psi \mapsto P^{(1,0)}\psi$ with $P^{(1,0)}\psi(p) = \psi(p^P)$. Note that the measure $\mu_m(\mathbf{p})$ defined in §4.2 is invariant under this transformation. We now define the space $\mathcal{W}_{(0,1)}$ as the image of $P^{(1,0)}$, that is the elements of $\mathcal{W}_{(0,1)}$ are obtained by applying $P^{(1,0)}$ to every element in $\mathcal{W}_{(1,0)}$. Explicitly we have

$$\mathcal{W}_{(0,1)} = \left\{\psi : (\psi,\psi)_{\mathcal{W}_{(0,1)}} < \infty\right\},$$

where

$$(\psi_1,\psi_2)_{\mathcal{W}_{(0,1)}} = \int_{O_m^+} (\psi_1(p), (\underset{\sim}{p}/m)\psi_2(p))_{\mathbb{C}^2}\, \mathrm{d}\mu_m(p), \quad \psi_1,\psi_2 \in \mathcal{W}_{(0,1)}.$$

The map $P^{(1,0)}$ is by definition an isometric isomorphism of $\mathcal{W}_{(1,0)}$ with $\mathcal{W}_{(0,1)}$. We can therefore pushforward the representation $U_{(1,0)}$ to a representation $U_{(0,1)} \overset{\text{def}}{=} P^{(1,0)}U_{(1,0)}(P^{(1,0)})^{-1}$ on $\mathcal{W}_{(0,1)}$. Explicitly we have

$$U_{(0,1)}(x,A)\psi(p) = e^{\mathrm{i}x_\mu p^\mu} A^{*-1}\psi(\Lambda(A)p),\ \psi \in \mathcal{W}_{(0,1)}, \tag{III.11}$$

where A^{*-1} denotes the complex-conjugate-transpose-inverse matrix, and $p \in O_m^+$, $(x,A) \in \mathbf{ISpin}^0(1,3)$. One can convince oneself that $U_{(0,1)}(x,A)$, $x \in \mathbb{R}^4$, $A \in \mathbf{SL}(2,\mathbb{C})$ is indeed unitary by noticing that (cf. [21]) $A^*\widetilde{p}A = \widetilde{p'}$, with $p' = \Lambda(A^{-1})p$ (where $\Lambda : \mathbf{SL}(2,\mathbb{C})_{\mathbb{R}} \to \mathbf{SO}^0(1,3)$ is the covering map), and taking complex-conjugate-transpose-inverse on both sides of this relation. We call this representation $(\mathcal{W}_{(0,1)}, U_{(0,1)})$ the $(\mathbf{0},\mathbf{1})$**-spinor representation**.

§ 4.12 Having defined both the $(1,0)$- and the $(0,1)$-spinor representations of $\mathbf{ISpin}^0(1,3)$, we can now take their direct sum $(1,0) \oplus (0,1)$. We denote this new *reducible* representation of $\mathbf{ISpin}^0(1,3)$ by $\left(\mathcal{W}_{(1,0)\oplus(0,1)}, U_{(1,0)\oplus(0,1)}\right)$. Explicitly we let

$$\mathcal{W}_{(1,0)\oplus(0,1)} = (\mathcal{W}_{(1,0)} \oplus \mathcal{W}_{(0,1)}).$$

The scalar product $(\cdot,\cdot)_{(1,0)\oplus(0,1)}$ on $\mathcal{W}_{(1,0)\oplus(0,1)}$ is given, for $\Psi_1,\Psi_2 \in \mathcal{W}_{(1,0)\oplus(0,1)}$, by

$$(\Psi_1,\Psi_2)_{(1,0)\oplus(0,1)} \overset{\text{def}}{=} \int_{O_m^+} (\Psi_1(p), \left(\gamma_0\widehat{p}/m\right)\ \Psi_2(p))_{\mathbb{C}^4}\, \mathrm{d}\mu_m(p). \tag{III.12}$$

Finally, we define the operators $U_{(1,0)\oplus(0,1)}(x,A)$, $(x,A)\in \mathbf{ISpin}^0(1,3)$, on $\Psi\in\mathcal{W}_{(1,0)\oplus(0,1)}$, by

$$U_{(1,0)\oplus(0,1)}(x,S)\Psi(p)=e^{\mathrm{i}x_\mu p^\mu}\,S\,\Psi(\Lambda(S)p),$$

where Λ is the covering map, now thought of as sending an element $S\in\mathbf{Spin}^0(1,3)\hookrightarrow\mathbb{C}\ell(4)$ to an element $\Lambda(S)\in\mathbf{SO}^0(1,3)$. Let us note that the embedding $\mathbf{Spin}^0(1,3)\cong\mathbf{SL}(2,\mathbb{C})_\mathbb{R}\hookrightarrow\mathbb{C}\ell(4)$ is explicitly given here by

$$A\mapsto\begin{pmatrix}A & 0\\ 0 & A^{*-1}\end{pmatrix},\, A\in\mathbf{SL}(2,\mathbb{C}) \tag{III.13}$$

where we have represented $\mathbb{C}\ell(4)$ as an algebra of 4-by-4 complex matrices acting on the space $\mathbb{C}^4$.

§ 4.13 As we saw, the map $P^{(1,0)}$ intertwines, by definition, the representation $(\mathcal{W}_{(1,0)},U_{(1,0)})$ with $(\mathcal{W}_{(0,1)},U_{(0,1)})$. On the space $\mathcal{W}_{(1,0)\oplus(0,1)}$ such an intertwining effect is obtained by the following map

$$(P^{(1,0)\oplus(0,1)}\Psi)(p)\overset{\text{def}}{=}B\Psi(p),\quad \Psi\in\mathcal{W}_{(1,0)\oplus(0,1)}\ p\in O_m^+,$$

where B is the block matrix

$$B=\begin{pmatrix}0 & \mathbb{I}_2\\ \mathbb{I}_2 & 0\end{pmatrix}.$$

Note that we are taking as representation of $\mathbf{Spin}^0(1,3)$ as 4-by-4 complex matrices acting on $\mathbb{C}^4$ given by the embedding (III.13).

Let us consider the group $\mathbf{G}_{B,\mathbf{ISpin}^0(1,3)}$ generated by the matrix B together with all the elements of $\mathbf{ISpin}^0(1,3)$ (where the $\mathbf{Spin}^0(1,3)$ "part" of $\mathbf{ISpin}^0(1,3)$ is realized by the representation just mentioned). Note also that the matrix B in fact represents the element $\gamma_0\in\mathcal{C}\ell(1,3)\hookrightarrow\mathbb{C}\ell(4)$ (cf. §**4.5**).

Now, by construction $P^{(1,0)\oplus(0,1)}$ gives a representation of B on the space $\mathcal{W}_{(1,0)\oplus(0,1)}$. Hence we see that on the space $\mathcal{W}_{(1,0)\oplus(0,1)}$ we have a representation of the whole group $\mathbf{G}_{B,\mathbf{ISpin}^0(1,3)}$.

The group $\mathbf{G}_{B,\mathbf{ISpin}^0(1,3)}$ is one of the possible groups obtained by lifting the parity transformation as an element of $\mathbf{O}(1,3)$ to one of the universal covers of $\mathcal{O}(1,3)$ (cf. §**2.12**). Not that the parity transformation $P\in\mathbf{O}(1,3)$ is lifted to *two* elements B and $-B$ in $\mathbf{G}_{B,\mathbf{ISpin}^0(1,3)}$. This means that *both* $P^{(1,0)\oplus(0,1)}$ and $-P^{(1,0)\oplus(0,1)}$ are *parity operators* on the space $\mathcal{W}_{(1,0)\oplus(0,1)}$, that is they both correspond to *the same* "physical" parity transformation P as an element of $\mathbf{O}(1,3)$.

By this discussion we see that we can extend the representation $\big(\mathcal{W}_{(1,0)\oplus(0,1)},U_{(1,0)\oplus(0,1)}\big)$ to a representation of $\mathbf{G}_{B,\mathbf{ISpin}^0(1,3)}$ by imposing that the matrix B (resp. $-B$), seen as an element of $\mathbf{G}_{B,\mathbf{ISpin}^0(1,3)}$, is represented by the operator $P^{(1,0)\oplus(0,1)}$ (resp. $-P^{(1,0)\oplus(0,1)}$). Let us denote this extension still by $\big(\mathcal{W}_{(1,0)\oplus(0,1)},U_{(1,0)\oplus(0,1)}\big)$. Then we have that the representation $\big(\mathcal{W}_{(1,0)\oplus(0,1)},U_{(1,0)\oplus(0,1)}\big)$ of $\mathbf{G}_{B,\mathbf{ISpin}^0(1,3)}$ is *reducible*. Indeed we can project onto the eigenspaces of the operator $P^{(1,0)\oplus(0,1)}$ (which corresponds to diagonalizing B). We construct the following projection operators acting on $\mathcal{W}_{(1,0)\oplus(0,1)}$:

$$\mathbb{Q}_+\Psi(p)\overset{\text{def}}{=}\frac{1}{\sqrt{2}}\big(\mathbb{I}_4-B\big)\Psi(p),\quad \mathbb{Q}_-\Psi(p)\overset{\text{def}}{=}\frac{1}{\sqrt{2}}\big(\mathbb{I}_4-B\big)\Psi(p),\quad \Psi\in\mathcal{W}_{(1,0)\oplus(0,1)},\ p\in O_m^+.$$

These projection operators project onto eigenspaces which are usually called respectively **even** and **odd parity** eigenspaces (cf. [23, §3.2.5, p. 93]).

We project the reducible representation $(\mathcal{W}_{(1,0)\oplus(0,1)},U_{(1,0)\oplus(0,1)})$ onto the *even parity eigenspace* obtaining an *irreducible* representation of the group $\mathbf{G}_{B,\mathbf{ISpin}^0(1,3)}$. We denote this new representation by $(\mathcal{W}^+_{(1,0)\oplus(0,1)},U^+_{(1,0)\oplus(0,1)})$ and we call it the **covariant 1-particle representation** of $\mathbf{ISpin}^0(1,3)$. Explicitly we have

$$\mathcal{W}^+_{(1,0)\oplus(0,1)}=\mathbb{Q}_+\mathcal{W}_{(1,0)\oplus(0,1)}.$$

The scalar product on $\mathcal{W}^+_{(1,0)\oplus(0,1)}$ is then

$$(\Psi_1,\Psi_2)_{1\mathrm{p}}=\int_{O_m^+}\Big(\Psi_1(p),\frac{1}{m}(\hat{p}\gamma_0+m\gamma_0)\Psi_2(p)\Big)_{\mathbb{C}^4}\mathrm{d}\mu_m(p),\quad \Psi_1,\Psi_2\in\mathcal{W}^+_{(1,0)\oplus(0,1)}. \tag{III.14}$$

5 Wightman and Schwinger functions

Comment. We link the "group theoretic" presentation up to this point with the Wightman theory in the case of free Dirac fields. In our presentation, we avoid the use of quantum fields. We want to show precisely why every step is necessary and justified from the basic principles without any unnecessary assumption.

We do *not* present the full axiomatic Wightman theory. We restrict ourselves to the special case of free Dirac fields which, we believe, to be only implicitly discussed throughout the literature.

The main aim of this section is to show that the "Schwinger function" (We will call it *Schwinger distribution* in the main part of the text) is non-ambiguously defined, in the case of free Dirac fields, starting from first principles. Moreover, we find that, even if the Schwinger function is uniquely prescribed, its interpretation is not. In fact, we claim that there remains some ambiguity in interpreting the Schwinger function as a bilinear form.

The importance of interpreting the Schwinger function as a bilinear form stems from the fact that this bilinear form is then interpreted as a 2-point function in the context of Euclidean field theory.

Distributions on the forward cone, analytic functions, and covariance

Conventions. We denote the proper, orthochronous, Lorentz group by $\mathbf{SO}^0(1,3)$. Let V denote a (real or complex) vector space. We denote the Minkowski metric by $g(\cdot,\cdot)$ with signature $(1,-1,-1,-1)$, hence we have

$$g(v,v) = v_0^2 - v_1^2 - v_2^2 - v_3^2, \quad v = (v_0, v_1, v_2, v_3) \in \mathbb{R}^4.$$

For vectors $\mathbf{k}, \mathbf{x} \in \mathbb{R}^3$ we denote by $\mathbf{x}\cdot\mathbf{k}$ the positive definite scalar product in $\mathbb{R}^3$ of $\mathbf{x}$ with $\mathbf{k}$. We denote by $\mathscr{S}(\mathbb{R}^4)$, respectively $\mathscr{S}(\mathbb{R}^4;V)$, the space of Schwartz test functions with values in $\mathbb{C}$, respectively in V. We denote by $\mathscr{S}'(\mathbb{R}^4)$, respectively $\mathscr{S}'(\mathbb{R}^4;V)$, the topological dual of $\mathscr{S}(\mathbb{R}^4)$, respectively $\mathscr{S}(\mathbb{R}^4;V)$.

§ 5.1 The **forward cone** V^+ is defined to be the open set of $\mathbb{R}^4$ given by

$$V^+ \stackrel{\text{def}}{=} \big\{ v = (v_0, \mathbf{v}) \in \mathbb{R}^4 : g(v,v) > 0, v_0 > 0 \big\},$$

where $g(\cdot,\cdot)$ is the Minkowski pseudo metric. That is, the forward cone denotes the interior of the "upper" light cone. We are thinking of vectors in V^+ as momentum 4-vectors and not as space-time 4-vectors. The **forward tube** $\mathscr{T}^+$ is defined as the open set of $\mathbb{C}^4$ given by

$$\mathscr{T}^+ \stackrel{\text{def}}{=} \mathbb{R}^4 - \mathrm{i}V_+.$$

That is, $\mathscr{T}^+ \subset \mathbb{C}^4$ consists of those complex four-vectors which can be written as a vector in $\mathbb{R}^4$ minus i times a vector inside the forward light cone.

§ 5.2 In our presentation, it will be convenient to define a reduced Wightman distribution for Dirac fields as a matrix valued distribution. We therefore introduce here the notion of a **vector valued tempered distribution**, that is a tempered distribution with values in a finite dimensional Hilbert space. Explicitly, let $(\Xi_\mu)_{\mu=1}^n$, $n \in \mathbb{N}$, be a set of tempered distributions, $\Xi_\mu \in \mathscr{S}'(\mathbb{R}^d)$, $d \in \mathbb{N}$. Let V be a finite dimensional Hilbert space of dimension n and $(e_k)_{k=1}^n$ an orthonormal basis of V. Then, we call $\Xi \stackrel{\text{def}}{=} \sum_{\mu=1}^n e_\mu \Xi_\mu$ a vector valued tempered distribution. Let us denote by $\mathscr{S}'(\mathbb{R}^d) \otimes V$ the space of vector valued tempered distributions with values in the finite dimensional Hilbert space V. If the vector space is also an algebra of matrices we will call the vector valued distribution a **matrix valued tempered distribution**. When the context is clear we will drop the term "tempered" and just say vector valued distribution.

§ 5.3 Let $\Xi \in \mathscr{S}'(\mathbb{R}^4) \otimes V$ be a vector valued tempered distribution with V be a finite dimensional Hilbert space. We say that Ξ is a **lower boundary value** of an analytic function F in the tube $\mathscr{T}^+$ with values in V when

$$\lim_{\eta_0 \downarrow 0}{}^{\mathscr{S}'(\mathbb{R}^4)} F(\cdot + \mathrm{i}\eta_0) = \Xi,$$

where $\lim^{\mathscr{S}'(\mathbb{R}^4)}$ denotes the limit in $\mathscr{S}'(\mathbb{R}^4)$, $\widetilde{F}(\cdot + \mathrm{i}\eta_0)$ denotes a function on $\mathbb{R}^4 = \mathfrak{R}\mathbb{C}^4$ (where $\mathfrak{R}$ means *real part of*) with values in V, and $\eta_0 \downarrow 0$ means η_0 goes to zero from the positive side.

§ 5.4 Proposition. *Let $\Xi = \mathscr{F}(\xi) \in \mathscr{S}'(\mathbb{R}^4) \otimes V$, V a finite dimensional Hilbert space, be the distributional Fourier transform of a measure valued distribution ξ supported on the closure $\overline{V}_+$ of the forward cone V_+. Then Ξ is the lower boundary value of an analytic function F defined on the forward tube $\mathscr{T}^+$. Moreover there exists a polynomial P and a positive integer N such that*

$$\|F(z)\|_V \leq \|P(z)\|_V (1 + \operatorname{dist}(\Im(z), \partial \overline{V}_+)^{-N}), \quad z \in \mathscr{T}^+,$$

where dist *denotes the distance of a point from a set and $\|\cdot\|_V$ denotes the norm on the finite dimensional Hilbert space V. Finally, the analytic function F on $\mathscr{T}^+$ is unique and is explicitly given by*

$$F(x - \mathrm{i}v) = \mathscr{F}(\exp_v(\cdot)\,\xi)(x), \quad v \in V_+, x \in \mathbb{R}^4 \text{ (hence } x - \mathrm{i}v \in \mathscr{T}_+),$$

where $\exp_\eta(\cdot)\,\xi$ denotes the distribution obtained by multiplying ξ by the function $\exp_v(p) \stackrel{\text{def}}{=} e^{-v\cdot p}$, $p \in \mathbb{R}^4$.

Proof. Cf. [16, Theorem IX.16, p. 23]. □

§ 5.5 Motivated by the previous proposition we make the following definitions. We call a Schwartz distribution $\Xi \in \mathscr{S}'(\mathbb{R}^4) \otimes V$, for a finite dimensional Hilbert space V, a **vector valued forward distribution** when it is the Fourier transform of a Schwartz distribution supported on the forward cone V_+. Moreover, we say that F is the **forward tube extension** of a distribution Ξ when Ξ is the lower boundary value of an analytic function F on the forward tube $\mathscr{T}^+$.

§ 5.6 Consider the complexification $\mathbf{SO}(4;\mathbb{C})$ of the proper, orthochronous, Lorentz group $\mathbf{SO}^0(1,3)$. We define the **extended tube** to be the subset of $\mathbb{C}^4$ given by

$$\mathscr{T}' \stackrel{\text{def}}{=} \left\{ z' \in \mathbb{C}^4 : \text{ there exist } z \in \mathscr{T}, L \in \mathbf{SO}(4,\mathbb{C}), \text{ such that } z' = Lz \right\}.$$

§ 5.7 Let $\widetilde{F}$ be an analytic function on $\mathscr{T}'$ with values in a complex vector space V. We say that $\widetilde{F}$ is an R-**covariant analytic function on** $\mathscr{T}'$ if there exists a finite dimensional[2] representation R of $\mathbf{SO}(4,\mathbb{C})$ on V such that

$$\widetilde{F}(z) = R(L)\widetilde{F}(Lz), \quad z \in \mathscr{T}', \; L \in \mathbf{SO}(4,\mathbb{C}).$$

§ 5.8 Let (V, R) be a finite dimensional representation of $\mathbf{SO}(4,\mathbb{C})$, and let $R_{\mathbf{SO}^0(1,3)}$ be its restriction to $\mathbf{SO}^0(1,3)$, where we consider $\mathbf{SO}^0(1,3)$ as embedded in $\mathbf{SO}(4,\mathbb{C})$ by the *standard embedding* defined in §**2.15**. We call $\Xi \in \mathscr{S}'(\mathbb{R}^4; V)$ an $R_{SO^0(1,3)}$**-covariant distribution** when it satisfies

$$\Xi = R_{\mathbf{SO}^0(1,3)}(\Lambda)\,\Xi\circ\Lambda, \quad \Lambda \in \mathbf{SO}^0(1,3),$$

where $\Xi\circ\Lambda$ is the distribution which satisfies, for any $f \in \mathscr{S}(\mathbb{R}^4; V)$, $\langle \Xi\circ\Lambda, f\rangle = \langle \Xi, f\circ\Lambda\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the $\mathscr{S}'(\mathbb{R}^4;V)$-$\mathscr{S}(\mathbb{R}^4;V)$ dual paring, and $f\circ\Lambda(x) = f(\Lambda x)$, $x \in \mathbb{R}^4$, $\Lambda \in \mathbf{SO}^0(1,3)$.

§ 5.9 Proposition. (Bargmann-Hall-Wightman) *Let $\widetilde{F}$ be an analytic function on $\mathscr{T}^+$. Let R be a finite dimensional representation of $\mathbf{SO}(4,\mathbb{C})$ on a vector space V. Let $R_{\mathbf{SO}^0(1,3)}$ a representation of $\mathbf{SO}^0(1,3)$ obtained by restricting the representation R to $\mathbf{SO}^0(1,3)$ by the standard embedding (where the standard embedding $\mathbf{SO}^0(1,3) \hookrightarrow \mathbf{SO}(4,\mathbb{C})$ is given in §**2.15**). Assume that $\widetilde{F}$ satisfies*

$$\widetilde{F}(z) = R_{\mathbf{SO}^0(1,3)(\Lambda)}\widetilde{F}(\Lambda z), \quad z \in \mathscr{T}^+, \; \Lambda \in \mathbf{SO}^0(1,3).$$

Then $\widetilde{F}$ admits a single valued extension to $\mathscr{T}^+$ and such an extension is unique if we require the extension to be an R-covariant analytic function on $\mathscr{T}'$.

Proof. Cf. anyone of the following: [21, Theorem 2–11, p. 66], [9, Chapter IV, section 4] [4, Theorem 9.1, p. 362]. □

[2] A finite dimensional representation of a non-compact, semisimple Lie group such as as $\mathbf{SO}(4,\mathbb{C})$ is unitary if and only if is trivial. Hence we are assuming here that our representation is either trivial or not unitary and in general reducible.

§ 5.10 Comment. The use that we make of this theorem is the following. We start from a $R_{\mathbf{SO}^0(1,3)}$-covariant forward distribution Ξ. First we use the fact that Ξ is assumed to be a vector valued *forward* distribution. This allows us, by §5.4, to extended Ξ by an analytic function F_Ξ to the forward tube $\mathscr{T}^+$. In the forward tube lie in particular points of the form $z = (\mathrm{i}t, \mathbf{x})$, for any $t > 0$ and any $\mathbf{x} \in \mathbb{R}^3$. Our final goal here is to pass to an Euclidean theory. For this reason we are interested in obtaining from the initial distribution Ξ a new distribution which is supported on the set of points $\{(\mathrm{i}t, \mathbf{x}) : t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3\}$. By extending Ξ we can obtain the analytic function F_Ξ which, as we said, is well defined for $z = (\mathrm{i}t, \mathbf{x})$, for any $t > 0$ and any $\mathbf{x} \in \mathbb{R}^3$. We would like to extend this functions also to points $z = (\mathrm{i}t, \mathbf{x})$, for any $t < 0$ and any $\mathbf{x} \in \mathbb{R}^3$. In general it is not clear how to analytically continue the function to points with "imaginary time" $\mathrm{i}t$ and $t < 0$. At this point if we use the assumption that Ξ is $R_{\mathbf{SO}^0(1,3)}$-covariant. By uniqueness of the extension of Ξ by the analytic function F_Ξ we can extend the $R_{\mathbf{SO}^0(1,3)}$-covariance to the function F_Ξ as long as we restrict to those transformations in $\mathbf{SO}^0(1,3)$ which preserve the forward tube $\mathscr{T}^+$, where F_Ξ has been defined. Now the important point of having the $R_{\mathbf{SO}^0(1,3)}$-covariance is that it implies that the function F_Ξ transforms under $\mathbf{SO}^0(1,3)$ by a representation which comes from a representation of the complexification $\mathbf{SO}(4,\mathbb{C})$ of $\mathbf{SO}^0(1,3)$. Inside $\mathbf{SO}(4,\mathbb{C})$ we have in particular a copy of $\mathbf{SO}(4)$ ($\mathbf{SO}(4) = \mathbf{SO}(4,\mathbb{R})$) and in particular we have a transformation $-\mathbb{I} \in \mathbf{SO}(4)$ which acts on $\mathbb{C}^4$ by

$$-\mathbb{I} : (z_0, z_1, z_2, z_3) \mapsto (-z_0, -z_1, -z_2, -z_3).$$

By this transformation we send $(\mathrm{i}t, -\mathbf{x})$ into $(-\mathrm{i}t, \mathbf{x})$, for any $t > 0$, $\mathbf{x} \in \mathbb{R}^3$. Hence, it can be used to reach points of the form $(\mathrm{i}\tau, \mathbf{x})$ with $\tau < 0, \mathbf{x} \in \mathbb{R}^3$ from points of the form $(\mathrm{i}t, \mathbf{y})$ with $t = -\tau > 0, \mathbf{y} \in \mathbb{R}^3$.

We therefore want to *define* a new function $\widetilde{F}_\Xi$ by imposing that on $\mathscr{T}^+$ it coincides with F_Ξ and is defined on the extended tube $\mathscr{T}'$ (that is on all points in $\mathbb{C}^4$ that can be reached from $\mathscr{T}^+$ by a transformation of $\mathbf{SO}(4,\mathbb{C})$) by imposing that

$$\widetilde{F}_\Xi(Lz^+) = R(L^{-1})F_\Xi(z^+), \quad z^+ \in \mathscr{T}^+, L \in \mathbf{SO}(4,\mathbb{C}).$$

This procedure is well defined because, given $z \in \mathscr{T}'$, regardless on how we reach it from $\mathscr{T}^+$ the value $\widetilde{F}_\Xi(z)$ will be the same. This fact is at the core of the proof of Bargmann-Hall-Wightman theorem we quoted in §5.9.

Now that we have obtained the analytic function $\widetilde{F}_\Xi$ on $\mathscr{T}'$ we can restrict it to $\mathscr{T}' \cap \{(\mathrm{i}t, \mathbf{x}) : t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3\}$.

We have thus motivated the next paragraph where we introduce the notion of Schwinger function.

§ 5.11 Consider the set $S \stackrel{\text{def}}{=} \left\{z' = (z'_0, \mathbf{z}') \in \mathscr{T}' \subset \mathbb{C}^4 : \Re(z'_0) = 0, \Im(\mathbf{z}) = 0\right\}$ of all points in the extended tube $\mathscr{T}'$ with *imaginary* first component and the other three components taken to be *real*. We call S the set of **Schwinger points** (not to be confused with *Jost points*). Moreover, let us define a map $\mathrm{Wick} : S \to \mathbb{R}^4$ by

$$\mathrm{Wick} : (\mathrm{i}t, x_1, x_2, x_3) \mapsto (t, x_1, x_2, x_3),$$

where $(\mathrm{i}t, x_1, x_2, x_3) \in S$. We call this map the **Wick rotation**. Let $\widetilde{F}$ be any *analytic* function on the *extended tube* $\mathscr{T}'$ with values in a vector space V. We define the **Schwinger function** associated to $\widetilde{F}$, which we denote by $\widetilde{S}_{\widetilde{F}}$, to be the restriction of $\widetilde{F}$ to S composed with the inverse of the Wick rotation, that is we define $\widetilde{S}_{\widetilde{F}}$ to be the function[3] on $\mathrm{Wick}(\mathscr{T}' \cap S)$ such that,

$$\widetilde{S}_{\widetilde{F}} = \widetilde{F}\restriction_S \circ \mathrm{Wick}^{-1},$$

where $\circ$ denotes the composition of functions. The reason of composing the restriction of $\widetilde{F}$ to S with the inverse of Wick rotation is that, this way, the Schwinger function is a function defined on the subset $\mathrm{Wick}(\mathscr{T}' \cap S)$ of $\mathbb{R}^4$ instead of being defined on $\mathscr{T}' \cap S$ which is a subset of $\mathrm{i}\mathbb{R} \times \mathbb{R}^3$.

[3]Note that our so defined Schwinger function is an analytic function on $\mathrm{Wick}(\mathscr{T}' \cap S)$. In the literature the term "Schwinger function" is often more broadly understood in the sense of what we shall call (in §5.13) *Schwinger distribution*.

§ 5.12 Following the notation in [15] let us denote by ${}^0\mathscr{S}(\mathbb{R}^4)$ to be the set of test functions in $\mathscr{S}(\mathbb{R}^4)$ which vanish with all their (partial) derivatives in zero equipped with the topology induced from $\mathscr{S}(\mathbb{R}^4)$. We define analogously ${}^0\mathscr{S}(\mathbb{R}^4; V)$ for a finite dimensional vector space V. Finally we denote by ${}^0\mathscr{S}'(\mathbb{R}^4)$, respectively ${}^0\mathscr{S}'(\mathbb{R}^4; V)$ the topological duals of ${}^0\mathscr{S}(\mathbb{R}^4)$, respectively ${}^0\mathscr{S}'(\mathbb{R}^4; V)$.

The following theorem describes the procedure by which we can *uniquely* construct a Schwinger function starting from an $R_{\mathbf{SO}^0(1,3)}$*-covariant, forward* distribution (the notion of forward (vector valued) distribution was introduced in §5.5).

§ 5.13 Theorem. *Let* (V, R) *be a finite dimensional analytic representation of* $\mathbf{SO}(4, \mathbb{C})$. *Let* $\Xi \in \mathscr{S}'(\mathbb{R}^4; V)$ *be an* $R_{\mathbf{SO}^0(1,3)}$*-covariant, forward distribution. Then*

1. *There exists a unique* F *analytic function on the forward tube* $\mathscr{T}^+$ *of which* Ξ *is the lower boundary value.*

2. *There exists a unique* R*-covariant analytic function on the extended tube* $\mathscr{T}'$ *which analytically continues* F.

3. *Finally, the Schwinger function* $\widetilde{S}_{\widetilde{F}}$ *associated to* $\widetilde{F}$ *uniquely defines a distribution* $\mathcal{S}_\Xi$ *in* ${}^0\mathscr{S}'(\mathbb{R}^4; V)$.

Wightman and Schwinger distributions for the Dirac field

§ 5.14 Let us generally call a *unitary* (in general *reducible*) representation $(\mathcal{W}, U)$ of $\mathbf{ISpin}^0(1, 3)$ a **free canonical** representation when

1. Let $\mathcal{M}$ be a manifold embedded in $\mathbb{R}^4$, equipped with a $\mathbf{Spin}^0(1, 3)$-action and an invariant measure[4]. The carrier Hilbert space $\mathcal{W}$ is a space of (equivalence classes of) functions defined (up to a set of measure zero) on $\mathcal{M}$ with values in a finite dimensional Hilbert space $\mathcal{H}$.

2. the action U of the Lie group $\mathbf{ISpin}^0(1, 3)$ is of the following form

$$U(x, A)f(p) = e^{\mathrm{i}p\cdot x}\sigma(A)f(\Lambda(A)p), \quad p \in \mathcal{M}, f \in \mathcal{W},$$

   where $(x, A) \in \mathbf{ISpin}^0(1, 3)$, that is $x \in \mathbb{R}^4$, $A \in \mathbf{Spin}^0(1, 3)$, Λ denotes the action of $\mathbf{Spin}^0(1, 3)$ on $\mathcal{M}$, and σ is any *finite dimensional* (in general reducible) representation of $\mathbf{Spin}^0(1, 3)$ (in particular, σ does *not* depend on the function f).

§ 5.15 Remarks.

1. Our definition of free canonical representation can describe what in theoretical Physics terminology are called *generalized free fields*. This is the case because we have required that the normal Abelian subgroup $\mathbb{R}^4$ of $\mathbf{ISpin}^0(1, 3) = \mathbb{R}^4 \rtimes \mathbf{Spin}^0(1, 3)$ be represented by the character $e^{\mathrm{i}p\cdot x}$. That is, we have required that the covariant representation, when restricted to $\mathbb{R}^4$, be *irreducible*. Hence, by the *Källén-Lehmann representation* (cf. e.g. [4, Section 8.3.B]) only generalized-free fields can correspond to such a representation.

2. Note that the Wigner representation is *not* a *free canonical* representation, in the sense of the above definition, because it violates the requirement that σ be a finite dimensional representation. The importance of the requirement of having a finite dimensional representation σ is to ensure the existence of an extension of the covariant representation of $\mathbf{ISpin}^0(1, 3)$ to a representation (in general *not* unitary) of the complexification $\mathbf{ISpin}(4; \mathbb{C}) = \mathbb{C}^4 \rtimes \mathbf{Spin}(4; \mathbb{C})$ of $\mathbf{ISpin}^0(1, 3)$.

3. The representations $(\mathcal{W}_{(1,0)}, U_{(1,0)})$ and $(\mathcal{W}^+_{(1,0)\oplus(0,1)}, U_{(1,0)\oplus(0,1)})$ are both free canonical representations.

[4] The term *measure* in this context will always mean *regular Borel measure* (cf. [10]).

§ 5.16 We have defined above the covariant 1-particle representation $(\mathcal{W}^{+}_{(1,0)\oplus(0,1)}, U^{+}_{(1,0)\oplus(0,1)})$ of $\mathbf{ISpin}^0(1,3)$. Let us simplify the notation and write

$$\mathcal{W}_{+} \equiv \mathcal{W}^{+}_{(1,0)\oplus(0,1)}, \quad U_{+} \equiv U^{+}_{(1,0)\oplus(0,1)}.$$

We call the carrier space $\mathcal{W}_{+}$ the **covariant 1-particle space**.

§ 5.17 Associated with the representation $(\mathcal{W}_{+}, U_{+})$ there is a notion which looks almost like a Fourier transform. Consider $\Psi \in \mathscr{S}(O^{+}_{m}; \mathbb{C}^4)$. Let

$$\check{\Psi}(x) = \int_{O^{+}_{m}} U_{+}(x,e)\Psi(p)\,\mathrm{d}\mu_m(p),. \quad x \in \mathbb{R}^4,$$

where $e \in \mathbf{Spin}^0(1,3)$ denotes the identity. If we parametrize the hyperboloid O^{+}_{m} by identifying it with $\mathbb{R}^3$ we can write $\check{\Psi}$ as follows

$$\check{\Psi}(x) = \int_{\mathbb{R}^3} e^{\mathrm{i}(\omega(\mathbf{p})t+\mathrm{i}\mathbf{p}\cdot\mathbf{x})}\,\Psi(\mathbf{p})\frac{1}{2\omega(\mathbf{p})}\,\mathrm{d}\mathbf{p},$$

where $\omega(\mathbf{p}) \stackrel{\text{def}}{=} \sqrt{\mathbf{p}^2 + m^2}$, and by abuse of notation, we have denoted by $\Psi(\mathbf{p})$ the function Ψ composed with the chart which identifies O^{+}_{m} with $\mathbb{R}^3$, and $x = (t, \mathbf{x}) \in \mathbb{R}^4$. Now we see that $\check{\Psi}$ is indeed very similar to an inverse Fourier transform. This notion of transform, very similar to the (inverse) Fourier transform, originates from the fact that $\mathbf{ISpin}^0(1,3)$ has a non-trivial, normal, Abelian subgroup. Let us denote by $\mathcal{T}$ the map $\mathcal{T} : \Psi \mapsto \check{\Psi}$.

§ 5.18 We denote by $\mathscr{F}$ the Fourier transform on the space $\mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$ of Schwartz test functions on $\mathbb{R}^4$ with values in $\mathbb{C}^4$. Let $f \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$. We define a map: $\mathscr{S}(\mathbb{R}^4; \mathbb{C}^4) \to \mathcal{W}_{+}$ by

$$f \mapsto \Psi_f \stackrel{\text{def}}{=} \mathbb{Q}_{+}\big(\mathscr{F}(f)\restriction_{O^{+}_{m}}\big),$$

where $\mathscr{F}(f)\restriction_{O^{+}_{m}}$ denotes the restriction to O^{+}_{m} of the Fourier transform $\mathscr{F}$ of f, and $\mathbb{Q}_{+}$ is the projection operator defined in §4.13. Note that, for $f \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$, $\mathscr{F}(f)\restriction_{O^{+}_{m}} \in \mathcal{W}_{(1,0)\oplus(0,1)}$. Therefore $\Psi_f \in \mathcal{W}_{+}$, and the map $f \mapsto \Psi_f$ is indeed a well defined map: $\mathscr{S}(\mathbb{R}^4; \mathbb{C}^4) \to \mathcal{W}_{+}$.

§ 5.19 Proposition. *The map* $w_D(\cdot,\cdot) : \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4) \times \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4) \to \mathbb{C}$,

$$w_D(f,g) \stackrel{\text{def}}{=} \big(\Psi_f, \Psi_g\big)_{(1,0)\oplus(0,1)}, \quad f, g \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4), \tag{III.15}$$

is a well defined bilinear form on $\mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$ *which we call* ***2-point Wightman distribution****. Moreover*

$$w_D(f,g) = \big(\Psi_f, \Psi_g\big)_{\mathcal{W}^{+}} \quad f, g \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4), \tag{III.16}$$

and we have, for $f, g \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$,

$$w_D(f,g) = \int_{O^{+}_{m}} \left(\frac{1}{(2\pi)^4} \int_{\mathbb{R}^4\times\mathbb{R}^4} \overline{f(x)} e^{-\mathrm{i}p_\mu x^\mu} \frac{1}{m}\big(\hat{p}\gamma^0 + m\gamma^0\big) e^{\mathrm{i}p_\mu y^\mu} g(y)\,\mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}\mu_m(p), \tag{III.17}$$

where we use Einstein notation $p_\mu x^\mu \stackrel{\text{def}}{=} p_0x_0 - p_1x_1 - p_2x_2 - p_3x_3$, *for* $p = (p_\mu)$, $x = (x_\mu)$, $x_\mu \in \mathbb{R}$, $p_\mu \in \mathbb{R}$, $\mu = 0,1,2,3$.

Proof. Let us first note that the integral in (III.17) converges because, by assumption, f, g are in $\mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$. Indeed, the integration $\mathrm{d}x$ and $\mathrm{d}y$ performs (modulo constant) a Fourier transform, that is, we have

$$w_{\mathrm{D}}(f,g) = \int_{O^{+}_{m}} \overline{\mathscr{F}(f)(p)} \frac{1}{m}\big(\hat{p} + m\,\mathbb{I}_4\big)\gamma^0 \mathscr{F}(g)(p)\,\mathrm{d}\mu_m(p),$$

where $\mathscr{F}$ denotes Fourier transform. Now the Fourier transform sends Schwartz test functions into Schwartz test functions hence $\mathscr{F}(f), \mathscr{F}(g) \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$. The matrix $(\widehat{p} + m\,\mathbb{I}_4)\gamma^0$ is positive definite for any $p \in O_m^+$. We now parametrize the hyperboloid O_m^+ by projecting it onto $\mathbb{R}^3$. Hence we write $p \in O_m^+$ as $p = (\omega(\mathbf{p}), \mathbf{p})$ where $\mathbf{p} \in \mathbb{R}^3$ and $\omega(\mathbf{p}) \stackrel{\text{def}}{=} \sqrt{\mathbf{p}^2 + m^2}$. Then we have,

$$w_{\mathrm{D}}(f, g) = \int_{\mathbb{R}^3} \overline{\mathscr{F}(f)(\omega(\mathbf{p}), \mathbf{p})} \frac{1}{m} \big(\gamma^0 \omega(\mathbf{p}) + \sum_{j=1}^{3} p_j \gamma_j + m\,\mathbb{I}_4\big) \gamma^0 \mathscr{F}(g)(\omega(\mathbf{p}), \mathbf{p}) \frac{1}{2\omega(\mathbf{p})}\, \mathrm{d}\mathbf{p}$$

Now the $\mathrm{d}\mathbf{p}$ integral converges because the kernel

$$\big(\gamma^0 \omega(\mathbf{p}) + \sum_{j=1}^{3} p_j \gamma_j + m\,\mathbb{I}_4\big) \frac{1}{\omega(\mathbf{p})}$$

is polynomially bounded and $\mathscr{F}(f), \mathscr{F}(g)$ are Schwartz test functions. Finally, the fact that the integral in (III.17) coincides with the definition of $w_{\mathrm{D}}(\cdot, \cdot)$ in (III.15) and the fact that (III.15) coincides with (III.16) follow directly from the definitions of the scalar products $(\cdot, \cdot)_{(1,0)\oplus(0,1)}$ and $(\cdot, \cdot)_{\mathcal{W}^+}$ given in (III.12) and (III.14). □

§ 5.20 Remark. Note that the bilinear form $w_{\mathrm{D}} : \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4) \times \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$ is *singular*, in the sense of having a non-trivial kernel. Indeed let $f \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$ and define

$$Q_- f \stackrel{\text{def}}{=} \frac{1}{m}\big(\widehat{\partial} - m\,\mathbb{I}\big) f, \quad f \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4),$$

where $\widehat{\partial} \stackrel{\text{def}}{=} \sum_{\mu=0}^{3} \gamma_\mu \frac{\partial}{\partial x_\mu}$. The map Q_- is a well defined map $\mathscr{S}(\mathbb{R}^4; \mathbb{C}^4) \to \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$ with non-zero image. The image $\operatorname{Ran} Q_-$ of Q_- is the kernel of the bilinear form w_{D} in the sense that for any fixed $f_- \in \operatorname{Ran} Q_-$, $w_{\mathrm{D}}(f_-, g) = 0$ for all $g \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$.

PROOF. This follows easily from the definition of w_{D} and of Q_- in §**4.13**. □

§ 5.21 Consider the real Clifford algebra $C\ell(1, 3)$ defined abstractly as the quotient of the real tensor algebra $\mathrm{T}(\mathbb{R}^4) \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} (\mathbb{R}^4)^{\otimes n}$ of $\mathbb{R}^4$ by the ideal $\mathscr{I}$ generated by the elements $v \otimes w + w \otimes v - g(v, w)\,\mathbb{I}$, $v, w \in \mathbb{R}^4$, where $g(\cdot, \cdot)$ denotes the Minkowski pseudo metric with signature $(1, -1, -1, -1)$. Now we realize $\mathbf{Spin}^0(1, 3)$ as a subgroup of $C\ell(1, 3)$ by first defining

$$\mathbf{Pin}(1, 3) \stackrel{\text{def}}{=} \big\{\varphi \in C\ell(1, 3) : \exists \varphi^{-1}, g(\varphi, \varphi) = \pm 1\big\}$$
$$\mathbf{Spin}(1, 3) \stackrel{\text{def}}{=} \mathbf{Pin}(1, 3) \cap C\ell^{\text{even}}(1, 3),$$

where $C\ell^{\text{even}}(1, 3)$ denotes the even part of the Clifford algebra $C\ell(1, 3)$, and then letting $\mathbf{Spin}^0(1, 3)$ be the component of $\mathbf{Spin}(1, 3)$ connected with the identity. Now denote by γ the canonical embedding $\mathbb{R}^4$ in the Clifford algebra $C\ell(1, 3)$, and define an action τ of $\mathbf{Spin}^0(1, 3)$ on $\mathbb{R}^4 \cong \gamma(\mathbb{R}^4)$ by

$$\tau(S) : \gamma(v) \mapsto S^{-1} \gamma(v) S, \quad v \in \mathbb{R}^4, S \in \mathbf{Spin}^0(1, 3).$$

Under the identification of $\mathbb{R}^4 \cong \gamma(\mathbb{R}^4)$ the action τ of $\mathbf{Spin}^0(1, 3)$ on $\gamma(\mathbb{R}^4)$ defines an action Λ of $\mathbf{Spin}^0(1, 3)$ on $\mathbb{R}^4$ by

$$\gamma(\Lambda(S)v) = \tau(S)(\gamma(v)), \quad v \in \mathbb{R}^4, S \in \mathbf{Spin}^0(1, 3).$$

We now want to introduce a complex Hilbert space into the picture because eventually it is needed when employing these definitions in the framework of representation theory. Hence, alongside $C\ell(1, 3)$, we consider the complex Clifford algebra $\mathbb{C}\ell(4)$ obtained by taking the quotient of the tensor algebra $\mathrm{T}(\mathbb{C}^4) = \mathrm{T}(\mathbb{R}^4) \otimes_{\mathbb{R}} \mathbb{C}$ by the ideal generated by the elements $v \otimes w + w \otimes v - v \cdot w\,\mathbb{I}$ where $v, w \in \mathbb{C}^4$, and $v \cdot w = \sum_{k=0}^{3} v_k w_k$. Note that the bilinear form $v \cdot w$, $v, w \in \mathbb{C}^4$ is *not* the standard Hermitian scalar product on $\mathbb{C}^4$ which would involve complex conjugation. We now embed $C\ell(1,3)$ in $\mathbb{C}\ell(4)$ and represent $C\ell(4)$ on $\mathbb{C}^4$ as an algebra of 4-by-4 complex matrices. It is a standard fact that $C\ell(4)$ is a *simple* algebra, that is $\mathbb{C}\ell(4) \cong \mathrm{Aut}(\mathbb{C}^4) \cong M_4(\mathbb{C})$, where $M_4(\mathbb{C})$ denotes the algebra of 4-by-4 complex matrices. We now equip $\mathbb{C}^4$ with the standard Hilbert space structure given by the Hermitian scalar product

$$(v, w)_{\mathbb{C}^4} \stackrel{\text{def}}{=} \sum_{k=0}^{3} \overline{v_k} w_k, \quad v, v \in \mathbb{C}^4,$$

where the over line denotes complex conjugation. Now look again at the embedding γ of $\mathbb{R}^4$ in $C\ell(1,3)$. We extend γ to an embedding of $\mathbb{R}^4$ in $\mathbb{C}\ell(4)$, by embedding $C\ell(1,3)$ in $\mathbb{C}\ell(4)$, and considering $\mathbb{C}\ell(4)$ as an algebra of matrices (that is operators) on the Hilbert space $(\mathbb{C}^4, (\cdot,\cdot)_{\mathbb{C}^4})$. Hence now $\gamma(x)$, $x \in \mathbb{R}^4$ is an operator: $\mathbb{C}^4 \to \mathbb{C}^4$. Since $x \in \mathbb{R}^4$ is by definition a *real* vector we would like perhaps its embedding into $\mathbb{C}\ell(4)$ to be directly an *Hermitian* operator but this is *not* the case, indeed we have

$$(v, \gamma(x)^* w)_{\mathbb{C}^4} = (\gamma(Px)v, w)_{\mathbb{C}^4}, \quad v, w \in \mathbb{C}^4, x \in \mathbb{R}^4$$

where we have defined, for $x = (x_0, x_1, x_2, x_3)$, $Px = (x_0, -x_1, -x_2, -x_3)$. For this reason it makes sense to define an embedding $\mathbb{R}^4$ in $C\ell(1,3)$ given by $x \mapsto \gamma^0\gamma(x)$, where $\gamma^0 \stackrel{\text{def}}{=} \gamma(e_0)$ where e_0 is the unit eigenvector of the Minkowski pseudo metric g with eigenvalue $+1$. Now, by the anticommutation rule for the gamma matrices, it is straightforward to show that the operator $\gamma^0\gamma(x)$ is indeed Hermitian with respect to the standard scalar product $(\cdot,\cdot)_{\mathbb{C}^4}$ of $\mathbb{C}^4$. Moreover we have that this embedding $x \mapsto \gamma^0\gamma(x)$ is "compatible" with respect of the action of $\mathbf{Spin}^0(1,3)$, that is

$$(v, \gamma^0\gamma(\Lambda(S)x)w)_{\mathbb{C}^4} = (v, S^*\gamma^0\gamma(x)Sw)_{\mathbb{C}^4} = (Sv, \gamma^0\gamma(x)Sw)_{\mathbb{C}^4},$$

for all $v, w \in \mathbb{C}^4$, $x \in \mathbb{R}^4$, $S \in \mathbf{Spin}^0(1,3)$ and where S^* denotes the matrix complex-conjugated transposed which is a well defined operation because we see $C\ell(1,3)$ as embedded in $\mathbb{C}\ell(4) \cong M_4(\mathbb{C})$.

§ 5.22 We define the **reduced Wightman (generalized)-function for a free Dirac particle** to be the matrix valued tempered distribution in $\mathscr{S}'(\mathbb{R}^4) \otimes (\mathbb{C}^4 \otimes \mathbb{C}^4)$ associated with the bilinear form $w_{\mathrm{D}}(\cdot,\cdot)$ via the nuclear theorem . We denote this distribution by W_{D}. Explicitly we have

$$W_{\mathrm{D}}(\varphi) = \int_{O_m^+} \left(\int_{\mathbb{R}^4} \frac{1}{m} \big(\gamma^0 \hat{p} + \gamma^0 m\big) e^{\mathrm{i} p_\mu y^\mu} \varphi(y)\, \mathrm{d}y \right) \mathrm{d}\mu_m(p), \quad \varphi \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}).$$

Note that $\varphi \in \mathscr{S}(\mathbb{R}^4; \mathbb{C})$ is a *scalar function*. For every $\varphi \in \mathscr{S}(\mathbb{R}^4; \mathbb{C})$, $W_{\mathrm{D}}(\varphi) \in C\ell(1,3)$ Hence, by in our terminology, W_{D} is a *matrix-valued* distribution. One can think of W_{D} as the distribution in $\mathscr{S}'(\mathbb{R}^4)$ obtained by Fourier transforming the positive definite, matrix valued, σ-finite measure

$$\frac{1}{m}\gamma^0 \big(\hat{p} + m\,\mathbb{I}_4\big)\, \mathrm{d}\mu_m.$$

Notice that the above (positive definite, matrix valued) measure is supported on the hyperboloid O_m^+ which, in turn, is contained in the (open) forward light-cone V^+.

§ 5.23 The reduced Wightman distribution $W_{\mathrm{D}}(\cdot)$ and the bilinear form $w_{\mathrm{D}}(\cdot,\cdot)$ are related, as a straight forward computation shows, by

$$w_{\mathrm{D}}(f, g) = \sum_{jk=0}^{3} W_{jk}(\overline{f_j} *^+ g_k), \quad f, g \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4),$$

where f_j, g_k, and W_{jk} denote the components of the vector functions f, g and the matrix-valued distribution W; moreover $*^+$ denotes the operator[5] of "convolution with the wrong sign", that is $(\varphi_1 *^+ \varphi_2)(x) = \int_{\mathbb{R}^4} \varphi_1(x+y)\varphi_2(y)\,\mathrm{d}y$, $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbb{R}^4; \mathbb{C})$.

[5] The operator $*^+$ has the interesting "anti-commutativity" property: $(\varphi_1 *^+ \varphi_2)(x) = (\varphi_2 *^+ \varphi_1)(-x)$.

§ 5.24 Proposition. *The reduced, matrix valued, Wightman distribution $W_D \in \mathscr{S}'(\mathbb{R}^4; \mathbb{C}^4 \otimes \mathbb{C}^4)$ is the boundary value of a function holomorphic on the forward tube $\mathscr{T}^+ \overset{\text{def}}{=} \mathbb{R}^4 - \mathrm{i}V^+ \subset \mathbb{C}^4$. That is, there exists a unique function $\widetilde{W}$, holomorphic on $\mathscr{T}^+$, such that*

$$\lim_{\eta_0 \downarrow 0}{}^{\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4)} \widetilde{W}_D(\cdot + \mathrm{i}\eta_0) = W_D,$$

where $\lim^{\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4)}$ denotes the limit in $\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4)$, $\widetilde{W}_D(\cdot + \mathrm{i}\eta_0)$ denotes a function on $\mathbb{R}^4 = \mathfrak{R}\mathbb{C}^4$ (where $\mathfrak{R}$ means real part of), and $\eta_0 \downarrow 0$ means that η_0 goes to zero from the positive side. Moreover, the analytic function $\widetilde{W}$, defined on the forward tube $\mathscr{T}^+$ is uniquely determined by W.

PROOF. This follows from §5.4 (Cf. also [16, Theorem IX.32, p. 67]). □

§ 5.25 From the definition in §5.22, the reduced, matrix valued, Wightman distribution W has the following symmetry, under the action of $\mathbf{Spin}^0(1,3)$,

$$W_{\mathrm{D}}(\varphi\circ\Lambda(S)) = S^*\, W_{\mathrm{D}}(\varphi)\, S, \quad \varphi \in \mathscr{S}(\mathbb{R}^4;\mathbb{C}), \tag{III.18}$$

where $S \in \mathbf{Spin}^0(1,3) \hookrightarrow \mathbb{C}\ell(4)$ is thought of as a 4-by-4 complex matrix, hence the complex-conjugated and transposed matrix S^* makes sense, Λ denotes the covering map from $\mathbf{Spin}^0(1,3)$ to $\mathbf{SO}^0(1,3)$, and $(f\circ\Lambda(S))(x) = f(\Lambda(S)x)$, $x \in \mathbb{R}^4$.

§ 5.26 Remark. The space $\mathbb{C}^4$ in $\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4)$, $\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4)$ comes from the representation of the group $\mathbf{ISpin}^0(1,3)$. In this representation we have identified $\mathbb{C}^4$ with the exterior algebra $\bigwedge\mathbb{C}^2$ to give a representation of the (real) Clifford algebra $C\ell(1,3)$ as embedded in the complex Clifford algebra $\mathbb{C}\ell(4) \cong \mathrm{End}(\bigwedge\mathbb{C}^2)$. For this reason the quadratic form $w_{\mathrm{D}}(\cdot,\cdot)$ can also be considered as a quadratic form on $\mathscr{S}(\mathbb{R}^4;\bigwedge\mathbb{C}^2)$. Moreover, the Wightman distribution $W_{\mathrm{D}}(\cdot)$ can also be seen as a distribution in $\mathscr{S}'(\mathbb{R}^4;\mathbb{C}\ell(4))$.

§ 5.27 Let us decompose the reduced, matrix valued, Wightman (matrix valued) distribution W_{D} into two parts $W_{\mathrm{D}} = W_1 + W_0$, where, for $\phi \in \mathscr{S}(\mathbb{R}^4;\mathbb{C})$,

$$W_1(\varphi) \overset{\text{def}}{=} \int_{O_m^+}\left(\int_{\mathbb{R}^4} \varphi(x)e^{\mathrm{i}x_\mu p^\mu}\frac{\gamma^0\widehat{p}}{m}\,\mathrm{d}x\right)\mathrm{d}\mu_m(p),\ \ W_0(\phi) \overset{\text{def}}{=} \int_{O_m^+}\left(\int_{\mathbb{R}^4} \varphi(x)e^{\mathrm{i}x_\mu p^\mu}\gamma^0\,\mathrm{d}x\right)\mathrm{d}\mu_m(p),$$

where we use, as above, the Einstein convention $x_\mu p^\mu = x_0p_0 - x_1p_1 - x_2p_2 - x_3p_3$. We have discussed in §5.21 that we can embed $\mathbb{R}^4$ in $\mathbb{C}\ell(4)$ in at least two ways. One is the canonical embedding which sends $k \in \mathbb{R}^4$ to $\widehat{p} = \gamma(p)$. Another is given by sending $p \in \mathbb{R}^4$ to $\widehat{p}\gamma^0 = \gamma(p)\gamma^0$. From the representation of $\mathbf{Spin}^0(1,3)$ as a subgroup of $\mathbb{C}\ell(4)$ we deduce the following transformation properties

$$\gamma(\Lambda(S)p) = S^{-1}\gamma(p)S, \quad \gamma^0\gamma(\Lambda(S)p) = S^*\,(\gamma^0\gamma(p))\,S, \quad p \in \mathbb{R}^4,$$

where $\Lambda : \mathbf{Spin}^0(1,3) \to \mathbf{SO}^0(1,3)$ is the covering map. From this and the transformation properties of the Wightman distribution W in (III.18) we see that W_1 transforms as a *vector* distribution whereas W_0 transforms as a scalar distribution. To be explicit, let us define the following vector and scalar distributions, for $\varphi \in \mathscr{S}(\mathbb{R}^4;\mathbb{C})$,

$$\Xi_1(\varphi) \overset{\text{def}}{=} \int_{O_m^+}\left(\int_{\mathbb{R}^4} \varphi(x)e^{\mathrm{i}x_\mu p^\mu}\frac{p}{m}\,\mathrm{d}x\right)\mathrm{d}\mu_m(p), \quad \Xi_0(x) \overset{\text{def}}{=} \int_{O_m^+}\left(\int_{\mathbb{R}^4} \varphi(x)e^{\mathrm{i}x_\mu p^\mu}\,\mathrm{d}x\right)\mathrm{d}\mu_m(p),$$

where $\Xi_1 \in \mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4)$, $\Xi_0 \in \mathscr{S}'(\mathbb{R}^4;\mathbb{C})$.

Now Ξ_1, respectively Ξ_0, transforms, under the orthochronous Lorentz group, as a vector, respectively scalar, distribution. That is, for any $\Lambda \in \mathbf{SO}^0(1,3)$, we have

$$\Xi_1(\varphi) = \Lambda F_1(\varphi\circ\Lambda^{-1}), \quad \Xi_0(\varphi) = F_0(\varphi\circ\Lambda^{-1}),$$

where $(\varphi\circ\Lambda^{-1})(x) = \varphi(\Lambda^{-1}x)$, $x \in \mathbb{R}^4$.

We summarize the above discussion in the following theorem.

§ 5.28 Theorem. *Let i denote the map $\mathbb{R}^4 \oplus \mathbb{R} \hookrightarrow C\ell(1,3)$ given by*

$$x \oplus m \mapsto \gamma^0\gamma(x) + m\gamma^0, \quad x \in \mathbb{R}^4, m \in \mathbb{R}.$$

Then i is an embedding and is canonical (in the sense that it does not depend on the choice of basis for $\mathbb{R}^4$ and $C\ell(1,3)$). Moreover, i extends uniquely to an embedding $i_{\mathbb{C}} : \mathbb{C} \oplus \mathbb{C} \hookrightarrow \mathbb{C}\ell(4)$. Let (V, R) be the finite dimensional representation of $\mathbf{SO}(4,\mathbb{C})$ given by

$$V \stackrel{\text{def}}{=} \mathbb{C}^4 \oplus \mathbb{C}, \quad R(L)\, z \oplus \lambda = (Lz) \oplus \lambda, \quad L \in \mathbf{SO}(4,\mathbb{C}), z \in \mathbb{C}^4, \lambda \in \mathbb{C}.$$

That is, R is the defining representation of $\mathbf{SO}(4,\mathbb{C})$ on $\mathbb{C}^4$ and the trivial representation on $\mathbb{C}$. Let $R_{\mathbf{SO}^0(1,3)}$ be the restriction of R to elements of $\mathbf{SO}^0(1,3)$ (where $\mathbf{SO}^0(1,3$ is embedded into $\mathbf{SO}(4,\mathbb{C})$ by the standard embedding defined in §2.15). Then:

1. *The Wightman distribution W_D is identified under the embedding $i_{\mathbb{C}}$ with the unique Schwartz distribution Ξ_D which satisfies, for all $\varphi \in \mathscr{S}(\mathbb{R}^4)$,*

$$W_D(\varphi) = i_{\mathbb{C}}\big(\Xi_D(\varphi)\big).$$

2. *The distribution Ξ_D is a $R_{\mathbf{SO}^0(1,3)}$-covariant, forward distribution (the notion of R-covariant distribution was defined in §5.8 and that of forward distribution in §5.5).* □

We now apply the theorem in §5.13 to Ξ_{D} in §5.28 obtaining the following corollary.

§ 5.29 Corollary. *By the construction given above, the reduced, matrix valued, Wightman matrix valued distribution W_D defines a unique Schwinger distribution $\mathcal{S}_D$ in ${}^0\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^5)$. Moreover $\mathcal{S}_D$ can be canonically extended to a tempered distribution in $\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^5)$.*

Proof. The only statement that perhaps still requires proof is the fact that we can extend $\mathcal{S}_{\mathrm{D}}$ to a tempered distribution. This can be done directly from the explicit expression of $\mathcal{S}_{\mathrm{D}}$. We have $\mathcal{S}_{\mathrm{D}}(\phi) = \mathcal{S}_1(\phi) \oplus \mathcal{S}_0(\phi)$ with

$$\mathcal{S}_\ell(\phi) = \int_{\mathrm{Wick}(\mathscr{T}'\cap S)} F_\ell(x)\phi(x)\,\mathrm{d}x, \quad \phi \in {}^0\mathscr{S}(\mathbb{R}^4), \ell = 0, 1,$$

where $\mathrm{Wick}(\mathscr{T}' \cap S)$ was defined in §5.11. Now, it not difficult to show that

$$F_1(x) = \int_{O_m^+} \Big(\theta(t)e^{-\omega(\mathbf{p})t - \mathrm{i}\mathbf{p}\cdot\mathbf{x}} - \theta(-t)e^{\omega(\mathbf{p})t + \mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big)\frac{1}{m}p\,\mathrm{d}\mu_m(p), \quad x = (t,\mathbf{x}), t \neq 0,$$

where θ denotes the Heaviside step function, and

$$F_0(x) = \int_{O_m^+} e^{-\omega(\mathbf{p})|t| - \mathrm{i}\mathbf{p}\cdot\mathbf{x}}\,\mathrm{d}\mu_m(p), \quad x = (t,\mathbf{x}), t \neq 0,$$

Now we extend $\mathcal{S}_\ell$, $\ell = 0, 1$, to tempered distributions by defining

$$\mathcal{S}_1(\varphi) \stackrel{\text{def}}{=} \int_{O_m^+}\bigg(\int_{\mathbb{R}^4} \varphi(x)\Big(\theta(t)e^{-\omega(\mathbf{p})t - \mathrm{i}\mathbf{p}\cdot\mathbf{x}} - \theta(-t)e^{\omega(\mathbf{p})t + \mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big)\frac{1}{m}p\,\mathrm{d}t\,\mathrm{d}\mathbf{x}\bigg)\,\mathrm{d}\mu_m(p)$$

and

$$\mathcal{S}_0(\varphi) \stackrel{\text{def}}{=} \int_{O_m^+}\bigg(\int_{\mathbb{R}^4} \varphi(x)e^{-\omega(\mathbf{p})|t| - \mathrm{i}\mathbf{p}\cdot\mathbf{x}}\frac{1}{m}p\,\mathrm{d}t\,\mathrm{d}\mathbf{x}\bigg)\,\mathrm{d}\mu_m(p),$$

where the integrals are easily seen to converge for any $\varphi \in \mathscr{S}(\mathbb{R}^4)$. Therefore, $\mathcal{S}_1$ and $\mathcal{S}_0$ are bounded linear functionals from $\mathscr{S}(\mathbb{R}^4)$ to $\mathbb{R}$, that is they are tempered distributions. □

Schwinger distributions and bilinear forms

§ 5.30 Theorem §**5.28** says that the Schwinger distribution $\mathcal{S}_{\mathrm{D}}$ is *uniquely* specified by the reduced Wightman distribution W_{D}. In our detailed analysis we have highlighted the fact that the reduced Wightman distribution W_{D} is in fact composed of *two* irreducible Wightman distributions. We have called these distributions Ξ_1 and Ξ_0, where Ξ_1 transforms *vectorially* and Ξ_0 as a scalar under the proper, orthochronous Lorentz group. Hence we identified the Wightman function W_{D} with an R-covariant, forward, matrix valued distribution *with values in* $\mathbb{C}^5$. One could say that the original meaning of a two-point Wightman distribution as a *bilinear form* given in §**5.19** is lost when one passes to the *reduced* Wightman distribution.

As a consequence, one could say that the (uniquely determined) Schwinger distribution, does *not* have an *a priori* interpretation as a *bilinear form*. To stress this fact, we have defined the Schwinger distribution $\mathcal{S}_{\mathrm{D}}$ as a tempered distribution *with values in* $\mathbb{C}^5$.

Nevertheless, in the context of Euclidean quantum electrodynamics in the sense of Schwinger ([17]), one wants to interpret the Schwinger distribution as *Euclidean invariant bilinear form*. Multiple methods have been proposed in the literature to achieve this goal (cf. [14, 31], [29] and reference therein). Every method leads to a "different looking" bilinear form. Sometimes the bilinear forms in the literature are equivalent some other time they are not . We claim that all these bilinear forms are in fact due to the ambiguity, *not* in what we called Schwinger distribution, but in the interpretation of such a distribution as a *bilinear form*.

In the remaining of this subsection we associate a bilinear form to, what we called, Schwinger distribution.

§ 5.31 Let $\mathcal{S}_{\mathrm{D}} \in \mathscr{S}'(\mathbb{R}^4) \otimes \mathbb{C}^5$ be the matrix valued, Schwinger distribution of §**5.29** and let $\mathbb{C}\ell(4)$ the complex Clifford algebra over $\mathbb{C}^4$. We want to interpret $\mathcal{S}_{\mathrm{D}}$ as a bilinear form on some appropriate space of test functions. The original 2-point Wightman distribution $w_{\mathrm{D}}(\cdot,\cdot)$ was defined in §**5.19** as a bilinear form on $\mathscr{S}(\mathbb{R}^4, \mathbb{C}^4) \times \mathscr{S}(\mathbb{R}^4, \mathbb{C}^4)$. From this 2-point Wightman distribution we obtained a reduced, matrix valued, Wightman distribution $W_{\mathrm{D}} \in \mathscr{S}'(\mathbb{R}^4) \otimes (\mathbb{C}^4 \otimes \mathbb{C}^4)$. To interpret the Schwinger distribution $\mathcal{S}_{\mathrm{D}}$ as a bilinear form, we can first consider it as a "reduced form" of a bilinear form, in the same way that the reduced, matrix valued, Wightman distribution W_{D} is a "reduced form" of the bilinear form w_{D}. In order to follow this approach, we first want to convert the *vector valued* Schwinger distribution $\mathcal{S}_{\mathrm{D}}$ into a *matrix valued* Schwinger distribution with values in some algebra of matrices. Let us choose as algebra the same algebra we had for the reduced, matrix valued, Wightman distribution, that is the complex Clifford algebra $\mathbb{C}\ell(4)$. Then, we have to embed the image of $\mathcal{S}_{\mathrm{D}}$ into $\mathbb{C}\ell(4)$, hence we have to embed $\mathbb{C}^5$ into $\mathbb{C}\ell(4)$. As remarked above there are multiple possible ways to embed a vector in $\mathbb{C}^5$ into $C\ell(4)$. We choose to embed $\mathbb{C}^5 = \mathbb{C}^4 \oplus \mathbb{C} \hookrightarrow \mathbb{C}\ell(4)$ by

$$\tau : \mathbb{C}^4 \oplus \mathbb{C} \hookrightarrow \mathbb{C}\ell(4), \quad \tau(z \oplus \lambda) = \gamma_{\mathcal{E}}(z) + \lambda\,\mathbb{I}, \quad z \oplus \lambda \in \mathbb{C}^4 \oplus \mathbb{C},$$

where $\gamma_{\mathcal{E}}$ is the canonical embedding of $\mathbb{C}^4 \in \mathbb{C}\ell(4)$. Note in particular that if we denote as before, for $v \in \mathbb{R}^4$,

$$\gamma(v) = \sum_{j=0}^{3} v_j \gamma_j,$$

where $\gamma_j \in \mathbb{C}\ell(4)$, $j = 0, 1, 2, 3$, are the gamma matrices in Dirac representation, then, letting $\mathbb{R}^4 \hookrightarrow \mathbb{C}^4$ be the standard embedding of $\mathbb{R}^4$ into $\mathbb{C}^4$, we have, for a $v \in \mathbb{R}^4$ seen as a vector in $\mathbb{C}^4$,

$$\gamma_{\mathcal{E}}(v) = \mathrm{i} v_0 \gamma_0 + \sum_{k=1}^{3} v_k \gamma_k.$$

Let us the define the **Euclidean Schwinger distribution (for Dirac fields)** to be the *matrix valued* distribution $S_{\mathcal{E}} \in \mathscr{S}'(\mathbb{R}^4) \otimes \mathbb{C}\ell(4)$ such that, for all $\varphi \in \mathscr{S}(\mathbb{R}^4)$, we have

$$S_{\mathcal{E}}(\varphi) \stackrel{\text{def}}{=} \tau(\mathcal{S}_{\mathrm{D}}(\varphi)).$$

Now, with a similar reasoning as in §5.23 we define a bilinear form $s_\mathcal{E} : \mathscr{S}(\mathbb{R}^4;\mathbb{C}^4)\times\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4)\to\mathbb{C}$ by

$$s_\mathcal{E}(f,g) = \sum_{jk=0}^{3}[\mathcal{S}_\mathrm{D}(\overline{f_j} *^+ g_k)]_{jk}, \quad f,g\in\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4),$$

where $[\mathcal{S}_\mathrm{D}(\overline{f_j} *^+ g_k)]_{jk}$, $j,k = 0,1,2,3$, denote, as above, the matrix components and $*^+$ denotes the "convolution with the wrong sign" as in §5.23.

A straightforward computation shows the following explicit form for $S_\mathcal{E}$. Let $\varphi\in\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4)$. Then we have

$$S_\mathcal{E}(\varphi) = \int_{\mathbb{R}^4}\left(\int_{\mathbb{R}^4}\varphi(x)e^{-\mathrm{i}p\cdot x}\frac{\mathrm{i}p_0\gamma_0+\sum_{k=1}^{3}p_k\gamma_k+m\,\mathbb{1}_4}{p^2+m^2}\,\mathrm{d}x\right)\mathrm{d}p, \tag{III.19}$$

where γ_j, $j = 0,1,2,3$, denote the gamma matrices in Dirac representation.

§ 5.32 Remark. Note that the Euclidean Schwinger distribution $S_\mathcal{E}$ in (III.19) gives rise, for every $\varphi\in\mathcal{S}(\mathbb{R}^4)$ to a *non definite*, but *Hermitian* (i.e. symmetric under the Hermitian scalar product in $\mathbb{C}^4$) matrix $S_\mathcal{E}(\varphi)\in\mathbb{C}\ell(4)$.

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IV

A note to Kupsch probabilistic setting for the Euclidean Dirac field

Abstract

The n-point functions of Euclidean Dirac quantum fields have been expressed by Kupsch as the expectation of a certain function of complex Gaussian random fields. We simplify his approach in three ways. First, by employing the Schwinger 2-point function for the Dirac quantum fields in the representation given in a paper by van Nieuwenhuizen and Waldron, we avoid the doubling of the number of spinor fields. Second, we use the chaos expansions of complex Gaussian processes in a way which, we believe, is better suited for further applications. Third, we use an isomorphism due to Friedrichs to relate the Fermionic Fock space and the Bosonic one. As a consequence of our simplified approach we can treat in a more unified way the antisymmetric properties of a collection of n-Fermions and the non-positive definiteness of the 2-point function.

Contents

1 Introduction

Mathematically speaking, one of the most appealing features of Euclidean techniques, when applied to Bosonic theories, is the following. At least after imposing some regularization, one can express the vacuum averages of the Euclidean quantum fields as integrals of some function with respect to a (σ-additive) probability measure. This picture, on one hand, removes the complexity of dealing with unbounded operators in favor of integration of a commuting algebra of functions with respect to a well defined probability measure. On the other hand, this probabilistic approach puts in evidence many properties of the given system, for example positivity, reflection positivity, and Markovianity that are useful also for the study of the corresponding relativistic quantum fields. In particular the positivity and σ-additivity of the probability measure which characterize the system (perhaps with some regularizations) are very strong properties which make directly applicable many results from analysis. Moreover many results from analysis of elliptic or parabolic systems become available through this approach. Due to well known connections between probability and analysis (e.g. in elliptic and parabolic differential equations) the Euclidean approach permits also the exploitation of strong related analytic methods.

Free Euclidean Bosonic field theories could be *defined*, in this context, as being those theories which correspond to a *centered, Gaussian, probability measure*. For example a free Euclidean Bosonic quantum field with positive mass, is completely determined by its 2-point function (i.e. the vacuum average of the Euclidean invariant bilinear form built from the quantum field and its Hermitian adjoint). Equivalently, in the probabilistic interpretation mentioned above, free Euclidean field theories with positive mass are completely determined by the covariance (or second moment) of the Gaussian measure describing them.

It would be desirable to extend such a measure theoretic, probabilistic setting to Fermionic theories (on this matter cf. e.g. [6]). Even when restricting to *free* Fermionic theories, one is usually faced with two main problems:

1. The presence of the Fermi-Dirac statistics,

2. The formulation of Euclidean covariance of a system with "half-integer" spin.

The Fermi-Dirac statistics implies that the operators describing Fermions satisfy anticommutation relations. In particular they cannot be related in a straight forward fashion to a commuting algebra of functions. Euclidean covariance, when the fields have half-integer spin, implies that the 2-point function is anti-symmetric (when described in an appropriate basis, more on this will be mentioned below), hence in particular they are *not* positive definite. The fact that Fermionic operators satisfy anticommutation relations and the fact the the 2-point function is anti-symmetric is a strong hint which suggests that these two problems are two faces of a single property. In many regards this is in fact true. One can think for example about the spin and statistics theorem in its various formulations (cf. e.g. [4]). Nevertheless, one could also argue that the nature of these two phenomena (statistics and Euclidean covariance) is quite different. The statistical property involves, by definition, more than one particle, whereas the Euclidean covariance is a property which can be described already at the one particle level.

Kupsch ([17]) describes a way to deal with the these two problems and, as a consequence, provides a probabilistic description of free Fermionic fields in terms of a Gaussian measure on a classical probability space.

The statistical problem is eliminated in [17] by relating the n-point functions of the theory, not directly with the moments of a Gaussian, but with a certain function of the moments of the same degree. This certain function encodes the combinatorial, anticommuting nature of Fermi-Dirac statistics.

The non-positivity of the 2-point function is handled in [17] first by doubling the number of Fermionic fields. Because of this doubling, the 2-point function is replaced by a different 2-point function (in terms of twice as many fields) which is antisymmetric (in the sense to be specified below). Then a standard property of antisymmetric bilinear forms is applied. Specifically, antisymmetric bilinear forms can be interpreted as the imaginary part of a *positive definite* Hermitian form. In turn, an Hermitian form can be taken as covariance of a *complex* Gaussian measure.

Combining these two ideas, Kupsch is able to describe Euclidean Fermionic n-point functions as the expectation of a, somewhat complicated, function of Gaussianly distributed *complex* random variables.

We give a simplified reformulation of Kupsch' result which, in our opinion, is better suited for further applications. The main points of our simplification are the following.

1. Kupsch, similarly to [19], doubles the number of Fermionic fields. In this way he obtains a new 2-point function, in twice as many fields, which is antisymmetric. Any 2-point function can be made antisymmetric by such a doubling. Hence it is not immediately clear why this would be a natural approach specific to Fermionic fields. We show that the doubling is not necessary to obtain an antisymmetric 2-point function, which can be indeed obtained effectively by choosing an appropriate representation for the quantum fields. In practice, we employ an idea from [23] where a special notion of Wick rotation is defined. Via this Wick rotation, the resulting Euclidean approach gives rise, without doubling, to an antisymmetric two point function.

2. We give a simple generalization of the treatment of *complex* Gaussian white noise, and of the corresponding chaos expansions, given in [12]. We need the formulas from this simple generalization, in conjunction with what we discuss in point 3. below, to treat the case of Euclidean Dirac field. We hope that our treatment, through these chaos expansions, will also have applications to the study of certain non Gaussian random fields. Our approach should simplify the treatment of non Gaussian random fields in future generalizations.

3. To treat the combinatorial, anticommuting nature of Fermionic fields, we use an isomorphism due to Friedrichs [5] which maps the Fermionic Fock space into the Bosonic one isometrically and respects the grading which corresponds to the number of particle operator (Let us mention the following further references regarding this isomorphism: from the physics literature we cite [16] and [8, 9, 10, 11]; from the mathematical literature we cite [15], [18]). This isomorphism avoids the need to choosing a basis of the physical Hilbert space. Nevertheless, it does not come free of choices. One still has to choose a fixed family of "*Friedrichs functions*" which have the task of "symmetrizing" the antisymmetric functions which constitute the elements of the Fermionic Fock space.

As a consequence of our approach we have the following interesting perspective. In the context of point 1. above, when dealing with the positivity of the 2-point function, we need to introduce a *real structure* (in conjunction with a *complex structure*), which allows us to relate the antisymmetric 2-point function with the imaginary part of a Hermitian, positive definite, 2-point function. In practice, the very fact of taking the imaginary part, corresponds to choosing a *real structure*. Similarly, the family of "*Friedrichs functions*", needed to define the Friedrichs isomorphism, could be considered as a new structure which generalizes the notion of real structure. The real structure is employed at the level of 2-point functions (hence at 1-particle level), whereas this new structure, built from Friedrichs functions, plays a similar role but at the level of $2n$-point functions, for all $n > 1$. We come back to this point in §**5.12**.

The structure of this note is as follows. In **section 2** we define the Euclidean 2-point function described in [23].

In **section 3** we convert the Euclidean 4-component complex Dirac spinors into Euclidean 8-component real Dirac spinors. In this representation the 2-point function becomes an anti-symmetric real operator.

In **section 4** we make use of the standard result that an anti-symmetric real bilinear form on a real Hilbert space can be obtained as the imaginary part of an Hermitian scalar product on a complex Hilbert space. There the main ingredient will be the introduction of a complex structure next to a real structure. We then describe the chaos expansions for complex Gaussian processes and, employing those, we construct a complex Gaussian random field with covariance given by this Hermitian scalar product.

In **section 5** we show how to recover the Euclidean Fermionic n-point functions from expectations of the complex Gaussian random field introduced in section 4. The main result concerning a probabilistic representation of free Euclidean Fermionic n-point functions is given in §**5.10**.

2 Euclidean Dirac fields via Wick rotation

We summarize the approach of [23] for an Euclidean version of the Dirac field .

Consider the four dimensional Minkowski space-time with metric $\eta = \mathrm{diag}(1, -1, -1, -1)$. We can think of the Euclidean imaginary time as a fifth real dimension. The Wick rotation is a rotation by $\pi/2$ in the plane formed by the original (real) time direction and the (imaginary) euclidean time. In [23] the authors proposed to perform a similar rotation in the spinor components alongside the Wick rotation in the time variable.

The result, after this prescription for Wick rotation, is to consider a Euclidean theory with 2-point function given, in momentum space, by

$$\widehat{S}_{\mathcal{E}}(p) = \gamma_0 \frac{\sum_{j=1,2,3} p_j \gamma_j + \mathrm{i} p_0 \gamma_5 + m\mathbb{1}}{p^2 + m^2}, \quad p \in \mathbb{R}^4, \quad m > 0,$$

where $p^2 = p_0^2 + p_1^2 + p_2^2 + p_3^3$ and the γ-matrices $(\gamma_\mu)_{\mu=0,1,2,3}$, $\gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ are the standard (Minkowski) γ-matrices which satisfy, for $\mu, \nu = 0, \dots, 4$,

$$\{\gamma_\mu, \gamma_\nu\} = \eta_{\mu\nu}\mathbb{1}, \quad \{\gamma_\mu, \gamma_5\} = 0, \quad (\gamma_5)^2 = \mathbb{1}, \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where the brackets $\{\cdot,\cdot\}$ denote the symmetrized product ($\{A, B\} = AB + BA$ for two 4-by-4 matrices A, B). We call $\widehat{S}_{\mathcal{E}}$ the Euclidean 2-point function.

For the sake of concreteness we chose in what follows the standard Dirac representation[1]:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ \gamma_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \ \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \ \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The point of this prescription of Wick rotation is that the Euclidean 2-point function is *Hermitian* in the sense that

$$\overline{\widehat{S}_{\mathcal{E}}(p)^{\mathrm{t}}} = \widehat{S}_{\mathcal{E}}(p), \quad p \in \mathbb{R}^4, \tag{IV.1}$$

where $\overline{\widehat{S}_{\mathcal{E}}(p)^{\mathrm{t}}}$ denotes the complex conjugation and transposition of the matrix $\widehat{S}_{\mathcal{E}}(p)$.

In the representation of γ-matrices that we chose we have, explicitly,

$$\widehat{S}_{\mathcal{E}}(p) = \frac{1}{p^2 + m^2} \begin{pmatrix} m & 0 & \mathrm{i}p_0 + p_3 & p_1 - \mathrm{i}p_2 \\ 0 & m & \mathrm{i}p_2 + p_1 & \mathrm{i}p_0 - p_3 \\ p_3 - \mathrm{i}p_0 & p_1 - \mathrm{i}p_2 & -m & 0 \\ \mathrm{i}p_2 + p_1 & -\mathrm{i}p_0 - p_3 & 0 & -m \end{pmatrix}.$$

Following the same derivation as in the Minkowski case [22], it can be shown that the function $\widehat{S}_{\mathcal{E}}(p)$ is the symbol (in momentum space) of a selfadjoint, bounded[2] operator $\mathrm{S}_{\mathcal{E}}$ on $L^2(\mathbb{R}^4; \mathbb{C}^4)$ given by

$$(\mathrm{S}_{\mathcal{E}} f)(x) \stackrel{\text{def}}{=} (S_{\mathcal{E}} * f)(x), \quad x = (x_\mu)_{\mu=0,1,2,3} \in \mathbb{R}^4,$$

where we define $S_{\mathcal{E}}$ as the inverse Fourier transform in the sense of distributions of $\widehat{S}_{\mathcal{E}}$, the $x \in \mathbb{R}^4$ being the conjugate variables of the $p \in \mathbb{R}^4$ via Fourier transform, $*$ denotes convolution, and $f \in \mathscr{S}(\mathbb{R}^4)$.

3 Real form of the fields

In this section we shall connect the Euclidean two point function(s) of the free Dirac field with vacuum averages of Fermionic fields in the standard, formal, notation of quantum field theory, cf. e.g. [1, 21, 20]. In particular $\Psi(x)$, $x\mathbb{R}^4$, will denote a quantum Fermionic field in the "*space-time representation*", where $x \in \mathbb{R}^4$ denotes a point of the Euclidean space-time. We shall employ the notation $\widehat{\Psi}(p)$, $p \in \mathbb{R}^4$, to denote a quantum Fermionic field in the "*momentum space representation*", where $p \in \mathbb{R}^4$ denotes a point of $\mathbb{R}^4$ thought as the space of Fourier variables.

With this notation we can give two notions of the 2-point function of a Euclidean free Dirac field, one in space-time and the other in momentum space. Let us denote by $\langle F(\Psi) \rangle$ the Berezin functional average of the function F of Ψ, that is its vacuum average. We can either consider the "*space-time representation*":

$$\left\langle \Psi(x)\Psi^\dagger(y) \right\rangle = -\left\langle \Psi^\dagger(y)\Psi(x) \right\rangle = \mathrm{i}S_{\mathcal{E}}(x-y), \quad x, y \in \mathbb{R}^4,$$

[1] Cf. e.g. [2].

[2] A direct computation shows that

$$\widehat{S}(p)^*\widehat{S}(p) = \widehat{S}(p)^2 = \frac{1}{p^2 + m^2}.$$

Hence, denoting by $\widehat{f}$ the four-dimensional Fourier transform of f, we get

$$\|\mathrm{S}_{\mathcal{E}} f\|^2_{L^2(\mathbb{R}^4;\mathbb{C}^4)} = \int_{\mathbb{R}^4} \|\widehat{S}(p)\widehat{f}(p)\|^2_{\mathbb{C}^4}\,\mathrm{d}p = \int_{\mathbb{R}^4} \frac{1}{p^2 + m^2} \|\widehat{f}(p)\|^2_{\mathbb{C}^4}\,\mathrm{d}p \lesssim \|f\|^2_{L^2(\mathbb{R}^4;\mathbb{C}^4)}, \tag{IV.2}$$

which proves that the operator $S_{\mathcal{E}}$ is bounded in $L^2(\mathbb{R}^4; \mathbb{C}^4)$.

or the "*momentum space representation*"

$$\left\langle \widehat{\Psi}(p)\widehat{\Psi}^{\dagger}(q)\right\rangle = -\left\langle \widehat{\Psi}^{\dagger}(p)\widehat{\Psi}(q)\right\rangle = \mathrm{i}\widehat{S}_{\mathcal{E}}(p)\,\delta(p-q), \quad p,q\in\mathbb{R}^4.$$

Note that, in both last formulas, the i on the right hand side makes the right hand side anti-Hermitian, cf. (IV.1).

We shall now define 8-component real fields with *real* anti-symmetric 2-point function. We perform the computations in the momentum space representation. One could also chose the space-time representation, in which case the computations would be analogous, mutatis mutandis.

Momentum space representation

With notations as above, we define

$$\widehat{\Gamma}(p) \stackrel{\text{def}}{=} \begin{pmatrix}\widehat{\Phi}(p)\\ \widehat{\Pi}(p)\end{pmatrix}, \quad \widehat{\Phi}(p) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(\widehat{\Psi}(p)+\widehat{\Psi}^{\dagger}(p)\right), \quad \widehat{\Pi}(p) \stackrel{\text{def}}{=} \frac{\mathrm{i}}{\sqrt{2}}\left(\widehat{\Psi}(p)-\widehat{\Psi}^{\dagger}(p)\right).$$

A straight forward computation shows that

$$\left\langle \widehat{\Gamma}(p)\widehat{\Gamma}(q)\right\rangle = \widehat{C}(p)\,\delta(p-q), \quad p,q\in\mathbb{R}^4$$

with

$$\widehat{C}(p) \stackrel{\text{def}}{=} \begin{pmatrix} I(p) & R(p)\\ -R(p) & I(p)\end{pmatrix}, \quad I(p) \stackrel{\text{def}}{=} \frac{\mathrm{i}}{2}\left(\widehat{S}_{\mathcal{E}}(p)-\widehat{S}_{\mathcal{E}}(p)^{\mathrm{t}}\right), \quad R(p) \stackrel{\text{def}}{=} \frac{1}{2}\left(\widehat{S}_{\mathcal{E}}(p)+\widehat{S}_{\mathcal{E}}(p)^{\mathrm{t}}\right). \tag{IV.3}$$

Explicitly, in the representation of γ-matrices we chose, we have

$$I(p) = \frac{1}{p^2+m^2}\begin{pmatrix} 0 & 0 & -p_0 & p_2\\ 0 & 0 & -p_2 & -p_0\\ p_0 & p_2 & 0 & 0\\ -p_2 & 2p_0 & 0 & 0\end{pmatrix}, \quad R(p) = \frac{1}{p^2+m^2}\begin{pmatrix} m & 0 & p_3 & p_1\\ 0 & m & p_1 & -p_3\\ p_3 & p_1 & -m & 0\\ p_1 & -p_3 & 0 & -m\end{pmatrix}.$$

Of course knowing $\widehat{C}(p)$ is equivalent to knowing $\widehat{S}_{\mathcal{E}}(p)$. Hence in what follows we will call $\widehat{C}(p)$ the **real (Euclidean) 2-point function**, whereas we will continue to call $\widehat{S}_{\mathcal{E}}(p)$ the **Euclidean 2-point function**.

In the usual theory of quantum fields one is interested in the n-point functions. In this Euclidean setting we call **Euclidean n-point functions** the following quantities

$$\widehat{S}^{(n)}_{\ell_1\ldots\ell_n}(p_1,\cdots,p_n) \stackrel{\text{def}}{=} \det_{1\le i,j\le n}\left(\widehat{S}_{\mathcal{E}}(p_i-p_j)\right), \quad n \text{ even}, \quad \ell_j = 1,\ldots,4, p_j\in\mathbb{R}^4, j=1,\ldots,n, \tag{IV.4}$$

and $\widehat{S}^{(n)} = 0$ for n odd. Alongside these Euclidean n-point functions we consider the **real Euclidean n-point functions** which we define to be

$$\widehat{C}^{(n)}_{r_1\ldots r_n}(p_1,\cdots,p_n) \stackrel{\text{def}}{=} \det_{1\le i,j\le n}\left(\widehat{C}_{r_ir_j}(p_i-p_j)\right), \quad n \text{ even}, \quad r_j = 1,\ldots,8, p_j\in\mathbb{R}^4, j=1,\ldots,n, \tag{IV.5}$$

and $\widehat{C}^{(n)} = 0$ for n odd.

4 Complex structure and complex Gaussian random field

Complex structure

We start with some general considerations about complex structures on real Hilbert spaces. Then we turn to the special case we are interested in.

§ 4.1 Complex structure. Let V be a *real* separable Hilbert space. We define a **complex structure** on V to be a linear anti-symmetric automorphism J of V such that $\mathrm{J}^2 = -\mathbb{1}$.

Consider the complexified Hilbert space $V_{\mathbb{C}} \stackrel{\text{def}}{=} V \otimes_{\mathbb{R}} \mathbb{C}$. Then we have the canonical decomposition

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1},$$

given by the two orthogonal projectors from $V_{\mathbb{C}}$ into $V^{1,0}$, respectively $V^{0,1}$, defined by:

$$P_{1,0} = \frac{1}{2}(\mathbb{1} + i\mathrm{J}), \quad P_{0,1} = \frac{1}{2}(\mathbb{1} - i\mathrm{J}).$$

J is the above complex structure. Consider the map $\phi : V \to V^{1,0}$ given by

$$\phi(v) \stackrel{\text{def}}{=} \frac{1}{2}(\mathbb{1} - \mathrm{iJ})v. \tag{IV.6}$$

that is, the map ϕ is just the projection $P_{1,0}$ restricted to V as a subspace of $V_{\mathbb{C}}$.
Proposition. The map ϕ is *complex* isomorphism of the Hilbert spaces[3] (V, J) and $(V^{1,0}, \mathrm{i})$.
Proof. First note that ϕ is $\mathbb{C}$-linear in the sense that

$$\phi(\mathrm{J}v) = \mathrm{i}\phi(v),$$

which also implies that, for any $w \in \operatorname{Ran}\phi$, we have $\mathrm{J}w = \mathrm{i}w$. Second ϕ is invertible. In fact let us define a linear map $\chi : V^{1,0} \to V$ by

$$\chi(w) = \Re w + \mathrm{J}\Im w, \quad w \in V^{1,0}.$$

A straight forward computation shows that, for ϕ as above, $\phi(\chi(w)) = w$, $w \in V^{1,0}$. Hence ϕ in invertible and χ is its inverse. Finally ϕ is an isometry because $\frac{1}{2}(\mathbb{1} + \mathrm{iJ})$ is an orthogonal projection. □

§ 4.2 We now turn to the special case which interests us. Consider the bounded operator C : $L^2(\mathbb{R}^4; \mathbb{R}^8) \to L^2(\mathbb{R}^4; \mathbb{R}^8)$

$$(\mathrm{C}f)(k) \stackrel{\text{def}}{=} \widehat{C}(k)f(k), \quad f \in \mathscr{S}(\mathbb{R}^4; \mathbb{R}^8),$$

where $\widehat{C}(k)$ is defined in (IV.3). Since C is selfadjoint, we have the following polar decomposition

$$\mathrm{C} = \mathrm{JP} = \mathrm{PJ}$$

with

$$\mathrm{P} = \sqrt{-\mathrm{C}^2} = \frac{1}{\sqrt{p^2 + m^2}}\mathbb{1}, \quad \mathrm{J} = \sqrt{p^2 + m^2}\,\mathrm{C}.$$

Here J is looked upon as a multiplication operator in $L^2(\mathbb{R}^4; \mathbb{R}^8)$. A straight forward computation, indeed, shows that $\mathrm{J}^2 = -\mathbb{1}$, hence J defines a complex structure on the space $L^2(\mathbb{R}^4; \mathbb{R}^8)$. The map χ above gives us an isomorphism between $(L^2(\mathbb{R}^4; \mathbb{C}^4), \mathrm{i})$ and $(L^2(\mathbb{R}^4; \mathbb{R}^8), \mathrm{J})$.

Now, given the bilinear, antisymmetric, real, quadratic form on $\mathscr{S}(\mathbb{R}^4; \mathbb{R}^8)$,

$$\omega(f, g) = \int_{\mathbb{R}^4} f(k)\widehat{C}(k)g(k)\,\mathrm{d}k, \quad f, g \in \mathscr{S}(\mathbb{R}^4; \mathbb{R}^8), \tag{IV.7}$$

we construct the following complex valued form

$$h(f, g) = \omega(\mathrm{J}f, g) - \mathrm{i}\omega(f, g).$$

[3] We denote by e.g. (V, J) the space V together with the complex structure J which makes V into a complex space.

The interesting case to us is when $h(\cdot,\cdot)$ takes the form

$$\begin{aligned} h(f,g) &= \int_{\mathbb{R}^4} f(k)\Big(\frac{1}{\sqrt{p^2+m^2}} - \mathrm{i}\widehat{C}(k)\Big) g(k)\,\mathrm{d}k \\ &= \sum_{ij=1}^{8} \int_{\mathbb{R}^4} \Big(\frac{1}{\sqrt{p^2+m^2}}\mathbb{1} - \mathrm{i}\widehat{C}(k)\Big)_{ij} \delta(p-k) f_i(k) g_j(p)\,\mathrm{d}k\,\mathrm{d}p, \quad f,g\in \mathscr{S}(\mathbb{R}^4;\mathbb{R}^8). \end{aligned} \tag{IV.8}$$

In the following we are interested in the following property, which follows from (IV.7),

$$\omega(f,g) = \mathrm{i}\sum_{ij=1}^{8} \int_{\mathbb{R}^4} \left(\frac{1}{\sqrt{p^2+m^2}} - \mathrm{i}\widehat{C}_{ij}(k)\right) \delta(p-q) \frac{1}{\sqrt{2}} (f\wedge g)_{ij}(k,p)\,\mathrm{d}p\,\mathrm{d}k, \tag{IV.9}$$

where $1/\sqrt{2}$ comes from the fact that $(f\wedge g)(p,k) \overset{\text{def}}{=} \frac{1}{2}(f_i(p)g_j(k) - f_j(k)g_i(p))$. Let us define

$$(\mathrm{K}f)_i(p) \overset{\text{def}}{=} \sum_{j=1}^{8} \int_{\mathbb{R}^4} \left(\frac{1}{\sqrt{p^2+m^2}} - \mathrm{i}\widehat{C}_{ij}(k)\right) \delta(p-k) f_j(k)\,\mathrm{d}k, \quad f\in\mathscr{S}(\mathbb{R}^4;\mathbb{R}^8),$$

as an operator which sends the real space $\mathscr{S}(\mathbb{R}^4;\mathbb{R}^8)$ into the complex space $L^2(\mathbb{R}^4;\mathbb{C}^8)$. We can write (IV.9) in a more abstract way as follows

$$\omega(f,g) = \mathrm{i}\mathrm{Tr}_{L^2(\mathbb{R}^4;\mathbb{R}^8)}\big(\mathrm{K}\,(f\wedge g)\big), \tag{IV.10}$$

where $\mathrm{Tr}_{L^2(\mathbb{R}^4;\mathbb{R}^8}$ denotes the trace of an operator in $L^2(\mathbb{R}^4;\mathbb{R}^8)$, and $\mathrm{K}(f\wedge g)$ denotes the product of the bounded operator K with the $f\wedge g$ interpreted as a trace class operator in $L^2(\mathbb{R}^4;\mathbb{R}^8)$.

§ 4.3 We now consider the map ϕ, which was defined in the general in (IV.6), as a map from $L^2(\mathbb{R}^4;\mathbb{R}^8)$ to $L^2(\mathbb{R}^4;\mathbb{C}^4)$. restricted to $\mathscr{S}(\mathbb{R}^4;\mathbb{R}^8)$, sends the bilinear form h into a bilinear form on $\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4)$ which we denote by $\vartriangle(\cdot,\cdot)$, that is, we define on $\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4)$ the bilinear form

$$\vartriangle(f,g) \overset{\text{def}}{=} h(\chi(f),\chi(g)), \quad f,g\in\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4).$$

In our case the form $\vartriangle(\cdot,\cdot)$ is

$$\vartriangle(f,g) = 2\int_{\mathbb{R}^4} \overline{f(k)} \frac{1}{\sqrt{k^2+m^2}} \mathbb{I} g(k)\,\mathrm{d}k, \quad f,g\in\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4), \tag{IV.11}$$

where $\mathbb{I}$ says for the unite matrix in $\mathbb{C}^4$. To show that this is the case it is enough to show that This computation shows that

$$\vartriangle(\phi(f),\phi(g)) = h(f,g), \quad f,g\in\mathscr{S}(\mathbb{R}^4;\mathbb{R}^8).$$

Now this follows from the following computation, where now $f, g \in \mathscr{S}(\mathbb{R}^4; \mathbb{R}^8)$ are *real*,

$$
\begin{aligned}
\vartriangle(\tfrac{1}{2}(\mathbb{1}+\mathrm{iJ})f, \tfrac{1}{2}(\mathbb{1}+\mathrm{iJ})g) &= \frac{1}{2}\int_{\mathbb{R}^4} \overline{(\mathbb{1}+\mathrm{iJ})f(k)} \frac{\mathbb{1}}{\sqrt{k^2+m^2}} (\mathbb{1}+\mathrm{iJ})g(k)\,\mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}^4} (\mathbb{1}-\mathrm{iJ})f(k) \frac{1}{\sqrt{k^2+m^2}} (\mathbb{1}+\mathrm{iJ})g(k)\,\mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}^4} \left(f(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}} g(k) + (\mathrm{J}f)(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}}(\mathrm{J}g)(k) \right) \mathrm{d}k \, + \\
&\qquad + \frac{\mathrm{i}}{2}\int_{\mathbb{R}^4} \left((\mathrm{J}f)(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}} g(k) - f(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}}(\mathrm{J}g)(k) \right) \mathrm{d}k \\
&= \frac{1}{2}\int_{\mathbb{R}^4} \left(f(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}} g(k) + f(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}}(-\mathrm{J}^2 g)(k) \right) \mathrm{d}k \, + \\
&\qquad + \frac{\mathrm{i}}{2}\int_{\mathbb{R}^4} \left(-f(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}} (\mathrm{J}g)(k) - f(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}}(\mathrm{J}g)(k) \right) \mathrm{d}k \\
&= \int_{\mathbb{R}^4} f(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}} g(k)\,\mathrm{d}k - \mathrm{i}\int_{\mathbb{R}^4} f(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}} (\mathrm{J}g)(k)\,\mathrm{d}k \\
&= \int_{\mathbb{R}^4} f(k)\frac{\mathbb{1}}{\sqrt{k^2+m^2}} g(k)\,\mathrm{d}k - \mathrm{i}\int_{\mathbb{R}^4} f(k)\widehat{C}(k) g(k)\,\mathrm{d}k,
\end{aligned}
$$

where in the fourth equality we use the anti-symmetry of J, in the fifth the property $\mathrm{J}^2 = -\mathbb{1}$, and in the last the definition of $\widehat{C}(k)$.

Using the fact that χ is the inverse of ϕ, shown in the proof of the proposition in paragraph §4, we have of course

$$\vartriangle(f, g) = h(\chi(f), \chi(g)), \quad f, g \in \mathscr{S}(\mathbb{R}^4; \mathbb{C}^4).$$

In the following we will deal with the form $\vartriangle(\cdot,\cdot)$ because of its easier form and because, differently than the Euclidean two-point function, the density $\frac{1}{\sqrt{k^2+m^2}}$, appearing in the definition (IV.11) of $\vartriangle(\cdot,\cdot)$, defines a *positive definite* multiplication operator. The original Euclidean two point function $\widehat{S}_{\mathcal{E}}$ can be recovered as follows. Through the construction explained in this subsection we can reconstruct from $\vartriangle(\cdot,\cdot)$ the form $h(\cdot,\cdot)$, and from $h(\cdot,\cdot)$ we can recover the antisymmetric form $\omega(\cdot,\cdot)$. Finally, form ω we recover the density $\widehat{C}$ (cf. (IV.7)) and as explained at the end of section 3, the Euclidean two-point function $S_{\mathcal{E}}$ is recovered from $\widehat{C}$.

It is a standard fact[4] that the bilinear form $\vartriangle(\cdot,\cdot)$ defines an Hermitian scalar product on $\mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)$ and therefore a Hilbertian norm which we denote by $\|\cdot\|_{\vartriangle}$.

We let

$$W_{\vartriangle} \stackrel{\mathrm{def}}{=} \mathrm{Closure}_{\|\cdot\|_{\vartriangle}} \left(\mathscr{S}(\mathbb{R}^4; \mathbb{C}^4)\right). \tag{IV.12}$$

Complex Gaussian random field

We now define a complex Gaussian random field[5] on $\mathscr{S}'(\mathbb{R}^4; \mathbb{C}^4)$ with covariance given by the scalar product $(\cdot,\cdot)$ defined above.

We introduce the Gel'fand triple

$$\mathscr{S}(\mathbb{R}^4; \mathbb{C}^4) \hookrightarrow L^2(\mathbb{R}^4; \mathbb{C}^4) \hookrightarrow \mathscr{S}'(\mathbb{R}^4; \mathbb{C}^4).$$

[4] Cf. e.g. [17], see also, e.g. [3], [14].

[5] In the sense for example of [12].

The scalar product $(f,g) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} \sum_{j=1}^4 \overline{f_j(k)} g_j(k)\,\mathrm{d}k$, for $f,g \in L^2(\mathbb{R}^4;\mathbb{C}^4)$, induces a paring $\langle\cdot,\cdot\rangle$: $\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4) \times \mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4) \to \mathbb{C}$.

Let T be a *positive definite*, *selfadjoint*, **bounded** operator on $L^2(\mathbb{R}^4;\mathbb{C}^4)$. We define the following Hermitian, complex valued, bilinear form on $\mathscr{S}(\mathbb{R}^4;\mathbb{C}^4) \times \mathscr{S}(\mathbb{R}^4;\mathbb{C}^4)$

$$\vartriangle_{\mathrm{T}}(\cdot,\cdot) : \mathscr{S}(\mathbb{R}^4;\mathbb{C}^4) \times \mathscr{S}(\mathbb{R}^4;\mathbb{C}^4) \to \mathbb{C}, \qquad \vartriangle_{\mathrm{T}}(f,g) \stackrel{\text{def}}{=} (f,Tg).$$

Now we specialize to the case where T is the operator in the definition of $\vartriangle$ in (IV.11), hence we set $\vartriangle_{\mathrm{T}}=\vartriangle$. Moreover let us set

$$W_{\mathrm{T}} \stackrel{\text{def}}{=} W_{\vartriangle},$$

where $W_{\vartriangle}$ was defined in (IV.12).

§ 4.4 Definition Let ν_{T} be the Borel probability measure on the nuclear space $\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4)$ defined by

$$\int_{\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4)} e^{\mathrm{i}\langle Z,f\rangle + \mathrm{i}\overline{\langle Z,g\rangle}}\,\mathrm{d}\nu_{\mathrm{T}}(Z) = e^{-(g,\mathrm{T}f)}.$$

This relation does in fact define a probability measure because of the Bochner-Minlos theorem generalized to this complex setting (cf. [13, Theorem 1.1, p. 2] for the standard statement of the Bochner-Minlos theorem). We call the probability measure ν_{T} a (**centered**) **complex Gaussian probability measure** and the collection of random variables $\langle\cdot, f\rangle$ for $f \in \mathscr{S}(\mathbb{R}^4;\mathbb{C}^4)$ on the probability space $(\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4), \mathfrak{B}, \nu_{\mathrm{T}})$ a (**standard**) **complex Gaussian random field** (**with zero mean**). As it is customary, we also write $\int_{\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4)} F(Z)\,\mathrm{d}\nu_{\mathrm{T}}(Z)$ as $\mathbb{E}[F(Z)]$ for a given integrable function $F : \mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4) \to \mathbb{C}$.

Remark The above definition is a straight forward generalization of [12, formula (6, 29) and Theorem 6.1]. We have the following formulas for the covariance

$$\mathbb{E}[\overline{\langle Z,f\rangle}\langle Z,g\rangle] = (f,\mathrm{T}g), \quad \mathbb{E}[\langle Z,f\rangle\langle Z,g\rangle] = \mathbb{E}[\overline{\langle Z,f\rangle\langle Z,g\rangle}] = 0.$$

In the proposition below give a simple generalization to our case of [12, Corollary of Theorem 6.4]. Following the notation in [12] we distinguish the space W_{T}, defined above, from $\overline{W}_{\mathrm{T}}$, which we define to be the space (canonically isomorphic to W_{T}) obtained by applying the antilinear map of complex conjugation to the whole space W_{T}.

§ 4.5 Proposition (complex Wiener-Itô-Hida-Segal isomorphism) We have the following isomorphism of complex Hilbert spaces

$$\mathscr{T} : L^2(\mathscr{S}'(\mathbb{R}^4;\mathbb{C}^4), \nu_{\mathrm{T}}) \xrightarrow{\cong} \mathrm{Closure}\left(\bigoplus_{n=0}^{\infty}\bigoplus_{k=0}^{\infty} \sqrt{(n-k)!k!}\; W_{\mathrm{T}}^{\odot(n-k)} \otimes \overline{W}_{\mathrm{T}}^{\odot k}\right),$$

where the $\mathscr{T}$-transform is defined by

$$(\mathscr{T}F)(f) \stackrel{\text{def}}{=} \int_{\mathscr{S}'(\mathbb{R}^d;\mathbb{C}^r)} \overline{F(Z)}\, e^{\mathrm{i}\langle Z,f\rangle + \mathrm{i}\overline{\langle Z,f\rangle}}\,\mathrm{d}\nu_{\mathrm{T}}(Z),\;\; f \in \mathscr{S}(\mathbb{R}^d,\mathbb{C}^r), F \in L^2(\mathscr{S}'(\mathbb{R}^d,\mathbb{C}^r)),$$

and where Closure$(\cdot)$ denotes the closure with respect to the scalar product induced by the scalar product on W_{T} (cf. §**5.1**). We have employed the tensor product symbol $\otimes$, respectively the symmetric tensor product symbol $\odot$, to denote respectively the *Hilbert* tensor product, respectively the *Hilbert* symmetric tensor product.

Proof. We show that this statement follows from [12, Corollary of Theorem 6.4]. Indeed in [12, Corollary of Theorem 6.4] the same statement is proved for $\mathrm{T} = \mathbb{1}$ and $r = 1$, in which case the space W_{T} coincides with $L^2(\mathbb{R}^d;\mathbb{C})$. Now the generalization to $r > 1$ is straight forward. Moreover when T is a general bounded, selfadjoint, positive operator in $L^2(\mathbb{R}^d;\mathbb{C}^r)$ its square root $\sqrt{\mathrm{T}}$ is also a well defined bounded, selfadjoint, positive operator in $L^2(\mathbb{R}^d;\mathbb{C}^r)$. We can now follow the same proof as in [12, Theorem 6.3] replacing the random variables $\langle z,\zeta\rangle$ and $\overline{\langle z,\zeta\rangle}$ by $\langle z,\sqrt{\mathrm{T}}\,\zeta\rangle$ and $\overline{\langle z,\sqrt{\mathrm{T}}\,\zeta\rangle}$. □

5 Euclidean n-point functions for Fermion fields

Jordan-Wigner-Friedrichs-Klauder isomorphism

We give a brief, yet self-contained, review of the isomorphism between Fermionic and Bosonic Fock spaces introduced in [5] (cf. also [16] for physical motivation, and [18]), which we will need below. The main result which we will use from this section is the **corollary** in §**5.5**.

§ 5.1 Fock spaces. Given a separable Hilbert space $\mathscr{H}$, let us denote by $\mathbb{F}_{\otimes}\mathscr{H}$, $\mathbb{F}_{\odot}\mathscr{H}$, and $\mathbb{F}_{\wedge}\mathscr{H}$ the **general Fock space**, the **Bosonic Fock space**, and the **Fermionic Fock space** respectively. For completeness we give the explicit definitions of these spaces. Let $\mathscr{N}$ be a nuclear space. We define

$$\Gamma_{\otimes}\mathscr{N} \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} \mathscr{N}^{\otimes n}, \quad \Gamma_{\odot}\mathscr{N} \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} \mathscr{N}^{\odot n}, \quad \Gamma_{\wedge}\mathscr{N} \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} \mathscr{N}^{\wedge n},$$

where $\mathscr{N}^{\otimes n}$, $\mathscr{N}^{\odot n}$, and $\mathscr{N}^{\wedge n}$ denote the n-th tensor power, the n-th *symmetric* tensor power, and the n-th *antisymmetric* tensor power of $\mathscr{N}$. Let $(\cdot,\cdot)$ be a given scalar product on $\mathscr{N}$ and let $\mathscr{H}$ be the Hilbert space obtained by closing $\mathscr{N}$ under $(\cdot,\cdot)$. We can lift the scalar product $(\cdot,\cdot)$ on $\mathscr{N}$ to a scalar product $(\!(\cdot,\cdot)\!)$ on $\Gamma_{\otimes}\mathscr{N}$ by extending by linearity the relations

$$(\!(f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes f_n)\!) \stackrel{\text{def}}{=} (f_1, g_1)\cdots(f_n, g_n), \quad f_j, g_j \in \mathscr{N},\ j = 1, \dots, n.$$

This scalar product naturally descends to a scalar product on $\Gamma_{\odot}\mathscr{N}$ and on $\Gamma_{\wedge}\mathscr{N}$. We denote the closures of $\Gamma_{\otimes}\mathscr{N}$, $\Gamma_{\odot}\mathscr{N}$, and $\Gamma_{\wedge}\mathscr{N}$ respectively by $\mathbb{F}_{\otimes}\mathscr{H}$, $\mathbb{F}_{\odot}\mathscr{H}$, and $\mathbb{F}_{\wedge}\mathscr{H}$. We call these three Hilbert space respectively the **general Fock space**, the **Bosonic Fock space**, and the **Fermionic Fock space**. Note that $\mathbb{F}_{\odot}\mathscr{H}$, and $\mathbb{F}_{\wedge}\mathscr{H}$ are Hilbert subspaces of $\mathbb{F}_{\otimes}\mathscr{H}$.

§ 5.2 Second quantization. Given a bounded operator T on a Hilbert space $\mathscr{H}$ we denote by ΓT or $\Gamma(T)$ the operator on $\mathbb{F}_{\otimes}\mathscr{H}$ extending by linearity the relations

$$(\Gamma T)(f_1 \otimes \cdots \otimes f_n) = (T f_1) \otimes \cdots \otimes (T f_n), \quad f_1, \cdots f_n \in \mathscr{H}, \quad n \in \mathbb{N}.$$

The operator ΓT restricts naturally to an operator on $\mathbb{F}_{\odot}\mathscr{H}$ and on $\mathbb{F}_{\wedge}\mathscr{H}$.

§ 5.3 Definition: Friedrichs-Klauder functions. Let $H^s((\mathbb{R}^d_x)^n)$ denote the Sobolev-Hilbert space with exponent $s \in \mathbb{R}$. The notation $\mathbb{R}^d_x$ is used to denote the d-dimensional Euclidean space $\mathbb{R}^d$ when we want to distinguish it from the momentum space, also isomorphic to $\mathbb{R}^d$, which we then denote by $\mathbb{R}^d_k$. Take $s < -d$. We call a family of functions $\epsilon^{(n)} \in H^s((\mathbb{R}^d_x)^n)$, $n \in \mathbb{N}$ a family of **Friedrichs-Klauder functions** when each $\epsilon^{(n)}$ satisfies the following properties

(i) the $(\mathbb{R}^d)^n$ Fourier transform $\hat{\epsilon}^{(n)}(k_1, \dots, k_n)$ of $\epsilon^{(n)}(x_1, \dots, x_n)$ satisfies

$$\left(\hat{\epsilon}^{(n)}(k_1, \dots, k_n)\right)^2 \stackrel{a.e.}{=} 1, \quad k_1, \dots, k_n \in \mathbb{R}^d_k;$$

(ii) $\hat{\epsilon}^{(n)}$ (and therefore also $\hat{\epsilon}^{(n)}$) is antisymmetric in the exchange of variables, that is under a permutation $\pi \in \mathfrak{S}_n$, $\pi : \{1, \dots, n\} \mapsto \{\pi(1), \dots, \pi(n)\}$, we require $\hat{\epsilon}^{(n)}$ to satisfy

$$\hat{\epsilon}^{(n)}(k_{\pi(1)}, \dots, k_{\pi(n)}) = \operatorname{sgn}(\pi)\, \hat{\epsilon}^{(n)}(k_1, \dots, k_n),$$

where $\operatorname{sgn}(\pi)$ denotes the sign of the permutation π.

§ 5.4 Proposition: JWFK isomorphism Let $E^0 = E^1 = 1$ be the identity map in $\mathbb{F}_{\otimes}L^2(\mathbb{R}^d_k)$. For $n \geq 2$, let E^n be the map

$$E^n : L^2((\mathbb{R}^d_k)^n) \to L^2((\mathbb{R}^d_k)^n) \quad \psi^{(n)} \mapsto \hat{\epsilon}^{(n)}\psi^{(n)},$$

where the product $\widehat{\epsilon}^{(n)}\psi^{(n)}$ is taken pointwise. We define the map $E : \Gamma_{\otimes} L^2(\mathbb{R}^d) \to \Gamma_{\otimes} L^2(\mathbb{R}^d)$ by

$$E \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} E^n.$$

Then E is *unitary* and descends to a *grading-preserving isometric isomorphism*

$$E : \Gamma_{\wedge} L^2(\mathbb{R}^d_k) \to \Gamma_{\odot} L^2(\mathbb{R}^d_k).$$

PROOF. First note that, since by hypothesis $\widehat{\epsilon}^{(n)}$ is antisymmetric, E^n maps symmetric functions into antisymmetric ones and antisymmetric function into symmetric ones.

We now show that the condition $(\widehat{\epsilon}^{(n)})^2 \stackrel{a.e.}{=} 1$ implies both that E^n is a partial isometry and that it is invertible. This means that E^n is unitary. Indeed

$$\begin{aligned}
\|\widehat{\epsilon}^{(n)}\psi^{(n)}\|^2_{\Gamma_{\otimes} L^2(\mathbb{R}^d_k)} &= \int\limits_{(\mathbb{R}^d_k)^n} |\widehat{\epsilon}^{(n)}(k_1, \dots, k_n)\, \psi^{(n)}(k_1, \dots, k_n)|^2 \, \mathrm{d}k_1 \cdots \mathrm{d}k_n \\
&= \int\limits_{(\mathbb{R}^d_k)^n} \widehat{\epsilon}^{(n)}(k_1, \dots, k_n)^2 \, |\psi^{(n)}(k_1, \dots, k_n)|^2 \, \mathrm{d}k_1 \cdots \mathrm{d}k_n \\
&= \|\psi^{(n)}\|^2_{\Gamma_{\otimes} L^2(\mathbb{R}^d_k}\,,
\end{aligned}$$

where we used the condition (i) in §5.4. This proves that E^n is a partial isometry. To see that it is invertible it is enough to note that

$$E^n(E^n\psi^{(n)}) = (\widehat{\epsilon}^{(n)})^2\psi^{(n)} \stackrel{a.e.}{=} \psi^{(n)},$$

which means that E^n is its own inverse. □

We shall use the following corollary of the previous proposition. We omit the proof which is, mutatis mutandis, the same as the one for the proposition.

§ 5.5 Corollary Consider a *bounded*, *selfadjoint*, *positive* operator T on the *complex* Hilbert space $L^2(\mathbb{R}^d; \mathbb{C}^r)$.

Let $\big(W_{\mathrm{T}}, (\cdot, \cdot)_{\mathrm{T}}\big)$ be the complex Hilbert space obtained by completing $\mathscr{S}(\mathbb{R}^d; \mathbb{C}^r)$ with respect to the Hermitian bilinear form $(\cdot, \cdot)_{\mathrm{T}} : \mathscr{S}(\mathbb{R}^d; \mathbb{C}^r) \times \mathscr{S}(\mathbb{R}^d; \mathbb{C}^r) \to \mathbb{C}$,

$$(f, g)_{\mathrm{T}} \stackrel{\text{def}}{=} (f, \mathrm{T}g), \quad f, g \in \mathscr{S}(\mathbb{R}^d; \mathbb{C}^r),$$

where $(\cdot, \cdot)$ denotes the Hermitian scalar product of $L^2(\mathbb{R}^d; \mathbb{C}^r)$.

Suppose that on $\Gamma_{\otimes}\mathscr{S}(\mathbb{R}^d; \mathbb{C}^d)$ the operators $\Gamma(\mathrm{T})$ and E commute. Then the assertions of the proposition in §5.4 hold also with $\big(W_{\mathrm{T}}, (\cdot, \cdot)_{\mathrm{T}}\big)$ in place of $(L^2(\mathbb{R}^d), (\cdot, \cdot)$. □

Remark A sufficient condition for the hypothesis, of the corollary in §5.5, that $\Gamma(\mathrm{T})$ and E commute when restricted to $\Gamma_{\otimes}\mathscr{S}(\mathbb{R}^d; \mathbb{C}^d)$, is the following. The operator T is a $d \times d$-matrix-valued multiplication operator, that is $\mathrm{T}f(k) = M(k)f(k)$ where $M(k)$, for $k \in \mathbb{R}^d$, is an appropriate matrix which makes T, on the Hilbert space $L^2(\mathbb{R}^d; \mathbb{C}^r)$: bounded, selfadjoint, and positive. This is the situation which we will need below.

We state in passing a useful combinatorial property of the maps $\widehat{\epsilon}^{(n)}$, $n \in \mathbb{N}$ cf. [7, Formula (3.6)]:

§ 5.6 Lemma Given a function $\widehat{\epsilon}(k, k')$ of two variables $k, k' \in \mathbb{R}^d$, antisymmetric in the permutation of the variables, we can construct a function

$$\widehat{\epsilon}^{(n)}(k_1, \dots, k_n) \stackrel{\text{def}}{=} \prod_{1 \le j < \ell \le n} \widehat{\epsilon}(k_j, k_\ell)$$

of n variables which is anti-symmetric in the permutation of any two variables. □

Representation of Euclidean Fermionic n-point functions

The objective in this subsection is to state and prove a result, related to the result in [17], which gives the representation of Fermionic n-point functions in terms of functionals of the chaos expansions of a complex Gaussian random field. This result will be stated and proved in §5.10. We begin with some preliminary lemmas dealing with easy combinatorial properties, the proofs of which are straight forward and therefore omitted.

§ 5.7 Lemma Define in the usual manner the **Pfaffian** of an arbitrary $2n \times 2n$ complex matrix $M = (m_{ij})_{ij=1}^{n2}$, as follows

$$\mathrm{pf}\, M \stackrel{\text{def}}{=} \frac{1}{2^n n!} \sum_{i_1 \dots i_{2n}} \varepsilon_{i_1 \dots i_{2n}} \prod_{\ell=1}^{n} m_{i_{2\ell-1} i_{2\ell}},$$

where $\varepsilon_{i_1 \dots i_{2n}}$ denotes the Levi-Civita symbol. Then, if we denote by $A(M) \stackrel{\text{def}}{=} (M - M^{\mathrm{t}})/2$ the *antisymmetric* part of M, we have

$$\mathrm{pf}(M) = \mathrm{pf}(A(M)). \qquad \square$$

§ 5.8 Lemma Let M be a complex valued n-by-n matrix. Let us denote by bold face letters $\mathbf{v}$, $\mathbf{w}$ vectors in $\mathbb{C}$. and by $\mathbf{v} \cdot M\mathbf{w}$ the bilinear form[6]

$$\mathbf{v} \cdot M\mathbf{w} \stackrel{\text{def}}{=} \sum_{ij=1}^{n} v_i M_{ij} w_j.$$

Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{C}^n$, $k \leq n$. Then we have

$$\begin{aligned} \mathop{\mathrm{pf}}_{1 \leq ij \leq k} [\mathbf{v}_i \cdot M\mathbf{v}_j] &\stackrel{\text{def}}{=} \sum_{i_1 \dots i_k = 1}^{k} \varepsilon_{i_1 \dots i_k} \prod_{\ell=1}^{k} \mathbf{v}_{i_{2\ell-1}} \cdot M\mathbf{v}_{i_{2\ell}} \\ &= \mathop{\mathrm{pf}}_{1 \leq ij \leq k} [\mathbf{v}_i \cdot A(M)\mathbf{v}_j], \end{aligned}$$

where, as above, $A(M) = \frac{M + M^{\mathrm{t}}}{2}$ is the antisymmetric part of the matrix M. $\square$

§ 5.9 Lemma Let $(V, \cdot)$ be a separable Hilbert space. Let $\mathbf{v}_1, \dots, \mathbf{v}_{2p} \in V$. Consider the p-form $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{2p} \in \Omega^{2p}(V) \subset \Gamma_\wedge V$. We can think of $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{2p} \in \Omega^{2p}(V) \subset \Gamma_\wedge V$ as a trace-class operator in $\mathrm{End}\,(\Omega^p(V))$ by considering its action to be

$$\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{2p}(\mathbf{w}_1 \wedge \cdots \wedge \mathbf{w}_p) \stackrel{\text{def}}{=} (\mathbf{v}_1 \wedge \mathbf{v}_2)(\mathbf{w}_1) \wedge \cdots \wedge (\mathbf{v}_{2p-1} \wedge \mathbf{v}_{2p})(\mathbf{w}_p),$$

where $(\mathbf{v}_i \wedge \mathbf{v}_j)(\mathbf{w}) \stackrel{\text{def}}{=} (\mathbf{v}_j \cdot \mathbf{w})\mathbf{v}_i - (\mathbf{v}_i \cdot \mathbf{w})\mathbf{v}_j$.

For $M \in \mathrm{End}(V)$, let $M^{\otimes p} \in \mathrm{End}(\Omega^p(V))$ be its p-th tensor power, $M^{\otimes p} \stackrel{\text{def}}{=} M \otimes \cdots \otimes M$ (p times). Finally, denote by $\mathrm{Tr}_{\mathrm{End}(\Omega^p(V))}(\cdot)$ the trace of a trace-class operator in $\mathrm{End}(\Omega^p(V))$.

Then we have

$$\mathrm{Tr}_{\mathrm{End}(\Omega^p(V))}\left(\left(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{2p}\right) \circ M^{\otimes p}\right) = \mathrm{Tr}_{\mathrm{End}(\Omega^p(V))}\left(\left(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{2p}\right) \circ A(M)^{\otimes p}\right),$$

where $\circ$ denotes the composition of operators and $A(M)$ denotes as before is the antisymmetric part of the operator M. $\square$

[6]Note that, in general, the bilinear form $\mathbf{v} \cdot M\mathbf{w}$ is neither symmetric nor Hermitian. In terms of the Hermitian scalar product $(\mathbf{v}, \mathbf{w}) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \overline{v_i} w_i$, where $\overline{v_i}$ denotes complex conjugation, we have

$$\mathbf{v} \cdot Mw = (\overline{\mathbf{v}}, M\mathbf{w}),$$

where $(\overline{\mathbf{v}})_i \stackrel{\text{def}}{=} \overline{v_i}$, for all $i = 1, \dots, n$.

Remark The functional

$$M \mapsto \mathrm{Tr}_{\mathrm{End}(\Omega^p(V))}\left(\left(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_{2p}\right) \circ M^{\otimes p}\right)$$

can be thought as a generalization of the concept of Pfaffian. Below, when it is clear in what space the trace is taken, we will simply write Tr.

Remark. We shall denote by M^{t} the *transpose* of the operator M. If the Hilbert space is complex this notion is different from the notion of Hilbert adjoint, which we denote by M^*.

The following proposition gives the procedure by which one recovers the Euclidean n-point functions from the complex Gaussian random field defined above.

§ 5.10 Theorem. Let the notation be as above. Then we can write compactly

$$C(\chi(f_1), \chi(g_1); \ldots; \chi(f_n), \chi(g_n)) = \\ = \mathrm{Tr}\left(\left(f_1 \wedge \overline{g}_1 \wedge \cdots \wedge f_n \wedge \overline{g}_n\right) \circ \left(E^{(n)} \otimes E^{(n)}\right) \circ \mathbb{E}\left[\overline{Z^{\odot n}} \otimes Z^{\odot n}\right]\right). \quad \text{(IV.13)}$$

Explicitly we have

$$C(\chi(f_1), \chi(g_1); \ldots; \chi(f_n), \chi(g_n)) = \\ = \int_{(\mathbb{R}^4)^{2n}} (\overline{g}_1 \wedge f_1 \wedge \cdots \wedge \overline{g}_n \wedge f_n)_{j_1\ell_1\ldots j_n\ell_n}(k_1, \ldots, k_n; p_1, \ldots, p_n) \times \\ \epsilon^{(n)}(k_1, \ldots, k_n)\epsilon^{(n)}(p_1, \ldots, p_n) \times \\ \times \mathbb{E}\left[Z_{j_1}(k_1)\overline{Z_{\ell_1}(p_1)} \cdots Z_{j_n}(k_n)\overline{Z_{\ell_n}(p_n)}\right] \mathrm{d}k_1\, \mathrm{d}p_1 \cdots \mathrm{d}k_n\, \mathrm{d}p_n, \quad \text{(IV.14)}$$

where

$$C(\chi(f_1), \chi(g_1); \ldots; \chi(f_n), \chi(g_n)) \stackrel{\mathrm{def}}{=} \sum_{r_1 s_1 \ldots r_n s_n = 1}^{8} \int_{(\mathbb{R}^4)^{2n}} \widehat{C}^{(2n)}_{r_1, s_1, \ldots, r_n, s_n}(k_1, p_1, \ldots, k_n, p_n) \times \\ \times (\chi g)_{r_1}(p_1) \cdots (\chi g)_{r_n}(p_n)(\chi f)_{s_1}(k_1) \cdots (\chi f)_{s_n}(k_n)\, \mathrm{d}p_1\, \mathrm{d}k_1 \cdots \mathrm{d}k_n\, \mathrm{d}p_n, \quad \text{(IV.15)}$$

and where the map χ is defined in section 4.

PROOF. The statement follows from the combinatorial lemmas above together with the decomposition in §**4.5**. The two expressions (IV.13) and (IV.14) follow from each other simply by applying the definitions of the maps $E^{(n)}$ and of the trace Tr. To show that, for example, (IV.13) holds we need to verify the following. First note that (IV.13) holds at the 2-point function level, that is, for $n = 1$. This is indeed true by construction. Then note that, for $n > 1$, it has the right symmetry property, that is is antisymmetric under any odd permutations among the set of functions $\{f_1, \overline{g}_1, \ldots, f_n, \overline{g}_n\}$. This is trivially satisfied because on the right hand side we have the wedge products of these functions. Finally note that, for a fixed n, the right hand side is *not* identically zero. Indeed the expression

$$\left(E^{(n)} \otimes E^{(n)}\right) \circ \mathbb{E}\left[\overline{Z^{\odot n}} \otimes Z^{\odot n}\right] \quad \text{(IV.16)}$$

is antisymmetric, because of the definition of the Friedrichs maps $E^{(n)}$, in the permutation of a pair $(f_j, \overline{g}_j)$ with another pair $(f_\ell, \overline{g}_\ell)$, $j, \ell \in \{1, \ldots, n\}$. Moreover, the expression (IV.16) is neither symmetric nor antisymmetric in the exchange of an f_j with a $\overline{g}_j$ with the same $j \in \{1, \ldots, n\}$. In particular it can be antisymmetrized and still give a *nonzero* result.

Finally, let $(e_j)_{j \in \mathbb{N}_+}$ be a basis for $L^2(\mathbb{R}^d; \mathbb{C}^r)$ composed of elements in $\mathscr{S}(\mathbb{R}^d; \mathbb{C}^r)$. Then one can trivially show, by explicit evaluation of the expectation $\mathbb{E}[\cdots]$, that the relation (IV.13) is satisfied when one chooses $e_1, \ldots, e_n$ for the functions $f_1, \ldots, f_n$ and $\overline{e}_1, \ldots, \overline{e}_n$ for the functions $\overline{g}_1, \ldots, \overline{g}_n$. Hence the proof is complete. □

We note that the wedge products in (IV.13) and (IV.14) in the theorem above are redundant, in the sense that we have the following corollary.

§ 5.11 Corollary. We can rewrite (IV.13) and (IV.14) respectively as

$$C(\chi(f_1), \chi(g_1); \dots ; \chi(f_n), \chi(g_n)) = \\ = \operatorname{Tr}\left(\left(f_1 \otimes \overline{g}_1 \otimes \cdots \otimes f_n \otimes \overline{g}_n\right) \circ \left(E^{(n)} \otimes E^{(n)}\right) \circ \mathbb{E}\left[\Im\left\{\overline{Z} \otimes Z\right\}^{\odot n}\right]\right), \quad \text{(IV.17)}$$

where the imaginary part $\Im\{\overline{Z} \otimes Z\}$ of the matrix $\overline{Z} \otimes Z$ is taken component wise, and as

$$C(\chi(f_1), \chi(g_1); \dots ; \chi(f_n), \chi(g_n)) = \\ = \int_{(\mathbb{R}^4)^{2n}} (\overline{g}_1 \otimes f_1 \otimes \cdots \otimes \overline{g}_n \otimes f_n)_{j_1\ell_1\dots j_n\ell_n}(k_1, \dots, k_n; p_1, \dots, p_n) \times \\ \epsilon^{(n)}(k_1, \dots, k_n)\epsilon^{(n)}(p_1, \dots, p_n) \times \\ \times \mathbb{E}\left[\Im\left\{Z_{j_1}(k_1)\overline{Z_{\ell_1}(p_1)}\right\} \cdots \Im\left\{Z_{j_n}(k_n)\overline{Z_{\ell_n}(p_n)}\right\}\right] \mathrm{d}k_1\, \mathrm{d}p_1 \cdots \mathrm{d}k_n\, \mathrm{d}p_n. \quad \text{(IV.18)}$$

Proof. We first show (IV.18) is indeed the same as (IV.14).

The antisymmetry of the permutation of a k_j with a k_ℓ, $j, \ell \in \{1, \dots, n\}$, $j \neq \ell$, is enforced by the Friedrichs function $\epsilon^{(n)}(k_1, \dots, k_n)$. Similarly, the antisymmetry of the permutation of the variable p_j with the variable p_ℓ, $j, \ell \in \{1, \dots, n\}$, is enforced by the Friedrichs function $\epsilon^{(n)}(p_1, \dots, p_n)$. Finally, the imaginary parts $\Im\{Z_{j_1}(k_1)\overline{Z_{\ell_1}(p_1)}\}$ inside the expectation enforce the antisymmetry under the permutation of a variable k_j with the variable p_j, $j \in \{1, \dots, n\}$. These antisymmetry properties imply also clearly the antisymmetry under the permutation of a variable k_j with a variable p_ℓ, $j, \ell \in \{1, \dots, n\}$, $j \neq \ell$. Hence we conclude that (IV.18) is indeed equivalent to (IV.14).

It now follows from (IV.18) that we can rewrite (IV.13) as follows Indeed, to obtain this last expression from (IV.18), it is enough to note that

$$\mathbb{E}\left[\overline{Z}^{\odot n} \otimes Z^{\odot n}\right] = \mathbb{E}\left[\left(\overline{Z} \otimes Z\right)^{\odot n}\right],$$

which follows from the symmetry property of the complex Gaussian probability measure. Indeed, for example, an expression of the form

$$\mathbb{E}\left[Z_{j_1}(k_1)\overline{Z_{\ell_1}(p_1)}Z_{j_2}(k_2)\overline{Z_{\ell_2}(p_2)}\right]$$

is symmetric under the permutation which exchanges (j_1, k_1) with (j_2, k_2).[7] □

§ 5.12 Concluding remark. The expressions (IV.17) and (IV.18) should clarify the comment made in the Introduction of this chapter, where we stated that the Friedrichs functions constitute a generalization of the notion of real structure to the whole Fock space. Indeed we see that taking the imaginary part (which means we have chosen a real structure which distinguishes between real and imaginary parts) antisymmetrizes at the level of the 2-point function, hence at the *one*-particle level. Whereas the Friedrichs functions $\epsilon^{(n)}$ plays a similar role at the n-particle level.

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V

Relativistic Fermions in 3 + 1 dimensions: complexified Poincaré spin group, and a Bosonic Fock space of Hilbert-Schmidt operators

Abstract

We construct a model for relativistic free Dirac Fermions which is realized canonically in a Bosonic Fock space. The starting point consist in considering the complexified space time and the induced representations of the complexified Poincaré spin group. Hence in our model we are describing both relativistic and Euclidean symmetries at once. Finally the Bosonic Fock space corresponds to a Fock space on a space of Hilbert-Schmidt operators or, which is the same, the 1-particle Hilbert space, on which the Fock space is constructed, is itself the (completed) tensor product of two other infinite dimensional (separable) Hilbert spaces. We conclude by giving a way to recover, from our model, the standard Fermionic Fock space for free relativistic Dirac fields.

Contents

1 Introduction

The motivation for this chapter comes from the desire of introducing probabilistic methods well adapted to the Euclidean Dirac field. The main two problems, when trying to adapt probability theory to the Euclidean Dirac field are

1. The (Euclidean) Schwinger 2-point function does not define a positive definite form, hence one cannot directly construct a Hilbert space of 1-particle states.

2. The state of n Dirac Fermions is antisymmetric (for every $n \in \mathbb{N}_+$. Hence it cannot be described as an n-th moments of a probability measure, because such moments correspond to symmetric quantities.

In the literature there exist methods to circumvent these two problems. We cite [17], [13], [14], [21] and reference therein. We believe that these methods have a common drawback. They solve the two problems mentioned above with a somewhat ad hoc technique, which works but adds very little to the already very robust functional analytic and algebraic formulation given either in terms of Fock spaces (cf. [20]) or in terms of C^* algebras (cf. [4]). Moreover, the approaches in the literature treat the two problems separately.

We give here a construction which solves the two problems mentioned above at the same time, starting from a basic generalization of the standard Dirac theory to complex space-time.

The complexified space time is a fundamental ingredient of Wightman theory. Indeed Wightman functions are considered as holomorphic functions in regions of the complexified spacetime.

The most relevant point for our purposes concerns the rigorous proof of the spin-statistics (*spin and statistics*) theorem in the context of Wightman axioms. There one uses a rotation in the complexified spacetime to prove that half-integer spin fields, if they commute for space separated points, then they are trivial. This theorem needs that the Wightman n-point functions be defined on the complexified spacetime. Moreover the Wightman functions are constructed in such a way that the action of the complexified Lorentz spin group $\mathbf{Spin}(4, \mathbb{C})$ is defined on them. Nevertheless, in Wightman theory, the complexified Lorentz spin group is *not* defined at the level of the quantum fields, it only acts on the analytically continued n-point functions. In the proof of this theorem, one considers the Wightman two-point function. Then one notices that, under a rotation by π in the plane determined by the x-axis and the *imaginary* time axis the two point function for half-integer fields is multiplied by -1. This means that this 2-point function is antisymmetric under this transformation, and similarly for the n-point functions. This is the main step in the proof which then implies the anticommutativity (at space separated points) of Fermionic fields in this axiomatic framework. This anticommutativity is what we recognized above as the second obstruction to a probabilistic description of relativistic Fermions.

We remark that here we shall basically be concerned with *free* Fermionic fields, whereas the content of the spin-statistics theorem applies in principle to any theory satisfying the Wightman axioms.

In fact, the antisymmetry of the Wightman two-point function under the rotation described above, is also the origin to the first obstruction we mentioned above. Indeed, this antisymmetry, under the rotation mentioned above of the Wightman two-point function, makes the Euclidean 2-point Schwinger function, in certain representations (cf. chapter III), correspond to an antisymmetric, Euclidean invariant, bilinear form. In particular, this form, being non-symmetric, cannot be positive definite.

On the other hand, the Wightman functions which correspond to Bosonic fields, are symmetric under the rotation mentioned above. This implies that, in particular, the Bosonic fields commute (for space like points) *and* that the Euclidean Schwinger two-point function is symmetric (and also positive-definite).

In our construction, instead of complexifying the variables of the Wightman functions by analytic continuation, we consider directly wavefunctions (which correspond to 1-particle states) on the complexified spacetime. Hence we define the action of the complexified Lorentz spin group (and in general of the complexified Poincaré spin group $\mathbb{C}^4 \rtimes \mathbf{Spin}(4, \mathbb{C})$) directly on the wavefunctions. This is a more general situation than the one treated by Wightman theory. To obtain the representation of the complexified Poincaré spin group on the wavefunctions we give a parallel construction to the standard Wigner-Mackey analysis of induced representations which is classically applied to the *real* Poincaré spin group.

In **section 2** we discuss, from a general point of view, the structure of the Bosonic Fock space constructed on a tensor product of two Hilbert spaces. We consider this structure as interesting on its own right because of its connection with the theory of Lie groups in infinite dimensions (cf. [16] and also [1, 2, 3, 12]). This fascinating connection between relativistic fields and analysis and probability on infinite dimensional Lie groups will be the subject of future investigation. We also discuss the relations between the Bosonic Fock space and the Fermionic Fock space, showing that the second can be embedded in the first one (proposition §**2.10**). The results described in section 2 are only used in our construction at the very end, in proposition §**5.10**.

In **section** 3 we introducing some notation and conventions regarding the complexified Poincaré spin group and some of its subgroups. In **section** 4 we describe how the theory of induced representations applies to our situation. In proposition §**4.3** we collect the most important points discussed in the section. In **section** 5 we start by explaining the construction a special unitary representation of the complexified Poincaré group with a certain degree of details to show how it corresponds to the standard construction of Dirac representations starting from Wigner representations. The point of arrival of our construction is the unitary representation of the complexified Poincaré group $\mathbb{C}^4 \rtimes \mathbf{Spin}(4, \mathbb{C})$ summarized in proposition §**5.8**.

Since we are representing the complexified Poincaré group, in particular we are representing the Euclidean rotation which is used in the proof of the spin-statistics theorem mentioned above. Moreover, since the representation which we construct is *unitary*, it is realized on a Hilbert space, therefore the Hermitian scalar product on this Hilbert space is an invariant positive definite bilinear form, which is also invariant, therefore symmetric, under the Euclidean rotation described above. By the discussion above regarding the spin-statistics theorem, a fortiori, we need to impose on our system the Bose-Einstein statistics, that is, we need to consider the Hilbert space of the representation as the 1-particle space inside a *Bosonic Fock space*.

The Bosonic Fock space, appropriate for our model, is described in the final part of the section.

An important point in our construction is that the Hilbert space of the representation which we constructed splits canonically into the (completed) tensor product of two Hilbert spaces. This allows us to use the results from section 2 to this case. In that final proposition we describe how one recovers the standard Fermionic Fock space for relativistic Dirac fields starting from the Bosonic model which we have constructed.

2 A Bosonic realization of the Fermionic Fock space

In this section we define a Bosonic Fock space over a Hilbert space of Hilbert-Schmidt operators. We will show that we can embed any Fermionic Fock space in a Bosonic Fock space of this type. The construction is interesting on its own right hence we discuss it in some details. This Bosonic Fock space will arise naturally in the realization of Dirac Fermions which we will give in the later sections and, as a result, will allow us to give a cogent, cohesive description of Dirac Fermions in terms of Bosonic objects.

Notation. We denote by $\mathcal{H}$ a separable Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A *basis* for $\mathcal{H}$ will always mean and *orthonormal* basis.

We denote by $\mathcal{H}^{\otimes n} \stackrel{\text{def}}{=} \mathcal{H} \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} \mathcal{H}$ the algebraic tensor product of n copies of $\mathcal{H}$ seen as a vector space, that is forgetting the Hilbert structure. We equip $\mathcal{H}^{\otimes n}$ with the scalar product induced by $\mathcal{H}$. For example we define on $\mathcal{H}^{\otimes 2}$ the scalar product obtained by extending the bilinear form

$$(f_1 \otimes f_2, g_1 \otimes g_2)_{\mathcal{H}^{\otimes 2}} \stackrel{\text{def}}{=} (f_1, g_1)_{\mathcal{H}} (f_2, g_2)_{\mathcal{H}} ,$$

defined first on monomials $f_1 \otimes f_2, g_1 \otimes g_2 \in \mathcal{H}$. This scalar product makes $\mathcal{H}^{\otimes n}$ into a pre-Hilbert space. We denote by $\mathcal{H}^{\widehat{\otimes}\, n}$ the completion of $\mathcal{H}^{\otimes n}$ with respect to the Hilbert norm associated to the scalar product just defined and call it *Hilbert n*-th tensor power of $\mathcal{H}$.

We denote by $\oplus_{n=0}^{\infty} \mathcal{H}_n$ the algebraic direct sum of a denumerable collection $(\mathcal{H})_{n \in \mathbb{N}}$ of (pre-)Hilbert spaces. If on the space $\oplus_{n=0}^{\infty} \mathcal{H}_n$ we have a notion of scalar product the direct sum is meant to be *orthogonal* with respect to it. In this case we denote by $\widehat{\bigoplus}_{n=0}^{\infty} \mathcal{H}_n$ its completion.

In general, we place a hat over a symbol to denote completion of a pre-Hilbert space with respect to its Hilbert norm.

We denote by $\mathbb{N}$ the set of *non-negative* integers and by $\mathbb{N}_+$ the set of positive integers.

Young symmetrizer, Young diagrams, and Schur functor

For completeness, we introduce in this subsections some standard combinatorial notions which will be necessary in the following subsections.

References. About Young symmetrizer, Young diagrams, Schur functor, etc..., cf. e.g. [15, 7, 22].

§ 2.1 For $k \in \mathbb{N}$, let $\lambda^{(k)} = (\lambda_1, \dots, \lambda_k)$ be a partition, that is a collection of non-negative integers $\lambda_1 \geq \dots \geq \lambda_k$, $\lambda_1 + \dots + \lambda_k = k$. A partition $\lambda^{(k)}$ can be visualized via its associated Young diagram, that is a collection of k boxes, arranged in the following "top-left corner" shape:

$$\begin{array}{|c|c|c|cc|c|c|}
\hline
1 & 2 & 3 & \cdots & & \lambda_1-1 & \lambda_1 \\
\hline
\lambda_1+1 & \lambda_1+2 & \lambda_1+3 & \cdots & & \lambda_2 & \\
\hline
\vdots & \vdots & \vdots & & & & \\
\hline
k-1 & & & & & & \\
\hline
k & & & & & & \\
\hline
\end{array}$$

where the boxes are numbered only to show how the boxes are arranged.

We will use the term Young diagram and partition interchangeably.

We denote by $\mathbb{S}_{\lambda^{(k)}}$ the Schur functor as a functor in the category of vector spaces. Explicitly we define $\mathbb{S}_{\lambda^{(k)}}\mathcal{H}$ to be the vector space obtained as follows. Consider a tensor F in the vector space $H^{\otimes k}$. Fix an orthogonal basis for $\mathcal{H}$, F has components[1] $F\big(j_1, \dots, j_k\big)$. Now we can picture the k indices $j_1, \dots, j_n$ as partitioned according to $\lambda^{(k)}$, that is

$$\underbrace{j_1, \dots, j_{\lambda_1}}_{\lambda_1}, \dots, \underbrace{j_{\lambda_1+\dots+\lambda_{k-1}+1}, \dots, j_k}_{\lambda_k}$$

and we can arrange these indices in the shape of the Young diagram associated to $\lambda^{(k)}$ as for example in the picture below.

$$\begin{array}{|c|c|c|cc|c|c|}
\hline
j_1 & j_2 & j_3 & \cdots & & j_{\lambda_1-1} & j_{\lambda_1} \\
\hline
j_{\lambda_1+1} & j_{\lambda_1+2} & j_{\lambda_1+3} & \cdots & & j_{\lambda_2} & \\
\hline
\vdots & \vdots & \vdots & & & & \\
\hline
j_{k-1} & & & & & & \\
\hline
j_k & & & & & & \\
\hline
\end{array}$$

Now denote by $\mathbf{c}_{\lambda^{(k)}}$ the **Young symmetrizer** which acts by symmetrizing the indices along the rows of the Young diagram and antisymmetrizing the indices along the columns, explicitly we define

$$\mathbf{c}_{\lambda^{(k)}} F\big(j_1, \dots, j_k\big) \stackrel{\text{def}}{=} \frac{1}{k!} \sum_{p \in H(\lambda^{(k)}), q \in V(\lambda^{(k)})} \epsilon_q \cdot (\pi_{pq} F)\big(j_1, \dots, j_k\big),$$

where $H(\lambda)$ denotes the set of permutations of k letters which permute only numbers in the same rows of λ, analogously $V(\lambda)$ denotes the set of permutations along the columns, ϵ_q is the sign of the permutation q, and π_{pq} denotes the standard representation of the composed permutation pq (where pq denotes the composition of the permutations p and q) on the space of tensors (that is permutations are represented by the operation of permuting the indices).

[1] We will denote the indices of a tensor F with "parenthesis notation" meaning that we write $F(j_1, \dots, j_k)$ in place of the more common $F_{j_1,\dots,j_k}$

§ 2.2 Definition: Schur functor. Given a Hilbert space $\mathcal{H}$, we let $\mathbb{S}_{\lambda^{(k)}}\mathcal{H}$ be the image of $\mathcal{H}^{\otimes n}$ under $\mathbf{c}_{\lambda^{(k)}}$. It is easy to see that $\mathbb{S}_{\lambda^{(k)}}$ can be made into an (endo)functor in the category of vector spaces which is usually called **Schur functor**.

We let $\widehat{\mathbb{S}}_{\lambda^{(k)}}\mathcal{H}$ be the closure of $\mathbb{S}_{\lambda^{(k)}}\mathcal{H}$ in the norm of $\mathcal{H}^{\hat{\otimes}\, k}$. The space $\widehat{\mathbb{S}}_{\lambda^{(k)}}\mathcal{H}$, equipped with the scalar product induced by $\mathcal{H}^{\hat{\otimes}\, n}$, is a Hilbert subspace of $\mathcal{H}^{\hat{\otimes}\, n}$.

§ 2.3 Remark. The vector space $\mathbb{S}_{\lambda^{(k)}}\mathcal{H}$ and the Hilbert space $\widehat{\mathbb{S}}_{\lambda^{(k)}}\mathcal{H}$ are well defined in the sense that they do *not* depend on the basis chosen for its construction.

PROOF. The construction of $\mathbb{S}_\lambda\mathcal{H}$ we have given used a choice of basis of $\mathcal{H}$ and the expression of a tensor $F \in \mathcal{H}^{\otimes n}$ in components. This is unnecessary[2]. The symmetric group $\mathfrak{S}_n$ acts naturally in $\mathcal{H}^{\otimes n}$ without any need of a choice of basis. Indeed, we have the representation $\pi\mathfrak{S}_n \to \mathrm{End}(\mathcal{H}^{\otimes n})$ given, on monomials $v_1 \otimes \cdots \otimes v_n \in \mathcal{H}^{\otimes n}$, by

$$\pi_\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad \sigma \in \mathfrak{S}_n.$$

Using this natural action of $\mathfrak{S}_n$ on $\mathcal{H}^{\otimes n}$ we can define the Schur functor $\mathbb{S}_\lambda$ as before. □

§ 2.4 Definitions: symmetric and antisymmetric powers. In case of the partition $\lambda^{(k)} = (k, 0, \dots, 0)$ we denote $\mathbb{S}_{\lambda^{(k)}}\mathcal{H}$ also by the shorthand $\odot^k\mathcal{H}$. Similarly for $\lambda^{(k)} = (1, \dots, 1)$ we employ the notation $\wedge^k\mathcal{H}$. When we complete these spaces with respect to the Hilbert norm of $\mathcal{H}^{\hat{\otimes}\, n}$ we write, respectively, $\mathcal{H}^{\hat{\odot}\, n}$ and $\hat{\wedge}^n\,\mathcal{H}$.

Note that we use the symbol $\wedge^n\mathcal{H}$ instead of a symbol like $\mathcal{H}^{\wedge n}$ which would be more consistent with the notations $\mathcal{H}^{\otimes n}$ and $\mathcal{H}^{\odot n}$. Through this difference in the notation we want express the fact that the tensor product and the symmetric tensor product *of vector spaces* are associative whereas the antisymmetric tensor product is *not*, that is

$$(\mathcal{H}^{\otimes n}) \otimes (\mathcal{H}^{\otimes m}) = \mathcal{H}^{\otimes(n+m)}, \quad (\mathcal{H}^{\odot n}) \otimes (\mathcal{H}^{\odot m}) = \mathcal{H}^{\odot(n+m)},$$

but

$$(\wedge^n\mathcal{H}) \wedge (\wedge^m H) \neq \wedge^{(n+m)}\mathcal{H}.$$

Bosonic Fock space and Fermions

§ 2.5 Definition: Bosonic Fock space (or symmetric Fock space). Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by $\Gamma_\odot\mathcal{H}$ the Hilbert space

$$\Gamma_\odot\mathcal{H} \stackrel{\text{def}}{=} \widehat{\bigoplus_{k=0}^{\infty}} \mathcal{H}^{\hat{\odot}\, k},$$

where $\mathcal{H}^{\hat{\odot}\, 0} \stackrel{\text{def}}{=} \mathbb{K}$ and the direct sum is an orthogonal direct sum in the sense of Hilbert spaces.

§ 2.6 Remark. Consider to separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. Note that the space $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ is naturally isomorphic with the space of Hilbert-Schmidt operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

§ 2.7 Definition. Taking into account the previous remark we call the Bosonic Fock space $\Gamma_\odot\left(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2\right)$ the **Bosonic Fock space of Hilbert-Schmidt operators**.

We now arrive at the main point in this section.

§ 2.8 Theorem. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two separable Hilbert spaces on the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ be their Hilbert tensor product. Then the Bosonic Fock space $\Gamma_\odot(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ over $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ decomposes into the following (orthogonal) direct sum*

$$\Gamma_\odot\left(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2\right) = \widehat{\bigoplus_{k=0}^{\infty}} \bigoplus_{\lambda^{(k)}} \left(\widehat{\mathbb{S}}_{\lambda^{(k)}}\mathcal{H}_1\right) \hat{\otimes} \left(\widehat{\mathbb{S}}_{\lambda^{(k)}}\mathcal{H}_2\right), \tag{V.1}$$

where for $k = 0$, $\lambda^{(0)} \stackrel{\text{def}}{=} \emptyset$ and $\mathbb{S}_\emptyset\mathcal{H}_j \stackrel{\text{def}}{=} \mathbb{K}$ $j = 1, 2$.

[2] We thank our friend W. Stern for pointing this out to us.

PROOF. We specialize the proof to the case $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. The general situation is proved similarly. Consider the definition

$$\Gamma_{\odot}\left(\mathcal{H}\,\widehat{\otimes}\,\mathcal{H}\right) = \bigoplus_{k=0}^{\infty} \odot^k(\mathcal{H}\,\widehat{\otimes}\,\mathcal{H}).$$

Fix a basis for $\mathcal{H}$ and thus a basis for $(\mathcal{H}\,\widehat{\otimes}\,\mathcal{H})^{\widehat{\odot}\,k}$. In this basis, an element $F^{(k)} \in (\mathcal{H}\,\widehat{\otimes}\,\mathcal{H})^{\widehat{\odot}\,k}$ will have components $F^{(k)}\left(j_1, \ell_1, \dots, j_k, \ell_k\right)$ and will be symmetric under any exchange of any pair j_r, ℓ_r with any other pair j_s, ℓ_s. Of course it does not have any symmetry property under, for example, the exchange of just a given j_s with another j_r. Given that F is symmetric under the exchange of the pairs if it so happens that F is also symmetric in the exchange of j_r with j_s then it has to be symmetric also in the exchange of ℓ_r with ℓ_s. Similarly if F is antisymmetric in the exchange of j_r with j_s then it has to be antisymmetric also in the exchange of ℓ_r with ℓ_s. Hence we can apply the Young symmetrizer $\mathbf{c}_{\lambda^{(k)}}$ to F seen just as a function of the j's. Because of what was said so far, the result of applying this Young symmetrizer to F is that also the ℓ's are forced to have the same symmetry. Moreover for any fixed pair j_r, ℓ_r the tensor F will be neither symmetric or antisymmetric. More precisely, the image under $\mathbf{c}_{\lambda^{(k)}}$ will consist of a tensor F' in the variables $j_1, \dots, j_k, \ell_1, \dots, \ell_k$ such that it has the specified symmetry in the permutation of the j's and separately the same symmetry under the permutation of the ℓ's, and will have no specified symmetry under the exchange of the j's with the k's. Said differently F' is an element of $\left(\mathbb{S}_{\lambda^{(k)}}\mathcal{H}\right) \otimes \left(\mathbb{S}_{\lambda^{(k)}}\mathcal{H}\right)$. Hence, we obtain the decomposition (and henceforth the statement of the theorem)

$$\odot^k(\mathcal{H}\,\widehat{\otimes}\,\mathcal{H}) = \bigoplus_{\lambda^{(k)}} \left(\mathbb{S}_{\lambda^{(k)}}\mathcal{H}\right) \otimes \left(\mathbb{S}_{\lambda^{(k)}}\mathcal{H}\right),$$

as soon as we prove orthogonality.

To show orthogonality fix a basis of $\mathcal{H}$. Pick elements $F \in \left(\mathbb{S}_{\lambda^{(k)}}\mathcal{H}\right) \otimes \left(\mathbb{S}_{\lambda^{(k)}}\mathcal{H}\right)$, $F' \in \left(\mathbb{S}_{\rho^{(k')}}\mathcal{H}\right) \otimes \left(\mathbb{S}_{\rho^{(k')}}\mathcal{H}\right)$ with $\lambda^{(k)} \neq \rho^{(k')}$. We have to show that F' is orthogonal to F. Note that we can restrict to $k = k'$ because for $k \neq k'$ we already have orthogonality by the orthogonal direct sum in the definition of the Bosonic Fock space.

By hypothesis F and F' have different transformation properties under the permutations of the indices of their components in the fixed basis. Moreover the scalar product in $\Gamma_{\odot}\left(\mathcal{H}\,\widehat{\otimes}\,\mathcal{H}\right)$ is invariant under the permutation of indices. This means that the action of permutation of the indices is unitary. And since by hypothesis F and F' have different transformation properties under the permutations of the indices we can find a unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that F and F' are both eigenvectors but with different eigenvalues. This implies that they are orthogonal. □

Fermionic part

§ 2.9 Definition: Fermionic Fock space (or antisymmetric Fock space). Let $\mathcal{H}$ be a Hilbert space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We denote by $\Gamma_{\wedge}\mathcal{H}$ the Hilbert space

$$\Gamma_{\wedge}\mathcal{H} \stackrel{\text{def}}{=} \widehat{\bigoplus_{k=0}^{\infty}} \hat{\wedge}^k\mathcal{H},$$

where $\hat{\wedge}^0\mathcal{H} \stackrel{\text{def}}{=} \mathbb{K}$ and the direct sum is an orthogonal direct sum in the sense of Hilbert spaces.

We want to show that how to embed the Fermionic Fock space $\Gamma_{\wedge}\mathcal{H}$ into the space $\Gamma_{\odot}(\mathcal{H}\,\widehat{\otimes}\,\mathcal{H})$. We can rewrite the decomposition (V.1) as follows

$$\Gamma_{\odot}\left(\mathcal{H}\,\widehat{\otimes}\,\mathcal{H}\right) = \bigoplus_{k=0}^{\infty} \left(\wedge^k\mathcal{H}\right) \otimes \left(\wedge^k\mathcal{H}\right) \;\oplus\; \bigoplus_{k=2}^{\infty} \bigoplus_{\lambda^{(k)}\neq\wedge^k} \left(\mathbb{S}_{\lambda^{(k)}}\mathcal{H}\right) \otimes \left(\mathbb{S}_{\lambda^{(k)}}\mathcal{H}\right), \tag{V.2}$$

where we have separated the original sum into two sums, one where we consider only completely anti-symmetric Young diagrams (cf. §2.4) and the second where all the other diagrams are considered. By $\lambda^{(k)} \neq \wedge^k$ in (V.2) we mean that $\lambda^{(k)}$ should not be equal to a completely antisymmetric Young diagram $\underbrace{(1, \dots, 1)}_{k \text{ times}}$.

For convenience we write

$$\mathbb{\Gamma}_{\odot}\left(\mathcal{H} \,\hat{\otimes}\, \mathcal{H}\right) = \mathcal{W}_{\wedge} \oplus \mathcal{W}_{\sharp}\,,$$

where

$$\mathcal{W}_{\wedge} \stackrel{\text{def}}{=} \bigoplus_{k=0}^{\infty} \left(\wedge^k \mathcal{H}\right) \otimes \left(\wedge^k \mathcal{H}\right), \quad \mathcal{W}_{\sharp} \stackrel{\text{def}}{=} \bigoplus_{k=2}^{\infty} \bigoplus_{\lambda^{(k)} \neq \wedge^k} \left(\mathbb{S}_{\lambda^{(k)}} \mathcal{H}\right) \otimes \left(\mathbb{S}_{\lambda^{(k)}} \mathcal{H}\right). \tag{V.3}$$

We shall call $\mathcal{W}_{\wedge}$ call the Fermionic part of the Bosonic Fock space $\mathbb{\Gamma}_{\odot}(\mathcal{H} \,\hat{\otimes}\, \mathcal{H})$.

We now embed the Fermionic Fock space $\mathbb{\Gamma}_{\wedge}\mathcal{H}$ inside the Fermionic part $\mathcal{W}_{\wedge}$.

For every $k \in \mathbb{N}$ pick a vector $v^{(k)} \in \wedge^k \mathcal{H}$. For example, we can fix an orthonormal basis e_j, $j = \mathbb{N}_+$, of $\mathcal{H}$, and define

$$v^{(0)} = 1, \quad v^{(k)} = e_1 \wedge \cdots \wedge e_k\,,\ k \in \mathbb{N}_+.$$

Denote by P_k the orthogonal projection onto the one dimensional span of the vector $v^{(k)}$.

We can now define the operator $P \in \operatorname{End}\left(\mathbb{\Gamma}_{\odot}\left(\mathcal{H} \,\hat{\otimes}\, \mathcal{H}\right)\right)$ by

$$P \stackrel{\text{def}}{=} \bigoplus_{k=0}^{\infty} P_k \otimes \mathbb{I}, \tag{V.4}$$

where $\mathbb{I}$ denotes the identity operator in $\mathcal{H}$, and where we extend $P_k \otimes \mathbb{I}$, $k \in \mathbb{N}$, to be operators from $\mathbb{\Gamma}_{\odot}\left(\mathcal{H} \,\hat{\otimes}\, \mathcal{H}\right)$ to itself by letting $P_k \mathbb{S}_{\lambda^{(k)}}(\mathcal{H}) = 0$, for all $\lambda^{(k)} \neq \wedge^k$.

By definition of the operator P, we see that the image of P is the Fermionic Fock space embedded in $\mathbb{\Gamma}_{\odot}\left(\mathcal{H} \,\hat{\otimes}\, \mathcal{H}\right)$. We have thus proved the following.

§ 2.10 Proposition.

$$P\mathbb{\Gamma}_{\odot}\left(\mathcal{H} \,\hat{\otimes}\, \mathcal{H}\right) \cong \mathbb{\Gamma}_{\wedge}\mathcal{H},$$

where $\cong$ denotes a canonical isomorphism of Hilbert spaces.

§ 2.11 Remark. The isomorphism $P\mathbb{\Gamma}_{\odot}\left(\mathcal{H} \,\hat{\otimes}\, \mathcal{H}\right) \cong \mathbb{\Gamma}_{\wedge}\mathcal{H}$ is canonical in the sense that it does *not* depend on the basis of $\mathcal{H}$. This does *not* mean that we *canonically embed* the Fermionic Fock space $\mathbb{\Gamma}_{\wedge}\mathcal{H}$ into $\mathbb{\Gamma}_{\odot}\left(\mathcal{H} \,\hat{\otimes}\, \mathcal{H}\right) \cong \mathbb{\Gamma}_{\wedge}\mathcal{H}$. This embedding depends on the arbitrary choice of the operator P.

§ 2.12 Discussion. The point of the last proposition is to show that indeed we can use the Bosonic space $\mathbb{\Gamma}_{\odot}(\mathcal{H} \,\hat{\otimes}\, \mathcal{H})$ as a replacement of the usual Fermionic Fock space. In particular, all the *Fermionic* observables can be defined in this *Bosonic* Fock space. When we want to compute physical quantities all it remains to do is to project onto the Fermionic Fock space embedded in our Bosonic Fock space.

3 The universal cover of the complexified Poincaré group and its subgroups

Definition of the complexified Poincaré group: ISpin$(4, \mathbb{C})$

§ 3.1 Let $\mathbb{C}\ell(4)$ be the complex Clifford algebra over $\mathbb{C}^4$. If we denote by $\mathcal{T}\mathbb{C}^4 \stackrel{\text{def}}{=} \oplus_{n \in \mathbb{N}} (\mathbb{C}^4)^{\otimes n}$ the full tensor algebra over $\mathbb{C}^4$, then

$$\mathbb{C}\ell(4) \stackrel{\text{def}}{=} \mathcal{T}\mathbb{C}^4 / \mathcal{I},$$

where $\mathcal{I}$ is the ideal in $\mathcal{T}$ generated by elements of the form $v \otimes v - (v \cdot v)\mathbb{1}$, for $v \in \mathbb{C}^4$, where $v \cdot v \stackrel{\text{def}}{=} \operatorname{Tr}(v \otimes v) \in \mathbb{C}$ denotes the extension to $\mathbb{C}^4$ of the standard scalar product in $\mathbb{R}^4$. We remark that, since on $\mathbb{C}^n$, $n \in \mathbb{N}$, all *non degenerate* forms are equivalent, we could have started with any ideal $\mathcal{I}_Q$ generated by elements of the form $v \otimes v - Q(v, v)\,\mathbb{1}$, $v \in \mathbb{C}^4$, for any *non degenerate* form Q, and we would have arrived at a complex Clifford algebra naturally isomorphic to $\mathbb{C}\ell(4)$. This point is relevant, because we are interested in two quadratic forms in $\mathbb{R}^4$, the standard one and the Minkowski one. Because of the previous remark, on $\mathbb{C}^4$ the distinction disappears hence we are in a sense treating both the Euclidean and Minkowski case at once.[3]

We can naturally identify $\mathbb{C}^4$ with its copy in the full tensor algebra $\mathcal{T}\mathbb{C}^4$. This identification is canonical in that does not depend on the basis. Under this identification we have that $\mathbb{C}^4$ is canonically considered as a subset of the complex Clifford algebra $\mathbb{C}\ell(4)$. The algebra structure of the Clifford algebra $\mathbb{C}\ell(4)$ is induced by that on $\mathcal{T}\mathbb{C}^4$. Let us denote just by juxtaposition the product on $\mathbb{C}\ell(4)$. Finally, let us extend the dot product $v \cdot w$ from $\mathbb{C}^4$ to the whole tensor algebra $\mathcal{T}\mathbb{C}^4$ and hence also to the subalgebra $\mathbb{C}\ell(4)$. For example, we have that $(v_1 v_2) \cdot (w_1 w_1) = (v_1 \cdot w_1)(v_2 \cdot w_2)$, with $v_1, v_2, w_1, w_2 \in \mathbb{C}^4$.

Let $\mathbf{Spin}(4,\mathbb{C}) \subset \mathbb{C}\ell(4)$ be the group, under the multiplication induced from the one of $\mathbb{C}\ell(4)$, defined as follows

$$\mathbf{Spin}(4,\mathbb{C}) \stackrel{\text{def}}{=} \left\{ \mathbb{1} v_1 \cdots v_r \in \mathbb{C}\ell(4) : v_j \cdot v_j = 1,\ v_j \in \mathbb{C}^4,\ j = 1, \dots, r,\ r \in \mathbb{N} \right\}.$$

The group $\mathbf{Spin}(4,\mathbb{C})$ has both the structure of an *analytic group* and of a (real) *Lie group*. When we want to stress that we are considering $\mathbf{Spin}(4,\mathbb{C})$ as a *real* Lie group we shall employ the notation $\mathbf{Spin}(4,\mathbb{C})_{\mathbb{R}}$.

The group $\mathbf{Spin}(4,\mathbb{C})$ is *not* simple, but just semisimple. In fact we have the following isomorphisms

$$\mathbf{Spin}(4,\mathbb{C}) \cong \mathbf{Spin}(3,\mathbb{C}) \times \mathbf{Spin}(3,\mathbb{C}), \quad \mathbf{Spin}(3,\mathbb{C}) \cong \mathbf{SL}(2,\mathbb{C}). \tag{V.5}$$

The reason we are interested in $\mathbf{Spin}(4,\mathbb{C})$ is because it is isomorphic to the complexification of the (real) Lie group $\mathbf{SL}(2,\mathbb{C})_{\mathbb{R}}$, where we denote by $\mathbf{SL}(2,\mathbb{C})_{\mathbb{R}}$ the group $\mathbf{SL}(2,\mathbb{C})$ equipped with the (real) Lie group structure to distinguish it from $\mathbf{SL}(2,\mathbb{C})$ equipped with the analytic group structure.

As remarked above we can canonically embed $\mathbb{C}^4$ in $\mathbb{C}\ell(4)$. Employing this embedding, we define the following canonical action λ of $C\ell(4)$ on $\mathbb{C}^4$:

$$\lambda(s)v \stackrel{\text{def}}{=} svs^{-1}, \quad v \in \mathbb{C}^4, \quad s \in \mathbf{Spin}(4,\mathbb{C}). \tag{V.6}$$

It is a standard result (cf. e.g. [19, Chapter 12, (1.43) p. 251][4]) that the action λ induces the covering map

$$\mathbf{Spin}(4,\mathbb{C}) \to \mathbf{SO}(4,\mathbb{C}).$$

Given the action λ of $\mathbf{Spin}(4,\mathbb{C}) \cong \mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})$ on $\mathbb{C}^4$ defined above, we define[5]

$$\mathbf{ISpin}(4,\mathbb{C}) \stackrel{\text{def}}{=} \mathbb{C}^4 \rtimes_{\lambda} \mathbf{Spin}(4,\mathbb{C}),$$

where $\rtimes_{\lambda}$ denotes the semidirect product with respect to the mentioned action λ.

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[3] A remark is in order. Here we shall treat *unitary* representations of the double cover of the Poincaré group. In particular we shall represent the translations by unitary phases. This will force us to introduce a *real structure* on $\mathbb{C}^4$ and therefore identify $\mathbb{C}^4$, via this real structure, with $\mathbb{R}^8$. We shall discuss this point further in §**4.1**.

[4] Let us make a remark between the conventions we employ here and those in [19]. In [19, Chapter 12, (1.40) p. 250] the superscript $\sharp$ denotes the conjugation on the real Clifford algebra $C\ell(n, Q)$ (notation as in [19]) obtained by composing the main anti-automorphism with the main involution. For $s \in \mathbf{Spin}(n,\mathbb{C})$, $n \in \mathbb{N}$, we have that $s^{-1} = s^{\sharp}$. Hence our definition of λ coincides with the definition of what in [19, Chapter 12, (1.43) p. 251] is denoted by θ.

[5] The notation **ISpin** is not standard in the literature. The symbol **ISpin** is meant as shorthand for *inhomogeneous Spin group* In employing this terminology we mimic a similar convention for the Euclidean group in n dimensions which is sometime denoted **ISO** for “inhomogeneous **SO** group”.

Little group and its embedding

§ 3.2 We call **little group** the subgroup of $\mathbf{Spin}(4,\mathbb{C})$ which leaves invariant, under the action of $\mathbf{Spin}(4,\mathbb{C})$ on $\mathbb{C}^4$, any vector of the form

$$\mathring{v} = \begin{pmatrix} \mu \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mu \in \mathbb{C}. \tag{V.7}$$

This choice generalizes to the complexified case the usual choice in the Wigner representations of the Poincaré group. We denote the little group by $\mathbf{W}$ (after Wigner), that is we let

$$\mathbf{W} \stackrel{\text{def}}{=} \left\{ s \in \mathbf{Spin}(4,\mathbb{C}) : s\mathring{v}s^{-1} = \mathring{v} \right\}. \tag{V.8}$$

For future use, it is convenient to define the Lie subgroup $\widetilde{\mathbf{L}}$ of $\mathbf{Spin}(4,\mathbb{C})_{\mathbb{R}}$ which double covers the Lie group $\mathbf{L}$ of *(real) Lorentz transformations*. In symbol, we define

$$\widetilde{\mathbf{L}} \stackrel{\text{def}}{=} \left\{ s \in \mathbf{Spin}(4,\mathbb{C}) : \gamma_0 s^* \gamma_0 = s^{-1} \right\},$$

where γ_0 is the image of the element $(1,0,0,0) \in \mathbb{C}^4$ under the canonical embedding $\mathbb{C}^4 \hookrightarrow \mathbb{C}\ell(4)$. Moreover, s^* denotes the complex-conjugate–transpose of an element $s \in \mathbf{Spin}(4,\mathbb{C})$ seen as a complex matrix.

§ 3.3 Weyl basis. Up to now every step has been natural, in the sense that we are not making any explicit choice. It is convenient at this point to choose a basis for $\mathbb{C}\ell(4)$. Let γ_μ^{Weyl}, $\mu = 0,1,2,3$, be the Weyl (or chiral) representation of Dirac's γ-matrices:

$$\gamma_0^{Weyl} = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \quad \gamma_j^{Weyl} = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \; j = 1,2,3,$$

where $\mathbb{I}_2$ is the unit 2-by-2 matrix, $\sigma_1 = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are the standard Pauli matrices. We call the basis of $\mathbb{C}\ell(4,\mathbb{C})$ obtained using these γ_μ^{Weyl}, $\mu = 0,1,2,3$, as generators, the **Weyl basis** of $\mathbb{C}\ell(4,\mathbb{C})$.

An element $s \in \mathbf{Spin}(4,\mathbb{C})$ can be written

$$s = \exp\Big\{ \sum_{0 \le \mu < \nu \le 3} w_{\mu\nu} \tfrac{1}{2} \gamma_\mu^{Weyl} \gamma_\nu^{Weyl} \Big\}, \quad w_{\mu\nu} \in \mathbb{C},\; 0 \le \mu < \nu \le 3.$$

By explicitly evaluating the generators $\frac{1}{2}\gamma_\mu\gamma_\nu$ of the Lie algebra $\mathfrak{so}(4,\mathbb{C})$ of $\mathbf{Spin}(4,\mathbb{C})$ one explicitly sees that any element $s \in \mathbf{Spin}(4,\mathbb{C})$, when written in this basis, is of the form

$$s = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

for some $A, B \in \mathbf{SL}(2,\mathbb{C})$, that is, in Weyl basis we have the identification

$$\mathbf{Spin}(4,\mathbb{C}) \stackrel{\text{Weyl basis}}{=} \mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C}).$$

This, in particular, proves the first isomorphism in (V.5).

Now consider the canonical embedding $\mathbb{C}^4 \hookrightarrow \mathbb{C}\ell(4,\mathbb{C})$ discussed above. Having now chosen a basis for $\mathbb{C}\ell(4,\mathbb{C})$ we denote this embedding when specialized to this choice of basis by γ^{Weyl}. Hence we have

$$\gamma^{Weyl} : \mathbb{C}^4 \hookrightarrow \mathbb{C}\ell(4,\mathbb{C}), \quad \gamma^{Weyl} : v \mapsto \gamma^{\text{Weyl}}(v) \stackrel{\text{def}}{=} \sum_{\mu=0}^{3} v_\mu \gamma_\mu^{Weyl},$$

where (v_μ) denote the components of the vector $v \in \mathbb{C}^4$. If we consider $\mathring{v}$ as given in §**3.2** (V.7), we have

$$\gamma^{Weyl}(\mathring{v}) = m\gamma_0^{Weyl}.$$

Employing this realization of $\mathring{v}$, a straightforward computation shows that, in this basis, the little group $\mathbf{W}$ is given by

$$\mathbf{W} \overset{\textit{Weyl basis}}{=} \mathbf{SL}(2,\mathbb{C})^{\mathrm{diag}} \overset{\mathrm{def}}{=} \big\{(A,B) \in \mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C}) : A = B\big\},$$

where $\mathbf{SL}(2,\mathbb{C})^{\mathrm{diag}}$ denotes the *diagonal subgroup* of the Cartesian product $\mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})$ (cf. next paragraph §3.4). Similarly we have

$$\widetilde{\mathbf{L}} \overset{\textit{Weyl basis}}{=} \big\{(A, A^{*-1}) \in \mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})\big\} \tag{V.9}$$

where A^{*-1} denotes the complex-conjugate–transpose–inverse of the matrix $A \in \mathbf{SL}(2,\mathbb{C})$, that is A^{*-1} is the image of A under the *Cartan involution* Θ (defined in the following paragraph). Finally, note that in Weyl basis the action λ of $\mathbf{Spin}(4,\mathbb{C})$ on $\mathbb{C}^4$ is explicitly given by

$$\lambda(s)v \overset{\textit{Weyl basis}}{=} \begin{pmatrix} 0 & A(v_0\,\mathbb{I}_2 + \sum_{j=1}^{3} v_j\sigma_j)B^{-1} \\ A(v_0\,\mathbb{I}_2 - \sum_{j=1}^{3} v_j\sigma_j)B^{-1} & 0 \end{pmatrix}, \tag{V.10}$$

where $s \in \mathbf{Spin}(4,\mathbb{C})$ is identified with $(A,B) \in \mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})$.

We have seen that $\mathbf{Spin}(4,\mathbb{C})$ decomposes into the Cartesian product $\mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})$ and that the *little group* can be identified with the diagonal subgroup of $\mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})$. We analyze this setting from from a more abstract perspective, with the hope of being clearer in the presentation.

§ 3.4 In general, for a topological group $\mathbf{G}$, we can consider the symmetric pair $(\mathbf{G} \times \mathbf{G}, \mathbf{G}^{\mathrm{diag}})$, where $\mathbf{G}^{\mathrm{diag}}$ denotes the diagonal subgroup of $\mathbf{G} \times \mathbf{G}$ (cf. [8, Chapter 4, §6, p. 223] and [9, Proposition (3.17), p. 51]). When $\mathbf{G}$ is a Lie group we have the following.

1. $\mathbf{G} \cong \mathbf{G}^{\mathrm{diag}}$, indeed $g \mapsto (g,g)$ is the canonical isomorphism of $\mathbf{G}$ with $\mathbf{G}^{\mathrm{diag}}$.

2. the division map $d : \mathbf{G}\times\mathbf{G} \to \mathbf{G}$, $d : (g_1,g_2) \mapsto g_1 g_2^{-1}$ has a right-inverse given by $c : \mathbf{G} \to \mathbf{G}\times\mathbf{G}$, $c : g \mapsto (g,e)$, where e denotes the identity element in $\mathbf{G}$.

3. the previous point actually shows that we have the following Lie group isomorphism

$$\mathbf{G} \times \mathbf{G}/(\mathbf{G}^{\mathrm{diag}}) \cong \mathbf{G}$$

   where $\mathbf{G}^{\mathrm{diag}}$ denotes the diagonal subgroup of $\mathbf{G} \times \mathbf{G}$.

4. the last two points give, in concrete terms, the following decomposition of an element $(g_1,g_2) \in \mathbf{G} \times \mathbf{G}$,

$$(g_1,g_2) = (g,e)(h,h), \quad g = g_1 g_2^{-1}, h = g_2, \tag{V.11}$$

   where by construction $(h,h) \in \mathbf{G}^{\mathrm{diag}}$ and (g,e) determines an element in $\mathbf{G} \times \mathbf{G}/\mathbf{G}^{\mathrm{diag}} \cong \mathbf{G}$.

5. If we consider $\mathbf{SL}(2,\mathbb{C})$ as a group of complex matrices we can define the *Cartan involution* as $\Theta : A \mapsto A^{*-1}$. The set of fixed points for this involution is the group $\mathbf{SU}(2)$ seen as a subgroup of $\mathbf{SL}(2,\mathbb{C})$. The symmetric space $\mathbf{SL}(2,\mathbb{C})/\mathbf{SU}(2)$ is *not* a Lie group, but is isomorphic to the familiar hyperboloid of “Lorentz boosts”. The decomposition of an element $A \in \mathbf{SL}(2,\mathbb{C})$ according to the Cartan involution Θ is the familiar *Cartan decomposition* which in this case actually corresponds to the *polar decomposition* $A = VP$, $V \in \mathbf{SU}(2)$, P positive definite, Hermitian.

6. Now consider $\mathbf{G} \times \mathbf{G}$. Clearly $\mathbf{G}^{\mathrm{diag}}$ is isomorphic (as a Lie group) to $\mathbf{G}$. The symmetric space $\mathbf{G} \times \mathbf{G}/\mathbf{G}^{\mathrm{diag}}$ is a Lie group and it is isomorphic to $\mathbf{G}$ as well (as we remarked in point 4.). If we define the involution $\mathfrak{S} : (g_1,g_2) \mapsto (g_2,g_1)$, then the set of fixed points for this involution is the Lie group $\mathbf{G}^{\mathrm{diag}}$ of diagonal elements $(h,h) \in \mathbf{G} \times \mathbf{G}$. Hence the decomposition we defined under point 4. is in fact the decomposition corresponding to the involution $\mathfrak{S}$

4 Inducing a representation of non-zero complex mass

§ 4.1 Real structure and characters. 1. In the semidirect product $\mathbf{ISpin}(4,\mathbb{C}) = \mathbb{C}^4 \rtimes_\lambda \mathbf{Spin}(4,\mathbb{C})$ the factor $\mathbb{C}^4$ denotes the *additive* group of translations in the four dimensional complex space (which is also denoted by $\mathbb{C}^4$).

2. The multiplicative characters of the additive group $\mathbb{C}^4$ are continuous group homomorphisms from the Abelian group $\mathbb{C}^4$ to the unit circle $\mathbb{T}$. Note: often one defines a multiplicative character as a group homomorphism of an Abelian group to $\mathbb{C}^\times$ instead of the circle $\mathbb{T}$. Since we care about *unitary representations* we *need* to map to the circle $\mathbb{T}$ (and *not* $\mathbb{C}^\times$). This forces us to define a **real structure** on $\mathbb{C}^4$, that is to define an identification of $\mathbb{C}^4$ with $\mathbb{R}^8$. This identification, in turn, forces us to consider $\mathbf{ISpin}(4,\mathbb{C})$ as a *real* Lie group. We shall employ the notation $\mathbf{ISpin}(4,\mathbb{C})_\mathbb{R}$ when we need to stress that we consider $\mathbf{ISpin}(4,\mathbb{C})$ as a real Lie group (technically speaking *real* Lie groups are just Lie groups, whereas *complex* Lie groups are more commonly called analytic groups. We use the redundant terminology *real Lie group* to be clearer).

3. The standard choice of *real structure* on $\mathbb{C}^4$ is the one which comes from taking the real and imaginary parts of each component of the vectors in $\mathbb{C}^4$. With this choice of real structure, the characters of $\mathbb{C}^4$ are
$$\chi_w(z) = \exp\left\{\mathrm{i}\sum_{\mu=0}^{3}(\Re w_\mu \Re z_\mu + \Im w_\mu \Im z_\mu)\right\},$$
where $w = (w_0, w_1, w_2, w_3) \in \mathbb{C}^4$, $z = (z_0, z_1, z_2, z_3) \in \mathbb{C}^4$ and $\Re$, $\Im$ denote the real and imaginary part respectively. We will denote $\mathbb{C}^4$ by $\mathbb{C}^4_\mathbb{R}$ when we need to stress that we are considering it as a *real* space (or a real Lie group) with respect to its standard real structure just described.

4. Note that an element $(A, A^{*-1}) \in \mathbf{L} \subset \mathbf{Spin}(4,\mathbb{C})$, which corresponds to a real Lorentz transformation, respects the real structure in the sense that a real vector $x = \Re z$, $z \in \mathbb{C}^4$, is mapped to a real vector $y = \lambda(A, A^{*-1})x$. This is compatible with our choice of action of $\mathbf{Spin}(4,\mathbb{C})$ on $\mathbb{C}^4$ (in (V.10)). Our choices so far have been aimed at considering the Lorentz group as the “fundamental object”. Another possibility of conventions would have been to consider as “fundamental object” the compact group $\mathbf{SO}(4)$.

5. The additive group $\mathbb{C}^4_\mathbb{R}$ is a closed normal Abelian subgroup of the locally compact group $\mathbf{ISpin}(4,\mathbb{C})_\mathbb{R}$. Let $\widehat{\mathbb{C}}^4$ denote the **Pontryagin dual group** of $\mathbb{C}^4_\mathbb{R}$. We identify $\mathbb{C}^4$ and $\mathbf{Spin}(4,\mathbb{C})$ with the respective (closed) subgroups in $\mathbf{ISpin}(4,\mathbb{C})$. Note: we defined the action λ of $\mathbf{Spin}(4,\mathbb{C})$ on $\mathbb{C}^4$ (in (V.10)). Now, in the semidirect product $\mathbf{ISpin}(4,\mathbb{C})$ the action λ of $\mathbf{Spin}(4,\mathbb{C})$ on $\mathbb{C}^4$ is by conjugation, that is
$$(z,e) \in \mathbb{C}^4 \;\mapsto\; (0,s)(z,e)(0,s)^{-1} = (\lambda(s)z, e) = (\sigma^{-1}(A\sigma(z)B^{-1}), e),$$
where $z \in \mathbb{C}^4$, $s = (A,B) \in \mathbf{Spin}(4,\mathbb{C})$, σ^{-1} is the inverse of σ restricted on the image of σ (so that it is invertible), e is the identity (of $\mathbf{Spin}(4,\mathbb{C})$).

6. We denote by $g \cdot \chi$ the action of an element $g \in \mathbf{ISpin}(4,\mathbb{C})_\mathbb{R}$ on an element $\chi \in \widehat{\mathbb{C}}^4_\mathbb{R}$, where $\mathbb{C}^4_\mathbb{R}$ denotes the additive group $\mathbb{C}^4$ as a real Lie group for the real structure defined above. Explicitly we have
$$(s \cdot \chi)(z) = \chi(\lambda(s)^{-1}z),$$
where $\lambda(s)^{-1}$ is by definition $\lambda(s^{-1})$. Disclaimer (!): If we had defined this action with $\lambda(s)$ in place of $\lambda(s^{-1})$ we would *not* have obtained a (left) action. Specifically, it would not satisfy $(s \cdot (r \cdot \chi) = ((s \cdot r) \cdot \chi)$, $s, r \in \mathbf{Spin}(4,\mathbb{C})$. Cf. [5, p. 504ff] v.s. [11, p. 145 (last line)].

§ 4.2 The representation of $\mathbf{ISpin}(4,\mathbb{C})$ induced from a representation of W and the character $\chi_{\mathring{v}}$. Consider the realization of $\mathbf{ISpin}(4,\mathbb{C})$ in Weyl basis. Then as seen above we have the identification
$$\mathbf{ISpin}(4,\mathbb{C}) \overset{\text{Weyl basis}}{=} \mathbb{C}^4 \rtimes_\lambda (\mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})).$$

Let us define

$$\mathbf{IW} \stackrel{\text{def}}{=} \mathbb{C}^4 \rtimes_\lambda \mathbf{W}.$$

In the Weyl basis we have the identification

$$\mathbf{IW} \stackrel{\text{Weyl basis}}{=} \mathbb{C}^4 \rtimes_\lambda \mathbf{SL}(2,\mathbb{C})^{\text{diag}}.$$

In this paragraph we shall look upon $\mathbf{ISpin}(4,\mathbb{C})$, $\mathbf{IW}$, $\mathbf{Spin}(4,\mathbb{C})$, and $\mathbf{W}$ always in this "Weyl basis realization". This shall cause no reduction of generality because we are going to defined a representation of $\mathbf{ISpin}(4,\mathbb{C})$ only *up to isomorphism*.

We want to define the representation $\operatorname{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}}\times\rho)$ of $\mathbf{ISpin}(4,\mathbb{C})$ induced from the representation $\chi_{\mathring{v}}\times\rho$ of $\mathbb{C}^4\rtimes_\lambda \mathbf{W} \stackrel{\textit{Weyl basis}}{=} \mathbb{C}^4\rtimes_\lambda \mathbf{SL}(2,\mathbb{C})^{\text{diag}}$. The induced representation $\operatorname{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}}\times\rho)$ is uniquely defined only up to isomorphism. Nevertheless, as in other similar situations, we refer to $\operatorname{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}}\times\rho)$ as *the* induced representation.

We break the construction in some steps.

1. Let $\mathcal{H}$ be a complex Hilbert space and $(\mathcal{H},\rho)$ a *unitary* representation of $\mathbf{W}\cong\mathbf{SL}(2,\mathbb{C})$ on $\mathcal{H}$. Consider the group $\mathbf{IW}$. As above, let $\chi_{\mathring{v}}$ be the character of $\mathbb{C}^4$ corresponding to the vector $\mathring{v}=(\mu,0,0,0)\in\mathbb{C}^4$. We define the representation $(\mathcal{H},\chi_{\mathring{v}}\times\rho)$ to be the representation of $\mathbf{IW}$ on $\mathcal{H}$ given by
$$(\chi_{\mathring{v}}\times\rho)(n,h)f \stackrel{\text{def}}{=} \chi_{\mathring{v}}(n)\rho(h)f,\quad f\in\mathscr{H},\quad (n,h)\in\mathbb{C}^4\rtimes_\lambda\mathbf{W}. \tag{V.12}$$

2. Let $\mathscr{O}_{\mathring{v}}$ denote the orbit of $\mathring{v}$ under[6] $\mathbf{ISpin}(4,\mathbb{C})$. If we consider the group $\mathbb{C}^4\rtimes_\lambda\mathbf{W}$ as a subgroup $\mathbf{ISpin}(4,\mathbb{C})=\mathbb{C}^4\rtimes_\lambda\mathbf{Spin}(4,\mathbb{C})$ then $\mathring{v}$ is invariant under it. Hence we have the first of the following isomorphisms of manifolds
$$\mathscr{O}_{\mathring{v}}\cong\mathbf{ISpin}(4,\mathbb{C})/(\mathbb{C}^4\rtimes_\lambda\mathbf{W})\cong\mathbf{Spin}(4,\mathbb{C})/\mathbf{W}\stackrel{\textit{Weyl basis}}{=}\mathbf{SL}(2,\mathbb{C})^{\text{diag}}\cong\mathbf{SL}(2,\mathbb{C}),$$
and the remaining isomorphisms follow from §3.4 and §3.3.

3. The Lie group $\mathbf{SL}(2,\mathbb{C})$ has a two-sided invariant measure (or Haar measure) which is unique up to multiplication by a constant. Let us fix μ to be one such a Haar measures. We shall refer to μ as *the* Haar measure of $\mathbf{SL}(2,\mathbb{C})$. The decomposition in (V.11), when applied to the case $\mathbf{G}=\mathbf{SL}(2,\mathbb{C})$, gives rise to an embedding
$$\imath:\mathbf{SL}(2,\mathbb{C})\cong\mathbf{SL}(2,\mathbb{C})\times\mathbf{SL}(2,\mathbb{C})/\mathbf{SL}(2,\mathbb{C})^{\text{diag}}\hookrightarrow\mathbf{SL}(2,\mathbb{C})\times\mathbf{SL}(2,\mathbb{C}).$$
That is, given an element $A\in\mathbf{SL}(2,\mathbb{C})$ this embedding defines an element $\imath(A)\in\mathbf{SL}(2,\mathbb{C})\times\mathbf{SL}(2,\mathbb{C})$. Explicitly, if we represent $\mathbf{SL}(2,\mathbb{C})\times\mathbf{SL}(2,\mathbb{C})$ as matrices of the form $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)$, $A,B\in\mathbf{SL}(2,\mathbb{C})$, then the embedding $\imath$ is
$$\imath(A)=\begin{pmatrix}A&0\\0&\mathbb{I}_2\end{pmatrix},\quad A\in\mathbf{SL}(2,\mathbb{C}),$$
where $\mathring{v}$ is as in (V.12) above. Let us then define the isomorphism $\jmath:\mathbf{SL}(2,\mathbb{C})\xrightarrow{\cong}\mathscr{O}_{\mathring{v}}$ as follows
$$\jmath: A\mapsto\imath(A)\mathring{v}.$$
This map is an isomorphism of manifolds and is therefore in particular measurable. We can therefore push forward the Haar measure μ on $\mathbf{SL}(2,\mathbb{C})$ along this map to obtain an invariant measure $\jmath_*\mu$

[6] Note that the action of $\mathbf{ISpin}(4,\mathbb{C})$ on $\mathbb{C}^4$ is by definition the action of $\mathbf{ISO}(4,\mathbb{C})$ of which $\mathbf{ISpin}(4,\mathbb{C})$ is the double cover.

on $\mathscr{O}_{\mathring{v}}$. We now give this measure explicitly. First we let μ to be the Haar measure on $\mathbf{SL}(2,\mathbb{C})$ explicitly given in the following parametrization (and normalization)

$$\mathrm{d}\mu(A) \stackrel{\text{def}}{=} \frac{1}{|a_{22}|^2}\,\mathrm{d}a_{12}\,\mathrm{d}a_{21}\,\mathrm{d}a_{22}\,\mathrm{d}\overline{a}_{12}\,\mathrm{d}\overline{a}_{21}\,\mathrm{d}\overline{a}_{22}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{V.13}$$

where the measure $\mathrm{d}a_{12}\,\mathrm{d}a_{21}\,\mathrm{d}a_{22}\,\mathrm{d}\overline{a}_{12}\,\mathrm{d}\overline{a}_{21}\,\mathrm{d}\overline{a}_{22}$ denotes the standard measure on $\mathbb{C}^3 \cong \mathbb{R}^6$. For the derivation of (V.13) we refer to [5, p. 68] or [18, p. 13]. Then the pushforward measure $j_*\mu$ can be represented as follows

$$\mathrm{d}(j_*\mu)(v) = \frac{1}{|\mu||v_{22}|^2}\,\mathrm{d}v_{12}\,\mathrm{d}v_{21}\,\mathrm{d}v_{22}\,\mathrm{d}\overline{v}_{12}\,\mathrm{d}\overline{v}_{21}\,\mathrm{d}\overline{v}_{22},$$

where $v = (v_{12}, v_{21}, v_{22}) \in \mathbb{C}^3$.

4. We define the carrier space for the representation $\mathrm{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}} \times \rho)$ ($\chi_{\mathring{v}}$ and ρ as in (V.12)). Let

$$\mathscr{H} \stackrel{\text{def}}{=} L^2(\mathbb{C}^3, j_*\mu; \mathcal{H}),$$

where $\mathcal{H}$ is the Hilbert space on which we are representing $\mathbf{W}$ (and also $\mathbf{IW}$).

5. To define the representation $\mathrm{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}}\times\rho)$ on $\mathscr{H}$ we need some notation for the decomposition in (V.11), specialized to the case $\mathbf{G} = \mathbf{SL}(2,\mathbb{C})$. This decomposition is a special case of what is often called **Mackey decomposition**. That decomposition gives, for any element $s = (A, B) \in \mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C})$, two elements $w(s)$, $q(s)$ which satisfy

$$s = q(s)w(s)$$

with $w(s) \in \mathbf{SL}(2,\mathbb{C})^{\mathrm{diag}}$. Explicitly, representing $s = (A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, we have

$$q(s) = \begin{pmatrix} AB^{-1} & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \qquad w(s) = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}. \tag{V.14}$$

6. Given $v \in \mathscr{O}_{\mathring{v}}$ we define the element $h_v \in \mathbf{Spin}(4,\mathbb{C})/\mathbf{W}$ such that

$$v = \lambda(h_v)\mathring{v}, \tag{V.15}$$

where λ is, as above, the action of $\mathbf{Spin}(4,\mathbb{C})$ on $\mathbb{C}^4$, and we are considering $\mathbf{Spin}(4,\mathbb{C})/\mathbf{W}$ as embedded in $\mathbf{ISpin}(4,\mathbb{C})$ by the embedding ι given in step 4. The element h_v is well defined and unique up to its sign (because the action λ forgets the sign). This ambiguity is resolved once we require that $h_{\mathring{v}} = \mathbb{I}$. Then, by the group structure, having specified the sign of one of these elements actually fixes the sign of all of them. Hence we end up with a *unique* h_v, for all $v \in \mathscr{O}_{\mathring{v}}$.

In the usual theory of Wigner representations of the (real) Lorentz group, h_v corresponds to what is usually called the **standard boost**.

7. We are now ready to define the representation $\left(\mathscr{H}, \mathrm{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}} \times \rho)\right)$. The standard theory[7] of induced representations produces the induced representation $\mathrm{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}} \times \rho)$, when realized on

[7] We give here the standard computation which leads to the definition we give in the text:

$$\begin{aligned}
\big(\pi(n,h)F\big)(v) &= \Big(\pi(0,h_v)\,\pi(0,h_v)^{-1}\,\pi(n,h)\,F\Big)(v) \\
&= \Big(\pi(0,h_v)\,\pi\big(\lambda(h_v^{-1})n, h_v^{-1}h\big)\,F\Big)(v) \\
&= \Big(\pi\big(\lambda(h_v^{-1})n, h_v^{-1}h\big)\,F\Big)(\mathring{v}) \\
&= \Big((\chi_{\mathring{v}} \times \rho)\big(\lambda(h_v^{-1})n, s^W(h_v^{-1}h)\big)\Big)F\big(\lambda(h_v^{-1}h)^{-1}\mathring{v}\big) \\
&= \Big(\chi_{\mathring{v}}\big(\lambda(h_v^{-1})n\big)\,\rho\big(s^W(h_v^{-1}h)\big)\Big)F\big(\lambda(h^{-1})v\big) \\
&= \Big(\chi_v\big(n\big)\,\rho\big(s^W(h_v^{-1}h)\big)\Big)F\big(\lambda(h^{-1})v\big).
\end{aligned}$$

the Hilbert space $\mathscr{H}$ defined above, in the following form

$$\left(\operatorname{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}} \times \rho)(n,h)F\right)(v) \stackrel{\text{def}}{=} \Big(\chi_v\big(n\big)\,\rho\big(w(h_v^{-1}h)\big)\Big)F\big(\lambda(h^{-1})v\big),$$
$$F \in \mathscr{H}, \quad (n,h) \in \mathbf{ISpin}(4,\mathbb{C}) = \mathbb{C}^4 \rtimes_\lambda \mathbf{Spin}(4,\mathbb{C}). \quad \text{(V.16)}$$

Note that, by writing the big parenthesis in $\Big(\chi_v\big(n\big)\,\rho\big(w(h_v^{-1}h)\big)\Big)$, we want to stress that the operator $\Big(\chi_v\big(n\big)\,\rho\big(w(h_v^{-1}h)\big)\Big)$ is applied to $F\big(\lambda(h^{-1}v)\big) \in \mathcal{H}$ and *not* to F as an element of $\mathscr{H}$.

8. We can simplify (V.16) employing the explicit form of $w(h_v^{-1}h)$ given in (V.14). Let us first define the following notation

$$\sigma(v) \stackrel{\text{def}}{=} v_0\,\mathbb{I}_2 + \sum_{j=1}^{3} v_j\sigma_j, \quad v \in \mathbb{C}^4.$$

Note that, when $v \in \mathscr{O}_{\mathring{v}} \subset \mathbb{C}^4$ then $\sigma(v) \in \mathbf{SL}(2,\mathbb{C})$. With this notation, we have, for $\mu \in \mathbb{C}$, $\mu \neq 0$,

$$h_v = \begin{pmatrix} \frac{1}{\mu}\sigma(v) & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \quad v \in \mathscr{O}_{\mathring{v}}, \quad \mathring{v} = (\mu,0,0,0). \quad \text{(V.17)}$$

Let us represent $h \in \mathbf{Spin}(4,\mathbb{C})$ in Weyl basis, that is let us set

$$h = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A, B \in \mathbf{SL}(2,\mathbb{C}).$$

Hence

$$h_v^{-1}h = \begin{pmatrix} (\frac{1}{\mu}\sigma(v))^{-1}A & 0 \\ 0 & B \end{pmatrix}.$$

Then we obtain from (V.14) that

$$q(h_v^{-1}h) = \begin{pmatrix} (\frac{1}{\mu}\sigma(v))^{-1}AB^{-1} & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \quad w(h_v^{-1}h) = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$

We can therefore rewrite (V.16) as follows, setting $\pi \stackrel{\text{def}}{=} \operatorname{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}} \times \rho)$,

$$\big(\pi(n,h)F\big)(v) = \big(\chi_v(n)\,\rho(B)\big)F\big(A\sigma(v)B^{-1}\big),$$
$$F \in \mathscr{H}, \quad n \in \mathbb{C}^4, \quad h = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad A, B \in \mathbf{SL}(2,\mathbb{C}), \quad \text{(V.18)}$$

where we have used the fact that in Weyl representation $h = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and, in this Weyl realization, we can identify $\lambda(h)v$ with $A\sigma(v)B^{-1}$ (cf. (V.10)), and by abuse of notation, we are evaluating F on the right hand side on $A\sigma(v)B^{-1} \in \mathbf{SL}(2,\mathbb{C})$. What we mean is to evaluate F on the four elements of such matrix, that is on $\sigma^{-1}(A\sigma(v)B^{-1})$, where σ^{-1} is the inverse image of σ and, as such, maps elements of $\mathbf{SL}(2,\mathbb{C})$ into $\mathbb{C}^4$.

This construction shows how to explicitly give one of the unitarily equivalent realization of the induced representation of $\mathbf{ISpin}(4,\mathbb{C})$ starting with the character $\chi_{\mathring{v}}$ and a unitary representation ρ of $\mathbf{ISpin}(3,\mathbb{C})$. We state what we found as a proposition.

For the standard theory of induced representations see, e.g. [11]. For computations similar to the one above see [6, (7.69), p. 283].

§ 4.3 Proposition. Let $\mathring{v} = (\mu, 0, 0, 0) \in \mathbb{C}^4$, $\mu \in \mathbb{C}$, and let $\chi_{\mathring{v}}$ be the character of the additive group $\mathbb{C}^4 \cong \mathbb{R}^8$,

$$\chi_{\mathring{v}}(n) = \exp\left\{\mathrm{i}\tfrac{1}{2}\left(\overline{\mathring{v}} \cdot n + \mathring{v} \cdot \overline{n}\right)\right\} = \exp\left\{\mathrm{i}\tfrac{1}{2}\left(\overline{\mu} n_0 + \mu \overline{n_0}\right)\right\}, \quad n = (n_0, n_1, n_2, n_3) \in \mathbb{C}^4,$$

where the over-line denotes complex conjugation and the $\cdot$ denotes the standard bilinear dot product in $\mathbb{C}^4$. Let $\mathbf{ISpin}(4, \mathbb{C})$, respectively $\mathbf{ISpin}(3, \mathbb{C})$, be the inhomogeneous complex spin groups in four, respectively three, dimensions. Let us identify $\mathring{v} \cong (\mathring{v}, e)$ with the element $(\mathring{v}, e) \in \mathbf{ISpin}(4, \mathbb{C})$, where e denotes the identity element in $\mathbf{Spin}(4, \mathbb{C})$. Let $\mathscr{O}_{\mathring{v}}$ be the orbit of $\mathring{v}$ under the action of $\mathbf{ISpin}(4, \mathbb{C})$ on itself. Finally, let $(\mathcal{H}(\rho), \rho)$ be a unitary representation of $\mathbf{Spin}(3, \mathbb{C}) \cong \mathbf{SL}(2, \mathbb{C})$ on a complex Hilbert space $\mathcal{H}(\rho)$. Then

1. the map $\chi_{\mathring{v}} \times \rho : \mathbf{ISpin}(4, \mathbb{C}) \to \mathbf{U}(\mathcal{H})$ into the unitary operators on $\mathcal{H}$ given by

   $$(\chi_{\mathring{v}} \times \rho)(n, h) f \overset{\text{def}}{=} \chi_{\mathring{v}}(n) \rho(s) f, \quad (n, h) \in \mathbf{ISpin}(3, \mathbb{C}), \quad n \in \mathbb{C}^4, h \in \mathbf{Spin}(3, \mathbb{C}), \quad f \in \mathcal{H}(\rho),$$

   defines a unitary representation of $\mathbf{ISpin}(3, \mathbb{C})$ on $\mathcal{H}(\rho)$.

2. The unitary representation, unique up to isomorphism, $(\pi, \mathscr{H})$, of $\mathbf{ISpin}(4, \mathbb{C})$, induced by the unitary representation $\chi_{\mathring{v}} \times \rho$ of $\mathbf{ISpin}(3, \mathbb{C})$, is realized, up to isomorphisms, as follows.

   $$\mathscr{H} \overset{\text{def}}{=} L^2\left(\mathscr{O}_{\mathring{v}}, j_* \mu_{\mathbf{SL}(2,\mathbb{C})} \,;\, \mathcal{H}(\rho)\right), \tag{V.19}$$

   where $\mu_{\mathbf{SL}(2,\mathbb{C})}$ denotes the, unique up to normalized Haar, measure of $\mathbf{SL}(2, \mathbb{C})$, and $j_* \mu_{\mathbf{SL}(2,\mathbb{C})}$ denotes the push forward measure along the map j which gives the isomorphism of smooth manifolds $j : \mathbf{SL}(2, \mathbb{C}) \xrightarrow{\cong} \mathscr{O}_{\mathring{v}}$.

   Let us identify $\mathbf{Spin}(4, \mathbb{C}) \cong \mathbf{SL}(2, \mathbb{C}) \times \mathbf{SL}(2, \mathbb{C})$ and denote by $\sigma : \mathbb{C}^4 \xrightarrow{\cong} \mathtt{M}_2(\mathbb{C})$ the vector space isomorphism of $\mathbb{C}^4$ with the space $\mathtt{M}_2(\mathbb{C})$ of 2-by-2 complex matrices given by

   $$\sigma(v) = v_0 \, \mathbb{I}_2 + \sum_{j=1}^{3} v_j \sigma_j, \quad v = (v_0, v_1, v_2, v_3) \in \mathbb{C}^4,$$

   where σ_j, $j = 1, 2, 3$, are the standard Pauli matrices. Then we have

   $$\left(\pi(n, s) F\right)(v) = \left(\chi_v(n)\, \rho(B)\right) F\left(A \sigma(v) B^{-1}\right), \tag{V.20}$$

   where

   $$F \in \mathscr{H}, \quad n \in \mathbb{C}^4, \quad s = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathbf{SL}(2, \mathbb{C}) \times \mathbf{SL}(2, \mathbb{C}) \cong \mathbf{Spin}(4, \mathbb{C}), \quad A, B \in \mathbf{SL}(2, \mathbb{C}).$$

§ 4.4 Remark. Note that, by abuse of notation, the function F on the left hand side it is evaluated on an element of $\mathbb{C}^4$ whereas on the right hand side is evaluated on a complex 2-by-2 matrix. This causes no ambiguity because we consider $\mathbb{C}^4$ identified with the space $\mathtt{M}_2(\mathbb{C})$ of 2-by-2 complex matrices via the vector space isomorphism $\sigma : \mathbb{C}^4 \xrightarrow{\cong} \mathtt{M}_2(\mathbb{C})$. □

5 A positive mass, 1/2-integer representation of the complexified Poincaré spin group and its application to the free Dirac field

§ 5.1 Choice of representation for W. In principle the most natural choice, for arriving at a representation of $\mathbf{ISpin}(4, \mathbb{C})$ in our setting, would be to pick an *irreducible* unitary representation of the little group $\mathbf{W}$. Since we want to connect with the standard theory of Dirac fields, it is simpler to chose instead a reducible unitary representation of $\mathbf{W}$. We now explain the choice of representation we are going to make. The important requirement on the representation that we are going to choose is that, when restricted to $\mathbf{SU}(2) \subset \mathbf{SL}(2, \mathbb{C})$, it should restrict to the irreducible unitary representation of $\mathbf{SU}(2)$ which corresponds to half-integer spin.

1. The group $\mathbf{W}$, as we discussed, is isomorphic to $\mathbf{SL}(2,\mathbb{C})$. Now, $\mathbf{SL}(2,\mathbb{C})$ can be seen as the double cover of the Lorentz group, that is, we have the following isomorphism

$$\mathbf{SL}(2,\mathbb{C}) \cong \mathbf{Spin}(1,3),$$

where $\mathbf{Spin}(1,3)$ is the spin group with respect to the Minkowski quadratic form with signature $(1,-1,-1,-1)$. Now, let us define

$$\mathbf{ISpin}(1,3) \stackrel{\text{def}}{=} \mathbb{R}^4 \rtimes_\lambda \mathbf{Spin}(1,3),$$

where λ is the same action as above just restricted to $\mathbf{Spin}(1,3) \subset \mathbf{Spin}(4,\mathbb{C})$.

2. The standard theory of Dirac fields (cf. chapter III) give us at least two Hilbert spaces on which to represent the whole inhomogeneous group $\mathbf{ISpin}(1,3)$. Loosely speaking, one of these Hilbert spaces is a space of functions in the "momentum variables" and the other Hilbert space is a space of functions in the "space-time variables". The one we are interested in, at the moment, is the one in the "momentum variables". Explicitly we define the representation $(\mathcal{H}, U)$ of $\mathbf{ISpin}(1,3)$ as follows.

First we consider the Hilbert space

$$\mathcal{W}_{(1,0)} \stackrel{\text{def}}{=} \left\{ \psi : \|\psi\|_{\mathcal{W}_{(1,0)}} \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \big(\psi(q), \tfrac{1}{m}\sigma(q)\,\psi(q)\big)_{\mathbb{C}^2} \frac{1}{2\sqrt{\mathbf{q}^2+m^2}}\,\mathrm{d}\mathbf{q} < +\infty \right\}, \tag{V.21}$$

where $q = (\sqrt{\mathbf{q}^2+m^2}, q_1, q_2, q_3)^{\mathrm{t}}$, $\mathbf{q} = (q_1,q_2,q_3) \in \mathbb{R}^4$, and $(\cdot,\cdot)_{\mathbb{C}^2}$ denotes the standard Hermitian scalar product in $\mathbb{C}^2$. Hence $\mathcal{W}_{(1,0)}$ is the closure of the space $C_c^\infty(\mathbb{R}^3;\mathbb{C}^2)$ with respect to the Hilbertian norm $\|\cdot\|_{\mathcal{W}_{(1,0)}}$ defined above. We take this space as the carrier space of the following representation of $\mathbf{SL}(2,\mathbb{C})$:

$$U_{(1,0)}(A)\psi(\mathbf{q}) = Af(\mathbf{\Lambda}_{A^{-1}}(\mathbf{q})), \quad A \in \mathbf{SL}(2,\mathbb{C}), \psi \in \mathcal{W}_{(1,0)}, \mathbf{q} \in \mathbb{R}^3,$$

where we have denoted by $\mathbf{\Lambda}_{A^{-1}}(\mathbf{p})$ the three dimensional vector obtained by applying $\lambda(B^{-1})$ to the four dimensional vector $\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$ and then restricting to the last three components. We refer to [20, §3.3.2, (3.89) p 95] or chapter III, §4.7 of the present work, for the construction of such a representation and the proof that it is the restriction of a well defined *irreducible*, unitary representation of $\mathbf{ISpin}(1,3)$ to $\mathbf{Spin}(1,3)$.

Now, we let

$$\mathcal{H} \stackrel{\text{def}}{=} \mathcal{W}_{(1,0)} \oplus \mathcal{W}_{(1,0)}, \quad \rho \stackrel{\text{def}}{=} U_{(1,0)} \oplus U_{(1,0)}. \tag{V.22}$$

Hence we have

$$\big(\rho(B)f\big)(\mathbf{q}) \stackrel{\text{def}}{=} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} f\big(\mathbf{\Lambda}_{B^{-1}}(\mathbf{q})\big),$$

$$B \in \mathbf{SL}(2,\mathbb{C}), \quad \mathbf{q} \in \mathbb{R}^3, \quad f \in \mathcal{H},.$$

We remark that the space $\mathcal{H}$ is the carrier space for the little group. It so just happens that the little group is isomorphic to a copy of the Lorentz group. The compact part of the little group, which is isomorphic to $\mathbf{SU}(2)$, should be identified with the double cover of the Lie group of 3-dimensional rotations. The "remaining" part, that is $\mathbf{SL}(2,\mathbb{C})/\mathbf{SU}(2)$, is isomorphic to the hyperboloid $q_0^2 - \mathbf{q}^2 = 1$, $q_0 \in \mathbb{R}$, $\mathbf{q} \in \mathbb{R}^3$. This part should be identified with the analytic continuation to imaginary angles of the 3-dimensional rotations.

3. We shall now combine the representation $(\mathcal{H}, \rho)$ defined in (V.22) with §4.2.4 and §4.2.7, where we defined the representation $(\mathscr{H}, \mathrm{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}} \times \rho)$. We start by considering the Hilbert space $\mathscr{H}$. By standard manipulation of Hilbert tensor products, we have

$$\begin{aligned} \mathscr{H} &= L^2(\mathbb{C}^3, j_*\mu; \mathcal{H}) \\ &= L^2(\mathbb{C}^3, j_*\mu) \,\widehat{\otimes}\, \mathcal{H} \end{aligned}$$

where $\widehat{\otimes}$ denotes the completed tensor product and $\otimes$ in the last line is used to denote the product measure.

We now turn to the definition of π in (V.18). With the choice of ρ given here in point 2. we get

$$\big(\mathrm{Ind}_{\mathbf{IW}}^{\mathbf{ISpin}(4,\mathbb{C})}(\chi_{\mathring{v}} \times \rho)(n,s)\Psi\big)\big(v, \mathbf{q}\big) = \chi_v(n) \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \Psi\big(A\sigma(v)B^{-1}, \mathbf{\Lambda}_{B^{-1}}(\mathbf{q})\big), \tag{V.23}$$

where

$$v \in \mathbb{C}^3,\ \mathbf{q} \in \mathbb{R}^3,\ \Psi \in L^2\big(\mathbb{C}^3, (j_*\mu)\big) \,\widehat{\otimes}\, \mathcal{H}; \mathbb{C}^4\big),$$

$$n \in \mathbb{C}^4,\ h = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},\ A, B \in \mathbf{SL}(2, \mathbb{C}).$$

§ 5.2 "Covariant realization". In the classical case of the real Poincaré group, one first defines Wigner representations. Then, to have a simpler transformation rule one can go from Wigner representation to a "covariant representation". We refer to chapter III for the details. The basic trick to go from the Wigner representation to the "covariant representation" is to apply to the spinor wavefunction in momentum space $f(\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^3$, a multiplication operator which multiplies by a *standard boost*, that is the matrix $\sqrt{\sigma(p)}$, $p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})^t$, where $\mathbf{p} \in \mathbb{R}^3$ coincides with the argument of the spinor wavefunction $f(\mathbf{p})$ (note that applying this "standard boost multiplication operator" is *not* equivalent with making an actual boost). For the details cf. [5, p. 523, (40)-(41)] or chapter III of the present work; in the latter, see in particular §4.7.

Here we are going to follow a similar path, but for $\mathbf{ISpin}(4, \mathbb{C})$ in place of the double cover $\mathbf{ISpin}^0(1, 3)$ of the real Poincaré group. We shall start with the representation $(\mathscr{H}, \pi)$ defined in (V.23), apply the "standard boost multiplication operator", and end up with a new representation, which we shall call $(\widetilde{H}, \widetilde{\pi})$ which has "nicer" transformation properties.

The "standard boost" for the case of $\mathbf{ISpin}(4, \mathbb{C})$ was defined in (V.15) and explicitly realized in (V.17). Hence we define a multiplication operator β by

$$(\beta F)(v, \mathbf{q}) \stackrel{\text{def}}{=} \begin{pmatrix} \frac{1}{\mu}\sigma(v) & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} F(v, \mathbf{q}), \quad v \in \mathscr{O}_{\mathring{v}}, \mathbf{q} \in \mathbb{R}^3,$$

where $\mathscr{O}_{\mathring{v}}$ is defined at point 2. We look at the operator β as an operator from $\mathscr{H}$ to, in general, the space of distributions $\mathscr{D}'(\mathscr{O}_v \times \mathbb{R}^3; \mathbb{C}^4)$. We then define a new space $\widetilde{\mathscr{H}}$ as the range of β in $\mathscr{D}'(\mathscr{O}_v \times \mathbb{R}^3; \mathbb{C}^4)$

$$\widetilde{\mathscr{H}} \stackrel{\text{def}}{=} \mathrm{Ran}\, \beta.$$

Since the matrix

$$\begin{pmatrix} \frac{1}{\mu}\sigma(v) & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}$$

is invertible for every $v \in \mathscr{O}_{\mathring{v}}$, the operator β is an isomorphism of $\mathscr{H}$ with $\widetilde{\mathscr{H}}$. Moreover we can define on $\widetilde{\mathscr{H}}$ a Hermitian scalar product by pushing forward the scalar product of $\mathscr{H}$ along the map β. In this way we make $\widetilde{\mathscr{H}}$ into a Hilbert space and β becomes an isometric isomorphism of Hilbert spaces. We push forward, along β, also the representation π on $\mathscr{H}$, obtaining a representation $\widetilde{\pi}$ on $\widetilde{\mathscr{H}}$. A straight forward computation shows that the representation $\widetilde{\pi}$ is given explicitly by the following

$$\big(\widetilde{\pi}(n,h)\widetilde{\Psi}\big)(v,\mathbf{q}) = \chi_v(n)\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}\Psi\big(A\sigma(v)B^{-1}, \mathbf{\Lambda}_{B^{-1}}(\mathbf{q})\big), \tag{V.24}$$

where

$$v\in\mathbb{C}^3,\ \mathbf{q}\in\mathbb{R}^3,\ \widetilde{\Psi}\in\widetilde{\mathscr{H}},\ \ n\in\mathbb{C}^4,\ h=\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix},\ A,B\in\mathbf{SL}(2,\mathbb{C}).$$

We remark that we chose the definition of the representation $(\mathcal{H},\rho)$, given in (V.22), in such a way as to obtain this final result. In particular, the form of the representation $(\widetilde{\mathscr{H}},\widetilde{\pi})$ forced us to consider as representation of the little group a direct sum of *two* copies of a 2-spinor representation.

We collect the properties of the representation $(\widetilde{\mathscr{H}},\widetilde{\pi})$ that we have constructed in a proposition which is parallel to the proposition in 4.3.

§ 5.3 Proposition. Let take the standard vector $\mathring{v} = \mathring{v}(m) \overset{\text{def}}{=} (m,0,0,0)$, $m\in\mathbb{R}$, and specialize the representation $(\mathscr{H},\pi)$ of 4.3 to this special case. Let $\widetilde{\beta} : \mathscr{H}\to \operatorname{Ran}\widetilde{\beta}$ be defined by

$$(\widetilde{\beta}F)(v,\mathbf{q}) \overset{\text{def}}{=} \begin{pmatrix} \sigma(v)\sqrt{\sigma(q)} & 0 \\ 0 & \sqrt{\sigma(q)}\end{pmatrix}\begin{pmatrix} F(v,\mathbf{q}) \\ F(v,\mathbf{q}\end{pmatrix}, \quad v\in\mathscr{O}_{\mathring{v}(m)},\quad q=(\sqrt{\mathbf{q}^2+m^2},\mathbf{q})\in\mathbb{R}^4,\quad \mathbf{q}\in\mathbb{R}^3. \tag{V.25}$$

The map $\widetilde{\beta}$ is indeed well defined for any $F\in\mathscr{H}$. It is invertible and it carries the unitary representation $(\mathscr{H},\pi)$ into the unitary representation $(\widetilde{\mathscr{H}},\widetilde{\pi})$ given by

$$\widetilde{\mathscr{H}} \overset{\text{def}}{=} \operatorname{Ran}\beta,\quad \widetilde{\pi}\overset{\text{def}}{=}\widetilde{\beta}\pi\widetilde{\beta}^{-1},$$

where $\mathscr{H}$ is a complex Hilbert space equipped with scalar product carried by the map $\widetilde{\beta}$ from $\mathscr{H}$ to $\widetilde{\mathscr{H}}$, that is

$$(\widetilde{\Psi}_1,\widetilde{\Psi}_2)_{\widetilde{\beta}^{-1}\mathscr{H}} \overset{\text{def}}{=} (\widetilde{\beta}^{-1}\widetilde{\Psi}_1,\widetilde{\beta}^{-1}\widetilde{\Psi}_2)_{\mathscr{H}},\quad \widetilde{\Psi}_1,\widetilde{\Psi}_2\in\mathcal{H}, \tag{V.26}$$

where $(\cdot,\cdot)_{\mathscr{H}}$ denotes the scalar product of $\mathscr{H}$. Explicitly we have

$$\big(\widetilde{\pi}(n,s)\widetilde{\Psi}\big)(v,\mathbf{q}) = \chi_v(n)\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}\widetilde{\Psi}\big(A\sigma(v)B^{-1}, \mathbf{\Lambda}_{B^{-1}}(\mathbf{q})\big), \tag{V.27}$$

where

$$v\in\mathbb{C}^3,\ \mathbf{q}\in\mathbb{R}^3,\ \widetilde{\Psi}\in\widetilde{\mathscr{H}},\ \ n\in\mathbb{C}^4,\ s=\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}\in\mathbf{SL}(2,\mathbb{C})\times\mathbf{SL}(2,\mathbb{C}),\ A,B\in\mathbf{SL}(2,\mathbb{C}).$$

Finally the scalar product on $\widetilde{\mathscr{H}}$ is explicitly given by

$$\begin{aligned}(\widetilde{\Psi}_1,\widetilde{\Psi}_2)_{\widetilde{\mathscr{H}}} &= \int_{\mathscr{O}_{\mathring{v}(m)}} \big((\beta^{-1}\widetilde{\Psi}_1)(v,\cdot),(\beta^{-1}\widetilde{\Psi}_2)(v,\cdot)\big)_{\mathcal{H}}\,\frac{1}{2\omega(\mathbf{q})}\,\mathrm{d}\mathbf{q}\,\mathrm{d}(j_*\mu_{\mathbf{SL}(2,\mathbb{C})})(v)\\ &= \int_{\mathscr{O}_{\mathring{v}(m)}} \left(\widetilde{\Psi}_1(v,\mathbf{q}),\begin{pmatrix}\frac{1}{m^3}\sigma(v)^{*-1}\sigma(q)^{-1}\sigma(v)^{-1} & 0\\ 0 & \frac{1}{m}\sigma(q)^{-1}\end{pmatrix}\widetilde{\Psi}_2(v,\mathbf{q})\right)_{\mathbb{C}^4}\times\\ &\qquad\times\frac{1}{2\omega(\mathbf{q})}\,\mathrm{d}\mathbf{q}\,\mathrm{d}(j_*\mu_{\mathbf{SL}(2,\mathbb{C})})(v),\end{aligned} \tag{V.28}$$

for $\widetilde{\Psi}_1,\widetilde{\Psi}_2\in\widetilde{\mathscr{H}}$ where $\mu_{\mathbf{SL}(2,\mathbb{C})}$ denotes the (properly normalized) Haar measure on $\mathbf{SL}(2,\mathbb{C})$ and $j_*\mu_{\mathbf{SL}(2,\mathbb{C})}$ was defined in point 3 of §**5.1**. □

§ 5.4 Remark. The representation $(\widetilde{\mathscr{H}}, \widetilde{\pi})$ given in the last proposition has already most of the features we desire. The wavefunction $\widetilde{\Psi} \in \widetilde{\mathscr{H}}$ depends on a variable $v \in \mathscr{O}_{\hat{v}(m)} \cong \mathbf{SL}(2,\mathbb{C})$ (see point 2. in §5.3) which transforms as a vector under $\mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C}) \cong \mathbf{Spin}(4,\mathbb{C})$. Moreover it depends on an additional variable $\mathbf{q}$ which comes out from the construction of the unitary representation but has not a direct meaning. Finally the wave function depends on a finite index $\mu = 0,1,2,3$ which gives to the wave function the correct bi-spinor nature. Indeed its bi-spinorial part transforms with the defining representation of $\mathbf{Spin}(4,\mathbb{C})$ (cf. chapter (V.27)). Nevertheless the scalar product in (V.28) is still somewhat cryptic. In particular it is not directly clear how to relate it to the usual scalar product which one gets in the standard relativistic Dirac fields (cf. III). We therefore are going to define (in §5.6) a change of variable which will simplify the structure of the representation and in particular of such a scalar product. To define this change of variables we need first to discuss in more details the Haar measure on $\mathbf{SL}(2,\mathbb{C})$ and its decomposition.

§ 5.5 Decomposition of the measure $j_*\mu$. 1. We start by decomposition μ. We recall that μ is the Haar measure (with a fixed normalization convention) on $\mathbf{SL}(2,\mathbb{C})$. With our convention of normalization the measure μ was given in (V.13). Consider now the polar decomposition $A = PU$ for a matrix $A \in \mathbf{SL}(2,\mathbb{C})$ in terms of a positive definite matrix $P \overset{\text{def}}{=} \sqrt{AA^*}$ and a unitary matrix of determinant one $U \in \mathbf{SU}(2)$. We now want to give the explicit form of the decomposition μ in terms of the measure ν_P on the set of Hermitian, positive definite 2-by-2 matrices and the Haar measure $\nu_{\mathbf{SU}(2)}$ of $\mathbf{SU}(2)$. We can parametrize the space of Hermitian, positive definite 2-by-2 matrices as follows

$$P = \begin{pmatrix} \omega_1(\mathbf{p}) - p_3 & p_2 + \mathrm{i}p_2 \\ p_2 - \mathrm{i}p_1 & \omega_1(\mathbf{p}) + p_3 \end{pmatrix},$$

where, as before, $\mathbf{p} \overset{\text{def}}{=} (p_1, p_2, p_3) \in \mathbb{R}^3$, and

$$\omega_1(\mathbf{p}) \overset{\text{def}}{=} \sqrt{\mathbf{p}^2 + 1}\,.$$

Hence, we want to decompose the measure μ given in (V.13) in terms of the Lebesgue measure $\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3$ on $\mathbb{R}^3$ and the Haar measure $\mu_{\mathbf{SU}(2)}$ on $\mathbf{SU}(2)$. This means that we need to compute the Jacobian J in the decomposition

$$\mathrm{d}\mu(A) = J\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3\,\mathrm{d}\mu_{\mathbf{SU}(2)}(U),$$

One way to compute this Jacobian is the following. First decompose a matrix $A \in \mathbf{SL}(2,\mathbb{C})$ according to the Iwasawa decomposition $A = KU$ (which holds almost everywhere with respect to the Haar measure on $SL(2,\mathbb{C})$):

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & z \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \qquad \lambda > 0, z \in \mathbb{C}.$$

The disintegration of the measure μ which corresponds to this decomposition of a matrix $A \in \mathbf{SL}(2,\mathbb{C})$ is well known (cf. [18, Appendix A–1, p. 282-283]. Also relevant is [10, Section 1.2]). One has, for μ normalized as in (V.13),

$$\mathrm{d}\mu(A) = \lambda\,\mathrm{d}\lambda\,\mathrm{d}(\Re z)\,\mathrm{d}(\Im z)\,\mathrm{d}\mu_{\mathbf{SU}(2)}(U), \quad \lambda > 0, z \in \mathbb{C}, U \in \mathbf{SU}(2), A \in \mathbf{SL}(2,\mathbb{C}),$$

where $\mu_{\mathbf{SU}(2)}$ denotes the Haar measure on $\mathbf{SU}(2)$ normalized to 1.

Then, we compute the Jacobian of the change of variables $(\lambda, \Re z, \Im z) \to (p_1, p_2, p_3)$. This change of variables is given by[8]

$$\lambda = \omega_1(\mathbf{p}) + p_3, \quad \lambda z = p_2 + \mathrm{i}p_3.$$

Performing this computation, after straight forward computations, one finds

$$\mathrm{d}\mu(A) = \frac{1}{2\omega_1(\mathbf{p})}\,\mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3\,\mathrm{d}\mu_{\mathbf{SU}(2)}(U),$$

with $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$, $U \in \mathbf{SU}(2)$, $A \in \mathbf{SL}(2,\mathbb{C})$.

[8]Cf. [5, §17.2.D, (36) p.522] where there is a small typo: one should replace $\frac{m}{2}$ with just m.

2. We now turn our attention to the measure $j_*\mu$ which, we recall, is the invariant measure on the manifold $\mathscr{O}_{\mathring{v}}$ obtained by pushing forward the (with a fixed normalization convention) Haar measure μ on $\mathbf{SL}(2,\mathbb{C})$. Recall that the vector $\mathring{v}$ is defined as $\mathring{v} \overset{\text{def}}{=} (\mu_0, 0, 0, 0)$ for a fixed complex number μ_0. Since we want to compare this model with the usual results concerning the half-spin representations of the *real* Poincaré group we now specialize to the situation where μ_0 is

$$\mu_0 = m > 0$$

for a given positive number m. In this case, one unsurprisingly finds the following decomposition for the measure $j_*\mu$:

$$\mathrm{d}(j_*\mu)(v) = \frac{1}{2\omega_m(\mathbf{p})}\,\mathrm{d}\mathbf{p}\,\mathrm{d}\mu_{\mathbf{SU}(2)}(U), \quad v \in \mathscr{O}_{\mathring{v}},\ \mathring{v} = (m,0,0,0),\ m > 0, \tag{V.29}$$

where $\mathrm{d}\mathbf{p} \overset{\text{def}}{=} \mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{d}p_3$ denotes the Lebesgue measure on $\mathbb{R}^3$ and

$$\omega_m(\mathbf{p}) \overset{\text{def}}{=} \sqrt{\mathbf{p}^2 + m^2}.$$

§ 5.6 Change of variables. Let us now go back to the representation $(\widetilde{\mathscr{H}}, \widetilde{\pi})$ introduced in §5.2. A function $\widetilde{\Psi} \in \widetilde{\mathscr{H}}$ is a vector valued functions of two variables, $v \in \mathscr{O}_{\mathring{v}(m)}$ and $\mathbf{q} \in \mathbb{R}^3$. Let us, as usual, identify $\mathbb{C}^4$ with the space $\mathtt{M}_2(\mathbb{C})$ of 2-bi-2 complex matrices via the vector space isomorphism σ. Under this this isomorphism the manifold $\mathscr{O}_{\mathring{v}(m)}$ is sent into the space of 2-by-2 complex matrices with determinant equal to m^2. This space is clearly isomorphic to $\mathbf{SL}(2,\mathbb{C})$ and corresponds to having "dilated" every element in $\mathbf{SL}(2,\mathbb{C})$ by multiplying it by m. The decomposition in (V.29) corresponds to seeing every variable $v \in \mathscr{O}_{\mathring{v}}$ as decomposed into two components:

$$v = (\mathbf{p}, U), \quad v \in \mathscr{O}_{\mathring{v}}, \quad \mathbf{p} \in \mathbb{R}^3, \quad U \in \mathbf{SU}(2).$$

With this decomposition in mind, we can think of a function $\widetilde{\Psi} \in \widetilde{\mathscr{H}}$ as a function of three variables: $\mathbf{p} \in \mathbb{R}^3$, $U \in \mathbf{SU}(2)$, $\mathbf{q} \in \mathbb{R}^3$. We can intuitively think of these variables as follows. The variable $\mathbf{p}$ parametrizes the physical mass hyperboloid, that is it has the same meaning in this model as in the conventional case. The variable U characterizes give the degrees of freedom which corresponds to "imaginary boosts" that are Euclidean rotations between the first (0-th) axis and the remaining axis. The variable $\mathbf{q}$ characterizes the degrees of freedom corresponding to "imaginary three-dimensional rotations".

We are now ready to describe the change of variables which we want to perform. Consider the formula (V.28). In the matrix on the right hand side we have the term

$$\tfrac{1}{m^3}\sigma(v)^{*-1}\sigma(q)^{-1}\sigma(v)^{-1} = \tfrac{1}{m^3}\big(\sigma(v)\sigma(q)\sigma(v)^*\big)^{-1}.$$

We observe that the transformation

$$\tfrac{1}{m}\sigma(q) \mapsto \tfrac{1}{m^3}\big(\sigma(v)\sigma(q)\sigma(v)^*\big)^{-1}, \quad q = (\sqrt{\mathbf{q}^2 + m^2}, \mathbf{q}) \in \mathbb{R}^4,\ \mathbf{q} \in \mathbb{R}^3, \quad v \in \mathscr{O}_{\mathring{v}(m)}$$

corresponds to performing on q a Lorentz transformation. In particular, since q is on the mass hyperboloid, this transformation will send it to a new vector q_v still on the same hyperboloid. As discussed above, we consider a function $\widetilde{\Psi} \in \widetilde{\mathscr{H}}$ as a function of $(\mathbf{p}, U, \mathbf{q})$, $\mathbf{p} \in \mathbb{R}^3$, $U \in \mathbf{SU}(2)$, and $\mathbf{q} \in \mathbb{R}^3$, where both the variables $\mathbf{p}$ and $\mathbf{q}$ parametrize a copy of the mass hyperboloid with mass m. Hence we can define the change of variables

$$(\mathbf{p}, U, \mathbf{q}) \mapsto (\mathbf{p}', U, \mathbf{q})$$

where $\mathbf{p}'$ is defined by requiring

$$\tfrac{1}{m}\sigma(p') = \tfrac{1}{m^3}\sigma(v)\sigma(q)\sigma(v)^* \tag{V.30}$$

where $p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p})$ and the other variables are as above. This is a well defined change of variables because, as we remarked, the right hand side defines a point on the mass hyperboloid.

§ 5.7 Let $(\mathbf{p}, U, \mathbf{q}) \mapsto (\mathbf{p}', U, \mathbf{q})$ be the change of variables explained in the last paragraph. We define an operator φ which implements on the functions in $\widetilde{\mathscr{H}}$ such change of variables, that is we let

$$\varphi : \widetilde{\Psi}(\mathbf{p}, U, \mathbf{q}) \mapsto \widetilde{\Psi}(\mathbf{p}', U, \mathbf{q}),$$

for $\widetilde{\Psi} \in \widetilde{\mathscr{H}}$, $\mathbf{p} \in \mathbb{R}^3$, $\mathbf{q} \in \mathbb{R}^3$, and $\mathbf{p}'$ defined by (V.30).

Let us define the Hilbert space[9] $\mathscr{H}_{\mathbb{C}\mathrm{D}}$ as the image of the map φ applied to every element in $\widetilde{\mathscr{H}}$:

$$\mathscr{H}_{\mathbb{C}\mathrm{D}} \stackrel{\text{def}}{=} \operatorname{Ran} \varphi.$$

The scalar product of $\mathscr{H}_{\mathbb{C}\mathrm{D}}$ is obtained by carrying over the scalar product of $\widetilde{\mathscr{H}}$ via the map φ. Similarly, the representation $\widetilde{\pi}$ of $\mathbf{ISpin}(4, \mathbb{C})$ on $\widetilde{\mathscr{H}}$ is carried by the map φ into the representation $\pi_{\mathbb{C}\mathrm{D}}$ of $\mathbf{ISpin}(4, \mathbb{C})$ on $\mathscr{H}_{\mathbb{C}\mathrm{D}}$. We now explicitly compute the form of the representation $\pi_{\mathbb{C}\mathrm{D}}$ and of the scalar product of $\mathscr{H}_{\mathbb{C}\mathrm{D}}$.

For the representation $\pi_{\mathbb{C}\mathrm{D}}$ we get, using the definition of φ,

$$\begin{aligned}
\big(\pi_{\mathbb{C}\mathrm{D}}(n, s)\Phi\big)(\mathbf{p}', U, \mathbf{q}) &= \big(\varphi \widetilde{\pi}(n, s)\varphi^{-1}\Phi\big)(\mathbf{p}', U, \mathbf{q}) \\
&= \big(\widetilde{\pi}(n, s)\varphi^{-1}\Phi\big)(\sigma(v)\sigma(q)\sigma(v)^*, U, \mathbf{q}) \\
&= \chi_v(n) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \big(\varphi^{-1}\Phi\big)\big(A\sigma(v)B^{-1}B\sigma(q)B^*B^{*-1}\sigma(v)^*A^*, U, \mathbf{\Lambda}_{\phi(B)^{-1}}(\mathbf{q})\big) \\
&= \chi_v(n) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \big(\varphi^{-1}\Phi\big)\big(A\sigma(v)\sigma(q)\sigma(v)^*A^*, U, \mathbf{\Lambda}_{B^{-1}}(\mathbf{q})\big) \\
&= \chi_v(n) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Phi\big(A\sigma(p')A^*, U, \mathbf{\Lambda}_{B^{-1}}(\mathbf{q})\big).
\end{aligned}$$

We note that, to pass from the second to the third line, we used the fact that, by construction, we have the following transformation properties

$$\sigma(v) \mapsto A\sigma(v)B^{-1}, \quad \sigma(q) \mapsto B\sigma(q)B^*$$

under the action of an element $s = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathbf{SL}(2, \mathbb{C}) \times \mathbf{SL}(2, \mathbb{C}) \cong \mathbf{Spin}(4, \mathbb{C})$.

Now we consider the scalar product on $\mathscr{H}_{\mathbb{C}\mathrm{D}}$. Because of the definition of change of variables in (V.30), it is easy to see that we get

$$\begin{aligned}
(\Phi_1, \Phi_2)_{\mathscr{H}_{\mathbb{C}\mathrm{D}}} &= (\varphi^{-1}\Phi_1, \varphi^{-1}\Phi_2)_{\widetilde{\mathscr{H}}} \\
&= \int_{\mathscr{O}_{\check{v}(m)}} \left(\Phi_1(\mathbf{p}', U, \mathbf{q}), \begin{pmatrix} \frac{1}{m}\sigma(p')^{-1} & 0 \\ 0 & \frac{1}{m}\sigma(q)^{-1} \end{pmatrix} \Phi_2(\mathbf{p}', U, \mathbf{q})\right)_{\mathbb{C}^4} \times \\
&\qquad \times \frac{1}{2\omega(\mathbf{q})}\, \mathrm{d}\mathbf{q} \frac{1}{2\omega(\mathbf{p}')}\, \mathrm{d}\mathbf{p}'\, \mathrm{d}\mu_{\mathbf{SU}(2)}(U).
\end{aligned}$$

where we used the fact that the measure $\frac{1}{2\omega(\mathbf{p}')}\,\mathrm{d}\mathbf{p}'$ is invariant under the change of variables φ because such a change of variables corresponds to a Lorentz transformation.

We collect the properties of this new representation $(\mathscr{H}_{\mathbb{C}\mathrm{D}}, \pi_{\mathbb{C}\mathrm{D}})$ in the following proposition.

§ 5.8 Proposition. Let $\sigma : \mathbb{C}^4 \xrightarrow{\cong} \mathtt{M}_2(\mathbb{C})$ be the vector space isomorphism of $\mathbb{C}^4$ with the space $\mathtt{M}_2(\mathbb{C})$ of 2-by-2 complex matrices defined above (cf. Remark in §4.3). Hence we identify $v \in \mathbb{C}^4$ with $\sigma(v) \in \mathtt{M}_2(\mathbb{C})$. Moreover let us identify $\mathbf{Spin}(4, \mathbb{C}) \cong \mathbf{SL}(2, \mathbb{C}) \times \mathbf{SL}(2, \mathbb{C})$ in such a way that it acts on $\mathbb{C}^4 \cong \mathtt{M}_2(\mathbb{C})$ by

$$\sigma(v) \mapsto \lambda(s, v) = A\sigma(v)B^{-1}, \quad s = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad v \in \mathbb{C}^4.$$

[9] The subscript $\mathbb{C}\mathrm{D}$ in $\mathscr{H}_{\mathbb{C}\mathrm{D}}$ is for "complexified Dirac".

Then we can define a unitary representation $(\mathscr{H}_{\mathbb{C}\mathrm{D}}, \pi_{\mathbb{C}\mathrm{D}})$ of $\mathbf{ISpin}(4,\mathbb{C})$ by the following. The complex Hilbert space $\mathscr{H}_{\mathbb{C}\mathrm{D}}$ is the completion of the space of $\mathbb{C}^4$-valued, smooth, compactly supported functions $C_0^\infty(\mathbb{R}^3 \times \mathbf{SU}(2) \times \mathbb{R}^3; \mathbb{C}^4)$ on $\mathbb{R}^3 \times \mathbf{SU}(2) \times \mathbb{R}^3$ under the Hermitian scalar product

$$(\Phi_1, \Phi_2)_{\mathscr{H}_{\mathbb{C}\mathrm{D}}} = \int_{\mathscr{O}_{\mathring{v}(m)}} \left(\Phi_1(\mathbf{p}', U, \mathbf{q}), \begin{pmatrix} \frac{1}{m}\sigma(p')^{-1} & 0 \\ 0 & \frac{1}{m}\sigma(q)^{-1} \end{pmatrix} \Phi_2(\mathbf{p}', U, \mathbf{q})\right)_{\mathbb{C}^4} \times$$
$$\times \frac{1}{2\omega(\mathbf{q})}\,\mathrm{d}\mathbf{q} \frac{1}{2\omega(\mathbf{p}')}\,\mathrm{d}\mathbf{p}'\,\mathrm{d}\mu_{\mathbf{SU}(2)}(U). \quad \text{(V.31)}$$

The representation $\pi_{\mathbb{C}\mathrm{D}}$ is given by

$$\big(\pi_{\mathbb{C}\mathrm{D}}(n,s)\Phi\big)(\mathbf{p}, U, \mathbf{q}) = \chi_v(n) \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Phi\big(A\sigma(p)A^*, U, B\sigma(\mathbf{q})B^*\big), \quad \text{(V.32)}$$

where we recall that $\chi_{\mathring{v}}$ denotes the character of the additive group $\mathbb{C}^4 \cong \mathbb{R}^8$ relative to the vector $\mathring{v} = (m, 0, 0, 0)$, $m > 0$ (cf. §**4.1** point 3.) and

$$\Phi \in \mathscr{H}_{\mathbb{C}\mathrm{D}}, \quad A, B \in \mathbf{SL}(2,\mathbb{C}), \quad p = (\sqrt{\mathbf{p}^2 + m^2}, \mathbf{p}),\ q = (\sqrt{\mathbf{q}^2 + m^2}, \mathbf{q}), \quad \mathbf{p}, \mathbf{q} \in \mathbb{R}^3. \qquad \square$$

§ 5.9 Remark. As we have discussed, if we denote by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and element of $\mathbf{SL}(2,\mathbb{C}) \times \mathbf{SL}(2,\mathbb{C}) \cong \mathbf{Spin}(4,\mathbb{C})$, then the subgroup of $\mathbf{Spin}(4,\mathbb{C})$ which corresponds to real Lorentz spin transformations is given by elements of the form $\begin{pmatrix} A & 0 \\ 0 & A^{*-1} \end{pmatrix}$, for $A \in \mathbf{SL}(2,\mathbb{C})$. The scalar product in (V.31) bears a similar relation with the scalar product which appears in the 1-particle Hilbert space for the standard Dirac field. Indeed, with notation as in (V.31)), if we fix the variable $\mathbf{q}$ such that it satisfy

$$\sigma(q) = \sigma(p')^{*-1},$$

we obtain, by a straight forward computation,

$$\begin{pmatrix} \frac{1}{m}\sigma(p')^{-1} & 0 \\ 0 & \frac{1}{m}\sigma(q)^{-1} \end{pmatrix} = \begin{pmatrix} \frac{1}{m}\sigma(p')^{-1} & 0 \\ 0 & \frac{1}{m}\sigma(p')^{*} \end{pmatrix} = m\,\mathbb{I}_4 + \omega(\mathbf{p}')\gamma_0^{Weyl} - \sum_{j=1}^{3} p'_j \gamma_j^{Weyl},$$

where $\omega(\mathbf{p}') = \sqrt{\mathbf{p}'^2 + m^2}$, $\mathbf{p}' = (p'_1, p'_2, p'_3) \in \mathbb{R}^3$ and γ_μ^{Weyl}, $\mu = 0, 1, 2, 3$, denote the Dirac γ-matrices in Weyl basis. Now, the rightmost term in the previous expression is precisely the matrix appearing in the scalar product which we encountered in §**4.12**, (III.12). The relation of that scalar product with the theory of Wightman and Schwinger 2-point functions is given in that chapter (cf. especially §**4.13** and §**5.19**).

Hence we see in what sense this construction generalizes Wightman theory for the free Dirac fields to complex spacetime and the complexified Poincaré spinor group.

Fock space

In this final subsection we finally introduce the Fock space on the 1-particle space $\mathscr{H}_{\mathbb{C}\mathrm{D}}$ which is the carrier space of the representation described in §**5.8**. Moreover we apply the results from section 2 to the present situation.

§ 5.10 Proposition. Let $(\mathscr{H}_{\mathbb{C}\mathrm{D}}, \pi_{\mathbb{C}\mathrm{D}})$ be the representation of $\mathbf{ISpin}(4,\mathbb{C})$ given in proposition §**5.8**. Then

1. The complex Hilbert space $\mathscr{H}_{\mathbb{C}\mathrm{D}}$ decomposes in a Hilbert tensor product $\mathscr{H}_{\mathbb{C}\mathrm{D}} = \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ where

$$\mathcal{H}_1 = L^2(\mathbf{SU}(2),\, \mathrm{d}\mu_{\mathbf{SU}(2)}),$$

and $\mathcal{H}_2$ is the completion of the space of $\mathbb{C}^4$-valued, smooth, compactly supported functions $C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{C}^4)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ with respect to the scalar product

$$(\psi_1, \psi_2)_{\mathcal{H}_2} \stackrel{\text{def}}{=} \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} \left(\psi_1(\mathbf{p}', \mathbf{q}), \begin{pmatrix} \frac{1}{m}\sigma(p')^{-1} & 0 \\ 0 & \frac{1}{m}\sigma(q)^{-1} \end{pmatrix} \psi_2(\mathbf{p}', \mathbf{q})\right)_{\mathbb{C}^4} \frac{1}{2\omega(\mathbf{q})}\, \mathrm{d}\mathbf{q} \frac{1}{2\omega(\mathbf{p}')}\, \mathrm{d}\mathbf{p}',$$

where $\psi_1, \psi_2 \in \mathcal{H}_2$ and the rest of the notation is as in §**5.8**.

2. Consider the Bosonic Fock space over $\mathscr{H}_{\mathbb{C}\mathrm{D}}$, namely the space

$$\Gamma_\odot \mathscr{H}_{\mathbb{C}\mathrm{D}}.$$

Let $(e_j)_{j\in\mathbb{N}_+}$ be an ordered orthonormal basis for $\mathcal{H}_1$ and let P be the projection of section 2 §**2.10**. Then the Fermionic Fock space for the free relativistic Dirac field is obtained by applying P to $\Gamma_\odot \mathscr{H}_{\mathbb{C}\mathrm{D}}$ and then restricting to functions for which $\sigma(q) = \sigma(p')^{*-1}$ in the sense of §**5.9**.

Proof. The first statement follows from the fact that the scalar product of $\mathscr{H}_{\mathbb{C}\mathrm{D}}$ does *not* mix the variable $U \in \mathbf{SU}(2)$ with the other variables. Hence $\mathscr{H}_{\mathbb{C}\mathrm{D}}$ has a canonical decomposition into the L^2 space with respect to the variable $U \in \mathbf{SU}(2)$ and the remaining Hilbert space in the remaining variables.

The second statement follows from the proposition §**2.10** in section 2 and the remark §**5.9**. □

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VI

On the Faddeev-Popov quantization of gauge theories and Euclidean quantum radiation field

Abstract

We first give a brief review on the standard aspects of classical Euclidean gauge theories with particular emphasis on some relatively recent results regarding the smooth structure of the space of gauge potentials. Then, we formulate a simple approach, in Euclidean space time, to rigorously define the procedure of Faddeev-Popov quantization, for simple gauge theories which do *not* need renormalization. We call this approach "*naive Faddeev-Popov quantization*". Finally we turn to the example of the free Euclidean electromagnetic radiation field. We describe its quantization in two ways. First we describe, in our terms, the approach in which one takes the quotient of the space of gauge potentials by the group of gauge transformations. Second we apply our "*naive Faddeev-Popov quantization*" to provide the quantization of the Euclidean electromagnetic radiation field.

Contents

1 Introduction

In the seminal work [23] Faddeev and Popov gave a formal prescription for the quantization of gauge theories in the context of perturbation theory. Their original interest was toward Yang-Mill gauge theory in Minkowski space-time and involved formal manipulations of Feynman path-integrals. Perhaps the most successful application in theoretical Physics of the Faddeev-Popov formal technique is the description of weak interactions (cf. e.g. the exposition by Georgi [26]).

Feynman path integral techniques are often studied in the Euclidean setting ([54], [27]) where one hopes to cast Feynman's original formal idea into a rigorous technique within the realm of measure theory. For this reason here we start directly from the Euclidean formulation of gauge theories.

The objective we want to achieve here is two fold. On one hand, we describe, in general terms, the quantization of gauge theories employing the Faddeev-Popov ideas stressing what can be retained in a rigorous measure theoretic context from the formal approach. Our discussion tries to be as natural as possible thus avoiding ad hoc constructions. Moreover we try to be "model agnostic", in such a way that the techniques described here would remain applicable in almost automatic way in different models. Our discussion assumes that very strong conditions are fulfilled by the gauge theory which is to be quantized. For these reason we like to call our approach a "naive Faddeev-Popov quantization". The case of Yang-Mills gauge theory, without regularizations remains outside the possibilities of the rigorous presentation we give here.

On the other hand, we apply our construction to the simplest of the physically relevant problems: the quantized Maxwell (free, electromagnetic) field in four (Euclidean) dimensions. This case we study in full detail.

This chapter serves as a starting point for future research where we plan to combine the Fermionic techniques explained in the previous chapters with techniques from (Bosonic) gauge theories, in particular those of the present chapter.

The geometry of *classical* Euclidean gauge theories is by now quite well understood (cf. e.g. [24, 22, 50]). The geometry of the space of configuration of the classical gauge field is of central importance in any attempt to quantization. The classical situation is investigated, among others, in [19, 1, 39, 37, 38, 2, 3] cf. also [21]. Perhaps of particular interest, is the fact that the space of (classical) gauge potentials has the structure of an infinite dimensional **G**-bundle, where the group **G** is the infinite dimensional Lie group of gauge transformations (cf. [15, 3]). Moreover the kinetic term of the Lagrangian density arises naturally as a Riemannian metric on the orbit space (that is the quotient of the space of (classical) gauge potentials and the group of gauge transformations). We touch upon these results in section 3.

The situation for *quantum* (Euclidean) gauge theories is much less understood (cf. however [34]). We cite the following references to give a feeling of the applicability of the ideas developed here and of the directions and generalizations which we intend to pursue in the future: [5, 6, 7, 8, 9, 10, 11, 13, 14, 18, 28, 40, 52]

The problem of "quantization" of the Euclidean Maxwell free field has been studied by numerous authors. In particular we mention [32, 25, 30, 20, 12, 33, 49, 29, 58, 41]. For more details regarding how the approaches in these reference relate to our "naive Faddeev-Popov" quantization compare sections 5 and 6.

This chapter roughly splits into two parts. The first part corresponds to sections 2, 3, and 4 and describes gauge theories in general. Sections 6 and 7 form the second part where we specialize to the Maxwell gauge theory. Finally section 5 connects the two parts.

In more details, the content is as follows. Section 2 and 3 describe the general settings for *classical Euclidean gauge theories*. In these sections we are necessarily elliptic in our presentation. Nevertheless, in section 3, we try to touch what we believe are important modern results regarding the topological and geometrical properties of the infinite dimensional manifold which describes the configuration space for gauge fields. Section 4 describes our naive Faddeev-Popov quantization. In section 5 we briefly explain the simplifications which take place when one specializes from a generic (non Abelian) gauge theory to the Abelian Maxwell gauge theory. In section 6 we describe an approach for the quantization of the Maxwell field in which one takes the quotient of the space of gauge fields by the action of the group of gauge transformations. This approach is arguably simpler, than the approach we discuss next, but is restricted, to our knowledge, to just the Maxwell gauge theory, where the space of gauge potentials factors into a Cartesian product. In section 7 we apply our naive Faddeev-Popov quantization to the quantization of the Euclidean Maxwell free field. The results here are, to our knowledge, entirely new. The main results are formulated in theorem §**7.15**, theorem §**7.18**, and theorem §**7.19**.

2 Differential geometric setting for gauge theories

In this section we fix some differential geometric notation which we need in the sequel.

§ 2.1 Let $\mathcal{E} = (\mathrm{E}, \pi, \mathrm{M}, \mathrm{F}, \mathbf{G}, \alpha)$ be **fiber G-bundle** with total space a manifold E, projection map π, base manifold M, standard fiber a manifold F, structure group a Lie group $\mathbf{G}$, and group action α. A fiber $\mathbf{G}$-bundle $\mathcal{E}$ is a fiber bundle $\pi : \mathrm{E} \to \mathrm{M}$ for which the transition functions take values in the image

$\operatorname{Ran}\alpha = \alpha(\mathbf{G})$ under a smooth homomorphism $\alpha : \mathbf{G} \to \operatorname{Diff}\mathrm{F}$ of a Lie group $\mathbf{G}$ into the diffeomorphisms $\operatorname{Diff}\mathrm{F}$ of the standard fiber F. A **vector G-bundle** is a fiber $\mathbf{G}$-bundle where the standard fiber is a vector space and the action of the structure group on the fiber is a linear representation. A **principal G-bundle** is a fiber $\mathbf{G}$-bundle where the standard fiber is a copy of the structure group and the action of the structure group on itself is by left translation. To distinguish between these three cases we will denote a generic *fiber* $\mathbf{G}$-bundle by $\mathcal{E}$, a generic *vector* $\mathbf{G}$-bundle by η, and a generic *principal* $\mathbf{G}$-bundle by $\mathcal{P}$.

§ 2.2 Let $\mathcal{P}$ be a principal $\mathbf{G}$-bundle over a manifold M. Let $\mathfrak{ad}\mathcal{P}$ be the **adjoint bundle** of $\mathcal{P}$, that is a *vector* $\mathbf{G}$-bundle with fiber the Lie algebra $\mathfrak{g}$ and as action of $\mathbf{G}$ the adjoint representation of $\mathbf{G}$ on $\mathfrak{g}$. Moreover let $\operatorname{Ad}\mathcal{P}$ denote the *fiber* $\mathbf{G}$-bundle with standard fiber a copy of the structure group and action given by conjugation (i.e. an element $g \in \mathbf{G}$ acts on another element $h \in \mathbf{G}$ by $h \mapsto ghg^{-1}$). A warning about notation. The representation of $\mathbf{G}$, by which $\mathbf{G}$ acts on the fibers in the adjoint bundle $\mathfrak{ad}\mathcal{P}$, is the adjoint representation of the Lie group which is usually denoted by Ad. Nevertheless the adjoint bundle is *not* usually denoted by $\operatorname{Ad}\mathcal{P}$. The notation $\operatorname{Ad}\mathcal{P}$ is usually reserved for the adjoint fiber $\mathbf{G}$ in which, since it is *not* a vector bundle, the action of $\mathbf{G}$ on the fiber does *not* constitute a (linear) representation.

§ 2.3 Let us denote by $\Omega^p(\mathrm{M},\eta)$, $p \in \mathbb{N}$, the (global, smooth) p-forms on the manifold M with values in the vector bundle η. That is, we let

$$\Omega^p(\mathrm{M},\eta) \overset{\text{def}}{=} \Gamma(\eta \otimes_{\mathrm{M}} \Lambda^p\mathrm{M}),$$

where the right hand side denotes the space of (global, smooth) sections of the tensor product bundle $\eta \otimes_{\mathrm{M}} \Lambda^p\mathrm{M}$. The tensor product bundle is the vector bundle over M with standard fiber the vector space $\mathrm{F} \otimes \bigwedge^p(\mathbb{R}^{\mathrm{dimM}})$, where F denotes the standard fiber of η and $\bigwedge^p(\mathbb{R}^{\mathrm{dimM}})$ denotes the vector space of p-forms on $\mathbb{R}^{\mathrm{dimM}}$.

§ 2.4 Let us denote by $\mathcal{C}(\mathcal{P})$ the space of connections over the principal $\mathbf{G}$-bundle $\mathcal{P}$. We will not need to work directly with this space, hence we limit ourselves to refer to [1] for its description. The space $\mathcal{C}(\mathcal{P})$ has an *affine space* structure. To go back to the usual *vector space* structure we distinguish between *connections* and *gauge-potentials*. We call **(smooth) gauge-potential** the difference of two connections. Then the **space of (smooth) gauge-potentials** is a *vector space* which we denote by $\mathcal{A}$. One has the standard identification

$$\mathcal{A} = \Omega^1(\mathrm{M}, \mathfrak{ad}\mathcal{P}).$$

of the space of gauge-potentials $\mathcal{A}$ with the vector space of one-forms over M with values in the vector bundle $\mathfrak{ad}\mathcal{P}$ over M. We will use this identification as our *definition* of $\mathcal{A}$.

§ 2.5 Let $\mathcal{P}$ be a principal bundle with structure group $\mathbf{G}$ and total space P. We denote by $\operatorname{Gau}\mathcal{P}$ the **group of gauge transformations**[1], that is the group (under fiber-wise multiplication) of $\mathbf{G}$-equivariant, *fiber preserving*, diffeomorphisms $\mathrm{P} \to \mathrm{P}$. One has the canonical identification (cf. e.g. [47, Lemma 4.1.2, p. 82])

$$\operatorname{Gau}\mathcal{P} = \Gamma(\operatorname{Ad}\mathcal{P}),$$

where $\Gamma(\operatorname{Ad}\mathcal{P})$ denotes the space of smooth sections of the fiber bundle $\operatorname{Ad}\mathcal{P}$. $\Gamma(\operatorname{Ad}\mathcal{P})$ is considered as a group under fiber-wise multiplication. Again, we will use this identification as our *definition* of the group $\operatorname{Gau}\mathcal{P}$ of gauge transformations.

§ 2.6 To describe the action of the group of gauge transformations $\operatorname{Gau}\mathcal{P}$ on the vector space of gauge potentials $\mathcal{A}$, we pass to the local picture. For the global description we refer to [3, 1]. Let $\{U_\alpha\}$ be an atlas for the base manifold M of the vector bundle $\mathfrak{ad}\mathcal{P}$ and $A \in \mathcal{A}$. Then, on a given chart U_α, the $\mathfrak{g}$-valued one form A is just a vector valued function:

$$A\restriction_{U_\alpha} : U_\alpha \to \mathfrak{g} \otimes \mathbb{R}^{\dim M}.$$

[1] In some of the literature one finds $\operatorname{Aut}\mathcal{P}$ denoting what we here denote by $\operatorname{Gau}\mathcal{P}$.

Similarly let $g \in \operatorname{Gau}\mathcal{P}$, then on a chart U_α the section g of $\operatorname{Ad}\mathcal{P}$ is just a $\mathbf{G}$-valued function

$$g{\restriction_{U_\alpha}} : U_\alpha \to \mathbf{G}.$$

We *define* an action $\alpha : \operatorname{Gau}\mathcal{P} \times \mathcal{A} \to \mathcal{A}$ by imposing its behavior chart-wise:

$$\alpha(g{\restriction_{U_\alpha}}, A{\restriction_{U_\alpha}})(x) = g{\restriction_{U_\alpha}}(x) A{\restriction_{U_\alpha}}(x) g{\restriction_{U_\alpha}}(x)^{-1} + \partial g{\restriction_{U_\alpha}}(x) g{\restriction_{U_\alpha}}(x)^{-1}.$$

3 Infinite dimensional manifold structure

In this section we briefly examine the smooth structure introduced by Michor ([43, 45, 44, 42] and [46]) on the set $C^\infty(\mathrm{M},\mathrm{N})$ of smooth maps between two (smooth, finite dimensional) manifolds M and N. Then we discuss the situations where, in place of $C^\infty(\mathrm{M},\mathrm{N})$, one specializes to the cases of the space $\mathcal{A}$ of gauge potentials and the group $\operatorname{Gau}\mathcal{P}$ of gauge transformations.

§ 3.1 Let $C^0(\mathrm{M},\mathrm{N})$ be the set of continuous functions between two manifolds M, N. On $C^0(\mathrm{M},\mathrm{N})$ the **locally finite open topology** (**LO-topology** for short) is the topology with basis

$$\{f \in C(\mathrm{M},\mathrm{N}) : f(L_\alpha) \subset U_\alpha\},$$

where $L = (L_\alpha)$ is a locally finite family of closed subsets $L_\alpha \subset \mathrm{M}$, and $U = (U_\alpha)$ is a family of open subsets of N with the same index set.

§ 3.2 Consider a function $f \in C^\infty(\mathrm{M},\mathrm{N})$, that is a smooth function from M to N. We denote by $T^\ell f$ the ℓ-th derivative of f. For example $T^0 f = f$ and $T^1 f = Tf$ denotes the push-forward (also called Jacobian) of f. For a fixed $k \in \mathbb{N}$, let us introduce the k-jet $j^k f(x)$ of $f \in C^\infty(\mathrm{M},\mathrm{N})$ at $x \in \mathrm{M}$:

$$j^k f(x) \overset{\text{def}}{=} [f, x]_k,$$

where $[f, x]_k$ denotes the equivalence class of all pairs (g, y) $g \in C^\infty(\mathrm{M},\mathrm{N})$, $y \in \mathrm{M}$, such that $y = x$ and and $T^\ell g = T^\ell f$, for all $0 \le \ell \le k$. The set of all k-jets from M to N is denoted by $J^k(\mathrm{M},\mathrm{N})$. Given $f \in C^\infty(\mathrm{M},\mathrm{N})$, we can identify the equivalence class $j_k f(x)$ with the list of Taylor coefficients which, by definition, characterize the class. The zero order coefficient is just a point in N. In this way, we see that one can consider $J(\mathrm{M},\mathrm{N})$ as a vector bundle over $\mathrm{M}\times\mathrm{N}$, with standard fiber the vector space of polynomial of order up to k without constant coefficient. In particular $J^0(\mathrm{M},\mathrm{N}) = \mathrm{M}\times\mathrm{N}$. As a consequence of this picture, we see that j^k is a map from $C^\infty(\mathrm{M},\mathrm{N})$ into $C^0(\mathrm{M}, J^k(\mathrm{M},\mathrm{N}))$.

Now, let us denote by $J^\infty(\mathrm{M},\mathrm{N})$ the projective limit (in the category of Hausdorff topological spaces) of the sequence

$$J^0(\mathrm{M},\mathrm{N}) \leftarrow J^1(\mathrm{M},\mathrm{N}) \leftarrow \cdots.$$

Then we have a well defined family of projections $\pi_k^\infty : J^\infty(\mathrm{M},\mathrm{N}) \to J^k(\mathrm{M},\mathrm{N})$. For $f \in C^\infty(\mathrm{M},\mathrm{N})$, $j^\infty f$ denotes the element in $J^\infty(\mathrm{M},\mathrm{N})$ such that $\pi_k^\infty j^\infty f(x) = j^k f(x)$, for all $0 \le k < \infty$, $x \in \mathrm{M}$. Similarly to the case of j^k, $0 \le k < \infty$, we interpret j^∞ as a map from $C^\infty(\mathrm{M},\mathrm{N})$ into $C^0(\mathrm{M}, J^\infty(\mathrm{M},\mathrm{N}))$.

§ 3.3 We define the $\mathscr{D}$-**topology** on $C^\infty(\mathrm{M},\mathrm{N})$ to be the topology induced by the embedding $j^\infty : C^\infty(\mathrm{M},\mathrm{N}) \to C^0(\mathrm{M}, J^\infty(\mathrm{M},\mathrm{N}))$ from the LO-topology (defined in §**3.1**).

Let $\mathscr{D}(\mathrm{M},\mathrm{N}) \overset{\text{def}}{=} C_c^\infty(\mathrm{M},\mathrm{N})$ be the space of smooth maps from M to N with compact support. Then $(\mathscr{D}(\mathrm{M},\mathrm{N}), \mathscr{D}$-topology) is a **nuclear LF-space** (that is LF-space which is also nuclear).

We now define Michor's $F\mathscr{D}$-topology on $C^\infty(\mathrm{M},\mathrm{N})$. To do so, we define the equivalence relation $f \sim g$, for $f, g \in C^\infty(\mathrm{M},\mathrm{N})$, by imposing f and g to be equivalent when the set $\{x \in \mathrm{M} : f(x) \neq g(x)\}$ is relatively compact in M.

The **fine** $\mathscr{D}$-**topology** (**F**$\mathscr{D}$-**topology** for short) is defined to be the coarsest topology on $C^\infty(\mathrm{M},\mathrm{N})$ which is finer than the $\mathscr{D}$-topology and for which the set of cosets of the equivalence relation above are open.

§ 3.4 Following [46, Chapter 9] we define the notion of C_c^∞-manifold as a Hausdorff topological space M equipped with a family $(U_\alpha, u_\alpha, E_\alpha)_{\alpha\in I}$ such that:

1. (U_α) is an open cover of M, and $u_\alpha : U_\alpha \to E_\alpha$, for all $\alpha \in I$, is a homeomorphism onto an open subset $\operatorname{Ran} u_\alpha$ of a locally convex topological vector space E_α;

2. for $U_\alpha \cap U_\beta \neq \emptyset$, the maps $u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}$ are C_c^∞ diffeomorphisms. As a consequence E_α and E_β, $\alpha, \beta \in I$ are linearly isomorphic.

By **NLF-manifold** we mean a C_c^∞-manifold where, for all $\alpha \in I$, the E_α are isomorphic to a nuclear LF-space E.

We now are ready to quote (without proof) two results which apply this general theory of infinite dimensional manifolds to the cases we need: the group $\operatorname{Gau}\mathcal{P}$ of gauge transformations and the space $\mathcal{A}$ of gauge potentials.

§ 3.5 Proposition ([19, 2]). Let $\mathcal{P}$ be a principal bundle with base manifold a (not necessarily compact) finite dimensional smooth manifold M and with structure group a (not necessarily compact) Lie group G. Then, the group of gauge transformations $\operatorname{Gau}\mathcal{P}$ equipped with the $\mathscr{D}$-topology is an NLF-manifold.

§ 3.6 Proposition ([46, Theorem 10.4]). Let $\mathcal{P}$ be a principal bundle with base manifold a (not necessarily compact) finite dimensional smooth manifold M and with structure group a (not necessarily compact, finite dimensional) Lie group G. Then the space $\mathcal{A}$ of gauge potentials equipped with $F\mathscr{D}$-topology is an NFL-vector space (that is an NFL-manifold which is also a topological vector space). Moreover the space $\mathcal{A}_c$ of gauge potentials with compact support, equipped with the $\mathscr{D}$-topology is a Fréchet nuclear vector space.

4 A naive Faddeev-Popov quantization

In this section is we give a "naive" but rigorous interpretation of the formal Faddeev-Popov ideas. By "naive", we mean that the rigorous approach described here can be expected to work only whenever *no* renormalization is needed.

§ 4.1 Remark. Loosely speaking, in classical mechanics, we are presented with a space of states and an action functional defined on it. The classical trajectories are determined by minimizing the action functional. The idea of Feynman can be rephrased informally in the following way. Starting with the action functional, we construct a notion of integration, or of averaging, on the space of states. The averaging process should have the property that we can recover the classical minimization process as a "linear approximation". In the context of gauge theories we can consider the space $\mathcal{A}$ of gauge potentials as "**total space of states**". It is very convenient, both from mathematical and physical reasons, to consider theory which are gauge invariant. This implies that the space of gauge potentials includes states (i.e. gauge fields) which are gauge equivalent, that is, they are connected by a gauge transformation. In defining a notion of averaging in the space of gauge potentials, one is thus faced with the problem of how to account for these equivalent states. We have (at least) two options to define a notion of averaging in this context.

One is to quotient out the action of the group $\operatorname{Gau}\mathcal{P}$ of gauge transformations and to consider the **orbit space** $\mathcal{A}/\operatorname{Gau}\mathcal{P}$ as "true configuration space" . The geometry of this "true configuration space" is studied e.g. in [15]. The issue with this space is that in general it has a complicated structure (cf. e.g. [3]) whereas the space $\mathcal{A}$ of gauge potentials is a "simple" topological vector space . The second option is to keep the total space $\mathcal{A}$ of gauge potentials (or the space $\mathcal{A}_c$ of compactly supported gauge potentials) and develop a notion of integration there. The formal approach of Faddeev-Popov opts for this second path.

§ 4.2 To make the ideas described in the previous paragraph rigorous, we need a well defined notion of integration on an infinite dimensional space. Within the context of measure theory, one possible way to attack this problem is via the powerful Kolmogorov theorem.

Thanks to this theorem, we can give a "naive" but rigorous formulation of the idea of Faddeev and Popov on how to implement Feynman quantization approach for gauge theories which satisfy some restrictive hypothesis which allow for the program to go through. The formulation that we discuss here is rigorous but limited in scope, hence the epithet "naive".

In the last section we will apply the approach described here to a basic though interesting example: the quantization of the Maxwell field in Euclidean four-dimensional space-time. It would be interesting to apply this formulation of Faddeev-Popov quantization to other examples, where the geometry of gauge invariance would play a more prominent role.

§ 4.3 The notion of action functional is of central importance in theoretical as well as mathematical physics. Let us give here a definition of this notion, which simply adapts to our setting. We call **action functional** a nonlinear functional (or, in other words, a function) $S : \mathcal{A} \to \mathbb{R}$. We call an action functional S **gauge invariant** when

$$S(\alpha(g, A)) = S(A), \quad \text{for all } g \in \operatorname{Gau}\mathcal{P} \text{ and } A \in \mathcal{A},$$

where $\alpha : \operatorname{Gau}\mathcal{P} \times \mathcal{A} \to \mathcal{A}$ denotes the action of the group $\operatorname{Gau}\mathcal{P}$ of gauge transformations on the space $\mathcal{A}$ of gauge potentials. Note that since the space $\mathcal{A}$ is a topological *vector* space, the notion of action as a (nonlinear) functional is globally well defined.

§ 4.4 Let X be a topological, locally convex, vector space. Following e.g. [16], we call a sequence $(e_n)_{n\in\mathbb{N}}$ os elements of X a **topological basis** for X when, for every $x \in X$, there exists a unique sequence of numbers $(c_n(x))_{n\in\mathbb{N}}$ such that $x = \sum_{n\in\mathbb{N}} c_n(x)e_n$, where the series converges in X. If all functionals $x \mapsto c_n(x)$ are continuous, then $(e_n)_{n\in\mathbb{N}}$ is called a **Schauder bases** for X.

§ 4.5 Let $(E^N, \mathscr{B}^N, \pi_{N,N+1})_{N\in\mathbb{N}}$ be a projective system of measurable spaces. For concreteness sake, we assume $E^N \subset E^{N+1}$ to be an increasing sequence of topological vector spaces, and $\mathscr{B}^N$ the Borel σ-algebra on E^N. Then $(\pi_{N,N+1})_{N\in\mathbb{N}}$ is the canonical family of continuous projections $E^{N+1} \to E^N$. Consider a sequence (μ^N) of measures where, for each $N \in \mathbb{N}$, μ^N is a Borel measure on E^N. The family $(\mu^N)_{N\in\mathbb{N}}$ is said to be **self-consistent** when

$$\pi_{N,N+1}\circ\mu^{N+1} = \mu^N, \quad \text{for all } N \in \mathbb{N},$$

where $\pi_{N,N+1}\circ\mu^{N+1}$ denotes the push-forward of the measure μ^{N+1} under the map $\pi_{N,N+1}$.

We state a simple version of the celebrated Kolmogorov theorem which serves our purposes (cf. e.g. [57, Corollary p. 39]).

§ 4.6 Proposition (Kolmogorov). Let $(E^N)_{N\in\mathbb{N}}$ be an increasing sequence of complete, separable, metric spaces. Then every self-consistent family of Borel, probability measures can be extended to a σ-additive probability measure μ on the projective limit measurable space $\varprojlim E^N$.

§ 4.7 Faddeev-Popov quantization. We are now ready to describe what we call "naive Faddeev-Popov quantization". The goal is to define an appropriate notion (cf. §**4.1**) of integration on the space $\mathcal{A}$ of gauge transformation. To construct such a notion of integration, within the context of measure theory, we have at our disposal the powerful tool given by the Kolmogorov theorem. So we will proceed by defining measures on "smaller spaces" and then use Kolmogorov to pass to the limit and define a measure on the full space $\mathcal{A}$ of gauge transformation.

We therefore need to define first a notion of "smaller spaces". To do so we follow the classical procedure in quantum field theory to introduce *two* regularization: an *infrared* regularization and an *ultraviolet* regularization.

Infrared regularization: In this approach to infrared regularization we use the topological structure of $\mathcal{A}$ to define an infrared regularization. As discussed above we can assume $\mathcal{A}$ to be equipped with a nuclear LF-space topology. In particular, being an LF-space, we will have a set of indices $\mathscr{I}$, and a family

$(\mathcal{A}_K)_{K\in\mathscr{I}}$ of Fréchet spaces, such that $\mathcal{A}$ is their direct limit $\mathcal{A} = \lim_{\rightarrow} \mathcal{A}_K$. We shall call the procedure of restricting the functionals and measures, yet to be constructed, to each $\mathcal{A}_K$, an infrared regularization. This type of infrared regularization, in section 7, will correspond to an *infrared cut-off*. We also note that in the section 7 it will become important to impose extra conditions on the family of spaces $(\mathcal{A}_K)_{K\in\mathscr{K}}$. In particular $\mathscr{I}$ will denote there a family of absorbing, connected, convex subsets of $\mathbb{R}^4$ with smooth boundaries.

Ultraviolet regularization: To introduce in a convenient way an ultraviolet regularization we introduce an extra structure, on the spaces $\mathcal{A}_K$. We assume thus, that we are give an Hilbert scalar product on each $\mathcal{A}_K$ and we denote by $\mathcal{H}_K$ the completion with respect to such scalar product. The appropriate choice of Hilbert structure will depend in particular on the action functional. For each $K \in \mathscr{I}$ we choose a Hilbert basis $(e_n^K)_{n\in\mathbb{N}_+}$ of $\mathcal{H}_K$. We choose as ultraviolet regularization the restriction from the (infinite dimensional) Hilbert spaces $\mathcal{H}_K$, $K \in \mathscr{I}$, to the finite dimensional subspaces

$$V_K^N \stackrel{\text{def}}{=} \operatorname{Span}\big(e_1^K, \dots, e_N^K\big).$$

Let $P_{N,K} : \mathcal{H}_K \to V_K^N$ be the orthogonal projection on the finite dimensional subspace V_K^N. Denote by $\mathcal{G}$ the group Gau $\mathcal{P}$ of gauge transformations when considered as a transformation group, that is, denoting by $\alpha : \operatorname{Gau}\mathcal{P}\times\mathcal{A} \to \mathcal{A}$ the action of Gau $\mathcal{P}$ on $\mathcal{A}$, we let

$$\mathcal{G} \stackrel{\text{def}}{=} \big\{\alpha(g,\cdot) : g \in \operatorname{Gau}\mathcal{P}\big\},$$

where, $\alpha(g,\cdot)$, for $g \in \operatorname{Gau}\mathcal{P}$, denotes a map $\mathcal{A} \to \mathcal{A}$, and the (group) composition law is given by letting $\alpha(g,\cdot)\circ\alpha(g',\cdot) \stackrel{\text{def}}{=} \alpha(gg',\cdot)$. In the following we will identify Gau $\mathcal{P}$ with $\mathcal{G}$. In particular, by abuse of notation, we will denote elements of $\mathcal{G}$ by $g, g', \dots$, as we do for Gau $\mathcal{P}$, and we will denote the group product by simple juxtaposition $(g,g') \mapsto gg'$, as we do for Gau $\mathcal{G}$. Define the **truncated gauge transformation groups** $\mathcal{G}_K^N$ by

$$\mathcal{G}_K^N \stackrel{\text{def}}{=} P_{N,K}\,\mathcal{G}\,P_{N,K} = \{P_{N,K}\,g\,P_{N,K} : g \in \mathcal{G}\},$$

where $g \in \mathcal{G}$ is now understood as a transformation on $\mathcal{A}$. Note that, for $N \in \mathbb{N}$, $K \in \mathscr{I}$, $\mathcal{G}_K^N$ is a Lie group under the **truncated product**: $(g_1g_2)^N \stackrel{\text{def}}{=} P_K^N g_1 g_2 P_K^N$, $g_1, g_2 \in \operatorname{Gau}\mathcal{P}$.

Let us denote by $A^{K,N}$ the projection of $A \in \mathcal{A}$ on the subspace V_K^N, that is $A^{K,N} \stackrel{\text{def}}{=} P_{N,K}AP_{N,K}$. Moreover denote by $\mathrm{d}A^{K,N}$ the Lebesgue measure on V_K^N and by $\mathrm{d}g^{K,N}$ the Haar measure on $\mathcal{G}_K^N$.

Faddeev-Popov functional: Let us introduce a nonlinear functional $F_{\text{FP}} : \mathcal{A} \to [0,+\infty)$, which we shall call **Faddeev-Popov functional**, which is assumed to satisfy the following condition:

$$\lim_{N\to+\infty} \frac{\int_{\mathcal{G}_K^N} F_{\text{FP}}^{K,N}(g^{K,N},0)\,\mathrm{d}g^{K,N}}{\int_{\mathcal{G}_K^N} F_{\text{FP}}^{K,N}(g^{K,N},A^{K,N}))\,\mathrm{d}g^{K,N}} < +\infty, \quad \text{for all } A^{K,N} \in V_K^N, \tag{VI.1}$$

where

$$F_{\text{FP}}^{K,N}(g^{K,N},A^{K,N}) \stackrel{\text{def}}{=} F_{\text{FP}}\left(P_{K,N}\,\alpha(g^{K,N},A^{K,N})\right),$$
$$A^{K,N} \in V_K^N,\, g^{K,N} \in \mathcal{G}_K^N,\, N \in \mathbb{N}_+,\, K \in \mathscr{I}, \tag{VI.2}$$

denote the “finite dimensional approximations” of F_{FP}. where $\alpha : \operatorname{Gau}\mathcal{P}\times\mathcal{A} \to \mathcal{A}$ denotes the action of Gau $\mathcal{P}$ on $\mathcal{A}$.

Projective system of measures: We define the following family of positive measures

$$\mathrm{d}\nu^{K,N} \stackrel{\text{def}}{=} \Bigg(\frac{\int_{\mathcal{G}_K^N} F_{\text{FP}}^{K,N}(g^{K,N},0)\,\mathrm{d}g^{K,N}}{\int_{\mathcal{G}_K^N} F_{\text{FP}}^{K,N}(g^{K,N},A^{K,N})\,\mathrm{d}g^{K,N}}\Bigg) F^{K,N}(A^{K,N})e^{-S^{K,N}(A^{K,N})}\,\mathrm{d}A^{K,N}.$$

Each of these $\mathrm{d}\nu^{K,N}$ is a well defined measure on the finite dimensional vector space $V^{K,N}$. The family of measures $\nu^{K,N}$, $N \in \mathbb{N}$, $K \in \mathscr{I}$, still needs to be normalized. We therefore define

$$\mathrm{d}\mu^{K,N} \stackrel{\text{def}}{=} \frac{\mathrm{d}\nu^{K,N}}{|\nu^{K,N}|\big(V_K^N\big)},$$

where $|\nu^{K,N}|$ denotes the total variation of the measure $\nu^{K,N}$. We now want to remove first the *ultraviolet regularization* (which will correspond to a projective limit $N \to \infty$) and second the *infrared regularization* (projective[2] limit in $\mathscr{I}$).

First,if we assume that the family of measures $(\nu^{K,N})_{N\in\mathbb{N}_+}$ defines a compatible system of probability measures for every $K \in \mathscr{I}$, then

by Kolmogorov theorem (§4.6), for every $K \in \mathscr{I}$, there exists a probability measure μ^K on the projective limit $\mathcal{B} \stackrel{\text{def}}{=} \lim_{\leftarrow} \mathcal{A}^N$ such that it agrees with each $\mu^{K,N}$ when projected on each $V^{K,N}$, $N \in \mathbb{N}$.

Second, if we assume that the new measures μ^K, $K \in \mathscr{I}$, also define a compatible system of measures, then, we apply Kolmogorov a second time to obtain a measure μ on the projective limit space.

This construction of a measure μ is what we call "**naive Faddeev-Popov quantization**". We shall apply this construction to a concrete example in the following section.

5 Quantization of Maxwell field in the Euclidean four-dimensional space-time

§ 5.1 In this section we will make two basic assumptions to be able to carry out our study at this stage.

The first assumption is that the base space M is assumed to be $\mathbb{R}^4$. From the homotopy perspective this is, of course, a great simplification because $\mathbb{R}^4$ is a contractible (paracompact, Hausdorff) space. Hence any fiber bundle is homotopy equivalent to the trivial (product) bundle (cf. [Corollary p. 102][53]). From the functional analysis perspective this assumption is not so much a simplification. In fact, $\mathbb{R}^4$ being non compact, it forces to deal with operators (e.g. the Laplacian) with continuous spectrum. From the physical perspective it is quite natural to assume the base manifold to be $\mathbb{R}^4$. In fact, it is often useful to make the working assumption that the large scale structure of space time does *not* play an important role in the description of elementary particle physics (although in might be necessary to relax this physical assumption in the future).

The second assumption, is to consider the structure group to be $U(1)$ (which of course means excluding, e.g., Yang-Mills fields). Let us mention that, starting with any principal bundle $\mathcal{P}$ with Abelian structure group, we obtain an adjoint vector bundle $\mathfrak{ad}\mathcal{P}$ equivalent to the trivial vector bundle $\mathcal{M} \times \mathfrak{g}$, where M is the base manifold of $\mathcal{P}$ and $\mathfrak{g}$ is the (trivial) Lie algebra of the Abelian structure group of $\mathcal{P}$. The most important effect of considering an Abelian structure group is that the space of connections (or the space of gauge potentials) factors with respect to the action of the group of gauge transformations. That is we have

$$\mathcal{A} = \mathrm{Gau}\,\mathcal{P} \times \mathcal{A}/\mathrm{Gau}\,\mathcal{P}.$$

Indeed, the group $\mathrm{Gau}\,\mathcal{P}$ of gauge transformations act on the vector space $\mathcal{A}$ of gauge potentials by translation by an exact 1-form:

$$\alpha(g, A) = A + d\alpha, \quad g \in \mathrm{Gau}\,\mathcal{P}, A \in \mathcal{A},$$

where $\lambda = \partial g g^{-1}$ is a zero-form and $g = \exp\lambda$, for some real valued smooth function λ on $\mathbb{R}^4$.

§ 5.2 As a result of the assumptions described in the last paragraph we are left with the following situation.

$$\mathcal{A} \cong C^\infty(\mathbb{R}^4; \mathbb{R}^4), \quad \mathrm{Gau}\,\mathcal{P} \cong \{d\lambda : \lambda \in C^\infty(\mathbb{R}^4; \mathbb{R})\},$$

[2]Note that $\mathcal{A}$ is assumed to be the *inductive* limit of the A^K, $K \in \mathscr{I}$. But, when we remove the regularization we want to look for a measure on the *projective* limit.

where we are identifying $\mathbb{R}^4$ with the space of one forms $\bigwedge^1\mathbb{R}^4$ over $\mathbb{R}^4$. Moreover, we are identifying Gau $\mathcal{P}$ with the space of (smooth) exact 1-forms on $\mathbb{R}^4$, whence, in this identification, Gau $\mathcal{P}$ is a topological vector subspace of $\mathcal{A}$.

Because of the simple geometry in this example, we can employ, in place of the approach we described in §**4.7**, a simpler approach which employs Hilbert space techniques. We will explain this in the following section (section 6). We go back to the more general approach described in §**4.7** in section 7.

6 Quantization of the Euclidean radiation field by taking the quotient of the "state space"

Literature. The "quantization procedure" which we describe in this subsection is closely related to the discussion in existing literature: [32, 25, 30, 20, 12, 33, 49, 29, 58, 41]. The main difference is that we will consider the gauge potentials one-forms as fundamental whereas in the literature cited above often the main focus is on the field two-forms.

§ 6.1 Let M be a *geodesically complete*, C^∞, Riemann manifold (for example $\mathbb{R}^4$). Let us denote by $C_0^\infty(\Lambda^k T^*\mathrm{M})$, $k\in\mathbb{N}$, the space of smooth k-forms over M with compact support. Similarly denote by $L^2(\Lambda^k T^*\mathrm{M}, \sqrt{\det g(x)}\,\mathrm{d}x)$ the space of square integrable k-forms on M with respect to the Riemannian volume measure $\sqrt{\det g(x)}\,\mathrm{d}x$. Let

$$\mathcal{H} \overset{\text{def}}{=} \widehat{\bigoplus_{k\in\mathbb{N}}} L^2(\Lambda^k T^*\mathrm{M}, \sqrt{\det g(x)}\,\mathrm{d}x),$$

where $\widehat{\bigoplus}$ denotes the (orthogonal, free) Hilbert direct sum. We denote by d the exterior differential defined on a dense domain $\mathcal{D}$ given by the union in $\mathcal{H}$ of the $C_0^\infty(\Lambda^k T^*\mathrm{M})$ for all $k\in\mathbb{N}$. We employ both standard notations d^* and δ to denote the formal adjoint in $\mathcal{H}$ of d. Let $\Delta \overset{\text{def}}{=} dd^* + d^*d$ be the Hodge-Laplacian[3]. We take $\mathcal{D}$ (as defined above) as dense domain of the whole algebra of differential operators generated by d, d^*. We assume that on $\mathcal{D}$ the Hodge Laplacian is essentially selfadjoint Let Δ_k, for each fixed $k = 0, 1, \dots, \dim \mathrm{M}$, be the restriction of Δ on k-forms (and restricting its dense domain accordingly) and by Δ_k^{cl} its unique selfadjoint closure in $\mathcal{H}$. Then we have the following *weak L^2-Hodge decomposition*.

§ 6.2 Hodge-Kodaira-Friedrichs decomposition. We have the following orthogonal decomposition (notation as above)

$$L^2(\Lambda^k T^*\mathrm{M}, \sqrt{\det g(x)}\,\mathrm{d}x) = H_{k,2}(\mathrm{M}) \oplus \overline{dC_c^\infty(\Lambda^{k-1}T^*\mathrm{M})} \oplus \overline{d^*C_c^\infty(\Lambda^{k+1}T^*\mathrm{M})},$$

where $H_{k,2} = \ker \Delta_k^{\mathrm{cl}}$ and the over-line denotes closure in $L^2(\Lambda^k T^*\mathrm{M}, \sqrt{\det g(x)}\,\mathrm{d}x)$.

PROOF. The proof goes back to Kodaira [36]. □

§ 6.3 Let us now specialize to the case $\mathrm{M} = \mathbb{R}^4$. From Kodaira proof, specialized to this case, we obtain the following facts. The following operators are orthogonal projection operators on $\mathcal{H}$,

$$P = \Delta^{-1}d^*d, \quad Q = \Delta^{-1}dd^*, \tag{VI.3}$$

where first we define P and Q on the dense domain $\mathcal{D}$ and then extend to the whole of $\mathcal{H}$. Note that $P + Q = \mathbb{I}_{\mathcal{H}}$. The weak L^2 decomposition of the proposition above, when $\mathrm{M} = \mathbb{R}^4$, becomes

$$L^2(\mathbb{R}^4; \Lambda^k\mathbb{R}^4) = \overline{dC_c^\infty(\mathbb{R}^4, \Lambda^{k-1}\mathbb{R}^4)} \oplus \overline{d^*C_c^\infty(\mathbb{R}^4, \Lambda^{k+1}\mathbb{R}^4)},, \tag{VI.4}$$

because we do not have harmonic forms in $L^2(\mathbb{R}^4; \Lambda^k\mathbb{R}^4)$. Moreover we have

$$\operatorname{Ran} P = \ker Q = \overline{d^*C_c^\infty(\mathbb{R}^4, \Lambda^{k+1}\mathbb{R}^4)}, \quad \operatorname{Ran} Q = \ker P = \overline{dC_c^\infty(\mathbb{R}^4, \Lambda^{k+1}\mathbb{R}^4)}. \tag{VI.5}$$

[3] In this convention (from complex geometry) the Hodge-Laplacian when evaluated on functions coincides with *minus* the standard Laplacian.

§ 6.4 Let us define the space of **Schwartz gauge potentials** $\mathcal{A}_S$ and the group $\mathcal{G}_S$ of **Schwartz gauge transformations** to be respectively

$$\mathcal{A}_S \stackrel{\text{def}}{=} \mathscr{S}(\mathbb{R}^4;\mathbb{R}^4), \quad \mathcal{G}_S \stackrel{\text{def}}{=} \operatorname{Gau}\mathcal{P}\cap\mathcal{A}_S.$$

The action functional S in this case is a (nonlinear) functional on $\mathcal{A}_S$. The standard gauge invariant action for the electromagnetic field is, for $A\in\mathcal{A}_S$,

$$S(A) \stackrel{\text{def}}{=} \frac{1}{4}\int_{\mathbb{R}^4}\sum_{\mu,\nu=0}^{3} F_{\mu\nu}(A)(x)^2\,\mathrm{d}x, \quad F_{\mu\nu}(A)(x) \stackrel{\text{def}}{=} [dA]_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),$$
$$\mu,\nu = 0,1,2,3,$$

where, by *gauge invariant*, we mean that S is invariant under the gauge transformation $A\mapsto A+d\lambda$, for $A\in\mathcal{A}_S$ and $d\lambda\in\mathcal{G}_S$. Because of the gauge invariance, the action S defines a quadratic form $\mathcal{A}_S\times\mathcal{A}_S\to\mathbb{R}$ which has a *non* zero kernel.

§ 6.5 The space $\mathcal{B}_S$. The decomposition (VI.4) suggests how to deal with this problem. We let $\mathcal{B}_S$ be the space obtained closing the space $d^*C_c^\infty(\mathbb{R}^4;\Lambda^2\mathbb{R}^4)$ in the topology of $\mathcal{A}_S$. Then $\mathcal{B}_S$, being a closed subspace of the nuclear subspace of $\mathcal{A}_S$, is itself nuclear. Moreover, since $\mathcal{A}_S$ is dense in $L^2(\mathbb{R}^4,\Lambda^1\mathbb{R}^4)$, the vector space $\mathcal{B}_S$ is a dense subset in the quotient space

$$\overline{d^*C_c^\infty(\mathbb{R}^4,\Lambda^2\mathbb{R}^4)} = \operatorname{Ran}Q = L^2(\mathbb{R}^4;\Lambda^1\mathbb{R}^4)\ominus\overline{dC_c^\infty(\mathbb{R}^4;\Lambda^0\mathbb{R}^4)}$$

of square integrable 1-forms modulo exact (=closed here) 1-forms. Let us denote by $S_Q \stackrel{\text{def}}{=} S\circ Q$ the action S restricted to $\mathcal{B}_S$, that is we let

$$S_Q(\widetilde{A}) \stackrel{\text{def}}{=} \int_{\mathbb{R}^4}\big(d\widetilde{A}(x), d\widetilde{A}(x)\big)_{\Lambda^1\mathbb{R}^4}\,\mathrm{d}x, \quad \widetilde{A}\in\mathcal{B}_S,$$

where $(\cdot,\cdot)_{\Lambda^1\mathbb{R}^4}$ denotes the standard product on $\Lambda^1\mathbb{R}^4\cong\mathbb{R}^4$. By definition, we have that S_Q defines a non-degenerate, strictly positive definite, quadratic form $\langle\cdot,\cdot\rangle : \mathcal{B}_S\times\mathcal{B}_S\to\mathbb{R}$. To see this, note that we have, for every $\widetilde{A}\in\mathcal{B}_S$,

$$\begin{aligned} S_Q(\widetilde{A}) &= \int_{\mathbb{R}^4}\big(\widetilde{A}(x), d^*d\widetilde{A}(x)\big)_{\Lambda^1\mathbb{R}^4}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^4}\big(\widetilde{A}(x), \Delta\, Q\,\widetilde{A}(x)\big)_{\Lambda^1\mathbb{R}^4}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^4}\big(\widetilde{A}(x), \Delta\widetilde{A}(x)\big)_{\Lambda^1\mathbb{R}^4}\,\mathrm{d}x, \end{aligned} \tag{VI.6}$$

where the first equality follows by definition of Hilbert adjoint, the second by definition of the projection Q, and the last by the fact that for $\widetilde{A}\in\mathcal{B}_S = Q\mathcal{A}_S$ we have $Q\,\widetilde{A}=\widetilde{A}$, since Q is a projection.

From the fact that S_Q is a positive definite, non-degenerate, quadratic we see that S_Q defines a norm on $\mathcal{B}_S$, actually a *Hilbertian norm*, that is a norm which comes from a scalar product.

§ 6.6 As we noted above, the space $\mathcal{B}_S$, being a closed subspace of the nuclear subspace $\mathcal{A}_S$ is itself a nuclear space. Let $\mathcal{B}'_S$ denote the dual of $\mathcal{B}_S$ with respect to the duality induced by the scalar product in $L^2(\mathbb{R}^4;\mathbb{R}^4)$. We have then the following *Gel'fand triple*

$$\mathcal{B}_S \hookrightarrow Q(L^2(\mathbb{R}^4;\mathbb{R}^4)) \hookrightarrow \mathcal{B}'_S,$$

where $Q(L^2(\mathbb{R}^4;\mathbb{R}^4))$ denotes the image of $L^2(\mathbb{R}^4;\mathbb{R}^4)$ under Q, that is $Q(L^2(\mathbb{R}^4;\mathbb{R}^4)) = \overline{d^*C_c^\infty(\mathbb{R}^4;\Lambda^2\mathbb{R}^4)}$. We can therefore apply Bochner-Minlos theorem, which we now quote.

§ 6.7 Proposition. Let C be a complex valued function on a countably Hilbert nuclear space $\mathcal{N}$ and assume that C is continuous, positive definite, and satisfies $C(0) = 1$. Then there exists a unique probability measure μ_C on the topological dual $\mathcal{N}'$ of $\mathcal{N}$ such that, for all $f \in \mathcal{N}$,

$$\int_{\mathcal{N}'} \exp\{\mathrm{i}\langle x, f\rangle\} \, \mathrm{d}\mu_C(x) = C(f),$$

where $\langle\cdot,\cdot\rangle$ denotes the $\mathcal{N}', \mathcal{N}$ dual pairing. Moreover, let $(\mathcal{N}_n, |\cdot|_n)$, $n \in \mathbb{N}$ be a family of pre-Hilbert spaces such that $\mathcal{N}$ is the inductive limit $\mathcal{N} = \lim_{\rightarrow} \mathcal{N}_n$. If, for a fixed $k \in \mathbb{N}$, C is continuous with respect to the Hilbertian norm $|\cdot|_k$ and the injection $i_k^n : \mathcal{N}_n \to \mathcal{N}_k$ is of Hilbert-Schmidt type, then $\mu_C(\mathcal{N}_{-k}) = 1$, where $\mathcal{N}_{-k}$ is the dual of $\mathcal{N}_k$ with respect to the $\mathcal{N}'$-$\mathcal{N}$ pairing $\langle\cdot,\cdot\rangle$.

PROOF. Cf. e.g. [31, Theorem 1.1, p. 2-3]. □

§ 6.8 Because of (VI.6) , we see that we should define a random field with covariance given by Δ^{-1}. Note that $\mathcal{B}_S$ (as defined in §**6.5**) is *not* in the domain of Δ^{-1}. For example Δ^{-1} does *not* map compactly supported functions to square integrable functions. Nevertheless, denoting by $\langle\cdot,\cdot\rangle$ the $\mathscr{D}'(\mathbb{R}^4, \Lambda^1) - \mathscr{D}(\mathbb{R}^4, \Lambda^1)$ dual paring the expression $\langle\Delta^{-1}A, A\rangle$, $A \in \mathcal{B}_S$, is a well defined quadratic form. Moreover the function $\mathcal{B}_S \ni A \mapsto \langle\Delta^{-1}A, A\rangle \in \mathbb{R}$ is continuous (with respect to the topologies of $\mathcal{B}_S$ and $\mathbb{R}$) because Δ^{-1} is continuous as an operator from $\mathcal{B}_S$ to $\mathscr{D}'(\mathbb{R}^4; \Lambda^1\mathbb{R}^4)$.

We are therefore justified to apply the Bochner-Minlos theorem with $\mathcal{N} = \mathcal{B}_S$ and

$$C(f) = \exp\left\{\int_{\mathbb{R}^4} \big(f(x), \Delta^{-1}f(x)\big) \, \mathrm{d}x\right\}, \quad f \in \mathcal{B}_S.$$

We obtain, by Bochner-Minlos' theorem, a well defined random field on $\mathcal{B}'_S$ with covariance operator given by Δ^{-1}. We collect this result in the following statement.

§ 6.9 Theorem. The action S_Q, defined on $\mathcal{B}_S$, is a non-singular quadratic form induced by the essentially selfadjoint operator $\Delta_{1,Q}$, that is the restriction of the Hodge Laplacian Δ to 1-forms which are in the image of the projection operator Q defined in (VI.3). The inverse $\Delta_{1,Q}^{-1}$ of $\Delta_{1,Q}$ is a well defined bounded operator in $Q\big(L^2(\mathbb{R}^4; \Lambda^1\mathbb{R}^4)\big)$.

The function $C(f) = \exp\left\{\int_{\mathbb{R}^4} \big(f(x), \Delta^{-1}f(x)\big) \, \mathrm{d}x\right\}$, for $f \in \mathcal{B}_S$ defines a probability measure $\mu_{\Delta^{-1}}$ on $\mathcal{B}'_S$, which means that we have a well defined Gaußian random field $X : \mathcal{B}'_S \to \mathbb{R}$ (on the Borel probability measure space $(\mathcal{B}'_S, \mathscr{B}(\mathcal{B}'_S), \mu_{\Delta_{1,Q}^{-1}})$) with zero mean and covariance $\mathbb{E}\big[X(f)X(g)\big] = \big(f(x), \Delta_{1,Q}^{-1}g(x)\big)_{\Lambda^1\mathbb{R}^4}$. □

7 Faddeev-Popov quantization of Euclidean radiation field

Our objective here is to give an application of the method explained in §**4.7**.

To achieve our objective we first need to characterize the model from the "classical" (i.e. non quantum) standpoint. In particular we need to specify

1. The spaces of "classical gauge potentials" and of "classical gauge transformations".
2. The action functional and the Faddeev-Popov functional as functionals on the space of "classical gauge potentials" and which have well defined transformation properties under the "classical gauge transformations".

After this "classical" description, we introduce the infrared and ultraviolet regularizations.

These regularizations will allow us to introduce a family of finite dimensional measures, as described in §**4.7**, parametrized by the regularizations.

The final step will be to show that, when we remove the regularizations, this family of finite dimensional measures converges to a well defined measure in a nuclear (infinite dimensional) space.

Along the way we will introduce several facts borrowed from the Hodge theory.

We will often employ the following shorthand in dealing with function spaces. For example for the space $L^2(\mathbb{R}^4; \Lambda^p\mathbb{R}^4)$, $p \in \mathbb{N}$, will be employ the shorthand notation $L^2(\mathbb{R}^4, \Lambda^p)$.

§ 7.1 Spaces of "classical gauge potentials" and "classical gauge transformations" Let us first, as we did in the last section, restrict the state of gauge potentials and the group of gauge transformations.

We take as "classical state space" the space $\mathcal{A}_c$ of smooth, compactly supported, gauge potentials and as group of gauge transformations the subgroup $\mathcal{G}_c$ of smooth, compactly supported, gauge transformations. Explicitly we take

$$\mathcal{A}_c \stackrel{\text{def}}{=} \mathscr{D}(\mathbb{R}^4, \Lambda^1\mathbb{R}^4), \quad \mathcal{G}_c \stackrel{\text{def}}{=} \mathcal{A}_c \cap \big\{ d\lambda : \lambda \in C^\infty(\mathbb{R}^4; \mathbb{R}) \big\}.$$

§ 7.2 Gauge invariant action functional. On the space $\mathcal{A}_c$ we define the following gauge-invariant action

$$S(A) \stackrel{\text{def}}{=} \frac{1}{4} \int_{\mathbb{R}^4} \big(dA(x), dA(x) \big)_{\Lambda^1\mathbb{R}^4} \,\mathrm{d}x, \quad A \in \mathcal{A}_c.$$

We will often identify $\Lambda^1\mathbb{R}^4 \cong \mathbb{R}^4$. Hence, a gauge potential $A \in \mathcal{A}_c$ will be considered often as a vector valued function (instead of 1-form valued). Under this identification, we have

$$\begin{aligned} S(A) &= \frac{1}{4} \int_{\mathbb{R}^4} \sum_{\mu,\nu=0}^{3} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \,\mathrm{d}x \\ &= \frac{1}{2} \int_{\mathbb{R}^4} \sum_{\mu,\nu=0}^{3} \big(- A_\nu \partial_\mu \partial_\mu A_\nu + (\partial_\mu A_\mu)(\partial_\nu A_\nu) \big) \,\mathrm{d}x \\ &= -\frac{1}{2} \int_{\mathbb{R}^4} A \cdot (\Delta + \nabla \otimes \nabla) A \,\mathrm{d}x, \qquad A \in \mathcal{A}_c, \end{aligned}$$

where in the second line we have used integration by parts and in the last line $\cdot$ denotes the standard scalar product in $\mathbb{R}^4$.

§ 7.3 Faddeev-Popov functional. We choose a Faddeev-Popov functional, that is a function $F_{\mathrm{FP}} : \mathcal{A}_c \to [0, +\infty)$ which satisfies the assumptions given in §**4.7**.

This functional, loosely speaking, defines a measure on the fibers of equivalent potentials, that is gauge potentials which are transformed into one another by a gauge transformation. A "true" gauge fixing would corresponds to a Dirac delta measure on the fibers. By analogy, any Faddeev-Popov functional is often referred to as a gauge fixing even though, more strictly speaking it is a "gauge averaging".

A standard choice of F_{FP} is (cf. e.g. [56, (15.5.22), (15.5.24)])

$$\begin{aligned} F_{\mathrm{FP}}(A) &\stackrel{\text{def}}{=} \exp\left\{ -\frac{1}{\xi} \int_{\mathbb{R}^4} \big(\delta A(x), \delta A(x) \big)_{\Lambda^1\mathbb{R}^4} \,\mathrm{d}x \right\} \\ &= \exp\left\{ -\frac{1}{\xi} \int_{\mathbb{R}^4} \big(\operatorname{div} A(x) \big)^2 \,\mathrm{d}x \right\}, \quad A \in \mathcal{A}_c,\ \xi \in (0, +\infty), \end{aligned} \tag{VI.7}$$

where, in the first line we used the notation δ to denote the formal adjoint d^* of the exterior derivative d and in the second line $\operatorname{div} A(x) \stackrel{\text{def}}{=} \sum_{\mu=0}^{3} \partial_\mu A_\mu(x)$. Note that this Faddeev-Popov functional is actually a family of functionals parametrized by a constant $\xi \in (0, +\infty)$ ("gauge parameter").

Below, we will define infrared and ultraviolet regularizations and show (cf. Lemma below) that this functional does (trivially) satisfy the assumption (VI.1) in §**4.7** with respect to those regularizations.

Before introducing the infrared and ultraviolet regularization we need some general results regarding the Hodge theory.

§ 7.4 Theorem: Hodge-Friedrichs-Kodaira decomposition for manifold with boundary. Let $\mathcal{M}$ be a compact smooth oriented Riemannian manifold with (smooth) boundary $\partial\mathcal{M}$. Let $\Omega^k(\mathcal{M})$ denote the space of k-forms on $\mathcal{M}$. Moreover we define

$$\begin{aligned}
\Omega_t^k(\mathcal{M}) &\stackrel{\text{def}}{=} \{\alpha \in \Omega^k(\mathcal{M}) : \alpha \text{ is tangent to } \partial\mathcal{M}\},\\
\Omega_n^k(\mathcal{M}) &\stackrel{\text{def}}{=} \{\alpha \in \Omega^k(\mathcal{M}) : \alpha \text{ is tangent to } \partial\mathcal{M}\},\\
\mathcal{H}^k(\mathcal{M}) &\stackrel{\text{def}}{=} \{\alpha \in \Omega^k(\mathcal{M}) : d\alpha = \delta\alpha = 0\}.
\end{aligned}$$

We remark, following [4], that $d\alpha = \delta\alpha = 0$ is in general stronger than $\Delta\alpha = 0$. Nevertheless it is customary to employ the term *harmonic forms* to denote the elements in $\mathcal{H}^k(\mathcal{M})$. Then we have the following orthogonal decompositions:

$$\begin{aligned}
\Omega^k(\mathcal{M}) &= d\Omega_t^{k-1}(\mathcal{M}) \oplus \delta\Omega_n^{k+1}(\mathcal{M}) \oplus \mathcal{H}^k(\mathcal{M}),\\
L^2(\mathcal{M}, \Lambda^k) &= \overline{dC_c^\infty(\mathcal{M}, \Lambda^{k-1})} \oplus \overline{\delta C_c^\infty(\mathcal{M}, \Lambda^{k+1})} \oplus \mathcal{H}^k(\mathcal{M}),
\end{aligned} \tag{VI.8}$$

where the over-line denotes the closure in L^2. Moreover the operators d, δ, $\Delta = d\delta + \delta d$, defined on $C_c^\infty(\mathcal{M}, \Lambda^k)$, are extended to closed operators d_R, δ_R, d_A, δ_A, Δ_R, Δ_A, defined on the domains

$$\begin{aligned}
\mathrm{Dom}(d_R) = \mathrm{Dom}(\delta_R) &= \bigoplus_{k\in\mathbb{N}} H_R^1(\mathcal{M}, \Lambda^k),\\
\mathrm{Dom}(d_A) = \mathrm{Dom}(\delta_A) &= \bigoplus_{k\in\mathbb{N}} H_A^1(\mathcal{M}, \Lambda^k),\\
\mathrm{Dom}(\Delta_R) &= \bigoplus_{k\in\mathbb{N}} H_R^2(\mathcal{M}, \Lambda^k),\\
\mathrm{Dom}(\Delta_A) &= \bigoplus_{k\in\mathbb{N}} H_A^2(\mathcal{M}, \Lambda^k),\\
H_R^1(\mathcal{M}, \Lambda^k) &\stackrel{\text{def}}{=} \{\alpha \in H^1(\mathcal{M}, \Lambda^k) : \nu \wedge \alpha\restriction_{\partial\mathcal{M}} = 0\},\\
H_A^1(\mathcal{M}, \Lambda^k) &\stackrel{\text{def}}{=} \{\alpha \in H^1(\mathcal{M}, \Lambda^k) : \nu \wedge (\star\alpha)\restriction_{\partial\mathcal{M}} = 0\},\\
H_R^2(\mathcal{M}, \Lambda^k) &\stackrel{\text{def}}{=} \{\alpha \in H^2(\mathcal{M}, \Lambda^k) : \nu \wedge \alpha\restriction_{\partial\mathcal{M}} = 0, \nu \wedge (\delta\alpha)\restriction_{\partial\mathcal{M}} = 0\},\\
H_A^2(\mathcal{M}, \Lambda^k) &\stackrel{\text{def}}{=} \{\alpha \in H^2(\mathcal{M}, \Lambda^k) : \nu \wedge (\star\alpha)\restriction_{\partial\mathcal{M}} = 0, \nu \wedge (\delta \star \alpha)\restriction_{\partial\mathcal{M}} = 0\},
\end{aligned}$$

where $\star$ denotes the Hodge star and ν denotes the one form orthogonal to the boundary $\partial\mathcal{M}$ pointing in the outward direction and of constant length equal to one.

The domains $H_b^0(\mathcal{M}, \Lambda^k)$ and $H_b^1(\mathcal{M}, \Lambda^k)$, respectively for $b = R, A$, correspond to what are usually called (cf. [55, Chapter 5, §9]) respectively *regular* and *absolute* boundary conditions.

We now specialize to *regular* boundary conditions. Let us denote the kernel of the (closed, selfadjoint, positive semi definite) operator Δ_R by $\mathcal{H}_R(\mathcal{M})$. Then the space $\mathcal{H}_R(\mathcal{M})$ is finite dimensional and is equal to the span of the eigenvector of Δ_R corresponding to the zero eigenvalue. Following [55, Chapter 5, §9], let us denote by G_R the operator Δ_R which annihilates $\mathcal{H}_R(\mathcal{M})$ and inverts Δ_R on the orthogonal complement of its kernel, that is G_R satisfies

$$G_R\Delta_R = (1 - P_h^R),$$

where P_h denotes the projection operator onto the space $\mathcal{H}_R(\mathcal{M})$. Finally we have the following orthogonal decomposition of a function $u \in L^2(\mathcal{M}, \Lambda^k)$:

$$u = d\delta G_R u + \delta d G_R u + P_h^R u. \tag{VI.9}$$

PROOF. We took the first decomposition in (VI.8), $\Omega^k(\mathcal{M}) = d\Omega_t^{k-1}(\mathcal{M}) \oplus^\perp \delta\Omega_n^{k+1}(\mathcal{M})$, from [4, 8.5.5 Theorem, p. 515.] where it is given without proof. The explicit proof follows, for example, from the result proved in [48, Theorem 7.7.8(ii)].

The second decomposition in (VI.8) follows by closure in L^2 of the first decomposition. For the decomposition in (VI.9) cf. e.g. [55, Chapter 5, Proposition 9.8, p. 367]. □

Corollary 1. Let $P_d^R = d\delta G^R$, $P_\delta^R \stackrel{\text{def}}{=} \delta d G^R$. Moreover let P_h be the projection operator onto the finite dimensional space $\bigoplus_{k=0}^{\dim \mathcal{M}} \mathcal{H}^k(\mathcal{M})$ of harmonic forms.

All the bounded operators P_d^R, P_δ^R, P_h, G^R, commute among themselves. Moreover the unbounded operators $\delta^R d^R$, $d^R\delta^R$, and $\Delta^R = \delta^R d^R$, $\delta^R d^R$ strongly commute among themselves.

PROOF. The operators P_d^R, P_δ^R, and P_h^R all commute because they are mutually orthogonal projection operators. We show that P_d^R commutes with G^R. The remaining commutation relations are proved similarly. Let $x, y \in L^2(\mathcal{M}, \Lambda^k)$ and denote by $(\cdot,\cdot)$ the scalar product in $L^2(\mathcal{M}, \Lambda^k)$. Then

$$\begin{aligned}(x, P_d^R G^R y) = (P_d^R x, G^R y) = (d\delta G^R x, G^R y) =\\ = (G^R x, d\delta G^R y) = (G^R x, P_d^R y) = (x, G^R P_d^R y),\end{aligned}$$

where: the equalities on the first line follow from the symmetry of P_d^R and its definition; to go from the first line to the second we used the fact that the form (du, dv) is a symmetric closed form with domain which contains the range of G^R; finally the remaining equalities follow from the definition of P_d^R and the fact that G^R is symmetric. From this computation we see that for every $x, y \in L^2(\mathcal{M}; \Lambda^k)$ $(x, P_d^R G^R y) = (x, G^R P_d^R y)$, that is, $P_d^R G^R = G^R P_d^R$. □

Corollary 2. The operators $d^R\delta^R$, $\delta^R d^R$, $\Delta^R = d^R\delta^R + \delta^R d^R$ defined on the domain $H_R^2(\mathcal{M}, \Lambda^k$ are selfadjoint operators which strongly commute with each other. In particular, they have a common basis of eigenvectors.

PROOF. Since we are considering a compact manifold $\mathcal{M}$ with smooth boundary we obtain that the spectrum of the Hodge Laplacian Δ^R with relative boundary conditions is discrete. In particular we have a basis of eigenvectors for Δ^R. By the previous corollary we obtain that each eigenvector belongs to the image of precisely one of the projection operators P_d^R, P_δ^R, and P_h.

Moreover every eigenvector of the Hodge Laplacian is in the domain of both the operators $d^R\delta^R$, $\delta^R d^R$. This can be sees for example by noting the following. Let $v_\lambda \in L^2(\mathcal{M}, \Lambda^k)$ is an eigenvector for the Hodge Laplacian with eigenvalue $\lambda \neq 0$. Then $P_d v_\lambda = d\delta G^R v_\lambda = \frac{1}{\lambda} d\delta v$. Now since P_d is bounded, $P_d v_\lambda$ is in $L^2(\mathcal{M}, \Lambda^k)$. But then also $\frac{1}{\lambda} d\delta v_\lambda$ is in $L^2(\mathcal{M}, \Lambda^k)$, which implies $d\delta v_\lambda \in L^2(\mathcal{M}, \Lambda^k)$. Therefore v_λ is in the domain of $d^R\delta^R$. Similar proofs hold in the case of $\lambda = 0$ and for the operator $\delta^R d^R$.

Finally, by the algebraic identities $d^2 = 0$, $\delta^2 = 0$, the eigenvectors in the range of P_d^R are annihilated by $\delta^R d^R$ and those in the range of P_δ^R are annihilated by $d^R\delta^R$. From this and the definition $\Delta^R = \delta^R d^R + d^R\delta^R$ it follows that the eigenvectors of Δ^R are also eigenvectors of $\delta^R d^R$ and $d^R\delta^R$.

Now by the spectral theorem for unbounded selfadjoint operators we know that the projection valued measures of the operators Δ^R, $\delta^R d^R$, and $d^R\delta^R$, are formed from the same projection operator which projects onto the subspaces spanned by these common eigenvalues. Hence these spectral families commute, that is the operators strongly commute. □

§ 7.5 Remark. The convenience of the regular and absolute boundary conditions is seen for example by this last corollary. If we had imposed Dirichlet boundary conditions (component wise on the differential forms) then the strong commutativity of the operators in this last corollary would not hold in general.

We need one last result.

§ 7.6 Proposition. Let $\mathcal{M}$ be a smooth compact manifold with boundary $\partial\mathcal{M}$. If the boundary is convex then the finite dimensional vector space $\mathcal{H}_R(\mathcal{M})$ defined as the space of the 0-eigenvectors of the Hodge Laplacian with relative boundary conditions is equal to zero.

PROOF. This result follows from e.g. [51, Theorem 2.6.4, p. 106]. □

We can now turn our attention to the infrared and ultraviolet regularizations.

§ 7.7 Remark. Perhaps the less trivial point, at this stage, is that the regularizations applied to the space of "classical gauge potentials" (to be defined below) will induce, in turn, respective regularizations on the space of "classical gauge transformations".

We first introduce the infrared regularization.

§ 7.8 Infrared regularization. Essentially we introduce as infrared regularization a cut-off in the Euclidean space-time $\mathbb{R}^4$. Let Ω be a open convex domain in $\mathbb{R}^4$ with smooth boundary. We can now define the spaces of gauge potentials and gauge transformations which are smooth and compactly supported inside Ω:

$$\mathcal{A}_c(\Omega) \stackrel{\text{def}}{=} \left\{A \in \mathcal{A}_c : \operatorname{supp} A \subset\subset \Omega\right\},$$

$$\mathcal{G}_c(\Omega) \stackrel{\text{def}}{=} \left\{d\lambda \in \mathcal{G}_c : \operatorname{supp} \lambda \subset\subset \Omega\right\}.$$

§ 7.9 Completion and boundary conditions. Before we introduce the ultraviolet regularization we need to introduce boundary conditions. This is required in our approach because we want to express the ultraviolet regularization in terms of eigenvector expansion. Hence we want to enlarge our spaces $\mathcal{A}_c(\Omega)$ and $\mathcal{G}_c(\Omega)$ in such a way as to make them Hilbert spaces.

Now, the definition of the action functional and the Faddeev-Popov functional depend on two quadratic forms which we denote respectively by Q_S and Q_{FP}, explicitly we have

$$Q_S(f,g) = (df, dg), \quad Q_{FP}(f,g) = (\delta f, \delta g), \quad f,g \in \mathcal{A}_c(\Omega).$$

These two quadratic forms are both symmetric. They can be extended to different closed symmetric forms. We chose one specific extension which is convenient in our setting. Let

$$\mathcal{A}^1(\Omega) \stackrel{\text{def}}{=} H^1_R(\Omega; \Lambda^1).$$

Then the two quadratic forms Q_S and Q_{FP}, extended to the domain $\mathcal{A}^1(\Omega)$ are closed. This can be seen directly from the fact that the space $H^1_R(\Omega; \Lambda^1)$ is closed in the Sobolev space $H^1(\Omega; \Lambda^1)$ and the two quadratic forms define norms that are equivalent to the one of the Sobolev space $H^1(\Omega; \Lambda^1)$.

This choice of extension is convenient because it comes from the closure of the Hodge exterior differential operator d. Hence in particular it allows the application of the results from the Hodge theory quoted above.

It remains to extend also the space of gauge transformations $\mathcal{G}_c(\Omega)$. Now, since we want the action to be gauge invariant, we have to impose that $d\lambda$ be in the domain of the form Q_S, that is in $\mathcal{A}^1(\Omega)$. This is the case when we take

$$\mathcal{G}^2(\Omega) \stackrel{\text{def}}{=} H^2_R(\Omega, \Lambda^0).$$

Indeed the exterior differential operator d preserves the relative boundary conditions. Moreover it maps the Sobolev space $H^2(\Omega, \Lambda^0)$ continuously into the Sobolev space $H^1(\Omega, \Lambda^1)$.

Finally let us define

$$\mathcal{G}^1(\Omega) \stackrel{\text{def}}{=} dH^2_R(\Omega, \Lambda^0).$$

That is for every gauge transformation $\lambda \in \mathcal{G}^2(\Omega)$, we have $d\lambda \in \mathcal{G}^1(\Omega)$. The space $\mathcal{G}^1(\Omega)$ is convenient because it is a subspace of the space $\mathcal{A}^1(\Omega)$. In particular a gauge transformation acts simply by translating a vector in $\mathcal{A}^1(\Omega)$ by a vector in $\mathcal{G}^1(\Omega)$.

The last point we want to make is that the elements in $\mathcal{G}^1(\Omega)$ are in one-to-one correspondence with the elements of $\mathcal{G}^2(\Omega)$, that is the map d from $\mathcal{G}^2(\Omega)$ to $\mathcal{G}^1(\Omega)$ is a bijection. The surjectivity is clear from the definition of $\mathcal{G}^1(\Omega)$. The injectivity follows from the fact that d is bounded and has kernel equal to zero. Indeed, since $\lambda \in \mathcal{G}^2(\Omega)$ are zero forms, the regular boundary conditions, which we have imposed, *coincide* with the Dirichlet boundary conditions. Therefore $d\lambda = 0$ implies $\lambda = 0$, that is, $\ker d = \{0\}$.

Having introduced the Hilbert spaces $\mathcal{A}^1(\Omega)$ and $\mathcal{G}^1(\Omega)$, we can now easily define an ultraviolet regularization.

§ 7.10 Ultraviolet regularization. We consider the Hodge Laplacian Δ_Ω^R with regular boundary conditions on a bounded, convex, smooth domain $\Omega \subset \mathbb{R}^4$. The Hodge Laplacian Δ_Ω^R has domain $\bigoplus_{k\in\mathbb{N}} H^2(\Omega, \Lambda^k)$, it is selfadjoint and has discrete spectrum. Moreover, since we have chosen a convex bounded smooth domain Ω we have that the kernel of Δ_Ω^R coincides with the harmonic forms and, by the proposition in §7.6, it is zero. We consider the restriction of Δ_Ω^R on the space of one forms, that is on the domain $\mathcal{A}^1(\Omega)$ and we denote such restriction still by Δ_Ω^R. Let $(e_k^\Omega)_{k\in\mathbb{N}_+}$ be the set of eigenvectors of Δ_Ω^R. Then we define the following "ultraviolet regularized" finite dimensional spaces

$$\mathcal{A}^1(\Omega, N) \stackrel{\text{def}}{=} \operatorname{span}(e_1^\Omega, \dots, e_N^\Omega)$$
$$\mathcal{G}^1(\Omega, N) \stackrel{\text{def}}{=} \mathcal{A}^1(\Omega, N) \cap \mathcal{G}^1(\Omega).$$

§ 7.11 Regularized action functional and Faddeev-Popov functional. Let as before $S : \mathcal{A}_c \to \mathbb{R}$ and $F_{\mathrm{FP}} : \mathcal{A}_c \to \mathbb{R}$ be respectively the action and Faddeev-Popov functionals defined on the "classical space of gauge potentials" $\mathcal{A}_c$.

These functionals restrict trivially to functionals on $\mathcal{A}_c(\Omega)$, where $\mathcal{A}_c(\Omega)$ is the space of gauge potentials with support in Ω. Let us denote by S^Ω and F^Ω such restrictions. Moreover, let us denote by $\widetilde{S}^\Omega$ and $\widetilde{F}^\Omega$ the extensions of S^Ω and F^Ω from the space $\mathcal{A}_c(\Omega)$ to the domain $\mathcal{A}^1(\Omega)$.

Finally we denote by $\widetilde{S}^{\Omega,N}$ and $\widetilde{F}^{\Omega,N}$ the restrictions of $\widetilde{S}^\Omega$ and $\widetilde{F}^\Omega$ to the finite dimensional spaces $\mathcal{A}^1(\Omega, N)$. We call $\widetilde{S}^{\Omega,N}$ and $\widetilde{F}^{\Omega,N}$ respectively the regularized action and Faddeev-Popov functionals.

We are now ready to define our family of probability measures.

§ 7.12 Finite dimensional measure. Let $\mathscr{I}$ be a family of convex, bounded, smooth domains $\Omega_1 \subset \Omega_2 \subset \cdots \nearrow \mathbb{R}^4$. We define a family of non-normalized[4] measures $(\nu^{\Omega,N})_{\Omega\in\mathscr{I},N\in\mathbb{N}}$, each $\nu^{\Omega,N}$ supported on $\mathcal{A}(\Omega, N)$:

$$\mathrm{d}\nu^{\Omega,N} \stackrel{\text{def}}{=} \frac{\int_{\mathcal{G}^1(\Omega,N)} \widetilde{F}_{\mathrm{FP}}^{\Omega,N}(0, \lambda^{\Omega,N})\,\mathrm{d}\lambda^{\Omega,N}}{\int_{\mathcal{G}^1(\Omega,N)} \widetilde{F}_{\mathrm{FP}}^{\Omega,N}(A^{\Omega,N}, \lambda^{\Omega,N})\,\mathrm{d}\lambda^{\Omega,N}} \widetilde{F}_{\mathrm{FP}}^{\Omega,N}(A^{\Omega,N}) e^{-\widetilde{S}^{\Omega,N}(A^{\Omega,N})}\,\mathrm{d}A^{\Omega,N},$$
$$A^{\Omega,N} \in V^{\Omega,N}, \lambda^{\Omega,N} \in G^{\Omega,N}, \Omega \in \mathscr{I}, N \in \mathbb{N}, \quad \text{(VI.10)}$$

where $\mathrm{d}A^{\Omega,N}$ and $\mathrm{d}\lambda^{\Omega,N}$ denote respectively the Lebesgue measure on $\mathcal{A}^1(\Omega, N)$ and on $\mathcal{G}^1(\Omega, N)$, $\Omega \in \mathscr{I}$, $N \in \mathbb{N}$.

As a result of our choice of regularizations we have the following result which simplifies the form of the family of probability measures.

§ 7.13 Lemma. Consider the family of measure $\nu^{\Omega,N}$ defined in (VI.10). Then the fraction appearing on the right hand side of (VI.10) is identically equal to one. That is, we have

$$\frac{\int_{\mathcal{G}^1(\Omega,N)} \widetilde{F}_{\mathrm{FP}}^{\Omega,N}(0, \lambda^{\Omega,N})\,\mathrm{d}\lambda^{\Omega,N}}{\int_{\mathcal{G}^1(\Omega,N)} \widetilde{F}_{\mathrm{FP}}^{\Omega,N}(A^{\Omega,N}, \lambda^{\Omega,N})\,\mathrm{d}\lambda^{\Omega,N}} = 1,$$

for all $A^{\Omega,N} \in \mathcal{A}^1(\Omega, N)$, $\Omega \in \mathscr{I}$, $N \in \mathbb{N}$.

Proof. Identifying $\mathcal{A}^1(\Omega, N)$ with $\mathbb{R}^N$ by passing to the basis $(e_n)_{n\in\mathbb{N}_+}$ of eigenvalues of Δ_Ω^R we have that the integrals are Gaußian integrals. Moreover, the $(e_n)_{n\in\mathbb{N}_+}$ being also eigenvectors of

[4] In §7.15 we will define the normalized version of these measures.

$(d\delta)^R_\Omega$, the covariant matrices of the numerator and denominator are diagonal. Let us denote by R^N the matrix

$$R^N_{jk} \stackrel{\text{def}}{=} (e_j, \nabla \otimes \nabla e_k)_{L^2(\mathbb{R}^4;\Lambda^1\mathbb{R}^4}, \quad j,k \in \{1,\dots,N\}, N \in \mathbb{N}_+.$$

Then we get

$$\frac{\int_{\mathbb{R}^M} \widetilde{F}^{\Omega,N}_{\text{FP}}(0,\lambda^{\Omega,N})\,\mathrm{d}\lambda^{\Omega,N}}{\int_{\mathbb{R}^M} \widetilde{F}^{\Omega,N}_{\text{FP}}(A^{\Omega,N},\lambda^{\Omega,N})\,\mathrm{d}\lambda^{\Omega,N}} = \exp\left\{x^{\mathrm{T}} R^N x\right\} \exp\left\{-(J(x))^{\mathrm{T}}(B^N)^{-1} J(x)\right\},$$

where x denotes the infinite dimensional vector with components

$$x_j \stackrel{\text{def}}{=} (A^{\Omega,N}, e_j),$$

$[J(x)]_k \stackrel{\text{def}}{=} \sum_{j\in\mathbb{N}_+} J_{kj} x_j$, and, for $j,k = 1,\dots,N$,

$$B^N_{jk} \stackrel{\text{def}}{=} (h_j, \Delta^2 h_k)_{L^2(\mathbb{R}^4;\Lambda^0)},$$
$$J^N_{jk} \stackrel{\text{def}}{=} (\nabla\Delta e_j, h_k)_{L^2(\mathbb{R}^4;\Lambda^1)},$$

with $(h_j)_{j\in\mathbb{N}_+}$ the family of scalar functions uniquely defined by $\nabla h_j = e_j$, $j \in \mathbb{N}_+$. The fact that these h_j are uniquely defined follows from §7.6.

Substituting the definitions we obtain

$$\sum_{k,\ell=1}^{N} [J(x)^{\mathrm{T}}]_k [(B^N)^{-1}]_{k\ell} [J(x)]_\ell = \sum_{k,\ell=1}^{N} x^{\mathrm{T}}_k R^N_{k\ell} x_\ell.$$

Hence the statement of the lemma holds. □

§ 7.14 Remark. This lemma is an embodiment of the usual formal physical procedure of removing the "Faddeev-Popov" determinants when they do not explicitly depend on the gauge potential.

We now need to remove the regularizations. We start by removing the ultraviolet regularization.

§ 7.15 Theorem. With notations as above (μ and $\mathscr{I}$ were defined in §7.12), let, for every $\Omega \in \mathscr{I}$, $N \in \mathbb{N}_+$,

$$\mu^{\Omega,N} \stackrel{\text{def}}{=} \frac{\nu^{\Omega,N}}{|\nu^{\Omega,N}|\big(\mathcal{A}^1(\Omega,N)\big)},$$

Then, for a fixed $\Omega \in \mathscr{I}$, $\mu^{\Omega,N}_{N\in\mathbb{N}}$ converges weakly in $\mathscr{D}'(\mathbb{R}^4;\Lambda^1\mathbb{R}^4)$, as $N \to \infty$, to a measure μ^Ω supported on $\mathscr{D}'(\mathbb{R}^4,\Lambda^1\mathbb{R}^4)$.

Proof. Denote by $\langle\cdot,\cdot\rangle$ the dual pairing $\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)$-$\mathscr{D}(\mathbb{R}^4;\mathbb{R}^4)$ which restrict to the scalar product in $L^2(\mathbb{R}^4;\mathbb{R}^4)$ when we consider $L^2(\mathbb{R}^4;\mathbb{R}^4)$ as embedded in $\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)$. Consider, for every $\Omega \in \mathscr{I}$, $\mathcal{A}^1(\Omega)$ as embedded in $L^2(\Omega;\mathbb{R}^4)$. Then, if we extend every function in $\mathcal{A}^1(\Omega)$ by zero we have an embedding of $\mathcal{A}^1(\Omega)$ in $\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)$. Now, for every $\Omega \in \mathscr{I}$, $\mathcal{A}^1(\Omega,N)$, $N \in \mathbb{N}_+$ is embedded in $\mathcal{A}^1(\Omega)$ hence $\mathcal{A}^1(\Omega,N)$ embeds in $\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)$. Now we can push forward the measure $\nu^{\Omega,N}$ via this embedding $\mathcal{A}^1(\Omega,N) \hookrightarrow \mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)$ to obtain a measure $\widetilde{\nu}^{\Omega,N}$ on $\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)$.

Let $(e_n)_{n\in\mathbb{N}_+}$ be a basis of $L^2(\Omega;\mathbb{R}^4)$ of eigenvalues of the selfadjoint Hodge Laplacian Δ^R_Ω defined above. Moreover, let $D^{\Omega,N}$ be the following N-by-N matrix

$$D^{\Omega,N}_{jk} \stackrel{\text{def}}{=} (e_j, \Delta^R_\Omega e_k)_{L^2(\Omega;\mathbb{R}^4)}, \quad j,k \in \{1,\dots,N\}.$$

We fix a basis of $\mathcal{A}^1(\Omega, N)$ and denote by $a_j^{\Omega,N}$, for $j \in \{1, \ldots, N\}$, the components of $A^{\Omega,N} \in \mathcal{A}^1(\Omega, N)$ in this basis. Then, by definition of $\widetilde{\nu}^{\Omega,N}$ and the lemma above we have the following explicit representation of the measure $\widetilde{\mu}^{\Omega,N}$:

$$\int_{\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)} \exp\left\{\mathrm{i}\langle X, A\rangle\right\} \mathrm{d}\nu^{\Omega,N}(X) =$$

$$= \int_{\mathbb{R}^N} \exp\left\{\mathrm{i}\sum_{j=1}^{N} x_j^N a_j^{\Omega,N} + \sum_{j,k=1}^{N} x_j D_{jk}^{\Omega,N} x_k\right\} \mathrm{d}x^N,$$

where $\mathrm{d}x^N$ denotes the Lebesgue measure on $\mathbb{R}^N$.

Now note that $D^{\Omega,N}$ is a symmetric non-degenerate matrix. Indeed it is symmetric by definition. Moreover it is non-degenerate because if it had a zero eigenvalues then the corresponding eigenvector could be extended to a vector in $L^2(\Omega, \mathbb{R}^4)$ which would be an eigenvector of Δ_Ω^R with zero eigenvalues. But this is a contradiction because Δ_Ω^R has no zero-eigenvalues on $L^2(\Omega, \mathbb{R}^4)$. In particular $D^{\Omega,N}$ is diagonalizable by an orthogonal change of basis in $\mathbb{R}^N$ which, therefore, leaves the Lebesgue measure invariant. We therefore see that, for every fixed $\Omega \in \mathscr{I}$ and $N \in \mathbb{N}_+$, the measure $\nu^{\Omega,N}$ is in fact a product measure

$$\nu^{\Omega,N} = \nu_1^\Omega \otimes \cdot \otimes \nu_N^\Omega$$

of appropriate (not yet normalized) Gaußian measures ν_j^Ω, $j = 1, \ldots, N$. If we now consider, as in the statement of the theorem, the normalized measure $\mu^{\Omega,N} = \nu^{\Omega,N}/|\nu^{\Omega,N}|(V^{\Omega,N})$, then we see that $\mu^{\Omega,N}$ is a product of normalized Gaußian measures μ_j^Ω, $j = 1, \ldots, N$. We can now pass to the limit $N \to \infty$ and define the measures μ^Ω $\mathscr{D}'(\Omega; \mathbb{R}^4)$ as the weak limit (that is the limit in $\mathscr{D}'(\Omega; \mathbb{R}^4)$) $\mu^\Omega \overset{\text{def}}{=} \text{w-lim}_{N\to\infty} \mu^{\Omega,N}$. □

§ 7.16 Remark. We note that, by definition, the Faddeev-Popov functional F_{FP} depends on a parameter $\xi > 0$ which we consider fixed once and for all. Because of this dependence all the quantity which are defined in terms of F_{FP} depend on this parameter. In particular, for example, the measures μ^Ω, $\Omega \in \mathscr{I}$, depend on ξ. We have chosen to hide the dependence on ξ everywhere because we feel it would contribute to the clutter of the notations without extra gain. In principle ξ could be chosen equal to one everywhere. Nevertheless we keep the hidden dependence on ξ because we aim at arriving at the formula (VI.11) below, which is usually given in term of the parameter ξ.

Finally we remove the infrared regularization.

Before turning to the main theorem in this section we quote the following standard result. We use the following result in the proof below in place of a direct application of Kolmogorov theorem.

§ 7.17 Lévy-Fernique theorem. Let $\mathcal{D}$ be a nuclear space and $\mathcal{D}'$ its topological dual. Let μ_M, $N \in \mathbb{N}_+$, and μ be probability measures on $\mathcal{D}'$. If

$$\lim_{M\to+\infty} \int_{\mathcal{D}'} e^{\mathrm{i}\langle X,f\rangle} \mathrm{d}\mu_M(X) = \int_{\mathcal{D}'} e^{\mathrm{i}\langle X,f\rangle} \mathrm{d}\mu(X)$$

for each $f \in \mathcal{D}$, then μ_N converges weakly in $\mathcal{D}'$ to μ as $M \to \infty$.

Proof. Cf. [17, Théorème 4.5]. □

§ 7.18 Theorem. The probability measures μ^Ω on $\mathscr{D}'(\mathbb{R}^4; \mathbb{R}^4)$, given by Theorem §7.15, converge weakly in $\mathscr{D}'(\mathbb{R}^4; \mathbb{R}^4)$, as $\Omega \nearrow \mathbb{R}^4$, to a probability measure μ on $\mathscr{D}'(\mathbb{R}^4; \mathbb{R}^4)$. Moreover, the measure μ is supported on $\mathscr{S}'(\mathbb{R}^4; \mathbb{R}^4) \subset \mathscr{D}'(\mathbb{R}^4; \mathbb{R}^4)$ and its Fourier transform is given by

$$\int_{\mathscr{S}'(\mathbb{R}^4;\mathbb{R}^4)} \exp\left\{\mathrm{i}\langle X, A\rangle\right\} \mathrm{d}\mu(X) =$$

$$= \exp\left\{-\int_{\mathbb{R}^4} \widehat{A}(-k) \cdot \left(\frac{|k|^2 \mathbb{I}_4 + (\xi - 1)k \otimes k}{|k|^4}\right) \widehat{A}(k)\, \mathrm{d}k\right\}, \quad \text{(VI.11)}$$

where $\langle\cdot,\cdot\rangle$ denotes the $\mathscr{S}'(\mathbb{R}^4;\mathbb{R}^4)$-$\mathscr{S}(\mathbb{R}^4;\mathbb{R}^4)$ pairing, $\widehat{A}$ denotes the Fourier transform of $A\in\mathscr{S}(\mathbb{R}^4;\mathbb{R}^4)$, and $\xi>0$ is the "gauge" parameter on which also μ depends (cf. the remark in §7.16).

PROOF. By pushing forward the measures μ^Ω, though the embedding of $\mathscr{D}'(\Omega;\mathbb{R}^4)$ in $\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)$, we get a measure on $\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)$ which we still denote by μ^Ω. Hence we can now consider the limit of the family of (probability) measures μ^Ω, $\Omega\in\mathscr{I}$, as a (weak) limit in $\mathscr{D}'(\mathbb{R}^4;\mathbb{R}^4)$, as Ω expands to $\mathbb{R}^4$.

Consider an element $A\in\mathscr{D}(\mathbb{R}^4;\mathbb{R}^4)$. Then, for a $\Omega'\in\mathscr{I}$ big enough to contain the support of A, we have that A is contained in $\mathscr{D}(\Omega';\mathbb{R}^4)$. Hence we have

$$\int_{D'(\mathbb{R}^4;\mathbb{R}^4)} e^{\mathrm{i}\langle X,A\rangle}\,\mathrm{d}\mu^{\Omega'}(X)=\exp\left\{\Big(A,\big[\big(\delta d+\tfrac{1}{\xi}d\delta\big)^R_K\big]^{-1}A\Big)_{L^2(\mathbb{R}^4;\mathbb{R}^4)}\right\},$$

for every Ω' is assumed big enough so that $A\in\mathscr{D}(\mathbb{R}^4;\mathbb{R}^4)$ is supported on a set strictly contained in Ω'.

Now we want to pass to the limit $\Omega'\nearrow\mathbb{R}^4$. By the Lévy-Fernique theorem above we have the result provided the covariance

$$\frac{1}{2}\big[\big(\delta d-\tfrac{1}{\xi}d\delta\big)^R_K\big]^{-1}=\big[(-\Delta+(1-\tfrac{1}{\xi})\nabla\otimes\nabla)^R_K\big]^{-1},\quad \xi>0,$$

converges in the weak operator topology to the operator

$$\big[(-\Delta+(1-\tfrac{1}{\xi})\nabla\otimes\nabla)^R_{\mathbb{R}}\big]^{-1},$$

where $(-\Delta+(1-\frac{1}{\xi})\nabla\otimes\nabla)^R_{\mathbb{R}}$ denotes the selfadjoint extension in $L^2(\mathbb{R}^4;\mathbb{R}^4)$, corresponding to regular boundary conditions, of the positive definite ($\xi>0$) operator in parenthesis initially defined on $\mathscr{D}(\mathbb{R}^4;\mathbb{R}^4)$, and similarly for $(-\Delta+(1-\frac{1}{\xi})\nabla\otimes\nabla)^R_K$. The proof of this convergence is given below where, for clarity, we state this result as a separate theorem.

The final statement in the present theorem (§7.18) follows by representing the covariance of the limit in Fourier transform and noting that is continuous on $\mathscr{S}(\mathbb{R}^4;\mathbb{R}^4)$ hence, by Bochner-Minlos' theorem, corresponds to a probability measure supported on $\mathscr{S}'(\mathbb{R}^4;\mathbb{R}^4)$. □

§ 7.19 Theorem. Let Ω be a bounded, convex, open domain in $\mathbb{R}^4$. Consider the two operators with the same form and different domains:

$$L^R_\Omega\overset{\text{def}}{=}(-\Delta+(1-\tfrac{1}{\xi})\nabla\otimes\nabla),\quad \xi>0,\qquad \mathrm{Dom}(L^R_\Omega)\overset{\text{def}}{=}H^2_R(\Omega,\Lambda^1),$$
$$L\overset{\text{def}}{=}(-\Delta+(1-\tfrac{1}{\xi})\nabla\otimes\nabla),\quad \xi>0,\qquad \mathrm{Dom}(L)\overset{\text{def}}{=}H^2(\mathbb{R}^4,\Lambda^1).$$

These operators are selfadjoint strictly positive definite operators. They obviously depend on a $\xi>0$ but we choose to hide such dependence in the notation (cf. remark in §7.16). Moreover they have well defined, selfadjoint inverses $(L^R_\Omega)^{-1}$ (that is also bounded) and L^{-1} (which is unbounded). The operator L^{-1} admits an extension to an operator $\mathring{L}^{-1}:L^2(\mathbb{R}^4,\Lambda^1)\to\mathscr{D}'(\mathbb{R}^4,\Lambda^1)$. Furthermore for every $\varphi\in C^\infty_c(\mathbb{R}^4,\Lambda^1)$, we have

$$\lim_{\Omega\nearrow\mathbb{R}^4}\big((L^R_\Omega)^{-1}\varphi\,,\,\varphi\big)_{L^2(\mathbb{R}^4;\mathbb{R}^4)}=\langle\mathring{L}^{-1}\varphi\,,\,\varphi\rangle,\tag{VI.12}$$

where the operator L^{-1} $\langle\cdot,\cdot\rangle$ denotes the $\mathscr{D}'(\mathbb{R}^4,\Lambda^1)$-$\mathscr{D}(\mathbb{R}^4,\Lambda^1)$ pairing[5].

PROOF. The fact that the operator L^R_Ω is selfadjoint on $H^2_R(\Omega,\Lambda^1)$ was discussed above. The operator L can be obtained by Friedrichs extension of the real quadratic form $q_L(\varphi,\varphi)\overset{\text{def}}{=}(\varphi,L\varphi)$

[5] We need to employ the dual pairing $\langle\cdot,\cdot,\rangle$ in place of the L^2 scalar product $(\cdot,\cdot)_{L^2(\mathbb{R}^4,\Lambda^1)}$ because for $\varphi\in C^\infty_c(\mathbb{R}^4,\Lambda^1)$, $L^{-1}\varphi$ needs *not* be in $L^2(\mathbb{R}^4,\Lambda^1)$ but, as will be seen from the proof, in will still be an element of $\mathscr{D}'(\mathbb{R}^4,\Lambda^1)$.

on the initial domain $C_c^\infty(\mathbb{R}^4;\Lambda^1)$ which makes it densely defined in $L^2(\mathbb{R}^4,\Lambda^1)$. Note that the operator $L+1$ induces in $C_c^\infty(\mathbb{R}^4;\Lambda^1)$ a norm equivalent to the norm of $H^2(\mathbb{R}^4,\Lambda^1)$. This shows that $H^2(\mathbb{R}^4,\Lambda^1)$ is indeed the domain of selfadjointness, as an operator in $L^2(\mathbb{R}^4;\Lambda^1\mathbb{R}^4)$.

Both the operators L_Ω^R and L are (strictly) positive definite. The operator L can be shown to be strictly positive and invertible (on its range) directly from its symbol (Fourier transform). If we denote by $\mathscr{F}$ the unitary Fourier operator in $L^2(\mathbb{R}^4,\Lambda^1)$, we have that

$$(\mathscr{F}^{-1}L^{-1}f)(k) = \frac{1}{2\pi}\frac{|k|^2\,\mathbb{I}_4 + k\otimes k}{|k|^4}(\mathscr{F}f)(k), \quad k\in\mathbb{R}^4. \tag{VI.13}$$

The we call the density

$$\frac{|k|^2\,\mathbb{I}_4 + k\otimes k}{|k|^4}, k\in\mathbb{R}^4,$$

the *symbol* of the operator L^{-1}.

Note that the function of $k\in\mathbb{R}^4$ on the right hand side of (VI.13) is a well defined function in $L^2(\mathbb{R}^4,\Lambda^1)$, if f is in the range of L. On the other hand, if instead of a function f in the range of L we take a function $\phi\in C_c^\infty(\mathbb{R}^4,\Lambda^1)$ the function on the right hand side of (VI.13), with f replaced by φ, is still a well defined integrable function because near the origin the symbol $\frac{|k|^2\,\mathbb{I}_2+k\otimes k}{|k|^4}$ behaves, in norm, as const.$\frac{1}{|k|^2}$ and far from the origin $\mathscr{F}\varphi$ decays fast enough because, for $\varphi\in C_c^\infty(\mathbb{R}^4,\Lambda^1)$, $\mathscr{F}\varphi$ is at least in $\mathscr{S}(\mathbb{R}^4,\Lambda^1)$. Nevertheless the resulting function will fail to be in $L^2(\mathbb{R}^4,\Lambda^1)$. Indeed, in the L^2 norm of $L^{-1}\varphi$, we have a density which near the origin behaves as $\frac{1}{|k|^4}$ which is *not* integrable in four dimensions. In any case, as we shall see shortly, we can extend L^{-1}, to an operator $\mathring{L}^{-1} : C_c^\infty(\mathbb{R}^4,\Lambda^1)\to\mathscr{D}'(\mathbb{R}^4,\Lambda^1)$ with the same symbol. Then we have that $\mathring{L}^{-1}\varphi$ is a well defined element in $\mathscr{D}'(\mathbb{R}^4,\Lambda^1)$ for $\varphi\in C_c^\infty(\mathbb{R}^4,\Lambda^1)$. Or, more accurately (as was stated in the body of the theorem), the operator L^{-1} admits an extension $\mathring{L}^{-1}$ with range in $\mathscr{D}'(\mathbb{R}^4,\Lambda^1)$.

Similarly, let us denote by $\mathring{L}$ the operator with the same symbol as L, but defined as a weak differential operator in $\mathscr{D}'(\mathbb{R}^4,\Lambda^1)$ (that is, as a differential operator defined on distributions).

To see that $\mathring{L}^{-1}\varphi$ is a well defined element in $\mathscr{D}'(\mathbb{R}^4,\Lambda^1)$, note that by definition of the dual paring $\langle\cdot,\cdot\rangle$, we have

$$\langle\mathring{L}^{-1}\varphi\,,\,\psi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^4}(\mathscr{F}\varphi)(k)\frac{|k|^2\,\mathbb{I}_4 + k\otimes k}{|k|^4}(\mathscr{F}\psi)(k)\,\mathrm{d}k, \tag{VI.14}$$

which is integrable for any $\varphi,\psi\in C_c^\infty(\mathbb{R}^4,\Lambda^1)$ because, as we noted above, the density $\frac{|k|^2\,\mathbb{I}_4+k\otimes k}{|k|^4}$ is integrable near the origin, bounded far from the origin, and far from the origin the whole integral converges because $\mathscr{F}\varphi$ and $\mathscr{F}\psi$ are in $\mathscr{S}(\mathbb{R}^4,\Lambda^1)$ when $\varphi,\psi\in C_C^\infty(\mathbb{R}^4,\Lambda^1)\subset\mathscr{S}(\mathbb{R}^4,\Lambda^1)$.

We now turn to the operator L_Ω^R. Note that it has zero kernel because of our assumption on Ω and the proposition in §**7.6**. Since the operator is selfadjoint, this implies that the operator invertible. Moreover the inverse is a bounded selfadjoint operator.

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We turn to the proof of (VI.12), the only remaining step to conclude the proof of Theorem §**7.19**. In the following we use the following notation.

Notation: For the remaining of the proof, we define u_Ω^R to be the solution in $H_R^2(\Omega,\Lambda^1)$, that is with *relative* boundary conditions, to

$$L_\Omega^R u_\Omega^R = f,, \tag{VI.15}$$

for $f\in C_c^\infty(\mathbb{R}^4,\Lambda^1)$ with $\operatorname{supp} f \subset\subset \Omega$.

We, correspondingly, define

$$u = \mathring{L}^{-1} f, \quad f \in C_c^\infty(\mathbb{R}^4, \Lambda^1).$$

Note that at worst $u \in H^{-2}(\mathbb{R}^4, \Lambda^1) \subset \mathscr{D}'(\mathbb{R}^4, \Lambda^1)$, as seen e.g. from the form of the Fourier transform of u (cf. (VI.14)). We break the proof of (VI.12) in the following steps.

1. There exists an extension $\widetilde{u}_\Omega$ of u_Ω^R, to the whole of $\mathbb{R}^4$, such that

$$\|\nabla \widetilde{u}_\Omega\|_{L^2(\mathbb{R}^4,(\Lambda^1)^{\otimes 2})} \leq C \|\nabla u_\Omega^R\|_{L^2(\Omega,(\Lambda^1)^{\otimes 2})} \tag{VI.16}$$

with a constant C *independent* of Ω. Such extension exists for any function in $H^2(\Omega, \Lambda^1)$, regardless of the choice of boundary conditions, cf. [35, Theorem 1.] (where such an extension is constructed for more general domains).

2. As above, let us denote by $\langle \cdot, \cdot \rangle$ the dual pairing $\mathscr{D}'(\mathbb{R}^4, \Lambda^1)$-$\mathscr{D}(\mathbb{R}^4, \Lambda^1)$. We have, by the definition of L, for any $w \in H^1(\mathbb{R}^4, \Lambda^1)$,

$$\langle Lw, w \rangle = (\nabla w, \nabla w)_{L^2(\mathbb{R}^4,(\Lambda^1)^{\otimes 2})} + \frac{1}{\xi}(\operatorname{div} w, \operatorname{div} w)_{L^2(\mathbb{R}^4,\Lambda^0)} \geq \|\nabla w\|^2_{L^2(\mathbb{R}^4,(\Lambda^1)^{\otimes 2})}.$$

Note that in the left hand side we had to use the dual pairing $\langle \cdot, \cdot \rangle$, but on the right hand side of the equality sign, all the L^2 scalar products are well defined because w is assumed to be in $H^1(\mathbb{R}^4, \Lambda^1)$.

Notation. From this point on we write simply $L^p(\Omega)$, for some $p > 0$, to denote either $L^p(\Omega, \Lambda^0)$ or $L^p(\Omega, (\Lambda^1)^{\otimes 2})$.

3. Since u_Ω^R is supported in $\overline{\Omega}$ we have

$$\langle L u_\Omega^R, u_\Omega^R \rangle = (L_\Omega^R u_\Omega^R, u_\Omega^R). \tag{VI.17}$$

To see that this is the case we note that one can approximate u_Ω^R with functions $(\varphi_j)_{j \in \mathbb{N}_+}$ compactly supported in Ω. If we replace u_Ω^R in (VI.17) by φ_j, $j \in \mathbb{N}_+$, then the expression certainly holds for every $j \in \mathbb{N}_+$. Now taking the limit as $j \to \infty$ we have that $(L_\Omega \varphi_j, \varphi_j)$ converges to the right hand side of (VI.17). But then also the left hand side must converge to the left hand side of (VI.17), that is (VI.17) holds.

Now, by (VI.17), we have

$$\langle L u_\Omega^R, u_\Omega^R \rangle = (L_\Omega^R u_\Omega^R, u_\Omega^R) = (f, u_\Omega^R) \leq \|f\|_{L^{4/3}(\Omega)} \|u_\Omega^R\|_{L^4(\Omega)},$$

where the second equality follows from (VI.15) and the last inequality is the standard Hölder inequality with $p = 4, q = 4/3$.

4. We now use the fact, which again follows from [35, Theorem 1.], that the extension $\widetilde{u}_\Omega$ is bounded in $L^p(\mathbb{R}^4)$ whenever u_Ω^R is bounded in $L^p(\Omega)$. From this we get the first inequality in the following:

$$\|u_\Omega^R\|_{L^4(\Omega)} \leq \|\widetilde{u}_\Omega\|_{L^4(\mathbb{R}^4)} \leq C_1 \|\nabla \widetilde{u}_\Omega\|_{L^2(\mathbb{R}^4)} \leq C_1 C \|\nabla u_\Omega^R\|_{L^2(\Omega)}$$

where the second inequality is the Gagliardo-Nirenberg-Sobolev inequality, and the last inequality follows from point 1. Note that the constant C which comes from Gagliardo-Nirenberg-Sobolev inequality is independent on the domain Ω (this is the reason we are using this inequality).

5. Combining the previous points we obtain, setting $C_2 = C_1 C$,

$$\langle \mathring{L} u_\Omega^R, u_\Omega^R \rangle \leq \|f\|_{L^{4/3}(\Omega)} \|u_\Omega^R\|_{L^4(\Omega)} \leq C_2 \|f\|_{L^{4/3}(\Omega)} \|\nabla u_\Omega^R\|_{L^2(\Omega)} \leq$$
$$\leq C_2 \|f\|_{L^{4/3}(\Omega)} \sqrt{(u_\Omega^R, L u_\Omega^R)},$$

where the first inequality comes from point 3., the second from point 4., and the last inequality follows from point 2. and the fact that $u_\Omega^R \in H_R^2(\mathbb{R}^4, \Lambda^1)$, hence in $H_R^1(\mathbb{R}^4, \Lambda^1)$.

6. Noting that by definition of u^R_Ω we have $\langle (L^R_\Omega)^{-1} f\,,\, f\rangle = \langle L u^R_\Omega, u^R_\Omega\rangle$, we obtain from the previous point that, for f as in (VI.15),

$$\langle (L^R_\Omega)^{-1} f\,,\, f\rangle = \langle L u^R_\Omega, u^R_\Omega\rangle \le C(f)$$

where $C(f) \stackrel{\text{def}}{=} C_2\|f\|_{L^{4/3}(\Omega)}$ is *independent* on the (size of the) domain Ω, for f fixed with support in the interior of Ω.

7. Note first that, by definition of u^R_Ω, we have $\langle (L^R_\Omega)^{-1} f\,,\, f\rangle = \langle u^R_\Omega, f\rangle$. Now, since the bound obtained in point 6. is uniform in Ω we have that for any fixed $f \in C^\infty_c(\mathbb{R}^4, \Lambda^1)$ and any sequence of (open, bounded, smooth, convex) domains $\Omega_1 \subset \Omega_2 \subset \cdots \nearrow \mathbb{R}^4$ there exists a subsequence $(\Omega_{n(k)})_{k\in\mathbb{N}_+}$ which converges, that is:

$$\lim_{k\to\infty} (u^R_{\Omega_{n(k)}}, f) = \langle u^*, f\rangle,$$

where in general u^* will depend on the sequence $(\Omega_n)_{n\in\mathbb{N}_+}$. Without loss of generality we can assume $\operatorname{supp} f \subset\subset \Omega$ for some $\Omega \subset \Omega_1 \subset \Omega_2 \subset \cdots$. Now, note that each $u^R_{\Omega_j}$ satisfies weakly (by hypothesis) $\mathring{L} u^R_{\Omega_j} = f$ on Ω, that is, for every $g \in C^\infty_c(\Omega, \Lambda^1)$, we have $\langle L u^R_{\Omega_j}, g\rangle = (f, g)$. Hence, passing to the limit $j \to \infty$, we also have $\langle \mathring{L} u^*, g\rangle = (g, f)$ for any $g \in C^\infty_c(\Omega, \Lambda^1)$.

Now the point is that the relation $\langle L u^*, g\rangle = (f, g)$ can be extended to hol for any $f, g \in C^\infty_c(\mathbb{R}^4, \Lambda^1)$. Indeed, first note that $\langle L u^*, g\rangle = (f, g)$ holds for every $g \in C^\infty_c(\mathbb{R}^4, \Lambda^1)$ (and $f \in C^\infty_c(\Omega, \Lambda^1)$ still fixed). This is true because for any $g \in C^\infty_c(\mathbb{R}^4, \Lambda^1)$ we can find a domain Ω_{n_0} in the sequence $(\Omega_n)_{n\in\mathbb{N}_+}$ such that $\operatorname{supp} g \subset\subset \Omega_n$. Hence for any $n > n_0$ we have $\langle \mathring{L} u^R_{\Omega_n}, g\rangle = (f, g)$. In particular in the limit we will still have $\langle \mathring{L} u^*, g\rangle = (f, g)$. Finally we want to show that $\langle \mathring{L} u^*, g\rangle = (f, g)$ holds also if we change the support of f. To see this note that we could have chosen (at the beginning) any f in $C^\infty_c(\mathbb{R}^4, \Lambda^1)$. Going through the same construction we would have obtained a different u^*_f, where we now have explicitly displayed the dependence on f. But for any $f \in C^\infty_c(\mathbb{R}^4, \Lambda^1)$ we have, in the sequence $(\Omega_n)_{n\in\mathbb{N}_+}$, an element Ω_{n_0} such that $\operatorname{supp} f \subset\subset \Omega_{n_0}$. And, as before, for any $n > n_0$, $\langle \mathring{L} u^R_{\Omega_n}, g\rangle = (f, g)$. Hence we conclude that, indeed,

$$\langle \mathring{L} u^*_f, g\rangle = (f, g), \quad f, g \in C^\infty_c(\mathbb{R}^4, \Lambda^1). \tag{VI.18}$$

Since $\mathring{L}^{-1}$ exists in $\mathscr{D}'(\mathbb{R}^4, \Lambda^1)$, the relation (VI.18) implies that u^*_f is the solution to the problem $Lu = f$ in $\mathscr{D}'(\mathbb{R}^4, \Lambda^1)$, for $f \in C^\infty_c(\mathbb{R}^4, \Lambda^1)$, which is unique and coincides with $\mathring{L}^{-1} f$.

Hence we have that for any sequence $(\Omega_n)_{n\in\mathbb{N}_+}$ the sequence $((L^R_{\Omega_n})^{-1} f\,,\, f) = (u^R_{\Omega_n}, f)$, for any $f \in C^\infty_c(\mathbb{R}^4, \Lambda^1)$, has a well defined limit and the limit is $\langle L^{-1} f, f\rangle$, in formulas

$$\lim_{n\in\mathbb{N}_+} ((L^R_{\Omega_n})^{-1} f\,,\, f) = \langle L^{-1} f, f\rangle,$$

which concludes the proof of the theorem. □

References

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