Uniform Boundedness of the Pole Order of General Eisenstein Series

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Abstract

This work consists of two parts. In the first part we give a general introduction to the chapter 7 of [L1], and settle down some bases needed for the second chapter in which we prove that the order of the poles of a residual Eisenstein series on an arbitrary reductive group G which satisfies the conditions of the chapter 1 of this work is uniformly bounded by a constant which depends only on the number of elements of a subgroup of the Weyl group of G via the methods developed in [F1] and [F2]. Having a general understanding of the main assertions and difficulties that Langlands had faced and solved through his treatment of Eisenstein series is crucial in understanding [F1] and [F2], on them this work has been built, consequently we start this work with an introduction to Eisenstein series and afterwards in chapter 1 we review Eisenstein systems, and in chapter two we will prove the main claim of this work.

Chapter 0

Introduction

(0.1) The spectral decomposition of the regular representation of certain topological groups G on the Hilbert spaces of the form $L^2(\Gamma \setminus G)$, in which Γ is a discrete subgroup of G, lies at the intersection of several disciplines of mathematics such as number theory, functional analysis, and the theory of algebraic groups. The heart of spectral decomposition is the study of the Eisenstein series which is the starting point of the theory of Automorphic forms, which came out to be one of the fruitful branches of mathematics in the past decades with far reaching applications and deep conjectures.
In this introduction we will try to give an overview of the origins of the central problems that we are going to consider in this work, and to do so we start with a review of the classical Eisenstein series and after that we give the adelic interpretation of the classical situation due to Langlands which follows by a short discussion of the main problem considered in the second chapter this thesis.

We begin our discussion by fixing some notation.
We denote by $\mathfrak{H} = \{z \in \mathbb{C} | \Im(z) > 0\} \cong \mathrm{PSL}(2,\mathbb{R})/\mathrm{SO}(2)$ the upper half plane model of hyperbolic plane. The group $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathfrak{H}$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z \mapsto \frac{az+b}{cz+d}$. We will fix a fundamental domain $\mathfrak{F}$ for this action. The Laplacian on $\mathfrak{H}$ will be denoted by $\Delta_{\mathfrak{H}} = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$.
Now we can introduce the main object of this theory in the simplest setting, i.e., the Eisenstein series defined on a Fuchsian subgroup Γ of $\mathrm{PSL}(2,\mathbb{R})$ of the first kind. This means that Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$ with finite covolume and such that that every point of $\mathbb{R} \cup \infty$ is a limit point for the action of Γ on $\mathfrak{H}$. This implies that Γ has a finite complete set $\{\kappa_1, ...\kappa_n\} \subset \mathbb{R} \cup \infty$ of inequivalent **cusps**. By definition a point $\kappa \in \mathbb{R} \cup \infty$ is a cusp for Γ if the subgroup Γ_κ of Γ defined by $\Gamma_\kappa = \{\gamma \in \Gamma \mid \gamma \cdot \kappa = \kappa\}$ is conjugate to the subgroup $N(\mathbb{Z}) = \left\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \mid n \in \mathbb{Z}\right\}$. By definition, a **parabolic** subgroup is a subgroup that fixes a cusp. We will give the general definition of them in the next chapter. From now on we assume without los of generality that we have only one cusp κ. Fix an element $\sigma_\kappa \in \mathrm{SL}(2,\mathbb{R})$ such that $\sigma_\kappa(\infty) = \kappa$ and $\sigma_\kappa \Gamma_\kappa \sigma_\kappa^{-1} \cong N(\mathbb{Z})$. Then the **Eisenstein series** for the cusp κ is the

series

$$E(s,z)=\sum_{\gamma\in\Gamma_\kappa\backslash\Gamma}\Im(\sigma_\kappa^{-1}\cdot\gamma\cdot z)^s, \qquad (z\in\mathfrak{H},\ s\in\mathbb{C}).$$

The main properties of this series proved by Selberg in [S1] are:
(**a**) $E(s,z)$ converges absolutely and uniformly in the region $\Re s>1$ to an analytic function.
(**b**) $E(s,z)$ has meromorphic continuation to the whole plane with values in the Frechet space $C^\infty(\Gamma\setminus\mathfrak{H})$. The possible poles of $E(s,z)$ in $\{s\in\mathbb{C}\mid\Re s\geq\frac{1}{2}\}$ all lie in the interval $(\frac{1}{2},1]$. These poles are simple and the residue of $E(s,z)$ with respect to them is an element of $L^2(\Gamma\setminus\mathfrak{H})$.
(**c**) For $\Re s>1$ this series is an eigenfunction of the Laplacian $\Delta_{\mathfrak{H}}E(s,z)=s(1-s)E(s,z)$, the analytic continuation of this series to the whole plane satisfies this functional equation too, and if $s\in\frac{1}{2}+\mathrm{i}\mathbb{R}$, the Eisenstein series are generalized eigenfunctions of $\Delta_{\mathfrak{H}}$.
For the proof of these we refer to [B] theorems 10.4, 11.4 and 11.9.

(0.2) If $s\in\frac{1}{2}+\mathrm{i}\mathbb{R}$ then the Eisenstein series $E(s,z)$ play the same role for the spectrum of Laplace operator $\Delta_{\mathfrak{H}}$ on the space $L^2(\Gamma\setminus\mathfrak{H})$ as it is played by the continuous family of functions $\{e^{2\pi\mathrm{i}\lambda x}\mid\lambda\in\mathbb{R}\}$ for the spectrum of the operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $L^2(\mathbb{R})$. In other words the function $E(s,z)$ is the generalization of exponential function on $\mathbb{R}$ to locally symmetric spaces of the group $\mathrm{SL}(2,\mathbb{R})$. Like the exponential function, although they are the building blocks of the L^2 spectrum, they are not square integrable.
These properties were first observed by A.Selberg in his seminal paper [S], in which he was mainly concerned with the analytical properties his famous trace formula. Since then there was an effort to generalize the ideas of Selberg to more general groups and also under the adelic language which is important in number theory and has applications in physics too. This happened to be a major challenge which was done by R.Langlands in [L1]. In that paper Langlands generalized the results of Selberg to the discrete subgroups of real reductive groups of finite covolume. To gain a glimpse of this theory in the sense of Langlands we will work in the context of a general reductive algebraic group G defined over $\mathbb{Q}$ and its adelization $G(\mathbb{A})$, since in addition to several enhancements, it makes the exposition of the theory much easier. With this principle at hand The general form of the Langlands-Eisenstein series will look like

$$E(\lambda,f)(g)=\sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})}e^{\langle\lambda+\rho_P,H_P(\gamma g)\rangle}f(\gamma g),$$

in which the function $f(g)$ belongs to a space of quadratic integrable automorphic forms $A^2(P)$, defined in **(1.4)**. In the above expression P is a parabolic subgroup of G, λ a parameter that belongs to the Lie algebra $\check{\mathfrak{a}}_P$ of the maximal split torus of G, $H_P(.)$ denotes a height function on $G(\mathbb{A})$ with values in the maximal $\mathbb{Q}$-split torus of the center of the Levi component $L\subset P$, and ρ_P is the half sum of the positive roots with respect to P. All these terms will be defined in the paragraphs **(1.1)** and **(1.3)** of the next chapter.

One of the difficulties of the Langlands theory of Eisenstein series relies on the fact, that a direct approach to these series, at least with the methods developed by Langlands, seems to be impossible. Langlands developed his theory by starting from **cusp** forms $f(.)$ on the Levi component L of P and from them he created a family of Eisenstein series that he called the **cuspidal Eisenstein series**. See chapter 1 paragraphs **(1.1)** and **(1.3)**. He proved that these series satisfy similar properties **(a)** to **(c)** of the classical situation (mentioned in **(0.1)**) in a much more general setting. If we drop the cuspidality condition and assume that $f(.)$ is a general quadratic integrable automorphic form, we will obtain the most general form of Eisenstein series. For these series there is no known direct way of meromorphic continuation which enables us to generalize the property **(a)** to **(c)** to this more general setting, but such a generalization is vital for the spectral decomposition and also for the applications in the trace formula. In the lack of a direct way, Langlands developed an approach to generate most general Eisenstein series by treating these series as the iterating residues of cuspidal Eisenstein series and called them the **residual Eisenstein series**. One of his remarkable achievements was that he showed that in this way one obtains all the Eisenstein series needed to exhaust the complete spectral decomposition of the space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$. Langlands showed also that affine hyperplanes which carry the parameters of the Langlands-Eisenstein series are real[1] and also that the residual series are holomorphic on the unitary axis consequently generalized the first part of **(b)** to the residual series. But the second property **(b)**, the realness of the poles for residual series, was not directly obtainable from the methods of Langlands. The generalization of the second property in **(b)** was first proved by J.Franke as the theorem 1 in [F2]:

Let $H \subset (\check{\mathfrak{a}}_P)_{\mathbb{C}}$ be singular hyperplane of a residual Eisenstein series $E(\lambda, f)$ such that $H \cap (\check{\mathfrak{a}}_G + \mathrm{i}\, \check{\mathfrak{a}}_P + \overline{\check{\mathfrak{a}}_P^{G+}}) \neq \emptyset$. Then H is real and meets $\check{\mathfrak{a}}_P^{G+}$.[2]

The third part of the property **(b)**, the simplicity of the poles was proved to be wrong for the residual series. Langlands computed a counterexample in the case of the group G_2 in the appendix (III) of [L1] and showed that there exists poles of residual Eisenstein series which are not simple and the real part of them lie in the positive Weyl chamber. A natural question arises here that what are the possible orders of these poles? In the second chapter of this work we give a partial answer to this question by proving the following theorem in paragraph **(2.2)** of this work:

[1] see the definition of Eisenstein systems in **(1.11)** of the next chapter
[2] All the terms and spaces used here are defined in the next chapter.

Theorem. Let H be singular hyperplane of a residual Eisenstein series such that $H \cap (\check{\mathfrak{a}}_G + \mathrm{i}\,\check{\mathfrak{a}}_P + \overline{\check{\mathfrak{a}}_P^{G+}}) \neq \emptyset$. Then the order of H is at most $\max_{Q \in \{\mathrm{P}\}} \#\Omega(P, \chi, \psi, Q)$.

In other words, the order of the poles of a residual Eisenstein series is bounded by a constant which depends on the order of the Weyl group of G and consequently to the group G itself. The proof will follow from the remark 3 in [F2]:

...assume that [the singular hyperplane] H meets $\check{\mathfrak{a}}_P^{G+}$, let $k > 0$ and let $\omega \in \Omega(H, P, \chi, \boldsymbol{\psi}, Q)$ be such that $N_k(\omega, \lambda) \not\equiv 0$ and such that if $\tilde{\omega} \in \Omega(H, P, \chi, \boldsymbol{\psi}, Q)$ and $N_k(\tilde{\omega}, \lambda) \not\equiv 0$ then $|\ (\omega(x))_+\ | \geq |\ (\tilde{\omega}(x))_+\ |$ for all x in an open subset of $H \cap \check{\mathfrak{a}}_P^{G+}$. Then $N_j(\omega, \lambda) \equiv 0$ for $j > k$.

We reformulate this remark as the theorem 1 of the chapter two and define all the needed terms in chapter one of this work. To prove this theorem we have to prove that for an quadratic integrable automorphic form $f(.)$ defined on the quotient $G(\mathbb{Q})A_G(\mathbb{R})^\circ \backslash G(\mathbb{A})$ and a standard parabolic subgroup Q of G, the $f_{Q,\lambda}(H_Q(g))$ (defined in **(2.3)**) operators, which are known to be polynomials (lemma 4.2 of [L1]), are actually monomials. This is done in Lemma 1 of the chapter 2. To prove this lemma we need some structural results which rely on some results of Harish-Chandra and Helgason. Afterwards it will be an easy consequence that the $f_{Q,\lambda}(H_Q(g))$ operators are actually harmonic polynomials with respect to the subgroup W_λ of the Weyl group $W(G, A)$ of G which leaves λ invariant. After these preparations the machinery of the Eisenstein systems leads us to obtain the same result for the $N(.,.)$ operators (defined in **(2.1)**) attached to residual Eisenstein series. Then through the techniques developed in [F2] we prove the above mentioned upper bound on the order of the poles of a residual Eisenstein series.

Chapter 1

A Review of Eisenstein Systems

(1.1) In this chapter we follow two goals. First of all, since we are going to use a part of the machinery of Eisenstein systems in the proof of our main lemma in the second chapter of this work, we have to give an overview of this concept which was introduced in chapter 7 and appendix II of [L1], following the presentations of [A1], [A2], and [OW] in the language of Adeles. Our second goal is to show that how the axioms given in 5.2 of [F1] (which served as a black box for the proofs given there) are deduced from the lemmas and the main theorem of the chapter 7 of [L1]. This will be done at the end of this chapter in paragraph **(1.20)**. We use a combination of the notations of [F1], [F2] and [M2] which is almost identical with [A1] and [A2].

Let G be a reductive algebraic group defined over $\mathbb{Q}$ with the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})$. We denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ admits the so called Levi decomposition of $\mathfrak{g} = \mathfrak{rad}(\mathfrak{g}) \oplus \mathfrak{l}$, with $\mathfrak{l}$ the Levi subalgebra and $\mathfrak{rad}(\mathfrak{g})$ the radical, i.e., the largest solvable ideal of $\mathfrak{g}$. A Levi subalgebra $\mathfrak{l}$ is always semisimple since the natural projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$ maps isomorphically any Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ onto the semisimple lie algebra $\mathfrak{s} = \mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$. Reductive Lie algebras admit the decomposition $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ with respect to the Killing form, in which the center $\mathfrak{z}(\mathfrak{g})$ consist of semi simple elements and the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is a semisimple subalgebra. We will denote the Killing form of $\mathfrak{g}$ by $\langle x, y\rangle$ which satisfies the identity $\langle [x, y], z]\rangle$ $+\langle y, [x, z]\rangle$=0. In what follows, the dual subspace of a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ with respect to the Killing form will be denoted by $\check{\mathfrak{a}}$.
Suppose for the moment that $\mathfrak{g}$ is a complex semisimple Lie algebra. Let us fix a Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$, and decompose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with respect to this involution to the eigenspaces corresponding to ± 1 eigenvalues respectively. We recall that a **Cartan subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is by definition a maximal nilpotent subalgebra of $\mathfrak{g}$ invariant under θ, which coincides with its normalizer $\mathfrak{n}(\mathfrak{h})$. We can extend the definition of Cartan subalgebra from the complex semisimple case to complex reductive Lie algebras by just adjoining the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ to a Cartan subalgebra of the $[\mathfrak{g}, \mathfrak{g}]$, the semisimple

part of $\mathfrak{g}$. Then we can extend this definition to the real reductive Lie algebras $\mathfrak{g}$ by requiring that a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a Cartan subalgebra if its complexification $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of the $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$.

We define a **parabolic subalgebra** $\mathfrak{p} \subseteq \mathfrak{g}$ to be any subalgebra of the real reductive Lie algebra $\mathfrak{g}$ that contains a maximal solvable subalgebra of $\mathfrak{g}$, i.e., a **Borel subalgebra** $\mathfrak{b}$ of $\mathfrak{g}$. It is clear that $\mathfrak{rad}(\mathfrak{g}) \subset \mathfrak{b}$. Let us denote by $\mathfrak{n}_P$ the maximal normal subalgebra of $\mathfrak{p} \cap [\mathfrak{g}, \mathfrak{g}]$ whose image in ad$\mathfrak{g}$ consists of nilpotent elements, and let $\mathfrak{m}_P$ denote the maximal subalgebra of $\mathfrak{p}$ whose image in ad$\mathfrak{g}$ is reductive. Hence we have the decomposition $\mathfrak{p} = \mathfrak{m}_P + \mathfrak{n}_P$. Let us denote by $\mathfrak{a}_P$ the maximal diagonalizable subalgebra of $\mathfrak{z}(\mathfrak{m}_P \cap [\mathfrak{g}, \mathfrak{g}])$ and denote by $\mathfrak{m}_P^1$ the orthogonal complement of $\mathfrak{a}_P$ in $\mathfrak{m}_P$ with respect to the Killing form. Then we have $\mathfrak{a}_P \cap \mathfrak{m}_P^1 = \{0\}$ and these considerations lead us to the **Langlands decomposition** $\mathfrak{p} = \mathfrak{m}_P^1 + \mathfrak{a}_P + \mathfrak{n}_P$. We call the subalgebra $\mathfrak{m}_P = \mathfrak{m}_P^1 + \mathfrak{a}_P$ the **Levi subalgebra**, and the subalgebra $\mathfrak{a}_P$ the **split component** of $\mathfrak{p}$. We will give a characterization of these subspaces soon.

On the other hand we define a **parabolic subgroup** P of G to be a Zariski closed subgroup which contains a Borel subgroup. By definition, a **Borel subgroup** of G is a subgroup whose Lie algebra is a Borel subalgebra $\mathfrak{b}$ as defined above. There is a correspondence between the Borel subalgebras of $\mathfrak{g}$ and Borel subalgebras of $\mathfrak{g}/\mathfrak{rad}(\mathfrak{g})$ since by maximal solvability we have $\mathfrak{rad}(\mathfrak{g}) \subset \mathfrak{b}$ for each Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. The maximal solvability of the subalgebras $\mathfrak{b}$ implies that the quotient space G/P is complete, which is a consequence of the well known Borel fixed point theorem. Again by virtue of the solvability we deduce the existence of parabolic subgroups which are minimal in the quasi projective variety of parabolic subgroups of G. By the definition it is clear that the minimal parabolic subgroups are Borel subgroups. We fix once and for all a minimal parabolic subgroup $P_\circ$ of G and call a parabolic subgroup P **standard** if it contains $P_\circ$. For the reasons that will be clarified soon, we deal only with standard parabolic subgroups in this work.

Each parabolic subgroup P can be decomposed as $P = M_P N_P$, which is the analog of the Langlands decomposition of the Lie algebras given above. The subgroup M_P is the **Levi subgroup** of P and the subgroup N_P is the **unipotent radical**. It is clear that $\mathfrak{m}_P$ and $\mathfrak{n}_P$ are the Lie algebras of M_P and N_P. As in the discussion about the Lie algebras above we can decompose the Levi component further as $M_P = M_P^1 A_P$, in which A_P is the $\mathbb{Q}$-**split torus** (or the split component) of the center of M_P with the Lie algebra $\mathfrak{a}_P$. The **rank** of a Parabolic subgroup is the dimension of its split component A_P over $\mathbb{Q}$ and is denoted by $\mathrm{rank}_{\mathbb{Q}}(P)$.

The corresponding Langlands decomposition for the minimal parabolic subgroup $P_\circ$ will be $M_\circ^1 A_\circ N_\circ$. Observe that $A_P \subset A_\circ$ and $M_\circ \subset M_P$ for all the standard parabolic subgroups $P \subset G$. The inclusion $A_P \subset A_\circ$ defines an inclusion $\mathfrak{a}_P \to \mathfrak{a}_\circ$ which gives us the direct sum decomposition $\mathfrak{a}_\circ = \mathfrak{a}_P \oplus \mathfrak{a}_\circ^P$

for $\mathfrak{a}_\circ^P$ the orthogonal complement of $\mathfrak{a}_P$ in $\mathfrak{a}_\circ$ with respect to the Killing form. There is also a decomposition of the corresponding dual subspaces as $\check{\mathfrak{a}}_\circ = \check{\mathfrak{a}}_P \oplus \check{\mathfrak{a}}_\circ^P$. More generally for the parabolic subgroups $P \subset Q$ we will have a decomposition of the $\mathbb{Q}$-split torus as $\mathfrak{a}_P = \mathfrak{a}_Q \oplus \mathfrak{a}_P^Q$, in which $\mathfrak{a}_P^Q$ could be characterized either as the intersection of $\mathfrak{a}_P$ and $\mathfrak{a}_\circ^Q$ in $\mathfrak{a}_\circ$ or as the orthogonal complement of $\mathfrak{a}_Q$ in $\mathfrak{a}_P$. We have also the analog decomposition $\check{\mathfrak{a}}_P = \check{\mathfrak{a}}_P \oplus \check{\mathfrak{a}}_Q^P$ for the dual spaces. For an intrinsic characterization of these subgroups and subspaces appearing in the Langlands decomposition in the real situation we need to review briefly the root systems.

Let $X(A_\circ)_\mathbb{Q} \subset \check{\mathfrak{a}}_\circ$ denote the group of $\mathbb{Q}$-rational characters of $A_\circ$. We denote by $\Phi_\circ \subset X(A_\circ)_\mathbb{Q}$ the set of roots of $A_\circ$ in $\mathfrak{g}$. The set of roots of the pair (P, A_P) will be denoted by $\Phi(\mathfrak{n}_P)$, the positive roots by $\Phi^+(\mathfrak{n}_P)$, the subset of simple roots by Δ_P^+, and their duals by $\check{\Phi}(\mathfrak{n}_P)$, $\check{\Phi}^+(\mathfrak{n}_P)$ and $\check{\Delta}_P^+$. The corresponding set of roots with respect to $(P_\circ, A_\circ)$ will be denoted by $\check{\Delta}_\circ^+$ etc. For standard parabolic subgroups $P \subset R$ the decomposition $\check{\mathfrak{a}}_P = \check{\mathfrak{a}}_R \oplus \check{\mathfrak{a}}_P^R$ gives us the corresponding subsets $\Phi^+(\mathfrak{n}_P^R)$, $\check{\Delta}_P^{R+}$ and Δ_P^{R+}. Then the set $\Phi^+(\mathfrak{n}_P^R)$ will be the set of positive roots which occur in $\mathfrak{n}_P$ but not in $\mathfrak{n}_R$. The half sum of the positive roots of the pair (P, A_P) will be denoted by $\rho_P = \frac{1}{2}\sum_{\alpha\in\Phi^+(\mathfrak{n}_P)} \check{\alpha}$. Then for parabolic subgroups $P \subset R$, ρ_R is the projection of ρ_P on the $\check{\mathfrak{a}}_P^R$.

Now we can characterize the components of the Langlands decomposition for the Lie subalgebra $\mathfrak{p}$ as follows. For each root $\alpha \in \Phi(\mathfrak{n}_P)$ we define a root subalgebra $\mathfrak{n}_\alpha = \{X \in \mathfrak{n} \mid [H, X] = \alpha(H)X, \forall H \in \mathfrak{a}_P\}$ which yields the decomposition $\mathfrak{n}_P = \oplus_{\alpha\in\Phi(\mathfrak{n}_P)}\mathfrak{n}_\alpha$ of the Lie algebra of the unipotent radical N_P of P. The split component $\mathfrak{a}_P$ (or A_P) is distinguished by the property that $\mathrm{tr}(\mathrm{ad}(Y)|\mathfrak{n}_\alpha) = 0$ for all $Y \in \mathfrak{m}_P^1$ and all $\alpha \in \Phi(\mathfrak{n}_P)$, and the subgroup M_P^1 is characterized by the property that it consists of the elements $m \in M_P$ such that $\det(\mathrm{Ad}(m)|\mathfrak{n}_\alpha) = \pm 1$ for all $\alpha \in \Phi(\mathfrak{n}_P)$. We can also obtain this characterizations by introducing $X(M_P)_\mathbb{Q}$, the group of rational characters of M_P defined over $\mathbb{Q}$, and observing that $\mathfrak{a}_P \cong \mathrm{Hom}_\mathbb{Z}(X(M_P)_\mathbb{Q}, \mathbb{R})$, and also the dual isomorphism $\check{\mathfrak{a}}_P \cong X(M_P)_\mathbb{Q} \otimes_\mathbb{Z} \mathbb{R}$ with respect to the Killing form. The transfer from the real situation to rational situation is obvious now.

As in the introduction, the group of adeles of $\mathbb{Q}$ will be denoted by $\mathbb{A}$. The groups over $\mathbb{Q}$ or adelic points will be denoted by $G(\mathbb{Q})$ or $G(\mathbb{A})$ etc. We also fix once and for and all a compact subgroup $\mathbb{K} = KK_f \subset G(\mathbb{A})$ in which $K_f = \prod_{\nu<\infty} K_\nu$ for $K_\nu \subset G(\mathbb{Q}_\nu)$ a maximal compact open subgroup and $K \subseteq K_\infty$ a compact subgroup such that the **Iwasawa decomposition** $G(\mathbb{A}) = P_\circ(\mathbb{A})\mathbb{K}$ holds. This decomposition is also valid for P any standard parabolic subgroup, which gives us the decomposition $G(\mathbb{A}) = M_P^1(\mathbb{A})A_P(\mathbb{R})^0N_P(\mathbb{A})\mathbb{K}$ and any $g \in G(\mathbb{A})$ can be decomposed as $g = namk$ for $n \in N(\mathbb{A})$, $a \in A(\mathbb{R})^0$, $m \in M^1(\mathbb{A})$, and $k \in K$. If we denote the Lie algebra of K_∞ by $\mathfrak{k}$ then this decomposition is supposed to satisfy the **admissibility** condition $\langle \mathfrak{k}, \mathfrak{a}_P \rangle = 0$, for $\mathfrak{a}_P$ the Lie algebra of the subgroup A_P of P.

Following [A1] we define for the Levi subgroup M_P of a parabolic subgroup P a homomorphism $H_{M_P}(m)$ from $M_P(\mathbb{A})$ to the additive group $\mathfrak{a}_{M_P}$ as follows. Let $m = \Pi_\nu m_\nu \in M_P(\mathbb{A})$, for ν places of $\mathbb{Q}$, and let χ be any rational character of M_P. Then $H_{M_P}(m)$ is the vector in $\mathfrak{a}_M$ which satisfies the relation $e^{\langle \chi, H_{M_P}(m) \rangle} = |\chi(m)| = \Pi_\nu |\chi(m_\nu)|_\nu$. The kernel of the homomorphism $H_{M_P} : M_P(\mathbb{A}) \to \mathfrak{a}_{M_P}$ is the subgroup $M_P^1(\mathbb{A})$. Then $H_{M_P}(.)$ factors through $M_P^1(\mathbb{A}) \setminus M_P(\mathbb{A})$ and also $H_{M_P}(M_P^1(\mathbb{A}) \setminus M_P(\mathbb{A})) = \Re(\mathfrak{a}_{M_P})$. We can extend this homomorphism to a homomorphism $H_P(g)$ on $G(\mathbb{A})$ by setting $H_P(g) = H_{M_P}(ma) = H_{M_P}(a)$. Observe that $|\det(\mathrm{Ad} p)|\mathfrak{n}_P(\mathbb{A})| = e^{\langle 2\rho_P, H_P(p) \rangle}$. Then for parabolic subgroups $P \subset Q$ the additive group $\mathfrak{a}_P^Q$ is the image of $M_Q(\mathbb{A})$ under $H_P(.)$.

It is well-known that any rational parabolic subgroup $P \subseteq G$ is conjugate to a standard parabolic subgroup via an element of $G(\mathbb{Q})$. For two parabolic subgroups P and Q of G let $\Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)$ denote the set of linear transformations from $\check{\mathfrak{a}}_P$ to $\check{\mathfrak{a}}_Q$ obtained by restricting $\mathrm{Ad}\, g$ to $\check{\mathfrak{a}}_P$ for $g \in G$. It is a subquotient set of the Weyl group of G. If $\Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)$ happens to be non empty then P and Q are called **associated** parabolic subgroups. For the groups defined over $\mathbb{R}$ the relation of being associated breaks the set of parabolic subgroups of G into finitely many equivalence classes. Alternatively two parabolic subgroups P and Q are associated if and only if their Levi components are conjugate over $\mathbb{Q}$, or equivalently if and only if P and xQx^{-1} have a common Levi subgroup for some $x \in G(\mathbb{Q})$. Therefore the associated classes are also detectable through the Levi subgroups of parabolic subgroups. Any two Levi subgroups of a parabolic subgroup P are also conjugate via an element of $P(\mathbb{Q})$. Since in the adelic situation we have to deal with only one cusp, we need only to consider one class of conjugate Levi subgroups of parabolic subgroups. We fix such an associated class $\{\mathrm{P}\}$ of standard parabolic subgroups of G whose Levis are conjugate to each other via the conjugation by an element of $\Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)$, i.e. if $P, Q \in \{\mathrm{P}\}$, then for each element of $\omega \in \Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)$ there exists an element $s = s_\omega \in G(\mathbb{Q})$ such that $sM_Ps^{-1} = M_Q$.
We mention two consequences of the above definitions which will be important for us in the definition of Eisenstein systems:
(A) Let P and Q be parabolic subgroups of G. If P in conjugate to Q and if $P \cap Q$ is a parabolic subgroup then $P = Q$.
(B) Conjugate parabolic subgroups of G are associated, but the converse is not generally true.
For example in $\mathrm{SL}(3, \mathbb{R})$ two parabolic subgroups with respect to the decompositions $3 = 2 + 1$ and $3 = 1 + 2$ are associated but not conjugate. To justify **(A)**, suppose that for some $x \in G$ we have $P = xQx^{-1}$. We first

observe that P is self-normalizer in G, i.e., $N_G(P) = P$, and let us take a Borel subgroup B of P. Then the subgroups B and xBx^{-1} are both Borel subgroups of P. Since B is contained in P^0, the connected component of P, we observe that xBx^{-1} is also a Borel subgroup of P^0. Since all the Borel subgroups are conjugate to each other, we see that there is a $y \in P^0$ such that $xBx^{-1} = yBy^{-1}$. Since the Borel subgroups are also self normalized, this implies that yx^{-1} lies in B and hence $x \in P^0$. Consequently x lies in P and $P = Q$. For more comments on these properties we refer for example to [War] section 1.2. Minimal parabolic subgroups are associated if and only if they are conjugate. Consequently, without loss of generality, we can restrict our attention to the standard parabolic subgroups which contain the minimal parabolic subgroup $P_\circ$ fixed above. From now on we assume that all the parabolic subgroups appearing in this work are standard.

After this discussion of parabolic subgroups we consider some subspaces which will be important for us in what follows. Let $F \subset \Delta_P^+$. We call the subspace

$$\mathfrak{c}_F = \{\lambda \in (\check{\mathfrak{a}}_P)_{\mathbb{C}} |\ \alpha(\lambda) = 0 \text{ for all } \alpha \in F\}$$

a **distinguished subspace** of $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$. We write $\check{\mathfrak{a}}_0$ instead of $\mathfrak{c}_F$ if F is the fixed subset of the roots which define a parabolic subalgebra $\mathfrak{p}_0$ and hence associate a parabolic subgroup P_0 to this distinguished subspace $\check{\mathfrak{a}}_0$.

Let

$$\mathfrak{a}_P^+ = \{\lambda \in \mathfrak{a}_P \mid \alpha(\lambda) > 0\ \forall \alpha \in \Delta_P^+\},$$

and

$${}^+\mathfrak{a}_P = \{\lambda \in \mathfrak{a}_P \mid \check{\alpha}(\lambda) > 0\ \forall \alpha \in \check{\Delta}_P^+\}.$$

Then we have $\mathfrak{a}_P^+ \subset {}^+\mathfrak{a}_P$. If we denote the Cartan subalgebra $\mathfrak{h}$ fixed above by $\mathfrak{a}_G$ (i.e., by regarding the group G as a parabolic subgroup in itself), we can form the subspaces $\mathfrak{a}_\circ^G$ and $\check{\mathfrak{a}}_\circ^G$ with respect to $P_\circ$. Then the subspaces $\mathfrak{a}_\circ^{G+} \subset \mathfrak{a}_\circ^G$ and $\check{\mathfrak{a}}_\circ^{G+} \subset \check{\mathfrak{a}}_\circ^G$ are called the open positive **Weyl Chambers**, and the subspaces ${}^+\mathfrak{a}_\circ^G \subset \mathfrak{a}_\circ^G$ and ${}^+\check{\mathfrak{a}}_\circ^{G+} \subset \check{\mathfrak{a}}_\circ^G$ are the open positive cones dual to the positive Weyl chambers with respect to the simple positive roots $\Delta_\circ^+$.

For a constant $c \in \mathbb{R}_{>0}$ let also

$$A_P^+(c) = \{\lambda \in \mathfrak{a}_P \mid e^{\langle \alpha, \lambda \rangle} > c\ \forall \alpha \in \Delta_P^+\},$$

and

$${}^+A_P(c) = \{\lambda \in \mathfrak{a}_P \mid e^{\langle \check{\alpha}, \lambda \rangle} > c\ \forall \check{\alpha} \in \check{\Delta}_P^+\}.$$

Fix a compact subset $\omega \in M_P N_P$, we define a **Siegel Domain** associated to P to be the set

$$\mathfrak{S}_P(c) = \Big\{ g = nmak \mid mn \in \omega, a \in \exp\big(A_{M_\circ}^{M_P+}(c)\big), k \in K \Big\}.$$

The space $A^{M_P+}_{M_\circ}(c)$ is defined just like above with respect to the roots in $\Delta^{M_P+}_{M_\circ}$. Since in the adelic formalism we can consider all the cusps at once, we will fix once and for all a Siegel domain $\mathfrak{S}_\circ$ with respect to the fixed minimal parabolic subgroup $P_\circ$ and a positive parameter $c = \min_{m\in M_\circ,\alpha\in\Delta_\circ}\{e^{\langle\check{\alpha},a_\circ(m)\rangle}\}$ and $A^+_{M_\circ}$. We will return to the subject of Siegel domains in **(1.3)** when we discuss the reduction theory.

Let $P \subset R$ be two standard parabolic subgroups. There is a geometrical decomposition of the elements of $\check{\mathfrak{a}}_P$ introduced by Langlands which will be crucial for us in the next chapter. We introduce the basis $\hat{\Delta}^{R+}_P = \{\mathfrak{w}^R_{\check{\alpha}} | \alpha \in \Delta^{R+}_P\}$ of $\check{\mathfrak{a}}^R_P$ which is dual to the basis $\check{\Delta}^{R+}_P$. We can define the elements $\check{\mathfrak{w}}^R_{\check{\alpha}} \in \mathfrak{a}^R_P$ similarly. We can now state our decomposition:
Let $\lambda \in \check{\mathfrak{a}}^R_P$. Then according to the theorem 2.3 of [L1], there exists a parabolic subgroup $R(\lambda)$ which satisfies $P \subseteq R(\lambda) \subseteq R$ and a subset $P(\lambda) \subseteq \Delta^{R+}_P$ such that

$$\lambda = \sum_{\alpha\in\Delta^{R+}_P - P(\lambda)} a_\alpha \mathfrak{w}^R_{\check{\alpha}} - \sum_{\beta\in P(\lambda)} b_\beta \beta$$

with $a_\alpha > 0$ and $b_\beta \geq 0$. We will write

$$(\lambda)_+ = \sum_{\alpha\in\Delta^{R+}_P - P(\lambda)} a_\alpha \mathfrak{w}^R_{\check{\alpha}} \in \check{\mathfrak{a}}^{R+}_{R(\lambda)},$$

and

$$(\lambda)_- = -\sum_{\beta\in P(\lambda)} b_\beta \beta \in -{}^+\check{\mathfrak{a}}^{R(\lambda)}_P.$$

For the proof we refer to [W] p.164.

Finally we fix a height function $\| \, . \, \|$ on G coming from the Killing form and satisfies the properties mentioned in I.2.2 of [MW].

(1.2) In this subsection we discuss shortly the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$ which plays a pivotal role in the theory of Eisenstein series and automorphic forms. As a general reference for this subsection we refer to [L1] chapters 4, [H1] chapter IV and [H2]. We denote by $\mathfrak{B}$ the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$. Then the center $\mathcal{Z}(\mathfrak{g})$ of $\mathfrak{B}$ is noetherian and will be identified with the algebra of polynomials on $\mathfrak{h}_\mathbb{C}$ invariant under the Weyl group W_G of G and further to the algebra of the left invariant differential operators on G. For a parabolic subgroup $P \subset G$ the decomposition $\mathfrak{p} = \mathfrak{m}^1_P + \mathfrak{a}_P + \mathfrak{n}_P$ yields a decomposition in the enveloping algebras as follows. Let $\mathfrak{n}^-_P$ denote the negative of $\mathfrak{n}_P$ such that $\mathfrak{g}_\mathbb{C} = \mathfrak{n}^-_{P\mathbb{C}} + \mathfrak{m}^1_{P\mathbb{C}} + \mathfrak{a}_{P\mathbb{C}} + \mathfrak{n}_{P\mathbb{C}}$. Let us denote by $\mathfrak{N}^-$, $\mathfrak{M}$, $\mathfrak{A}$, and $\mathfrak{N}$ the corresponding sub algebras of $\mathfrak{B}$ respectively. Then we have an isomorphism $\mathfrak{N}\otimes\mathfrak{M}\otimes\mathfrak{A}\otimes\mathfrak{N}^- \to \mathfrak{B}$. If we identify $1\otimes\mathfrak{A}\otimes\mathfrak{M}\otimes 1$ with $\mathfrak{A}\otimes\mathfrak{M}$ and denote the center of $\mathfrak{M}$ by $\mathcal{Z}(\mathfrak{m}_P)$, then we can identify each element of $\mathcal{Z}(\mathfrak{g})$ with an element of $\mathfrak{A}\otimes\mathcal{Z}(\mathfrak{m}_P)$

modulo $\mathfrak{n}_{P\mathbb{C}}\mathfrak{B}$.
There is a distinguished element of $\mathcal{Z}(\mathfrak{g})$ called the **Casimir** operator of $\mathfrak{g}$, which will play a crucial role for us in the second part of this work. We will define it first under the restriction that $\mathfrak{g}$ is semisimple, then we will show that the reductive situation is a slight generalization of the semisimple situation. Let us choose a basis $X_1, ..., X_n$ for $\mathfrak{g}_{\mathbb{C}}$ and put $g_{ij} = \langle X_i, X_j \rangle$ and let g^{ij} denote the corresponding elements of the inverse matrix. Then the element $\omega_{\mathfrak{g}} = \sum_{i,j} g^{ij} X_i X_j$ is the Casimir element of $\mathfrak{g}_{\mathbb{C}}$ which lies in $\mathcal{Z}(\mathfrak{g})$. More precisely, we take the Cartan involution θ of $\mathfrak{g}$ which gives us the Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ fixed at **(1.1)**. We fix a fundamental system of roots for the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, then, for each fundamental root α, we choose a pair of normalized bases $\{X_\alpha\}$ and $\{H_\alpha\}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$, where H_α are the elements of $\mathfrak{h}$ such that $\langle H, H_\alpha \rangle = \alpha(H)$ for all $H \in \mathfrak{h}$, which gives us the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha > 0} (\mathfrak{g}_{\alpha\mathbb{C}} + \mathfrak{g}_{-\alpha\mathbb{C}})$. Fix a basis $\{H_1, ..., H_m\}$ of $\mathfrak{h}$ over $\mathbb{R}$ such that $\langle H_i, H_j \rangle = \delta_{ij}$ and such that $\{H_1, ..., H_l\}$ is a basis of $\mathfrak{h} \cap \mathfrak{p}$, and $\{H_{l+1}, ...H_m\}$ is a basis of $\mathfrak{h} \cap \mathfrak{k}$ over $\mathbb{R}$. Then the Casimir operator $\omega_{\mathfrak{g}}$ of $\mathfrak{g}_{\mathbb{C}}$ can be written as

$$\begin{aligned}
\omega_{\mathfrak{g}} &= H_1^2 + ... + H_m^2 + \sum_{\alpha \in \Phi^+(\mathfrak{n}_P)} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) \\
&= H_1^2 + ... + H_m^2 + 2 \sum_{\alpha \in \Phi^+(\mathfrak{n}_P)} X_\alpha X_{-\alpha} - \sum_{\alpha \in \Phi^+(\mathfrak{n}_P)} H_\alpha.
\end{aligned}$$

We put $\omega_{\mathfrak{h}} = H_1^2 + ... + H_m^2$.
More generally, suppose that the algebra $\mathfrak{g}$ is reductive. Then we have a decomposition $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}'$, in which $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $\mathfrak{g}'$ is, as usual, the derived subalgebra of $\mathfrak{g}$. We fix a basis $C_1, ..., C_r$ of $\mathfrak{c}$ over $\mathbb{R}$. Then we have the decomposition $\mathcal{Z}(\mathfrak{g}) = \mathfrak{C} \otimes \mathcal{Z}(\mathfrak{g}')$ with the obvious notation. This shows that the Casimir element of $\mathfrak{g}$ is the sum of the Casimir elements of $\mathfrak{C}$ and of $\mathcal{Z}(\mathfrak{g}')$. If we put $\omega_{\mathfrak{c}} = C_1^2 + ... + C_r^2$ we can finally write the Casimir element of $\mathfrak{B}$ as

$$\omega_{\mathfrak{g}} = \omega_{\mathfrak{g}'} - \omega_{\mathfrak{c}},$$

in which we compute $\omega_{\mathfrak{g}'}$ (with respect to the Cartan subalgebra $\mathfrak{h}'$ such that $\mathfrak{h}'_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \cap [\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}]$) via the element $\omega_{\mathfrak{h}'}$, just like the above construction in the semisimple case. The above discussion clarifies that this procedure could be repeated if we restrict ourselves to the reductive subalgebras $\mathfrak{a}_P + \mathfrak{m}_P$ since it contains a Cartan subalgebra. Consequently we can associate to each standard parabolic subgroup $P \subseteq G$ a Casimir element in $\mathfrak{A} \otimes \mathcal{Z}(\mathfrak{m}_P)$ modulo $\mathfrak{n}_{P\mathbb{C}}\mathfrak{B}$ which we denote by $\omega_{\mathfrak{p}}$. This finishes our discussion about Casimir element .

Now we introduce a finite subset of affine transformations $\Omega(P, \chi, \boldsymbol{\psi}, Q)$ which we need later. Let Q and P be associated parabolic subgroups. We call two characters $\psi : \mathcal{Z}(\mathfrak{m}_Q) \to \mathbb{C}^*$ and $\chi : \mathcal{Z}(\mathfrak{m}_P) \to \mathbb{C}^*$associated if the following condition holds. There is a $g \in G(\mathbb{A})$ which satisfies $\mathrm{Int}(g)\mathfrak{m}_P = \mathfrak{m}_Q$

for the inner automorphism Int(.) of $\mathfrak{g}$, and such that g identifies χ with ψ. This is an equivalence relation and we denote equivalence class of such associated infinitesimal characters by $\boldsymbol{\psi}$. We will denote by $\boldsymbol{\psi}_Q$ the subset of $\boldsymbol{\psi}$ which consists merely of characters of $\mathcal{Z}(\mathfrak{m}_Q)$. For P, Q, χ and $\boldsymbol{\psi}$ we have a finite set $\Omega(P,\chi,\boldsymbol{\psi},Q)$ of affine transformations from $\check{\mathfrak{a}}_P$ to $\check{\mathfrak{a}}_Q$ such that for each $\omega \in \Omega(P,\chi,\boldsymbol{\psi},Q)$ the linear part $\hat{\omega}$ of ω is the restriction to $\check{\mathfrak{a}}_P$ of an element of the Weyl group W_G of G, and $\omega(0)$ is orthogonal to $\hat{\omega}(\check{\mathfrak{a}}_P)$. If $Q \subseteq P$ let $\Omega_\circ(P,\chi,\boldsymbol{\psi},Q) \subset \Omega(P,\chi,\boldsymbol{\psi},Q)$ denote the subset of all affine transformations $\omega \in \Omega(P,\chi,\boldsymbol{\psi},Q)$ such that $\hat{\omega}$ is the identity embedding of $\check{\mathfrak{a}}_P$ to $\check{\mathfrak{a}}_Q$. Observe that it is possible that $\Omega_\circ(P,\chi,\boldsymbol{\psi},Q) = \varnothing$.

In what follows we denote by $S(V)$ the symmetric algebra on a vector space V. Then $S(V^*)$, the symmetric algebra over the dual space of V, is isomorphic to the polynomial ring $\mathfrak{P}(V^*)$ on V in indeterminates that are basis vectors for V^*.

(1.3) In this section we introduce the type of functions which we will consider in the rest of this work. We will follow [A1], [A2] and [F2]and [B1] in this presentation. The central object throughout this review will be the space $L^2(Z_G(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A}))$ of square integrable functions $f : G(\mathbb{Q}) \setminus G(\mathbb{A}) \to \mathbb{C}$ modulo the center and $L^2\big(M_P(\mathbb{Q})N_P(\mathbb{A})A_P(\mathbb{R})^0 \setminus G(\mathbb{A})\big)$ and also subspaces of $L^2(\mathbb{K})$. They will be explained in this and the next two subsections.

For the rest of this work we fix once and for all a finite set $F_{\mathbb{K}}$ of irreducible representations of $\mathbb{K}$ on a vector space V, and let Γ denote the space of functions on $\mathbb{K}$ spanned by the matrix elements of the representations in $F_{\mathbb{K}}$. We say a function $f(.)$ is **Γ-finite** (or of type Γ) if $f(gk)$ for $k \in \mathbb{K}$ belongs to Γ for almost all $g \in G$. In what follows the subscript Γ under a space (like $L^2(.)_\Gamma$ and so on) means the subspace of functions in the space under consideration which satisfy the property just explained. The set of all such equivalence classes Γ will be denoted by $\{\Gamma\}$. The set of all equivalence classes of finite dimensional irreducible representations of $\mathbb{K}$ will be denoted by $\mathcal{E}_{\mathbb{K}}$.

Let $f : G(\mathbb{A}) \to \mathbb{C}$ be a function. A function is smooth if it is smooth at the archimedean places and locally constant on an open neighborhood of the non-Archimedean places. The space of smooth functions on $G(\mathbb{A})$ will be denoted by $C^\infty(G(\mathbb{A}))$. We say that a continuous function $f : G(\mathbb{A}) \to \mathbb{C}$ has **moderate growth** if there is a constant $r \in \mathbb{R}$ such that $|f(g)| \leq \parallel g \parallel^r$ for all $g \in G(\mathbb{A})$. The notion of moderate growth could be extended to the space $C^\infty(G(\mathbb{A}))$ as follows. Let $X \in \mathfrak{g}$ and $f \in C^\infty(G(\mathbb{A}))$, the element X acts on the right on this space by the rule $X \cdot f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(g.\mathrm{exp\ tX}) \mid_{t=0}$. We extend this action from $\mathfrak{g}$ to all $X \in \mathfrak{B}$ by the universal property. A function $f \in C^\infty(G(\mathbb{A}))$ has **uniform moderate growth** if there is a $r \in \mathbb{R}$ and for each $X \in \mathfrak{B}$ a constant c_X such that $|X \cdot f(g)| \leq c_X \parallel g \parallel^r$. We extend these definitions to the other quotient spaces by trivial modifications.

In what follows we will use also the notion of the **constant term**. To define it we fix a parabolic subgroup $P = M_P N_P$ of G and let f be a measurable locally L^1 function on $N_P(\mathbb{Q}) \setminus G(\mathbb{A})$. Then the constant term of f will be the measurable locally L^1 function on $N_P(\mathbb{A}) \setminus G(\mathbb{A})$ defined by

$$f_{N_P}(g) = \int_{N_P(\mathbb{Q})\setminus N_P(\mathbb{A})} f(ng)\mathrm{d}n.$$

This function has the property that if $f(g)$ is left $G(\mathbb{Q})$-invariant, smooth and of moderate growth, then $f_{N_P}(g)$ will be left $M_P(\mathbb{Q})$-invariant, smooth and of moderate growth. If for a function $f(g)$ we have $f_{N_P}(g) = 0$ for all parabolic subgroups $P \subsetneqq G$, we call $f(g)$ a **cusp form**.

(1.4) Let P be a parabolic subgroup of G. Let us consider the space of functions $\phi : M_P(\mathbb{Q})N_P(\mathbb{A})A_P(\mathbb{R})^\circ \setminus G(\mathbb{A}) \to \mathbb{C}$ of moderate growth which satisfy the following conditions:

1) $\int_{\mathbb{K}} \int_{M_P(\mathbb{Q})\setminus M_P^1(\mathbb{A})\times\mathbb{K}} |\phi(mk)|^2 \mathrm{d}m\mathrm{d}k < \infty$.

2) There is a character $\xi : Z_{M_P}(\mathbb{A}) \to \mathbb{C}^*$ such that we have $\phi(zg) = e^{\langle \rho_P+\xi,z\rangle}\phi(g)$ for all $g \in G(\mathbb{A})$ and all $z \in Z_{M_P}(\mathbb{A})$.

3) The space spanned by $\{k \cdot \phi(g) = \phi(gk) \mid k \in \mathbb{K}\}$ is finite dimensional and contains only irreducible representations of $\mathbb{K}$ equivalent to those lying in $F_{\mathbb{K}}$ of **(1.3)**.

4) The space spanned by $\{X \cdot \phi(g) \mid X \in \mathcal{Z}(\mathfrak{g})\}$ is finite dimensional.

This space will be denoted by $A^2(P)$ and will be called the space of square integrable **automorphic forms** on $M_P(\mathbb{Q})N_P(\mathbb{A})A_P(\mathbb{R})^\circ\setminus G(\mathbb{A})$. The corresponding subspace of cuspidal automorphic forms will be denoted by $A^2_{\mathrm{cusp}}(P)$. This definition is related to the classical situation of functions defined on the quotient space $M_P(\mathbb{Q})\setminus M_P(\mathbb{A})$ by corresponding $\phi \in A^2(P) \mapsto \phi_k = e^{-\langle\rho_P,H_P(m)\rangle}\phi(mk) \in A^2(M_P(\mathbb{Q}) \setminus M_P(\mathbb{A}))$ for $m \in M_P(\mathbb{A})$. The space $A^2(M_P(\mathbb{Q}) \setminus M_P(\mathbb{A}))$ satisfies analog of the conditions **1)-4)** above.

Let us denote the orbit of a fixed character $\chi : \mathcal{Z}(\mathfrak{m}_P) \to \mathbb{C}^*$ under the Weyl group W_G of G by $\boldsymbol{\chi}$. Then the subspace of $A^2(P)$ of functions which satisfy the extra conditions $X \cdot \phi(g) = \chi(X)\phi(g)$, for $\chi \in \boldsymbol{\chi}$, and $\phi(gk) \in \Gamma$ for all $g \in G$ will be denoted by $A^2(P, \boldsymbol{\chi}, \Gamma)$. We denote the subspace of the cusp forms by $A^2_{\mathrm{cusp}}(P, \boldsymbol{\chi}, \Gamma)$. All the spaces $A^2(P)$, $A^2(M_P(\mathbb{Q}) \setminus M_P(\mathbb{A}))$, $A^2(P, \boldsymbol{\chi}, \Gamma)$ etc. are finite dimensional, according to [H1].

(1.5) We now introduce some spaces which are basic to what follows. Let P be a parabolic subgroup and let $\boldsymbol{\chi}$ be as in **(1.4)**. Let

$$L^2\big(M_P(\mathbb{Q})N_P(\mathbb{A}) \setminus G(\mathbb{A})\big)_{\boldsymbol{\chi},\Gamma}, \tag{1.1}$$

denote the space of quadratic integrable functions of type Γ such that for every $g \in G(\mathbb{A})$ and $l \in M_P(\mathbb{A})A_P(\mathbb{R})^0$ the function $f(lg)$ is an eigenfunction of $\mathfrak{A} \otimes \mathcal{Z}(\mathfrak{m}_P)$ associated to some element of the orbit $\boldsymbol{\chi}$, if $\mathfrak{A}$ denotes the universal enveloping algebra of $\mathfrak{a}_P$ and $\mathcal{Z}(\mathfrak{m}_P)$ the center of the universal enveloping algebra $\mathfrak{M}_P$ of the Levi component $\mathfrak{m}_P$ of $\mathfrak{p}$ as it is described in **(1.4)**. In other words if $X \in \mathfrak{A} \otimes \mathcal{Z}(\mathfrak{m}_P)$ is a differential operator then $X \cdot f(g) = \chi(X)f(g)$ for all f in (1.1) and a character $\chi \in \boldsymbol{\chi}$.

Let Q be a parabolic subgroup associated to P and let us fix a class $\{\mathrm{P}\}$ of associated subgroups, let the character χ be as in paragraph **(1.4)**, and let $\boldsymbol{\psi}$ be as in **(1.2)**. The spaces relevant for us to construct Eisenstein series will be the following subspace of (1.1):

$$L^2\big(P(\mathbb{Q})N_P(\mathbb{A})A_P(\mathbb{R})^o \setminus G(\mathbb{A})\big)_{\chi,\Gamma}, \tag{1.2}$$

and the subspace:

$$L^2\big(P(\mathbb{Q})N_P(\mathbb{A})A_P(\mathbb{R})^o \setminus G(\mathbb{A})\big)_{\chi,\{\mathrm{P}\},\boldsymbol{\psi},\Gamma}, \tag{1.3}$$

of (1.2) which we define as follows. By definition (1.3) is the space of functions $f(.)$ such that their constant term $f_{N_Q}(g) = \int_{N_Q(\mathbb{Q})\setminus N_Q(\mathbb{A})} f(ng)\mathrm{dn}$ has the property that if $k \in \mathbb{K}$ and $l \in M_Q(\mathbb{A})A_Q(\mathbb{A})$ then $f_{N_Q}(lk)$ is orthogonal to the space of cusp forms if $Q \notin \{\mathrm{P}\}$ and is a sum of cusp forms transforming under infinitesimal characters of $\mathcal{Z}(\mathfrak{m}_Q)$ which belong to $\boldsymbol{\psi}_Q$ if $Q \in \{\mathrm{P}\}$. If P contains no element of $\{\mathrm{P}\}$ then (1.3) is zero by this definition and Lemma 3.7 in [L1].

The space of Γ-finite cusp forms on $G(\mathbb{Q})\backslash G(\mathbb{A})$ belonging to the character $\xi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}^*$ will be denoted by

$$L^2_{\mathrm{cusp}}\big(G(\mathbb{Q}) \setminus G(\mathbb{A})\big)_{\xi,\Gamma}. \tag{1.4}$$

The C^∞-cuspidal functions that lie in a Sobolov subspace of (1.4) are of rapid decay. For each parabolic subgroup P there is a bijection between the set of parabolic subgroups of M_P and the set of those parabolic subgroups of G contained in P ([H1] lemma 2). Consequently in the above discussion we can define the analog of this space for the Levi subgroups and the characters $\chi : \mathcal{Z}(\mathfrak{m}_P) \to \mathbb{C}^*$, which we denote by

$$L^2_{\mathrm{cusp}}\big(M_P(\mathbb{Q})) \setminus M_P^1(\mathbb{A})\big)_{\chi,\Gamma}. \tag{1.5}$$

(1.6) Now let f belong to $A^2(P)$. For $\lambda \in (\check{\mathfrak{a}}_P)_{\mathbb{C}}$, the **Eisenstein series** attached to f is by definition the series

$$E_P^G(\lambda, f)(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda + \rho_P, H_P(\gamma g)\rangle} f(\gamma g) \tag{1.6}$$

which is known to converge uniformly and absolutely on compact subsets of the Cartesian product of the **domain of absolute convergence**

$$\mathfrak{A}_P = \{\lambda \in (\mathfrak{a}_P)_{\mathbb{C}} \mid \Re(\lambda) \in \rho_P + \mathfrak{a}_P^+\},$$

and $G(\mathbb{A})^\circ$. Then $E_P^G(\lambda, f)(g)$ is an infinitely differentiable function with respect to λ and g and analytic with respect to λ for fixed g which is an automorphic form on $G(\mathbb{Q}) \setminus G(\mathbb{A})$.

For the sake of the induction argument used in theorem 7.1 in [L1], Langlands defined a new kind of Eisenstein series which are suitable for descent arguments. They are constructed as follows. Fix two standard parabolic subgroups $P \subseteq R$ and for $f(.) \in A^2_{\text{cusp}}(P, \chi, \Gamma)$ define the Eisenstein series:

$$E_P^R(\lambda, f)(g) = \sum_{\gamma \in P(\mathbb{Q})\setminus R(\mathbb{Q})} e^{\langle \lambda + \rho_P, H_P(\gamma g)\rangle} f(\gamma g),$$

which converges for a suitable $\lambda \in \mathfrak{A}_P$ and can be meromorphically continued. It is clear that $E_P^G(\lambda, f)(g) = \sum_{\delta \in R(\mathbb{Q})\setminus G(\mathbb{Q})} E_P^R(\lambda, f)(\delta g)$. These new series are introduced in the discussion following the theorem 4.1 of [L1], and their main properties are proved there. For non-cuspidal functions $f(.)$ in $A^2(P)$ their existence are guarantied by the theorem 7.1 in [L1].

In what follows we will need the constant term of Eisenstein series (1.6) computed along parabolic subgroups of G. The concept of constant term is introduced in section **(1.3)** above. Suppose that P and Q are parabolic subgroups of G. Then the constant term of the Eisenstein series $E_P^G(\lambda, f)(g)$ along Q is the integral

$$(E_P^G(\lambda, f)(g))_Q = \int_{N_Q(\mathbb{Q})\setminus N_Q(\mathbb{A})} E_P^G(\lambda, f)(ng)\mathrm{d}n.$$

The cuspidal component of this integral is zero if P and Q are not associated, or better said, it is orthogonal to the space of cusp forms over Q. On the other hand if P and Q are associated then for each $\lambda \in \mathfrak{A}_P$ such that λ is not a pole of $E_P^G(.,.)$, and for each $\omega \in \Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)$ there exist linear operators $N(\omega, \lambda) : A^2_{\text{cusp}}(P, \chi, \Gamma) \to A^2_{\text{cusp}}(Q, \omega\chi, \Gamma)$ such that

$$(E_P^G(\lambda, f)(g))_Q = \sum_{\omega \in \Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)} e^{\langle \rho_Q + \omega\lambda, H_Q(\gamma g)\rangle} (N(\omega, \lambda) f(g)).$$

For the sake of the functional equation of Eisenstein series given in **(1.20)** below, we modify the Haar measure on Q such that $\mathrm{vol}(N_Q(\mathbb{Q}) \setminus N_Q(\mathbb{A})) = 1$, which gives then $N(1, \lambda) = \mathrm{id}$.
For the partial Eisenstein series $E_P^R(\lambda, f)(g)$, the constant term along Q will be

$$\left(E_P^R(\lambda, f)(g)\right)_Q = \sum_{\substack{\omega \in \Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q) \\ \tilde{\omega}|\check{\mathfrak{a}}_R = \mathrm{Id}}} e^{\langle \rho_Q + \omega\lambda, H_Q(\gamma g)\rangle} (N(\omega, \lambda) f(g)).$$

Although we have not yet defined the residual Eisenstein series, but here is a good place to introduce their constant term to show the contrast between the two situations. Intuitively they are built out of cuspidal Eisenstein series given above by taking residues on the intersection of their singularities with certain affine subspaces of $\check{\mathfrak{a}}_P$. These singularities will be proved to be hyperplanes. The exact meaning of them will be clear soon when we investigate Eisenstein systems in **(1.11)** below.

Let P and Q be like above and let $S\big((\check{\mathfrak{a}}_Q^G)_{\mathbb{C}}\big)$ denote as usual the symmetric algebra over $(\check{\mathfrak{a}}_Q^G)_{\mathbb{C}}$. If f belongs to $A^2(P)$ then there is a meromorphic functions $N(\omega,\lambda)$ from $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ to the $\mathbb{K}$-equivariant linear transformations from the space (1.3) to the space

$$\mathrm{Hom}_{\mathbb{K}}\left(S\big((\check{\mathfrak{a}}_Q^G)_{\mathbb{C}}\big),\ A^2_{\mathrm{cusp}}(Q,\chi,\Gamma)\right), \tag{1.7}$$

(notation just like **(1.2)**) such that

$$\big(E_P^G(\lambda,f)\big)_Q(g)=\int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} E_P^G(\lambda,f)\big)(ng)dn=$$

$$\sum_{\omega\in\Omega(P,\chi,\psi,Q)} e^{\langle\rho_Q+\omega\lambda,H_Q(g)\rangle}\Big(\big(N(\omega,\lambda)f\big)(H_Q(g))\Big)(g). \tag{1.8}$$

It is known from lemma 7.2 of [L1] (proved in lemma 7.5 there) that the operators $N(\omega,\lambda)$ are independent of λ if $\omega\in\Omega_\circ(P,\chi,\boldsymbol{\psi},Q)$.

(1.7) Fix a parabolic subgroup $P\subset G$ and let χ have the same meaning as in **(1.4)**. We fix once and for all a constant R such that $R>\langle\rho,\rho\rangle^{\frac{1}{2}}$. Let us denote by $\mathcal{PW}\big((\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}\big)$ the space of complex valued holomorphic functions f defined on $(\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}$ which satisfy the growth condition

$$\sup_{\lambda\in(\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}} |f(\lambda)|e^{-r\|\Im\lambda\|}(1+\|\lambda\|)^n<\infty,\ \exists\ r>0,\ \forall n\in\mathbb{N}.$$

We call this space the space of **Paley-Wiener** functions on $(\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}$.

More generally, let us denote by $\mathcal{PW}_R\big((\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}\big)$ the space of complex valued holomorphic functions $\Phi(\lambda)$ defined on the strip

$$\mathrm{Str}_{M_P}^G(R)=\{\lambda\in(\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}|\ \|\Re\lambda\|<R\},$$

which satisfy the growth condition

$$\sup_{\lambda\in\mathrm{Str}_{M_P}^G(R)} |\Phi(\lambda)|(1+\|\lambda\|)^n<\infty$$

for all $n\in\mathbb{N}$. This implies that such functions decay faster than any polynomial in the direction of the imaginary axis, i.e., $\|p(\Im\lambda)\Phi(\lambda)\|_{L^2}$ is bounded on $\mathrm{Str}_{M_P}(R)$ for each polynomial $p(.)$ on $\mathrm{Str}_{M_P}^G(R)$. Each such function $\Phi(.)$ defines an element $\mathrm{d}\Phi(.)$ of the space (1.7) via developing it as a Taylor series $\eta\longmapsto\Phi(\eta+\lambda)$ in a small neighborhood of the origin.

With these spaces at hand we can define a subspace of functions on $(\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}$ with values in the space

$$A^2_{\text{cusp}}(P,\chi,\Gamma)\otimes\mathcal{PW}\big((\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}\big).$$

Analogously, we will consider the subspace of functions defined on $\mathrm{Str}^G_{M_P}(R)$ with values in the space

$$A^2_{\text{cusp}}(P,\chi,\Gamma)\otimes\mathcal{PW}_R\big((\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}\big).$$

We call the functions belonging to these spaces again Paley-Wiener functions.

Let us consider the space $\mathcal{PW}\big((\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}\big)$. These Paley-Wiener functions are characterized by the property that they are Fourier transforms of C^∞ functions defined on $M_P(\mathbb{Q})Z_G(\mathbb{A})\backslash G(\mathbb{A})$ with compact support. We explain this property more precisely.
Let $\mathcal{D}(M_P)_{\boldsymbol{\chi},\Gamma}$ denote the set of continuous $\Gamma-$ and $\chi-$finite functions $\varphi(g)$ on $M_P(\mathbb{Q})N_P(\mathbb{A})A_p(\mathbb{R})^\circ\setminus G(\mathbb{A})$ such that the projection of their support on $M_P^1(\mathbb{A})Z_G(\mathbb{A})\setminus M_P(\mathbb{A})$ is compact and $\varphi(mg)\in A^2_{\text{cusp}}(P,\chi,\Gamma)$ for all $g\in G$.

Fix $\lambda_0\in\check{\mathfrak{a}}_{M_P}^G$. Each $\varphi(.)\in\mathcal{D}(M_P)_{\boldsymbol{\chi},\Gamma}$ can be represented as a **Fourier integral**

$$\varphi(g)=\left(\frac{1}{2\pi\mathrm{i}}\right)^{\dim(\check{\mathfrak{a}}_{M_P}^G)}\int_{\Re(\lambda)=\lambda_0}e^{\langle\lambda+\rho_P,H_P(g)\rangle}\Phi(g,\lambda)\mathrm{d}\lambda,$$

where, for $\lambda\in(\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}$, $\Phi(g,\lambda)$ is a well-defined holomorphic function on $(\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}$, which in general does not belong to the space $A^2_{\text{cusp}}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A}))_{\chi,\Gamma}$ since the property of $M_P(\mathbb{A})\cap\mathbb{K}$-finiteness is lost at the archimedean places. But this function belongs to the subspace of the right $M-$translations of $A^2_{\text{cusp}}(M_P(\mathbb{Q})\setminus M_P(\mathbb{A}))_{\chi,\Gamma}$ which we denote by $\tilde{A}^2_{\text{cusp}}(M_P(\mathbb{Q})\setminus M_P(\mathbb{A}))_{\chi,\Gamma}$. We call $\Phi(.)$ the **Fourier transform** of $\varphi(.)$. Let $g=nam'k\in G(\mathbb{A})$. Then the function $\varphi(m'mk)$ is well defined on $L_P(\mathbb{A})\times\mathbb{K}$, take values in the space $\tilde{A}^2_{\text{cusp}}(M_P(\mathbb{Q})\setminus M_P(\mathbb{A}))_{\chi,\Gamma}$. Then for $\lambda\in(\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}$

$$\Phi(m'k,\lambda)=\int_{M_P^1(\mathbb{A})Z_G(\mathbb{A})\backslash M_P(\mathbb{A})}e^{-\langle\lambda+\rho_P,H_P(m)\rangle}\varphi(m'mk)\mathrm{d}m,$$

is a Paley-Wiener function on $(\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}$.

This discussion (see also II.1.3 in [MW]) shows that there is an isomorphism between

$$\mathcal{D}(P)_{\chi,\Gamma}\cong A^2_{\text{cusp}}(P,\chi,\Gamma)\otimes C_c^\infty(M_P^1(\mathbb{A})Z_G(\mathbb{A})\setminus M_P(\mathbb{A}))$$

and

$$A^2_{\text{cusp}}(P,\chi,\Gamma)\otimes\mathcal{PW}\big((\check{\mathfrak{a}}_{M_P}^G)_{\mathbb{C}}\big)$$

defined by

$$(*)\qquad\sum(\psi_j\otimes\phi_j)(g)(m)\tilde{\longrightarrow}\sum\psi_j(mg)\otimes\Phi_j(m,H_P(g))e^{-\rho_P(m)},$$

if $g \in G(\mathbb{A})$.

For the space $\mathcal{PW}_R\big((\check{\mathfrak{a}}^G_{M_P})_{\mathbb{C}}\big)$ the situation is more complicated since unlike the space $\mathcal{PW}\big((\check{\mathfrak{a}}^G_{M_P})_{\mathbb{C}}\big)$ the Fourier transform of the functions in $\mathcal{PW}_R\big((\check{\mathfrak{a}}^G_{M_P})_{\mathbb{C}}\big)$ do not have compact support in general. To achieve an isomorphism like $(*)$ we have to restrict ourselves to the subspace of functions of exponential type, like the classical Paley-Wiener theorem (see [R] chapter 19). Consider the subspace $\widetilde{\mathcal{PW}}_R\big((\check{\mathfrak{a}}^G_{M_P})_{\mathbb{C}}\big) \subseteq \mathcal{PW}_R\big((\check{\mathfrak{a}}^G_{M_P})_{\mathbb{C}}\big)$ consisting of complex valued functions f defined on $(\check{\mathfrak{a}}^G_{M_P})_{\mathbb{C}}$ such that for all $m \in M^1_P(\mathbb{A})Z_G(\mathbb{A}) \setminus M_P(\mathbb{A})$ there is a constant c and for R fixed above we have $|f(H^G_P(m))| \leq c.e^{-R\langle H^G_P(m), H^G_P(m)\rangle}$. Then there is a isomorphism, which we denote by $(**)$, between

$$A^2_{\mathrm{cusp}}(P, \chi, \Gamma) \otimes \widetilde{\mathcal{PW}}_R\big((\check{\mathfrak{a}}^G_{M_P})_{\mathbb{C}}\big)$$

and the set of functions

$$\phi : G(\mathbb{A}) \to \tilde{A}^2_{\mathrm{cusp}}(M_P(\mathbb{Q}) \setminus M^1_P(\mathbb{A}))_{\chi,\Gamma},$$

which are of exponential type, i.e., there is a λ which satisfies $\Re\lambda > \lambda_0$ and a constant c such that $|\phi(g)| \leq c.e^{\langle\lambda, H_P(g)\rangle}$ for all $g \in \mathfrak{S}_P$. This is the isomorphism we sought and will use further on.

(1.8) Let $\phi(.)$ lie in the image of the one of the isomorphisms $(*)$ or $(**)$ given just above. It is proved in theorem 3.6 of [L1] that the so called **pseudo theta series**

$$\sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma g)$$

converge absolutely to a function $\widehat{\phi}(g)$ which satisfies the growth condition

$$|\ \widehat{\phi}(g)\ | \leq \max_{\alpha \in \Phi^+(\mathfrak{n}_P)} e^{r\alpha(H_\circ(g))},$$

in a Siegel domain $\mathfrak{S}_\circ$ associated to the minimal parabolic subgroup $P_\circ$ (fixed in **(1.1)**) for a real number r and $g \in \mathfrak{S}_\circ$.

If $\phi(.)$ lies in the image of $(*)$ then we have $\widehat{\phi}(g) \in L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$. This justifies the interchange of integration and summation in the following computations:

$$\widehat{\phi}(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\gamma g) =$$

$$\sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \left(\frac{1}{2\pi \mathrm{i}}\right)^{\dim(\check{\mathfrak{a}}_P)} \int_{\Re\lambda=\lambda_0} \Phi(\gamma m, \lambda) e^{\langle\lambda+\rho_P, H_P(g)\rangle} \mathrm{d}\lambda =$$

$$\left(\frac{1}{2\pi \mathrm{i}}\right)^{\dim(\check{\mathfrak{a}}_P)} \int_{\Re\lambda=\lambda_0} \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \Phi(\gamma m, \lambda) e^{\langle\lambda+\rho_P, H_P(g)\rangle} \mathrm{d}\lambda =$$

$$\left(\frac{1}{2\pi \mathrm{i}}\right)^{\dim(\check{\mathfrak{a}}_P)} \int_{\Re\lambda=\lambda_0} E(g,\Phi,\lambda)\mathrm{d}\lambda.$$

This computation shows that the L^2 closure of the space spanned by the functions $\widehat{\phi}(.)$, i.e.,

$$\overline{\big\langle \widehat{\phi} \mid \phi \in \mathcal{D}(P)_{\boldsymbol{\chi},\Gamma}, \text{ for all } P \in \{\mathrm{P}\}\big\rangle}$$

is the closed subspace $L^2(Z_G(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A}))_{\boldsymbol{\chi},\{\mathrm{P}\},\Gamma}$ of $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$, which we denote by

$$L^2\big(\{\mathrm{P}\},\boldsymbol{\chi},\Gamma\big). \tag{1.9}$$

So we achieve the decomposition of $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ if we decompose the spaces (1.9) or equivalently the space $L^2\big(\{\mathrm{P}\},\chi,\Gamma\big)$. We try to give an intuitive description of how to achieve this goal here and more comprehensively in the next sections. Suppose we have $\alpha(\lambda_0) > \langle \alpha, \rho\rangle$ for all $\alpha \in \Phi^+(\mathfrak{n}_P)$. Then if in the integral

$$\widehat{\phi}(g) = \left(\frac{1}{2\pi \mathrm{i}}\right)^{\dim(\check{\mathfrak{a}}_P)} \int_{\Re(\lambda)=\lambda_0} E_P(g,\Phi(\lambda),\lambda)\mathrm{d}\lambda,$$

we could shift the contour $\Re(\lambda) = \lambda_0$ to $\Re(\lambda) = 0$ we obtain all functions $\widehat{\phi}(g)$ which lie in (1.9) and eventually produce the whole space. This is the main result of the chapter 7 of [L1]. By shifting this contour we have to deal with singularities and residues of Eisenstein series which show several unwanted behavior including having poles of higher order and that during this shifting the contour will leave the domain of absolute convergence $\mathfrak{A}_P$ and the behavior of the intertwining operators are unknown beyond this domain. Some of these difficulties are partially explained in the only known example of the group G_2 in [L1].

(1.9) The above considerations show us that if we decompose the space $L^2(Z_G(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A}))_{\boldsymbol{\chi},\{\mathrm{P}\},\Gamma}$, which as we saw, is the closure in the space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))_\Gamma$ of the space generated by pseudo theta series, into the subspaces under the action of the regular representation of G, we achieve our goal of decomposing $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))_\Gamma$. One of the first results in this direction is the theorem 4.6 of [L1] which gives the direct sum

$$L^2(G(\mathbb{Q}) \setminus G(\mathbb{A})) = \bigoplus_{\boldsymbol{\chi}} \bigoplus_{\Gamma\in\mathcal{E}_{\mathbb{K}}} \bigoplus_{\{\mathrm{P}\}} L^2\big(\{\mathrm{P}\},\boldsymbol{\chi},\Gamma\big).$$

The subscript $\{\mathrm{P}\}$ means that the sum is taken over all equivalence classes of associated parabolic subgroups of G. The $\mathcal{E}_{\mathbb{K}}$ and Γ and $\boldsymbol{\chi}$ have the same meaning as already introduced at **(1.3)** and **(1.4)**. Now to achieve a finer decomposition it is necessary to decompose each of the subspaces

$$L^2\big(\{\mathrm{P}\},\boldsymbol{\chi},\Gamma\big).$$

This goal can be equivalently achieved if we decompose the closure of the space spanned by the pseudo theta series defined on the Levi components of the parabolic subgroups belonging to {P}.

More precisely, let us fix a parabolic $P \in \{\mathrm{P}\}$ and a parabolic subgroups Q such that $P \subseteq Q$, and let us assume that the function $\phi(.)$ comes from a Paley-Wiener function $\Phi \in \mathcal{PW}_R\big((\check{\mathfrak{a}}_P^Q)_{\mathbb{C}}\big)_{\boldsymbol{\chi},\Gamma}$ in the sense that we described in **(1.7)**, and denote the closure of the space spanned by such functions

$$\widehat{\phi}(g) = \sum_{\gamma \in P(\mathbb{Q})\backslash Q(\mathbb{Q})} \phi(\gamma g),$$

in the space

$$L^2\big(M_Q(\mathbb{Q}) \setminus M_Q^1(\mathbb{A})\big)_{\boldsymbol{\chi},\{\mathrm{P}\},\Gamma},$$

by

$$^{Q}L^2\big(\{\mathrm{P}\}, \boldsymbol{\chi}, \Gamma\big), \tag{1.10}$$

as P varies over all parabolic subgroups contained in Q. Then if we decompose (1.10) we actually achieve the decomposition of the space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$.

(1.10) We now take a closer look at the problem of spectral decomposition and how it leads to the idea of Eisenstein systems. The Hilbert space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ with the action of the regular representation admits a decomposition $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A})) = L^2_{\text{dis}}(G(\mathbb{Q}) \setminus G(\mathbb{A})) \oplus L^2_{\text{cont}}(G(\mathbb{Q}) \setminus G(\mathbb{A}))$. The $L^2_{\text{dis}}(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ part is the closure of the space spanned by closed irreducible invariant subspaces of $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$. The $L^2_{\text{cont}}(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ is generated by the direct integrals of representations which are induced from the Levi factors of standard parabolic subgroups of G.

Let us denote by $L^2_{\text{cus}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ the subspace of $L^2_{\text{dis}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ consisting merely of cuspidal functions. The orthogonal complement of $L^2_{\text{cus}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ in $L^2_{\text{dis}}(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ consists of residual functions, which we will denote by $L^2_{\text{res}}(G(\mathbb{Q}) \setminus G(\mathbb{A}))$. This space is not 0 since it is known that the constant functions are belonging to it. We have the decomposition

$$L^2(G(\mathbb{Q}) \setminus G(\mathbb{A})) = L^2_{\text{cont}}(G(\mathbb{Q}) \setminus G(\mathbb{A})) \oplus L^2_{\text{cus}}(G(\mathbb{Q}) \setminus G(\mathbb{A})) \oplus L^2_{\text{res}}(G(\mathbb{Q}) \setminus G(\mathbb{A})),$$

into closed invariant subspaces. The last two constitute the discrete spectrum and the last one is generated by the residual forms, which we call for brevity residual spectrum, but it does not have anything to do with the usual residual spectrum of Hilbert spaces, as it is defined for example in [Y] page 209 and should not be mistaken with it. The main building block of the residual spectrum consists of the square integrable functions $\Phi : G(\mathbb{A}) \to \mathbb{C}$ such that for a parabolic subgroup $P \in \{\mathrm{P}\}$ satisfy the conditions:

1) $\Phi(.)$ is left invariant under the action of the group $P(\mathbb{Q})N_P(\mathbb{A})A_P(\mathbb{R})^\circ$.

2) for any $g \in G$, the function $k \to \Phi(gk)$ belongs to $\Gamma \subset L^2(\mathbb{K})$.

3) for any $g \in G$, the function $m \to \Phi(mg)$ belongs to $L^2_{\text{res}}(M_P(\mathbb{Q})\backslash M_P^1(\mathbb{A}))_\chi$.

We will denote this subspace by $A^2_{\rm res}(P,\chi,\Gamma)$. The same reasoning as it is done in [H1] for the subspace $A^2_{\rm cus}(P,\chi,\Gamma)$ shows the finite dimensionality of $A^2_{\rm res}(P,\chi,\Gamma)$. See also chapter 6 of [OW]. It is also evident that this space is invariant under the left convolution with functions $\alpha \in C_c^\infty(G(\mathbb{A}))$ satisfying $\alpha(kxk^{-1}) = \alpha(x)$ for all $k \in \mathbb{K}$ and $x \in G(\mathbb{A})$. If we attach an Eisenstein series $E_P^G(\lambda,\Phi)(m)$, $\lambda \in \rho_P + \check{\mathfrak{a}}_P^{G+}$, to each element $\Phi \in A^2_{\rm res}(P,\chi,\Gamma)$ and continue it meromorphically from the complex tube over the positive Weyl chamber $\check{\mathfrak{a}}_P^{G+} + {\rm i}\check{\mathfrak{a}}_P^G$ to all $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ then we can generate the subspace $L^2_{\rm res}(M_P(\mathbb{Q}) \setminus M_P^1(\mathbb{A}))_\chi$. The same discussion holds for the whole discrete part $L^2_{\rm dis}(M_P(\mathbb{Q}) \setminus M_P^1(\mathbb{A}))_\chi$ with the subspace $A^2_{\rm dis}(P,\chi,\Gamma)$, which is defined through the analog of the properties **1)** to **3)** above plus the meromorphic continuation from the complex tube over the positive Weyl chamber to all $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$. In turns out that with some extra effort we can (roughly saying) generate the space $L^2_{\rm dis}(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ by going through all the elements of $\{\mathrm{P}\}$, $\boldsymbol{\chi}$ and Γ. It was a marvelous insight of Langlands that this goal could be reached by taking successive residues of Eisenstein series attached to cusp forms along their singular hyperplanes, and he proved that this process exhausts the space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ completely. These residues satisfy a set of axioms which are called **Eisenstein Systems** which we discuss in the following paragraph.

(1.11) We have to lay the geometrical foundations on which the Eisenstein systems are constructed. To begin with let us fix a parabolic subgroup $P \in \{\mathrm{P}\}$. Let $\mathfrak{s} \subset (\check{\mathfrak{a}}_P)_{\mathbb{C}}$ denote an affine subspace which is defined by equations of the form

$$\alpha(\lambda) = \mu, \quad \text{for } \alpha \in \Phi^+(\mathfrak{n}_B), \lambda \in (\check{\mathfrak{a}}_\circ)_{\mathbb{C}}, \text{ and } \ \mu \in \mathbb{C},$$

with a distinguished normal vector $X(\mathfrak{s})$ such that

$$\mathfrak{s} = X(\mathfrak{s}) + \tilde{\mathfrak{s}},$$

in which the $\tilde{\mathfrak{s}}$ is a subspace of $\check{\mathfrak{a}}_\circ$ defined by the real linear equations like above for $\mu = 0$. In this work we will consider only affine subspaces of this type and denote them by $\mathfrak{s}$ or $\mathfrak{t}$.

Starting from these spaces, we can consider a sequence of affine subspaces of $\check{\mathfrak{a}}_P$

$$S: \qquad \mathfrak{t}_0 \subset ... \subset \mathfrak{t}_r \subset \check{\mathfrak{a}}_P,$$

in which $\mathfrak{t}_r$ is defined by the equations of the form above and each $\mathfrak{t}_i$ is defined recursively by

$$\tilde{\mathfrak{t}}_i = \{\lambda \in \tilde{\mathfrak{t}}_{i+1} | \alpha(\lambda) = 0, \alpha \in \Phi^+(\mathfrak{n}_P)\},$$

and for each $\mathfrak{t}_i$ a distinguished normal vector $X(\mathfrak{t}_i)$ like above. The set of all such finite resolutions S along with the distinguished unit normal vectors will be denoted by

$$S_P(\mathfrak{t}).$$

The minimum affine subspace $\mathfrak{t}_0 = X(\mathfrak{t}_0) + \tilde{\mathfrak{t}}_0$ is of special importance for the residual process which lies at the heart of the Langlands construction. If we restrict our situation to residual Eisenstein series then $\tilde{\mathfrak{t}}_0$ will be a distinguished subspace (defined in **(1.1)**) of $\check{\mathfrak{a}}_P$ and there is a parabolic subgroup Q containing P such that $\tilde{\mathfrak{t}}_0 = \check{\mathfrak{a}}_Q$. But for the general situation of theorem 7.1 of [L1] (see **(1.18)** below), this is not the case since there are subspaces $\mathfrak{s} \subset \mathfrak{h}$ that carry an Eisenstein system which are not in general distinguished subspaces. We will comment on this when we investigate the lemma 7.2 of [L1] below in **(1.13)**.

With these spaces at hand we try to motivate the definition of Eisenstein Systems by considering a simpler situation. Let us fix parabolic subgroups $P \subseteq Q$ for $P \in \{\mathrm{P}\}$. Recall from **(1.6)** that the Langlands Eisenstein series have been defined as the in $\mathfrak{A}_P^Q$ convergent series

$$E_P^Q(g, \phi, \lambda) = \sum_{\gamma \in P(\mathbb{Q}) \backslash Q(\mathbb{Q})} e^{\langle \lambda + \rho_P^Q, H_P(\gamma g) \rangle} \phi(\gamma g),$$

for any cuspidal functions $\phi(.) \in A^2_{\text{cusp}}(P, \chi, \Gamma)$. Since all the other Eisenstein series appearing in the spectral decomposition are constructed from these Eisenstein series by taking residues along suitable hyperplanes, we need also the concept of local residues of meromorphic functions which we explain briefly as follows and more detailed in the next chapter. Let $\Phi_P(\lambda)$ be a meromorphic function defined on $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ with values in a locally convex vector space with singularities along the hyperplanes of the form given above and let us denote by $\mathfrak{t}$ a hyperplane of $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ not necessarily a singular hyperplane of $\Phi_P(\lambda)$. We choose a unit real normal vector H_0 to $\mathfrak{t}$ and define a meromorphic function

$$\operatorname{Res}_{\mathfrak{t}} \Phi_P(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{C_\epsilon} \Phi_P(\lambda + zH_0) \mathrm{dz},$$

for a small circle C_ϵ in $\mathbb{C}$ about the origin. ϵ is chosen such that no other singular hyperplane of $\Phi_P(\lambda + zH_0)$ cuts the circle C_ϵ. We will refine this definition in the next chapter of this work.

Now take Φ_P to be an analytic function defined on $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ with values in $A^2_{\text{cusp}}(P, \chi, \Gamma)$. Then for each $g \in M_Q(\mathbb{Q})N_Q(\mathbb{A}) \backslash G(\mathbb{A})$ the series $E_P^Q(g, \Phi_P, \lambda)$ is a cuspidal Eisenstein series which is meromorphic on $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$. It is shown in chapter 6 and 7 of [L1] that the singularities of $E_P^Q(g, \Phi_P, \lambda)$ are lying along hyperplanes of the form given above. This discussion shows that the singular hyperplanes of $\operatorname{Res}_{\mathfrak{t}} E_P^Q(g, \Phi_P, \lambda)$ and their intersections define a resolution S like above and we can define the function

$$\phi(x) = \sum_{S \in \tilde{S}_P(\mathfrak{t})} \operatorname*{Res}_{\lambda \to X(\mathfrak{t}_0)}{}_S \; E_P^Q(x, \Phi_P(\lambda), \lambda),$$

($\lambda \in (\mathfrak{t}_S)_{\mathbb{C}}$ and $x \in G(\mathbb{A})$) in which the set $\tilde{S}_P(\mathfrak{t}) \subseteq S_P(\mathfrak{t})$ is dependent on the path taken in the process of shifting the contour of integration to $X(\mathfrak{t}_0)$

(which is indicated by $\lambda \to X(\mathfrak{t}_0)$) and eliminating the parameters by taking residues successively until we reach the zero dimensional spectrum and this sum will represent an element of $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$. These residues will not generate the whole space $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}))$ unless we could include all affine hyperplanes of $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ which carry some Eisenstein series in the above sense. It will be shown that there are only a finite number of such a affine hyperplanes, but the choices are not canonical. Consequently to generate all Eisenstein series which are relevant to the spectral decomposition one has to proceed as follows.

Fix a parabolic subgroup $P \in \{\mathrm{P}\}$ and a choose parabolic subgroup $Q \supseteq P$. Let F_P be a function from $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ to $A^2_{\mathrm{cusp}}(P, \chi, \Gamma)$ defined and analytic in a neighborhood of the point $\lambda \in (\check{\mathfrak{a}}^Q_P)_{\mathbb{C}}$ and let us choose an affine subspace $\mathfrak{s}$ defined by root equations like above such that $\tilde{\mathfrak{s}}$ contains $\check{\mathfrak{a}}_Q$. Let us denote by $\mathfrak{s}^{\perp}$ the orthogonal complement of $\tilde{\mathfrak{s}}$ in $\check{\mathfrak{a}}^Q_P$. If we identify $S(\mathfrak{s})$ with a subalgebra of holomorphic differential operators with constant coefficients on $(\check{\mathfrak{a}}^Q_P)_{\mathbb{C}}$ then there is a linear map $\mathrm{d}F_P(\lambda) = \mathrm{d}_S F_P(\lambda) \in \mathrm{Hom}\big(S(\mathfrak{s}), A^2_{\mathrm{cusp}}(P, \chi, \Gamma)\big)$ obtained by developing the analytic function

$$\eta \longmapsto F_B(\eta + \lambda), \eta \in (\check{\mathfrak{a}}^Q_P)_{\mathbb{C}} \cap \mathfrak{s}^{\perp}$$

as a Taylor series around $\eta = 0$. The least degree of the nonzero terms occurring in this expansion is called the degree of F. Langlands has shown that all the Eisenstein series which are appearing in the spectral decomposition are constructed starting from such a cuspidal functions and then taking their residues at suitable affine subspaces introduced above. This process will lead us to the definition of Eisenstein systems. To proceed to the definition we need a finite set of complex affine subspaces of the form

$$\mathfrak{s} = X(\mathfrak{s}) + \tilde{\mathfrak{s}},$$

of $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ such that, as claimed in the lemmas 7.2 and proved in 7.5 of [L1] (see **(1.13)** and **(1.15)** below), there is a finite (non-canonical) sequence

$$S_P(Q, \mathfrak{s}) : \quad \mathfrak{s}_0 \subset ... \subset \mathfrak{s}_r \subset \check{\mathfrak{a}}_P,$$

of affine subspaces $\mathfrak{s}$ of $\check{\mathfrak{a}}_P$ with the distinguished normal as above such that $\tilde{\mathfrak{s}}_0 \supseteq \check{\mathfrak{a}}_Q$. These sets are clearly non canonical. But we can construct a canonical system of functions on this set of non-canonical subspaces which in turn generate the residual spectrum. This is mainly done in the course of the proof of the theorem 7.1 of [L1] (see paragraph **(1.18)** below) and we will define them in a moment. To proceed to the definition, let us fix a filtration S and write $\Phi_{\mathfrak{s}} = \mathrm{d}F_P(\lambda)$ for a function $F_P(\lambda)$ like above. For the affine subspace $\mathfrak{s}$ defined by the root equations given above and satisfy the condition $\check{\mathfrak{a}}_Q \subseteq \tilde{\mathfrak{s}} \subseteq \check{\mathfrak{a}}_P$, let us denote by $\mathfrak{s}^Q_P$ the projection of $\mathfrak{s}$ on $\check{\mathfrak{a}}^Q_P$. Then an **Eisenstein System** is by definition a collection of functions

$$\big\{E^Q_{\mathfrak{s},P}(x, \Phi_{\mathfrak{s}}, \lambda) : Q \supset P,\ \check{\mathfrak{a}}_Q \subseteq \tilde{\mathfrak{s}}\big\}$$

which are defined on the set

$$N_Q(\mathbb{A})M_Q(\mathbb{Q})A_Q(\mathbb{R})^+ \setminus G(\mathbb{A}) \times \operatorname{Hom}\big(S(\mathfrak{s}), A^2_{\mathrm{cusp}}(P,\chi,\Gamma)\big) \times (\mathfrak{s}^Q_P)_{\mathbb{C}}, \quad (1.11)$$

which satisfy the following **(i)** to **(v)** conditions:

(i). Fix parabolic subgroups $P \in \{\mathrm{P}\}$ and $Q \supseteq P$. For each $g \in G$ and $F_P \in \operatorname{Hom}\big(S(\mathfrak{s}), A^2_{\mathrm{cusp}}(P,\chi,\Gamma)\big)$ the function $E^Q_{\mathfrak{s},P}(g, F_P, \lambda)$ is meromorphic on $\mathfrak{s}^Q_P$. Moreover if λ_0 is any point of $\mathfrak{s}^Q_P$ there is a polynomial $p(\lambda) = \prod_{\alpha\in\Phi^+(\mathfrak{n}^Q_P)} \big(\alpha(\lambda) - \mu_\alpha\big)^{k_\alpha}$, for $\mu_\alpha \in \mathbb{C}$ and $k_\alpha \in \mathbb{N}\cup\{0\}$, which dos not vanish identically on $(\mathfrak{s}^Q_P)_{\mathbb{C}}$, and a neighborhood U of λ_0 such that $p(\lambda)E^Q_{\mathfrak{s},P}(g, F_P, \lambda)$ is a continuous function on $N_Q(\mathbb{A})M_Q(\mathbb{Q})A_Q(\mathbb{R})^+\setminus G(\mathbb{A})\times U$ which is analytic on U for each g and such that if $\mathfrak{S}_\circ$ is a Siegel domain associated to the minimal parabolic subgroup $P_\circ$ (fixed in **(1.1)**) and F_P as above then there are constants b and c such that

$$|p(\lambda)E^Q_{\mathfrak{s},P}(mk, F_P, \lambda))| \leq c \max_{\alpha\in\Phi^+(\mathfrak{n}^P_{P_\circ})} e^{b\langle\alpha, a_\circ(m)\rangle},$$

for all $m \in \mathfrak{S}_\circ$, $k \in K$ and all $\lambda \in U$. The function $E^Q_{\mathfrak{s},P}(g, F_P, \lambda)$ is for each g and λ a linear function of F_P and there is an n such that it vanishes identically if the order of F_P is bigger than n. $\diamond$

To proceed to the next definition we need some new notation. Let $P \in \{\mathrm{P}\}$ and let us choose an arbitrary parabolic subgroup R. For a fixed affine subspace $\mathfrak{s} \subseteq \check{\mathfrak{a}}_P$ let $\Omega(\mathfrak{s}, \check{\mathfrak{a}}_R)$ denote the set of distinct linear transformations from $\mathfrak{s}$ to $\check{\mathfrak{a}}_R$ obtained by restricting the elements of $\Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_R)$ to $\mathfrak{s}$. The linear part of an affine transformation $\omega \in \Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_R)$ will be denote by $\tilde{\omega}$. We can now proceed further.

(ii). Let $P \in \{\mathrm{P}\}$ and $Q \supseteq P$. Let us choose another parabolic subgroup P' such that $\check{\mathfrak{a}}_Q \subseteq \check{\mathfrak{a}}_{P'}$.

Let $\omega \in \Omega(\mathfrak{s}, \check{\mathfrak{a}}_{P'})$ be such that $\tilde{\omega} \mid \check{\mathfrak{a}}_Q = \mathrm{Id}$ and put

$$\mathfrak{s}_\omega = \{-\overline{(\omega\lambda)} | \lambda \in \mathfrak{s}\},$$

$\mathfrak{s}_\omega$ is a complex affine subspace of $\check{\mathfrak{a}}_{P'}$. Then for such ω there is a function $N(\omega, \lambda)$ on $\mathfrak{s}^Q_P$, the projection of $\mathfrak{s}$ on $(\check{\mathfrak{a}}^Q_P)_{\mathbb{C}}$, with values in the space of linear transformations from $\operatorname{Hom}\big(S(\mathfrak{s}), A^2_{\mathrm{cusp}}(P,\chi,\Gamma)\big)$ to $S(\mathfrak{s}_\omega) \otimes A^2_{\mathrm{cusp}}(P', \omega\chi, \Gamma)$ such that if F belongs to $\operatorname{Hom}\big(S(\mathfrak{s}), A^2_{\mathrm{cusp}}(P,\chi,\Gamma)\big)$ and F' belongs to $S(\mathfrak{s}_\omega)\otimes A^2_{\mathrm{cusp}}(P', \omega\chi, \Gamma)$, the function $(N(\omega,\lambda)F, F')$ is meromorphic on $\mathfrak{s}^Q_P$. If λ_0 is a point of $\mathfrak{s}^Q_P$ there is a polynomial $p(\lambda)$ and a neighborhood U of λ_0 in $\mathfrak{s}^P_B$ such that $p(\lambda)(N(\omega,\lambda)F, F')$ is analytic on U for all F and F' and there is an integer n such that $(N(\omega,\lambda)F, F') \equiv 0$ if the order[1] of F or F' is greater

[1]The order of F is the degree of the lowest nonzero term which appears in the power series expansion of F.

than n. Finally

$$\int_{N_{P'}(\mathbb{Q})\backslash N_{P'}(\mathbb{A})} E^{Q}_{\mathfrak{s},P}(nmk,F,\lambda)\mathrm{d}n =$$

$$\begin{cases} \text{is orthogonal to the space of cusp forms,} & \text{if } P' \notin \{\mathrm{P}\} \\ \sum_{\substack{\omega\in\Omega(\mathfrak{s},\check{\mathfrak{a}}_{P'})\\ \tilde{\omega}|\check{\mathfrak{a}}_{Q}=\mathrm{Id}}} \big(e^{\langle \rho_{P'}+\omega\lambda, H_{P'}(m)\rangle}\big) N(\omega,\lambda)F(mk), & \text{if } P' \in \{\mathrm{P}\}.\diamond \end{cases}$$

(iii). Let $P \in \{\mathrm{P}\}$ and let R and Q be a parabolic subgroup such that $P \subseteq R \subseteq Q$ and let P' be a parabolic subgroup such that $\check{\mathfrak{a}}_{P'} \supseteq \check{\mathfrak{a}}_{Q}$. Let $\mathfrak{s}$ be an affine subspace of $(\check{\mathfrak{a}}_{P})_{\mathbb{C}}$ such that $\check{\mathfrak{a}}_{Q} \subseteq \check{\mathfrak{a}}_{R} \subseteq \tilde{\mathfrak{s}} \subseteq \check{\mathfrak{a}}_{P}$. Suppose that F belongs to $\mathrm{Hom}\big(S(\mathfrak{s}), A^2_{\mathrm{cusp}}(P,\chi,\Gamma)\big)$. Consdier two Eisenstein functions $E^{Q}_{\mathfrak{s},P}(g,F,\eta)$ and $E^{R}_{\mathfrak{s},P}(g,F,\nu)$ on $A_Q(\mathbb{R})^0 N_Q(\mathbb{A}) M_Q(\mathbb{Q}) \setminus G(\mathbb{A}) \times \mathfrak{s}^{Q}_{P}$ and $A_R(\mathbb{R})^0 N_R(\mathbb{A}) M_R(\mathbb{Q}) \setminus G(\mathbb{A}) \times \mathfrak{s}^{R}_{P}$ respectively, in which $\mathfrak{s}^{Q}_{P}$ and $\mathfrak{s}^{R}_{P}$ are the projections of $\mathfrak{s}$ on $\check{\mathfrak{a}}^{Q}_{P}$ and $\check{\mathfrak{a}}^{R}_{P}$ respectively. Thus $\nu \in \mathfrak{s}^{Q}_{P}$, and if $\theta \in \mathfrak{s}^{Q}_{P}$ then $\theta = \theta^{Q}_{R} + \theta^{R}_{P}$ which $\theta^{Q}_{R} \in (\check{\mathfrak{a}}^{Q}_{R})_{\mathbb{C}}$ and $\theta^{R}_{P} \in \mathfrak{s}^{R}_{P}$. Let

$$\theta \in \bigcup_{\substack{\check{\mathfrak{a}}_{P'}\\ \check{\mathfrak{a}}_{Q}\subseteq\check{\mathfrak{a}}_{P'}}} \; \bigcup_{\substack{\omega\in\Omega(\check{\mathfrak{a}}_{P},\check{\mathfrak{a}}_{P'})\\ \tilde{\omega}|\check{\mathfrak{a}}_{Q}=\mathrm{Id}}} \omega^{-1}\mathfrak{A}^{Q}_{P'}, \tag{1.12}$$

in which

$$\mathfrak{A}^{Q}_{P'} = \{\lambda \in (\mathfrak{a}^{Q}_{P'})_{\mathbb{C}} \mid \Re(\lambda) \in \rho^{Q}_{P'} + \mathfrak{a}^{Q+}_{P'}\}.$$

Then if θ lies in the convex hull of the right hand side of (1.12) there is a decomposition

$$E^{Q}_{\mathfrak{s},P}(g,F,\theta) = \sum_{\gamma\in R(\mathbb{Q})\backslash Q(\mathbb{Q})} e^{\langle \theta^{Q}_{R}+\rho^{Q}_{R}, H_Q(\gamma g)\rangle} E^{R}_{\mathfrak{s},P}(\gamma g,F,\theta^{R}_{P}), \tag{1.13}$$

if $E^{R}_{\mathfrak{s},P}(\gamma g,F,\theta^{R}_{P})$ is analytic at θ^{R}_{P}. The convergence of the right hand side of (1.13) follows from the discussions after the theorem 4.1 in [L1]. Moreover, for θ like above, if we take the constant term of $E^{Q}_{\mathfrak{s},P}(g,F,\theta)$ along the parabolic subgroup P' (as it is explained in **(ii)** above), we obtain

$$N'(\omega,\theta) = N^{R}_{P}(\omega,\theta^{R}_{P}) \tag{1.14}$$

if $\omega \in \Omega(\mathfrak{s},\check{\mathfrak{a}}_{P})$ is such that $\tilde{\omega} \mid \check{\mathfrak{a}}_{Q} = \mathrm{Id}.\diamond$

(iv) Since as we mentioned already in **(A)** in **(1.1)**, in adelic setting, associated parabolic subgroups which are also conjugate through an element of $G(\mathbb{Q})$ are equal, this definition (which is originally formulated for the real groups) will be trivial in the language of adeles. See the page 169 of [L1] for the original definition.$\diamond$

To state the part **(v)** we need some definitions which we give them in a footnote since they are only relevant to this definition in order to not to interrupt the main text.[2]

(v). Let $k \in K$ and $F \in A^2_{\text{cusp}}(P, \chi, \Gamma)$ and $\theta \in \check{\mathfrak{a}}_P$. Then $k.E(g, F, \theta) = E(g, k.F, \theta)$ for the usual action of the subgroup K. Let f be a continuous function with compact support on $G(\mathbb{A})$ such that $f(gk^{-1})$ and $f(k^{-1}g)$ both belong to the space spanned by the matrix elements of an irreducible representation of $\mathbb{K}$ (i.e. f belongs to Γ, for Γ defined at **(1.3)**). Then $f * E(g, F, \theta) = E(g, \mathrm{d}\big(\pi(f, \theta)\big) * F, \theta)$.$\diamond$

(1.12) Now we can explain the main construction of [L1]. We recall that for the parabolic subgroups $P \subseteq Q$ and affine subspace $\mathfrak{s}$ of $(\check{\mathfrak{a}}^Q_P)_{\mathbb{C}}$ such that $\check{\mathfrak{a}}_Q \subseteq \tilde{\mathfrak{s}} \subseteq \check{\mathfrak{a}}_P$, the symmetric algebra $S(\mathfrak{s})$ is isomorphic to a subalgebra of the algebra of holomorphic differential operators with constant coefficients on $(\check{\mathfrak{a}}^Q_P)_{\mathbb{C}}$. Now, at the first step suppose that in the above setting we have $\mathfrak{s} = \check{\mathfrak{a}}_\circ$, the split component of the minimal parabolic $P_\circ$ which was fixed at **(1.1)**. Then $S(\mathfrak{s})$ consists of constant functions, and for all standard parabolic subgroups $P \in \{\text{P}\}$ with split component $\check{\mathfrak{a}}_P = \check{\mathfrak{a}}_\circ$ and for all the functions $F_P : (\check{\mathfrak{a}}_P)_{\mathbb{C}} \to A^2_{\text{cusp}}(P, \chi, \Gamma)$ as in **(1.11)**, the map $F_P \longmapsto F_P(1)$ is an isomorphism between $\operatorname{Hom}\big(S(\mathfrak{s}), A^2_{\text{cusp}}(P, \chi, \Gamma)\big)$ and $A^2_{\text{cusp}}(P, \chi, \Gamma)$. Let us choose a parabolic subgroup Q satisfying $P \subseteq Q$ and a parameter $\lambda \in \mathfrak{A}^Q_P = \{\lambda \in (\check{\mathfrak{a}}^Q_P)_{\mathbb{C}} \mid \Re(\lambda) \in \rho^Q_P + \check{\mathfrak{a}}^{Q+}_P\}$. Then we can construct

[2] In the definition **v)** below we need to define two operators $\mathrm{d}(.)$ and $\pi(.,.)$ and the convolution of functions defined on $G(\mathbb{A})$. Let us begin with convolution. Let ϕ be a locally integrable function on $G(\mathbb{A})$ and $f(.) \in C^N_c(G(\mathbb{A}))$. Then we can form the convolution of these two functions as $(f * \phi)(x) = \int_{G(\mathbb{A})} f(y)\phi(xy)\mathrm{d}y$. The convolution is a compact operator on the space of cuspidal automorphic forms. More precisely, if $\phi(.) \in A^2_{\text{cusp}}(P, \chi, \Gamma)$ then the map $\phi \to (f * \phi)(x)$ on $A^2_{\text{cusp}}(P, \chi, \Gamma)$ is a compact operator. This fact is proved in the corollary of the lemma 3.1 of [L1], see also [H1] page 14.
To define $\mathrm{d}(.)$ suppose that R and Q are parabolics such that $\check{\mathfrak{a}}_R \subseteq \tilde{\mathfrak{s}} \subseteq \check{\mathfrak{a}}_Q$ and let $\theta \in S(\mathfrak{s})$. Then we have $\theta_R = \theta \otimes \theta^Q_R \in S(\mathfrak{s}_R) = S(\mathfrak{s}) \otimes S(\mathfrak{s}^R_Q)$. Let F a function from $\check{\mathfrak{a}}^R_Q$ to $\operatorname{Hom}\big(S(\mathfrak{s}), A^2_{\text{cusp}}(P, \chi, \Gamma)\big)$ which is analytic in a neighborhood of a point λ, then the d operator is defined as $\mathrm{d}F(\lambda)(\theta \otimes \theta^R_Q) = \mathrm{D}(\theta)F(\lambda)(\theta^R_Q)$, for $\lambda \in \check{\mathfrak{a}}^R_Q$. It is clear from the definition that $\mathrm{dd}F(\lambda)(.) = \mathrm{d}F(\lambda)(.)$.
Let us proceed to define the operator $\pi(y, \theta)$. Fix a standard parabolic subgroup P and a function $\phi(x) \in A^2_{\text{cusp}}(P, \chi, \Gamma)$. Let us consider functions of the form $\phi(x)e^{\langle \theta + \rho_P, H_P(x) \rangle}$ for $\theta \in (\check{\mathfrak{a}_P})_{\mathbb{C}}$ and $x \in G(\mathbb{A})$. If $y \in G(\mathbb{A})$ then there is a function $\varphi(x) \in A^2_{\text{cusp}}(P, \chi, \Gamma)$ such that

$$\phi(xy)e^{\langle \theta + \rho_P, H_P(xy) \rangle} = \varphi(x)e^{\langle \theta + \rho_P, H_P(x) \rangle}.$$

We will write $\varphi(x) = \pi(y, \theta)\phi(xy)$. Then $\pi(y, \theta)$ is a bounded linear operator from $A^2_{\text{cusp}}(P, \chi, \Gamma) \to A^2_{\text{cusp}}(P, \chi, \Gamma)$, $\pi(xy, \theta) = \pi(x, \theta).\pi(y, \theta)$ and $\pi(1, \theta) = \text{id}$. It is readily seen that there are constants c and N such that $\| \pi(y, \theta) \|_{op} \leq c.(1 + \theta)^N$ which implies that $\pi(y, \theta)$ is a strongly continuous representation of $A^2_{\text{cusp}}(P, \chi, \Gamma)$. We can define the convolution for the operators $\pi(y, \theta)$ as follows. Let $\phi(.) \in A^2_{\text{cusp}}(P, \chi, \Gamma)$ and $f(.)$ be a continuous function with compact support on G. Then $\pi(f, \theta) * \phi = \int_{G(\mathbb{A})} f(y)\pi(y, \theta)\phi\mathrm{d}y$.

the cuspidal Eisenstein series

$$E_P^Q(\lambda, F)(g) = \sum_{\gamma\in P(\mathbb{Q})\backslash Q(\mathbb{Q})} e^{\langle\lambda+\rho_P^Q, H_P(\gamma g)\rangle} F(1)(\gamma g).$$

The collection $\{E_P^Q(\lambda, F)|\ Q \supset P \text{ for } P \text{ fixed}\}$ defines an Eisenstein system which can be checked directly from the definitions **(i)** to **(v)** in **(1.11)**. At the second step it has to be shown that the residues of these cuspidal Eisenstein series satisfy the definition of the Eisenstein Systems and and all the other Eisenstein series appearing in the spectral decomposition are obtained from systems of this type by taking residues. This is done through the lemmas 7.5, 7.6 and theorem 7.1 in [L1]. This construction can be explained intuitively as follows. We proceed from the above situation by dropping the assumption that P is minimal and let P and Q be standard parabolic subgroups such that $P \in \{\mathrm{P}\}$ and $P \subseteq Q$. Then we construct the disjoint union $S_P(Q) = \bigcup_{\mathfrak{s}} S_P(Q, \mathfrak{s})$ over all the subspaces $\mathfrak{s}$ which carry an Eisenstein system and satisfy $\check{\mathfrak{a}}_Q \subseteq \mathfrak{s} \subseteq \check{\mathfrak{a}}_P$. Note that the spaces $S_P(Q, \mathfrak{s})$ were defined at the beginning of **(1.11)**. Let the function

$$F_P : (\check{\mathfrak{a}}_P)_{\mathbb{C}} \to A^2_{\mathrm{cusp}}(P, \chi, \Gamma)$$

be regular at the point $X(\mathfrak{s}_0)$ for $S \in S_P(Q)$. Then the vector

$$\Phi = \bigoplus_P \bigoplus_{S\in S_P(Q)} (\mathrm{d}_S F_P)(X(\mathfrak{s}_0))$$

belongs to a subspace of

$$\bigoplus_P \bigoplus_{S\in S_P(Q)} \mathrm{Hom}\left(S(\mathfrak{s}), A^2_{\mathrm{cusp}}(P, \chi, \Gamma)\right).$$

Then the main construction of the theorem 7.1 of [L1] shows that the functions defined by

$$\phi(x) = \sum_{\{P\in\{\mathrm{P}\}:P\subset Q\}} \sum_{S\in S_P(Q)} \mathrm{Res}_{S,\lambda\to X(\mathfrak{s}_0)} E^Q(x, F_P(\lambda), \lambda),$$

($X(\mathfrak{s}_0)$ as in **(1.11)**) lies is $A^2(Q, \chi, \Gamma)$ and the whole space is spanned by such a functions, and the map $\Phi \to \phi$ is a surjective linear map, which he had proved to be actually an isomorphism.

This construction is done mainly in the course of the proof of the above mentioned theorem, but before reaching that goal, Langlands had to prove that the relevant subspaces satisfy some properties which are stated in lemma 7.2 and settled down in lemmas 7.4, 7.5 and 7.6, which we give an overview of them.

(1.13) Let us fix a parabolic subgroup $P \in \{\mathrm{P}\}$ and choose a parabolic subgroup $Q \supset P$. Let ${}^{Q}S^{P}_{m}$ denote a collection of distinct affine subspaces $\mathfrak{s}$ of $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ of dimension m such that $\check{\mathfrak{a}}_Q \subseteq \tilde{\mathfrak{s}} \subseteq \check{\mathfrak{a}}_P$ as in **(1.11)**, and an Eisenstein system defined on each of them. Set $S^{P}_{m} = \cup_{\{Q|Q\supseteq P\}} {}^{Q}S^{P}_{m}$. This collection is locally finite, as it is proved in lemma 7.5 of [L1], which we will explain briefly in **(1.15)** below. For $\omega \in \Omega_G$ and the affine subspace $\mathfrak{s}$ like above set $\mathfrak{s}_\omega = \{-\overline{\omega\lambda}|\lambda \in \mathfrak{s}\}$. Let P' be a parabolic subgroup satisfying $\check{\mathfrak{a}}_Q \subseteq \check{\mathfrak{a}}_{P'}$, and let $\mathfrak{t}$ be an affine subspace of $\check{\mathfrak{a}}_{P'}$ such that $\check{\mathfrak{a}}_Q \subseteq \mathfrak{t} \subseteq \check{\mathfrak{a}}_{P'}$ and let $\Omega_Q(\tilde{\mathfrak{s}}, \tilde{\mathfrak{t}})$ be the subgroup of the elements $\omega \in \Omega_G$ which leave $\check{\mathfrak{a}}_Q$ pointwise fixed and which map $\tilde{\mathfrak{s}} \to (\check{\mathfrak{a}}_{P'})_{\mathbb{C}}$ linearly and satisfying the property that $\tilde{\mathfrak{s}}_\omega = \tilde{\mathfrak{t}}$. We call two elements $\mathfrak{s}$ and $\mathfrak{t}$ of S^{P}_{m} equivalent if $\Omega_R(\tilde{\mathfrak{s}}, \tilde{\mathfrak{t}})$ is not empty. We will denote by $\Omega(\mathfrak{s}, \mathfrak{t})$ the set of linear transformations $\mathfrak{s} \to \{\lambda| -\overline{\lambda} \in \mathfrak{t}\}$. Then for each $\mathfrak{s} \in S^{P}_{m}$ there is an element $s^\circ \in \Omega(\mathfrak{s}, \mathfrak{s})$ such that

$$s^\circ(X(\mathfrak{s}) + \lambda) = -\overline{X(\mathfrak{s})} + \lambda$$

for all $\lambda \in \tilde{\mathfrak{s}}$. [3] To construct the general residual Eisenstein series and proving their functional equations, Langlands had to prove that:

1) The collection of relevant hyperplanes are locally finite.

2) Let $P \in \{\mathrm{P}\}$ and let $R \supset P$ be a parabolic subgroup. Then $\mathfrak{s} = -\omega\check{\mathfrak{a}}_R$ is a hyperplane along which an Eisenstein system is defined. Suppose Q is the smallest parabolic subgroup containing P such that $\check{\mathfrak{a}}_Q \subset \hat{\omega}\check{\mathfrak{a}}_R$. Then $\Re(X(\mathfrak{s})) \in {}^{+}\check{\mathfrak{a}}^{Q}_{P}$ and and lies in a compact subset of $\check{\mathfrak{a}}_P$.

3) For each relevant hyperplane $\mathfrak{t}$ like above, there is an element $\omega_0 \in \Omega(\mathfrak{t}, \mathfrak{t})$ which leaves the $\tilde{\mathfrak{t}}$ point-wise fixed.

4) The operators $N(\omega, \lambda)$ defined for $\lambda \in \mathfrak{t}$ are non zero only if there is a hyperplane $\mathfrak{s}$ such that $\mathfrak{t}_\omega = \mathfrak{s}$.

These items are stated in the lemma 7.2 of [L1] as assumptions, and they imply that the collection S as above is finite (proved in lemma 7.5), and for each $\mathfrak{s}$ the point $X(\mathfrak{s})$ is real (proved at the very end of the proof of the theorem 7.1), and for any choice of $\check{\mathfrak{a}}_R$, every equivalence class in S contains an element $\mathfrak{s}$ such that each $\tilde{\mathfrak{s}}$ is the complexification of a split component of a parabolic subgroup, which is a subspace of $\mathfrak{h}$ (proved in lemma 7.5). Furthermore $\mathfrak{s}^\perp$, the orthogonal complement of $\tilde{\mathfrak{s}} \cap \check{\mathfrak{a}}_R$ in $\check{\mathfrak{a}}_P$ is spanned by the vector $X(\mathfrak{s})$. Item **1**) is also a consequence of the theorem 7.5 which proves that the singularities of the Eisenstein series relevant to the spectral decomposition are lying along root hyperplanes and consequently locally finite.

To prove **4**) Langlands uses the lemma 7.3 which we state here as:

[3] There is a slight ambiguity in the notation here. we use letters s, s°, t etc. to denote the elements of the Weyl group when they are supposed to act on the hyperplanes rather than subspaces, for the subspaces instead we use the letters ω, σ etc. Since there may (and do) exist subspaces in the relevant (shifted) hyperplanes, this notation would mix up at some places but it wont lead to confusion.

Lemma 7.3. *Let ϕ belongs to the space $L^2\big(\{\mathrm{P}\}, \boldsymbol{\chi}, \Gamma\big)$ and suppose that there are distinct points $\{\lambda_1, ..., \lambda_n\}$ in $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ such that*

$$\int_{N_P(\mathbb{Q})\backslash N_P(\mathbb{A})} \phi(ng)\, dn = \sum_{k=1}^{n} e^{\langle \lambda_k + \rho_P, H_P(g)\rangle} \phi_{P,\lambda_k}(H_P(g))(g),$$

for $\phi_{P,\lambda_k}(H_P(g))(g)$ products of some polynomials and some $N_P(\mathbb{A})$-invariant functions. If $\phi_{P,\lambda_k}(H_P(g))(g) \not\equiv 0$ then the points λ_k are all real.

The significance of this lemma can be explained as follows. The Eisenstein system $E_P^G(g, \Phi_{\mathfrak{s}}, \lambda)$, defined on a hyperplane $\mathfrak{s} = -\omega\check{\mathfrak{a}}_R$ for a function $\Phi_{\mathfrak{s}}(.) = \mathrm{d}F_P(.)$ (see the beginning of **(1.11)**), does not belong to the space (1.2) for general λ unless $\lambda = X(\mathfrak{s})$. Then this lemma and the equation (1.8) (given in **(1.8)**) imply that the intertwining functions $N(\omega, \lambda_k)\Phi_{\mathfrak{s}}$ are not vanishing only if $-\Re(\omega\lambda) = \Re(X_{\mathfrak{s}_\omega}) \in {}^+\check{\mathfrak{a}}_P^Q$ for Q the largest parabolic subgroup such that $\check{\mathfrak{a}}_Q \subset \hat{\omega}\check{\mathfrak{a}}_R$.

(1.14) The lemma 7.3 plays a pivotal role in lemma 7.4 which in turn proves the functional equation for the general Eisenstein series which is based on the functional equation for the cuspidal Eisenstein series. To state it let us fix like in **(1.13)** a parabolic subgroups $P \in \{\mathrm{P}\}$, a parabolic subgroup Q such that $P \subseteq Q$, and another parabolic subgroup P' satisfying $\check{\mathfrak{a}}_Q \subseteq \check{\mathfrak{a}}_{P'}$, affine subspaces $\mathfrak{s}$ and $\mathfrak{t}$ such that $\check{\mathfrak{a}}_Q \subseteq \tilde{\mathfrak{s}} \subseteq \check{\mathfrak{a}}_P$, and $\check{\mathfrak{a}}_Q \subseteq \tilde{\mathfrak{t}} \subseteq \check{\mathfrak{a}}_{P'}$. Let us fix an equivalence class C in ${}^Q S_m = \cup_{\{P|Q\supseteq P\}} {}^Q S_m^P$ under the action of the Weyl group of G, Ω_G, and fix an element $\mathfrak{s}$ in C such that $\tilde{\mathfrak{s}}$ is a distinguished subspace of $\check{\mathfrak{a}}_\circ$ and let $\Omega_Q(\mathfrak{s}, \mathfrak{t})$ be the subset of such a linear transformations that leave $(\check{\mathfrak{a}}_Q)_{\mathbb{C}}$ fixed. Let $\Omega(\mathfrak{s}, C) = \cup_{\mathfrak{t}\in C}\Omega_Q(\mathfrak{s}, \mathfrak{t})$ and let $\Omega_0(\mathfrak{s}, C)$ denote the set of elements in $\Omega(\mathfrak{s}, C)$ which leave each point of $\tilde{\mathfrak{s}}$ fixed. If $\mathfrak{t}_1$ and $\mathfrak{t}_2$ belong to C and s° is as defined in **(1.13)**, then every element of $\Omega(\mathfrak{t}_1, \mathfrak{t}_2)$ can be written as $ts^\circ s^{-1}$ for $s \in \Omega(\mathfrak{s}, \mathfrak{t}_1)$ and $t \in \Omega(\mathfrak{s}, \mathfrak{t}_2)$. If $\lambda \in \check{\mathfrak{a}}_P^Q$ then we have a matrix

$$M(\lambda) = \Big(M(ts^\circ s^{-1}, ss^\circ\lambda) | s, t \in \Omega(\mathfrak{s}, C) \Big)$$

which is a meromorphic function of λ, and a matrix

$$M = \Big(M(ts^\circ s^{-1}, ss^\circ\lambda) | s, t \in \Omega_0(\mathfrak{s}, C) \Big),$$

which has constant coefficients. These matrices have finite dimensional range and the dimension of their range is their rank. Then the lemma 7.4 states that the rank of these two matrices are equal. This implies the functional equation of all Eisenstein series as we will see it soon.
They are linear transformations between the following two spaces:

$$\bigoplus_{s\in\Omega'(\mathfrak{s},C)} \mathrm{Hom}_{\mathbb{K}}\left(S(\mathfrak{s}_s), A^2_{\mathrm{cusp}}({}^sP, s\chi,\Gamma)\right) \\ \to \bigoplus_{s\in\Omega'(\mathfrak{s},C)} S(\mathfrak{s}_s)\otimes A^2_{\mathrm{cusp}}({}^sP, s\chi,\Gamma) \quad (1.15)$$

such that if $\Phi(.)$ belongs to the left hand side of (1.15), then the component of $M(s,\lambda)\Phi(.)$ on the right hand side of (1.15) is $N(s,\lambda)\Phi(.)$. By ${}^sP$ we denote the parabolic subgroup whose Levi component is sM_Ps^{-1}, and $\Omega'(\mathfrak{s},C)$ stands for $\Omega(.,.)$ or $\Omega_0(.,.)$ respectively for $M(\lambda)$ and M. The key feature to prove the functional equation of all Eisenstein series, which is a consequence of the equality of the ranks of the above matrices, is that if $\oplus_{s\in\Omega_0(\mathfrak{s},C)}M(ts^os^{-1}, ss^o\lambda)\phi_s$ belong to the range of $M(\lambda)$ in (1.15) and and $\oplus_{t\in\Omega_0(\mathfrak{s},C)}\phi'_t$ be a function lying in that range, and $\oplus_{s\in\Omega_0(\mathfrak{s},C)}\oplus_{t\in\Omega_0(\mathfrak{s},C)} \left(M(ts^os^{-1}, ss^o\lambda)\phi_s, \phi'_t\right)$ vanishes, then it vanishes for all $s,t\in\Omega(\mathfrak{s},C)$. A direct consequence of this is that if we have $E(g,\oplus_s\Phi_s,\lambda)=0$ for some $\oplus_s\Phi_s$ lying in (1.15), then $\Phi_s=0$ for all $s\in\Omega(\mathfrak{s},C)$. In particular, if Φ belongs to a space of cusp forms, then for $\Psi(\lambda)=M(\lambda)\Phi-M\Phi$ we have $E(g,\Psi,\lambda)=0$ and consequently $\Psi(\lambda)\equiv 0$. The functional equation of the cuspidal Eisenstein series (given in lemma 6.1 of [L1]) says that we have $E(g,M(s,\lambda)\Phi,s\lambda)=E(g,\Phi,\lambda)$. Then the above vanishing result implies that for the function $\Psi_s(\lambda)=(M(\lambda)-M)\Phi(\lambda)$ we have $E(g,\Psi_s(\lambda),\lambda)=0$ for all $s\in\Omega_0(\mathfrak{s},C)$, consequently $\Psi_s(\lambda)=0$ for all $s\in\Omega(\mathfrak{s},C)$ identically. Now we can prove the functional equation of residual Eisenstein series. Let Φ be like above. Then by taking residues, we will obtain

$$\mathrm{Res}_{\mathfrak{s}}\,E(g,M(s,\lambda)\Phi,s\lambda)=\mathrm{Res}_{\mathfrak{s}}\,E(g,M(s,s^{-1}\lambda')\Phi,\lambda')\mid_{\lambda'=s\lambda} \\ =\mathrm{Res}_{\mathfrak{s}_s}\,E(g,M\Phi,\lambda)=\mathrm{Res}_{\mathfrak{s}}\,E(g,\Phi,\lambda). \quad (1.16)$$

The first equality comes from the above mentioned vanishing result. The second one is a consequence of lemma (7.2) which says that the functions $M(s,\lambda)$ are zero unless $\mathfrak{s}_s=\mathfrak{t}$ for some $\mathfrak{t}\in {}^QS^P_m$.

This implies also that in the process of taking residue of product of pseudo theta series we can bring the residue inside the product. Since lemma 7.5 in [L1] shows that the residues of Eisenstein series are holomorphic in the complex tube above the positive Weyl chamber, the problem of analytic continuation (which was tightly interrelated to the functional equation in the cuspidal case) is automatically achieved in the residual situation. An important consequence of this lemma, which is the content of the corollary 1 to the lemma 7.4, is that the highest residues of Eisenstein series are also eigenfunctions for the Casimir operator. As a consequence the functions defined by Eisenstein systems are representable as the residues of cuspidal Eisenstein series. Then the corollary 2 to this lemma (page 202 in [L1]) says that for each subspace $\check{\mathfrak{a}}_R\subseteq\tilde{\mathfrak{s}}$ and each function F in (1.2) the collection ${}^QS^P_m$ and the associated Eisenstein system are uniquely determined if $E(g, F, X(\mathfrak{s}))$ (which lie in $L^2(G(\mathbb{Q}) \setminus G(\mathbb{A})))$ are given. This corollary gives the meromorphic continuation of residual Eisenstein series almost immediately, so that for $\Phi(.) \in A^2_{\mathrm{res}}(P, \chi, \Gamma)$ and $\lambda \in \mathfrak{A}_P + \mathrm{i}\check{\mathfrak{a}}_P$ the Eisenstein series $E(g, \Phi, \lambda)$ can be meromorphically continued to $\check{\mathfrak{a}}_P + \mathrm{i}\check{\mathfrak{a}}_P$.

(1.15) From now on we fix the collection S^P_m as defined in **(1.13)**. The lemma 7.5 is the analytic heart of the Langlands construction. It singles out a (non-canonical) collection of hyperplanes S like what we have mentioned above, which are appearing in the spectral decomposition of theorem 7.1. The main analytical properties of residual Eisenstein series are settled down in this lemma. To state it we need some definitions from the spectral theory. Fix a parabolic subgroup P with Levi component $M_P A_p$. Then there is a decomposition of the Cartan algebra of $(\mathfrak{h}_G)_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ as

$$(\mathfrak{h}_G)_{\mathbb{C}} = (\check{\mathfrak{a}}_P)_{\mathbb{C}} \oplus (\mathfrak{h}_{M_P})_{\mathbb{C}},$$

for $\mathfrak{h}_{M_P}$ the Cartan subalgebra of $\mathfrak{m}_P$. Suppose $X \in \mathcal{Z}(\mathfrak{m}_P) \otimes \mathfrak{A}(\mathfrak{a}_G)$. A famous theorem of Harish-Chandra shows that there is an isomorphism $X \to p_X(H)$ between the center of universal enveloping algebra $\mathcal{Z}(\mathfrak{g})$ of $\mathfrak{g}$ and the algebra of polynomials on $\mathfrak{h}_G$ invariant under the Weyl group Ω_G of $\mathfrak{g}$. Let this isomorphism map $Y \in \mathcal{Z}(\mathfrak{g})$ to p_Y, then a simple calculation given in chapter 4 of [L1] shows that for any $X \in \mathcal{Z}(\mathfrak{g})$ and $\phi \in A^2(P, p_X(.), \Gamma)$ there is a point $Z \in \mathfrak{h}_{M_P}$ such that

$$X \cdot E(g, \phi, H) = p_X(Z) E(g, \phi, H),$$

for the action $X\cdot$ of the universal enveloping algebra defined as $X \cdot f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(g.\mathrm{exp}tX)|_{t=0}$.
In particular, for the Casimir operator ω_G of $\mathfrak{g}$ and for a function $\phi \in A^2(P, \chi, \Gamma)$ we will have

$$\omega_G \cdot E(g, \phi, H) = \{\langle H, H\rangle - \langle \rho, \rho\rangle + \langle \chi, \chi\rangle\} E(g, \phi, H).$$

Suppose that $\phi(g)$ is the Fourier transform of a Paley-Wiener function $\Phi(g, \lambda)$. Suppose also that for a constant $R > \langle \rho, \rho\rangle$ the points H are belonging to $\mathrm{Str}_P(R)$. Then the values of $\langle H, H\rangle$ are bounded from above by R and the pointwise multiplication

$$\Phi = (\Phi_1, ..., \Phi_r) \to \Psi(H) = (\langle H, H\rangle \Phi_1, ..., \langle H, H\rangle \Phi_r)$$

of functions $\Phi(H)$ lying in $\mathrm{Str}_P(R)$ defines an unbounded self adjoint linear operator

$$\widehat{\phi} \to \hat{\psi} = A\widehat{\phi},$$

on the space (1.2) which is one to one with dense range on (1.1). These A operators are of type mentioned in **iv)** of Eisenstein systems in **(1.11)**.

The resolvent of the operator A will be denoted by $R(\mu, A) = (\mu - \langle\lambda, \lambda\rangle)^{-1}$ and is analytic for μ off the interval $(-\infty, R^2]$. Let Q denote an orthogonal projection from (1.9) to a subspace which commutes with the right multiplication by bounded analytic functions $f(.) = (f_R(.))_{\check{\mathfrak{a}}_R \subset \check{\mathfrak{a}}_P}$ defined on the tube $\mathrm{Str}_P^Q(R)$ satisfying the property $f_R(\lambda) = f_{R'}(\omega\lambda)$ if $\omega \in \Omega(\check{\mathfrak{a}}_R, \check{\mathfrak{a}}_{R'})$ for R and R' are associated parabolic subgroups such that $\check{\mathfrak{a}}_Q$ is contained in $\check{\mathfrak{a}}_R$ and $\check{\mathfrak{a}}_{R'}$. Then the lemma 7.5 states that for each collection S_m^P of hyperplanes defined as above, if $\check{\mathfrak{a}}_Q$ is the distinguished subspace contained in $\tilde{\mathfrak{s}}_\omega = -\omega\check{\mathfrak{a}}_R$ for $\omega \in \Omega(\check{\mathfrak{a}}_R, \check{\mathfrak{a}}_P)$ then $-\omega 0 = \Re X(\mathfrak{s}) \in {}^+\check{\mathfrak{a}}_P^Q$ and $X(\mathfrak{s}) \in \mathrm{Str}_P(R)$, and if only a finite number of the elements of S_m^P meet each compact subset of $(\check{\mathfrak{a}}_P)_{\mathbb{C}}$, and if, for arbitrary Φ (chosen so that it vanishes to a sufficiently high degree on the singular hyperplanes of $N(\omega, \lambda)$ meeting the domain of integration below), the differences

$$\big(R(\mu, A)Q\hat{\phi}, \hat{\psi}\big) -$$

$$\sum_{\mathfrak{s}\in S} \sum_{\omega\in\Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)} \int_{\substack{X(\mathfrak{s})+\mathrm{i}\,\check{\mathfrak{a}}_P^Q,\\ \|\Im(\lambda)\|<a}} \Big(N(\omega, \lambda)\mathrm{d}\Big(\frac{\Phi(\lambda)}{\mu - (\lambda, \lambda)}\Big), \mathrm{d}\Psi(-\omega\overline{\lambda})\Big)\mathrm{d}\lambda,$$

and

$$\big(Q\hat{\phi}, R(\overline{\mu}, A)\hat{\psi}\big) -$$

$$\sum_{\mathfrak{s}\in S} \sum_{\omega\in\Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)} \int_{\substack{X(\mathfrak{s})+\mathrm{i}\,\check{\mathfrak{a}}_P^Q,\\ \|\Im(\lambda)\|<a}} \Big(N(\omega, \lambda)\mathrm{d}\Phi(\lambda), \mathrm{d}\Big(\frac{\Psi(-\omega\overline{\lambda})}{\mu - (-\omega\overline{\lambda}, -\omega\overline{\lambda})}\Big)\Big)\mathrm{d}\lambda,$$

are analytic for $\Re(\mu) > R^2 - a^2$, for a positive number a satisfying $a < \sqrt{R^2 + \| \Re(\mu) \|^2}$, then there is an element $s_0 \in \Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_P)$ which leaves each point of $\tilde{\mathfrak{s}}$ pointwise fixed, and the functions $N(.,.)$ are vanishing identically unless there is a hyperplane $\mathfrak{t}$ in this class such that $\mathfrak{s}_\omega = \mathfrak{t}$, and then

$$N(\omega, \lambda) = \overline{N(\omega^{-1}, -\omega\overline{\lambda})}.$$

The collection of affine subspaces fixed by the lemma 7.5 will be used in the lemma 7.6 to produce functions which generate the spectral decomposition. It will be shows in the lemma 7.6 that these functions are integrals of residual Eisenstein Series on some unitary axes. To explain it we need some definitions from the spectral theory. For the proofs we refer to [Y] or [S].

(1.16) Let A be a self adjoint operator on a Hilbert space $\mathfrak{H}$ and let $R(z, A) = (A - z\,\mathrm{Id}_{\mathfrak{H}})^{-1}$, $z \in \mathbb{C}$ denote its resolvent. Let $\mathcal{A}$ be the $\sigma-$Algebra of Borel functions on $\mathbb{R}$. Then there exists a positive measure $\mu = \mu_A$ which maps every $X \in \mathcal{A}$ to a self adjoint projection $\mu_A(X) \in \mathrm{Hom}(\mathfrak{H}, \mathfrak{H})$ such that the following properties are satisfied:
a. For all $X \in \mathcal{A}$ the operator $\mu(X) \in \mathrm{Hom}(\mathfrak{H}, \mathfrak{H})$ is self-adjoint and we have $\langle\mu(X)v, v\rangle \in [0, \infty)$.
b. For all $X, Y \in \mathcal{A}$ such that $X \cap Y = \emptyset$ we have $\mu(X \cup Y) = \mu(X) + \mu(Y)$. In particular $\mu(0) = 0$.

c. $\mu(\mathbb{R}) = \mathrm{Id}_{\mathfrak{H}}$ and $\| \mu(X)v \| \leq \| \mu(\mathbb{R}) \| \, |v|^2 = |v|^2$. In particular $\mu(t)\mu(s) = \mu(s)\mu(t) = \mu(t)$ if $t < s$ and
d. Let $X_1 \subseteq X_2 \subseteq ...$, $X_i \in \mathcal{A}$ and $X = \cup_{i=1}^{\infty} X_i$. Then for all $v \in \mathfrak{H}$ we have $\mu(X) = \mathrm{s.}\lim_{i\to\infty} \mu(X_i)$ in the strong operator topology, as well in the weak-topology.
Then this collection of measures defines for each σ-measurable function f on $\mathbb{R}$ a self adjoint operator $\mathrm{I}_f(X)$ on $\mathfrak{H}$ by $\langle \mathrm{I}_f(X)v, v\rangle = \int_X f(t)\mathrm{d}\langle \mu(t)v, v\rangle$ for each $X \in \mathcal{A}$, with the domain $\mathcal{D}_{\mathrm{I}_f} = \{v \in \mathfrak{H} | \int_{\mathbb{R}} f^2(t)\mathrm{d}\langle \mu(t)v, v\rangle < \infty\}$. Let $u, v \in \mathfrak{H}$. There is a canonical unique representation of the operator A via the spectral measure μ as $\langle u, Av\rangle = \int_{\mathbb{R}} t\mathrm{d}\langle \mu(t)v, v\rangle$ and $\langle (A - \lambda \,\mathrm{Id}_{\mathfrak{H}})^{-1}u, v\rangle = \int_{\mathbb{R}} \frac{1}{t-\lambda}\mathrm{d}\langle \mu(t)u, v\rangle$. Observe that then $\mathcal{D}_{\mathrm{I}_f} = \mathcal{D}_A$. An important consequence of this construction is the famous formula

$$\langle (\mathrm{Id}_{\mathfrak{H}} - \mu(\lambda))u, v\rangle = \lim_{\delta\to 0} \frac{1}{2\pi\,\mathrm{i}} \int_{C(\alpha,\beta,\gamma,\delta)} \langle R(z, A)u, v\rangle \mathrm{d}z$$

for $\alpha < (\rho, \rho) < \beta$, $\delta < \gamma$ and $\alpha = \lambda$, due to Stone (in the form which is used by Langlands). The Contour consists of two oriented polygonal ways with vertices at $\alpha + \mathrm{i}\,\delta$, $\alpha + \mathrm{i}\,\gamma$, $\beta + \mathrm{i}\,\gamma$, $\beta + \mathrm{i}\,\delta$, and $\alpha - \mathrm{i}\,\delta$, $\alpha - \mathrm{i}\,\gamma$, $\beta - \mathrm{i}\,\gamma$, $\beta - \mathrm{i}\,\delta$. We usually take $\mathfrak{H}$ to be a subspace of (1.0).

(1.17) Now we are ready to state the lemma 7.6. Fix as usual $P \in \{\mathrm{P}\}$ and $P \subseteq Q$. Let $C_1, ..., C_u$ be equivalence classes of m-dimensional affine hyperplanes in ${}^{Q}S_m^P$ under the action of the Weyl group. For each $1 \leq n \leq u$ choose an element $\mathfrak{s}^n \in C_n$ such that $\tilde{\mathfrak{s}}^n$ contains the complexification of a distinguished subspace $\check{\mathfrak{a}}^n$ of $\mathfrak{h}$. If $P \in \{\mathrm{P}\}$ and $\check{\mathfrak{a}}_Q \subseteq \check{\mathfrak{a}}_P$ and if $\Phi(.)$ is like above then

$$\sum_{\omega\in\Omega_n^P(\mathfrak{s}^n, C_n)} E(g, \mathrm{d}\Phi(\omega\omega_n^\circ\lambda), \omega\omega_n^\circ\lambda)$$

is analytic on the unitary axis $X(\mathfrak{s}_\omega^n) + \mathrm{i}\,\omega\check{\mathfrak{a}}_P^n$. The space $\check{\mathfrak{a}}_P^n$ is the orthogonal complement of $\check{\mathfrak{a}}^n$ in $\check{\mathfrak{a}}_P$. The subset $\Omega_n^P(\mathfrak{s}^n, C_n)$ is defined like our discussion about the lemma 7.4 above to be the subset of $\Omega_n(\mathfrak{s}, C) = \cup_{\mathfrak{t}\in C}\Omega_n(\mathfrak{s}, \mathfrak{t})$, restrictions to $\mathfrak{s}_n$ of elements of $\Omega(\mathfrak{s}, C)$, such that for some $\mathfrak{t} \in {}^{\circ}S_m^P$ we have $\mathfrak{s}_s = \mathfrak{t}$. Furthermore, if $\omega_{\mathfrak{s}^n} = \#\Omega_n^P(\mathfrak{s}^n, C_n)$, then the function

$$\phi_T(g) = \sum_{n=1}^{u} \frac{1}{\omega_{\mathfrak{s}^n}(2\pi\,\mathrm{i})^m} \int E(g, \mathrm{d}\Phi(\omega\lambda), \omega\lambda)\mathrm{d}\lambda,$$

in which the integration is taken over the unitary axis

$$\mathrm{U}(\mathfrak{s}_\omega^n, X(\mathfrak{s}_\omega^n), T) = \{X(\mathfrak{s}_\omega^n) + \mathrm{i}\,\lambda | \; \lambda \in \tilde{\mathfrak{s}}_\omega^n \cap \check{\mathfrak{a}}_P^n, \;\; \| \lambda \| < T\},$$

belongs to the closed subspace

$${}^{Q}L_m^2(\{\mathrm{P}\}, \chi, \Gamma) \subseteq {}^{Q}L^2(\{\mathrm{P}\}, \chi, \Gamma),$$

spanned by the functions of the form $(\mathrm{Id}_{\mathfrak{H}} - \mu(t))Q\hat{\phi}$ for the correspondence $\phi(g) \leftrightarrow \Phi(\lambda)$ described above as we discussed the Paley-Wiener functions, and the projection of $\hat{\phi}(.)$ on this subspace is equal to $\lim_{T\to\infty} \phi_T$. Furthermore for functions $\Phi(.)$ and $\Psi(.)$ and $Q \supseteq P$, the inner product

$$\sum_{\theta\in\Omega_n^Q(\mathfrak{s}^n,C_n)} \sum_{\omega\in\Omega_n^P(\mathfrak{s}^n,C_n)} \big(N(\theta\omega_n^\circ\omega^{-1},\lambda)\mathrm{d}\Phi(\omega\omega_n^\circ\lambda), \mathrm{d}\Psi(-\theta\overline{\lambda})\big)$$

is analytic on the unitary axis $X(\mathfrak{s}_\omega^n) + \mathrm{i}\,\omega\check{\mathfrak{a}}_Q^n$.

There is an important corollary to the lemma 7.6, which we have mentioned previously and we state it as follows. We use the notation introduced in **(iii)** of **(1.11)**.
Suppose that $P \in \{\mathrm{P}\}$ and let $P \subseteq Q$ be a parabolic subgroup such that $\check{\mathfrak{a}}_Q \subseteq \tilde{\mathfrak{s}} \subseteq \check{\mathfrak{a}}_P$ for $\mathfrak{s} \in S_Q(P)$, and let $F_Q(.) \in \mathrm{Hom}\,\big(S\big((\mathfrak{s}_Q^P)_{\mathbb{C}}\big), A^2(Q,\chi,\Gamma)\big)$. Suppose that $\check{\mathfrak{a}}_Q$ is the largest distinguished subspace which is contained in $\tilde{\mathfrak{s}}$, and also suppose that $\mathfrak{r}$ denotes the inverse image in $\mathfrak{s}$ of a singular hyperplane of the function $E(g, F_Q(\lambda), \lambda)$, $\lambda \in X(\mathfrak{s}) + \mathfrak{s}_P^Q$, which meets the unitary axis $X(\mathfrak{s}) + \mathrm{i}\,\check{\mathfrak{s}}_P^Q$. Then $\tilde{\mathfrak{r}}$ contains $\check{\mathfrak{a}}_Q$.
This corollary implies that the singular hyperplanes considered in lemmas 7.5 and 7.6 wont meet the unitary axis $X(\mathfrak{s}) + \mathrm{i}\,\check{\mathfrak{a}}_Q$.

(1.18) The subspace ${}^Q L_m^2\big(\{\mathrm{P}\},\chi,\Gamma\big)$ constructed in lemma 7.6 are exactly those subspaces appearing in the decomposition of the space ${}^Q L^2\big(\{\mathrm{P}\},\chi,\Gamma\big)$. We can now state the main theorem of chapter 7:

Theorem 7.1. *Let $P \in \{\mathrm{P}\}$ and let $P \subseteq Q$. Let $p = \dim(\check{\mathfrak{a}}_P)$ and consider the $p+1$ collection $S_m = \cup_{\{Q|P\subseteq Q\}} {}^Q S_m^P$, $0 \le m \le p$, of affine subspaces of $\check{\mathfrak{a}}_P$ of dimension m and the unique Eisenstein systems belonging to S_m, which satisfy the hypothesis of the lemma* 7.5*. Consider the subspace*

$${}^Q L_m^2\big(\{\mathrm{P}\},\chi,\Gamma\big)$$

which is a closed subspace of

$${}^Q L^2\big(\{\mathrm{P}\},\chi,\Gamma\big)$$

associated to S_m by lemma 7.6*. Then there is an orthogonal decomposition*

$${}^Q L^2\big(\{\mathrm{P}\},\chi,\Gamma\big) = \bigoplus_{m=\dim A_Q}^{p} {}^Q L_m^2\big(\{\mathrm{P}\},\chi,\Gamma\big),$$

in which $L_s^2(.)$ and $L_r^2(.)$ are orthogonal if $r \neq s$.

(1.19) Since the Eisenstein series which were used in the spectral decomposition may have singular hyperplanes which intersect the complex tube above the region of absolute convergence, we have to settle down a kind of Cauchy theorem suitable for the situation. This is the content of the lemma 7.1 which is used only in the proof of the theorem 7.1 to deal with the three kind of singularities introduced at the beginning of the theorem 7.1. Through the proof of this theorem it was also proved implicitly that the singularities of Eisenstein systems which are relevant to the spectral decomposition are only of these three kinds. We review them briefly. To begin with, beside the collection S_m introduced above we have to consider the collection T_m of not necessarily distinguished subspaces of $\check{\mathfrak{a}}_P$ of dimension $m-1$. Suppose that only a finite number of these affine subspaces intersect any compact subspace of $\check{\mathfrak{a}}_P$. Moreover, the points $X(\mathfrak{s})$ are lying in $\mathrm{Str}_P(R)$ and $\Re(X(\mathfrak{s}))$ lies in the ${}^+\check{\mathfrak{a}}_P^0$ if $\check{\mathfrak{a}}_0$ is the largest distinguished subspace contained in $\tilde{\mathfrak{s}}$. For each affine subspace $\mathfrak{s} \subset \check{\mathfrak{a}}_P$ there is a subset $\Phi^+(\mathfrak{s}) \subset \Phi^+(\mathfrak{n}_P)$ and constants $\mu \in \mathbb{C}$ such that this space can be defined by the equations of the form $\alpha(\lambda) = \mu$ for $\lambda \in \check{\mathfrak{a}}_P$ and $\alpha \in \Phi^+(\mathfrak{s})$. For $\mathfrak{s} \in S_m$ put ${}^+\check{\mathfrak{a}}(\mathfrak{s}) = \{\lambda \in \check{\mathfrak{a}}_P |\ \alpha(\lambda) > 0 \text{ for } \alpha \in \Phi^+(\mathfrak{s})\}$. The unitary axis above $\mathfrak{s}$ will be denoted by

$$\mathrm{U}(\mathfrak{s}, X(\mathfrak{s}), T) = \{X(\mathfrak{s}) + \mathrm{i}\,\lambda |\ \lambda \in \tilde{\mathfrak{s}} \cap \check{\mathfrak{a}}_P, \quad \| \lambda \| < T\},$$

for T a non-negative real number possibly infinite. Associated with these unitary axes there are open convex cones defined as follows. Let $0 < a < 1$ be a real number. Let $\mathrm{V}(\mathfrak{s}, a)$ denote a non-empty open convex cone in $X(\mathfrak{s}) + (\tilde{\mathfrak{s}} \cap \check{\mathfrak{a}}_P)$ with vertex $X(\mathfrak{s})$ and base U an open subset of a sphere of radius ϵ in $\tilde{\mathfrak{s}} \cap \check{\mathfrak{a}}_P$:

$$\mathrm{V}(\mathfrak{s}, a) = \{X(\mathfrak{s}) + (1-x)\lambda | a < x < 1, \lambda \in U\}.$$

These cones fulfill the condition $\mathrm{V}(\mathfrak{s}, a_1) \subseteq \mathrm{V}(\mathfrak{s}, a_2)$ if $a_1 > a_2$, which is explained in the Lemma 7.1 of [L1].

Let also that $\mathrm{C}(\mathfrak{s}, \epsilon, a)$ denote a cylinder

$$\mathrm{C}(\mathfrak{s}, \epsilon, a) = \{X(\mathfrak{s}) + \lambda |\ \lambda \in \tilde{\mathfrak{s}}, \quad \| \Re\lambda \| < \epsilon, \quad \| \Im\big(X(\mathfrak{s}) + \lambda\big) \| < a\}.$$

Then the singularities of relevant Eisenstein systems carried by affine hyperspaces are of three kinds:

Type A. Let $\mathfrak{s} \in S_m$. A hyperplanes of type A satisfies the property that each singular hyperplane of the associated Eisenstein system which meets the closure of $\mathrm{C}(\mathfrak{s}, \epsilon, a)$ meets the closure of the unitary axis $\mathrm{U}(\mathfrak{s}, X(\mathfrak{s}), a)$ too, but no singular hyperplane meets the closure of $\mathrm{U}(\mathfrak{s}, Z, a)$ if $Z \in \mathrm{V}(\mathfrak{s}, a)$, and so that the closure of $\mathrm{V}(\mathfrak{s}, a)$ lies in $\mathrm{Str}_P(R)$.[4]

Type B. Let $\mathfrak{t} \in T_{m-1}$. Let $\check{\mathfrak{a}}_0 \subset \tilde{\mathfrak{t}}$ be the largest distinguished subspace, R

[4]There is an implicit restriction on the values of a. More precisely, let us write $X(\mathfrak{s}) = \Re X(\mathfrak{s}) + \mathrm{i}\,\Im X(\mathfrak{s})$. then $\Re\langle X(\mathfrak{s}), X(\mathfrak{s})\rangle = \| \Re X(\mathfrak{s}) \|^2 - \| \Im X(\mathfrak{s}) \|^2$. Then in the lemmas 7.5 and 7.6 it was assumed that $\Re\langle X(\mathfrak{s}), X(\mathfrak{s})\rangle > R^2 - a^2$. Since $\| \Re X(\mathfrak{s}) \|^2 < R^2$ this implies that $\| \Im X(\mathfrak{s}) \|^2 < a^2$.

and $\tilde{R}$ be parabolic subgroups such that $\check{\mathfrak{a}}_R \subset \check{\mathfrak{a}}_P$ and $\check{\mathfrak{a}}_{\tilde{R}} \subset \check{\mathfrak{a}}_R$, and finally let $F \in \operatorname{Hom}\big(S\big((\check{\mathfrak{t}}_R^{\tilde{R}})_{\mathbb{C}}\big), A^2(R,\chi,\Gamma)\big)$. Then the hyperplane $\mathfrak{t}$ is of type B if the inverse image in $\mathfrak{t}$ of a singular hyperplane $\mathfrak{r}$ of the function $E(.,F,.)$ which is defined on $\check{\mathfrak{a}}_R^{\tilde{R}}$ and which meets the unitary axis $\mathrm{U}(\check{\mathfrak{a}}_R^{\tilde{R}}, X(\mathfrak{t}), \infty)$ then $\check{\mathfrak{a}}_0 \subset \tilde{\mathfrak{r}}$.
Type C. Let $\mathfrak{t} \in T_{m-1}$ and $f \in A^2(P,\chi,\Gamma)$. Then $\mathfrak{t}$ is of type C if no singular hyperplane of $E(.,f,.)$ meets the closure of $\mathrm{U}(\mathfrak{t},Z,a)$ if $Z \in \mathrm{V}(\mathfrak{t},a)$ and such that the set $\{\Re Z | Z \in \mathrm{V}(\mathfrak{t},a)\}$ contained in the interior of the convex hull of $\check{\mathfrak{a}}_P^+$ and ${}^+\check{\mathfrak{a}}(\mathfrak{t})$.

The proof of the theorem 7.1 is incomplete unless one shows that how we can have to shift the contour of the integration to $X(\mathfrak{t}_0)$ (as it explained in **(1.12)**) and collect the residues. This is done via a variant of the Cauchy theorem, stated in lemma 7.1. We state a variant of it due to Langlands (given in page 181 of loc.cit. above) which is more suitable for the root spaces. Suppose we have an Eisenstein system $\{E(.,.,.)\}$ belonging to $\mathfrak{s}$ and let $\mathfrak{t}_1, ...\mathfrak{t}_l$ be its singular hyperplanes.

Let us fix two cones $\mathrm{V}_i(\mathfrak{s},a)$, $i = 1,2$, such that no singular hyperplane meets the closure of $\mathrm{U}(\mathfrak{s},W,a)$ if $W \in \mathrm{V}_i(\mathfrak{s},a)$ and $\mathrm{V}_i(\mathfrak{s},a_1) \subseteq \mathrm{V}_i(\mathfrak{s},a_2)$ if $a_1 \geq a_2$. Fix also a cone $\mathrm{C}(.,.,.)$ like above and suppose that for all a each singular hyperplane of the Eisenstein system which meets the closure of it meets the closure of the unitary axis $\mathrm{U}(\mathfrak{s},X(\mathfrak{s}),a)$. There is also a subset T of the set of singular hyperplanes such that $\Re X(\mathfrak{s}) = X(\mathfrak{t})$ and two nonempty convex cones $\mathrm{W}_i(\mathfrak{s},a_1) \subseteq \mathrm{V}_i(\mathfrak{s},a_2)$ with vertex $X(\mathfrak{t})$ for all $\mathfrak{t} \in T$. Choose and an arbitrary point $\mathrm{W}(\mathfrak{t}) \in \mathrm{V}(\mathfrak{t},a)$. Then there exists an open subset U' of some real subspace of dimension $m' = \dim \tilde{\mathfrak{s}} - \dim \check{\mathfrak{a}}_0$ of the space $\check{\mathfrak{a}}_P^0$ which is either $\mathfrak{s}$ itself or a singular hyperplane $\mathfrak{t}$ of the Eisenstein system such that $\tilde{\mathfrak{t}}$ contains $\check{\mathfrak{a}}_0$, and is contained in $\{\lambda \in \check{\mathfrak{a}}_P^0 | \parallel \Im\lambda \parallel > a\}$. Now, if one (and hence each, according to the corollary of the lemma 7.6) of the (shifted) singular hyperplanes $\tilde{\mathfrak{t}}_k$, $k = 1,...,l$, does not contain $\check{\mathfrak{a}}_P^0$ then

$$\frac{1}{(2\pi\,\mathrm{i})^m}\int_{\mathrm{U}(\check{\mathfrak{a}}_P^0,W_1,a)}\phi\mathrm{d}\lambda - \frac{1}{(2\pi\,\mathrm{i})^m}\int_{\mathrm{U}(\check{\mathfrak{a}}_P^0,W_2,a)}\phi\mathrm{d}\lambda =$$

$$\frac{1}{(2\pi\,\mathrm{i})^{m'}}\int_{U'}\phi\mathrm{d}\lambda,$$

and if one of (and consequently all of) them contains $\check{\mathfrak{a}}_P^0$ then

$$\frac{1}{(2\pi\,\mathrm{i})^m}\int_{\mathrm{U}(\check{\mathfrak{a}}_P^0,W_1,a)}\phi\mathrm{d}\lambda - \frac{1}{(2\pi\,\mathrm{i})^m}\int_{\mathrm{U}(\check{\mathfrak{a}}_P^0,W_2,a)}\phi\mathrm{d}\lambda -$$

$$\sum_{k=1}^{l}\frac{1}{(2\pi\,\mathrm{i})^{m-1}}\int_{\mathrm{U}(\check{\mathfrak{a}}_P^0,W_k,a)}\operatorname{Res}_{\mathfrak{t}_k}\phi\mathrm{d}\lambda =$$

$$\frac{1}{(2\pi\,\mathrm{i})^{m'}}\int_{U'}\phi\mathrm{d}\lambda,$$

in which ϕ stands for either $E(g, \Phi, \lambda)$ or $(N(\omega, \lambda)\mathrm{d}\Phi(\lambda), \mathrm{d}\Psi(-\omega\overline{\lambda}))$. The points W_i are belonging to $\mathrm{W}_i(\mathfrak{s}, a)$ and W_k to $\mathrm{V}(\mathfrak{t}_k, a)$.

This finishes our description of the main constructions given in the chapter 7 of loc. cit. above. The proof of these statements are very involved and a detailed account will be found in [MW] or [OW].

(1.20) We wont need the full force of the above construction, but rather a black-box view given in [F1] pages 38-42 will serve us as well. We summarize the main points here and show where they are related to our previous discussion.

First of all, the lemma 7.4 mentioned in **(1.14)** gives the functional equations of the general Eisenstein series and the intertwining operators. To explain it let $t \in \Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)$ like above. For this t we choose a representative in $G(\mathbb{Q})$ which we denote by ω. For λ lying in the domain of holomorphy $\mathfrak{A}_P$ defined above we define the intertwining operator $M(\omega, \lambda)$ by an absolutely convergent integral

$$\big(M(t,\lambda)\phi\big)(g) = \int_{N_Q(\mathbb{A})\cap\omega N_P(\mathbb{A})\omega^{-1}\backslash N_Q(\mathbb{A})} \phi(\omega^{-1}ng)e^{\langle\lambda+\rho_P, H_P(\omega^{-1}ng)\rangle}e^{-\langle\omega\lambda+\rho_Q, H_Q(g)\rangle}\mathrm{d}n,$$

with values in the space of linear operators from $A^2(P, \chi, \sigma)$ to $A^2(Q, \chi, \sigma)$ which admits an analytic continuation to a meromorphic function of $\lambda \in (\check{\mathfrak{a}}_P)_{\mathbb{C}}$.

Let R be another parabolic subgroup associated to Q. Then the lemma 7.4, discussed in **(1.14)**, guaranties that for $\lambda \in (\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$ and $\sigma \in G(\mathbb{Q})$ a representative of a transformation $w \in \Omega(\check{\mathfrak{a}}_Q, \check{\mathfrak{a}}_R)$ we have the functional equations:

Fun 1 $\quad E(\omega\lambda, M(\omega, \lambda)\phi) = E(\lambda, \phi).$

Fun 2 $\quad M(\omega\sigma, \lambda) = M(\omega, \sigma\lambda)M(\sigma, \lambda)$

for $\omega, \sigma \in G(\mathbb{Q})$. More generally, we decompose $\omega = \omega_{\alpha_n}...\omega_{\alpha_1}$ such that if $\omega_i = \omega_{\alpha_i}...\omega_{\alpha_1}$ then $M_i = \omega_i M \omega_i^{-1}$ is the standard Levi of a parabolic subgroup P_i with unipotent radical N_i, and α_{i+1} is a simple root of $\Phi(\mathfrak{n}_i)^+$ such that $\omega_i^{-1}(\alpha_{i+1}) > 0$. Then we can decompose the above integral over $N_n(\mathbb{A}) \cap \omega N_P(\mathbb{A})\omega^{-1} \setminus N_n(\mathbb{A})$ to n integrals over

$$\big(N_n(\mathbb{A}) \cap \omega\omega_{i-1}^{-1}N_{i-1}(\mathbb{A})\omega_{i-1}\omega^{-1}\big) \setminus \big(N_n(\mathbb{A}) \cap \omega\omega_i^{-1}N_i(\mathbb{A})\omega_i\omega^{-1}\big)$$

to obtain the general decomposition

$$M(\omega, \lambda)\phi = M(\omega_{\alpha_n}, \omega_{n-1}\lambda)M(\omega_{\alpha_{n-1}}, \omega_{n-2}\lambda)\cdots M(\omega_{\alpha_1}, \lambda)\phi.$$

$\diamond$

We can also attach a generalized symmetry s_α to each simple root in the sense of page 13 of [MW], and obtain that

Fun 3 $M(s_\alpha, \lambda)$ depends only on $\langle \check{\alpha}, \lambda \rangle$.

$\diamond$

For a general partial Eisenstein series (analog of the cuspidal ones were defined in **(1.6)**), the part **(iii)** of the definition of Eisenstein system (see **(1.11)**) implies that if $P \subseteq R$ and Q associated to P then

Fun 4 $\big(E_Q^R(\phi, \lambda)\big)_P(g) = \sum_{\substack{\omega \in \Omega(\check{\mathfrak{a}}_Q, \check{\mathfrak{a}}_P) \\ \check{\omega}|\hat{\mathfrak{a}}_R = \mathrm{Id}}} e^{\langle \omega\lambda + \rho_P, H_P(g) \rangle} \big(N(\omega, \lambda)\phi\big)(H_p(g))(g).$

The theorem 7.2 (discussed in **(1.13)**) implies that if there is no affine function $\omega : \check{\mathfrak{a}}_Q \to \check{\mathfrak{a}}_P$ such that $\omega^{-1}(\alpha) > 0 \; \forall \alpha \in \Phi^+(\mathfrak{n}_P)$ and $\omega M_Q(\mathbb{A})\omega^{-1}$ is not a standard Levi of $M_P(\mathbb{A})$ then the right hand side of the above equation is zero.

$\diamond$

In the case of cuspidal Eisenstein series the $N(.,.)$ and $M(.,.)$ operators coincide, in the general case they are different but related by the functional equation

Fun 5 $N(\omega\sigma, \lambda) = N(\omega, \sigma\lambda)M(\sigma, \lambda)$

for σ and λ like above. To see this we have to combine the computation (1.16) given in **(1.14)** and the process of taking the constant term of a general Eisenstein series $E_Q^R(\phi, \lambda)$ along a parabolic subgroup $P \subseteq R$, which is be the right hand side of the **Fun 4**. On the other hand we know from **(1.14)** that if $\sigma \in \Omega(\check{\mathfrak{a}}_{\tilde{Q}}, \check{\mathfrak{a}}_Q)$ for a parabolic subgroup $\tilde{Q} \subseteq R$ associated to P then $E_{\tilde{Q}}^R(M(\sigma, \lambda)\phi, \sigma\lambda) = E_Q^R(\phi, \lambda)$, which gives

$$\big(E_{\tilde{Q}}^R(M(\sigma, \lambda)\phi, \sigma\lambda)\big)_P(g) = \sum_{\substack{\omega \in \Omega(\check{\mathfrak{a}}_{\tilde{Q}}, \check{\mathfrak{a}}_P) \\ \check{\omega}|\hat{\mathfrak{a}}_R = \mathrm{Id}}} e^{\langle \omega\sigma\lambda + \rho_P, H_P(g) \rangle} \big(N(\omega, \sigma\lambda)M(\sigma, \lambda)\phi\big)(H_p(g))(g).$$

The left hand side of the **Fun 4** and this equation are meromorphic functions which by the meromorphic continuation of Eisenstein series are coincide on the convex hull of the disjoint union of the cones of absolute convergence of their corresponding Eisenstein series, and consequently the right hand side of them are identical which gives the functional equation **Fun 5**.

$\diamond$

Let $\omega \in \Omega(\check{\mathfrak{a}}_R, \check{\mathfrak{a}}_P)$, $R \supseteq P$, and let Q be the smallest parabolic subgroup containing P subject to the condition that $\check{\mathfrak{a}}_Q$ is contained in $\hat{\omega}\check{\mathfrak{a}}_R$. We have shown above that the affine subspace $\mathfrak{s} = -\omega\check{\mathfrak{a}}_R$ is an affine subspace along which an Eisenstein system is defined and $X(\mathfrak{s}) = -\omega 0$ is the point of minimum norm in that subspace. One of the main features of the subspaces carrying Eisenstein systems is the fact proven in the second to the last page of [L1] (page 230) that

Fun 6 $\omega 0 \in -{}^{+}\check{\mathfrak{a}}_P^Q$.

This fact was part of the assumptions of the lemma 7.2 mentioned in **(1.13)** and is proved only after the main structure of the induction in the proof of the theorem 7.1 has been settled down.

$\diamond$

This finishes our review of the main features of the Eisenstein systems which we need in the next section.

Chapter 2

Uniform Boundedness of the Pole Order

(2.1) In this section we prove the remark (3) in [F2], which says simply that the order of the poles of an Eisenstein series defined on a reductive group G is bounded by a constant which depends on the group G. Suppose that h is a meromorphic function on $\check{\mathfrak{a}}_P^G$ with a singularity along a hyperplane $H \subset \check{\mathfrak{a}}_P^G$, $\mathfrak{t}_0$ a unit vector normal to H and $z \in H$ a generic point. We will denote by $\big(\mathrm{Res}_{H,k}(h)\big)(z)$ the $-k$-th coefficient in the Laurent series $h(z+\tau\mathfrak{t}_0) = \sum_{k\gg\infty}^{\infty} a_k \tau^k$. Let us fix parabolic subgroups P and Q such that the set $\Omega(P,\chi,\psi,Q) \neq \varnothing$ and recall the definition of λ_+ given in **(1.1)**.

Our goal is to prove the following

Theorem 1 (Remark 3 in [F2])**.** *Let H be a singular hyperplane of the Eisenstein series*

$$E_P^G(\lambda, f)(g) = \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} e^{\langle \lambda+\rho_P, H_P(\gamma g)\rangle} f(\gamma g)$$

which meets

$$\check{\mathfrak{a}}_G + \mathrm{i}\,\check{\mathfrak{a}}_P + \overline{\check{\mathfrak{a}}_P^{G+}}.$$

For an affine function $\sigma : H \to (\check{\mathfrak{a}}_Q^G)_{\mathbb{C}}$, let $N_i(\sigma, .)$ be the function on $\check{\mathfrak{a}}_Q^G$ with values in the space of linear transformations from (1.3) *to*

$$S\big((\check{\mathfrak{a}}_Q^G)_{\mathbb{C}}\big) \otimes A^2_{cusp}(Q, \boldsymbol{\chi}, \Gamma),$$

defined by

$$\Big(N_i(\sigma,\lambda)f\Big)(x) = e^{-\langle\sigma(\lambda),x\rangle} \sum_{\omega|H=\sigma} \Big(\mathrm{Res}_{H,i}\, e^{\langle\omega(\eta),x\rangle} N(\omega,\eta) f\Big)(\lambda), \qquad (2.1)$$

for a parameter η which lies in a small convex neighborhood of x in $(\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$. Let $\Omega_i(H,P,\chi,\psi,Q)$ be the set of linear transformations σ such that $N_i(\sigma, .)$

does not vanish identically and let $\Omega(H,P,\chi,\psi,Q)$ be the union of these sets over i. Let $k>0$ and let $\omega\in\Omega(H,P,\chi,\psi,Q)$ be such that $N_k(\omega,\lambda)\not\equiv 0$ and such that if $\tilde\omega\in\Omega(H,P,\chi,\psi,Q)$ and $N_k(\tilde\omega,\lambda)\not\equiv 0$ then [1]

$$\left|\big(\omega(x)\big)_+\right| \geq \left|\big(\tilde\omega(x)\big)_+\right|$$

for all x in some open subset of $H\cap\check{\mathfrak{a}}_P^{G+}$. Then $N_j(\omega,\lambda)\equiv 0$ for $j>k$.

(2.2) We explain how this theorem yields the uniform boundedness of the pole order of general Eisenstein series. Let H be a singular hyperplane of the Eisenstein series (1.3) like above. Recall that by definition **(ii)** of Eisenstein systems given in **(1.11)** we have

$$\int_{N_Q(\mathbb{Q})\backslash N_Q(\mathbb{A})} \mathrm{Res}_{H,i}\, E(nmk,F,\lambda)\mathrm{d}n =$$

$$\begin{cases} \text{is orthogonal to the space of cusp forms,} & \text{if } Q\notin\{\mathrm{P}\} \\ \sum_{\omega\in\Omega_i(H,P,\chi,\psi,Q))}\left(e^{\langle\rho+\omega\lambda,H_Q(m)\rangle}\right)\big(\mathrm{Res}_{H,i}\,N(\omega,\lambda)F\big)(mk), & \text{if } Q\in\{\mathrm{P}\}\end{cases}$$

Then the lemma 3.7 of [L1] implies that the residue of the residual Eisenstein series should vanish if its constant term vanishes for all $Q\in\{\mathrm{P}\}$.

Using this fact, we reformulate the theorem as follows. Write the i-th term of the principal part of the Laurent expansion of the constant term of the Eisenstein series as

$$(\mathrm{Res}_{H,i}\,E_P^G f)_Q(\lambda) = \sum_{\sigma\in\Omega_i(H,P,\chi,\psi,Q)} \big((N_i(\sigma,\lambda)f)(H_Q(g))\big)(g),$$

and suppose that the other assumptions of the theorem 1 are fulfilled. Our claim is that the leading summand in the constant term of the i-th residue of the above Eisenstein series with parameters lying in the positive Weyl chamber may not contribute to the constant terms of the residues of order $> i$. This implies that the order of a singular hyperplane intersecting the positive Weyl chamber is bounded by a maximum over $Q\in\{\mathrm{P}\}$ and the number of elements of $\Omega(H,P,\chi,\psi,Q)$ which in turn is bounded by a constant depending only on G. To see this, observe first, as mentioned above, that for a function $f\in$(1.1) there are only finitely many i such that $(\mathrm{Res}_{H,i}\,E_P^G f)_Q(\lambda)\not\equiv 0$ and also there are finitely many $\omega\in\Omega(H,P,\chi,\psi,Q))$ such that $N_i(\omega,\lambda)f\not\equiv 0$ (lemma 7.2 of [L1]). Choose a neighborhood V like in the proof of the main theorem of [F2], page 314. There is a constant T, such that for the character χ fixed in **(1.1)** we have $\mid\chi\mid^2+\mid\lambda\mid^2< T$ for all $\lambda\in V$. There are only finitely many characters χ such that the space (1.1) does not vanish, which implies that there are only finitely many such a

[1]Which is actually the condition $\omega_0\in\Omega_x^{\mathrm{lead}}$ in [F2] which implies that $H=\{x\in(\check{\mathfrak{a}}_P^G)_{\mathbb{C}}\mid\,\omega_0 x\in\sigma(\check{\mathfrak{a}}_R)_{\mathbb{C}}\}$.

neighborhoods V. Between these neighborhoods there is at least one neighborhood such that for all $x \in H \cap \check{\mathfrak{a}}_P^{G+}$ which lie in that neighborhood there is an ω such that the inequality $\left|\big(\omega(x)\big)_+\right| \geq \left|\big(\tilde{\omega}(x)\big)_+\right|$ is valid for all other $\tilde{\omega} \in \Omega(H,P,\chi,\psi,Q)$ such that $N_k(\tilde{\omega},\lambda) \not\equiv 0$. If there is no such a ω then we will have $\omega \in \Omega_0(H,P,\chi,\psi,Q))$ which implies that $N(\omega,\lambda)$ is independent of λ and consequently $N_i(\omega,\lambda) \equiv 0$ for all i which is a contradiction. If all these assumptions were fulfilled then the theorem 1 says that we will have $N_j(\omega,\lambda) \equiv 0$ for all $j > i$, which means that the leading summand in the constant term of the $i-$th residue of an Eisenstein series in the positive Weyl chamber does not contribute in the constant terms of the residues of order greater than i. This means that this ω wont contribute to the terms of order higher than i in the Laurent expansion of the constant term of $E_P^G(f,\lambda)$ in the direction of Q. Then if we move to the next term in the principal part of the Laurent expansion we see that either for some $\omega' \in \Omega(H,P,\chi,\psi,Q)\setminus\{\omega\}$ the next term $N_{i+1}(\omega',\lambda)$ of the Laurent expansion does not vanish identically and the inequality $\left|\big(\omega'(x)\big)_+\right| \geq \left|\big(\tilde{\omega}(x)\big)_+\right|$ and $N_{i+1}(\tilde{\omega},\lambda) \not\equiv 0$ are valid, or it is identically zero and this procedure could be repeated until all the members of the $\Omega(H,P,\chi,\psi,Q)$ are exhausted. Say it simply, to each non-zero term of the principal part of the Laurent expansion (which is an analytic object) we have associated an element of the Weyl group (which is an algebraic object), which wont appear in the higher non-zero terms of the principal part and since the order of the Weyl group in finite the pole order of the Laurent series is universally bounded.

Consequently the pole order depends on the order of the polynomials $N(\omega,\lambda)$, the length of the gaps between the non-zero terms of the Laurent expansion and the number of elements in $\Omega(P,\chi,\psi,Q)$. If we could show that these polynomials are actually monomials (which is the content of lemma 1 in **(2.3)**), and the length of the gaps in the principal part of the Laurent expansion is at most 1(which is the content of proposition 4 at **(2.6)**), we can state our main theorem:

Theorem 2. *For P,Q and k as in the theorem 1 we have*

$$k \leq \max_{Q\in\{\mathrm{P}\}} \#\Omega(P,\chi,\psi,Q).$$

Example. Consider the only known computed example of the phenomenon of non-simple singularities, G_2. The set of positive roots with respect to a maximal torus $T \subset \mathrm{SU}(3)$ will be denoted by $\Phi_+ = \{\alpha,\beta,\alpha+\beta,2\alpha+\beta,3\alpha+\beta,3\alpha+2\beta\}$ and the subset of simple roots by $\Delta = \{\alpha,\beta\}$, in which α is the short and β is the long root.

The Weyl group of G_2 is

$$W_G = \{1,\rho_\alpha,\rho_\beta,\rho_{\alpha+\beta},\rho_{3\alpha+2\beta},\rho_{2\alpha+\beta},\rho_{3\alpha+\beta},\sigma(\frac{\pi}{3}),\sigma(\frac{2\pi}{3}),\sigma(\pi),\sigma(\frac{4\pi}{3}),\sigma(\frac{5\pi}{3})\},$$

in which the $\rho_.$ denotes the reflection with respect to the roots and $\sigma(.)$ the rotation with the indicated angle. G_2 contains two maximal parabolic subgroups P and Q with Levi components $M_P \cong M_Q \cong \mathrm{GL}(2,\mathbb{R})$, associated to the parabolic subalgebras

$$\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+2\beta} \oplus \mathfrak{g}_{3\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{-\alpha},$$

with split component $(\check{\mathfrak{a}}_P)_{\mathbb{C}} = X(M_P) \otimes \mathbb{C} = \mathbb{C}(3\alpha + 2\beta)$, and

$$\mathfrak{q} = \mathfrak{h} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+2\beta} \oplus \mathfrak{g}_{3\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\beta},$$

with split component $(\check{\mathfrak{a}}_Q)_{\mathbb{C}} = X(M_Q) \otimes \mathbb{C} = \mathbb{C}(2\alpha + \beta)$; and we have a Borel subgroup B^+ associated to the Borel subalgebra

$$\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+2\beta} \oplus \mathfrak{g}_{3\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta} \oplus \mathfrak{g}_\alpha,$$

with split component $(\check{\mathfrak{a}}_{B^+})_{\mathbb{C}} = X(M_{B^+}) \otimes \mathbb{C} = \mathbb{C}(2\alpha + \beta) + \mathbb{C}(\alpha + \beta)$. The subalgebra $\mathfrak{h}$ is isomorphic to $\mathfrak{gl}_2$.

To compute the set $\Omega(.,.)$ for each of these parabolic subgroups we have to compute the constant term of the relevant Eisenstein series. In what follows we will write B and G instead of B^+ and G_2. We start with P. Let $\phi \in A^2(P, \chi, \Gamma)$, $\lambda \in (\check{\mathfrak{a}}_P)_{\mathbb{C}}$, $\rho_P = \frac{9}{2}\alpha + 3\beta$, and set

$$E_P^G(\lambda, \phi, g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\langle \lambda + \rho_P, H_P(\gamma g) \rangle} \phi(\gamma g),$$

with the constant term

$$\begin{aligned}\big(E_P^G(\lambda, \phi)\big)_P(g) &= e^{\langle \lambda + \rho_P, H_P(g) \rangle} \big((N(1, \lambda)\phi)(H_P(g))\big)(g) \\ &\quad + e^{\langle \rho_{3\alpha+2\beta}\lambda + \rho_P, H_P(g) \rangle} \big((N(\rho_\alpha, \lambda)\phi)(H_P(g))\big)(g),\end{aligned}$$

in the direction of P, which implies that $\Omega(P, \chi, \boldsymbol{\psi}, P) = \{1, \rho_{3\alpha+2\beta}\}$. Doing the same computations in the direction of Q gives that $\big(E_P^G(\lambda, \phi)\big)_Q(g)$ is orthogonal to the space of cusp forms and $\Omega(P, \chi, \boldsymbol{\psi}, Q) = \varnothing$. The same calculation for the group Q, in which we have $\rho_Q = 5\alpha + \frac{5}{2}\beta$, shows that $\Omega(Q, \chi, \boldsymbol{\psi}, Q) = \{1, \rho_{2\alpha+\beta}\}$.

For the Borel subgroup B we have the Eisenstein series

$$E_B^G(\lambda, \phi, g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} e^{\langle \lambda + \rho_B, H_B(\gamma g) \rangle} \phi(\gamma g),$$

in which $\rho_B = 5\alpha + 3\beta$ and $\phi \in A^2(B, \chi, \Gamma)$, $\lambda \in (\check{\mathfrak{a}}_B)_{\mathbb{C}}$, which gives the constant terms

$$\big(E_B^G(\lambda, \phi)\big)_B(g) = \sum_{\omega \in W_G} e^{\langle \omega\lambda + \rho_B, H_B(\gamma g) \rangle} \big((N(\omega, \lambda)\phi)(H_B(g))\big)(g),$$

in the direction of B which gives us $\Omega(B, \chi, \boldsymbol{\psi}, B) = W_G$. We also see that $\Omega(B, \chi, \boldsymbol{\psi}, P)$ and $\Omega(B, \chi, \psi, Q)$ are empty. This implies that for the group

G we have $k \leq 12$, which is a bad estimate, since the computations of Langlands show that $k = 2$.

(2.3) To begin the proof of the theorems 1 and 2 we need the following lemma. This lemma is true if we restrict ourselves to the representations in the $A^2(P)$ spaces defined in **(1.4)** in chapter 1. The situation wont change a lot if we consider a more general setting of the space

$$S_\infty\big(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A})\big)$$

which is introduced in [F1] page 18, and which is a generalization of the $A^2(P)$ spaces. We state this lemma in this more general setting to stay in coherence with [F1] and [F2] which are the basic frames for this work. To define it we introduce for each $\lambda \in \check{\mathfrak{a}}_\circ^G$ a weight function ρ_λ on $G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A})/\mathbb{K}$ as follows. Let χ be a C^∞ function which is equal to zero on $(-\infty, D-1]$ and is equal to one on $[D, \infty)$. Define

$$\rho_\lambda(g) = \sum_{\gamma \in P_\circ(\mathbb{Q})\setminus G(\mathbb{Q})} e^{\langle \lambda, H_{P_\circ}(\gamma g)\rangle} \prod_{\alpha \in \Delta_{P_\circ}} \chi(\langle \alpha, H_{P_\circ}(\gamma g)\rangle).$$

Let us denote the space of $\mathbb{K}$-finite functions ϕ on $G(\mathbb{Q}) \setminus G(\mathbb{A})$ such that $\int_{G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A})} \rho_\lambda(g)^2 |(X\cdot\phi)(g)|^2 \mathrm{d}g < \infty$ for all $X \in \mathfrak{B}$ by $S_{\rho_\lambda}\big(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A})\big)$. Now we define $S_\infty(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A})) = \cup_{\rho_\lambda} S_{\rho_\lambda}(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A}))$. Then the theorem 4.2 of [L1] shows that the constant terms of the elements of this space are combination of polynomial and exponential functions and hence are of desired form for our main aim in the next lemma.

Lemma 1. *Fix a standard parabolic subgroup $Q \in \{\mathrm{P}\}$. Let the automorphic form $f(.)$ which lies in the space*

$$S_\infty(G(\mathbb{Q})A_G(\mathbb{R})^\circ \setminus G(\mathbb{A})) \tag{2.2}$$

be a simultaneous eigenfunction of $\mathcal{Z}(\mathfrak{g})$. Let us write the constant term of f along N_Q as a sum with but finitely many non-vanishing summands

$$f_{N_Q}(g) = \int_{N_Q(\mathbb{Q})\setminus N_Q(\mathbb{A})} f(ng)dn = \sum\nolimits_{\lambda \in (\check{\mathfrak{a}}_Q)_\mathbb{C}} e^{\langle \lambda + \rho_Q, H_Q(g)\rangle} f_{Q,\lambda}(H_Q(g))(g),$$

in which $g \in G(\mathbb{Q})A_G(\mathbb{R})^o \setminus G(\mathbb{A})$ and $f_{Q,\lambda}$ belongs to

$$S\big((\check{\mathfrak{a}}_Q^G)_\mathbb{C}\big) \otimes S_\infty\big(N_Q(\mathbb{A})A_Q(\mathbb{R})^\circ Q(\mathbb{Q}) \setminus G(\mathbb{A})\big), \tag{2.3}$$

and are products of polynomials on $\mathfrak{a}_Q$ with values in the space of cusp forms on $N_Q(\mathbb{A})Q(\mathbb{Q})A(\mathbb{R})^\circ \setminus G(\mathbb{A})$ and functions in

$$C^\infty_{umg}\big(N_Q(\mathbb{A})A_Q(\mathbb{R})^\circ Q(\mathbb{Q}) \setminus G(\mathbb{A})\big),$$

([F1] proposition 2.3.2) . Then these polynomials are actually sums of monomials which are products of $\xi \in (\check{\mathfrak{a}}_Q)_\mathbb{C}$ that are orthogonal to $\lambda \in (\check{\mathfrak{a}}_Q)_\mathbb{C}$ with respect to the dual of the Killing form of $\mathfrak{g}$.

We prove this lemma for the situation $\mathfrak{h} = \mathfrak{a}_Q$, the general situation is similar.

Proof. We fix a basis $\{H_1, ..., H_m\}$ of $\mathfrak{h}_\mathbb{C}$ which satisfies $\kappa(H_i, H_j) = \delta_{ij}$ such that $\{H_1, ..., H_l\}$ is a basis for $\mathfrak{h}_\mathbb{C} \cap (\mathfrak{a}_Q)_\mathbb{C}$ and $\{H_{1+1}, ..., H_m\}$ a basis of $\mathfrak{h}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C}$, and for each $\alpha \in \Phi^+(\mathfrak{n}_Q)$ choose a $X_\alpha \in (\mathfrak{g}_\mathbb{C})_\alpha$ and $X_{-\alpha} \in (\mathfrak{g}_\mathbb{C})_{-\alpha}$ such that $\kappa(X_\alpha, X_{-\alpha}) = 1$. Then following the procedure explained in **(1.2)** we write the Casimir operator as

$$\omega_\mathfrak{q} = H_1^2 + ... + H_m^2 + \sum_{\alpha \in \Phi^+(\mathfrak{n}_P)} (X_\alpha X_{-\alpha} + X_{-\alpha} X_\alpha) - \omega_\mathfrak{c}$$

$$= H_1^2 + ... + H_m^2 + 2 \sum_{\alpha \in \Phi^+(\mathfrak{n}_P)} X_\alpha X_{-\alpha} - \sum_{\alpha \in \Phi^+(\mathfrak{n}_P)} H_\alpha - \omega_\mathfrak{c}.$$

If we apply it to the function $F_\lambda(g) = e^{\langle \lambda + \rho_Q, H_Q(g) \rangle} f_{Q,\lambda}(H_Q(g))$ we obtain an equation

$$\sum_{i=1}^m \Big(2\langle \lambda, H_i \rangle H_i \cdot f_{Q,\lambda}(H) + H_i^2 \cdot f_{Q,\lambda}(H) \Big) = \{\langle \lambda, \lambda \rangle - \langle \rho, \rho \rangle\} f_{Q,\lambda}(H).$$

Our claim follows if we could show that the term

$$\sum_{i=1}^m 2\langle \lambda, H_i \rangle H_i \cdot f_{Q,\lambda}(H)$$

identically vanishes.

To show this we have to exploit the fact that $f(g)$ (and consequently $f_{N_Q}(g)$) is a simultaneous eigenfunction of not only Casimir operator but the whole of $\mathcal{Z}(\mathfrak{g})$. To do this we have to interpret the action of $\mathcal{Z}(\mathfrak{g})$ on (2.3) as the action of $S(\mathfrak{h})^{W_G}$ on it via the Harish-Chandra isomorphism

$$\xi_G : \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{h})^{W_G}$$

through the corresponding differential operators. In what follows to avoid confusion we will denote the operator $X\cdot$ defined above by D_X.

For each $X \in \mathfrak{h}$ there corresponds a differential operator D_X which acts on the space $C^\infty(\mathfrak{h})$ by $\mathrm{D}_X(f(Y)) = \frac{\mathrm{d}}{\mathrm{d}t} f(Y + tX)|_{t=0}$. Then the Killing form induces an identification between $\mathfrak{h}$ and a space of functions on $\mathfrak{h}$ because of its bilinearity. The mapping $X \to \mathrm{D}_X$ extends trivially to an algebraic isomorphism between $S(\mathfrak{h})$ and the algebra of differential operators with constant coefficients on $\mathfrak{h}$. In this way each $p \in S(\mathfrak{h})$ is mapped canonically to a differential operator D_p and for a polynomial q, $\mathrm{D}_p(q)$ is the value of the derivative of q with respect to p at zero. Then for the polynomials $p(X_1, ..., X_n) = \sum_{\nu=0}^N a_\nu X_1^{m_1^\nu} ... X_n^{m_n^\nu}$ $(X_i \in \mathfrak{h})$, and $q(X_1, ..., X_n) = X_1^{m_1^\nu} ... X_n^{m_n^\nu}$ we have $\mathrm{D}_q(p) = m_1^\nu! ... m_n^\nu! a_\nu$ which implies that if $\mathrm{D}_q(p) = 0$

for all $q \in S(\mathfrak{h})$ then $p = 0$. This implies that $\mathrm{D}_q(p) = \mathrm{D}_p(q)$, and hence $\mathrm{D}_1(q) \equiv 0$ for all polynomials $q \neq 1$ and $\mathrm{D}_1(1) = \mathrm{id}$. Moreover if $s \in W_G$ and if we denote its action on the polynomials in $S(\mathfrak{h})$ by $s.q(X) = q(s^{-1}X)$, then the Weyl-group invariance of the Killing form extends to this situation and implies that $s.\big(\mathrm{D}_q(p)\big) = \mathrm{D}_{s.q}(s.p) = \mathrm{D}_q(p)$ and consequently $\mathrm{D}_q(p)$ will be invariant under the W_G-action and hence it defines a symmetric bilinear form on $S(\mathfrak{h})$ whose restriction to $\mathfrak{h}$ will be the usual Killing form of $\mathfrak{g}$, i.e., for $X, Y \in \mathfrak{h}$, $\mathrm{D}_Y(X) = \kappa(Y, X)$. All these are taken from [H3].

For each λ appearing on the right hand side of (2.2) let W_G^λ denote the subgroup of W_G which leaves λ fixed. Let $\mathcal{I}_\lambda \subseteq S(\mathfrak{h})$ denote the subalgebra of differential operators with constant coefficients which are left fixed under the action of W_G^λ. Then $S(\mathfrak{h})^{W_G} \subseteq \mathcal{I}_\lambda \subseteq S(\mathfrak{h})$. This subalgebra is bigger than the image of the Harish-Chandra isomorphism but it consists of a direct sum of isomorphic images of $\mathcal{Z}(\mathfrak{g})$. To see this let $r = [W_G : W_G^\lambda]$. Then according to lemma 8 of [H4] there are homogeneous elements $\omega_1 = 1, \omega_2, ..., \omega_r$ in $\mathcal{I}_\lambda$ such that there is a direct sum

$$\mathcal{I}_\lambda = \bigoplus_{k=1}^{r} \big(S(\mathfrak{h})^{W_G}\big)\omega_k, \tag{+}$$

which is what we looked for. Now at first we deal with the situation that we have a single $F_\lambda(g) = e^{\langle \lambda + \rho_Q, H_Q(g)\rangle} f_{Q,\lambda}(H_Q(g))$, $\lambda \in (\mathfrak{a}_Q)_{\mathbb{C}}$, on the right hand side of the $f_{N_Q}(g)$ above which we supposed to be an eigenfunction of $S(\mathfrak{h})^{W_G}$. We will show in a moment for function of this form that being an eigenfunction of $S(\mathfrak{h})^{W_G}$ is equivalent to be an eigenfunction of $\mathcal{I}_\lambda$, consequently we suppose that $f_{N_Q}(g)$ is an eigenfunction of $\mathcal{I}_\lambda$. Let $q \in \mathcal{I}_\lambda$. Since $F_\lambda(g)$ is an eigenfunction, there is a polynomial[2] $\kappa(q, \lambda)$ such that

$$\mathrm{D}_q\big(e^{\langle \lambda, H_Q(g)\rangle} f_{Q,\lambda}(g)\big) = \kappa(q, \lambda)\big(e^{\langle \lambda, H_Q(g)\rangle} f_{Q,\lambda}(g)\big).$$

The operators D_q were defined only on the $S(\mathfrak{h})$. Hence to compute the left hand side we have to extend them as follows. Observe first that the map $q \to e^{-\langle \lambda, H_Q(g)\rangle}\mathrm{D}_q e^{\langle \lambda, H_Q(g)\rangle}$ is a homomorphism of $S(\mathfrak{h})$ into $\mathrm{D}S(\mathfrak{h})$, the image of $S(\mathfrak{h})$ under the operators D_q with $q(.)$ as above. It is clear from the Weyl group invariance of D_q that this homomorphism maps $\mathcal{I}_\lambda$ onto itself. Consequently there is an element $\tilde{q}(.) \in \mathcal{I}_\lambda$ such that $e^{-\langle \lambda, H_Q(g)\rangle}\mathrm{D}_q e^{\langle \lambda, H_Q(g)\rangle} = \mathrm{D}_{\tilde{q}}$. If we compute both sides at the point 0 on a basis of $S(\mathfrak{h})$ we see that $\tilde{q}(0) = q(\lambda)$, in other words $\mathrm{D}_{\tilde{q}(H)} = \mathrm{D}_{q(H+\lambda)}$ on each basis element H.

This justifies the following computation:

$$\mathrm{D}_q\big(e^{\langle \lambda, H_Q(g)\rangle} f_{Q,\lambda}(g)\big) = e^{\langle \lambda, H_Q(g)\rangle} e^{-\langle \lambda, H_Q(g)\rangle}\mathrm{D}_q\big(e^{\langle \lambda, H_Q(g)\rangle} f_{Q,\lambda}(g)\big) =$$

$$= e^{\langle \lambda, H_Q(g)\rangle}\big(\mathrm{D}_{q(H_Q(g)+\lambda)} f_{Q,\lambda}(g)\big).$$

[2] If $q(.) = X \in \mathfrak{A}(\mathfrak{h}) \otimes \mathcal{Z}(\mathfrak{m}_G)$ then the polynomial $\kappa(q, \lambda)$ is the same as $p_X(Z_i)$ of the lemma 4.2 of Langlands.

Now if we write $q(H_Q(g)+\lambda)$ as $q(H_Q(g)+\lambda)-q(\lambda)+q(\lambda)$, the above expression is equal to

$$e^{\langle\lambda,H_Q(g)\rangle}\big(q(\lambda)f_{Q,\lambda}(g)+\mathrm{D}_{q(\cdot+\lambda)-q(\lambda)}f_{Q,\lambda}(\cdot)\big).$$

Since the term $q(H_Q(g)+\lambda)-q(\lambda)$ has no constant term the total degree of $\mathrm{D}_{q(\cdot+\lambda)-q(\lambda)}f_{Q,\lambda}(\cdot)$ is absolutely less than the total degree of $f_{Q,\lambda}(g)$ which implies that $f_{Q,\lambda}(g)$ is an eigenfunction if and only if

$$\mathrm{D}_{q(\cdot+\lambda)-q(\lambda)}f_{Q,\lambda}(\cdot)=0, \tag{++}$$

for all $q\in\mathcal{I}_\lambda$. In particular for the orthonormal basis $\{H_1,...,H_l\}$ of $\mathfrak{a}_Q$ given above, we have $\lambda=\lambda_1H_1+...+\lambda_lH_l$, and interpret it as a linear polynomial in $S(\mathfrak{h})^{W_G}$, it belongs to $\mathcal{I}_\lambda$, so then we have $\mathrm{D}_\lambda\big(f_{Q,\lambda}(g)\big)=\sum_{i=1}^{l}\lambda_i\mathrm{D}_{H_i}\big(f_{Q,\lambda}(g)\big)=0$ which is what we were looking for. It is evident from the construction that if $W_G^\lambda=\{1\}$ then $f_{Q,\lambda}(g)\in\mathbb{C}$.

To finish the proof of the lemma we need to show that if $F_\lambda(g)=e^{\langle\lambda+\rho_Q,\cdot\rangle}f_{Q,\lambda}(\cdot)$ is an eigenfunction of $S(\mathfrak{h})^{W_G}$ then it is an eigenfunction of $\mathcal{I}_\lambda$ too. This result goes back originally to Harish-Chandra which we give a proof through the following propositions. We start by fixing some definitions. Let

$$\mathcal{E}^\lambda(\mathfrak{h})=\{f\in C^\infty(\mathfrak{h})\mid \mathrm{D}_q(f)=q(\lambda)f \text{ for all } q\in S(\mathfrak{h})^{W_G},\ \lambda\in\check{\mathfrak{h}}_{\mathbb{C}}\}.$$

We denote the space of **harmonic** polynomials on $\mathfrak{h}$ by

$$\mathcal{H}(\mathfrak{h})=\{f\in C^\infty(\mathfrak{h})\mid f \text{ is a polynomial such that } \mathrm{D}_q(f)=0 \text{ for all } q\in S(\mathfrak{h})^{W_G}\}.$$

Let also

$$\mathcal{H}_\lambda(\mathfrak{h})=\{f\in C^\infty(\mathfrak{h})\mid f \text{ is a polynomial such that } \mathrm{D}_q(f)=0 \text{ for all } q\in\mathcal{I}_\lambda\}.$$

We will call the functions in $\mathcal{H}_\lambda(\mathfrak{h})$ the W_G^λ-**harmonic** polynomials. We will denote by $\mathcal{I}_\lambda^+$ the subalgebra of the homogeneous elements of (+) of degree ≥ 1. Let $\mathcal{H}_{s\cdot\lambda}(\mathfrak{h})$ and $\mathcal{I}_{s\cdot\lambda}$ denote the analogs of $\mathcal{H}_\lambda(\mathfrak{h})$ and $\mathcal{I}_\lambda$ for the vectors $s\cdot\lambda$ when s varies over a complete set $\mathfrak{s}=\{s_0=1,s_1,...s_{r-1}\}$ of representatives of the cosets of W_G^λ in W_G.

We will need the following two propositions which are theorem III.3.11 and lemma III.3.13 of [Hel].

Proposition 1. *Any homomorphism $\chi_\lambda:S(\mathfrak{h})^{W_G}\to\mathbb{C}$ is given by the evaluation map $S(\mathfrak{h})^{W_G}\ni q(.)\mapsto q(\lambda)$ for some $\lambda\in\check{\mathfrak{h}}_{\mathbb{C}}$. Two such homomorphisms χ_μ and χ_λ coincide if and only if $\mu\in W_G\cdot\lambda$.*

Proof. This is actually the famous theorem of Harish-Chandra on the characters of the $S(\mathfrak{h})^{W_G}$ which could be found in [Hu] page 129. □

Proposition 2. *Let the set of polynomials* $\{p_t(.) \mid t \in W_G^{s\cdot\lambda}\}$ *constitute a basis of* $\mathcal{H}_{s\cdot\lambda}(\mathfrak{h})$. *Then the set of functions* $\{p_t(.)e^{\langle s\cdot\lambda,.\rangle} \mid s \in \mathfrak{s},\ t \in W_G^{s\cdot\lambda}\}$ *is a basis for* $\mathcal{E}^\lambda(\mathfrak{h})$. *More over* $\dim(\mathcal{E}^\lambda(\mathfrak{h})) \leq \#W_G$.

In particular, if the element λ is regular the polynomials $\{p_t(.) \mid t \in W_G^{s\cdot\lambda}\}$ are all constants.

Proof. First we prove the second statement. Let $l = \dim(\mathfrak{h})$ and let the elements $j_1, ..., j_l$ are linearly independent elements of $S(\mathfrak{h})^{W_G}$. The map $\mathfrak{h}_\mathbb{C} \ni \theta \to (j_1(\theta), ..., j_l(\theta)) \in \mathbb{C}^l$ is a bijection from $\mathfrak{h}_\mathbb{C}/W_G \to \mathbb{C}^l$. In fact the proposition 1 shows that the map $S(\mathfrak{h})^{W_G} \to \mathbb{C}^l$ given by $(j_1, ..., j_l) \to (\xi_1, ..., \xi_l) \in \mathbb{C}^l$, $j_i \mapsto j_i(\theta) = \xi_i, \theta \in \mathfrak{h}_\mathbb{C}$, gives this bijection via the evaluation at a point $\theta \in \mathfrak{h}_\mathbb{C}$. Let $\{p^t(.) \mid t \in W_G\}$ be a basis of $\mathcal{H}(\mathfrak{h})$. To these polynomials we can correspond a set of polynomials $\{P^t(.) \mid t \in W_G\}$, $P^t(.) \in S(\mathfrak{h})^{W_G}$. For $f \in \mathcal{E}^\lambda(\mathfrak{h})$ we put $c_t(f) = (\mathrm{D}_{P_t}(f))(\theta)$ for $t \in W_G, \theta \in \mathfrak{h}$. Since $S(\mathfrak{h}) = S(\mathfrak{h})^{W_G}\mathcal{H}(\mathfrak{h})$ (theorem III.1.2 of loc.cit.) we see that if $c_t(f) = 0$ for all $t \in W_G$ then $f \equiv 0$. Consequently the mapping $f \mapsto (c_t(f))_{t\in W_G}$ is a one to one into mapping $\mathcal{E}^\lambda(\mathfrak{h}) \to \mathbb{C}^{\#W_G}$. This implies that $\dim(\mathcal{E}^\lambda(\mathfrak{h})) \leq \#W_G$. No we prove the first statement. Since the polynomials $p_t(.)$ are harmonic the equation (++) shows that we have $\mathrm{D}_q(p_t(.)) = 0$ for all $q(.) \in S(\mathfrak{h})^{W_G}$. This implies that the functions $\{p_t(.)e^{\langle s\cdot\lambda,.\rangle} \mid s \in \mathfrak{s},\ t \in W_G^{s\cdot\lambda}\}$ are belong to $\mathcal{E}^\lambda(\mathfrak{h})$. From the theorem III.3.4 of loc.cit we know that $\dim(\mathcal{H}_{s\cdot\lambda}(\mathfrak{h})) = \#W_G^{s\cdot\lambda}$. Since the functions $\{e^{\langle s\cdot\lambda,H\rangle} \mid s \in \mathfrak{s}\}$ are linearly independent over $S(\mathfrak{h})$ and since $W_G = \cup_{s\in\mathfrak{s}} W_G^{s\cdot\lambda}$, the functions $\{p_t(.)e^{\langle s\cdot\lambda,.\rangle} \mid s \in \mathfrak{s},\ t \in W_G^{s\cdot\lambda}\}$ constitute a basis of $\mathcal{E}^\lambda(\mathfrak{h})$.

□

We can now prove our claim about the equivalence of the eigenfunctions of $S(\mathfrak{h})^{W_G}$ and $\mathcal{I}_\lambda$. For $s \in \mathfrak{s}$ let

$$\mathcal{E}_{s\cdot\lambda}(\mathfrak{h}) = \{p(H)e^{\langle s\cdot\lambda,H\rangle} \mid H \in \mathfrak{h},\ p(.) \in S(\mathfrak{h}) \text{ such that } \mathrm{D}_q(p) = 0 \text{ for all } q \in \mathcal{I}^+_{s\cdot\lambda}\}.$$

Since we are concerned with the function $F_\lambda(g)$ we will fix λ in what follows and denote by $\mathcal{E}(\mathfrak{h})$ the subspace of $\mathcal{E}^\lambda(\mathfrak{h})$ for this fixed λ. Observe first that since the algebra $\mathcal{I}_\lambda$ is invariant under the translation $H \longmapsto H+\lambda$, the equation (++) implies that $\mathrm{D}_{q(.)} f_{Q,\lambda}(\cdot) = 0$ for all homogeneous polynomials $q \in \mathcal{I}^+_\lambda$ of degree greater than or equal 1. consequently if $F_\lambda(g)$ is an eigenfunction of $\mathcal{I}_\lambda$ it has to be an eigenfunction of $\mathcal{I}^+_\lambda$ and vice versa. From the lemma 3.4 of [H2] we know that $\dim(\mathcal{E}_{s\cdot\lambda}(\mathfrak{h})) = \dim\left(\frac{S(\mathfrak{h})}{S(\mathfrak{h})\mathcal{I}^+_{s\cdot\lambda}}\right) = \#W_G^{s\cdot\lambda} = \#W_G^\lambda$. Since $S(\mathfrak{h})^{W_G} \subset \mathcal{I}_{s\cdot\lambda}$ we will have $\mathcal{E}_{s\cdot\lambda}(\mathfrak{h}) \subset \mathcal{E}(\mathfrak{h})$. From the proposition 2 above we know that $\dim(\mathcal{E}(\mathfrak{h})) \leq \#W_G$. Since the functions $\{e^{\langle s\cdot\lambda,H\rangle} \mid s \in \mathfrak{s}\}$ are linearly independent over $S(\mathfrak{h})$ the sum $\sum_{k=0}^{r-1} \mathcal{E}_{s_k\cdot\lambda}(\mathfrak{h})$ is direct. Since $\sum_{s\in\mathfrak{s}} \dim(\mathcal{E}_{s\cdot\lambda}(\mathfrak{h})) = r.(\#W_G^\lambda) \geq \#W_G \geq \dim(\mathcal{E}(\mathfrak{h}))$ this direct sum exhausts the whole space $\mathcal{E}(\mathfrak{h})$. Hence we will have $\mathcal{E}(\mathfrak{h}) = \bigoplus_{k=0}^{r-1} \mathcal{E}_{s_k\cdot\lambda}(\mathfrak{h})$. Now suppose that $F_\lambda(g)$ is an eigenfunction of $S(\mathfrak{h})^{W_G}$. Then by (++) we have $\mathrm{D}_{q(\cdot+\lambda)-q(\lambda)} f_{Q,\lambda}(\cdot) = 0$ for all $q(.) \in S(\mathfrak{h})^{W_G}$. Since $S(\mathfrak{h})^{W_G}$ is invariant under the translation $H \to H + \lambda$ we have $\mathrm{D}_{q(\cdot)} f_{Q,\lambda}(\cdot) = 0$ for all $q(.) \in S(\mathfrak{h})^{W_G}$ and hence $F_\lambda(g) \in \mathcal{E}(\mathfrak{h})$. Since the functions $\{e^{\langle s\cdot\lambda, H\rangle} \mid s \in \mathfrak{s}\}$ are linearly independent over $S(\mathfrak{h})$ the function $F_\lambda(g)$ can lie in only one summand of the direct sum $\mathcal{E}(\mathfrak{h}) = \bigoplus_{k=0}^{r-1} \mathcal{E}_{s_k\cdot\lambda}(\mathfrak{h})$. Since for $s \in \mathfrak{s}$, $s \neq s_0$ we have $s \cdot \lambda \neq \lambda$ this summand should be $\mathcal{E}_{s_0\cdot\lambda}(\mathfrak{h})$, hence $F_\lambda(g) \in \mathcal{E}_\lambda(\mathfrak{h})$ which is our claim. □

The above proof is valid for the case of a single function $F_\lambda(g) = e^{\langle \lambda+\rho_Q, H_Q(g)\rangle}$ $f_{Q,\lambda}(H_Q(g))$ on the right hand side of the $f_{N_Q}(g)$. In the general case there are several but finite number of them on the right hand side. To extend the above result to this situation we proceed as follows. Recall the correspondence $X \longleftrightarrow \mathrm{D}_X$ given above with the usual action on $C^\infty(\mathfrak{g})$, we see that for each $F_\lambda(g)$ there is a $d \in \mathbb{N}$ such that $\big(\mathrm{D}_X - \langle \lambda, X\rangle - \rho(X)\big)^d F_\lambda(g) = 0$. Now consider the function

$$f_{N_Q}(g) = \sum\nolimits_{\lambda \in (\check{\mathfrak{a}}_Q)_\mathbb{C}} e^{\langle \lambda+\rho_Q, H_Q(g)\rangle} f_{Q,\lambda}(H_Q(g))(g)$$

given above. Chinese remainder theorem implies that for each λ appearing on the right hand side there is a single variable polynomial Π_λ with constant coefficients such that $\Pi_\lambda(\mathrm{D}_X) f_{N_Q}(g) = e^{\langle \lambda+\rho_Q, H_Q(g)\rangle} f_{Q,\lambda}(H_Q(g))(g)$. Since the operators $\Pi_\lambda(\mathrm{D}_X)$ commute with the operators in $\mathcal{Z}(\mathfrak{g})$ we can apply the above result for a single $F_\lambda(g)$ to the case of multiple $F_\lambda(g)$s now. This finishes the proof of lemma 1. □

(2.4) It is evident from the proof of the lemma 1 that

Theorem 3. *For the automorphic form $f(.)$ that satisfies the conditions of lemma 1 the length $\parallel \lambda \parallel$ of the parameters λ appearing on the right hand side of $f_{N_Q}(g)$ are equal and $\parallel \lambda \parallel -\langle \rho, \rho\rangle - \langle \chi, \chi\rangle$ is the eigenvalue of $f_{N_Q}(g)$ under the Casimir operator. Moreover, the polynomials $f_{Q,\lambda}(H_Q(g))$ are separable and W_G^λ-harmonic.* □

Corollary 1. *For a C^∞ function $f \in L^2(Q(\mathbb{Q})N_Q(\mathbb{A})A_Q(\mathbb{R})^+\backslash G(\mathbb{A}))_{\chi,\mathbb{K}-finite}$ such that $f(mg)$, $m \in M_Q(\mathbb{R})$, is an eigenfunction of $\mathcal{Z}(\mathfrak{m}_Q)$ with character χ the term $(N(\omega, \lambda)f)(H_Q(g))$ in* (1.8) *is a polynomial on $\mathfrak{a}_Q$ (with values in a specific space of cusp forms), which can be represented by monomials, which are products of $\theta \in (\check{\mathfrak{a}}_Q)_\mathbb{C}$ which are orthogonal to $\omega\lambda \in (\check{\mathfrak{a}}_Q)_\mathbb{C}$ with respect to the dual of the Killing form.*

Proof. Since the Eisenstein series $E_Q(f, \lambda)(g)$ are $\mathcal{Z}(\mathfrak{g})$-eigenfunctions if λ lies inside the domain of holomorphy

$$\mathfrak{A}_Q = \{\lambda \in (\mathfrak{a}_Q)_\mathbb{C} \mid \Re(\lambda) \in \rho_Q + \mathfrak{a}_Q^+\},$$

the above Lemma shows that the constant term of this Eisenstein series along Q is a $\mathcal{Z}(\mathfrak{m}_Q)$ eigenfunction as long as we remain in this domain. □

Remark. In the case of cuspidal Eisenstein series the functions $N(\omega,\lambda)$ and $M(\omega,\lambda)$ (given in **(1.20)**) are equal and the functional equation of Eisenstein series implies that for the associated parabolic subgroups P,Q and R we have

$$N(\omega\sigma,\lambda) = N(\omega,\sigma\lambda)N(\sigma,\lambda),$$

for $\omega \in \Omega(\check{\mathfrak{a}}_P, \check{\mathfrak{a}}_Q)$ and $\sigma \in \Omega(\check{\mathfrak{a}}_Q, \check{\mathfrak{a}}_R)$. This implies also that the polynomials $N(.,.)$ are decomposable as a product of the monomials with respect to the elements of the Weyl group of length one. In general these functions are related by the functional equation $N(\omega\sigma,\lambda) = N(\omega,\sigma\lambda)M(\sigma,\lambda)$.

The following corollary is an immediate consequence of the definition (iii) of the Eisenstein systems given in **(1.11)**, equation (1.9) and the fact that if $R \subset \tilde{R}$, then $\rho_R = \rho_{\tilde{R}} + \rho_R^{\tilde{R}}$.

Corollary 2. *If $R \subset \tilde{R}$ and suppose that $Q, R \in \{\mathrm{P}\}$ and $\hat{\omega} \mid \check{\mathfrak{a}}_{\tilde{R}} = \mathrm{Id}$, then the terms in the polynomial $N(\omega,\lambda)$ are monomials which belong to the symmetric algebra $S(\check{\mathfrak{a}}_Q^{\tilde{R}})$.*

We will apply these corollaries (1) and (2) to theorem (1), but before that we discuss the concept of local residue theorem tailored by Langlands for the Eisenstein systems. This machinery is crucial for the main argument of the theorem 1 which will be given soon.

(2.5) We start our discussion about the local residues by defining root hyperplane arrangements. By a **root hyperplane arrangemant** we mean a locally finite set of hyperplanes of $H \in (\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$ defined by

$$H = \{\lambda \in (\check{\mathfrak{a}}_P^G)_{\mathbb{C}} \mid \langle\alpha,\lambda\rangle = \langle\alpha,\mathfrak{t}\rangle, \alpha \in \check{\Phi}^+(\mathfrak{n}_P)\},$$

in which $\mathfrak{t}$ is a normal vector to the hyperplane H. For such a hyperplane, following the notation of **(1.11)** we put

$$\tilde{H} = \{\lambda \in (\check{\mathfrak{a}}_P^G)_{\mathbb{C}} \mid \langle\alpha,\lambda\rangle = 0, \alpha \in \check{\Phi}^+(\mathfrak{n}_P)\},$$

i.e. $H = \mathfrak{t} + \tilde{H}$. Local finiteness means that only a finite number of these hyperplanes intersect each compact subset $K \subset (\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$. We denote such a locally finite root hyperplane arrangement by by $\mathcal{H}$. We will fix for each H and $\mathfrak{t}$ as above a unit normal vector to H as $\mathfrak{t}_0 = \frac{\mathfrak{t}}{|\mathfrak{t}|}$. The open (and connected) components of the complement of this hyperplane arrangement will be denoted by $\mathrm{reg}\big((\check{\mathfrak{a}}_P^G)_{\mathbb{C}}, \mathcal{H}\big)$, the subset of regular subsets of the subspace $(\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$.

Let $\mathcal{M}\big((\check{\mathfrak{a}}_P^G)_{\mathbb{C}}, \mathcal{H}\big)$ denote the set of meromorphic functions with singularities along these hyperplanes. For a function $f \in \mathcal{M}\big((\check{\mathfrak{a}}_P^G)_{\mathbb{C}}, \mathcal{H}\big)$ we define the residue $\mathrm{Res}_H f$ along the hyperplane H to be the function

$$\mathrm{Res}_H f(\lambda) = \frac{1}{2\pi i}\int_{C_\epsilon} f(\lambda + z\mathfrak{t}_0)\mathrm{d}z \tag{2.4}$$

for $\lambda \in (\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$ and $\mathfrak{t}_0$ as above, and z belongs to the circle C_ϵ around 0 with the radius ϵ which is small enough such that the function $f(\lambda + z\mathfrak{t}_0)$ has no other singularities in it. Then for each hyperplane H there is a natural number $n = n(H)$ and a polynomial $R_H(\lambda) = \langle \alpha, \lambda - \mathfrak{t} \rangle$ and a compact subset $K \in (\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$ such that $K \cap H \neq \emptyset$ and such that $R_H(\lambda)^n f(\lambda)$ is a holomorphic function as long as $\lambda \in K$. Note that H is the null set of $R_H(\lambda)$. Then just like the classical residue theorem, we have

Proposition 3. *For $f \in \mathcal{M}\big((\check{\mathfrak{a}}_P^G)_{\mathbb{C}}, \mathcal{H}\big)$ and n as above we have*

$$\mathrm{Res}_H f(\lambda) = \frac{2\pi}{(n-1)!} \frac{1}{\langle \alpha, \mathfrak{t}_0 \rangle^{n-1}} D(\mathfrak{t}_0)^{n-1} \big\{ R_H(\lambda)^n f(\lambda) \big\}, \tag{2.5}$$

where $D(\mathfrak{t}_0)$ belongs to the symmetric algebra $S\big((\check{\mathfrak{a}}_P^G)_{\mathbb{C}}\big)$, which the action on a function f is defined as usual by $D(\mathfrak{t}_0) f(\lambda) = \frac{d}{dt} f(\lambda + t\mathfrak{t}_0) \mid_{t=0}$ and $D(\mathfrak{t}_0)^0 f(\lambda) = \lim_{t \to 0} f(\lambda + t\mathfrak{t}_0)$.

Proof. Suppose that $\lambda \in \mathrm{reg}\big(H, \mathcal{H}'\big)$, in which $\mathcal{H}'$ is the subset of elements of $\mathcal{H}$ which intersect H. For each such a λ define a function $F_\lambda(z)$ on $\mathbb{C}$ by $F_\lambda(z) = f(\lambda + z\mathfrak{t}_0)$. Then according to the one dimensional residue theorem we have

$$\mathrm{Res}_{z=0} F_\lambda(z) = \frac{2\pi}{(n-1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{n-1} \big\{ z^n F_\lambda(z) \big\} \mid_{z=0}$$

which is equal to

$$\frac{2\pi}{(n-1)!} \frac{1}{\langle \alpha, \mathfrak{t}_0 \rangle^{n-1}} D(\mathfrak{t}_0)^{n-1} \big\{ R_H(\lambda)^n f(\lambda) \big\},$$

since by our definition $R_H(\lambda + z\mathfrak{t}_0) = z\langle \alpha, \mathfrak{t}_0 \rangle$. □

For a meromorphic function f in $\mathcal{M}\big((\check{\mathfrak{a}}_P^G)_{\mathbb{C}}, \mathcal{H}\big)$, the $-k$-th coefficient in the Laurent series $f(\lambda + z\mathfrak{t}_0) = \sum_{n \gg -\infty}^{\infty} a_n z^n$ is given by

$$\big(\mathrm{Res}_{H,k}(f)\big)(\lambda) = \int_{C_\epsilon} f(\lambda + z\mathfrak{t}_0) R_H(\lambda + z\mathfrak{t}_0)^{k-1} \frac{\mathrm{d}z}{\mathrm{i}}.$$

The meaning of $\mathfrak{t}_0$ and C_ϵ is like above. Then if the function f has a pole of order n along the hyperplane H, the above proposition gives for $k \leq n$ the $-k$-th coefficient as

$$\mathrm{Res}_{H,k} f(\lambda) = \frac{2\pi}{(n-k)!} \frac{1}{\langle \alpha, \mathfrak{t}_0 \rangle^{n-k}} D(\mathfrak{t}_0)^{n-k} \big\{ R_H(\lambda)^n f(\lambda) \big\} \tag{2.6}$$

(2.6) Before we prove the theorem 1 in the next section we prove the important claim given in **(2.2)** (just before the theorem 2), that there is no gap between the successive terms of the Laurent expansion of a general Eisenstein series.

Proposition 4. *Let $f \in A^2(P)$ and let H be a singular hyperplane of the residual Eisenten series $E_P^G(g, f, \lambda)$ which intersects the closure of the positive Weyl chamber. Then there is no gap between the nonzero terms of the Laurent expansion of $E_P^G(g, f, \lambda)$ around H.*

Proof. Let $\lambda \in H$ be a generic point and $\mathfrak{t}_0$ be a unit normal vector to H, and $z \in \mathbb{C}$. We develop the $E_P^G(g, f, \lambda + z\mathfrak{t}_0)$ as a meromorphic function in a small neighborhood of λ such that no other singular hyperplane of $E_P^G(f, g, \lambda)$ intersects this neighborhood. The Laurent expansion around H is

$$E_P^G(g, f, \lambda + z\mathfrak{t}_0) = \sum_{k=-N}^{\infty} z^k E_k(g).$$

Recall the definition of Casimir operator $\omega_{\mathfrak{g}}$ in **(1.2)**, we apply it to the both sides of this expansion and obtain

$$\omega_{\mathfrak{g}} E_P^G(g, f, \lambda + z\mathfrak{t}_0) = \{\langle \lambda + z\mathfrak{t}_0, \lambda + z\mathfrak{t}_0 \rangle + \langle \rho, \rho \rangle + \langle \chi, \chi \rangle\} E_P^G(g, f, \lambda + z\mathfrak{t}_0) =$$

$$(az^2 + bz + c) E(g, f, \lambda + z\mathfrak{t}_0) = (az^2 + bz + c) \sum_{k=-N}^{\infty} z^k E_k(g),$$

in which $a = \langle \mathfrak{t}_0, \mathfrak{t}_0 \rangle^2$, $b = \langle \lambda, \mathfrak{t}_0 \rangle$ and $c = \langle \lambda, \lambda \rangle + \langle \rho, \rho \rangle + \langle \chi, \chi \rangle$. We observe that a is non zero and real since according to the main result of [F2] H and (hence) $\mathfrak{t}_0$ are real, b is also nonzero since otherwise this hyperplane will intersect the unitary axis, which is impossible for the residual Eisenstein series supported on distinguished subspaces (shown in the proof of the lemma 7.6 in [L1]), in contrast to the Eisenstein systems described in the theorem 7.1 of [L1], which are not necessarily supported on distinguished subspace. We observe also that for λ being in general position c wont vanish identically. Now if we compare the Laurent coefficients of the both sides of the above identity we see that for all $2 \leq n \leq N$ we have

$$\omega_{\mathfrak{g}} E_n = a.E_{n-2} + b.E_{n-1} + c.E_n,$$

which we write it as

$$(\omega_{\mathfrak{g}} - c) E_n = a.E_{n-2} + b.E_{n-1}.$$

We call this last identity $(*)$.
Suppose that k is the smallest index such that $E_k \neq 0$. Then $(*)$ implies that $(\omega_{\mathfrak{g}} - c) E_k = 0$. We claim that for all $l \in \mathbb{N}$ we have $(\omega_{\mathfrak{g}} - c)^{l+1} E_{k+l} = 0$, the claim being true for $l = 0$. For the induction step we apply $(\omega_{\mathfrak{g}} - c)^l$ to the both sides of $(*)$ for $n = k + l$ and obtain

$$(\omega_{\mathfrak{g}} - c)^{l+1} E_{k+l} =$$

$$a.(\omega_{\mathfrak{g}} - c)(\omega_{\mathfrak{g}} - c)^{l-1} E_{k+l-2} + b.(\omega_{\mathfrak{g}} - c)^l E_{k+l-1}.$$

By induction hypothesis these two terms on the right hand side of the above equality vanish and our claim follows. Now we apply successively $(\omega_{\mathfrak{g}}-c)^{l-j}$, for $j=1,...,l-1$, to $(*)$ for $n=k+l$ and performing a downward induction, we obtain

$$(\omega_{\mathfrak{g}}-c)^{l}E_{k+l}=b^{l}.E_{k}\neq 0,$$

since b^l and E_k are non-zero. We conclude that the Laurent coefficients $E_{k+l}\not\equiv 0$ for all $0\leq l\leq -k$. In other words there is no gap between the nonzero coefficients of the Laurent expansion of residual Eisenstein series which is our claim. □

(2.7) Now we consider Theorem 1 in which we are interested in the poles of higher order. The main theorem of [F2], shows that if $H\subset(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ is a singular hyperplane of the Eisenstein series $E_P^G(\lambda,f)$ which intersects the complex strip above the positive Weyl chamber, then it is real. Suppose that the function $E_P^G(\lambda,f)$ has a pole of order n at the singular hyperplane H.

Since H is a singular hyperplane by our assumption, there is a $k>0$ such that $\mathrm{Res}_{H,k}\,E_P^G(f,\lambda)\neq 0$. As it is shown in the proof of the theorem 1 of [F2], there is a generic point $\lambda\in\overline{\check{\mathfrak{a}}_P^{+}\cap H}$, a parabolic subgroup R, a function g in (1.2), an element $\theta\in{}^{+}\check{\mathfrak{a}}_R$ and a $\sigma\in\Omega_0(R,\chi_R,\psi,Q)$ such that $\sigma\theta=\omega_0\lambda$, that satisfy the relations $\omega_0H\subseteq\sigma(\check{\mathfrak{a}}_R)_{\mathbb{C}}$ and $\hat{\omega}_0\lambda=\lambda$, and such that the leading terms of

$$\mathrm{Res}_{H,k}\,E_P^G(f,\lambda),\tag{2.7}$$

and according to (2.6), a linear combination of the derivatives of

$$E_R^G(g,\eta),\tag{2.8}$$

are coincide at the point $x=\lambda$, for a parameter η which lies in a small convex neighborhood of x in $(\check{\mathfrak{a}}_P^G)_{\mathbb{C}}$. Since we have, as indicated in formula (2.1),

$$(\mathrm{Res}_{H,k}\,E_P^G f)_Q(\lambda)=\sum_{\sigma\in\Omega_k(H,P,\chi,\psi,Q))}\big((N_k(\sigma,\lambda)f)(H_Q(g))\big)(g),$$

the leading term of (2.7) is a sum of the derivatives with respect to the free parameter η of the functions $e^{\langle .,.\rangle}N(.,.)$. The set $\Omega_k(H,P,\chi,\psi,Q)$ is the set of linear transformations σ such that $N_k(\sigma,\lambda)$ does not identically vanish.

Now suppose that $\mathfrak{t}$ is a generator of $H^{\perp}$ and α a positive root like **(2.5)**, and suppose that $N(\omega_0,\lambda)f$ has a pole of order n at H. We will apply the formula (2.6) to constant term of (2.8) in the direction of Q to compute the k-th term $(\mathrm{Res}_{H,k}\,E_P^G f)_Q(\lambda)$. Suppose that the there is at least an ω_0 such that the k-th term $N_k(\omega_0,\lambda)\not\equiv 0$ at $H\subset(\check{\mathfrak{a}}_P)_{\mathbb{C}}$ and such that the inequality $\left|\big(\omega_0(x)\big)_+\right|\geq\left|\big(\tilde{\omega}(x)\big)_+\right|$ is fulfilled for all $x\in H\cap\check{\mathfrak{a}}_P^{G+}$. We can suppose that there is only one such an ω_0. We have to show that $N_j(\omega_0,\lambda)\equiv 0$ for $j>k$, which means that for this ω_0 the constant term of the $k-$th residue of an Eisenstein series in the positive Weyl chamber may not contribute to the constant terms of the residues of order $> k$. To show this we compute the term of j−th order, $j > k$, by the formula (2.6):

$$\frac{1}{(n-j)!}\frac{\mathrm{d}^{n-j}}{\mathrm{d}z^{n-j}}\big\{\langle\alpha, \lambda + z\mathfrak{t}_0 - \mathfrak{t}\rangle^n e^{\langle\omega_0(\lambda+z\mathfrak{t}_0),x\rangle} N(\omega_0, \lambda + z\mathfrak{t}_0)\big\}|_{z=0},$$

using Leibniz rule $(f.g.h)^{(j)} = \sum_{l+m+n=j}\binom{j}{l,m,n} f^{(l)}.g^{(m)}.h^{(n)}$, and the fact that the operators $N(.,.)$ are monomials in the orthogonal complement of $\omega_0\lambda$, we see[3] that the leading term is a constant multiple of

$$\langle\hat{\omega}_0\mathfrak{t}_0, \tilde{H}\rangle^j F_j + ... + \langle\hat{\omega}_0\mathfrak{t}_0, \tilde{H}\rangle F_1 + F_0, \tag{2.9}$$

in which $H = \mathfrak{t} + \tilde{H}$ is as in **(2.5)** and the functions F_i (according to the formula (1.15) of the previous chapter) belong to various subspaces of

$$S\big(\hat{\omega}_0(\check{\mathfrak{a}}_P)^{\perp}\big) \otimes S_\infty\big(N_Q(\mathbb{A})A_Q(\mathbb{R})^\circ Q(\mathbb{Q}) \setminus G(\mathbb{A})\big).$$

Now we consider the expression (2.9) and show that in order for it to be comptible with [F1, Theorem 14] we have to have $\hat{\omega}_0\mathfrak{t}_0 \perp \sigma 0$, for σ an affine transformation like in the proof of the main theorem of [F2].

Let $\hat{\sigma}$ denote the linear part of the affine transformation σ. By taking derivatives of (2.8) with respect to the free parameter η at the point x some factors of the form $\langle x, H\rangle$ would be produced, in which $x \in \hat{\sigma}(\check{\mathfrak{a}}_R)$. It is shown in the page 230 of [L1] that $\sigma 0$ is real and orthogonal to $\hat{\sigma}(\check{\mathfrak{a}}_R)$ and belongs to $-{}^{+}\check{\mathfrak{a}}_R^Q$ in which Q is the smallest parabolic subgroup containing R such that $\check{\mathfrak{a}}_Q$ is contained in $\hat{\omega}\check{\mathfrak{a}}_R$. As we know from corollary 1, the polynomial $N(\sigma, \theta)$ consists of monomials from the $S(\omega\theta)$ at the point θ for ω such that $\omega|H = \sigma$. Then this monomials are consisting of the elements of the orthogonal complement of $\lambda = \sigma\lambda - \sigma 0$ since we have $\hat{\sigma}\lambda = \lambda$ (because $\hat{\omega}_0\lambda = \lambda$) and consequently consisting of the elements from the orthogonal complement of $\sigma 0$. This observation in accordance to lemma 1 yields that the leading coefficient of (2.7) has the form (2.9) only if

$$\hat{\omega}_0\mathfrak{t}_0 \perp \sigma 0. \tag{2.10}$$

Then to prove that $N_j(\omega_0, \lambda) \equiv 0$ for $j > k$ we are reduced to prove that (2.10) cannot occur.

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[3]For example, for $n = 2, j = 1$ the calculation is

$$\frac{\mathrm{d}}{\mathrm{d}z}\big\{\langle\alpha, \lambda + z\mathfrak{t}_0 - \mathfrak{t}\rangle^2 e^{\langle\omega_0(\lambda+z\mathfrak{t}_0),x\rangle} N(\omega_0, \lambda + z\mathfrak{t}_0)\big\}|_{z=0} =$$

$$\begin{aligned}&\big\{2\langle\alpha, \mathfrak{t}_0\rangle.R_H(\lambda + z\mathfrak{t}_0)e^{\langle\omega_0(\lambda+z\mathfrak{t}_0),x\rangle}.N(\omega_0, \lambda + z\mathfrak{t}) +\\ &\qquad R_H(\lambda + z\mathfrak{t}_0)^2.\langle\omega_0\mathfrak{t}_0, x\rangle.e^{\langle\omega_0(\lambda+z\mathfrak{t}_0),x\rangle}.N(\omega_0, \lambda + z\mathfrak{t})+\\ &\qquad\qquad R_H(\lambda + z\mathfrak{t}_0)^2.e^{\langle\omega_0(\lambda+z\mathfrak{t}_0),x\rangle}.N'(\omega_0, \lambda + z\mathfrak{t})\big\}|_{z=0},\end{aligned}$$

which gives the leading term $\langle\hat{\omega}_0\mathfrak{t}_0, \tilde{H}\rangle F_2 + F_1$. Then the proof of the general case is a simple induction.

Suppose that (2.10) is true. We remind that $\hat{\sigma}\check{\mathfrak{a}}_R \perp \sigma 0$. It was also proved in the main theorem of [F2] that $\hat{\sigma}\check{\mathfrak{a}}_R \supseteq \omega_0 H$. These facts along with (2.10) imply that

$$\hat{\omega}_0(\check{\mathfrak{a}}_P) \perp \sigma 0. \tag{2.11}$$

Since $\omega_0(y) \in \sigma(0) + \check{\mathfrak{a}}_R^{G+}$ and we know that $\sigma(0) \perp \check{\mathfrak{a}}_R^G$ we will have $(\omega_0\lambda)_- = \sigma 0$. Then we will have

$$\left|\sigma 0\right|^2 = \langle(\omega_0\lambda)_-, (\omega_0\lambda)_-\rangle =$$

$$\langle(\omega_0\lambda)_-, \omega_0\lambda\rangle = \langle(\omega_0\lambda)_-, \omega_0 0\rangle \leq \left|(\omega_0\lambda)_-\right|\left|\omega_0 0\right| = \left|\sigma 0\right|\left|\omega_0 0\right|$$

which implies $\left|\sigma 0\right| \leq \left|\omega_0 0\right|$.
On the other hand we choose a point λ as above we have

$$\left|\omega_0\lambda\right|^2 = \left|(\omega_0\lambda)_+ + \sigma 0\right|^2 = \left|(\omega_0\lambda)_+\right|^2 + \left|\sigma 0\right|^2 < \left|\lambda\right|^2 + \left|\sigma 0\right|^2.$$

Here we used the inequality $\left|(\omega_0\lambda)_+\right| < \left|\lambda\right|$ which is proved in the lemma 1 of [F2]. This together with the equality

$$\left|\omega_0\lambda\right|^2 = \left|\omega_0 0\right|^2 + \left|\lambda\right|^2$$

implies $\left|\sigma 0\right| > \left|\omega_0 0\right|$ which is a contradiction. This proves that $N_j(\omega_0, \lambda) \equiv 0$ for $j > k$ which implies that the pole order of a of a general Eisenstein series at a singular hyperplane which intersects the positive Weyl chamber is bounded by the number $\max\limits_{Q\in\{\mathrm{P}\}} \#\Omega(P, \chi, \psi, Q)$.

$\square$

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