Low Dimensional Gauge Theories and Quantum Geometry

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

von
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aus
Lucknow (Indien)

Bonn, July 2020

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion:	02.10.2020
Erscheinungsjahr:	2020

Abstract

This thesis explores the connection between supersymmetric gauge theories on curved spaces of dimensions two and three and geometries that arise in string theory. It does so by studying quantum field theoretic objects such as the partition function and correlation functions on the gauge theory and finding links to geometric information of its target space. An exact calculation of the path integral is made possible using the localisation technique for supersymmetric gauge theories.

A 2d gauge theory with $\mathcal{N} = (2,2)$ supersymmetry on the Ω-deformed two-sphere subject to the A-twist is the first object of study. Here we find that the correlation functions of the twisted chiral field fulfil certain universal and non-trivial relations. These relations can be interpreted as quantum operators that govern the moduli dependence of the ground state of the gauge theory in a Hilbert-space picture. Furthermore, the relations can be imparted a representation as differential operators that are shown to annihilate Givental's cohomology-valued I-function on the target space of the gauge theory. This is a consequence of the fact that the 2d gauge theory provides an ultraviolet model for quantum cohomology on a manifold. In particular, for gauge theories with Calabi-Yau target spaces, these operators coincide with Picard-Fuchs operators in algebraic geometry. For a certain class of Calabi-Yau manifolds, we turn the argument around and express the Picard-Fuchs operators in terms of a finite number of correlators in the gauge theory.

In 3d we study $\mathcal{N} = 2$ gauge theories on the solid torus $D^2 \times_q S^1$, where q is the twist in the fibration of D^2 over S^1, with Grassmannian manifolds as target spaces. These theories are ultraviolet models for quantum K-theory on their target spaces. We compute the partition function and extract from it Givental's I-function of permutation symmetric quantum K-theory. This facilitates a calculation of the algebra of Wilson loops which, for different values of the Chern-Simons levels, is shown to be isomorphic to either the quantum K-theoretic ring of Schubert structure sheaves on the Grassmannian or the Verlinde algebra. Additionally, we evaluate difference equations that annihilate this I-function. A limit where the S^1 contracts to a point, all the computed quantities are shown to coincide with the corresponding objects encountered in 2d.

Acknowledgements

I am immensely thankful to Dr. Hans Jockers for providing a thoughtful and tireless supervision, constant approachability and an ever-present support in all matters alike. I will be forever grateful for the many interesting avenues of research I was introduced to and for all that I learnt as a doctoral student under his supervision. I would also like to thank Prof. Albrecht Klemm for admitting me into his research group and acquainting me with several fascinating ideas in physics and mathematics during seminars and informal discussions.

I am very grateful for the fruitful and engaging collaborations I had with Dr. Andreas Gerhardus, Prof. Peter Mayr, Dr. Mauricio Romo and Alexander Tabler. My interactions with all the present and past members of the Klemm group have been helpful and educational, for which I am thankful. I would especially like to thank Fabian Fischbach and Christoph Nega for assistance with the manuscript as well as Dr. Andreas Gerhardus and Dr. Thorsten Schimannek for scientific guidance.

I would like to thank the Bonn-Cologne Graduate School of Physics and Astronomy whose financial support made several valuable academic trips possible.

I express my gratitude to members and alumni of the BCTP who have provided levity in times of stress and motivation in times of inactivity. I am particularly grateful to Christa Börsch, Dagmar Fassbender, Petra Weiss, Dr. Andreas Wisskirchen and Patricia Zündorf for easing the many logistical hurdles one faces as a doctoral student. I also owe a lot to the friends I have made while studying physics and who have been pillars, mentors and educators alike.

I am indebted to my uncle Dr. Ravi Sinha whose infectious love for fundamental physics has been instrumental in the direction my education and research took. Finally, an inexpressible thanks to Aai, Baa and Paul for much more, everything.

List of Publications

The original work presented in this thesis is based on the following publications of the author:

1. A. Gerhardus, H. Jockers, and U. Ninad, *The Geometry of Gauged Linear Sigma Model Correlation Functions*, Nucl. Phys. B **933** (2018) 65-133, `arXiv:1803.10253 [hep-th]`.

2. H. Jockers, P. Mayr, U. Ninad, and A. Tabler, *Wilson loop algebras and quantum K-theory for Grassmannians*, `arXiv:1911.13286 [hep-th]`.

Contents

CHAPTER 1

Introduction

Point Particle Physics

Einstein's theory of gravity is encapsulated in the Einstein-Hilbert action,

$$S \sim \frac{1}{G_N} \int d^4x \, \sqrt{g} \, \mathcal{R} \; , \tag{1.1}$$

expressed in natural units, where G_N is the Newton's gravitational constant. Here g is the determinant of the metric tensor of spacetime $g_{\mu\nu}$ and $\mathcal{R}$ is a dynamical scalar that captures the curvature of spacetime, known as the Ricci scalar. This theory was considered a complete description of the gravitational force and inspired landmark findings at the cosmological scale until a microscopic description of the electromagnetic force came along. For an exposition to general relativity, see for instance [1, 2].

Quantum field theory is a formalism that elevates classical fields to quantum fields, going beyond a (first) quantisation of particles and earning itself the term second quantisation. The architects of quantum field theory set out trying to unify quantum mechanics of multi-particle states with special relativity and were faced with explaining particle creation and annihilation. This ultimately led them to postulate the wave-functions associated to particles as quantum-mechanical objects themselves and gave birth to a vast network of ideas that we today call quantum field theory. The first consistently constructed quantum field theory was that of electrodynamics which paved way for the quantisation of weak and strong interactions. Some comprehensive and pedagogical reviews include [3–6].

Soon after its conception, quantum field theory was faced with the problem of infinities. Assuming that the principles of quantum field theory are applicable for infinite energy scales, physical observables such as the mass and strength of interaction are also rendered infinite by the underlying mathematical framework. This led to the birth of renormalisation, an idea whereby the fields are renormalised in a way that restores the finiteness of physical observables. While the original idea was conceived in the context of quantum electrodynamics, it was generalised to a general Yang-Mills theory in four dimensions. This immense success of quantum field theory as a mathematical framework was further boosted by the fact that it lent itself to non-trivial state-of-the-art experimental checks. This ultimately led to the formulation of the standard model of particle physics, a framework that describes matter and the three fundamental interactions: electromagnetic, weak and strong, quantum mechanically.

Attempts to fit the fourth fundamental interaction, gravity, into the framework of quantum field theory failed for the reason that while one could conceive of the notion of a quantum-mechanical particle, called the graviton, that mediates gravitational interaction, the interaction in itself is non-renormalisable. This is evident from equation (1.1) where the strength of interaction G_N has units of (length)^2, and thus the dimensionless coupling constant $G_N \cdot (\text{energy})^2$ diverges in the ultraviolet regime. This negative mass dimension (or positive length dimension) of the gravitational coupling constant is in contrast to the dimensionlessness of Yang-Mills coupling constants. This early hurdle in quantising gravity led to the belief that quantum field theory may be an effective theory and while it estimates the interactions at energy scales much less than the Planck scale with near accuracy, it fails to be applicable for gravitational interaction, which only comes into play in the vicinity of the Planck scale. A shift in the preconceived relation of matter and interactions to spacetime was precedented.

String Theory

String theory postulates that the fabric of spacetime is constituted not of points, which are zero-dimensional, but of strings, which are one-dimensional. In other words, the fundamental objects that make up spacetime are one-dimensional strings which are characterised by their tension T. Strings propagating in spacetime trace out a 'worldsheet', as opposed to the worldline of a point particle. The advantage of a radical change of this sort in the way we view and model spacetime is that not only does this theory give rise to a spin-two excitation of the string, which is dubbed the graviton, but it does so in a way that causes no ultraviolet divergences. This can be understood heuristically as a consequence of the fact that ultraviolet divergences arise in point particle quantum theory as a result of phenomena at infinitesimal distances. In string theory however the finite length of strings, given by $\sqrt{1/T}$, that constitute spacetime forbid such a limit, in effect eliminating the region of configuration space that could lead to ultraviolet divergences. Moreover, the tension T is the only free parameter in this theory and, in principle, if this theory is to be the consistent unification of all known forces and matter into one framework, then all the known parameters of the standard model as well as the Newton's gravitational constant G_N must arise therefrom. For a pedagogical introduction to string theory, see [7–11].

The spacetime in which a string theory can be consistently defined, i.e., in which strings can consistently propagate without violating any of the fundamental symmetries of the theory, is known as the target space. The dimension of the target space is constrained by the requirement that the symmetries of the worldsheet theory, diffeomorphism- and Weyl-invariance[1], remain anomaly-free upon quantisation. If we assume that the target space is flat for simplicity, then for a bosonic string this critical dimension turns out to be 26 whereas for the superstring, i.e., a string whose modes fall into a representation of the supersymmetry algebra[2], this dimension is ten. The discrepancy between the critical dimension of superstring theory and that of observable spacetime inspired the proposal that the six extra dimensions span a small and compact six dimensional manifold. The length scale of this compactification manifold must be small enough to remain unprobed in modern day experiments. Lifting the assumption of the flatness of target space leads one to study the action of the string in curved spacetime and impose conservation

[1] Diffeomorphism invariance implies that a reparametrisation of the coordinates leaves the action unchanged and Weyl invariance implies that a local scaling of the metric leaves the action unchanged.

[2] Supersymmetry links particles with integer spin to particles with half integer spins by a symmetry transformation. For an introduction to supersymmetry see [4, 12, 13].

of worldsheet symmetries. The consequences of this approach will be briefly touched upon in forthcoming section on the target space approach to study spacetime geometry.

The diffeomorphism- and Weyl-invariance of the 2d worldsheet theory imply the existence of conformal symmetry[3]. It was shown that a string theory the string worldsheet equipped with $\mathcal{N} = 2$ supersymmetry in addition to the conformal symmetry can give rise to a supersymmetry on the target space by constructing an operator, known as spectral flow, in the worldsheet superconformal algebra that generates spacetime supersymmetry. Upon equipping both the left- and right-handed sector of the closed superstring with $\mathcal{N} = 2$ superconformal symmetry and requiring a tachyon-free spectrum, achieved by a projection of states into specific left- and right-handed fermion numbers known as the GSO projection, leads to two distinct theories known with the massless spectra of type IIA and type IIB supergravity. These are known as type IIA/B string theories, respectively. Type IIB string theory is chiral and upon imposing parity symmetry on the left and right handed sector one obtains type I string theory. This theory has half the supersymmetry of the type IIB string and a non-orientable worldsheet. Finally, conjuring a theory with $\mathcal{N} = 2$ superconformal symmetry in the left handed-sector and no supersymmetry in the right-handed sector results in the heterotic string theory. The 26-dimensional right handed bosonic string must be compactified down to 10 dimensions to match the left-handed sector dimensionally. The two self-dual 16-dimensional lattices along which this can be done consistently lead to heterotic string theory with the gauge groups $\mathrm{Spin}(32)/\mathbb{Z}_2$ and $\mathrm{E}_8 \times \mathrm{E}_8$ [17, 18]. Summarising, the five consistent superstring theories with varying spacetime spectra, gauge group and supersymmetry in ten dimensions are type I, type IIA, type IIB, heterotic with gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$ and $\mathrm{E}_8 \times \mathrm{E}_8$, respectively. These distinct theories are all connected by a web of dualities and believed to be different limits of an eleven-dimensional theory known as M-theory [19].

In the following discussion we will narrow our focus down to the type II string theories and we will give a lightning-fast review of some of the foundational ideas. The fundamental string can be open or closed. The simplest boundary conditions that the open string can have at its ends are Neumann (N) or Dirichlet (D), where the Neumann boundary conditions denote the subset of spacetime in which the ends of the open string can freely move whereas the Dirichlet boundary conditions denote the subset in which the ends of the open string must remain stationary. The latter are interpreted to span extended objects of dimension $p + 1$, corresponding to the number of N boundary conditions, and are known as Dp-branes. Dp-branes are dynamical objects that propagate in the subset of spacetime spanned by the D boundary conditions. The fermionic sector of the closed string can either be Ramond (R) or Neveu-Schwarz (NS) depending on whether the condition on the fermionic oscillator modes upon circling the string is periodic or anti-periodic, respectively. The closed sector consists of left and right-moving sectors which can either be Ramond or Neveu-Schwarz, giving rise to two bosonic sectors, (R,R) and (NS,NS), and two fermionic sectors, (R,NS) and (NS,R), respectively. The bosonic spectrum of these theories along with the amount of symmetry they have is detailed in Table 1.1. The fermionic sectors consist of the superpartners of the bosonic sector, modulo the GSO projection which ensures a tachyon-free spectrum and causes the (non-)chirality of the theories. As is detailed in Table 1.1, the graviton is a natural consequence of all the 10d superstring theories, which together with the UV complete nature of string theory makes it a viable candidate for the quantum theory of gravity.

[3] Conformal symmetry consists of those transformations of the spacetime coordinates that leave the metric invariant up to an overall scale factor. For a pedagogical exposition see [14–16].

Superstring theory in 10d	# Supercharges ; Chirality $\mathcal{N}=(N_L,N_R)$	Massless spectrum (bosonic) (NS,NS)	(R,R)	Gauge Group
Type I	16 ; $\mathcal{N}=(1,0)$	$(\varphi,,g_{\mu\nu})$	$(b_{\mu\nu})$	SO(32)
Type IIA	32 ; $\mathcal{N}=(1,1)$	$(\varphi,b_{\mu\nu},g_{\mu\nu})$	$(A_\mu,C_{\mu\nu\rho})$	–
Type IIB	32 ; $\mathcal{N}=(2,0)$	$(\varphi,b_{\mu\nu},g_{\mu\nu})$	$(\phi,B_{\mu\nu},D_{\mu\nu\rho\sigma})$	–
Heterotic $E_8\times E_8$	16 ; $\mathcal{N}=(1,0)$	$(\varphi,b_{\mu\nu},g_{\mu\nu})$		$E_8\times E_8$
Het. Spin(32)/$\mathbb{Z}_2$	16 ; $\mathcal{N}=(1,0)$	$(\varphi,b_{\mu\nu},g_{\mu\nu})$		Spin(32)/$\mathbb{Z}_2$

Table 1.1: The table shows the list of the five consistently defined superstring theories in 10d along with the number of spacetime supercharges, their chirality denoted by the amount of left or right supersymmetry (N_L,N_R) and the gauge group in 10d spacetime. The bosonic sector splits into (NS,NS) and (R,R) sectors, respectively, in all but the heterotic theories where only the left handed sector is a superstring. The dilaton φ, anti-symmetric tensor $b_{\mu\nu}$ and graviton $g_{\mu\nu}$ appear in all theories and the type II theories additionally have $p-$forms A,B,C and D in the (R,R) sector. The gauge group of the type I theory is a consequence of a stack of 32 D9-branes from the open string sector carrying Chan-Paton indices. The gauge groups of the heterotic theories are a consequence of the 26-dimensional right-handed bosonic sector being compactified along the 16 dimensions, to match the ten-dimensional left-handed sector, along a 16d self-dual lattice.

Dp-branes in type II theories are understood [20] to be stable due to a balance between the attractive gravitational force from coupling to the graviton and the repulsive electromagnetic force from coupling with the $p-$form fields in the (R,R) sector. The 32 spacetime filling D9-branes in type I theory on the other hand are required by the Green-Schwarz anomaly cancellation mechanism [21] to cancel the non-zero tadpole contribution from the spacetime orientifold planes, which are non-oriented counterparts of D-branes.

The analysis of the compactification manifold can be done in the context of any of the above-mentioned string theories, where further constraints come from requiring that compactifying the extra dimensions results in a 4d theory with the desired properties of observable spacetime. The geometric properties of the compactification manifold determine how much of the supersymmetry of the ten-dimensional theory survives in four dimensions. For instance, compactifying along an $n-$dimensional torus preserves all of the supersymmetry. A certain class of compact complex Kähler manifolds known as Calabi-Yau manifolds preserve a quarter of the supersymmetry. For instance, upon compactifying a heterotic string theory in 10d along a 6d Calabi-Yau manifold, also known as a Calabi-Yau threefold because of complex dimension 3, one gets a 4d theory with 4 supercharges. Such compactifications are attractive from a phenomenological perspective and as they give rise to supersymmetric extensions of chiral gauge theories with standard model-like spectra.

Two perspectives to study spacetime geometry

The quantum theory of maps from a specific worldsheet Σ to the total spacetime $\widetilde{X}$ is known as a non-linear sigma model (NLSM). Note that in a physicist's parlance the quantum theory of a map from a worldline to flat 4d Minkowski space is a quantum field theory. For simplicity one imposes that the total spacetime is a direct product of the observable 4d spacetime and the internal space X, i.e., $\widetilde{X} = \mathcal{M}^{1,3} \times X$. Furthermore, usually one restricts these maps to the internal space X, whose nature one wishes to probe further. The internal manifold inside the total spacetime manifold is commonly referred to as the NLSM target space for simplicity. One can adopt either a worldsheet or a target space approach to studying the NLSM target space geometry. We emphasise that this is a novel pursuit as these geometries differ fundamentally from point-like geometries that are the foundation of classical Riemannian geometry. The geometries that arise in string theory are sometimes called *quantum geometries*, where the 'quantum' alludes to their quantum as well as string-like nature.

The *worldsheet perspective* calls for a study of families of superconformal field theories (SCFTs) on the worldsheet. The central charge, which is an inherent property of an SCFT, is fixed by requiring the cancellation of an anomaly in the Weyl symmetry, which is a subset of the conformal symmetry. The worldsheet approach places constraints only of the nature of the SCFT, that this SCFT admits a geometric description need not be required. For instance, an NLSM mapping into either a geometrically well-defined Calabi-Yau manifold or a non-Lagrangian non-geometric theory is equally interesting to study from a worldsheet perspective if they are both equipped with the necessary SCFT characteristics. We mention as an aside that the $\mathcal{N} = 2$ superconformal algebra in 2d contains an operator which is instrumental in generating a supersymmetric spacetime spectrum. A supersymmetric spectrum in spacetime is phenomenologically attractive to those that strive to seek a proof of supersymmetry in the observable world. To those for whom this is not a sturdy justification for the pursuit of spacetime supersymmetry in string theory, one might cite radical computational control as the driving reason for adhering to this setup.

The *target space perspective* on the other hand involves studying the geometry of the compactification manifold via studying the action of the superstring, known as the Polyakov action, perturbatively in the NLSM parameter $\alpha' \sim 1/T$. A requirement of the vanishing $\beta-$functions at first order in α', needed for the superconformal symmetry to be maintained at the quantum level, imposes definite conditions on the spacetime geometry. If the string length scale $\sim \sqrt{\alpha'}$ is small enough or, alternatively, the spacetime has a large enough volume, then this one-loop analysis offers a good estimation of the spacetime geometry. The low energy effective action is obtained by constructing a Lagrangian in 10d (or whatever the consistent spacetime dimension we are working with is) which reproduces the vanishing $\beta-$function equations of motion for spacetime. One can then study this effective action to learn more about the internal manifold geometry.

Either approach comes with its merits and drawbacks. The target space approach is powerful in the large volume regime and captures the non-perturbative effects of D-branes. The worldsheet approach, while oblivious to D-brane instanton effects, captures the quantum stringy nature of spacetime as opposed to the target space approach which is typically only accessible in the supergravity (large volume) approximation.

A specific worldsheet approach: Gauge theories

We will adopt a specific worldsheet approach to studying quantum geometries where instead of studying worldsheet SCFTs that map to relevant NLSM target spaces, we study supersymmetric gauge theories which are certain UV limits of these worldsheet SCFTs. These theories were originally devised in [22] in 2d with 4 supercharges and termed gauged linear sigma models (GLSMs). In the recent years higher dimensional supersymmetric gauge theories have become actively researched on as well because of non-trivial connections to NLSM targets spaces.

The reason these gauge theories are more accessible as opposed to studying NLSM target spaces directly has to do with the idea of *localisation*. Localisation was first conceived in the context of NLSMs by Witten in [23]. The $\mathcal{N}=2$ superconformal algebra on the worldsheet of type II theories can be 'twisted', i.e., deformed by taking appropriate linear combinations of the generators of the algebra, in a way that results in a topological theory and makes calculating the otherwise untameable path integral possible. This twisting can be done in two ways resulting in distinct theories known as the A- and B-model. It is commonly said that the observables of the A-model can be computed by counting holomorphic maps from the worldsheet to the target space whereas those of the B-model can be computed by integrating over certain differential forms on the target space. In essence, the A-model reduces all interesting physics captured in the infinite dimensional path integral to integrals over the moduli space of instantons of the NLSM and the B-model does so to integrals over the target space. As a side note, mirror symmetry establishes a duality between these two models by identifying the A-model invariants of a target space X with the B-model invariants of its mirror $\widetilde{X}$. Thus a knowledge of mirror pairs can help reduce the computationally challenging A-model integrals of one space to the classical B-model integrals of the other.

The idea behind localisation was then applied also to supersymmetric gauge theories to calculate the partition functions and certain observables exactly [24]. Despite there not being an obvious connection to string theory, they have fascinating connections to NLSM target spaces in string theory that we will discuss in this thesis. Another powerful application of localisation in quantum field theory is that various duality webs that were conjectured for gauge theories that flow to the same conformal field theory in the infrared, in the same vein as the original electric-magnetic duality in 4d [25], were proven by comparing observables made exactly calculable by localisation. This was a giant step in the direction of classifying all possible worldsheet theories that give rise to the same spacetime physics. For a comprehensive review on recent developments in on application of localisation to supersymmetric gauge theories see [26].

In this thesis we explore certain 2d and 3d supersymmetric gauge theories in search of novel connections to quantum geometries.

In Chapter 2 we will review the technique of localisation to compute certain path integrals exactly. We will begin in Section 2.1 by demonstrating the main idea behind why and how this technique works. In Section 2.2, we will review the original work [23, 27] for the computation of the partition function of a non-linear sigma model with a topological twist, i.e., a twist in the supersymmetry generators on the worldsheet that renders the theory independent of the metric on the target space. The topological twist on a worldsheet with $\mathcal{N}=(2,2)$ supersymmetry comes in the form of an A- or a B-twist, which will be individually analysed. In Section 2.3 we will review the application of localisation to a supersymmetric gauge theories in two and three dimensional curved spaces with 4 supercharges. We will discuss the form of the supersymmetric Lagrangian and state the main results for the partition function for the two dimensional case. We will postpone the localisation results for three dimensions to Chapter 2 where they will be explicitly employed.

In Chapter 3 we will focus on certain 2d $\mathcal{N}=(2,2)$ gauge theories called gauged linear sigma models [22]. In Section 3.1 we will explain the importance of these gauge theories for a study of target spaces relevant for string theory. We will underline that these gauge theories flow in the infrared to $\mathcal{N}=(2,2)$ superconformal field theories which can made to take the form of interesting target space geometries. In Section 3.2 we will review the results of applying localisation to these gauge theories and specifically state the form of the localised path integral for certain correlators [28, 29]. Sections 3.3 and 3.4 are based on the original work of the author [30] which explores various relations among the correlators and their pertinence as operators governing a moduli-dependent ground state in a Hilbert space interpretation and as generators of a differential ideal in the ring of correlators. For Calabi-Yau target spaces we will interpret the relations in a differential representation as the well-known Picard-Fuchs operators. For a Calabi-Yau threefolds with one Kähler parameter we will give universal expressions for the Picard-Fuchs operators entirely in terms of correlators, but the interested reader is invited to look at similar results for two parameter threefolds as well as one parameter fourfolds in [30]. Finally, in Section 3.5 we will give an an elementary overview of some of the mathematics underlying the A-model of string theory, whose physics is intimately related to most of the results discussed in this chapter. In particular, we will define moduli spaces of stable maps, genus zero Gromov-Witten invariants and Givental's cohomological I-function and review the explicit connection of the latter to gauged linear sigma model correlators. The derived correlators relations will then be shown to annihilate the cohomological I-function in the differential operator representation.

The object of interest in Chapter 4 will be 3d $\mathcal{N} = 2$ gauge theories. Starting with a Lagrangian description, we will emphasise the appearance of Chern-Simons terms which become important for later results. On shoulders of several important works on such theories which will be cited in the chapter, in Section 4.2 we will compute the localised expression for the partition function of $\mathcal{N} = 2$ theories on a twisted solid torus $D^2 \times_q S^1$ with a Grassmannian target space and extract the permutation equivariant I-function of quantum K-theory [31]. The content of Section 4.3 is based on the original work [32]. We first derive the algebra of Wilson line operators on the 3d geometry and show that for specific Chern-Simons levels this algebra is isomorphic to the quantum K-theory ring of Schubert structure sheaves on the target space or to the Verlinde algebra for unitary groups, which itself is isomorphic to quantum cohomology on the Grassmannian. We also compute difference equations annihilating the K-theoretic I-function, which can be understood as a lift of the differential equation annihilating the cohomological I-function. Finally, at the end of Section 4.3 we take the 2d limits of the partition function, the I-function and the difference equations obtained from 3d gauge theory techniques and find agreement with their 2d counterparts.

We conclude with a summary and a perspective on the several outlooks for the direction of research embarked on in this thesis in Chapter 5. The Appendix A collects standard results on Grassmannians, including important characteristic classes and generators of quantum cohomology ring.

CHAPTER 2

Localisation of Gauge Theories

In the recent years supersymmetric gauge theories have become a powerful probe for *quantum geometries*, which is a commonly-used term for distinguishing the geometries arising in string theory from those that arise in point-particle physics. They have certain advantages over the original worldsheet approach to studying target spaces which will be highlighted in a later section. Before delving into the anatomy of these gauge theories, we will summarise the powerful technique of localisation which was first conceptualised to tame partition functions of non-linear sigma models by deforming them into topological theories which are simpler by virtue of their sole dependence on moduli spaces of the target space [23, 33]. This idea was then extended to supersymmetric gauge theories and has made possible an exact calculation of observables in these theories [24, 27]. The theories of interest to us are a subset of these gauge theories and ramifications of localisation computations therein form a large part of this thesis.

In Section 2.1 we will sketch the central idea behind localisation and how it enables an exact computation of infinite dimensional path integrals. In Section 2.2 we will explain the application of localisation to non-linear sigma models and motivate the topological twisting of the supersymmetry algebra on the worldsheet, leading to the A- and B-model. The first two sections follow the work [33] closely. In Section 2.3 we will review the application of localisation to supersymmetric gauge theories of dimensions two and three with four supercharges [34–39], which will be the key players of the subsequent chapters.

2.1 The Main Idea

The central idea behind localisation is to capitalise on the symmetry of a theory to reduce the physical configuration space to be considered for a computation. It dates back to a theorem of Lefschetz [40] to calculate the number of fixed points, i.e., points which are mapped to themselves, of a mapping of a space to itself. This foundational fixed point theorem has several generalisations, most famously the Atiyah-Bott fixed point theorem which proves the integral of a closed form over a compact manifold to be a certain discrete sum over the fixed points of a U(1) symmetry of the manifold [41–43].

In physics, localisation becomes a powerful tool to exactly compute path integrals of supersymmetric theories. This is because under fitting circumstances the supersymmetry generator can be made globally nilpotent — commonly referred to as a BRST generator[1] in this context —

[1] This terminology is borrowed from the BRST formalism of quantisation of non-Abelian gauge theories, for a pedagogical introduction see [4].

and can be utilised to localise a path integral in the following way. Given a theory with a BRST group G action and a configuration space $\mathcal{C}$ of all paths, the physical domain of integration can be reduced to those paths that are distinct up to all possible G-transformations, i.e.,

$$\int_{\mathcal{C}} e^{iS}\mathcal{O} = \mathrm{vol}(G) \int_{\mathcal{C}/G} e^{iS}\mathcal{O} \ .$$

Here S stands for the action of the theory and $\mathcal{O}$ for a product of operator insertions. The volume of G denoted by $\mathrm{vol}(G)$ quantifies the number of paths equivalent up to a G-transformation. For instance, when $G = \mathrm{U}(1)$, $\mathrm{vol}(G) = \int_0^{2\pi} d\phi = 2\pi$. Note that in the Fadeev-Popov method of quantum field theory we only consider the symmetry-inequivalent factor by dividing the total path integral by the volume of the symmetry group. This is so because the symmetry in question is a gauge symmetry which signals a redundancy in the system, as opposed to the BRST-symmetry in question for localisation.

Since the generator of a BRST symmetry group G is nilpotent, the infinitesimal variable of G is given by Grassmann variable θ and,

$$\mathrm{vol}(G) = \int d\theta(1) = 0 \ ,$$

because integration and differentiation work equivalently for Grassmann variables. This implies that for a theory with a BRST symmetry all path integrals will always identically vanish. However, if the group G has fixed points in the configuration space, the extraction of volume factor in the path integral is incorrect as that was done under the assumption that all distinct paths appear with the same weight given by $\mathrm{vol}(G)$. The configuration space $\mathcal{C}$ can be split in to a 'smooth' space $\mathcal{C}'$, i.e., one containing no fixed points, and a set of all fixed points. A path integral over $\mathcal{C}'$ will identically vanish due to the argument above and the residual path integral will amount to a sum over the set of all the fixed points of G. The set of fixed points of the symmetry may not necessarily be discrete. The form of the BRST symmetries we will encounter in the forthcoming sections will have a finite dimensional locus of fixed points in the configuration space. In other words, the path integral will *localise* to a finite dimensional integral and thus become exactly calculable.

2.2 Localisation in Non-Linear Sigma Models

In this section we will describe the theory in which the concept of localisation first found a physical application, the non linear sigma model [23]. The results reviewed in this section will not be called upon in the later chapters presenting the original work of the author, and can be skipped if the reader so wishes.

The core idea behind localisation is that infinite dimensional integrals can be made to localise on to a locus of finite dimensional integrals if the theory has a BRST symmetry. Recall that a non-linear sigma model is the space of maps from the worldsheet Σ to the target space,

$$\varphi : \Sigma \to \mathcal{M}^{1,3} \times X \ ,$$

where the target space is assumed to be a direct product of the observable 4d Minkowski space $\mathcal{M}^{1,3}$ and the internal compactification space X. We would like to focus on restriction of these maps to the internal space X in order to probe this space independently. In the following, target space will refer to the internal space X and not the total space $\widetilde{X}$. Additionally we would like to assign $\mathcal{N}=(2,2)$ supersymmetry to the worldsheet as our primary focus would be type II theories. As briefly explained in the introduction, in order to get type II supergravity spectrum in 10d, one must equip the worldsheet with $\mathcal{N}=(2,2)$ supersymmetry. The action of such an non-linear sigma model is given by,

$$S_{\text{NLSM}} \sim \frac{1}{\alpha'} \int d^2z \; g_{\mu\nu} \; (\partial_z \phi^\mu \; \partial_{\bar{z}} \phi^\nu + i\psi^\mu_- \; D_z \psi^\nu_- + i\psi^\mu_+ \; D_{\bar{z}} \psi^\nu_+) + \cdots \; . \tag{2.1}$$

Here the measure $d^2z = dzd\bar{z}$ is given in terms of the so-called 'light-cone' coordinates $(z,\bar{z})$ on the worldsheet. These coordinates are linear combinations of the canonical coordinates on the worldsheet (σ,τ), with σ parametrising the position on the string and τ the propagation of this string in time. The metric on the worldsheet has been gauge fixed to the Minkowski metric $\eta_{\alpha\beta}$ using the Weyl symmetry, with which the $(z,\bar{z})$ indices have been contracted. The target space metric $g_{\mu\nu}$ captures the curvature of the target space. The fields ϕ are scalars on the worldsheet and can be understood as the coordinates of the target space and thus are indexed by μ, ν. The fields ψ_+ and ψ_- are left- and right-handed fermionic superpartners of the scalars ϕ on the worldsheet and vectors on the target space. The operators D_z and $D_{\bar{z}}$ are pull-backs of the partial derivatives acting on elements of the tangent bundle TX of X. The ellipsis $\cdots$ in the action stand for the contributions from other couplings to which the string can be coupled. For instance, from the (NS,NS) sector these terms would correspond to coupling of the string to the antisymmetric $b-$field and the dilaton.

At the outset this appears like a theory with $\mathcal{N}=(1,1)$ supersymmetry on the worldsheet, however a duplicity in supercharges can be explained by noting that for a complex target space X, the tangent bundle of X splits as $TX = T^{(1,0)}X \oplus T^{(0,1)}$, implying a doubling of supercharges corresponding to the holomorphic and anti-holomorphic sectors, respectively. The worldsheet fermions can be projected on to either of these bundles and thus the action (2.1) can be written as,

$$S_{\text{NLSM}} \sim \frac{1}{\alpha'} \int d^2z \; g_{\mu\nu} \; \partial_z \phi^\mu \; \partial_{\bar{z}} \phi^\nu + ig_{\bar{i}j} (\psi^{\bar{i}}_- \; D_z \psi^j_- + \psi^{\bar{i}}_+ \; D_{\bar{z}} \psi^j_+) + \cdots \; . \tag{2.2}$$

The idea behind a *topological twist* is to note that in terms of the canonical bundle [2] K on Σ and the tangent bundle on X the $\psi^{\bar{i}}_+ \in \Gamma(K^{1/2} \otimes \varphi^*(T^{(0,1)}X))$ and $\psi^j_+ \in \Gamma(K^{1/2} \otimes \varphi^*(T^{(1,0)}X))$. The fermion kinetic term in the Lagrangian is left unchanged with deforming the fermions such that they are sections of slightly different bundles,

$$\begin{aligned} \psi^{\bar{i}}_+ \; D_{\bar{z}} \; \psi^j_+ \; &\in \; \Gamma(K^{1/2} \otimes \varphi^*(T^{(0,1)}X)) \otimes \Gamma(K^{-1}) \; \otimes \; \Gamma(K^{1/2} \otimes \varphi^*(T^{(1,0)}X)) \\ &\xrightarrow{\text{def.}} \; \Gamma(K \otimes \varphi^*(T^{(0,1)}X)) \otimes \; \Gamma(K^{-1}) \; \otimes \; \Gamma(\varphi^*(T^{(1,0)}X)) \; . \end{aligned}$$

This deformation is termed as a '+' twist. One can equally well fashion a '−' twist by taking $\psi^{\bar{i}}_+ \in \Gamma(\varphi^*(T^{(0,1)}X))$ and $\psi^j_+ \in \Gamma(K \otimes \varphi^*(T^{(1,0)}X))$. One can similarly deform the $\psi_- D_z \psi_-$ term to a '+' and '−' twisted term respectively. By doing such a deformation the Lagrangian remains changed however the four infinitesimal parameters governing the each supersymmetry transformation that were previously spinors, i.e., sections of the square-root of the canonical

[2] For a complex manifold X with $\dim_{\mathbb{C}}(X) = n$, the tangent bundle has the decomposition $TX = T^{(1,0)}X \oplus T^{(0,1)}X$ into holomorphic and anti-holomorphic parts. The k^{th} exterior power of the cotangent bundle is a k-form. The canonical bundle consists of $(n,0)$ forms and the anti-canonical line bundle of $(0,n)$ forms.

bundle, are now split into scalars and sections of inverse-canonical bundles. Heuristically speaking we have split the product of two spinors into a vector and a scalar.

There are two combinations of the twists which are relevant for the following discussion, namely the '$(+,-)$' or the A-twist and '$(-,-)$' or the B-twist. The first entry of $(\cdot,\cdot)$ denotes the sign of the twist on the ψ_+ fermion and the second entry denotes the sign of the twist on ψ_- fermion. The other two twists '$(-,+)$' and '$(+,+)$' are isomorphic, up to a reversal of the complex structure of the target space, to the A- and B-twists, respectively. One can also consider the so-called *half twist* which, as the name suggests, twists either one of the ψ_+ or ψ_- fermions. The half twist is interesting in its own right however we will not study it in this thesis.

We will now discuss the ramifications of the A- and B-twist. Before delving into each of these individually, let us first discuss their commonalities. In both theories, half the supersymmetry parameters are scalars and can be set to constant values while the other half which are sections of the inverse-canonical bundles can be set to zero. By this procedure the existing supersymmetries are globally defined on the Riemann surface, and hence become BRST symmetries of the theory which can be deployed to localise the path integrals.

Let the generator of this BRST symmetry be denoted by $\mathcal{Q}_{A/B}$ for the A- and B-twist, respectively. The essence of doing the topological twist now becomes evident. The computation of the path integral, with possible operator insertions, can be simplified by considering the path integral over the BRST-inequivalent contributions and multiplying it by the volume of the symmetry group. As discussed before, the computation of the path integral localises to the fixed point locus of the BRST symmetry.

The fixed point locus differs for the A- and B-model given their distinct BRST symmetry generators, however for both the A- and B-model the specific choice of supersymmetry parameters leads to a simplification where the non-linear sigma model Lagrangian can be split into a BRST-exact term and a model dependent term. That is,

$$\mathcal{L}^{A(B)}_{\text{NLSM}} = \{\mathcal{Q}_{A(B)}, V_{A(B)}\} + \mathcal{L}_{A(B)} \ . \tag{2.3}$$

The fixed point locus over which the path integral is to be computed in each of the model in addition in to the simplification of the Lagrangian above goes a long way in the exact calculation of the partition function.

Let us now discuss the specificities of the A- and B-model.

2.2.1 A-Model

Because of the $(+,-)$ twist in the A-model, the worldsheet fermions are now sections of the following bundles,

$$\begin{aligned}
\psi^i_+ &\in \Gamma(\varphi^*(T^{(1,0)}X)) \quad ; \quad \psi^{\bar{i}}_+ \in \Gamma(K \otimes \varphi^*(T^{(0,1)}X)) \quad (+ \text{ in } (+,-)) \\
\psi^{\bar{i}}_- &\in \Gamma(\varphi^*(T^{(0,1)}X)) \quad ; \quad \psi^i_- \in \Gamma(\overline{K} \otimes \varphi^*(T^{(1,0)}X)) \quad (- \text{ in } (+,-)) \ .
\end{aligned}$$

The infinitesimal parameters accompanying ψ^i_+ and $\psi^{\bar{i}}_-$ in the supersymmetry transformation laws are thus scalars and those accompanying $\psi^{\bar{i}}_+$ and ψ^i_- are sections of the inverses of canonical and anti-canonical bundles, i.e., objects of the form a^z and $a^{\bar{z}}$, respectively. As discussed previously, the former can be set to constant values whereas the latter to zero. The existing scalars parameterise a BRST symmetry whose generator we denote as $\mathcal{Q}_A$. The fixed points of this symmetry can be found by looking at the transformation laws of all the fields and setting them to zero. In this case they happen to lie along the locus of holomorphic curves from Σ to X, i.e.,

$$\partial_{\bar{z}}\phi^i = 0 \;\;;\;\; \partial_z\phi^{\bar{i}} = 0 \;, \tag{2.4}$$

to which the previously infinite dimensional measure of the partition function localises. Recall that the ϕ's are coordinates of the target space and hence constraints on them imply constraints on the maps from the worldsheet to the target space.

Also the action (2.2) can be rewritten for the A-model as,

$$S^{\mathrm{A}}_{\mathrm{NLSM}} \sim \frac{1}{\alpha'}\int d^2z \;\left(\{\mathcal{Q}_A, V_A\} + \varphi^* K\right) \;. \tag{2.5}$$

Here $V_A = ig_{\bar{i}j}\left(\psi^{\bar{i}}_+\partial_{\bar{z}}\phi^j + \partial_z\phi^{\bar{i}}\psi^j_-\right)$ and φ^*K is the pullback of the Kähler form K on the target space, $K = -ig_{i\bar{j}}\; dz^i \wedge dz^{\bar{j}}$. We have simplified the Lagrangian to a sum of a term dependent only on the Kähler form and a term that is BRST-exact.

2.2.2 B-Model

The $(-,-)$ twist of the B-model leads the worldsheet fermions to be the sections of the following bundles,

$$\begin{aligned} &\psi^i_+ \in \Gamma(K\otimes\varphi^*(T^{(1,0)}X)) \;\;;\;\; \psi^{\bar{i}}_+ \in \Gamma(\varphi^*(T^{(0,1)}X)) \quad (\text{ First ` } - \text{ ' in } (-,-)) \\ &\psi^{\bar{i}}_- \in \Gamma(\varphi^*(T^{(0,1)}X)) \;\;;\;\; \psi^i_- \in \Gamma(\overline{K}\otimes\varphi^*(T^{(1,0)}X)) \quad (\text{ Second ` } - \text{ ' in } (-,-)) \;. \end{aligned}$$

In this case the infinitesimal parameters accompanying $\psi^{\bar{i}}_+$ and $\psi^{\bar{i}}_-$ in the supersymmetry transformation laws are scalars and those accompanying ψ^i_+ and ψ^i_- are sections of the inverses of canonical and anti-canonical bundles. Similar to the case of the A-model, the scalar parameters can be set to constant values whereas the parameters which are sections of inverse anti-canonical bundles can be set to zero. The surviving supersymmetry becomes a globally defined BRST symmetry of theory and whose generator we term $\mathcal{Q}_B$. The fixed points of this symmetry are found to lie along constant maps from the worldsheet Σ to the target space X, i.e.,

$$d\phi^i = 0 \;. \tag{2.6}$$

The B-model action also simplifies to the form,

$$S^{\mathrm{B}}_{\mathrm{NLSM}} \sim \frac{1}{\alpha'}\int d^2z \;\left(\{\mathcal{Q}_B, V_B\} + W\right) \;, \tag{2.7}$$

as predicted in (2.3). Here $V_B = ig_{i\bar{j}}\left(\psi^i_+\partial_{\bar{z}}\phi^{\bar{j}} + \partial_z\phi^{\bar{j}}\psi^i_-\right)$ and W is a term that depends only on the complex structure of the target space.

Computing the path integral with the (2.5) and (2.7) for the A and B-model, we note that the $\mathcal{Q}$-exact term can be evaluated in the limit where $1/\alpha'$ is very large and the only contribution to the integral are from the saddle points of the $\{\mathcal{Q}, V\}$ term. This is akin to taking the classical limit $\hbar \to 0$ in quantum mechanics and requiring that the Euler-Lagrange equations of motion be exactly satisfied, as opposed to taking the path integral over all configurations in the quantum case. For both A- and B-models, the saddle points of the $\mathcal{Q}$-exact terms coincide with the fixed point loci of the symmetry transformation, as one can quickly check by requiring $V_{A(B)}$ to vanish.

Thus the saddle points of the action in the large volume (or small α') limit match the localised configuration space due to the fixed point contributions. This may, however, not always be the case. As we will soon encounter in the case of supersymmetric localisation, the BRST-exact terms' saddle points can be made to further restrict the configuration space to aid the exact evaluation of the partition function. In principle, an addition of $\mathcal{Q}$-exact terms to the action should not alter the value of the partition function, or any other $\mathcal{Q}$-closed observable for that matter. We will underline that various ways of restricting the configuration space lead to the same result upon integration.

Finally, a brief explanation of the terminology 'topological'-twist for the above methodology of simplifying the non-linear sigma model in two ways is to note that for the A-model the final result after integration over the localised configuration space depends only the Kähler structure of the target space whereas for the B-model the result depends only on the complex structure of the target space X. Neither of these models depend on the complex structure of the worldsheet Σ or the precise metric $g_{\mu\nu}$ on X. The topologically twisted models have the advantage of certain observables, that are related to untwisted physical model, being exactly computable. Additionally mirror symmetry postulates that A-model on X is equivalent to B-model on the mirror $\widetilde{X}$. This connection can be deployed to compute the complicated instanton sums of the A-theory by instead doing the simpler 'classical' computation in the B-theory in the large volume limit on the mirror manifold.

2.3 Localisation in Supersymmetric Gauge Theories

In the previous section we reviewed localisation in the context of non-linear sigma models to calculate partition functions and certain observables exactly. This was indeed the subject where idea behind localisation, which dates back to the Lefschetz and Atiyah-Bott fixed point theorems of topology, found its first application in physics. For non-linear sigma models the topological twist served the purpose of constructing a globally defined BRST symmetry, the fixed points of which the configuration space localised to. The twist of the superconformal algebra of the worldsheet was necessary because the infinitesimal supersymmetry parameters are do not have global sections for a general Riemann surface of genus g.

In [24] localisation found an application in computing path integrals of supersymmetric Yang-Mills theories on S^4. Applying supersymmetric localisation to gauge theories on curved spaces without the aid of a topological twist, to maintain the physicality of the theory, is not without complications. In contrast to supersymmetry on flat space, where the infinitesimal supersymmetry parameters are covariantly constant spinors, defining sections of the inverse of square root canonical bundle must be handled with care on curved spaces. The early works on application of localisation to the realm of gauge theories dealt with theories that also exhibited superconformal symmetry [24, 36]. This was done by considering the action of the generators of the superconformal algebra on the fields in the theory with the parameters being required to specify conformal Killing equations. The work in [44] developed a general methodology to put supersymmetry on curved spaces focussing on 4d manifolds with four supercharges. The general idea therein is to couple the corresponding supersymmetric theory in flat space to the supergravity multiplet, which consists of the graviton, gravitino and two auxiliary fields, and fix the metric to that of the desired curved space. Following that the supersymmetry can be made 'rigid', i.e., decoupled to the fluctuations of the gravitational field, by taking the Planck mass to infinity. It must be highlighted that developing techniques to equip curved spaces with supersymmetry is an indispensable task in the localisation project as infinite dimensional integrals such as the partition function are subject to an infra-red cutoff on compact spaces. This infrared cutoff along with the aid of a BRST symmetry makes such infinite dimensional integrals exactly calculable. For a detailed review of the localisation project in supersymmetric gauge theories of various dimensions see [26].

The ability to compute partition functions and special observables exactly using localisation trickled soon after to supersymmetric gauge theories in lower dimensions. In [38], the results of localisation of superconformal Chern-Simons theories on S^3 with four supercharges was extended to theories without conformal symmetry. In [34] and [35], these results were extended to $\mathcal{N} = (2,2)$ supersymmetric gauge theories on S^2.

Localisation results are also applicable to manifolds with boundaries as was done in [45] for $\mathcal{N} = 2$ gauge theories on $D^2 \times S^1$ and in [46,47] for $\mathcal{N} = (2,2)$ gauge theories on D^2. While boundary theories have the additional complexity from the choice of boundary conditions for various fields, most simply Dirichlet or Neumann boundary conditions and higher dimensional lifts thereof, for the theories stated above the D^2 provides a flat space on which to define a covariantly constant spinor thus eliminating the need of aforementioned curved space manipulations to consistently define supersymmetry.

An exact calculation of certain observables serves many useful purposes in gauge theory. They are playgrounds for worldsheet theories of non-linear sigma models, as we will delve into in later sections, thus providing a channel to study relevant string theoretic target space geometries. They provide tools to perform checks of Seiberg-like dualities [25,48,49] in various dimensions by comparing partition functions on conjectured dual theories. They have highly non-trivial connections with mathematical concepts such as quantum cohomology, quantum K-theory and elliptic cohomology to name a few. These reasons, amongst many others, provide a strong motivation to closely study supersymmetric gauge theories in the context of localisation techniques.

2.3.1 2d

In this subsection we will first review the specific gauge theories in 2d that will be our focus in the forthcoming sections. These are known as *gauged linear sigma models (GLSMs)* and are a product of the work of Witten [22] which established a relation between these gauge theories and non-linear sigma models. We will first describe the spectrum and Lagrangian of these theories and their connection to worldsheet theories, we will discuss localisation of these gauge theories to compute the partition function and discuss its significance.

Field Content and Lagrangian

We will start by considering an $\mathcal{N} = (2,2)$ gauge theory on a flat 2d space with the Euclidean metric. Since this theory has four supercharges, it can be obtained by starting with a 4d $\mathcal{N} = 1$ gauge theory and compactifying two dimensions. For a pedagogical introduction to $\mathcal{N} = 1$ gauge theories in 4d see [5,12]. The superfields of the 2d theory are chiral and vector multiplets obtained from the dimensional reduction of $\mathcal{N} = 1$ 4d chiral and vector multiplets, respectively.

The differential operators generating the supersymmetry algebra, $\mathcal{Q}_\pm$ and $\overline{\mathcal{Q}}_\pm$, can be used to the construct the usual operators $\mathcal{D}_\pm$ and $\overline{\mathcal{D}}_\pm$ which define the chiral field as a superfield that obeys $\overline{\mathcal{D}}_\pm \Phi = 0$. The chiral superfield consists of a complex scalar ϕ, fermions ψ_+ and ψ_-

of opposite chirality, and an auxiliary scalar F. In superfield notation,

$$\Phi(y^\pm, \theta, \bar{\theta}) = \phi(y^\pm) + \theta^\alpha \psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm) \ ,$$

where the the argument $y^\pm$ is a linear combination spacetime coordinate in the lightcone basis $x^\pm$ and Grassmann coordinates $\theta^\pm$ of the form $y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$. In a generic situation the chiral field is in a reducible representation $\mathcal{R}$ of the gauge group G of the theory.

The vector multiplet V is characterised by the reality condition, $V = V^\dagger$, and the fact that it is in the adjoint representation of G. We will always work in Wess-Zumino gauge where V consists of a vector v^μ, a complex scalar σ, 2-component fermions λ and $\bar{\lambda}$ which contain left and rigt handed components, and the auxiliary scalar field D.

In addition to the chiral and vector superfield there exists a twisted chiral field Σ in 2d which satisfies the condition $\overline{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0$. The field strength superfield of the vector multiplet V is a twisted chiral superfield, i.e., $\frac{1}{2}\{\overline{\mathcal{D}}_+, \mathcal{D}_-\} =: \Sigma$. Thus supersymmetric gauge theory in 2d is hinged on the dynamics of chiral and twisted chiral superfields. This terminology is analogous to chiral and twisted-chiral primaries in $\mathcal{N} = (2,2)$ superconformal algebra[3].

The Lagrangian of the gauged linear sigma model gets contribution from the kinetic term of the chiral supermultiplet with the vector multiplet V in the exponent being given by $V = V_a T^a_{\mathcal{R}}$ where $T^a_{\mathcal{R}}$ are the generators of G in the representation $\mathcal{R}$ defined by the chiral multiplet,

$$\mathcal{L}_{\text{chiral}} = \int d^2x d^4\theta \ \left(\overline{\Phi} e^V \Phi\right) \ , \tag{2.8}$$

the kinetic term of the vector supermultiplet,

$$\mathcal{L}_{\text{vector}} = \frac{1}{g_s^2} \int d^2x d^4\theta \ \text{tr}_{\text{adj.}} \left(\overline{\Sigma}\Sigma\right) \ , \tag{2.9}$$

where g_s is the gauge coupling and $\Sigma = \Sigma_a T^a$ with T_a's being the generators of G in the adjoint representation, and the superpotential,

$$\mathcal{L}_W = \int d^2x d\theta^+ d\theta^- \ W(\Phi)|_{\bar{\theta}_\pm = 0} + \ \text{h.c.} \quad , \tag{2.10}$$

which is a gauge invariant holomorphic function of the chiral fields Φ. The chiral multiplet can be assigned twisted masses $\mathfrak{m}$ by the shift of the twisted chiral scalar $\sigma \to \sigma + \mathfrak{m}$. The relevance of the twisted masses, which are essentially expectation values of the complex scalar σ_F corresponding to a background flavour symmetry F of the theory, will be explored further in Chapter 3. In addition to these terms if the the gauge group G is such that $U(1)^\ell \subset G$, then corresponding to each $U(1)_s$ factor there exists a Fayet-Iliopolous(FI) term for $\xi_s \in \mathbb{R}$,

$$\mathcal{L}_{\text{FI}} = -i\xi_s \int d^2x d^4\theta \ \text{tr}_s(V) = -i\xi_s \int d^2x \ \text{tr}_s(D) \ , \tag{2.11}$$

and a topological ϑ-term,

$$\mathcal{L}_{\text{top.}} = -i\frac{\vartheta_s}{2\pi} \int d^2x \ \text{tr}_s(F_{01}) \ , \tag{2.12}$$

where ϑ is a parameter periodic in 2π and $F_{01} = \partial_0 v_1 - \partial_1 v_0$ is the scalar component of the field

[3] For a pedagogical introduction of the $\mathcal{N} = (2,2)$ superconformal algebra see [11].

strength tensor. The $\mathrm{tr}_s(\ldots)$ in the Fayet-Iliopolous and ϑ-term is to be taken so as to pick up the component of the adjoint valued field (V, D or F_{01}) corresponding to the $U(1)_s$ factor in question. In fact, these terms can be neatly grouped together using the twisted chiral superfield because of the property, $\int d^2x d\theta^+ d\bar{\theta}^-\ \mathrm{tr}\Sigma \sim \int d^2x\ \mathrm{tr}(D - iF_{01})$, in the following way,

$$\mathcal{L}_{\mathrm{FI+top.}} = i\tau_s \int d^2x d\theta^+ d\bar{\theta}^- \Sigma_s + \mathrm{h.c.}\ , \tag{2.13}$$

with the complexified Fayet-Iliopolous parameter $\tau_s = i\xi_s + \frac{\vartheta_s}{2\pi}$. Thus in total there will be ℓ such complexified Fayet-Iliopolous terms for $U(1)^\ell \subset G$. The Σ_s denotes a trace that picks up the s^{th} $U(1)$ factor in Σ, i.e., for $\Sigma = \Sigma_\alpha T^\alpha$ where T^α are generators of G in the adjoint representation, this trace will retain only those Σ_α's corresponding to $U(1)_\alpha \subset G$. Equation (2.13) is a special form of twisted chiral superpotential term $\widetilde{W}(\Sigma)$, which is a term akin to superpotential for chiral fields. The parameter ξ is classically marginal parameter which renormalised due to 1-loop corrections to the D-field.

The theory also has a $U(1)_R$ symmetry. An R-symmetry is an outer automorphism of the supersymmetry algebra which commutes with the gauge symmetry generators. The left and right moving R-symmetries of the theory can be combined into vector and axial R-symmetries. We will return to the topic of R-symmetries and renormalisation of the Fayet-Iliopolous parameter in Chapter 3.

Localisation of Partition Function

We now review the application of supersymmetric localisation to gauged linear sigma models as was performed in [34] and [35]. The Lagrangian we have looked at so far has lived on the flat $\mathbb{R}^2$, however for the purpose of localisation this theory must be placed on a compact space. The 2d compact Riemann surface of choice will be the two-sphere S^2 for the remaining treatment.

For employing localisation without needing to topologically twist the theory requires a subset of spinors ϵ (where the chirality and holomorphicity indices have been supressed), that parameterise infinitesimal transformations under the supersymmetry generators $\mathcal{Q}$ to be globally defined. As explained in the previous section, a globally defined fermionic generator serves to define a BRST symmetry of the theory. With the BRST symmetry at hand the path integral localises to the fixed points of this theory. In order to do so one starts with the superconformal algebra of an $\mathcal{N} = (2, 2)$ theory on S^2 with spinors ϵ satisfying the most general conformal Killing spinor equations. Then precisely those solutions to these equations that preserve all but the conformal symmetry are considered. The Lagrangian must also be consistently deformed to be defined on the curved space. This can be done by using the techniques of [44] as summarised previously. These steps suffice in constructing a supersymmetric gauge theory on a curved space. Under a specific choice of supercharge $\mathcal{Q}$, the fixed point equations schematically read,

$$\delta_{\mathcal{Q}}(\mathrm{fermions}) = 0\ . \tag{2.14}$$

We only consider transformations of fermionic fields as it is these equations that assign expectation values to scalars of theory, thus maintaining Lorentz invariance.

Furthermore, we are free to deform the theory with $\mathcal{Q}$-exact terms as they are irrelevant for all path integral integral computations. This can be made evident by noting that for a deformation term $t\{\mathcal{Q}, V\}$, tuned by the parameter t, deforms the partition function Z as,

$$Z_{\text{def.}}(t) = \int [\mathcal{D}\phi][\mathcal{D}\psi]\ e^{S+t\{\mathcal{Q},V\}}\ . \tag{2.15}$$

Differentiating with t and using that the action is $\mathcal{Q}$-closed we conclude,

$$\frac{dZ_{\text{def.}}(t)}{dt} = \int [\mathcal{D}\phi][\mathcal{D}\psi]\ \{\mathcal{Q}, V\}\ e^{S+t\{\mathcal{Q},V\}} = \int [\mathcal{D}\phi][\mathcal{D}\psi]\ \{\mathcal{Q}, V\ e^{S+t\{\mathcal{Q},V\}}\} = 0\ , \tag{2.16}$$

i.e., the partition function is insensitive to fluctuations in t. This implies,

$$Z = Z_{\text{def.}}(0) = Z_{\text{def.}}(\infty)\ ,$$

i.e., we are free to take the limit $t \to \infty$ to compute the original partition function Z. In this limit the partition function further localises to the saddle points of the deformation term $\{\mathcal{Q}, V\}$, which in the non-generic case will not exactly coincide with the fixed points of the BRST symmetry and thus prove to simplify the calculation further. This tool will be deployed in the computation of the partition function.

Specifically the $\mathcal{Q}$-exact deformation that is the most useful for this computation is,

$$\{\mathcal{Q}, V\} = \mathcal{L}_{\text{chiral}} + \mathcal{L}_{\text{vector}}\ , \tag{2.17}$$

which is only possible because the vector and chiral multiplet contributions to the Lagrangian happen to already be $\mathcal{Q}$-exact under the choice of supercharge.

The fixed point equation (2.14) yields a system of equations that interpolates between BPS[4] vortex and anti-vortex solutions at the north and nouth pole of the S^2, respectively. With the appropriate choice of deformation terms (2.17) the vortex and anti-vortex solutions are rendered non-supersymmetric, i.e., not on the saddle points of the deformation term. This leads to the localisation of the path integral onto the so-called *Coulomb Branch* locus. With a different choice of the deformation term it is also possible to selectively take only the vortex and anti-vortex solutions into account and yield a different formula for the partition function. This representation of the localised path integral is known as *Higgs Branch* localisation. The Coulomb and Higgs branch loci must not be confused with the Higgs and Coulomb branch vacua of the supersymmetric gauge theory, although this nomenclature is motivated from the supersymmetric vacua picture. Heuristically this can be explained by noting that the specific scalar fields, corresponding to the vector and chiral multiplet, that acquire expectation values in each of the two loci. Since ultimately adding the distinct deformation terms dies not physically alter the theory, the values of the localised partition function along each of the two loci should, and does, match.

To summarise, the two consecutive steps for localising the infinite dimensional partition function are:

- Choice of the supercharge $\mathcal{Q}$ that becomes the BRST generator whose fixed points the path integral localises to ;
- Choice of the $\mathcal{Q}$-exact deformation $\{\mathcal{Q}, V\}$ to the action, the saddle points of which (possibly) further localise the path integral.

[4] BPS states in a supersymmetric theory refer to the states that satisfy the BPS bound and consequently preserve a fraction of the bulk supersymmetry. For an exposition, see [5].

In effect the total partition function reduces to an integration over the intersection of the fixed point and saddle point locus with the integrand being a product of the classical value of the original integrand at this locus, termed Z_{cl}, and a measure factor from the quadratic fluctuations around this locus commonly known as the one-loop determinant $Z_{\text{1-loop}}$. For an $\mathcal{N} = (2,2)$ gauge theory with R-symmetry and gauge group G defined on a S^2 localised to the the Coulomb branch locus discussed previously, this partition function is given by,

$$Z_{S^2}(\vec{k},\tau) = \frac{1}{|\mathcal{W}_G|} \sum_{\vec{k}\in\gamma_{\mathfrak{m}}} \int \left(\prod_{a=1}^{\text{rk}(G)} \frac{d\sigma_a}{2\pi i} \right) Z_{\text{cl}}(\sigma,\vec{k},\tau)\ Z_{\text{1-loop}}(\sigma,\vec{k})\ , \tag{2.18}$$

where $|\mathcal{W}_G|$ is the order of the Weyl group of G, σ_a correspond to $\text{rk}(G)$ scalars of the vector multiplet which is now in the adjoint representation of $\text{U}(1)^{\text{rk}(G)} \subset G$, i.e., the maximal torus that G breaks down to on the Coulomb branch. This phenomenon of the gauge group breaking down to its Cartan group will be discussed in detail in Chapter 3. The $\gamma_{\mathfrak{m}} \simeq \mathbb{Z}^{\text{rk}(G)} \subset \mathfrak{h}$ is the magnetic charge lattice of G with $\mathfrak{h}$ denoting the Cartan subalgebra of G. The classical and 1-loop factors inside the partition function are explicitly given by,

$$\begin{aligned} Z_{\text{cl}}(\sigma,\vec{k},\tau) &= e^{-4\pi i\langle \xi, \text{tr}(\sigma)\rangle - i\langle\vartheta,\vec{k}\rangle}\ , \\ Z_{\text{1-loop}}(\sigma,\vec{k}) &= Z_{\text{chiral}}(\sigma,\vec{k}) \cdot Z_{\text{vector}}(\sigma,\vec{k})\ , \end{aligned} \tag{2.19}$$

where Z_{chiral} and Z_{vector} are the contributions by the chiral and vector multiplet Lagrangians to the 1-loop determinant given by,

$$\begin{aligned} Z_{\text{chiral}}(\sigma,\vec{k}) &= \prod_{\rho_i\in\text{Irrep}(\rho)} \prod_{\beta\in w(\rho_j)} \frac{\Gamma\left(\frac{\mathfrak{q}_{\text{j}}}{2} - i\langle\beta,\sigma\rangle - \frac{1}{2}\langle\beta,\vec{k}\rangle\right)}{\Gamma\left(1 - \frac{\mathfrak{q}_{\text{j}}}{2} + i\langle\beta,\sigma\rangle - \frac{1}{2}\langle\beta,\vec{k}\rangle\right)}\ , \\ Z_{\text{vector}}(\sigma,\vec{k}) &= \prod_{\alpha>0} \left(\frac{\langle\alpha,\vec{k}\rangle^2}{4} + \langle\alpha,\sigma\rangle^2 \right)\ . \end{aligned} \tag{2.20}$$

Here the finite product in Z_{chiral} runs over the set of weights $w(\rho)$ of all the available irreducible representations $\text{Irrep}(\rho)$ of the gauge group G present in the chiral matter spectrum. The $\mathfrak{q}_j$ are the R-charge assignments of the chiral multiplet containing the irreducible representation ρ_j. The inner product $\langle\cdot,\cdot\rangle$ follows from the canonical pairing of the elements in the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ of G. The finite product in Z_{vector} runs over the set of positive roots α of the gauge group G.

One application of the formula (2.18) for the partition function along the Coulomb branch locus is that it can be used to verify conjectured dual gauged linear sigma models, for instance Seiberg-like dual theories [48, 49], by comparing their partition functions. This was done dual gauge theories with unitary gauge groups in [34]. We will return to other applications of the localisation technique on these gauge theories in the next chapter where we will in particular focus on computing certain correlators exactly its consequences.

2.3.2 3d

The focus of this subsection is to summarise the application of localisation to supersymmetric gauge theories on 3d manifolds. As with the case of the 2d theories, we will postpone the discussion of the relevance of these gauge theories from a string theory point of a view for later, see Chapter 4. Although it must be stated that 3d gauge theories are interesting in their own right due to new phenomena such as the appearance of Chern-Simons terms and connection to knot theory [50].

Field Content and Lagrangian

In the presence of a gauge symmetry there is a new term in the Lagrangian on a 3d manifold M known as the Chern-Simons(CS) term which adds interesting structure to the theory. For a non-supersymmetric theory with a gauge group G and its vector potential A, the CS term in the Lagrangian takes the form,

$$\mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi}\int\limits_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \ , \tag{2.21}$$

where the constant k is known as the Chern-Simons level of the theory. While the term Lagrangian (2.21) is classically gauge invariant, upon quantisation gauge covariance is only maintained for the quantisation condition $k \in \mathbb{Z}$. This is because the third homotopy group $\pi_3(G)$ is non-trivial for a compact Lie group G leading to disconnected components corresponding to non-zero winding numbers. For a non-zero winding number the Lagrangian shifts by constant multiple of $2\pi k$ which preserves the action $S \sim e^{i\mathcal{L}}$ for $k \in \mathbb{Z}$.

The superfield content of 3d $\mathcal{N} = 2$ theories can be obtained, as in case of 2d $\mathcal{N} = (2, 2)$ theories, by dimensional reduction from 4d $\mathcal{N} = 1$ theories since these are all theories with four supercharges [51, 52]. It consists of chiral multiplet Φ and vector multiplet V and as in case of 4d and 2d, the outer automorphism group of the supersymmetry algebra is $U(1)_R$ symmetry. In addition one can define a 'linear' multiplet Σ whose lowest component is is the real scalar in the vector multiplet corresponding to the component of the 4d vector multiplet gauge potential in the extra direction. This is achieved by defining $\Sigma := \epsilon^{\alpha\beta}\overline{D}_\alpha D_\beta V$, such that $D^2\Sigma = \overline{D}^2\Sigma = 0$, where D is usual differential operator on superspace. The matter and vector Lagrangian takes the form,

$$\mathcal{L}_{\text{chiral+vector}} = \int d^3x d^4\theta \, \left(\overline{\Phi} e^V \Phi + \frac{1}{g_s^2}\mathrm{tr}_{\mathrm{adj.}}(\Sigma^2)\right) \ . \tag{2.22}$$

Here in the first term the vector multiplet V in the exponent is given by $V = V_a T^a_{\mathcal{R}}$ where $T^a_{\mathcal{R}}$ are the generators of G in the representation $\mathcal{R}$ defined by the chiral multiplet. In the second term g_s is the gauge coupling and $\Sigma = \Sigma_a T^a$ with T_a's being the generators of G in the adjoint representation. There is also a superpotential contribution to the Lagrangian as in 4d and 2d,

$$\mathcal{L}_W = \int d^3x d^2\theta \ W(\Phi) + \ \text{h.c.} \ , \tag{2.23}$$

and Fayet-Iliopolous term ζ for each $U(1)$ factor in the gauge group G,

$$\mathcal{L}_{\mathrm{FI}} = i\zeta \int d^3x d^4\theta \ \mathrm{tr}(V) \ . \tag{2.24}$$

The supersymmetric Chern-Simons term for $G = \mathrm{U}(1)^r$ takes the form,

$$\mathcal{L}_{\mathrm{CS}} = \kappa_{ij} \int d^3x d^4\theta \ \mathrm{tr}\left(\Sigma_i V_j\right) \ , \tag{2.25}$$

where κ_{ij} is the Chern-Simons level. The indices i, j could either label the dynamical gauge fields or the background gauge fields corresponding to the flavour symmetries of the theory which include the R-symmetry. We also postpone the discussion of the non-Abelian Chern-Simons levels, the quantisation of Chern-Simons level in order to preserve gauge invariance, and the corrections to them by heavy fermions running a loop and the subsequent parity anomaly to Chapter 4.

Localisation

Historically the first implementation of localisation to a gauge theory in 4d [24] was followed by tackling superconformal Chern-Simons matter theories on S^3 with $\mathcal{N} = 2$ supersymmetry [36]. The authors of [36] computed the partition function and the expectation value of certain Wilson loop operators exactly. The work of [38, 39] extended this result to the case of supersymmetric gauge theory without conformal symmetry using the techniques of [44] of equipping curved spaces with Killing spinors and adding appropriate $\mathcal{O}(r^{-1})$ terms to the the Lagrangian, where r is the radius of the S^3. The supercharge chosen to be the BRST generator $\mathcal{Q}$ for localisation in these studies was such that the matter and vector multiplet Lagrangians were found to be $\mathcal{Q}$-exact as in the case of 2d, as we saw in the previous section. This implies that the partition function or other indices have no dependence on the gauge coupling g_s. The Chern-Simons and the Fayet-Iliopolous terms have non-zero values at the saddle points of the chosen supercharge and thus the resultant partition function depends on the corresponding parameters, i.e., the Chern-Simons level and the Fayet-Iliopolous parameter. The finite dimensional integral is over the Cartan subalgebra of the total group algebra $\mathfrak{g}$ as for the 2d case.

In [45] localisation was employed to compute a certain index to count the BPS operators of superconformal Chern-Simons matter theories on $S^1 \times S^2$, which could be employed to test the AdS$_4$/CFT$_3$ duality amongst other things. It was conjectured in [53] that the partition of functions of 3d $\mathcal{N} = 2$ CS theories on the the squashed sphere S^3_b and $S^1 \times S^2$ decomposes into a product of 'holomorphic blocks' which are partition functions of $\mathcal{N} = 2$ CS theories on $S^1 \times D^2$, i.e., the solid torus. This was one the motivations to compute the partition function on the $S^1 \times D^2$ using localisation techniques in [54] so as to find agreement with the holomorphic block conjecture. Actually both in [53] and [54], certain general twisted products of the geometries were considered where the S^2(or D^2) is fibered over the S^1 with a holonomy $\log(q)$. We will expand on the relevance of this twisting parameter in Chapter 4.

Since the method to localise 3d theories to compute the partition function exactly does not vary greatly compared to the method in 2d outlined before, besides the inclusion of CS terms which have a similar form as the Fayet-Iliopolous terms, we will not review it here again. We will come back to the results of [54] and utilise them to derive non-trivial connections to certain target space geometries. The broad connection of 2d and 3d supersymmetric theories to quantum stringy geometries will also be addressed in detail in the forthcoming chapters.

CHAPTER 3

2d Gauge Theories and Relations of Correlators

The $\mathcal{N} = (2,2)$ gauge theories in 2d that will be the focus of this chapter are known as Gauged Linear Sigma Models (GLSMs) [22]. In Section 2.3 of Chapter 2 we reviewed the field content and Lagrangian of such theories. In this chapter we begin in Section 3.1 by explaining their raison d'être, in particular how these theories are connected to non-linear sigma models (NLSMs) in string theory. This will shed some light on their pertinence as tools to probe quantum geometries relevant for string theory. In Section 3.2 we review the application of localisation techniques summarised in Chapter 2 to compute certain correlators of GLSMs exactly [28, 29]. This equips us to deal with the central content of the author's work [30] in Sections 3.3 and 3.4 where the previously highlighted results are utilised to compute universal relations between the correlators and extract therefrom certain characteristic differential operators corresponding to target spaces of the associated NLSMs. In Section 3.5 we review the explicit relation between the correlators and Givental's cohomology-valued I-function on the target space and employ it to interpret the relations of correlators as differential operators annihilating the I-function.

3.1 GLSM to NLSM

A gauged linear sigma model is a 2d supersymmetric theory whose spectrum is symmetric w.r.t a gauge group G which is generically a product of U(1) factors and semi-simple gauge groups H_j modulo a discrete normal group Γ, i.e.,

$$G = \left(\mathrm{U}(1)^{\ell} \times H_1 \times \ldots \times H_m\right) / \ \Gamma \ . \tag{3.1}$$

The spectrum consists of chiral mutiplets Φ_i in a representation $\mathcal{R}_i$ of G with $i = 1, \ldots, N$, a vector multiplet V in the adjoint representation of G. In addition, the theory is equipped with an R-symmetry of the supersymmetry algebra. The Lagrangian of this theory consists of a kinetic term of the chiral matter fields, a kinetic term of the vector superfield which can be expressed entirely in term of a twisted chiral multiplet Σ, a term for the superpotential W which is a holomorphic function of the chiral fields and a complexified Fayet-Iliopolous (FI) term composed of a real term and a topological ϑ term. The Lagrangian density, which we consistently refer to as the Lagrangian in what follows, is explicitly given by,

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{(\text{matter kin.+gauge kin.})} + \mathcal{L}_W + \mathcal{L}_{\text{FI},\vartheta} \\ &= \left(\int d^4\theta \left(\sum_{i=1}^{N} \overline{\Phi}_i e^V \Phi_i + \frac{1}{g_s^2}\text{tr}_{\text{adj.}}(\overline{\Sigma}\Sigma)\right)\right) + \left(\int d\theta^+ d\theta^- W(\Phi)|_{\bar{\theta}_\pm=0} + \text{ h.c.} \right) \\ &+ \sum_{i=1}^{\ell} \left(i\tau_i \int d\theta^+ d\bar{\theta}^- (\Sigma_i)|_{\theta^-=\bar{\theta}^+=0} + \text{h.c.}\right) \ .\end{aligned} \tag{3.2}$$

In first term, which is the kinetic term for the matter multiplet, the vector multiplet V in the exponent is given by $V = V_a T^a_{\mathcal{R}}$ where $T^a_{\mathcal{R}}$ are the generators of G in the representation $\mathcal{R}$ defined by the chiral multiplet. The parameter g_s is the gauge coupling constant. The Σ_i denotes a trace that picks up the i^{th} U(1) factor in Σ, i.e., for $\Sigma = \Sigma_a T^a$ where T^a are generators of G in the adjoint representation, this trace will retain only those Σ_a's corresponding to $\text{U}(1)_a \subset G$. The τ_i is the complexified Fayet-Iliopolous parameter, s.t., $\tau_i = i\xi_i + \frac{\vartheta_i}{2\pi}$ corresponding to the i^{th} U(1) factor. The last term corresponding to the complexified Fayet-Iliopolous coupling is a special case of the so-called twisted chiral superpotential $\widetilde{W}(\Sigma)$ term. As is evident, for the classical Lagrangian we need only consider a linear dependence on thew twisted chiral field Σ of the twisted superpotential. However we will observe that quantum correction to the classical theory induces a more interesting dependence on Σ.

3.1.1 Infrared Dynamics of GLSMs

We will now discuss three major aspects of gauged linear sigma models that will interest us, the anomalous R-symmetry, quantum corrections to the classical Lagrangian and finally the infrared dynamics of the theory which will highlight their connection to non-linear sigma model target spaces.

R-symmetry

The supersymmetry algebra has an outer automorphism group action under which the Grassmann variables $\theta^+, \theta^-, \bar{\theta}^+$ and $\bar{\theta}^-$ of the algebra transform non-trivially but the Lagrangian is invariant. This group action corresponds to a symmetry which is global but doesn't commute with supersymmetry transformations, which is unlike generic flavour symmetries in a supersymmetric theory. It is of course also possible to construct supersymmetric theories without an R-symmetry but for the gauge theories in question we require the existence of an R-symmetry for the reason that we wish these theories to flow to $\mathcal{N} = (2,2)$ superconformal field theories (SCFTs) in the infrared which necessarily have an R-symmetry generator in their algebra. This point will be fleshed out in the discussion on infrared dynamics.

Gauged linear sigma models can be constructed by dimensional reduction of $\mathcal{N} = 1$ supersymmetric theory in 4d which has a U(1) R-symmetry. For a 2d $\mathcal{N} = (2,2)$ theory where the left- and right-handed supercharges are independent this becomes a $\text{U}(1)_L \times \text{U}(1)_R$ symmetry corresponding to the left- and right-handed supercharge transformations, respectively. In particular the following $\text{U}(1)_L \times \text{U}(1)_R$ charges can be assigned to the Grassmann variables,

$$\rho(\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-) := ((0,1), (1,0), (0,-1), (-1,0)) \ , \tag{3.3}$$

where the left and right arguments of $(\cdot,\cdot)$ corresponds to the $\text{U}(1)_L$ and the $\text{U}(1)_R$ charge, respectively. Various components of a supermultiplet thus have R-charges so as to preserve the R-charge neutrality of the supermultiplet as a whole and that of the Lagrangian.

Quantisation of a supersymmetric theory renders the left- and right-handed R-symmetries anomalous. Their respective anomalies $\mathcal{A}$, for the gauge group defined in (3.1), are such that [28],

$$\mathcal{A}_L = -\mathcal{A}_R \sim \sum_{i=1}^{N} \mathrm{Tr}_{\mathcal{R}_i} F_{01} \sim \sum_{i=1}^{N} \sum_{j=1}^{\mathrm{rk}(G)} \rho^i_j \ , \tag{3.4}$$

where F_{01} is the adjoint-valued field strength tensor, ρ^i_j is the charge of the i^{th} matter field, of which there are N, under the j^{th} factor of the maximal torus of the gauge group G given by $\mathrm{U}(1)^{\mathrm{rk}(G)}$. The charge assignments of the chiral fields under the maximal torus can be given by considering the weights of the occurring irreducible representations. Each independent weight represents the charge under a $\mathrm{U}(1) \subset \mathrm{U}(1)^{\mathrm{rk}(G)}$ and the components of the weight representing the charges under the now-Abelianised chiral fields resulting from each irreducible representation. The charge assignment of the matter fields under the Abelianised gauge group will become more clear in Section 3.2, specifically the Table 3.2 summarises the spectrum of the Abelianised theory.

We can instead consider linear combinations, $J_V := J_L + J_R$ and $J_A := J_L - J_R$, of the left and right R-symmetry generators J_L and J_R. They are also known as the vector and axial R-symmetry generators, respectively. From the relation (3.4) it can be concluded that the vector R-symmetry group $\mathrm{U}(1)_V$ is non-anomalous while the anomaly resides in axial R-symmetry group $\mathrm{U}(1)_A$. To impose anomaly cancellation the condition,

$$\sum_{i=1}^{N} \sum_{j=1}^{\mathrm{rk}(G)} \rho^i_j = 0 \ , \tag{3.5}$$

on the gauge charges of the N matter fields must be ensured. We will later note that this is the criterion for conformal invariance in the infrared and thus the condition for obtaining a Calabi-Yau target manifold as the target space.

Additionally, it is evident from (3.2) and the R-charge assignments of the Grassmann variables that the superpotential must have a charge 2 under the $\mathrm{U}(1)_V$ symmetry to ensure R-invariance of the Lagrangian. We will return to this requirement in the discussion of the infrared dynamics.

Quantum Corrections and the Discriminant Locus

An essential advantage of any supersymmetric theory is the accompanying non-renormalisation theorems that protect certain quantities, for instance the superpotential, from getting renormalised [55,56]. The $\mathcal{N} = (2,2)$ gauged linear sigma models of interest to us are super-renormalisable, i.e., all potentially renormalisable quantities only get renormalised up to finite loop order. Specifically, the only term to receive a quantum correction that exhibits a divergence is the real Fayet-Iliopolous parameter ξ. This correction is only to 1-loop order, with all higher correction vanishing, which will be our focus as it will encode important information about the supersymmetric vacua of the theory.

The 1-loop correction to the real Fayet-Iliopolous parameter ξ_i comes from the expectation value of the D-field which it multiplies to in the action. For the gauge group defined in (3.1), there will be ℓ Fayet-Iliopolous parameters ξ_i of the theory. Since the D-field is an auxiliary field of the theory required for the supersymmetry algebra to also be closed off-shell, it can be solved for by the Euler-Lagrange equations of motion. In order to do that let's underline those terms in the Lagrangian (3.2), after integrating over all the Grassmann variables, that are dependent on the D-field,

$$\mathcal{L} \supset \frac{1}{2g_s^2} D_a D_a + D_a \sum_{i=1}^{N} \bar{\phi}_i T_i^a \phi_i - \sum_{j=1}^{\ell} \xi_j D_j \ , \tag{3.6}$$

where N are the total number of chiral fields and ℓ are number of U(1) factors in G. Recall that like all other fields in the vector multiplet, the D-field is adjoint valued, i.e., $D_{ij} = T_{ij}^a D_a$, where $i, j = 1, \ldots, \dim(G)$ and T_{ij}^a are the generators of G in the adjoint representation labelled by index $a = 1, \ldots \dim(G)$ which are normalised with respect to the usual Cartan-Killing form. In the second term the generators T_i^a are in the representation defined by the chiral field ϕ_i. In the last term, the D_i denotes a trace that picks up the i^{th} U(1) factor from $D_{ij} = T_{ij}^a D_a$, i.e., the coefficients of the U(1) generator.

Hence, in effect, there are two independent subsets of the Lagrangian to be extremised, those that stem from U(1) factors ($\mathcal{L}_{D_1}$) in G versus those that stem from semi-simple subgroups ($\mathcal{L}_{D_2}$) in G. They are given by,

$$\begin{aligned}
\mathcal{L}_{D_1} &= \sum_{j=1}^{\ell} \left(\frac{1}{2g_s^2} D_j D_j + D_j \sum_{i=1}^{N} \rho_j^i \ \bar{\phi}_i \phi_i - \xi_j D_j \right) \ , \\
\mathcal{L}_{D_2} &= \sum_{k=1}^{m} \left(\sum_{\alpha=1}^{\dim(H_k)} \left(\frac{1}{2g_s^2} D_\alpha^k D_\alpha^k + D_\alpha^k \sum_{i=1}^{N} \bar{\phi}_i T_{k,i}^\alpha \phi_i \right) \right) \ ,
\end{aligned} \tag{3.7}$$

where recall from (3.1) that $H_k \subset G$ is one semi-simple group in G and there are a total of m of them. Here ρ_j^i is the charge of the i^{th} chiral field under the j^{th} U(1)-factor in G. The generator $T_{k,i}^\alpha$ denotes generator of the k^{th} semi-simple subgroup H_k of G in the representation $\mathcal{R}_i$ defined by chiral field ϕ_i. From the first equation in (3.7), we get,

$$D_j = -g_s^2 \left(\sum_{i=1}^{N} (\rho_j^i \ \bar{\phi}_i \phi_i) - \xi_j \right) \ , \tag{3.8}$$

and from the second we get,

$$D_\alpha^k = -g_s^2 \left(\sum_{i=1}^{N} \bar{\phi}_i T_{k,i}^\alpha \phi_i \right) \ . \tag{3.9}$$

Now, since the Fayet-Iliopolous-terms only make an appearance for the 'Abelian components' of the D-term (3.8), those will be the relevant ones for the quantum corrections. In any case, the expectation value of the D-term for semi-simple gauge groups will vanish and thus not contribute. The 1-loop correction to this expectation value of the D-term would come from additional 'mass' terms generated for the chiral fields whose source will be discussed in the next subsection on the infrared dynamics of these theories. For this discussion it suffices to assume that these additionally generated masses are given by μ_i for the chiral field scalar ϕ_i, i.e., a term of the form $\mu_i \bar{\phi}_i \phi_i$ is generated for all chiral fields. Such a term would contribute to (3.8) to alter the expectation value of the D-term. The 1-loop integral diverges and can be regularised using Pauli-Villars technique by subtracting from a propagator with mass term with mass Λ_i.

This results in the 1-loop behaviour of the D-term being given by,

$$\langle D_j \rangle_{1-\text{loop}} \sim -g_s^2 \sum_{i=1}^{N} \rho_j^i \ \ln\left(\frac{\Lambda_i}{\mu_i}\right) \ . \tag{3.10}$$

As we will note shortly, in the infrared such a mass term is generated from the term in the potential V stemming from $\mathcal{L}_{\text{chiral}} = \overline{\Phi}_i e^V \Phi_i \supset \overline{\Phi}_i V^2 \Phi_i$ in the Lagrangian. There is an unfortunate degeneracy in the notation for the potential and the vector multiplet in the literature, however we will be explicit when using the symbol V. The mass term in the potential V is explicitly given by,

$$V \supset \sum_{i=1}^{N} \ \bar{\phi}_i \ \left\{\sigma(\rho^i)^\dagger, \sigma(\rho^i)\right\} \phi_i \ , \tag{3.11}$$

where $\sigma(\rho)$ denotes the canonical pairing between elements of the Lie algebra $\mathfrak{g}$ and the dual Lie algebra $\mathfrak{g}^*$, respectively . When σ acquires an expectation value this is becomes a mass term for ϕ_i, i.e.,

$$\mu_i = \left\{\sigma(\rho^i)^\dagger, \sigma(\rho^i)\right\} \ . \tag{3.12}$$

Now the correction to the D-term can be reinterpreted as a cutoff-dependent quantum correction in the real Fayet-Iliopolous parameter. Recall from Chapter 2 that the Fayet-Iliopolous term is a special case of a general twisted chiral superpotential term in the Lagrangian,

$$\mathcal{L}_{\widetilde{W}} = \int d\theta^+ d\bar{\theta}^- \widetilde{W}(\Sigma)|_{\theta^- = \bar{\theta}^+ = 0} + \text{h.c.} \ , \tag{3.13}$$

where the twisted superpotential $\widetilde{W}$ is linear in Σ, specifically,

$$\widetilde{W}(\Sigma) = \frac{i}{2} \sum_{i=1}^{\ell} \tau_i \ \Sigma_i \ . \tag{3.14}$$

The 1-loop correction to the Fayet-Iliopolous coupling can then be dressed as correction to the twisted superpotential which takes the form of a logarithmic dependence of Σ such that the new twisted superpotential reads,

$$\widetilde{W}(\Sigma) \sim \sum_{j=1}^{\ell} \frac{\Sigma_j}{2} \left(i\tau_j - \frac{1}{2\pi} \left(\sum_{i=1}^{N} \rho_j^i \ \ln\left[\frac{\sum_{b=1}^{\ell} \rho_b^i \Sigma_b}{\Lambda_i} \right] \right) \right) \ , \tag{3.15}$$

which clarifies how $\widetilde{W}$ deforms away from linearity. We will note that as the theory flows to the infrared, the complexified Fayet-Iliopolous parameters τ_j span the moduli space of supersymmetric vacua. This moduli space exhibits a singular locus, i.e., values of the Fayet-Iliopolous parameter at which the space becomes singular due to non-compact directions opening up. This singular locus can be solved for by noting that the contribution of the twisted superpotential to the potential V must vanish in the infrared,

$$\frac{\partial \widetilde{W}(\sigma)}{\partial \sigma} = 0 \qquad \Longrightarrow \qquad e^{2\pi i \tau_j} = \prod_{i=1}^{N} \left(\sum_{b=1}^{\ell} \rho_b^i \sigma_b \right)^{\rho_j^i} \ . \tag{3.16}$$

This equation on the moduli space of vacua is commonly known *discriminant locus*. In the work of [28] this result was elemental in proving that in correlation functions of A-twisted gauged linear sigma models the singular locus is the sole source of all singularities. Strictly speaking this derivation of the discriminant locus is applicable for models with non-Abelian gauge groups along the so-called Coulomb branch locus where the gauge group breaks to the maximal torus, i.e., $G \to \mathrm{U}(1)^{\mathrm{rk}G}$. For ease of calculation one can introduce auxiliary Fayet-Iliopolous parameters for all the newly generated U(1)'s and have the chiral fields be charged under the Abelianised gauge group. In order to rid the condition on $\widetilde{W}$ of the sliding scale Λ_i the condition,

$$\sum_{i=1}^{N} \sum_{j=1}^{\ell} \rho_j^i = 0 \ ,$$

has been employed. This condition coincides with the condition derived in (3.5) for superconformal invariance. As it happens, models for which this condition is not satisfied do not flow to a superconformal theory and do not exhibit a discriminant locus. In the examples of Section 3.3, a projective space and a Grassmannian manifold, this phenomenon of there being no discriminant locus will be illustrated explicitly.

Infrared Dynamics

Finally, we address explicitly how the gauged linear sigma models, whose properties we have studied so far, are interesting from the perspective of string theory. The defining ingredients of these models, i.e., the matter spectrum, the gauge group, R-charges and the superpotential, can be tuned such that in the renormalisation group flow to the infrared this gauge theory flows to a superconformal field theory (SCFT) with $\mathcal{N} = (2,2)$ supersymmetry. The family of SCFTs in the infrared are parametrised by the Fayet-Iliopolous parameter τ which is classically marginal and gets corrected to one-loop order. The effective value then serves as a marginal parameter that parametrises an SCFT for each distinct value that it takes. For special values of τ the corresponding SCFT corresponds to geometric spaces that arise as target spaces of non-linear sigma models. Not all SCFTs in the family of supersymmetric vacua enjoy a description as geometric spaces. For special values of τ they might take the form of other renowned SCFTs such as Landau-Ginzburg orbifolds, however, most generically a clear Lagrangian description of these theories evades us. We will note that (a priori) very distinct theories lie at various ends of the moduli space spanned by the Fayet-Iliopolous parameter τ corresponding to a certain gauged linear sigma model. Moreover there exist distinct models with moduli space of infrared fixed points that can be identified by relating the Fayet-Iliopolous parameters of the two theories. Such gauged linear sigma models are with the same infrared physics are said to be *dual* [49]. For this analysis, we will broadly refer to [22,48]. The idea of dualities of gauged linear sigma models is parallel to the electric-magnetic duality of Seiberg for $\mathcal{N} = 2$ theories in 4d [25]. Finally, we note that for models with a geometric target space in the infrared, the Fayet-Iliopolous parameter plays the role of the Kähler modulus of this target space and the family of infrared theories form the Kähler moduli space. This terminology does not coincide exactly with the usual Kähler modulus of a Kähler manifold because the Fayet-Iliopolous parameter can also be negative.

To observe the nature of the theory in the infrared of a gauged linear sigma model, we first note that the scalar potential V in the Lagrangian receives contributions from the chiral kinetic term,

$$\mathcal{L} \supset \sum_{i=1}^{N} \left(\bar{\phi}_i \ \left\{ \sigma(\rho^i)^\dagger, \sigma(\rho^i) \right\} \phi_i + F_i^\dagger F_i \right) \ , \tag{3.17}$$

and from the gauge kinetic term,

$$\mathcal{L} \supset \frac{1}{2g_s^2} \left(\mathrm{tr}_{\mathrm{adj.}}[\sigma, \sigma^\dagger] + D_a D_a \right) \ , \tag{3.18}$$

where the D^a is the component of the adjoint valued D-term corresponding to the a^{th} generator T^a_{ij}, i.e., $D_{ij} = T^a_{ij} D_a$, with, $i, j, a = 1, \ldots, \dim(G)$. The F and D fields are auxiliary which means they can be integrated out of the Lagrangian. For the F_i-field corresponding to the chiral scalar ϕ_i this implies,

$$F_i = \frac{\partial W}{\partial \phi_i} \ , \tag{3.19}$$

whereas for the D-field, these expectation values were given in (3.8) and (3.9).

In the infrared we require the scalar potential to vanish which can facilitated by noting that each term therein is positive definite and vanishes independently. We split V into three parts, the contribution of the D-field, F-field and σ-dependent terms, respectively,

$$V = V_D + V_F + V_\sigma \ , \tag{3.20}$$

and observe what the vanishing of each term implies. Before looking at each term individually a brief discussion of the *phase* that the theory in in is warranted. The value that the Fayet-Iliopolous parameters τ_i take defines an SCFT in the infrared. There exists more structure to the family of SCFTs in the infrared determined by the parameter space of τ_i being split into sectors, each of which corresponds to a phase of the theory. The phase boundaries are determined by the values of τ_i at which the moduli space of infrared theories becomes singular. These in turn are given by the roots of the discriminant locus (3.16), the derivation of which was discussed above. Descriptions of the theory across phase boundaries are topologically distinct from one another. Moreover, not all the phases of the theory correspond to a geometric NLSM target space description as they could be non-smooth quotient spaces or admit an entirely non-geometric description. The conception of a gauged linear sigma model that flows to a non-linear sigma model target space in a particular phase of the theory thus also yields dualities between the target space and the attained theories in other phases of the moduli space. An illustrative example touching upon these concepts will be outline after the analysis of the vanishing scalar potential. We now successively study the vanishing of the terms V_D, V_F and V_σ and the ramifications thereof.

- D-term and symplectic quotients :

  The vanishing of the D-field contribution, V_D, to the total potential V serves to achieve a *symplectic quotient* of the target space Y spanned by the complex scalars of the theory. Symplectic quotients are defined for manifolds with a compact group action that are equipped with a non-degenerate closed 2-form ω. The group action is said to be symplectic when the symplectic form ω is left invariant under the group action on the manifold. For the gauged linear sigma model the natural group action stems from the gauge group G of the theory. From the form of the value of the D-field (3.8) it can be inferred that it plays the role of the moment map μ which is a map from the symplectic manifold Y spanned by the scalars to the dual Lie algebra $\mathfrak{g}^*$ of the compact group G, i.e.,

$$\mu : Y \to \mathfrak{g}^* \ , \tag{3.21}$$

where we note that the real Fayet-Iliopolous parameters correspond to the U(1) factors in G. Mathematically this can be stated as,

$$\xi \in \mathrm{Ann}([\mathfrak{g}, \mathfrak{g}]) \subset \mathfrak{g}^*.$$

The vanishing of V_D restricts the space Y to the space,

$$V(\xi) := \mu^{-1}(\xi) / G \ , \tag{3.22}$$

where the quotient by the gauge group G is required by gauge invariance of the theory. The properties of the space $V(\xi)$ are dependent on the value of ξ and the broadly on the phase that the theory is in.

Focussing on that phase of the family of SCFTs which yields the desired NLSM target space, an important observation to make is that upon taking the symplectic quotient gauged linear sigma models with purely Abelian gauge groups beget toric varieties as target spaces whereas those with non-Abelian gauge groups beget general determinantal varieties.

- F-term and hypersurfaces :

  The F-term contribution to the scalar potential is given by,

$$V_F = \sum_{i=1}^{N} \left(F_i^\dagger F_i \right) \ . \tag{3.23}$$

While the D-field necessarily takes a non-trivial value upon being subject to equations of motion, the same is only true of the F-field if the superpotential $W(\Phi)$ is non-vanishing as is clear from the relation (3.19). We recall that the superpotential is gauge-invariant holomorphic function of the chiral fields with a $\mathrm{U}(1)_V$ R-charge assignment that ensures the R-charge neutrality of the Lagrangian. For the convention (3.3) this would correspond to the $\mathrm{U}(1)_V$ R-charge of $+2$ for the superpotential and thus it is only when the available chiral fields possess non-trivially distributed $\mathrm{U}(1)_V$ R-charges that can be arranged in gauge invariant way with the required R-charge that a superpotential can be constructed. For the class of models for which a superpotential exists, the vanishing of the F-term contribution to the scalar potential V_F imposes a further condition on the symplectic quotient $V(\xi)$, see (3.22), obtained from the vanishing D-term contribution.

The relevant target space X is given by,

$$X(\xi) := V(\xi) \cap \left(\frac{\partial W}{\partial \phi_i} = 0 \right) \ , \tag{3.24}$$

i.e., an intersection locus of the symplectic quotient space and the vanishing F-term locus. In cases where $V(\xi)$ is a toric variety or a determinantal variety this intersection locus might correspond to hypersurfaces or complete intersection varieties inside the ambient variety $V(\xi)$.

- Sigma-term and branches of supersymmetric vacua :

  The σ-term contribution to the scalar potential reads,

$$V_\sigma = \sum_{i=1}^{N} \left(\bar{\phi}_i \, \left\{ \sigma(\rho^i)^\dagger, \sigma(\rho^i) \right\} \phi_i \right) + \frac{1}{2g_s^2} \left(\mathrm{tr}_{\mathrm{adj.}} [\sigma, \sigma^\dagger] \right) \; . \tag{3.25}$$

The effect of the vanishing σ varies with respect to the value of the real Fayet-Iliopolous parameter ξ. In order to understand the consequences of the vanishing V_σ contribution better we first return to a couple of open threads from the previous sections, namely, the discriminant locus and phase boundaries in the moduli space parametrised by Fayet-Iliopolous couplings.

The phase boundary is defined by the vanishing of the quantum-corrected ξ_{eff} which in turn is given by the values of the bare ξ corresponding to the roots of the discriminant locus. Inside the moduli space far enough away from any phase boundary not all the chiral scalars can vanish in order to maintain the vanishing D-term contribution to the scalar potential. Looking at the first term in V_σ this implies that σ must vanish, which automatically ensures the vanishing of the second term in V_σ. Such a configuration where (at least some) scalars of the chiral multiplet are bound to take non-zero expectation values whereas as the scalars of the twisted chiral multiplet are required to take vanishing expectation values is known as a *Higgs branch* the moduli space of supersymmetric vacua. The convention of calling this a 'Higgs' branch is borrowed from 4d $\mathcal{N}=2$ theories where expectation values of the hypermultiplet, composed of two $\mathcal{N}=1$ chiral multiplets, are non-zero. Heuristically it can also be noted that a chiral scalar acquiring an expectation value leads to the gauge group G breaking to a discrete group (at best) by the Higgs mechanism, hence the name: Higgs branch.

On the other hand at the phase boundary the constraint on a subset of chiral scalars to be non-zero is lifted leading to no constraint on the expectation values of σ from the first term of V_σ. Thus the twisted chiral scalars can acquire generic expectation values which are in accordance with the vanishing of the second term in V_σ. Since this term only exists for non-Abelian gauge groups G, for gauged linear sigma models with purely Abelian gauge groups σ can acquire arbitrarily large expectation values. This phenomenon is known as the emergence of non-compact *Coulomb branch.* For non-Abelian gauge groups the vanishing of the second term implies,

$$[\sigma, \sigma^\dagger] = 0 \; , \tag{3.26}$$

which means that the gauge group G breaks down to the corresponding maximal torus $\mathrm{U}(1)^{\mathrm{rk}(G)}$. Again the nomenclature 'Coulomb' branch is inspired from 4d $\mathcal{N}=2$ theories where when the expectation values of the vector multiplet, composed of one chiral multiplet and one vector multiplet of an $\mathcal{N}=1$ theory in 4d, are non-zero this is termed as the Coulomb branch of the moduli space of supersymmetric vacua. Since the gauge group breaks down to an Abelian group, this theory is a generalisation of the Coulomb interaction corresponding to gauge group $\mathrm{U}(1)$, hence the name: Coulomb branch.

At this point we could also make connection to (3.11) which was assumed in the derivation of the quantum correction at 1-loop lever for ξ to be the mass generated for the chiral scalars. Looking at V_σ the mechanism of this mass generation becomes evident. On the

Higgs branch,

(i) the chiral scalars with vanishing expectation values, i.e., those that do not correspond to the coordinates of the symplectic quotient space $V(\xi)$, acquire masses from the $V_F \sim |dW|^2$ term in the scalar potential ;

(ii) the twisted chiral scalars acquire masses proportional to the non-vanishing chiral scalar expectation values the due to the first term in (3.25).

On the Coulomb branch,

(i) all the chiral scalars acquire masses proportional to the expectation value of σ as stated in (3.11) ;

(ii) the twisted chiral scalars remain massless.

To conclude, the infrared dynamics of a gauged linear sigma model holds key to the specific NLSM target space we seek to make a connection to via the spectrum and superpotential of the gauge theory. The desired NLSM target space in the infrared is only part of the story because of the intricate structure of the family of SCFTs in the infrared. The moduli space of supersymmetric vacua splits into the Higgs branch, Coulomb branch and/or mixed Coulomb-Higgs branch depending on the phase structure and phase boundaries of the theory. These branches are characterised by the specific subsets of chiral and twisted chiral scalars acquiring expectation values.

3.1.2 Illustrative Example: Hori-Tong GLSM

We will now overview many of the above mentioned concepts in the context of the specific example of the gauged linear sigma model designed in [48] to realise two Calabi-Yau threefolds as NLSM target spaces in different phases of the same model. One of these target space is the complete intersection X_{1^7} in a Grassmannian $\mathrm{Gr}(2,7)$ and the other a Pfaffian threefold which is realised as a rank constraint on a matrix defined using the homogenous coordinates of $\mathbb{P}^6$. A Grassmannian manifold $\mathrm{Gr}(k,N)$ is the vector space of a complex k-planes in $\mathbb{C}^N$ and $X_{1^7} \subset \mathrm{Gr}(2,7)$ denotes the complete intersection therein from 7 hypersurfaces of degree 1. These threefolds were first studied in [57] and were proven to satisfy a derived equivalence at the level of derived categories of coherent sheaves on the respective spaces in [58, 59].

(i) Spectrum: This theory is characterised by the gauge group U(2) which has the isomorphism,

$$\mathrm{U}(2) \sim (\mathrm{SU}(2) \times \mathrm{U}(1))\,/\mathbb{Z}_2\ , \tag{3.27}$$

and matter fields ϕ_i and P^j whose gauge and R-charge under the $\mathrm{U}(1)_V$ R-symmetry are summarised in Table 3.1.

(ii) Superpotential: Given the spectrum, a gauge invariant superpotential $W(\Phi)$ of all the chiral fields Φ with the R-charge $+2$ can be constructed as,

$$W(\Phi) = A^k_{ij} P^k \left(\epsilon_{\alpha\beta}\phi^\alpha_i \phi^\beta_j\right)\ , \tag{3.28}$$

where A^k_{ij} are coefficients required to contract flavour indices of the matter fields and the ϵ-tensor contracts the non-Abelian gauge indices α, β of ϕ_i. The variables $\epsilon_{\alpha\beta}\phi^\alpha_i\phi^\beta_j$ are

Chiral Field	U(2) Representation	$\mathrm{U}(1)_V$ Charge
$\phi_i\ ,\ i=1,\ldots,7$	$\square_{+1}$	0
$P^j\ ,\ j=1,\ldots,7$	$\mathbf{1}_{-2}$	2

Table 3.1: This table shows the matter spectrum and the corresponding gauge representation of the Hori-Tong GLSM with gauge group $\mathrm{U}(2) \sim (\mathrm{SU}(2)\times\mathrm{U}(1))/\mathbb{Z}_2$ for the Rødland Calabi-Yau threefold target spaces. The representation $\square_{+1}(\mathbf{1}_{-2})$ denotes the fundamental (trivial) representation under the $\mathrm{SU}(2)\subset\mathrm{U}(2)$ and charge $+1(-2)$ under $\mathrm{U}(1)\subset\mathrm{U}(2)$. Additionally the non-trivial R-charges under the $\mathrm{U}(1)_V$ R-symmetry are also displayed.

also known as Plücker coordinates as they correspond to the so-called Plücker embedding of a Grassmannian $\mathrm{Gr}(k,N)$ into a projective space $\mathbb{P}^{\binom{N}{k}-1}$. For a mathematical definition of the Grassmannian and the Plücker embedding, see the Appendix A. In the case of the fundamentals ϕ_i of the Hori-Tong model, $k=2$ and $N=7$.

(iii) Infrared Dynamics: From the rewriting (3.27) of the gauge group it is evident that there is one complexified Fayet-Iliopolous parameter τ and a corresponding real Fayet-Iliopolous parameter ξ corresponding to the $\mathrm{U}(1)\subset\mathrm{U}(2)$. Consequently, there are two phases in the infrared that can be studied corresponding to the Higgs branch branch vacua, given by $\xi \gg 0$ and $\xi \ll 0$, which are separated by the Coulomb branch vacuum directions. We will now study the specific form of the target spaces in each of the phases and the locus of singularities, the zeroes of which correspond to the emergence of the non-compact Coulomb branches in the moduli space of infrared theories.

Phase $\xi \gg 0$:

The D-term contributions to the scalar potential (3.8),

$$V_D = \sum_{i=1}^{7}\left(\bar{\phi}_i\phi_i - 2|P^i|^2\right) - \xi\ , \tag{3.29}$$

when required to vanish in the $\xi \gg 0$ phase requires the expectation value of the P fields to vanish in order to ensure no contribution from $\left|\frac{\partial W}{\partial \phi_i}\right|^2 \subset V_F$ to the scalar potential. This results in a symplectic quotient space inside the target space $Y\sim\mathbb{C}^7$ of the fundamentals ϕ^i. It is given by the space of 2-planes therein because of the fundamental **2**-representation of the ϕ's, also denoted as the $\square$-representation. Therefore the space (3.22) becomes a $\mathrm{Gr}(2,7)$. It is worth noting that the ϕ's acquiring expectation values implies that the gauge group U(2) breaks down completely in this phase.

The F-term contribution to the scalar potential,

$$V_F = \sum_{i\in\text{chirals}} F_i^\dagger F_i = \sum_{i\in\text{chirals}} \left|\frac{\partial W}{\partial \Phi_i}\right|^2\ , \tag{3.30}$$

upon vanishing results in the complete intersection of 7 hyperplanes given by the equations,

$$A_{ij}^k\left(\epsilon_{\alpha\beta}\phi_i^\alpha\phi_j^\beta\right) = 0\ , \tag{3.31}$$

for $k \in \{1, \ldots, 7\}$, inside the symplectic quotient space $\mathrm{Gr}(2,7)$. These hyperplanes are sections of the dual determinant line bundle of the $\mathrm{Gr}(2,7)$. For more details, see the Appendix A.

The σ-term contribution to the scalar potential gives a trivial contribution because the non-trivial expectation values of the ϕ's impose a vanishing expectation value on the σ. This is clear upon looking at the first term of V_σ in (3.25).

To conclude, in the $\xi \gg 0$ or the 'large volume' phase, termed so because of the large value of the Kähler modulus which quantifies the volume of the target space, the infrared limit is the geometric target space $X_{1^7} \subset \mathrm{Gr}(2,7)$.

Phase $\xi \ll 0$:

The infrared dynamics of this phase is slightly more subtle as opposed to the large volume phase. The vanishing of the scalar potential from the D-field (3.29) in this phase requires the expectation value of all P's to not simultaneously vanish. The symplectic quotient space is determined by the 7 P's providing homogenous coordinates on a $\mathbb{P}^6$. The vanishing of the scalar potential from the F-field required the expectation values of the ϕ's to vanish. In contrast to the large volume phase where the gauge group breaks down entirely, in this case the P's which are trivially charged under the $\mathrm{SU}(2) \subset \mathrm{U}(2)$ break only the $\mathrm{U}(1)$ subgroup. Since at least some of the chiral scalars acquire expectation values this forces the twisted chiral scalars σ to acquire a vanishing expectation value so as to ensure the vanishing of the V_σ contribution to the scalar potential (3.20).

Furthermore, the rank of the 7×7 matrices $A(P) := (A^k P^k)_{ij}$ have constraints imposed on their rank from the masses that the ϕ's acquire due P getting an expectation value. Since the k^{th} matrix A^k_{ij} is antisymmetric in the (ij) indices it must have even rank, i.e., $0, 2, 4,$ or 6. The ranks 0 and 2 are excluded because we choose the matrix A to be generic enough so as to impose the infrared flow to a smoothly defined Grassmannian in the large volume phase. The rank 6 is excluded because this would correspond to one massless flavour and hence no supersymmetric ground state [49]. This implies that in the $\xi \ll 0$ phase the space $\mathbb{P}^6$ localises to the $\mathrm{rk}(A(P)) = 4$ locus. This space is a Pfaffian Calabi-Yau threefold.

Coulomb Branch:

In the large volume and strong coupling phases studied above Higgs branch dynamics was at play given the chiral field scalars were the ones to acquire non-zero expectation values whereas the twisted chiral scalars had vanishing expectation values. When the Fayet-Iliopolous parameter lies on one of the roots of the discriminant locus, discussed below, the unbounded expectation values that the σ's can acquire correspond to a non-compact Coulomb branch opening up.

Specifically, the expression for V_σ (3.25) the vanishing of the first term is ensured by the vanishing expectation values of the chiral scalars, which also implies the vanishing of D- and F-term contributions to the scalar potential. The vanishing of the second term (3.25) implies that the gauge group breaks down to the maximal torus group $\mathrm{U}(1)^2$ as is typical of a pure Coulomb branch.

(iv) Discriminant locus: The equations (3.16) can be solved to extract the locus of singularities on the quantum Kähler moduli space. This system of equations was originally devised for purely Abelian gauge groups. However we can employ the trick of 'Abelianisation' to extend the application to non-Abelian gauged linear sigma models. To facilitate this computation we introduce auxiliary Fayet-Iliopolous parameters in addition to the original Fayet-Iliopolous parameters for the extra U(1)'s that emerges along the Coulomb branch corresponding to the rank of the semi-simple gauge group inside the total gauge group. The Abelianisation also imposes all the chiral fields to acquire consistent charges under the maximal torus group as well as for the twisted chiral field to become an adjoint representation of the maximal torus group. This purely Abelian theory is sometimes referred to as the Cartan theory and will be studied in detail in Section 3.2. The discriminant locus can be calculated in the Cartan theory and the auxiliary Fayet-Iliopolous parameters can be set to one later.

In the case of the Rødland model this would mean introducing one extra Fayet-Iliopolous parameter, upon doing this we get a system of two equations,

$$\begin{aligned} Q &= \left((-2\sigma_1)^2(\sigma_1+\sigma_2)(\sigma_1-\sigma_2)\right)^7 \ , \\ Q_{\text{aux}} &= \left((\sigma_1+\sigma_2)(\sigma_1-\sigma_2)^{-1}\right)^7 \ , \end{aligned} \tag{3.32}$$

where $Q = e^{2\pi i\tau}$ and Q_{aux} is the auxiliary parameter in the Cartan theory. The twisted chiral scalars σ_1 and σ_2 correspond to the components of the twisted chiral under each of the two U(1)'s in the maximal torus $\text{U}(1)^2$ on the Coulomb branch. The ratio $\left(\frac{\sigma_1}{\sigma_2}\right)$ can be eliminated and Q_{aux} can be set to one to get the discriminant locus on the quantum Kähler moduli space,

$$(1+57Q-289Q^2-Q^3)=0 \ . \tag{3.33}$$

We note that upon eliminating the ratio $\left(\frac{\sigma_2}{\sigma_1}\right)$ instead of the inverse thereof, a spurious pole on the moduli space appears at $Q=\frac{1}{128}$, i.e., the solution to the discriminant locus equations yield,

$$(1-128Q)(1+57Q-289Q^2-Q^3)=0 \ . \tag{3.34}$$

We postulate that the appearance of this spurious pole can be understood the violation of some genericness condition on the σ's as described in [48]. The genericness conditions ensure the emergence of a pure Coulomb branch at the poles of the discriminant locus, thus this spurious pole might correspond to a mixed Coulomb-Higgs branch.

3.2 Partition Functions and Correlators of GLSMs

Gauged linear sigma models provide a fertile playground for an exact computation of the partition function and correlators which in turn give an insight into the enumerative properties of the corresponding NLSM target space. Before the application of localisation techniques to compute these quantities, the authors of [28] devised a method to compute certain correlation functions of an A-twisted GLSM with Abelian gauge group exactly. Abelian GLSMs leads to NLSM target spaces that are intersections inside toric varieties, for instance, the quintic Calabi-Yau threefold $\mathbb{P}^4[5]$. The topological twist introduced in the first section of Chapter 2 in the context of NLSMs is applicable to the case of $\mathcal{N}=(2,2)$ supersymmetric gauge theories in 2d with $\text{U}(1)_R$ symmetry. This is so because at the level of the superconformal algebra the twist merely implies the shift of the spin generator by the R-symmetry generator with the sign of the shift determined by the type of twist, i.e., A or B twist. Since the GLSM is designed such that the in the infrared a superconformal algebra emerges, the requisite shift of generators can performed already in the ultraviolet to produce an A- or B-twisted GLSM which flows to an A- or B-twisted NLSM target space.

In the A-twisted GLSMs considered in the work of [28], the twisted chiral fields are non-trivial in the BRST cohomology. This implies that the relevant correlators are in the twisted chiral fields whereas all other fields are rendered BRST exact with vanishing correlators. The correlators are postulated and proven to holomorphic in the Fayet-Iliopolous parameters of the gauge theory which can be identified suitably with the quantum Kähler moduli parameters of the corresponding NLSM target space X. The computation relies on associating the twisted chiral fields of the gauge theory with the elements of $H^2(X)$ and computing n-point insertions of the σ using the properties of the moduli space of the A-twisted theory. This method restricts the number of insertions in the twisted chiral correlators to the $\dim_{\mathbb{C}}(X)$ with all higher insertions vanishing.

Mirror symmetry identifies the A-model on a Calabi-Yau manifold X to the B-model on the corresponding mirror manifold $\widetilde{X}$. The relevant correlators of the A-model receive instanton contributions, a closed formula for the summation of which does not exist, and the corresponding correlators on the B-model are classical and relatively easy to compute. This disparity in the solvability of A- and B-model leads mirror symmetry to assume an important role such that the A-model on X can be solved by constructing the mirror $\widetilde{X}$ and solving for the B-model on it and extracting the instanton contributions to the A-model correlators. The crux lies in the ability to construct the mirror manifold for all X's of interest and to translate the B-model correlators of $\widetilde{X}$ into the A-model correlators of X using a canonical 'mirror map'. Since the former are functions of the complex structure moduli parameters and the latter of the Kähler moduli parameters, the mirror map translates between these moduli parameters. Unfortunately, the method of computing A-twisted correlators of the gauged linear sigma model is unable to compute the mirror map directly.

The power of the technique employed in [28] lies in the fact that the construction of the mirror manifold to compute the B-model coupling is rendered unnecessary. Once the ultraviolet GLSM corresponding to the NLSM target space X is at hand, it can be A-twisted and the correlators of twisted chiral fields match exactly with the B-model correlators on the mirror $\widetilde{X}$ when the Fayet-Iliopolous parameter of the gauge theory is identified with the complex structure moduli parameter of $\widetilde{X}$. A formula to further translate this result in terms of the Kähler moduli parameters of X remains ambiguous from the correlator approach.

The exact result for the two-sphere partition function of a GLSM, Z_{S^2}, computed in [34, 35] and briefly reviewed in Chapter 2 Section 2.3 was employed in [60] to compute the mirror map and the Gromov-Witten invariants of Calabi-Yau threefold target spaces without the need to resort to the mirror manifold construction. This was facilitated by the conjecture in [60] that the Z_{S^2} yields the exact Kähler potential on the quantum Kähler moduli space of the target space. This was a landmark result in displaying the potential of GLSMs and localisation techniques to extract enumerative information of target spaces of interest which previously required the construction of a mirror manifold and the mysterious mirror map.

3.2.1 A-twisted GLSM correlators on S^2_Ω

We will now review the results of [29] which exactly computed correlators of A-twisted gauged linear sigma models on the Ω-deformed two-sphere S^2_Ω using localisation. This method has the two-fold advantage of applicability to higher-point correlators and non-Abelian models over the earlier work of [28]. The formula devised in [29] to compute twisted chiral field correlators will be of utmost relevance in the analysis and review of the work [30] of the author which will be done in the forthcoming sections.

A-twist The premise of the localisation technique applied to compute correlators in [29] was an $\mathcal{N} = (2,2)$ gauged linear sigma model topologically twisted to the A-model and defined on an Ω-deformed two-sphere S^2_Ω. As mentioned previously the topological twist assists in isolating the topological A- or B-sectors such that the two types of BRST-closed operators are twisted chiral and chiral fields, respectively. The topological twist serves to construct from the pre-existing supercharges a globally defined BRST charge. Localisation can be performed with respect to this BRST generator such that computation of correlators reduces to a finite-dimensional integral over the fixed point locus of the BRST symmetry.

Ω-deformation In addition to the A-twisting, the S^2 on which the gauged linear sigma model is defined is Ω-deformed. An Ω-deformation can be achieved by coupling the S^2 to an off-shell supergravity background, which is a method devised to define supersymmetry on curved spaces in [44], and giving a certain background value to the graviphoton field in the graviton supermultiplet. The idea behind such a deformation was introduced in the context of $\mathcal{N} = 2$ theories in 4d to compute the instanton contributions to the Seiberg-Witten prepotential [61, 62]. In effect this is a way of 'compactifying' the manifold on which the gauge theory is defined in order to perform localisation. This value is parameterised by a complex deformation parameter commonly denoted as ϵ_Ω. For the purpose of this thesis we will use the briefer notation ϵ. This deformation singles out the north and south pole of the S^2 as the fixed points of the isometry generated by the Killing vector field chosen to define the Ω-deformation. This implies that the theory shifts from being purely topological. The relevant correlators are those of the twisted chiral fields inserted at the north (N) and south (S) pole, respectively.

Field Content and Lagrangian The theory is defined by the non-Abelian gauge group G, as in (3.1), chiral fields ϕ_i, with $i \in 1, \ldots, N$, in representation $\mathcal{R}_i$ of G and a vector field V in the adjoint representation. The twisted chiral superfield Σ can be constructed as the field strength of V and has the scalar component σ, also in the adjoint representation of V. There are ℓ real and complexified Fayet-Iliopolous parameters, ξ_i and τ_i, and the exponentiated Fayet-Iliopolous parameters $Q_i = e^{2\pi i \tau_i}$. The Lagrangian (3.2) is deformed by relevant $\mathcal{O}(1/r)$ terms, where r is the length scale of the S^2, for consistently-defined supersymmetry [44, 63], and the appropriate Killing vector field is chosen for the Ω-deformation. The theory is A-twisted and the BRST-exact term to be added to the Lagrangian is chosen such that the theory localises on the Coulomb branch vacua. This means the infinite dimensional path integral measure localises to the finite dimensional integral over the expectation value of twisted chiral field σ.

Twisted Masses The final ingredient to be introduced before delving into the localisation formula are the twisted masses $\mathfrak{m}_i$ for chiral fields ϕ_i. The twisted masses were briefly alluded to in the discussion of the gauged linear sigma model in Chapter 2. If the gauge theory in question has a flavour symmetry then the background vector field, and correspondingly twisted chiral field, can be turned on for the flavour group and subsequently required to acquire an expectation value. In particular, the expectation value of the twisted chiral scalar σ_F corresponding to the flavour group F can be termed $\mathfrak{m}_F$ and the weights $\rho_i(\mathfrak{m}_F) = \rho_i \cdot \mathfrak{m}_F$ of the representation $\mathcal{R}_{F,i}$

of F that chiral field ϕ_i is in can be denoted as $\mathfrak{m}_i$. In practice this can be achieved by shifting the value of the twisted chiral field for the gauge group G such that for chiral field ϕ_i,

$$\rho_i \cdot \sigma \to \rho_i \cdot \sigma + \mathfrak{m}_i \ ,$$

where ρ are the weights under the representation $\mathcal{R}$ of the gauge group G. Here the pairing $\rho \cdot x$ comes from the canonical pairing between the elements of the dual Lie algebra $\mathfrak{g}^*$ and $\mathfrak{g}$ of G.

Localisation Formula The values assigned to twisted chiral fields inserted at the north and south pole, denoted σ_N and σ_S, are ϵ-deformation dependent as follows,

$$\sigma_N = \vec{\sigma} - \epsilon\frac{\vec{k}}{2} \quad ; \quad \sigma_S = \vec{\sigma} + \epsilon\frac{\vec{k}}{2} \ , \tag{3.35}$$

with $\vec{k} \in \gamma_m$, where $\gamma_m \simeq \mathbb{Z}^{\mathrm{rk}(G)}$ is the so-called magnetic charge lattice of G [64]. The vector $\vec{\sigma}$ represents $\mathrm{rk}(G)$-dimensional twisted chiral scalar on the maximal torus $\mathrm{U}(1)^{\mathrm{rk}(G)}$ that G breaks down to on the Coulomb branch. The final formula for the correlators that are functions $f(\cdot,\cdot)$ of twisted chiral fields inserted at the north and south pole, respectively, using the Coulomb branch localisation technique [29] is given by ,

$$\langle f(\sigma_N,\sigma_S)\rangle = \frac{1}{|\mathcal{W}_G|}\sum_{\vec{k}\in\gamma_m} \vec{Q}^{\vec{k}}\ \widetilde{Z}_{\vec{k}}(f(\sigma_N,\sigma_S),\epsilon,\mathfrak{m}_i) \ , \tag{3.36}$$

where $|\mathcal{W}_G|$ is the order of the Weyl group $\mathcal{W}_G$ of G. The Weyl group of G is defined as the set of all reflections of the roots of G. The Fayet-Iliopolous parameter vector $\vec{Q} = e^{2\pi i\vec{\tau}}$ is an ℓ-dimensional vector and $\vec{k}$ is an $\mathrm{rk}(G)$-dimensional vector. The pairing $\langle\vec{\tau},\vec{k}\rangle$ that defines the notation $\vec{Q}^{\vec{k}} = e^{2\pi i\langle\vec{\tau},\vec{k}\rangle}$ is the canonical pairing of elements in $\mathfrak{g}^*$ and $\mathfrak{g}$, with canonical embeddings of $\vec{\tau}$ and $\vec{k}$ into these algebras. The factor $\widetilde{Z}_{\vec{k}}$ is a specific residue of the contour integral along the Coulomb branch vacua given by,

$$\begin{aligned}\widetilde{Z}_{\vec{k}}(f(\sigma_N,\sigma_S),\epsilon,\mathfrak{m}_i) &= \oint\left(\prod_{a=1}^{\mathrm{rk}(G)}\frac{d\sigma_a}{2\pi i}\right) Z_{\text{1-loop}}(\vec{\sigma},\epsilon,\mathfrak{m}_i)\ f\left(\vec{\sigma}-\epsilon\frac{\vec{k}}{2},\vec{\sigma}+\epsilon\frac{\vec{k}}{2}\right) \ ,\\ &= \widetilde{\mathrm{Res}}^{\vec{\xi}}_{\vec{\sigma}}\left(Z_{\text{1-loop}}(\sigma,\epsilon,\mathfrak{m}_i)\ f\left(\vec{\sigma}-\epsilon\frac{\vec{k}}{2},\vec{\sigma}+\epsilon\frac{\vec{k}}{2}\right)\right) \ .\end{aligned} \tag{3.37}$$

Here the specific residue prescription denoted by $\widetilde{\mathrm{Res}}^{\vec{\xi}}_{\vec{\sigma}}$ will be explained shortly. The factor $Z_{\text{1-loop}}$ is a product from the 1-loop contributions from the chiral fields and the vector field, i.e.,

$$Z_{\text{1-loop}} = Z_{\text{chiral}} \cdot Z_{\text{vector}} \ \text{ with } ,$$

$$\begin{aligned} Z_{\text{chiral}}(\vec{\sigma},\mathfrak{m}_i,\epsilon) &= \prod_i^N Z^i(\vec{\sigma},\mathfrak{m}_i,\epsilon) \ ; \ Z^i(\vec{\sigma},\mathfrak{m}_i,\epsilon) = \epsilon^{\mathfrak{q}_i-\vec{\rho}\cdot\vec{k}-1}\frac{\Gamma\left(\frac{\vec{\rho}_i\cdot\vec{\sigma}+\mathfrak{m}_i}{\epsilon}+\frac{\mathfrak{q}_i-\vec{\rho}_i\cdot\vec{k}}{2}\right)}{\Gamma\left(\frac{\vec{\rho}_i\cdot\vec{\sigma}+\mathfrak{m}_i}{\epsilon}-\frac{\mathfrak{q}_i-\vec{\rho}_i\cdot\vec{k}}{2}+1\right)} \ ,\\ Z_{\text{vector}}(\vec{\sigma},\epsilon) &= \prod_{\vec{\omega}_\alpha>0}(-1)^{\vec{\omega}_\alpha\cdot\vec{k}+1}\left(\vec{\omega}_\alpha\cdot\left(\vec{\sigma}+\frac{\vec{k}}{2}\epsilon\right)\right)\left(\vec{\omega}_\alpha\cdot\left(\vec{\sigma}-\frac{\vec{k}}{2}\epsilon\right)\right) \ .\end{aligned} \tag{3.38}$$

Here $\mathfrak{q}_i$ are $\mathrm{U}(1)_V$ R-charges of chirals ϕ_i, the pairing $\vec{a} \cdot \vec{b}$ comes from the canonical pairing between the elements of the dual Lie algebra $\mathfrak{g}^*$ and the Lie algebra $\mathfrak{g}$ of G, respectively. The $\vec{\rho}_i$ is the charge vector of the i^{th} chiral field under the Cartan group $\mathrm{U}(1)^{\mathrm{rk}(G)}$, discussed in detail below, and $\vec{\omega}_\alpha$ are the roots of G.

Before explaining the residue $\widetilde{\mathrm{Res}}^{\vec{\xi}}_{\vec{\sigma}}$ to be taken the second equality of in (3.37), we make two assumptions about the form of the gauged linear sigma model in question:

(i) For GLSMs with non-Abelian gauge group G, we work with the respective Abelianised theory. This 'Abelianisation' trick was mentioned briefly when calculating the discriminant locus of the Hori-Tong GLSM at the end of Section 3.1. On the Coulomb branch of the theory, which is the relevant locus for the localisation computation that yields correlators, the gauge group G breaks down to $\mathrm{U}(1)^{\mathrm{rk}(G)}$ as discussed after Equation (3.26). This is because the previously G-adjoint-valued twisted chiral field σ acquires non-generic expectation values.

The chiral fields ϕ_i which were previously in representation $\mathcal{R}_i$ of G split into $\dim(\mathcal{R}_i)$ fields. These fields are charged under $\mathrm{U}(1)^{\mathrm{rk}(G)}$ with the charges $\vec{\omega}_i$ that are the weights of representations $\mathcal{R}_i$. The spontaneous breaking of the gauge group G yields massive W bosons from the vector multiplet, or equivalently the twisted chiral multiplet, with charges $\vec{\omega}_\alpha$ which are simply the roots of G [29]. These W-bosons carry a $\mathrm{U}(1)_V$ R-charge of $+2$. In fact this contributions from the W-bosons is what ultimately leads to the 1-loop contribution from the vector multiplet, Z_{vector}, in the Coulomb branch localisation formula. The spectrum in the Coulomb branch has been summarised in Table 3.2.

In evaluating the correlators we resort to the 'Cartan theory' where there are auxiliary Fayet-Iliopolous parameters corresponding to the all the unaccounted for $\mathrm{U}(1)$ subgroups of maximal torus $\mathrm{U}(1)^{\mathrm{rk}(G)}$. This provides ease of calculation of the discriminant locus Hori-Tong model and that of the summation over the magnetic charge lattice that counts the topological sectors. Additionally it facilitates the definition of a consistent residue prescription for for the Coulomb branch integral.

(ii) The twisted masses $\mathfrak{m}_i$ of the chiral fields ϕ_i are non-vanishing. This will assist in defining the residue prescription cleanly and, as we will note later, is indispensable in defining the correlator relations, to be defined in Section 3.3, of gauge theories that do not satisfy the Calabi-Yau condition, i.e., those with non-vanishing $\mathrm{U}(1)_A$-anomaly.

With these two broad facts in mind we define the residue $\widetilde{\mathrm{Res}}^{\vec{\xi}}_{\vec{\sigma}}$ that is taken in the Coulomb branch localisation formula. Firstly, this is a phase-dependent residue as is evident in the superscript $\vec{\xi}$, which determines the phase. The charge vectors $\vec{\omega}_{i_j}$ of the chiral fields ϕ_{i_j} of the Abelianised theory, see Table 3.2 for notation, take values in the electric charge lattice $\gamma_e \simeq \mathbb{Z}^{\mathrm{rk}(G)}$. The Fayet-Iliopolous parameter vector $\vec{\xi}$, containing the physical as well as the auxiliary parameters, takes values in $\mathbb{R}^{\mathrm{rk}(G)}$. A set of $\mathrm{rk}(G)$ charge vectors, $\vec{\omega}_{i_1}, \ldots, \vec{\omega}_{j_{\mathrm{rk}(G)}}$, that need not belong to the descendants of the same chiral multiplet of the unbroken theory, span a cone $\Pi^{\vec{\xi}}_{a_1}$ in the electric charge lattice γ_e. The Fayet-Iliopolous parameter vector $\vec{\xi}$ will be contained in a set of such cones $\{\Pi^{\vec{\xi}}_{a_i}\}$ depending on the phase of the theory.

The residue prescription is as follows: Let one of the cones in the electric charge lattice γ_e in the set of cones containing $\vec{\xi}$ be denoted as $\Pi^{\vec{\xi}}_{a_j}$. The residue $\widetilde{\mathrm{Res}}^{\vec{\xi}}_{\vec{\sigma}}$ must be taken for each topological sector that is numbered by $\vec{k}$. Furthermore, it must be taken such that the chiral

non-Abelian spectrum		Coulomb branch spectrum		$\mathrm{U}(1)_V$	twisted
non-Abelian fields	G-rep.	Abelian fields	$\mathrm{U}(1)^{\mathrm{rk}\,G}$-rep.	charge	mass
twisted chiral field σ	$\mathrm{adj}(G)$	chiral field W_1	$\vec{\omega}_{\alpha_1}$	2	0
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
		chiral field $W_{\dim(\mathrm{adj}(G))}$	$\vec{\omega}_{\alpha_{\dim(\mathrm{adj}(G))}}$	2	0
chiral field ϕ_i $i=1,\ldots,N$	$\mathcal{R}_i$	chiral field ϕ_{i_1}	$\vec{\omega}_{i_1}$	$\mathfrak{q}_i$	$\mathfrak{m}_i$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$
		chiral field $\phi_{i_{\dim(\mathcal{R}_i)}}$	$\vec{\omega}_{i_{\dim(\mathcal{R}_i)}}$	$\mathfrak{q}_i$	$\mathfrak{m}_i$

Table 3.2: The table shows the decomposition of the non-Abelian gauge theory spectrum into the Abelian spectrum of the Coulomb branch of the gauge theory, where the non-Abelian gauge group G is spontaneously broken to the maximal torus $\mathrm{U}(1)^{\mathrm{rk}(G)}$. The Abelian charge vectors $\vec{\omega}_{\alpha_i}$ and $\vec{\omega}_{i_j}$ are the weights of the non-Abelian representations of the multiplets that descend from the twisted chiral and the chiral multiplets charged under the unbroken G, respectively.

fields that define each cone $\Pi^{\vec{\xi}}_{a_j}$ in the set of cones $\{\Pi^{\vec{\xi}}_{a_i}\}$ contribute simultaneous poles to the one-loop determinant Z_{chiral} . This result must then be summed over contributions from all cones in $\{\Pi^{\vec{\xi}}_{a_i}\}$ for $\vec{\xi}$ in a certain phase. In other words, from (3.38),

$$\left(Z^i(\vec{\sigma},\mathfrak{m}_i,\epsilon)\right)^{-1} = 0 \ , \tag{3.39}$$

must hold simultaneously for all i corresponding to chiral fields ϕ_i that define the cone $\Pi^{\vec{\xi}}_{a_j}$ in the set $\{\Pi^{\vec{\xi}}_{a_i}\}$, and,

$$\widetilde{\mathrm{Res}}^{\vec{\xi}}_{\vec{\sigma}}(\ldots) = \sum_{\{\Pi^{\vec{\xi}}_{a_i}\}} \mathrm{Res}_{\vec{\sigma}}(\ldots) \ , \tag{3.40}$$

where the residue inside the summation is the usual Grothendieck residue, see for reference [65]. We note that this residue prescription is consistently defined for entirely generic twisted masses $\mathfrak{m}_i$ as it ensures that for a certain cone $\Pi^{\vec{\xi}}$ the pole sets defined by (3.39) are disjoint for all topological sectors.

The definition of the A-twisted Ω-deformed gauged linear sigma model correlators using localisation (3.37) is now complete. This concludes our discussion of the relevant results of [29] that will be employed heavily in sections Sections 3.3 and 3.4 that draw from the work of the author [30].

3.3 Relations of GLSM Correlators

The correlators defined in (3.36) are functions of the twisted chiral field inserted at the north and south pole of the S^2_Ω. In order to have uniformity of notation for such correlators with Abelian and well as non-Abelian gauge groups G, we establish the following two points for the entirety of this chapter:

1. Abelian Gauge Groups: For $G = \mathrm{U}(1)^\ell$, the $\langle f(\sigma_N, \sigma_S)\rangle$ in (3.36) is simply a product of arbitrary $\vec{\sigma}$ insertions at the north and south pole and hence can be written as,

$$\kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) := \langle \sigma_N^{\vec{n}} \sigma_S^{\vec{m}} \rangle \ , \tag{3.41}$$

where ℓ-dimensional vectors $\vec{n}, \vec{m}$ denote n_i, m_i insertions for i^{th} components of σ_N and σ_S, respectively.

2. Non-Abelian Gauge Groups: The correlators insertions are physical observables and are required to be gauge covariant. For general non-Abelian gauge groups of the form (3.1) this is a non-trivial statement. In particular, in (3.36) it implies for $\forall g \in G$,

$$\langle f(\sigma_N, \sigma_S)\rangle = \langle f(g^{-1}\sigma_N g, g^{-1}\sigma_S g)\rangle \ . \tag{3.42}$$

Because of its cyclicity property, the trace operator is considered to construct gauge invariant operators from twisted chiral operators and makes for an ideal function f in above equation. The correlators of interest to us will be those that are gauge invariant products of north and south pole insertions, i.e.,

$$\langle f(\sigma_N, \sigma_S)\rangle = \langle f_N(\sigma_N)\ f_S(\sigma_S)\rangle =: \kappa_{f_N, f_S}(\vec{Q}, \mathfrak{m}_i, \epsilon) \ . \tag{3.43}$$

The non-Abelian correlators $\kappa_{f_N, f_S}(\vec{Q}, \mathfrak{m}_i, \epsilon)$ defined in (3.43) in terms of functions f_N and f_S reduce to Weyl group $\mathcal{W}_G$-invariant polynomials of $\vec{\sigma}$ in the Abelianised theory. This is so because the Weyl group, which is the isometry group of the root vectors of G, preserves the maximal torus algebra of G by definition. Since on the Coulomb branch the gauge group G breaks down to the maximal torus subgroup $\mathrm{U}(1)^{\mathrm{rk}(G)}$, the correlators of $\vec{\sigma} \in \mathfrak{g}$ are Weyl group invariant.

Ultimately, in the Abelianised theory we can work with correlators that are $\mathcal{W}_G$-invariant insertions of σ_N and σ_S. Assuming this holds true, we can unambiguously rename the non-Abelian correlators,

$$\kappa_{f_N, f_S}(\vec{Q}, \mathfrak{m}_i, \epsilon) =: \kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_i, \epsilon) \ , \tag{3.44}$$

with $\vec{n}, \vec{m}$ being $\mathrm{rk}(G)$-dimensional vectors that again signify n_i, m_i insertions for i^{th} components of σ_N and σ_S, respectively.

For the correlators in Abelian and non-Abelian GLSMs alike, see (3.41), (3.43) and (3.44), there exists a selection rule [28, 29], that depends on the powers of the twisted masses, ϵ-deformation parameter, gauge charges and $\vec{\sigma}$-insertions into the correlator.

It is given by,

$$d + \#(\epsilon) + \#(\mathfrak{m}_\ell) + \sum_{k=1}^{\ell} \left(\sum_i^N \rho_k^i \right) \#(Q_k) \ \in \ \mathrm{pow}(f) \ . \tag{3.45}$$

Here the $\#(\cdot)$ the power of the specified argument when the correlator is considered as a power series in that argument, ρ_k^i is the gauge charge of the i^{th} chiral field under the k^{th} U(1) factor in G and $\mathrm{pow}(f)$ denotes set of degrees of all monomials in σ_N and σ_S that appear in $f(\sigma_N, \sigma_S)$. The d denotes the gravitational contribution to the $\mathrm{U}(1)_A$-anomaly which, if the GLSM admits a geometric interpretation, is the complex dimension of the target space. It is given by,

$$d = \sum_{i}^{N} (1 - \mathfrak{q}_i) \dim \rho_i - \dim \mathfrak{g} \ , \tag{3.46}$$

with $\mathfrak{q}_i, \rho_i$ denoting the $\mathrm{U}(1)_V$ R-charge and the gauge charge of the i^{th} chiral field, respectively, and $\mathfrak{g}$ the Lie algebra of the gauge group G.

3.3.1 Derivation of Relations

The form of the Coulomb branch localisation formula for correlators (3.36), (3.37) provides motivation for there to exist linear relations among these correlators. In this section we aim to deduce such relations, which are independent of the north pole insertions, among the correlators $\kappa_{\vec{n},\vec{m}}$. The derivation of relations among correlators applies Abelian and non-Abelian gauged linear sigma models alike following the notation in (3.41) and (3.44). For this purpose, we will consistently work in the Abelianised theory, whose spectrum is given by Table 3.2, where the gauge group is $\mathrm{U}(1)^{\mathrm{rk}(G)}$ and $\vec{Q}$ is a $\mathrm{rk}(G)$-dimensional vector appended by auxiliary Fayet-Iliopolous parameters that can be set to one later.

We will start with an ansatz for the relations, substitute into it the localisation formula for correlators, and by an appropriate change of the variable of integration and a Gamma-function identity we will arrive at an equation for the coefficients in the ansatz for the relations. We trace this derivation in the following steps:

1. The ansatz for the south-pole relations is given by,

$$R_S(\vec{Q}, \mathfrak{m}_l, \epsilon, \kappa_{\vec{n},\,\cdot}) \ = \ \sum_{\vec{m}=0}^{\vec{N}} c_{\vec{m}}(\vec{Q}, \mathfrak{m}_l, \epsilon) \kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_l, \epsilon) \ = \ 0 \ , \tag{3.47}$$

   with $\vec{N}$ denoting the highest south pole insertions. We require the coefficient functions $c_{\vec{m}}(\vec{Q}, \mathfrak{m}_l, \epsilon)$ to be polynomial in $\vec{Q}$. They can be expanded as,

$$c_{\vec{m}}(\vec{Q}, \mathfrak{m}_l, \epsilon) \ = \ \sum_{\vec{p}=0}^{\vec{s}} c_{\vec{m},\vec{p}}(\mathfrak{m}_l, \epsilon) \vec{Q}^{\vec{p}} \ = \ \sum_{p_1=0}^{s_1} \cdots \sum_{p_r=0}^{s_r} c_{\vec{m},p_1,\ldots,p_r}(\mathfrak{m}_l, \epsilon) Q_1^{p_1} \cdots Q_r^{p_r} \ , \tag{3.48}$$

   for some suitable finite vector $\vec{s}$.

2. The localisation formula for the correlators with the specific residue prescription defined in (3.40) is given by,

$$\kappa_{\vec{n},\vec{m}}(Q, \mathfrak{m}_l, \epsilon) \ = \ \sum_{\vec{k} \in \gamma_m} \vec{Q}^{\vec{k}} \ \widetilde{\mathrm{Res}}_{\vec{\sigma}}^{\vec{\xi}} \left[\left(\vec{\sigma} - \frac{\epsilon}{2}\vec{k}\right)^{\vec{n}} \left(\vec{\sigma} + \frac{\epsilon}{2}\vec{k}\right)^{\vec{m}} Z_{\vec{k}}(\vec{\sigma}, \mathfrak{m}_l, \epsilon) \right] \ . \tag{3.49}$$

   Recall that this residue prescription is consistently defined when the twisted masses are completely generic.

3. We now insert eq. (3.49) in the definition (3.47) and collect common powers of $\vec{Q}$. After the change of variables $\vec{w} = \vec{\sigma} + \epsilon\frac{\vec{k}+\vec{p}}{2}$, which maps the pole lattices, used in the definition

of $\widetilde{\mathrm{Res}}_{\vec{\sigma}}^{\vec{\xi}}$, to themselves, we arrive at,

$$0 = R_S(\vec{Q}, \kappa_{\vec{n},\cdot}) = \sum_{\vec{k}\in\gamma_m} \vec{Q}^{\vec{k}} \sum_{\vec{m}=0}^{\vec{N}} \sum_{\vec{p}=0}^{\vec{s}} c_{\vec{m},\vec{p}} \, \widetilde{\mathrm{Res}}_{\vec{w}}^{\vec{\xi}} \left[(\vec{w} - \epsilon\vec{k})^{\vec{n}} (\vec{w} - \epsilon\vec{p})^{\vec{m}} Z_{\vec{k}-\vec{p}}(\vec{w} - \epsilon\tfrac{\vec{k}+\vec{p}}{2}, \mathfrak{m}_l, \epsilon) \right] . \tag{3.50}$$

4. Note that the expression at the end of the previous step can only be zero if the coefficients of all powers of $\vec{Q}$ vanish separately. In order to separate the factors that are independent of $\vec{k}$ that determines the power of $\vec{Q}$ we use the Gamma function identity,

$$\Gamma(x-y) \; = \; \Gamma(x) \cdot \frac{\prod_{s=1+y}^{+\infty}(x-s)}{\prod_{s=1}^{+\infty}(x-s)} \; , \tag{3.51}$$

   obtain the constraint,

$$\begin{aligned} 0 = \widetilde{\mathrm{Res}}_{\vec{w}}^{\vec{\xi}} \Bigg[&(\vec{w} - \epsilon\,\vec{k})^{\vec{n}} Z_{\vec{k}}(\vec{w} - \epsilon\tfrac{\vec{k}}{2}, \mathfrak{m}_l, \epsilon) \\ &\sum_{\vec{p}=0}^{\vec{s}} \sum_{\vec{m}=0}^{\vec{N}} c_{\vec{m},\vec{p}} \, (\vec{w} - \epsilon\,\vec{p})^{\vec{m}} \prod_{l=1}^{M} \frac{\prod_{s=1}^{\infty} \left(\vec{w}\cdot\vec{\rho}_l + \mathfrak{m}_l + \epsilon\,(1 - \frac{\mathfrak{q}_l}{2} - s)\right)}{\prod_{s=1+\vec{\rho}_l\cdot\vec{p}}^{\infty} \left(\vec{w}\cdot\vec{\rho}_l + \mathfrak{m}_l + \epsilon\,(1 - \frac{\mathfrak{q}_l}{2} - s)\right)} \Bigg] . \end{aligned} \tag{3.52}$$

5. As the equation obtained above must hold for $\vec{k}$, it is necessary that the expression within the residue symbol vanishes itself. The constraint for a south pole relation thus takes the simple form,

$$0 \; = \; \sum_{\vec{p}=0}^{\vec{s}} \alpha_{\vec{p}}(\vec{w}, \mathfrak{m}_l, \epsilon) \cdot g_{\vec{p}}(\vec{w}, \mathfrak{m}_l, \epsilon) \; , \tag{3.53}$$

   in terms of the polynomials $\alpha_{\vec{p}}$ and rational functions $g_{\vec{p}}$ given by,

$$\begin{aligned} \alpha_{\vec{p}}(\vec{w}, \mathfrak{m}_l, \epsilon) \; &= \; \sum_{\vec{n}=0}^{\vec{N}} c_{\vec{n},\vec{p}}(\mathfrak{m}_l, \epsilon)\, (\vec{w} - \epsilon\,\vec{p})^{\vec{n}} \; , \\ g_{\vec{p}}(\vec{w}, \mathfrak{m}_l, \epsilon) \; &= \; \prod_{l=1}^{M} \frac{\prod_{s=1}^{\infty} \left(\vec{w}\cdot\vec{\rho}_l + \mathfrak{m}_l + \epsilon\,(1 - \frac{\mathfrak{q}_l}{2} - s)\right)}{\prod_{s=1+\vec{\rho}_l\cdot\vec{p}}^{\infty} \left(\vec{w}\cdot\vec{\rho}_l + \mathfrak{m}_l + \epsilon\,(1 - \frac{\mathfrak{q}_l}{2} - s)\right)} \; . \end{aligned} \tag{3.54}$$

   Since we aim to attain relations between south pole correlators, this expression must be manifestly independent of the north pole insertions, as is evident in (3.54).

6. We observe that the rational functions $g_{\vec{p}}$ are entirely fixed by the spectrum of the gauge theory under consideration. Since the $\alpha_{\vec{p}}$ contain all information appearing in the south pole relations R_S, determining these relations for a given gauge theory thus amounts to finding polynomials $\alpha_{\vec{p}}$ satisfying the constraints (3.53). This can be identified with a well-studied problem in commutative algebra: The set of polynomial solutions $\alpha_{\vec{p}}$ forms

the *syzygy module* over the polynomial ring $\mathbb{C}(\mathfrak{m}_l)[\vec{w},\epsilon]$ of the rational function $g_{\vec{p}}$. The elements of the polynomial ring $\mathbb{C}(\mathfrak{m}_l)[\vec{w},\epsilon]$ are polynomials in $\vec{w}$ and ϵ with coefficients in the field of complex rational functions in the twisted masses $\mathfrak{m}_l$, which is denoted by $\mathbb{C}(\mathfrak{m}_l)$.

From a given element $\alpha_{\vec{p}}$ in the south pole syzygy module we can then reconstruct the south pole correlator relation as,

$$R_S(\vec{Q},\mathfrak{m}_l,\epsilon,\kappa_{\vec{n},\cdot}) \;=\; \sum_{\vec{p}=0}^{\vec{s}} \vec{Q}^{\vec{p}} \left\langle \vec{\sigma}_N^{\vec{n}}\, \alpha_{\vec{p}}(\vec{\sigma}_S+\epsilon\vec{p},\mathfrak{m}_l,\epsilon) \right\rangle \;=\; 0 \ . \tag{3.55}$$

This concludes the derivation of south-pole relations R_S of correlators given by (3.53), (3.54) and (3.55). Note that these relations are independent of the north pole insertions, however, we could just as well have analogously defined north-pole relations R_N that are independent of south pole insertions. To work with the north or south pole relations is a matter of convention and we choose the latter for the purpose of this work.

3.3.2 Correlator Relations and the Differential Ideal

The moduli space of the gauged linear sigma model is parametrised by the physical Fayet-Iliopolous parameters of the theory. It is this moduli space that parametrises the infrared family of SCFTs and exhibits the Coulomb and Higgs branch vacua. We assume that at each point on this moduli space there exists a Hilbert space of states $\mathcal{H}_Q$ with a ground state $|\Omega(\vec{\xi},\vec{\vartheta})\rangle$. From this point of view the correlators of the twisted chiral fields enjoy an interpretation as matrix elements in this Hilbert space. This interpretation of gauge theoretic quantities from the point of view of a Hilbert space is motivated by [66] and the work on 3d supersymmetric theories by [53].

We can interpret the south pole correlator relations R_S derived in (3.55) as operators $\boldsymbol{R}_S$ annihilating the moduli-dependent ground state $\left|\Omega(\vec{\xi},\vec{\theta})\right\rangle$ of the gauge theory. Explicitly this can be written as,

$$\boldsymbol{R}_S(\vec{\boldsymbol{Q}},\vec{\boldsymbol{\sigma}}_S,\mathfrak{m}_l,\epsilon)\left|\Omega(\vec{\xi},\vec{\theta})\right\rangle \;=\; 0 \ . \tag{3.56}$$

The boldface letters $\boldsymbol{R}_S, \vec{\boldsymbol{Q}}$ and $\vec{\boldsymbol{\sigma}}_S$ indicate that they are operators acting on the Hilbert space of states $\mathcal{H}_Q$.

Additionally there exists a non-trivial commutation relation between operators $\boldsymbol{\sigma}_{S,i}$ and $\boldsymbol{Q}_j$,

$$[\boldsymbol{\sigma}_{S,i},\boldsymbol{Q}_j] \;=\; \delta_{ij}\epsilon\boldsymbol{Q}_j \ . \tag{3.57}$$

Here $i=1,\ldots,\mathrm{rk}(G)$ since we are working in the Abelianised theory. This can be seen in two steps that we now explain.

(i) We note that for every relation of the form (3.55) derived from solutions to $\alpha_{\vec{p}}(\vec{w},\mathfrak{m}_l,\epsilon)$

that satisfy (3.53), there exist descendant relations R'_S by defining $\alpha'_{\vec{p}}$ of the form,

$$\begin{aligned} 0 &= \sum_{\vec{p}=0}^{\vec{s}} \big(w_i \; \alpha_{\vec{p}}(\vec{w}, \mathfrak{m}_l, \epsilon)\big) \cdot g_{\vec{p}}(\vec{w}, \mathfrak{m}_l, \epsilon) \qquad ; \text{ where } i = 1, \ldots, \mathrm{rk}(G) \\ &= \sum_{\vec{p}=0}^{\vec{s}} \big(\alpha'_{\vec{p}}(\vec{w}, \mathfrak{m}_l, \epsilon)\big) \cdot g_{\vec{p}}(\vec{w}, \mathfrak{m}_l, \epsilon) \; =: \; R'_S(\vec{Q}, \mathfrak{m}_l, \epsilon, \kappa_{\vec{n},\cdot}) \; . \end{aligned} \tag{3.58}$$

(ii) From the relations R'_S we can define operators $\boldsymbol{R}'_S$ that act on $\mathcal{H}_Q$. These operators can also be interpreted as a multiplication of $\boldsymbol{\sigma}_{S,i}$ to the left of $\boldsymbol{R}'_S$,

$$\begin{aligned} \boldsymbol{R}'_S(\vec{\boldsymbol{Q}}, \vec{\boldsymbol{\sigma}}_S, \mathfrak{m}_l, \epsilon) &= \boldsymbol{\sigma}_{S,i} \sum_{\vec{p}=0}^{\vec{s}} \vec{\boldsymbol{Q}}^{\vec{p}} \alpha_{\vec{p}}(\vec{\boldsymbol{\sigma}}_S + \epsilon\vec{p}, \mathfrak{m}_l, \epsilon) \; = \; \sum_{\vec{p}=0}^{\vec{s}} \vec{\boldsymbol{Q}}^{\vec{p}} (\boldsymbol{\sigma}_{S,i} + \epsilon p_i) \alpha_{\vec{p}}(\vec{\boldsymbol{\sigma}}_S + \epsilon\vec{p}, \mathfrak{m}_l, \epsilon) \\ &= \sum_{\vec{p}=0}^{\vec{s}} \vec{\boldsymbol{Q}}^{\vec{p}} \alpha'_{\vec{p}}(\vec{\boldsymbol{\sigma}}_S + \epsilon\vec{p}, \mathfrak{m}_l, \epsilon) \; , \end{aligned} \tag{3.59}$$

where the second equality makes use of the commutation relation (3.57).

In case of non-Abelian GLSMs the commutation relation (3.57) consists of operators $\boldsymbol{\sigma}_{S,i}$, with $i \in \{1, \ldots, \mathrm{rk}(G)\}$, that are not gauge invariant in the sense of (3.42). We can construct from them commutation relations of linear gauge invariant operators of the from $\mathrm{tr}(\boldsymbol{\sigma}_S) \sim \sum_i \boldsymbol{\sigma}_{S,i}$ with the physical Fayet-Iliopolous parameter $\vec{Q}_{\text{phys}} (= \vec{Q}$ for Abelian gauge groups) by eliminating the auxiliary parameters (3.57). These commutation relations cannot be written, at least in this form, for the non-linear gauge-invariant operators, for instance, $\mathrm{tr}(\boldsymbol{\sigma}_S^2)$. However this observation is irrelevant for the mathematical consequences we wish to draw from the form of relations, as defined in (3.55), although achieving a commutation relation for non-linear gauge-invariant operators with $\vec{Q}_{\text{phys}}$ would be of interest for a more complete picture of the Hilbert space interpretation of correlator relations [67, 68].

Relations as Generators of a Differential Ideal The commutation relation (3.57) characterises the non-commutative south pole ring,

$$\boldsymbol{\mathcal{R}}_S \; = \; {\mathbb{C}(\mathfrak{m}_l)\langle \vec{\boldsymbol{Q}}, \vec{\boldsymbol{\sigma}}_S, \epsilon \rangle} \Big/ {[\boldsymbol{\sigma}_{S,i}, \boldsymbol{Q}_j] = \delta_{ij} \epsilon \boldsymbol{Q}_j} \; , \tag{3.60}$$

and the set of south pole operators $\boldsymbol{R}_S$ annihilating the ground state $\left|\Omega(\vec{\xi}, \vec{\theta})\right\rangle$, as in (3.56), forms a left ideal $\boldsymbol{\mathcal{I}}_S$ in this ring $\boldsymbol{\mathcal{R}}_S$, which according to eq. (3.53) is explicitly given by,

$$\boldsymbol{\mathcal{I}}_S \; = \; \left\{ \sum_{\vec{p}} \vec{\boldsymbol{Q}}^{\vec{p}} \alpha_{\vec{p}}(\vec{\boldsymbol{\sigma}}_S + \epsilon\vec{p}, \mathfrak{m}_l, \epsilon) \in \boldsymbol{\mathcal{R}}_S \; \middle| \; 0 = \sum_{\vec{p}} \alpha_{\vec{p}}(\vec{\sigma}_S, \mathfrak{m}_l, \epsilon) \cdot g_{\vec{p}}(\vec{\sigma}_S, \mathfrak{m}_l, \epsilon) \right\} \; . \tag{3.61}$$

An explicit representation for the ring $\boldsymbol{\mathcal{R}}_S$ that satisfies the commutation relation (3.57) can devised for Abelian and non-Abelian GLSMs separately. For Abelian GLSMs we note that an explicit representation is given by,

$$\boldsymbol{Q}_i \,=\, Q_i \;, \qquad \boldsymbol{\sigma}_{S,i} \,=\, \epsilon\, Q_i \frac{\partial}{\partial Q_i} \,=:\, \epsilon\, \Theta_i \;, \tag{3.62}$$

for $i = 1, \ldots, \mathrm{rk}(G)$, where $\mathrm{rk}(G) = \dim(G)$ for Abelian groups.

For the non-Abelian GLSMs, as discussed before, the commutation relations for the unbroken gauge group can only be written for linear gauge invariant operators of the form $\mathrm{tr}(\boldsymbol{\sigma}_S)$. The explicit representation of $\boldsymbol{Q}_i$ for a gauge group of the form (3.1) in this case is given by,

$$\boldsymbol{Q}_i = \begin{cases} Q_i & ;\ i = 1, \ldots, \ell \\ 1 & ;\ \text{otherwise}\ . \end{cases} \tag{3.63}$$

This representation may certainly change depending on the choice of basis for the Abelianised group on the Coulomb branch, as we will note to be the case for the example of the Grassmannian $\mathrm{Gr}(2,4)$ to be studied next.

The representation for the non-Abelian operator $\mathrm{tr}(\boldsymbol{\sigma}_S)$ is given by,

$$\mathrm{tr}_i(\boldsymbol{\sigma}_S) \,=\, \epsilon\, Q_i \frac{\partial}{\partial Q_i} \,=:\, \epsilon\, \Theta_i \;. \tag{3.64}$$

We refer to $\boldsymbol{\mathcal{I}}_S$ as the differential ideal, and the solutions to its differential equations,

$$\boldsymbol{R}_S(\vec{Q}, \epsilon\vec{\Theta}, \mathfrak{m}_l, \epsilon) f(\vec{Q}) = 0 \;,$$

capture the $\vec{Q}$-dependence of the gauge theory ground state $\left|\Omega(\vec{Q})\right\rangle$. For gauge theories with a geometric target space interpretation $\boldsymbol{\mathcal{I}}_S$ becomes the differential ideal governing the Gromov–Witten theory of the target space as we will note in the subsequent sections.

3.3.3 Examples

We now illustrate the aforementioned relations between correlators for an Abelian and a non-Abelian gauged linear sigma model. Their geometric target space description is given by the projective line $\mathbb{P}^1$ and the Grassmannian $\mathrm{Gr}(2,4)$, respectively. These are both, in fact, examples of Fano varieties. From a gauge theory perspective we denote a theory with an $\mathrm{U}(1)_A$ anomaly of the R-symmetry arising from $\sum_\ell \vec{\rho}_\ell > 0$ for all ℓ as a gauged linear sigma model with the Fano property, as will be the case for the forthcoming examples. In each of the examples we will begin by briefly discussing the geometric target space attained in the infrared of the gauge theory, the general structure of the correlators and the existence, or lack thereof, of the discriminant locus. We will then derive the relations among the correlators and the differential operators corresponding to the relations.

The Projective Line $\mathbb{P}^1$

We consider a gauged linear sigma model with the Abelian gauge group $\mathrm{U}(1)$ and charged matter spectrum as listed in Table 3.3.

Geometric Target Space In the infrared the classical scalar potential vanishes for a positive Fayet–Iliopoulos parameter, the symplectic quotient (3.22) yields the complex projective line

Chiral multiplets	U(1) charge	$\mathrm{U}(1)_V$ charge	twisted masses
ϕ_i, $i = 1, 2$	$+1$	0	$\mathfrak{m}_i$

Table 3.3: Matter spectrum of the U(1) gauged linear sigma model of the projective line $\mathbb{P}^1$.

$\mathbb{P}^1$ as classical target space geometry. Since the superpotential is trivially zero, because of a lack of chiral fields with non-vanishing $\mathrm{U}(1)_V$ R-charge, there appears no intersection atop the symplectic quotient $\mathbb{P}^1$, according to (3.24).

Discriminant Locus and Correlators Since this GLSM is of the Fano type, the infrared dynamics corresponding to the $\xi < 0$ phase does not correspond to geometric target space associated to an NLSM . Furthermore, there exists no solution to the equations (3.16) and thus there is no discriminant locus in the infrared theory. This implies that the correlators are polynomials in the parameters Q, ϵ and the twisted masses.

The first few south pole correlators can be solved using (3.36) and are given by,

$$\kappa_{0,0} = 0 \quad , \quad \kappa_{0,1} = 1 \quad , \quad \kappa_{0,2} = -(\mathfrak{m}_1 + \mathfrak{m}_2) \quad , \quad \kappa_{0,3} = (\mathfrak{m}_1^2 + \mathfrak{m}_1\mathfrak{m}_2 + \mathfrak{m}_2^2) + Q \ . \tag{3.65}$$

The correlators can be easily confirmed to satisfy the selection rule given by (3.45) for each monomial in the expression.

Relations of Correlators We now compute relations of the form (3.47). From the matter spectrum in Table 3.3 we determine the functions defined in eq. (3.54) as,

$$g_p(w, \mathfrak{m}_i, \epsilon) = \prod_{s=0}^{p-1} (w + \mathfrak{m}_1 - \epsilon s)(w + \mathfrak{m}_2 - \epsilon s) \ , \quad p = 0, 1, 2, \ldots \ . \tag{3.66}$$

The smallest one-dimensional vector $\vec{s}$ in (3.53) for which the constraint (3.53) can be solved equals two. Thus $g_0 = 1$ and $g_1 = (w + \mathfrak{m}_1)(w + \mathfrak{m}_2)$ and (3.53) is solved by the polynomials,

$$\alpha_0(w, \mathfrak{m}_i, \epsilon) = (w + \mathfrak{m}_1)(w + \mathfrak{m}_2) \ , \quad \alpha_1(w, \mathfrak{m}_i, \epsilon) = -1 \ . \tag{3.67}$$

Together with eq. (3.55) these determine the south pole correlator relation,

$$R_S(Q, \mathfrak{m}_i, \epsilon, \kappa_{n,\cdot}) = \langle \sigma_N^n (\sigma_S + \mathfrak{m}_1)(\sigma_S + \mathfrak{m}_2) \rangle - Q \langle \sigma_N^n \rangle \ , \tag{3.68}$$

for arbitrary north pole insertions of degree n.

Differential Ideal This relation can be interpreted as the differential ideal associated to the non-commutative polynomial ring defined by (3.60). According to eqs. (3.56) and (3.62) the corresponding differential operator can be written as,

$$\mathcal{L}(Q, \epsilon, \mathfrak{m}_i) = (\epsilon\Theta + \mathfrak{m}_1)(\epsilon\Theta + \mathfrak{m}_2) - Q \ , \tag{3.69}$$

given in terms of the logarithmic derivative $\Theta = Q\partial_Q$. It can be checked that this operator generates the entire differential ideal $\boldsymbol{\mathcal{I}}_S$ of the gauge theory. That is to say, other south pole correlator relations obtained from the higher degree polynomials (3.66) yield differential operators in the differential ideal generated by the above operator (3.69).

The Grassmannian $\mathrm{Gr}(2,4)$

We now consider the gauged linear sigma model with the non-Abelian gauge group U(2) and non-Abelian matter spectrum as displayed in Table 3.4.

Geometric Target Space In the infrared the scalar potential vanishes which leads to, for positive Fayet–Iliopoulos parameter ξ, the symplectic quotient (3.22) yielding the complex Grassmannian fourfold $\mathrm{Gr}(2,4)$ as classical target space geometry of this gauge theory. The matter fields ϕ_i span the vector space $\mathbb{C}^{4\times 2}$ as they transform under the $\square_{+1}$ of the U(2). The vanishing of the D-term in the scalar potential is equivalent to the vanishing of the inverse of the moment map μ from the ambient space $\mathbb{C}^{4\times 2}$ to the Lie algebra $\mathfrak{u}(2)$ corresponding to the action of the gauge group U(2) on the target space of ϕ_i's. This non-Abelian action yields the final target space $\mathrm{Gr}(2,4)$.

The Plücker embedding $\mathrm{Pl} : \mathrm{Gr}(2,4) \hookrightarrow \mathbb{P}(\Lambda^2\mathbb{C}^4)$ of the Grassmannian fourfold $\mathrm{Gr}(2,4)$ with its unique quadratic Plücker relation identifies this particular Grassmannian with a quadratic hypersuface in $\mathbb{P}^5$, i.e.,

$$\mathrm{Gr}(2,4) \simeq \mathbb{P}^5[2] \ . \tag{3.70}$$

We refer the reader to the Appendix A for details on the Plücker embedding.

non-Abelian gauge theory spectrum:			
Chiral multiplets	U(2) Representation	$\mathrm{U}(1)_R$ charge	twisted masses
ϕ_i, $i=1,\ldots,4$	$\square_{+1}$	0	$\mathfrak{m}_i$

Abelian Coulomb branch gauge theory spectrum:			
Chiral multiplets	$\mathrm{U}(1)\times\mathrm{U}(1)$ charge	$\mathrm{U}(1)_R$ charge	twisted masses
$\phi_i^{(1)}$, $i=1,\ldots,4$	$(+1,0)$	0	$\mathfrak{m}_i$
$\phi_i^{(2)}$, $i=1,\ldots,4$	$(0,+1)$	0	$\mathfrak{m}_i$
$W_\pm$	$(\pm1,\mp1)$	2	0

Table 3.4: The top part of the table shows the chiral matter multiplets of the U(2) gauged linear sigma model of the complex Grassmannian fourfold $\mathrm{Gr}(2,4)$, where the U(2) representation is specified in terms of the Young tableau of the non-Abelian subgroup SU(2) together with the charge of the diagonal U(1) subgroup as a subscript. The bottom part of the table lists the chiral spectrum in the Coulomb branch of the gauge theory, which comprises the decomposition of the non-Abelian matter multiplets into representations of the unbroken Abelian subgroup $\mathrm{U}(1)\times\mathrm{U}(1)$ together with the $W_\pm$ bosons that are part of the Abelianised spectrum.

The Coulomb branch gauge theory with the Abelian gauge group $\mathrm{U}(1)\times\mathrm{U}(1)$ has two Fayet–Iliopoulos parameters (ξ_1,ξ_2) corresponding to the parameters (Q_1,Q_2). These Abelianised parameters stem from the non-Abelian Fayet-Iliopolous parameter Q' such that in the fully non-Abelian theory $Q_1,Q_2\to Q'$. Note that in this basis for the Fayet–Iliopoulos parameters the formal parameters $\vec{Q}$ used in used in the subsection 3.3.2 correspond to,

$$\begin{aligned}
&\vec{Q} = (\sqrt{Q_1 Q_2}, \sqrt{Q_1/Q_2}) \quad \text{or}, \\
&\vec{\tau} = \left(\frac{\tau_1+\tau_2}{2}, \frac{\tau_1-\tau_2}{2}\right) \ .
\end{aligned} \tag{3.71}$$

Working in this basis of the maximal torus $\mathrm{U}(1)\times\mathrm{U}(1)$ has the advantage that under the action of the Weyl group $\mathcal{W}_G$ of $\mathrm{U}(2)$, given by $\mathbb{Z}_2$, the two $\mathrm{U}(1)$ factors of $\mathrm{U}(1)\times\mathrm{U}(1)$ get exchanged. At the level of the correlator insertions, this action permutes the σ_{S_i}, $i=1,2$, insertions in the Coulomb branch correlators.

In the Coulomb branch the matter spectrum decomposes into representations of the Abelian subgroup $\mathrm{U}(1)\times\mathrm{U}(1)$ together with the $W_\pm$ multiplets of the broken gauge group $\mathrm{U}(2)$, as listed in the second half of Table 3.4.

Discriminant Locus and Correlators Since this GLSM is of the Fano type, as discussed above, there exists no solution to the equations (3.16) and thus there is no discriminant locus in the infrared theory.

The first few non-zero correlators of this gauge theory for degree 4 insertions, i.e., $\kappa_{0,f_S}(Q',\mathfrak{m}_i,\epsilon)$, with the gauge invariant function f_S of degree 4 in the south pole insertions, are given by,

$$\left\langle \mathrm{tr}(\sigma_S)^4\right\rangle = 2 \quad , \quad \left\langle \mathrm{tr}(\sigma_S)^2\,\mathrm{tr}(\sigma_S^2)\right\rangle = 0 \quad , \quad \left\langle \mathrm{tr}(\sigma_S^2)^2\right\rangle = 2 \ , \tag{3.72}$$

and for degree 8 insertions are given by,

$$\begin{aligned}
\left\langle \mathrm{tr}(\sigma_S^2)^4\right\rangle &= 20(7\mathfrak{m}_1^4 - 20\mathfrak{m}_1^3\mathfrak{m}_2 + 30\mathfrak{m}_1^2\mathfrak{m}_2^2 - 20\mathfrak{m}_1\mathfrak{m}_2^3 + 7\mathfrak{m}_2^4) - 8Q' \xrightarrow{\mathfrak{m}_i\to 0} -8Q' \\
\left\langle \mathrm{tr}(\sigma_S)^2\,\mathrm{tr}(\sigma_S^2)^3\right\rangle &= 80\mathfrak{m}_1\mathfrak{m}_2(\mathfrak{m}_1^2+\mathfrak{m}_2^2) \xrightarrow{\mathfrak{m}_i\to 0} 0 \\
\left\langle \mathrm{tr}(\sigma_S)^4\,\mathrm{tr}(\sigma_S^2)^2\right\rangle &= 40\mathfrak{m}_1\mathfrak{m}_2(2\mathfrak{m}_1^2+5\mathfrak{m}_1\mathfrak{m}_2+2\mathfrak{m}_2^2) \xrightarrow{\mathfrak{m}_i\to 0} 0 \\
\left\langle \mathrm{tr}(\sigma_S)^6\,\mathrm{tr}(\sigma_S^2)\right\rangle &= 80\mathfrak{m}_1\mathfrak{m}_2(3\mathfrak{m}_1^2+5\mathfrak{m}_1\mathfrak{m}_2+3\mathfrak{m}_2^2) \xrightarrow{\mathfrak{m}_i\to 0} 0 \\
\left\langle \mathrm{tr}(\sigma_S)^8\right\rangle &= -140(\mathfrak{m}_1+\mathfrak{m}_2)^2(\mathfrak{m}_1^2-6\mathfrak{m}_1\mathfrak{m}_2+\mathfrak{m}_2^2) + 8Q' \xrightarrow{\mathfrak{m}_i\to 0} 8Q' \ .
\end{aligned} \tag{3.73}$$

The first three correlators, that are of degree four, compute the classical intersection numbers of the Grassmannian $\mathrm{Gr}(2,4)$, whereas the remaining correlators show the degree one contributions in some of the quantum products. Note that the quantum products (3.73) are in accord with the non-Abelian selection rule (3.45).

Relations of Correlators We first consider the Coulomb branch spectrum in order to arrive at the correlator relations of the non-Abelian gauge theory. The relevant polynomials (3.54) read,

$$\begin{aligned}
g_{p_1,p_2}(w_1,w_2,\mathfrak{m}_i,\epsilon) &= \prod_{i=1}^{4}\left[\prod_{s_1=0}^{p_1-1}(w_1+\mathfrak{m}_i-s_1\epsilon)\prod_{s_2=0}^{p_2-1}(w_2+\mathfrak{m}_i-s_2\epsilon)\right] \\
&\qquad\qquad \times (-1)^{p_1-p_2}\frac{w_1-w_2-(p_1-p_2)\epsilon}{w_1-w_2} \ .
\end{aligned} \tag{3.74}$$

These polynomials lead to the syzygy polynomials α_{p_1,p_2}. The two syzygies are given by the following equations, for varying $\vec{p}$ and $\vec{s}$ in (3.53),

$$\sum_{\vec{p}=(0,0)}^{\vec{s}=(1,0)} \alpha_{\vec{p}}(\vec{w},\mathfrak{m}_\ell,\epsilon)\cdot g_{\vec{p}}(\vec{w},\mathfrak{m}_\ell,\epsilon) \;=\; 0 \quad , \quad \sum_{\vec{p}=(0,1)}^{\vec{s}=(1,2)} \alpha_{\vec{p}}(\vec{w},\mathfrak{m}_\ell,\epsilon)\cdot g_{\vec{p}}(\vec{w},\mathfrak{m}_\ell,\epsilon) \;=\; 0 \ .$$

(i) For the first syzygy over $g_{0,0}$ and $g_{1,0}$ we find the solution,

$$\alpha_{0,0} \;=\; (w_1+\mathfrak{m}_1)\cdots(w_1+\mathfrak{m}_4)(w_1-w_2-\epsilon) \ , \quad \alpha_{1,0} \;=\; (w_1-w_2) \ . \tag{3.75}$$

This syzygy together with its Weyl orbit thus determines the Coulomb branch south pole correlator relations,

$$R_S^{(i)}(\kappa_{\vec{n},\cdot}) \;=\; \left\langle \vec{\sigma}_N^{\vec{n}}(\sigma_{S,i}+\mathfrak{m}_1)\cdots(\sigma_{S,i}+\mathfrak{m}_4)(\sigma_{S,i}-\sigma_{S,i+1}-\epsilon)\right\rangle + Q_i\left\langle \vec{\sigma}_N^{\vec{n}}(\sigma_{S,i}-\sigma_{S,i+1}+\epsilon)\right\rangle \ , \tag{3.76}$$

for $i=1,2$ and with the identification $\sigma_{S,3}\equiv\sigma_{S,1}$.

The relation obtained at this point is in the Abelianised theory and must be required to respect the Weyl symmetry group. Restricting to the non-Abelian physical parameter Q' by taking the limit $Q_1,Q_2\to Q'$ and by projecting to the $\mathcal{W}_G$-invariant part, which for the choice of basis implies a symmetry on the exchange of $\sigma_{S,1}$ and $\sigma_{S,2}$, we obtain from both relations (3.76) the $\mathbb{Z}_2$ invariant correlator relation,

$$\begin{aligned} R_S^{\mathbb{Z}_2} \;=&\; \frac{1}{2}\left\langle f_N(\vec{\sigma}_N)(\sigma_{S,1}+\mathfrak{m}_1)\cdots(\sigma_{S,1}+\mathfrak{m}_4)(\sigma_{S,1}-\sigma_{S,2}-\epsilon)\right\rangle \\ &+\frac{1}{2}\left\langle f_N(\vec{\sigma}_N)(\sigma_{S,2}+\mathfrak{m}_1)\cdots(\sigma_{S,2}+\mathfrak{m}_4)(\sigma_{S,2}-\sigma_{S,1}-\epsilon)\right\rangle + Q'\epsilon\left\langle f_N(\vec{\sigma}_N)\right\rangle \ . \end{aligned} \tag{3.77}$$

(ii) For the second syzygy over the rational functions $g_{0,1}$, $g_{1,0}$, $g_{1,1}$, and $g_{0,2}$ we get the solution,

$$\begin{aligned} \alpha_{0,1} \;=&\; (w_2+\mathfrak{m}_1)\cdots(w_2+\mathfrak{m}_4) - \epsilon\Bigg[2\left(\sum_{1\le i<j<k\le 4}\mathfrak{m}_i\mathfrak{m}_j\mathfrak{m}_k\right) + \left(\sum_{1\le i<j\le 4}\mathfrak{m}_i\mathfrak{m}_j\right)(w_1+3w_2) \\ &+\left(\sum_{i=1}^4\mathfrak{m}_i\right)\left(4w_2^2+w_1w_2+w_1^2\right) + \left(5w_2^3+w_1^3+w_1^2w_2+w_1w_2^2\right)\Bigg] \\ &+\epsilon^2\left[\left(w_1^2+2w_1w_2+9w_2^2\right) + \left(\sum_{i=1}^4\mathfrak{m}_i\right)(w_1+5w_2) + 2\left(\sum_{1\le i<j\le 4}\mathfrak{m}_i\mathfrak{m}_j\right)\right] \\ &-\epsilon^3\left[(w_1+7w_2)+2\left(\sum_{i=1}^4\mathfrak{m}_i\right)\right] + 2\epsilon^4 \ , \\ \alpha_{1,0} \;=&\; -(w_2+\mathfrak{m}_1)\cdots(w_2+\mathfrak{m}_4) \ , \qquad \alpha_{1,1} \;=\; -1 \ , \qquad \alpha_{0,2} \;=\; 1 \ . \end{aligned} \tag{3.78}$$

Analogous to the first syzygy, the second syzygy (3.78) yields the $\mathcal{W}_G$-invariant south pole correlator relation

$$\begin{aligned} T_S^{\mathbb{Z}_2} &= \langle f_N(\vec{\sigma}_N)(\sigma_{S,1}+\sigma_{S,2})(\sigma_{S,1}^2+\sigma_{S,2}^2)\rangle + \left(\sum_{i=1}^4 \mathfrak{m}_i\right)\langle f_N(\vec{\sigma}_N)(\sigma_{S,1}^2+\sigma_{S,1}\sigma_{S,2}+\sigma_{S,2}^2)\rangle \\ &+ \left(\sum_{1\le i<j\le 4}\mathfrak{m}_i\mathfrak{m}_j\right)\langle f_N(\vec{\sigma}_N)(\sigma_{S,1}+\sigma_{S,2})\rangle + \left(\sum_{1\le i<j<k\le 4}\mathfrak{m}_i\mathfrak{m}_j\mathfrak{m}_k\right)\langle f_N(\vec{\sigma}_N)\rangle \ . \end{aligned} \tag{3.79}$$

We have thus obtained two independent Weyl group invariant south pole correlator relations. From these relations, given by (3.77) and (3.79), we must still construct the G-invariant non-Abelian south pole correlator relations. This can be done by noting that the U(2)-invariant polynomial ring $\mathbb{C}[\mathfrak{u}(2)]^{\mathrm{U}(2)}$ is generated by the expressions $\mathrm{tr}(\sigma)$ and $\mathrm{tr}(\sigma^2)$, which map in the Coulomb branch to the symmetric polynomials $\sigma_1+\sigma_2$ and $\sigma_1^2+\sigma_2^2$, respectively. Thus, obtaining the non-Abelian correlator relations amounts to replacing the symmetric functions in two variables in terms of the U(2)-invariant generators $\mathrm{tr}(\sigma)$ and $\mathrm{tr}(\sigma^2)$.

We thus arrive at the non-Abelian south pole relations,

$$\begin{aligned} R_S^{\mathrm{U}(2)} &= \frac{1}{2}\left\langle f_N(\sigma_N)\left[\mathrm{tr}(\sigma_S)^3\,\mathrm{tr}(\sigma_S^2) - 2\,\mathrm{tr}(\sigma_S)\,\mathrm{tr}(\sigma_S^2)^2 - \epsilon\left(\frac{\mathrm{tr}(\sigma_S)^4}{2} - \mathrm{tr}(\sigma_S)^2\,\mathrm{tr}(\sigma_S^2) - \frac{\mathrm{tr}(\sigma_S^2)^2}{2}\right)\right]\right\rangle \\ &+ \left(\sum_{i=1}^4 \frac{\mathfrak{m}_i}{4}\right)\left\langle f_N(\sigma_N)\left[\mathrm{tr}(\sigma_S)^4 - \mathrm{tr}(\sigma_S)^2\,\mathrm{tr}(\sigma_S^2) - 2\,\mathrm{tr}(\sigma_S^2)^2 + \epsilon\left(3\,\mathrm{tr}(\sigma_S)\,\mathrm{tr}(\sigma_S^2) - \mathrm{tr}(\sigma_S)^3\right)\right]\right\rangle \\ &+ \left(\sum_{1\le i<j\le 4}\frac{\mathfrak{m}_i\mathfrak{m}_j}{2}\right)\left\langle f_N(\sigma_N)\left[\mathrm{tr}(\sigma_S)^3 - 2\,\mathrm{tr}(\sigma_S)\,\mathrm{tr}(\sigma_S^2) + \epsilon\,\mathrm{tr}(\sigma_S^2)\right]\right\rangle \\ &+ \left(\sum_{1\le i<j<k\le 4}\frac{\mathfrak{m}_i\mathfrak{m}_j\mathfrak{m}_k}{2}\right)\left\langle f_N(\sigma_N)\left[\mathrm{tr}(\sigma_S)^2 - 2\,\mathrm{tr}(\sigma_S^2) + \epsilon\,\mathrm{tr}(\sigma_S)\right]\right\rangle + \epsilon\left(\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3\mathfrak{m}_4 - Q'\right)\langle f_N(\sigma_N)\rangle \ , \end{aligned} \tag{3.80}$$

and,

$$\begin{aligned} T_S^{\mathrm{U}(2)} &= \langle f_N(\sigma_N)\,\mathrm{tr}(\sigma_S)\,\mathrm{tr}(\sigma_S^2)\rangle + \left(\sum_{i=1}^4\frac{\mathfrak{m}_i}{2}\right)\left\langle f_N(\sigma_N)\left[\mathrm{tr}(\sigma_S)^2 + \mathrm{tr}(\sigma_S^2)\right]\right\rangle \\ &+ \left(\sum_{1\le i<j\le 4}\mathfrak{m}_i\mathfrak{m}_j\right)\langle f_N(\sigma_N)\,\mathrm{tr}(\sigma_S)\rangle + \left(\sum_{1\le i<j<k\le 4}\mathfrak{m}_i\mathfrak{m}_j\mathfrak{m}_k\right)\langle f_N(\sigma_N)\rangle \ . \end{aligned} \tag{3.81}$$

The derived correlators relations enjoy a geometric interpretation as the relations defining the quantum Cohomology ring of the Grassmannian [69]. For simplicity we consider the limit of vanishing twisted masses $\mathfrak{m}_i = 0$, i.e., the non-equivariant case. For vanishing twisted masses, the correlator relation (3.81) generalises to,

$$\begin{aligned} T_S^{(k),\mathrm{U}(2)} &= \left\langle f_N(\sigma_N)\,\mathrm{tr}(\sigma_S)^k\,\mathrm{tr}(\sigma_S^2)\right\rangle \ , \quad k = 1,2,\dots \ , \\ T'^{\mathrm{U}(2)}_S &= \left\langle f_N(\sigma_N)\,\mathrm{tr}(\sigma_S)\,\mathrm{tr}(\sigma_S^2)^2\right\rangle + 2Q'\epsilon\,\langle f_N(\sigma_N)\rangle \ , \end{aligned} \tag{3.82}$$

which are obtained from the syzygy polynomials (3.78) after an overall multiplication with suitable powers of the $(w_1 + w_2)$ or $(w_1^2 + w_2^2)$. Combining these relations with the correlator relation (3.80), we obtain the modified correlator relation,

$$R'^{\mathrm{U}(2)}_S = \left\langle f_N(\sigma_N) \left(-\frac{\operatorname{tr}(\sigma_S)^4}{4} + \operatorname{tr}(\sigma_S)^2 \operatorname{tr}(\sigma_S^2) + \frac{\operatorname{tr}(\sigma_S^2)^2}{4} \right) \right\rangle + Q' \left\langle f_N(\sigma_N) \right\rangle \ . \tag{3.83}$$

In the Appendix A we discuss dictionary between the gauge invariant operator insertions and the Newton polynomials of the Chern roots of the universal subbundle of the $\mathrm{Gr}(2,4)$ that generate the quantum cohomology ring of the Grassmannian, see Appendix A for details. Using this dictionary we can see that the correlator relations $T_S^{(k),\mathrm{U}(2)}$ and $R'^{\mathrm{U}(2)}_S$ precisely realise the quantum cohomology relations . This demonstrates that the correlators of the studied non-Abelian gauged linear sigma model compute quantum cohomology products of the Grassmannian fourfold $\mathrm{Gr}(2,4)$.

Differential Ideal As discussed previously, the correspondence of the relations to the differential ideal of the ring generated by the correlators is limited to the $\operatorname{tr}\sigma_S$ operators for a non-Abelian gauge theory as in (3.64). We need to rewrite the relations purely in terms of such operators in order to make the identification (3.63), (3.64) so as to obtain a differential operator that generates the differential ideal $\boldsymbol{\mathcal{I}}_S$ of the theory.

Upon multiplying the first set of syzygy polynomials (3.75) with the overall factor $(w_1 + w_2)$ we arrive at a correlator relations of degree five in the adjoint insertion σ_S. Removing the quadratic insertions $\operatorname{tr}(\sigma_S^2)$ with the help of correlator relations of the type (3.82), it is straight forward to then deduce the degree five relation,

$$0 = \left\langle f_N(\sigma_N) \operatorname{tr}(\sigma_S)^5 \right\rangle - 2Q' \left\langle f_N(\sigma_N)(2\operatorname{tr}(\sigma_S) + \epsilon) \right\rangle \ , \tag{3.84}$$

which yields the differential operator as per (3.64),

$$\mathcal{L}(Q', \epsilon) = (\epsilon\,\Theta)^5 - 2Q'(2\epsilon\,\Theta + \epsilon) \ . \tag{3.85}$$

This result is in agreement with the work of [70] which was computed using the GKZ Hypergeometric system to compute the differential operator annihilating the I-function.

3.4 Picard-Fuchs Operators from Correlators

In Section 3.3 we studied the relations among south pole correlators of gauged linear sigma models. The form of the Coulomb branch localisation formula for correlators can be appropriately massaged to obtain these relations. Furthermore, equipped with the interpretation of these relations as operators that annihilate the quantum Kähler moduli dependent ground state in a Hilbert space of states, they could be given a representation in terms of differential operators whose solutions could be interpreted to capture the moduli dependence of the ground state. In this section we consider the gauged linear sigma models with Calabi-Yau target spaces. Calabi-Yau manifolds exhibit certain properties such that to each such manifold is an associated unique differential operator $\mathcal{L}$, known as the Picard-Fuchs differential operator, that annihilates the periods of the top-dimensional holomorphic form on this space. We will discuss the fundamental properties of Calabi-Yau manifolds and the Picard-Fuchs operators more closely in the upcoming subsection 3.4.1. For an overview on the Calabi-Yau manifolds and related properties we refer the reader to [71–75] .

In fact, the relations of the previous section happen to give rise to the Picard-Fuchs differential operator for the Calabi-Yau case. Aided with this fact we can now approach the problem of the relations of correlators from a different perspective. We assume that the relations of the form,

$$R_S(\vec{Q}, \mathfrak{m}_l, \epsilon, \kappa_{\vec{n},\cdot}) \;=\; \sum_{\vec{m}=0}^{\vec{N}} c_{\vec{m}}(\vec{Q}, \mathfrak{m}_l, \epsilon) \kappa_{\vec{n},\vec{m}}(\vec{Q}, \mathfrak{m}_l, \epsilon) \;=\; 0 \;, \tag{3.86}$$

are known to exist, as is certainly the case for gauged linear sigma models with Calabi-Yau target spaces due to the existence of the Picard-Fuchs operators. Then the finite number of coefficients $c_{\vec{m}}$ can be reconstructed in terms of correlators $\kappa_{\vec{n},\vec{m}}$ by considering a finite set of relations with increasing north pole insertions. We will explore this calculation for Calabi-Yau threefolds with one Kähler parameter, i.e., $\ell = 1$ in (3.1) at the level of the gauge theory, in subsection 3.4.2.

3.4.1 Calabi-Yau Manifolds

In the discussion of the source and cancellation of the U(1) axial anomaly (3.5) for gauged linear sigma models it was mentioned that the anomaly cancellation at the level of the gauge theory is akin to obtaining in the infrared a non-linear sigma model target space which is Calabi-Yau. Geometrically a Calabi-Yau manifold X of $\dim_{\mathbb{C}} = d$ is defined as a complex Kähler manifold with a $\mathrm{SU}(d)$ holonomy group. We now briefly explain each of the terms that go into defining a Calabi-Yau manifold.

A complex manifold M is defined to be equipped with a complex structure. A complex structure is induced from the so-called almost complex structure $\mathcal{J}$, which is an endomorphism on the tangent bundle of M, s.t., $\mathcal{J}^2 = -1$, under the condition that the Nijenhuis tensor acting on two vector fields on M vanishes. A manifold is said to be Kähler when the Kähler form, that is a $(1,1)$-form defined using the hermitian metric on the manifold, is closed. For a complex orientable manifold of dimension d the holonomy group is most generally given by $\mathrm{U}(d)$. However, for Calabi-Yau manifolds, the U(1) part of the connection vanishes, thus restricting the holonomy group to $\mathrm{SU}(d)$. In fact, the U(1) part of the holonomy is generated by the Ricci tensor and hence the condition for $\mathrm{SU}(d)$ holonomy is equivalent to the condition that the manifold admits a Ricci flat metric. Due to the theorem of [76] the requirement of Ricci flatness is tantamount to the vanishing of the first Chern class, $c_1(X) = 0$. The axial anomaly $\mathrm{U}(1)_A$ of a gauged linear sigma model can be shown to be related to the first Chern class of the target space geometry and the vanishing of one ensures the vanishing of the other. This is a heuristic explanation for why the anomaly cancellation condition is equivalently addressed as the Calabi-Yau condition.

A gauge theory that satisfies the Calabi-Yau condition where all the matter fields have trivial $\mathrm{U}(1)_V$ R-charge assignments yields a non-compact target space. This can be understood by noting that for trivial R-charges, the superpotential W is also trivial, as $\rho_{\mathrm{U}(1)_V}(W) = 2$. Thus the vanishing of the scalar potential implies a Calabi-Yau target space which is a vector bundle with the base space, that is a Fano variety, defined by the positively charged fields and the fibres defined by the negatively charged fields. The terminology of the positive and negative charges is applicable only to the case of Abelian gauge groups, however the technique of Abelianisation discussed in Section 3.2 can be applied to heuristically extend this discussion to the non-Abelian case as well. Contrarily, the Calabi-Yau condition in addition to non-trivial R-charge assignments ensures a non-trivial superpotential. In the infrared, the vanishing scalar potential traces an intersection inside the Fano variety defined by the positively charged fields. If the polynomials appearing in the superpotential condition satisfy certain rank conditions, sometimes known as 'transversality conditions', then the target space Calabi-Yau is compact. For the purpose of the discussion in this section we will adhere to the compact Calabi-Yau case, however, for a complete discussion the reader is invited to study the original work [30].

The geometric properties of the Calabi-Yau d-fold X ensure the existence of a unique holomorphic and nowhere-vanishing $(d, 0)$-form, usually called Ω, which is a section of the canonical bundle. This statement can be reformulated as the fact the vanishing of the first Chern class is tantamount to the triviality of the canonical bundle. For any d-cycle γ on X, one can define periods Π of the holomorphic $(d, 0)$-form Ω as $\Pi := \int_\gamma \Omega$. The periods satisfy a differential equation with respect to the complex structure moduli coordinate known as the Picard-Fuchs differential equation, i.e, $\mathcal{L}\ \Pi = 0$. The Picard-Fuchs differential equation can be calculated using several methods. The authors of [77] do this by an explicit computation of periods. Other methods such can be found in [78–83]. However for the manifolds of interest, these approaches require the construction of the mirror manifold $\widetilde{X}$, except the method of Givental in [83] which yields the mirror map by equating the cohomological I- and J-functions.

In the forthcoming section we will discuss the derivation of Picard-Fuchs operators using the gauged linear sigma model data alone, rendering obsolete the need to construct the mirror manifold and the periods thereof.

3.4.2 Relations of one parameter CY threefolds

The identification of the differential operator representation of relations of south pole correlators to Picard-Fuchs operators can be written as,

$$\mathcal{L}(\vec{Q}, \mathfrak{m}_\ell, \epsilon) \;=\; \boldsymbol{R}_S(\vec{Q}, \epsilon\,\vec{\Theta}, \mathfrak{m}_\ell, \epsilon) \;=\; \sum_{\vec{m}=0}^{\vec{N}} c_{\vec{m}}(\vec{Q}, \mathfrak{m}_\ell, \epsilon)\, \left(\epsilon\,\vec{\Theta}\right)^{\vec{m}} \;, \tag{3.87}$$

For this analysis for the case of non-Abelian gauge theories the notation for correlators follows (3.64). Thus subscripts to κ denote degrees of the trσ operator.

For simplicity we consider the case of vanishing twisted masses, $\mathfrak{m}_\ell = 0$. We aim to express the Picard–Fuchs operator $\mathcal{L}$ in terms of correlators. Any such operator $\mathcal{L}$ can be expanded as,

$$\mathcal{L}(\vec{Q}, 0, \epsilon) \;=\; \sum_{\vec{m}=0}^{\vec{N}} c_{\vec{m}}(\vec{Q}, 0, \epsilon)\, \left(\epsilon\,\vec{\Theta}\right)^{\vec{m}} \quad \text{with} \quad \sum_{\vec{m}=0}^{\vec{N}} c_{\vec{m}}(\vec{Q}, 0, \epsilon)\, \kappa_{\vec{n},\vec{m}}(\vec{Q}, 0, \epsilon) = 0 \tag{3.88}$$

for all possible north pole insertions $\vec{n}$. Assuming that we have the information of the order of this operator $\vec{N}$ we can construct descendant equations from (3.88) for ascending north pole insertion $\vec{m}$. This system of equation can be solved for the coefficients $c_{\vec{m}}(\vec{Q}, \epsilon)$ in the following schematic way, for a detailed methodology we refer the reader to the original paper [30]. We introduce the matrix M of correlators that appear in the system of equations,

$$M_{j,i} = (-1)^{|j|}\, \kappa_{j,i} \;. \tag{3.89}$$

The relation (3.88) can then be schematically described by coefficient vectors c_i's that satisfy,

$$M_{j,i} \cdot c_i = 0 \qquad \forall\, j\ . \tag{3.90}$$

In other words, the vector c_i of coefficient functions is in the kernel of M for all north pole insertions indexed by j.

We will now discuss the case in which the target space is a three-dimensional Calabi–Yau manifold with a single Kähler parameter as target space, i.e., $\dim_{\mathbb{C}} X = 3$ and $\ell = 1$ in (3.1). The correlators can be shown to obey,

$$\epsilon\Theta\, \kappa_{n,m} = \kappa_{n,m+1} - \kappa_{n+1,m}\ , \tag{3.91}$$

$$\kappa_{n,m} = (-1)^{1+n+m} \kappa_{m,n}\ , \tag{3.92}$$

$$\kappa_{n,m} = 0 \quad \text{for} \quad n+m < 3\ . \tag{3.93}$$

using the Coulomb branch localisation formula (3.36) and (3.37).

The knowledge of the order of the operator, i.e., the highest power of Θ in $\mathcal{L}$ must be a pre-requisite in order for this method to work. The variation of Hodge structures shows that the single Picard-Fuchs operator is of order $N = 4$. We therefore take the north pole insertions $\vec{n}$ in the set $I = \{0, 1, 2, 3, 4\}$ and consider the matrix,

$$M = \begin{pmatrix} 0 & 0 & 0 & \kappa_{0,3} & \kappa_{0,4} \\ 0 & 0 & -\kappa_{1,2} & -\kappa_{1,3} & -\kappa_{1,4} \\ 0 & \kappa_{1,2} & 0 & \kappa_{2,3} & \kappa_{2,4} \\ -\kappa_{0,3} & \kappa_{1,3} & -\kappa_{2,3} & 0 & -\kappa_{3,4} \\ -\kappa_{0,4} & \kappa_{1,4} & -\kappa_{2,4} & \kappa_{3,4} & 0 \end{pmatrix}, \tag{3.94}$$

which due to its antisymmetry is required to have a kernel. The solution to the equation $M \cdot c = 0$ is unique up to rescaling and leads to the Picard–Fuchs operator,

$$\begin{aligned}\mathcal{L} \ =& \kappa_{0,3}^2 (\epsilon\Theta)^4 - \kappa_{0,3}\kappa_{0,4}(\epsilon\Theta)^3 + (\kappa_{0,4}\kappa_{1,3} - \kappa_{0,3}\kappa_{1,4})\,(\epsilon\Theta)^2 \\ &+ (\kappa_{0,4}\kappa_{2,3} - \kappa_{0,3}\kappa_{2,4})\,(\epsilon\Theta) + (\kappa_{1,4}\kappa_{2,3} - \kappa_{1,3}\kappa_{2,4} - \kappa_{0,3}\kappa_{3,4})\ .\end{aligned} \tag{3.95}$$

By fully exploiting eq. (3.91) in order to reduce the number of required correlators in the above formula for the Picard-Fuchs operator, this can be rewritten as,

$$\begin{aligned}\mathcal{L} \ =& + \kappa_{0,3}^2 (\epsilon\Theta)^4 - 2\kappa_{0,3}\,(\epsilon\Theta\, \kappa_{0,3})\,(\epsilon\Theta)^3 + \Big[2\,(\epsilon\Theta\,\kappa_{0,3})^2 - \kappa_{0,3}\,\big(\epsilon^2\Theta^2\,\kappa_{0,3} + \kappa_{2,3}\big)\Big]\,(\epsilon\Theta^2) \\ &+ \left[2\kappa_{2,3}\,(\epsilon\Theta\,\kappa_{0,3}) - \kappa_{0,3}\,(\epsilon\Theta\,\kappa_{2,3})\right](\epsilon\Theta) \\ &+ \left[\kappa_{2,3}^2 - \kappa_{0,3}\kappa_{3,4} - (\epsilon\Theta\,\kappa_{0,3})\,(\epsilon\Theta\,\kappa_{2,3}) + \kappa_{2,3}\,\big(\epsilon^2\Theta^2\,\kappa_{0,3}\big)\right]\ .\end{aligned} \tag{3.96}$$

These formulae are valid for all three-dimensional Calabi–Yau manifolds with a single Kähler parameter.

The concept behind the derivation of the formula (3.96) remains the same for all Calabi-Yau manifolds, i.e., for a sufficiently high number of north pole insertions, the system of equations stemming from the relations of correlators can be employed to construct a matrix of correlators whose kernel yields the vector of coefficients. These coefficients then determine the Picard-Fuchs operator, as per (3.88).

3.4.3 Examples

We now consider two one-parameter gauged linear sigma models, Abelian and non-Abelian, with Calabi-Yau target spaces in order to illustrate the potential of equation (3.96) in computing the Picard-Fuchs operator directly.

The Quintic threefold $\mathbb{P}^4[5]$

The quintic hypersurface $\mathbb{P}^4[5]$ in the complex projective space $\mathbb{P}^4$ is the simplest non-trivial example of a compact Calabi–Yau threefold. The quintic arises as the target space of the gauged linear sigma model with the Abelian gauge group U(1) and the matter spectrum listed in Table 3.5.

Chiral multiplets	U(1) charge	$\mathrm{U}(1)_V$ charge	twisted masses
ϕ_i, $i = 1, \ldots, 5$	$+1$	0	$\mathfrak{m}_i$
P	-5	2	$\mathfrak{m}_P$

Table 3.5: Matter spectrum of the U(1) gauged linear sigma model of the quintic Calabi–Yau threefold $\mathbb{P}^4[5]$.

Geometric Target Space The infrared dynamics of this gauge theory was studied in detail in the original work of Witten [22]. This gauge theory is characterised by the non-trivial superpotential,

$$W(\Phi) = P \cdot G^{(5)}(\phi_i) \ , \tag{3.97}$$

with the degree 5 polynomial $G^{(5)}(\phi_i)$ that satisfies the 'transversality condition',

$$\frac{\partial G}{\partial \phi_i} = 0 \ , \ \forall i \quad \text{iff} \quad \phi_i = 0 \ , \ \forall i \ .$$

For the vanishing of the scalar potential in the $\xi > 0$ phase, the D-term ensures that the symplectic quotient space is given by $\mathbb{P}^4$ and the F-term imposes that the intersection locus in $\mathbb{P}^4$ is given by $G^{(5)}(\phi) = 0$, i.e., the final target space in the geometric phase is the quintic threefold $\mathbb{P}^4[5]$.

On the other hand in the $\xi < 0$ phase, the vanishing of the scalar potential results in the P field acquiring an expectation value. Due to the non-minimal charge of the P-field there is a residual $\mathbb{Z}_5$ symmetry and thus the symplectic quotient space is given by a Landau-Ginzburg orbifold $\mathbb{C}^5/\mathbb{Z}_5$, where the $\mathbb{C}^5$ corresponds to the target space of the ϕ's.

Discriminant Locus and Correlators The equation for the discriminant locus (3.16) is given by,

$$Q = (-5\sigma)^{-5}(\sigma)^5 \quad \Rightarrow \quad (1 + 5^5 Q) = 0 \ . \tag{3.98}$$

We also compute a few correlators, noting that for the formula (3.96) only the $\kappa_{0,3}$, $\kappa_{2,3}$ and $\kappa_{3,4}$ are required,

$$\kappa_{0,3} = \frac{5}{1+5^5Q} \ , \quad \kappa_{2,3} = \frac{-6250Q}{(1+5^5Q)^2}\epsilon^2 \ , \quad \kappa_{3,4} = \frac{100Q(-6+59375Q)}{(1+5^5Q)^3}\epsilon^4 \ . \tag{3.99}$$

Note that these correlators satisfy the selection rule (3.45).

Picard-Fuchs Operator We now discuss the derivation of the Picard-Fuchs operator using the techniques of section Sections 3.3 and 3.4.

From the method of Section 3.3 the south pole correlator relations R_S and the differential operators $\mathcal{L}$ can be computed and eventually yields the differential operator,

$$\mathcal{L}(Q,\epsilon,\mathfrak{m}_i,\mathfrak{m}_P) \;=\; (\epsilon\Theta+\mathfrak{m}_1)\cdots(\epsilon\Theta+\mathfrak{m}_5)+Q(5\epsilon\Theta-\mathfrak{m}_P+\epsilon)\cdots(5\epsilon\Theta-\mathfrak{m}_P+5\epsilon)\ . \tag{3.100}$$

In the limit of vanishing twisted masses we find the reduced differential operator,

$$\mathcal{L}(Q,\epsilon) \;=\; \Theta^4+5\,Q\,(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4)\ , \tag{3.101}$$

which is the well-known Picard–Fuchs differential operator of the quintic Calabi–Yau threefold. We should emphasis that from the reduced syzygy module M_S^0 we directly obtain the order four Picard–Fuchs operator for the quintic threefold. Other methods — for instance obtaining the GKZ system from the defining toric data of the quintic hypersurface — often yield the order five differential operator, which is given by eq. (3.100) in the limit of vanishing twisted masses. Arriving at the Picard–Fuchs operator of the desired minimal order is not a coincidence for specific examples of compact Calabi–Yau manifolds, but instead is a general feature of the presented approach.

We now illustrate the computation of the Picard–Fuchs differential operator from gauged linear sigma model correlators using methods of Section 3.4. Inserting the correlators (3.99) into the formula (3.96) yields the differential operator,

$$\mathcal{L} \;=\; \frac{5^2\epsilon^4}{(1+5^5Q)^3}\left[\Theta^4+5\,Q\,(5\Theta+1)(5\Theta+2)(5\Theta+3)(5\Theta+4)\right]\ , \tag{3.102}$$

which, up to a prefactor, agrees with the expected Picard–Fuchs operator (3.101) of the quintic Calabi–Yau threefold [77].

The Rødland Calabi–Yau threefold

As our next example we consider the non-Abelian gauged linear sigma model studied by Hori and Tong [48]. This gauge theory realizes as its two geometric phases the two derived-equivalent families of Calabi–Yau threefold target space varieties [58, 59], first constructed by Rødland [57]. The non-Abelian gauge group is U(2) together with the charged chiral matter spectrum listed in Table 3.6. Furthermore, the table shows the decomposition of the non-Abelian spectrum into Abelian chiral multiplets in the Coulomb branch spectrum with unbroken gauge group $\mathrm{U}(1)\times\mathrm{U}(1)$.

Geometric Target Space The geometric phases of this Hori–Tong gauged linear sigma model were summarised Section 3.1. A brief synopsis of the same reads: In the $\xi>0$ phase the geometric target space is a complete intersection Grassmannian threefold given by $X_{1^7}\subset\mathrm{Gr}(2,7)$ whereas in the $\xi<0$ phase the target space corresponds to a Pfaffian threefold corresponding to a rk 4 locus of the matrix A defined in terms of the coordinates of the $\mathbb{P}^6$.

non-Abelian gauge theory spectrum:			
Chiral multiplets	U(2) Representation	$\mathrm{U}(1)_R$ charge	twisted masses
ϕ_i, $i=1,\ldots,7$	$\square_{+1}$	0	$\mathfrak{m}_i$
P^j, $j=1,\ldots,7$	$\mathbf{1}_{-2}$	2	$\mathfrak{m}_P^j$

Abelian Coulomb branch gauge theory spectrum:			
Chiral multiplets	$\mathrm{U}(1)\times\mathrm{U}(1)$ charge	$\mathrm{U}(1)_R$ charge	twisted masses
$\phi_i^{(1)}$, $i=1,\ldots,7$	$(+1,0)$	0	$\mathfrak{m}_i$
$\phi_i^{(2)}$, $i=1,\ldots,7$	$(0,+1)$	0	$\mathfrak{m}_i$
P^j, $j=1,\ldots,7$	$(-1,-1)$	2	$\mathfrak{m}_P^j$
$W_\pm$	$(\pm1,\mp1)$	2	0

Table 3.6: The table shows the chiral matter multiplets of the U(2) Hori–Tong gauged linear sigma model for the Rødland Calabi–Yau threefold target spaces. The U(2) representation is specified in terms of the Young tableau of the non-Abelian subgroup SU(2) together with the charge of the diagonal U(1) subgroup as a subscript. Moreover, the table lists the chiral spectrum in the Coulomb branch of the gauge theory, where the Coulomb branch chiral fields fall into representations of $\mathrm{U}(1)\times\mathrm{U}(1)$.

Discriminant Locus and Correlators The discriminant locus for the Rødland model was also computed in Section 3.3 and is given by the equation (3.33).

Let us first connect the correlators of the Hori–Tong gauged linear sigma model to the geometry of the Calabi–Yau threefold X_{1^7}. The correlators of respectively gauge invariant south and north pole insertions arise from insertions of the type $\mathrm{tr}(\sigma)$ and $\mathrm{tr}(\sigma^2)$. There are two distinct correlators with south pole insertions cubic in the adjoint field σ (in the absence of north pole insertions),

$$\begin{aligned}\kappa_{0,3} &= \left\langle \mathrm{tr}(\sigma_S)^3\right\rangle = \frac{14(3+Q)}{1+57Q-289Q^2-Q^3}\ ,\\ \left\langle \mathrm{tr}(\sigma_S)\,\mathrm{tr}(\sigma_S^2)\right\rangle &= \frac{14(1-9Q)}{1+57Q-289Q^2-Q^3}\ .\end{aligned}\tag{3.103}$$

Apart from the Yukawa coupling correlator $\kappa_{0,3}$ the formula (3.96) to compute the Picard-Fuchs operators requires the correlators $\kappa_{2,3}$ and $\kappa_{3,4}$, which for the given example are calculated to be,

$$\begin{aligned}\kappa_{2,3} &= \frac{28Q(-51+787Q+75Q^2+Q^3)}{(1+57Q-289Q^2-Q^3)^2}\epsilon^2\ ,\\ \kappa_{3,4} &= \frac{14Q(-15+3340Q-32415Q^2+614760Q^3+20747Q^4-20Q^5+3Q^6)}{(1+57Q-289Q^2-Q^3)^3}\epsilon^4\ .\end{aligned}\tag{3.104}$$

Picard Fuchs Operator We want to extract the Picard–Fuchs differential equation of the quantum Kähler moduli space of the Rødland Calabi–Yau threefolds by using the universal correlator formula (3.96). This example demonstrates that correlator formulas for Picard–Fuchs equations are applicable and particularly powerful for non-Abelian gauged linear sigma models, as other methods are often more intricate to implement. Upon explicitly plugging in the values of these correlators, computed in (3.103) and (3.104), into formula (3.96), we find the Picard–Fuchs operator,

$$\begin{aligned}\mathcal{L} \,=\, &\frac{196\epsilon^4}{(1+57Q-289Q^2-Q^3)^3}\Big[(3+Q)^2(1+57Q-289Q^2-Q^3)\Theta^4 \\ &+4Q(3+Q)(85-867Q-149Q^2-Q^3)\Theta^3+2Q(408-7597Q-2353Q^2-239Q^3-3Q^4)\Theta^2 \\ &\quad+2Q(153-4773Q-675Q^2-87Q^3-2Q^4)\Theta+Q(45-2166Q-12Q^2-26Q^3-Q^4)\Big]\,,\end{aligned} \tag{3.105}$$

which is in agreement with the literature [57] up to an overall factor. Note that the Picard–Fuchs operator exhibits an apparent singularity at $Q=-3$. While this corresponds to a smooth point in moduli space with regular solutions, we observe that the Yukawa coupling $\kappa_{0,3}$ vanishes with the above discussed implications on the chiral ring. Such apparent singularities are a consequence of the universal form of the Picard–Fuchs operator (3.95), which implies either that $\kappa_{0,3}$ vanishes (as in the given example) or that the discriminant locus has a spurious singular component. Similarly as for the complex Grassmannian fourfold $\mathrm{Gr}(2,4)$, it is also possible to deduce non-Abelian correlator relations from the Abelian Coulomb branch spectrum summarised at the bottom of Table 3.6, which then — upon restricting to the linear gauge invariant insertions $\operatorname{tr}\sigma$ — yield differential operators. However for this higher dimensional case, the method universal formula for Calabi-Yau target spaces proves to come in more handy.

This example illustrates that simply computing the three correlators $\kappa_{0,3}$, $\kappa_{2,3}$, and $\kappa_{3,4}$ is a powerful approach to derive Picard–Fuchs operators of any Calabi–Yau threefolds with a single Kähler modulus — even for projective non-complete intersection varieties or complete intersections in non-toric varieties. The Rødland sigma model target space at the large volume point in the Kähler moduli space, $X_{1^7} \subset \mathrm{Gr}(2,7)$, is an example of the latter.

3.5 Givental's Cohomological I-function and GLSMs

We will now discuss the connection of the correlator relations as generators of a differential ideal over the quantum Kähler moduli space of the gauged linear sigma model target space X to the cohomology — of the target space X — valued formal function known as the Givental's cohomological I-function, referred to as simply the I-function in this section. Certain enumerative invariants known as Gromov-Witten invariants can be interpreted as entities that connect these two seemingly entirely distinct concepts. We will proceed in this section by giving a lightning-quick review of the Gromov-Witten invariants and the I-function to the extent required to understand the connection to gauged linear sigma model correlators and relations thereof. This summary, which is inspired from [71, 73], will be followed by an important result of [84, 85] which spells out an explicit relationship between the I-function and the A-twisted gauge theory correlators. In particular, their technique provides a methodology to obtain the latter entirely from the former. Finally, we state how this correspondence manifests itself in the context of the differential operator representation of the correlator relations.

3.5.1 Gromov-Witten Invariants and the I-function

Gromov-Witten invariants are invariants in enumerative geometry associated to a Riemann surface Σ and a target space X. They are a consequence of the study of the moduli space $\mathcal{M}_g$ of complex one-dimensional curves Σ_g, also know as Riemann surfaces, of genus g. For instance, the $g = 0$ curve is the two-sphere $S^2 \simeq \mathbb{P}^1$ and the $g = 1$ curve is the two-torus T^2. The terminology 'curve', which frequently used to denote Riemann surfaces, can understood by noting that $\dim_{\mathbb{C}}(\Sigma_g) = 1$. Although, strictly speaking, a complex curve corresponds to an oriented Riemann surface equipped with a complex strucure. A generalisation of the such a complex curve is to consider one with n marked points with the corresponding moduli space being denoted by $\mathcal{M}_{g,n}$. Furthermore, we define stable curves as curves with no continuous automorphism group. It is possible to consider the compactifications of these moduli spaces, denoted by $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$, by considering stable curves.

Moduli Space of Stable Maps

A further generalisation of the compactified moduli space of stable curves, $\overline{\mathcal{M}}_{g,n}$, can be achieved by considering the compactified moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of stable maps φ from a curve $\Sigma_{g,n}$ of genus g with n marked points to a target space X with $\beta \in H_2(X, \mathbb{Z})$. Here the map φ, with $\varphi : \Sigma_{g,n} \to X$, is such that $\varphi([\Sigma_{g,n}]) = \beta$. A special case is that of a map, from the curve to a point in X, which is the curve itself. From the perspective of physics, the map φ can be understood as a non-linear sigma model from the worldsheet to the target space.

The moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has a so-called *virtual* dimension,

$$\text{vdim } \overline{\mathcal{M}}_{g,n}(X, \beta) = \int_\beta c_1(T_X) + (\dim_{\mathbb{C}} X - 3)(1 - g) + n \ , \tag{3.106}$$

where $c_1(T_X) \in H^2(X)$ is the first Chern class of the tangent bundle T_X on X. The virtual dimension can be understood as the expected dimension of the moduli space as all deformations modulo the obstructions contribute to it.

Of primary interest to us are Riemann surfaces of genus $g = 0$ where the virtual dimension of the moduli spaces of stable maps vanishes, implying that heuristically the moduli space collapses to a countably finite set of points for a certain multiple of $\beta \in H_2(X, \mathbb{Z})$. For instance, this happens to be the case when X is a Calabi-Yau threefold for $\overline{\mathcal{M}}_{0,0}(X, \beta)$ or alternatively for $\overline{\mathcal{M}}_{0,3}^{D_1,D_2,D_3}(X, \beta)$, where D_1, D_2, and D_3 are divisors on X such that,

$$\overline{\mathcal{M}}_{0,3}^{D_1,D_2,D_3}(X, \beta) = \left\{ \left(\mathbb{P}^1, p_1, p_2, p_3, \varphi\right) \in \overline{\mathcal{M}}_{0,3}(X, \beta) \mid \varphi(p_i) \in D_i, \text{ for } i \in \{1, 2, 3\} \right\} \ . \tag{3.107}$$

It is for this reason that Calabi-Yau threefolds, that play an essential role in string theory as ideal candidates for internal compactification spaces, are of importance to Gromov-Witten theory as well.

The final ingredient are the n evaluation maps ev_i,

$$\text{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X \ , \tag{3.108}$$

such that $\text{ev}_i(\Sigma_{g,n}, p_1, \ldots, p_n, \varphi) = \varphi(p_i)$.

Gromov-Witten Invariants

Ultimately, the Gromov-Witten invariants corresponding to the moduli spaces $\overline{\mathcal{M}}_{g,n}(X,\beta)$ are defined using the pullback evaluation maps as,

$$\langle I_{g,n,\beta}\rangle(\alpha_1,\ldots,\alpha_n) = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\alpha_1)\cup \mathrm{ev}_2^*(\alpha_2)\cup\ldots\cup \mathrm{ev}_n^*(\alpha_n)\ , \tag{3.109}$$

where the $\alpha_1,\ldots\alpha_n \in H^*(X)$ are the Poincaré duals of the divisors $D_1,\ldots,D_n$ considered in (3.107) for the case $n=3$ and the $\cup$ is the usual cup product of the elements of a cohomology ring. The domain of integration is defined precisely in mathematics as the *virtual fundamental class* corresponding to the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$, such that $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}$ is a cycle of the expected dimension (3.106). Thus the virtual dimension, while physically intangible for generic moduli spaces, plays an important role in defining Gromov-Witten invariants.

Let us focus on the case described in (3.107) of a Calabi-Yau threefold X equipped with the complexified Kähler class $\omega = B + iJ$, where B is an antisymmetric two-form and J is the real Kähler form on X. In the A-model, the operators that are in the BRST cohomology correspond to elements of the cohomology group $H^{(1,1)}(X)$. Then the A-model correlator for the BRST-closed operators $\alpha_1,\alpha_2,\alpha_3 \in H^{(1,1)}(X)$ can be defined in terms of the Gromov-Witten invariants as,

$$\langle\alpha_1,\alpha_2,\alpha_3\rangle = \sum_{\vec{d}=0}^{\infty}\langle I_{0,0,\vec{d}}\rangle(\alpha_1,\alpha_2,\alpha_3)\vec{Q}^{\vec{d}}\ , \tag{3.110}$$

where,

$$\vec{Q}^{\vec{d}} := \exp\left(2\pi i\int_{\vec{d}}\omega\right)\ , \tag{3.111}$$

and $\vec{d}\in H_2(X,\mathbb{Z})$ with ω being expressed in the basis ω_k of $H^2(X)$ as $\sum_k t^k\,\omega_k$. The degeneracy in notation of this $\vec{Q}$ with the Fayet-Iliopolous parameter of the gauged linear sigma model is not entirely coincidental as both of them signify parameters on the complexified Kähler moduli space of the target space X. From the theory of topologically twisted non-linear sigma models [23], reviewed briefly in Chapter 2, the A-model correlators are expected to be of the form,

$$\langle\alpha_1,\alpha_2,\alpha_3\rangle = \int_X \alpha_1\wedge\alpha_2\wedge\alpha_3 + \sum_{\vec{d}} n_{\vec{d}}\int_{\vec{d}}\alpha_1\int_{\vec{d}}\alpha_2\int_{\vec{d}}\alpha_3\frac{\vec{Q}^{\vec{d}}}{1-\vec{Q}^{\vec{d}}}\ , \tag{3.112}$$

where $n_{\vec{d}}\in\mathbb{Z}_{\geq 0}$ counts the holomorphic curves corresponding to degree $\vec{d}$. The first term in this formal series corresponds to the classical intersection number of the 2-cycles dual to α_1,α_2 and α_3. Thus, the curve counting information stored in the numbers $n_{\vec{d}}$ directly yields the Gromov-Witten invariants defined in (3.109).

Specifically, for a Calabi-Yau threefold with a one dimensional Kähler moduli space, i.e., $\dim H^{1,1}(X) = 1$ the A-model correlator (3.110) simplifies in the following way. For $H\in H^{1,1}(X)$ denoting the hyperplane class,

$$\langle H,H,H\rangle = \int H\wedge H\wedge H + \sum_{d=0}^{\infty} n_d\, d^3\frac{Q^d}{1-Q^d}\ . \tag{3.113}$$

From a gauged linear sigma model perspective would correspond to a one-parameter model with $\ell = 1$ in (3.1).

Mirror symmetry postulates and proves an equivalence of A- and B-model correlators for mirror manifolds X and $\widetilde{X}$. Thus the A-model correlators of a target space X, and hence the corresponding Gromov-Witten invariants, can be computed given the B-model of the mirror manifold $\widetilde{X}$ and the so-called 'mirror map' between the Kähler moduli space parameter and the complex structure moduli space parameter or X and $\widetilde{X}$, respectively. As a starting point, the Riemann surfaces of interest to us are those with genus $g = 0$, i.e, two-spheres. The advantage of computing the correlators of an A-twisted gauged linear sigma model on a two-sphere is that these correlators yield precisely the B-model coupling of the mirror target space when the Fayet-Iliopolous parameter $\vec{Q}$ is identified the complex structure moduli space parameter of the mirror. Thus, this method circumvents the need to construct the mirror manifold which is a non-trivial procedure in itself [77, 86]. However, from the existing techniques the gauge theory has to yet been shown to yield the mirror map, which is needed for extracting the Gromov-Witten invariants from the B-model couplings.

Givental's Cohomological I-function

A seminal development in the field of mirror symmetry came with Givental devising the cohomological I- and J-function of a space X [83]. Givental's proof of the mirror theorem equates these functions in a way that indeed identifies A- and B-model correlators of mirror manifolds. While the J-function can be computed as a solution to the so-called 'quantum differential equations', the I-function is a formal function that can be constructed for Fano toric varieties and intersections therein. In [87] Ciocan-Fontanine, Kim and Maulik generalise the Givental I-function to more general GIT quotients than toric varieties. These geometries relate to gauged linear sigma model target spaces of non-Abelian gauge groups.

In order to provide a schematic explanation of [85], which we will cover in the forthcoming section, we start this discussion by introducing complete intersections in a compact weak Fano toric variety in the language of gauged linear sigma model target spaces. A weak Fano toric variety — denoted by $\mathbb{P}^d_\Delta$ — can be obtained from a $\mathrm{U}(1)^\ell$ gauge theory with chiral multiplets X_i, $i = 1, \ldots, d + \ell$, of gauge charge $\vec{\rho}^{\,x}_i$ and vanishing $\mathrm{U}(1)_R$ charge $\mathfrak{q}^x_\ell = 0$. The weak Fano condition implies that $\sum_\ell \rho^x_{\ell,s} \geq 0$ for all $s = 1, \ldots, r$. The gauge theory now realises the target space geometry $\mathbb{P}^d_\Delta$ as the vector space $\mathbb{C}^{d+\ell}$ spanned by the chiral multiplets X_i modulo the gauge transformations acting on the fields X_i. This geometric space is described precisely in terms of the symplectic quotient defined in (3.22). In particular the complete intersection of codimension n is given by $X = \mathbb{P}^d_\Delta(\vec{\rho}^{\,p}_1, \ldots, \vec{\rho}^{\,p}_n)$.

Givental introduced a cohomology-valued function known as the I-function in [83] for such spaces. The I-function is a formal function in terms of the 'input' $\vec{t} = (t_1, \ldots, t_\ell)$ — the coordinates on $H^2(X)$ with respect to the basis $\vec{p} = (p_1, \ldots, p_\ell)$ of $H^2(X)$ — and in the parameter $\hbar$. It maps to the (even) cohomology ring $H^{\mathrm{ev}}(X)$ and reads,

$$I_X(\vec{t}, \hbar) \;=\; e^{\vec{t}\cdot\vec{p}/\hbar} \sum_{\vec{k}} e^{\vec{t}\cdot\vec{k}} \frac{\prod_{i=1}^n \prod_{s=-\infty}^{\vec{k}(v_i)} (v_i + s\hbar) \prod_{j=1}^{d+\ell} \prod_{s=-\infty}^{0} (u_j + s\hbar)}{\prod_{i=1}^n \prod_{s=-\infty}^{0} (v_i + s\hbar) \prod_{j=1}^{d+\ell} \prod_{s=-\infty}^{\vec{k}(u_\ell)} (u_j + s\hbar)} \in H^{\mathrm{ev}}(X) \,. \qquad (3.114)$$

Here u_i, $i = 1, \ldots, d + \ell$, are the toric hyperplane classes of $\mathbb{P}^d_\Delta$ generating the ring $H^{\mathrm{ev}}(\mathbb{P}^d_\Delta)$, and v_i, $i = 1, \ldots, n$, are the first Chern classes of the non-negative line bundles $\mathcal{O}_\Delta(\vec{\rho}^{\,p}_i)$ associated with the complete intersection X. The sum runs over the semi-group of compact holomorphic curves $\vec{k}$ in the variety X, and $\vec{k}(\,\cdot\,)$ abbreviates the intersection pairing $\int_{\vec{k}} \cdot$. The I-function upon being suitably expanded in the basis of $H^{\mathrm{ev}}(\mathbb{P}^d_\Delta)$ yields the mirror-map as a ratio of the second to the first term, which coincide with the log-period and the fundamental period of target space. Furthermore, Givental's mirror theorem, which associates the I- and the J-function in a suitable way, also provides the B-model correlator directly from the I-function. With these two facts at hand, the I-function can yield Gromov-Witten invariants of the associated target space.

In [85], a formula for the cohomological I-function was also stated for the Grassmannian $\mathrm{Gr}(M,N)$ using localisation methods, which we present here now. A complex Grassmannian $\mathrm{Gr}(M,N) =: X$ has an M-dimensional universal subbundle with Chern roots given by $x_1,\ldots,x_M \in H^2(X)$, see appendix A for details on the standard bundles on the Grassmannian. In terms of an input t, the I-function is given by,

$$I_{\mathrm{Gr}(M,N)}(\vec{t},\hbar) \;=\; \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} (-1)^{(M-1)\sum_i^M d_i} e^{\sum_i^M (d_i + x_i/\hbar)t} \cdot \frac{\prod\limits_{1\leq i<j\leq M} (x_i - x_j + (d_i - d_j)\hbar)}{\prod\limits_{1\leq i<j\leq M} (x_i - x_j) \prod\limits_{i=1}^{M} \prod\limits_{l=1}^{d_i} (x_i + l\hbar)^N} \,. \tag{3.115}$$

We will recall this form of the I-function in Chapter 4 in order to compare it with its lift in 3d to the quantum K-theoretic I-function of Givental [83].

3.5.2 I-function and GLSM correlators

The work of [85] conjectures and proves for certain classes of examples a direct relationship between the Givental I-function and the gauged linear sigma model correlators. The authors define a function Φ as a bilinear pairing of Givental I-function as,

$$\Phi(\vec{t},\vec{t}',\hbar) \;=\; \int_X I_X(\vec{t},-\hbar) \cup I_X(\vec{t}',\hbar) \,. \tag{3.116}$$

The conjecture of [85] asserts that Φ is the generating function of the discussed gauge linear sigma model correlators.

Chiral multiplets	$\mathrm{U}(1)^r$ charge	$\mathrm{U}(1)_R$ charge	twisted masses
X_i, $i=1,\ldots,d+\ell$	$\vec{\rho}^{\,x}_i$	0	$\mathfrak{m}_i$
P^j, $j=1,\ldots,n$	$-\vec{\rho}^{\,p}_j$	2	$\mathfrak{m}^j_P$

Table 3.7: This table shows the matter spectrum of the $\mathrm{U}(1)^r$ gauged linear sigma model with the semi-classical large volume target space $X = \mathbb{P}^d_\Delta(\vec{\rho}^{\,p}_1,\ldots,\vec{\rho}^{\,p}_n)$. The $\mathrm{U}(1)^\ell$ charge vectors of the chiral fields X_i correspond to the one-dimensional cones in the fan Δ, realising the toric variety $\mathbb{P}^d_\Delta$ as the ambient space of X. Furthermore, the chiral fields P^j are responsible for the complete intersection locus $X \subset \mathbb{P}^d_\Delta$, which arises in the gauge theory from the F-terms of the superpotential.

We expect that the stated correlator conjecture (3.118) holds beyond the class of Abelian gauged linear sigma models. In fact the authors of [85] prove the correspondence for Grassmannian target spaces. For a further connection, appended by several non-trivial examples, between non-Abelian gauged linear sigma models and I-function on the target spaces, see [88]. However, in order to understand the essence of this relation, let us consider the Abelian gauged linear sigma model with the chiral matter spectrum displayed in Table 3.7. It realises the complete intersection X as its semi-classical target space in the large volume phase. Then, upon identifying the arguments of the Givental's I-function (3.114) with the gauge theory parameters according to the dictionary,

$$\epsilon = \hbar \,, \qquad e^{\vec{t}\cdot\vec{k}} = (-1)^{\sum_{i=1}^{n} \vec{\rho}_i^{\,p}} \vec{Q}^{\vec{k}} \,, \tag{3.117}$$

the correlators (3.41) of the gauged linear sigma model in Table 3.7 are given by,

$$\kappa_{\vec{n},\vec{m}}(\vec{Q},0,\epsilon) = \epsilon^{|\vec{m}|_1}(-\epsilon)^{|\vec{n}|_1} \left.\frac{\partial^{|\vec{n}|_1+|\vec{m}|_1}\Phi(\vec{t},\vec{t}',\epsilon)}{\partial t_1^{n_1}\cdots\partial t_r^{n_r}\partial {t'_1}^{m_1}\cdots\partial {t'_r}^{m_r}}\right|_{\vec{t}=\vec{t}'=\log(\pm\vec{Q})} . \tag{3.118}$$

For the correspondence we focus on the correlators for vanishing twisted masses $\mathfrak{m}_i$ and $\mathfrak{m}_P^j$. In fact, the twisted mass dependence can be quickly recovered by noting that the equivariant parameters of the toric $\mathbb{C}^*$-symmetries, Λ_i, $i=1,\ldots,d+\ell$, and those of the $\mathbb{C}^*$-symmetries of the line bundles $\mathcal{O}_\Delta(\vec{\rho}_j^{\,p})$, Λ'_j, $j=1,\ldots,n$, correspond precisely to the twisted masses $\mathfrak{m}_i$ and $\mathfrak{m}_P^j$, respectively. Thus it is straight-forward to restore the twisted masses in order to obtain the generalised correlator correspondence in the equivariant setting. Inserting the geometric definition (3.118) of the gauge theory correlators into a south pole correlator relation (3.47) and using eqs. (3.56) and (3.62), we find,

$$\begin{aligned} 0 = R_S(\vec{Q},\epsilon,\kappa_{\vec{n},\cdot}) &= (-1)^{|n|_1}\int_X (\epsilon\vec{\Theta})^{\vec{n}} I_X(\vec{Q},-\epsilon)\cup\left(\sum_{\vec{m}} c_{\vec{m}}(\vec{Q},\epsilon)(\epsilon\vec{\Theta})^{\vec{m}} I_X(\vec{Q},\epsilon)\right) \\ &= (-1)^{|n|_1}\int_X (\epsilon\vec{\Theta})^{\vec{n}} I_X(\vec{Q},-\epsilon)\cup\left(\boldsymbol{R}_S(\vec{Q},\epsilon\vec{\Theta},\epsilon) I_X(\vec{Q},\epsilon)\right) . \end{aligned} \tag{3.119}$$

Using the relation between $\vec{t}$ and $\vec{Q}$ as in (3.111) and (3.117), we express the Givental I-function in terms of the gauge theory parameters $\vec{Q}$ instead of the parameters $\vec{t}$. As the above relation holds for general $\vec{n}$, we conclude that the differential operators $\boldsymbol{R}_S$ of the south pole correlator relations annihilate the Givental I-function, i.e.,

$$\boldsymbol{R}_S(\vec{Q},\epsilon\vec{\Theta},\epsilon) I_X(\vec{Q},\epsilon) = 0 \,. \tag{3.120}$$

This result explicitly connects the differential operators obtained from the gauge theory correlator relations with the quantum cohomology of the target space geometry. The established relationship of the correlator relations to the Givental I-function also reflects the close relationship between the analysed correlators and the quantum A-periods of the A-twisted gauged linear sigma model considered in [89].

3.5.3 Examples

We now flesh out the ideas mentioned above by stating the I-functions for the gauged linear sigma model target spaces discussed in Sections 3.1, 3.3 and 3.4 from the literature and noting that these I-functions are annihilated by the differential operator representations of the respective correlators relations computed in the previous sections.

Projective Line $\mathbb{P}^1$

For the projective line the Givental I-function takes the form [83, 90],

$$I_{\mathbb{P}^1}(H,Q,\epsilon,\mathfrak{m}_i) \;=\; \sum_{k=0}^{\infty} \frac{1}{\prod_{\ell=1}^{k}(H+\mathfrak{m}_1+\ell\epsilon)(H+\mathfrak{m}_2+\ell\epsilon)} Q^{\frac{H}{\epsilon}+k} \;. \tag{3.121}$$

Here, H is the hyperplane divisor of $\mathbb{P}^1$ and the twisted masses $\mathfrak{m}_i$ correponds to the equivariant parameters of the $(\mathbb{C}^*)^2$-action canonically acting on the homogeneous coordinates of the projective line $\mathbb{P}^1$. The differential operator representation of the correlator relations given by (3.69),

$$\mathcal{L}(Q,\epsilon,\mathfrak{m}_i) = (\epsilon\Theta+\mathfrak{m}_1)(\epsilon\Theta+\mathfrak{m}_2) - Q \;, \tag{3.122}$$

annihilates the stated I-function as predicted in (3.120), with $\theta = Q\partial_Q$.

Quintic $\mathbb{P}^4[5]$

The Givental I-function of the quintic hypersurface for generic twisted masses is given by [83, 90],

$$I_{\mathbb{P}^4[5]}(H,Q,\epsilon,\mathfrak{m}_i,\mathfrak{m}_P) \;=\; \sum_{k=0}^{\infty} \frac{\prod_{\ell=1}^{5k}(5H-\mathfrak{m}_P+\ell\epsilon)}{\prod_{\ell=1}^{k}(H+\mathfrak{m}_1+\ell\epsilon)\cdots(H+\mathfrak{m}_5+\ell\epsilon)} (-Q)^{\frac{H}{\epsilon}+k} \;. \tag{3.123}$$

The differential operator given by (3.100) annihilates this I-function,

$$\mathcal{L}(Q,\epsilon,\mathfrak{m}_i,\mathfrak{m}_P) \;=\; (\epsilon\Theta+\mathfrak{m}_1)\cdots(\epsilon\Theta+\mathfrak{m}_5) + Q(5\epsilon\Theta-\mathfrak{m}_P+\epsilon)\cdots(5\epsilon\Theta-\mathfrak{m}_P+5\epsilon) \;, \tag{3.124}$$

namely $\mathcal{L} I_{\mathbb{P}^4[5]} = 0$ as expected.

Grassmannian $\mathrm{Gr}(2,4)$

For this example we focus on the isomorphism $\mathrm{Gr}(2,4) \simeq \mathbb{P}^5[2]$ from (3.70) attained from the Plücker embedding of the Grassmannian into a projective space along with a Plücker relation. The Givental I-function of quadratic hypersurfaces in $\mathbb{P}^5$ for vanishing twisted masses is given by [83, 90],

$$I_{\mathbb{P}^5[2]}(H,Q',\epsilon) \;=\; \sum_{k=0}^{\infty} \frac{\prod_{\ell=1}^{2k}(2H+\ell\epsilon)}{\prod_{\ell=1}^{k}(H+\ell\epsilon)^6} Q^{\frac{H}{\epsilon}+k} \;, \tag{3.125}$$

where H is the hyperplane class of $\mathbb{P}^5$. The differential operator that annihilates this I-function is given by,

$$\mathcal{L}(Q,\epsilon) = (\epsilon\,\Theta)^5 - 2Q(2\epsilon\,\Theta+\epsilon) \;, \tag{3.126}$$

and it evidently coincides with our computation (3.85).

CHAPTER 4

3d Gauge Theories and Wilson Loop Algebras

The focus of this chapter will be supersymmetric gauge theories in 3d, specifically $\mathcal{N} = 2$ theories on the solid torus $D^2 \times S^1$. The broad study of supersymmetric gauge theories with four supercharges in three dimensions was initiated by the works of [51, 91, 92], which were themselves followed by the study of $\mathcal{N} = 4$ theories in 3d in [93–96]. The comprehensive work in [97] established a correspondence between 3d gauge theories with $\mathcal{N} = 2$ supersymmetry and quantum K-theory on their target spaces, motivated in [98–100]. In particular, this work proposed that 3d $\mathcal{N} = 2$ gauge theories with a target space X serve as an ultraviolet lift of permutation equivariant quantum K-theory on X [31], much in the same way the 2d gauged linear sigma models of Chapter 3 are ultraviolet lifts of quantum cohomology on the target space. The results and implications of this work provide fodder for the original work of the author [32] which will be covered in this chapter. As in the case of the previous chapter on 2d, we will begin in Section 4.1 by briefly summarising the Lagrangian in flat 3d and we will note that while various key players of 2d theories make a reappearance, there are additional phenomena specific to 3d that provide this class of theories a further layer of complexity. In Section 4.2 we will hone in on theories defined on the solid torus $D^2 \times S^1$ and state the partition function obtained via localisation in [54], although other methods to compute the partition function exist [101, 102]. Furthermore, we will focus explicitly on the gauge theories with target spaces that are Grassmannian manifolds $\mathrm{Gr}(M, N)$ and compute their localised partition function. Motivated by the of 3d gauge theory/quantum K-theory correspondence of [97], from the partition function of gauge theories with Grassmannian target spaces we will extract and explain the I-function of Givental's permutation symmetric quantum K-theory. We will move on in Section 4.3 by first giving a brief introduction to objects of quantum K-theory that will be relevant to us. We will then evaluate the relations of the quantum K-theory ring on the Grassmannian and compute the so-called 'quantum difference equations' that annihilate this I-function. Finally, we will briefly study the 2d limits of each of the quantities computed in Section 4.3 to find agreement with the results from 2d computations in Chapter 3.

4.1 Lagrangian in 3d

Since we glimpsed the Lagrangian of gauge theories in 3d with four supercharges in Section 2.3 of Chapter 2, we will collect all the terms here. Recall that the major ingredients are the chiral multiplet Φ in a representation $\mathcal{R}$ of the gauge group G, vector multiplet V and the outer automorphism group of the supersymmetry algebra $U(1)_R$ symmetry. There is also a linear multiplet Σ, a cousin of the twisted chiral multiplet in 2d, whose lowest component is is the real scalar in the vector multiplet corresponding to the component of the 4d vector multiplet gauge potential in the extra direction. It is defined as $\Sigma := \epsilon^{\alpha\beta}\overline{D}_\alpha D_\beta V$, where D is usual differential operator on superspace. In addition there is a supersymmetric Chern-Simons term in 3d theories with a gauge group G, a descendant of Chern-Simons terms in non-supersymmetric theories. The gauge group G takes the generic form it did in the previous chapter, i.e., $G = \left(\mathrm{U}(1)^\ell \times H_1 \times \ldots \times H_m\right) / \Gamma$ where H_i's are semi-simple groups and Γ is a discrete normal subgroup. The Lagrangian consists of kinetic terms for the chiral and vector multiplet, a term for the superpotential W, a Fayet-Iliopolous (FI) term for each U(1) factor in G and finally a supersymmetric Chern-Simons (CS) term. It takes the form,

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{(\text{matter kin.+gauge kin.})} + \mathcal{L}_W + \mathcal{L}_{\text{FI}} + \mathcal{L}_{\text{CS}} \\ &= \left(\int d^4\theta\ (\overline{\Phi}e^V\Phi + \frac{1}{g_s^2}\text{tr}_{\text{adj.}}(\Sigma^2))\right) + \left(\int d^2\theta\ W(\Phi) + \text{ h.c. }\right) \\ &+ \sum_{a=1}^{\ell}\left(i\zeta_a \int d^4\theta\ V_a\right) + \mathcal{L}_{\text{CS}}\ .\end{aligned} \tag{4.1}$$

As for the 2d case, the expression above is actually the Lagrangian density, however we will consistently refer to it as the Lagrangian. As always, in the kinetic term for the matter multiplet, the vector multiplet V in the exponent is given by $V = V_a T^a_{\mathcal{R}}$ where $T^a_{\mathcal{R}}$ are the generators of G in the representation $\mathcal{R}$ defined by the chiral multiplet. The parameter g_s is the gauge coupling constant and ζ_a is the real Fayet-iliopolous parameter. In the gauge kinetic term $\Sigma = \Sigma_a T^a$ with T^a being the generators of G in the adjoint representation. Note that unlike the 2d case there is no topological theta term because the term in Lagrangian that corresponded to the theta-coupling is absent in 3d. In the Fayet-Iliopolous term, the V_a denotes a trace that picks up the a^{th} $U(1)$ factor in V, i.e., for $V = V_\alpha T^\alpha$ where T^α are generators of G in the adjoint representation, this trace will retain only those V_α's corresponding to $U(1)_\alpha \subset G$.

We now discuss the Chern-Simons term $\mathcal{L}_{\text{CS}}$ in the Lagrangian. Unlike the other terms in the Lagrangian, the Chern-Simons term does not look much simpler in the superspace notation, however the interested reader may refer to [103–107]. In terms of the components of the vector superfield it is given by,

$$\mathcal{L}_{\text{CS}} = \frac{\kappa}{4\pi}\text{tr}_{\text{adj.}}\left(A \wedge dA + \frac{2}{3}A\wedge A\wedge A - \bar{\lambda}\lambda + 2D\sigma\right)\ , \tag{4.2}$$

where A is the gauge connection, λ is the gaugino, D is the auxiliary D-field and σ is the real scalar in V obtained from the dimensional reduction of the gauge connection of the 4d $\mathcal{N}=1$ vector superfield. The overall constant κ is known as the Chern-Simons level and is subject to a quantisation condition in order to ensure invariance under large gauge transformations [50, 108]. Large gauge transformations refer to those elements of the gauge group G that are disconnected from the identity, which in 3d is given by the third homotopy group $\pi_3(G)$ of G. For instance, $\pi_3(\mathrm{SU}(N)) = \mathbb{Z}$, i.e., the winding numbers of $\mathrm{SU}(N)$ are integral.

We refine the notation of the Chern-Simons level κ by noting that there exists the pure Chern-Simons term κ_i for the Abelian as well as the non-Abelian factors in G labelled by i ,

$$\frac{\kappa_i}{4\pi}\ \text{tr}_{\text{adj.}}\left(A_i \wedge dA_i + \frac{2}{3}A_i\wedge A_i\wedge A_i\right)\ , \tag{4.3}$$

and the mixed Chern-Simon term κ_{ij} for just the U(1) factors in G,

$$\frac{\kappa_{ij}}{4\pi}\ \mathrm{tr}_{\mathrm{adj.}}\left(A_i \wedge dA_j + dA_i \wedge A_j\right)\ , \tag{4.4}$$

where we focus on the gauge connection dependent terms in (4.2) for simplicity. The indices i, j in (4.4) could either label the dynamical gauge fields or the background gauge fields corresponding to the flavour symmetries of the theory which include the R-symmetry. There are no mixed terms for Abelian and non-Abelian factors in G because the trace would render lone non-Abelian factors zero.

In supersymmetric Chern-Simons theories, such as the one we are dealing with, an additional subtlety arises because of the charged fermions. In the following we will refer to the Chern-Simons level κ in the Lagrangian (4.2) as the *bare* Chern-Simons level. Due to an argument of [109, 110], massive charged fermions that have been integrated out contribute to the bare Chern-Simons level as potentially half integer terms. The total Chern-Simons level, together with the contribution from the massive fermions, is known as the *effective* Chern-Simons level $\hat{\kappa}$. Gauge invariance requires $\hat{\kappa} \in \mathbb{Z}$, thus if the fermion contribution is an half-integer, so must the bare Chern-Simons level κ be. This case with non-vanishing Chern-Simons level leads to the so-called 'parity anomaly' of supersymmetric Chern-Simons theory. On the other hand, if one opts to preserve the parity symmetry then the gauge symmetry must necessarily be broken. Hence, in interest of preserving the gauge symmetry we adhere to the parity anomaly.

The renormalisation group flow of 3d theory with four supercharges functions analogously to the 2d case in the previous chapter, however the extra degree of freedom along the third dimension leads to more enumerative insight into the target space geometry X. For details on the infrared dynamics of the 3d theories in question, we refer the reader to [103]. Whereas in the case of the 2d, the resulting I-function was an element of the quantum cohomology ring on X, in 3d a new I-function will emerge from the gauge theory computation which takes values in the so-called quantum K-theory ring on X.

4.2 Partition Function on $D^2 \times_q S^1$

The partition function of a gauge theory in 3d with four supercharges was first done in [38] on a three-sphere S^3 and the squashed three-sphere S^3_b using localisation [39]. In [45] a certain index of an $\mathcal{N} = 2$ gauge theory on an $S^2 \times_q S^1$ was computed in order count the BPS states, i.e., those states that preserve half of the supersymmetry. Here q is a twisting parameter that quantifies the holonomy along the S^1 and appears in the index formula as the chemical potential for the combined $\mathrm{U}(1)_R$ and S^2 rotations. Following these works Yoshida and Sugiyama computed the partition function on the $D^2 \times_q S^1$ with $\mathcal{N} = 2$ supersymmetry. It is this reference that we will closely follow in this section.

The methodology to localise the supersymmetric partition function on a curved space is as discussed in Chapters 2 and 3. One tries to find a solution to the conformal Killing spinor equations corresponding to the curved space metric that preserve the desired amount of supersymmetry and subsequently computes the supersymmetric transformation of all the fields with respect to the Killing spinor solutions. This helps append the flat space Lagrangian with the necessary curved space terms of order $1/r$ and $1/r^2$, where r is the characteristic length scale of the manifold. The solution to the Killing spinor equations is constrained by the chosen BRST generator with respect to which the localisation is to be performed. We briefly note that in order to perform a twist on the supersymmetry generator in 3d, akin to the topological twist in 2d introduced in Chapter 2, there must be at least 8 supercharges, i.e., twice the amount that we are interested in.

The manifold in question, i.e., $D^2 \times S^1$, has a two-dimensional boundary $S^1 \times S^1$. Supersymmetric theories on manifolds with boundaries exhibit subtleties as the entire supersymmetry algebra doesn't close on the boundary. The boundaries of open manifolds can preserve at most half of the supersymmetry that exists in the bulk. The subset of the supersymmetry that is preserved on the boundary can be chosen and one choice that is of particular interest from a string theory perspective is the 2d $\mathcal{N} = (0, 2)$ algebra. This is so, amongst other reasons, because a 2d $\mathcal{N} = (0, 2)$ is the worldsheet theory of a ten-dimensional heterotic string which has various phenomenological applications. Since the boundary supersymmetry is chiral, these boundary conditions are also known as those of the B-type. Alternatively a choice of the generators that preserves an $\mathcal{N} = (1, 1)$ supersymmetry in 2d corresponds to a boundary condition of the A-type. Equipping an open curved manifold with supersymmetry must also preserve the chosen supersymmetry on the boundary, in this case that of the B-type. This leads to a choice amongst a 'Dirichlet' versus a 'Neumann' boundary condition for the chiral as well as the vector multiplet. The nomenclature has been borrowed from the boundaries on a 2d worldsheet because, heuristically, for the chiral multiplet in 3d the Neumann condition corresponds to the vanishing of the derivative of the chiral field scalar whereas the Dirichlet condition corresponds to the vanishing of the chiral scalar itself. This holds similarly for the scalar σ of the linear multiplet Σ in 3d. In general all the chiral multiplets and the vector multiplets can be assigned any permutation of Neumann and Dirichlet boundary conditions. Since the original work of the author [32] deals with models where all fields have Neumann boundary conditions, that will be the focus for the course of this thesis. That is also to say that all contributions to the partition function will only be stated for Neumann b.c. without mention, for details on Dirichlet b.c. we refer the reader to [54, 102].

Furthermore, the localisation computation must be done so as to preserve the BRST symmetry on the boundary $S^1 \times S^1$ as well. On the saddle point of localisation, the component of the gauge potential $\mathcal{A}_\mu$ along the S^1 is Cartan subalgebra $\mathfrak{h}$-valued and the components of along the D^2 are trivial. The localised path integral reduces to a finite dimensional integral over the U(1) Wilson line z along the S^1 which takes values in the Cartan subgroup $\mathrm{U}(1)^{\mathrm{rk}(G)} \subset G$, i.e.,

$$
\begin{aligned}
z &= \mathcal{P} e^{i \int \mathcal{A}_\mu dx^\mu} = \mathcal{P} e^{i \int_{S^1} \mathcal{A}_3 dx^3} \ , \\
&= \mathcal{P} e^{i \left(\sum_{c=1}^{\mathrm{rk} G} a^c T^c \right) \int_{S^1} dx^3} = \mathcal{P} e^{i\beta r \left(\sum_{c=1}^{\mathrm{rk} G} a^c T^c \right)} \ , \\
&=: e^{\vec{\sigma}} \ ,
\end{aligned}
\tag{4.5}
$$

where $\mathcal{P}$ is the path ordering operator required to define a Wilson loop and in the second line r is the radius of the D^2 and βr is the length of the perimeter of the S^1. In the last equation we introduce the Cartan subalgebra $\mathfrak{h}$-valued field $\vec{\sigma}$ which is a $\mathrm{rk}(G)$-dimensional vector. All the other fields are trivial at the saddle point of localisation. Finally, the path integral takes the form [54] [1],

$$
Z_{D^2 \times_q S^1} = \frac{1}{|\mathcal{W}_G|} \oint \left(\prod_{a=1}^{\mathrm{rk}(G)} \frac{d\sigma_a}{2\pi i} \right) Z_{\mathrm{cl}}(\zeta, \vec{\sigma}, q) \ Z_{\text{1-loop}}(\vec{\sigma}, q) \ , \tag{4.6}
$$

[1] q here is square of the one that appears in [54].

where $\mathcal{W}_G$ is the Weyl group of G. We now state the individual classical and one-loop contributions and the ingredients therein.

(i) The classical contribution to the path integral Z_{cl} at the saddle point of localisation locus is given by Fayet-Iliopolous and Chern-Simons terms in the Lagrangian, i.e.,

$$Z_{\text{cl}} = Z_{\text{FI}} \cdot Z_{\text{CS}} \ .$$

The Fayet-Iliopolous term is given by,

$$Z_{\text{FI}} = e^{-2\pi \sum\limits_{i=1}^{\ell} \frac{\zeta_i}{\log(q)} \ \text{tr}_i(\vec{\sigma})} \ , \tag{4.7}$$

where the sum runs over the U(1) factors in G and $\text{tr}_i(\vec{\sigma})$ is the component of $\vec{\sigma}$ corresponding to the i^{th} U(1) generator. The Chern-Simons term is given by,

$$Z_{\text{CS}} = e^{\frac{1}{\log(q)} \sum\limits_{i,j} \kappa_{ij} \text{tr}(\vec{\sigma}_i \vec{\sigma}_j)} \ . \tag{4.8}$$

Here κ_{ij} denotes all possible Chern-Simons levels, including both the pure and the mixed ones. In particular, the entry of this vector corresponds to the subgroups of G paired with themselves and mixed terms for the U(1) factors paired with the $\text{U}(1)_R$ symmetry as introduced in in (4.1). For instance, for $G = \text{U}(2) \simeq (\text{SU}(2) \times \text{U}(1))/\mathbb{Z}_2$, the Chern-Simons coefficient vector would have three non-trivial entries, $\kappa_{ij} = (\kappa_S, \kappa_A, \kappa_R)$, where the levels κ_S and κ_A are for the self-paired (SU(2),SU(2)) and (U(1),U(1)) terms, respectively, and κ_R is the mixed $(\text{U}(1), \text{U}(1)_R)$ level.

(ii) The one loop contribution $Z_{\text{1-loop}}$ to the partition function around the saddle point is a product of the chiral and the vector multiplet terms, i.e.,

$$Z_{\text{1-loop}} = Z_{\text{chiral}} \cdot Z_{\text{vector}} \ .$$

For a gauge theory with N chiral multiplets ϕ_i with the charge vector $\vec{\rho}_i$ under the maximal torus $\text{U}(1)^{\text{rk}(G)}$ of G, see Chapter 3 Section 3.2, $\text{U}(1)_R$ charge Δ_i and charge f_{ir} under the flavour symmetry group with the chemical potential y_r,

$$Z_{\text{chiral}} = \prod_{i=1}^{N} e^{\left(-\frac{\log(q)}{24} - \frac{1}{4}\left(u_i^2 - u_i\right)\right)} \frac{1}{\left(e^{\vec{\rho}_i \cdot \vec{\sigma}} q^{\Delta_i/2} y_r^{f_{ir}}, q\right)_\infty} \ , \tag{4.9}$$

where the product $\vec{\rho} \cdot \vec{\sigma}$ arises from the natural pairing between the elements of the dual Lie algebra $\mathfrak{h}^*$ and the Lie algebra $\mathfrak{h}$, respectively. The flavour charges can be reinterpreted as a shift in the value of $\vec{\sigma}$ by real masses. The u_i is defined as,

$$q^{u_i} = e^{\vec{\rho}_i \cdot \vec{\sigma}} q^{\Delta_i/2} y_r^{f_{ir}} \ . \tag{4.10}$$

The vector multiplet contribution is a product over the non-zero root vectors $\vec{\alpha}$ of G corresponding to the charges of the W-bosons that appear in the abelianised theory,

$$Z_{\text{vector}} = \prod_{\alpha} e^{-\frac{(\vec{\alpha} \cdot \vec{\sigma})^2}{4 \log q}} \left(e^{\vec{\alpha} \cdot \vec{\sigma}}, q\right)_\infty \ , \tag{4.11}$$

see Table 3.2 in Chapter 3 for details on the Abelianised spectrum.

In both the chiral and vector 1-loop determinants, (4.9) and (4.11), the q-Pochhammer symbol makes an appearance and is defined as,

$$(x,q)_k = \prod_{n=0}^{k-1} (1 - xq^n) \ . \tag{4.12}$$

The inverse of the q-Pochhammer symbol $(x,q)_k^{-1}$ acquires poles when $x = q^{-d}$, where $d \in \mathbb{Z}_{\geq 0}$ and $d < k$. Additional properties of the q-Pochhammer symbol will be discussed in the forthcoming subsection on the example of the Grassmannian target space.

4.2.1 Grassmannian $\mathrm{Gr}(M, N)$ Target Space

The aim of this section is to evaluate the partition function (4.6) for a gauge theory with gauge group $G = \mathrm{U}(M)$ with the matter content displayed in Table 4.1. A gauge theory with the characteristics as displayed in this table has a Grassmannian manifold $\mathrm{Gr}(M, N)$ as its target space. The content presented here is dominantly motivated from the work of the author [111].

non-Abelian gauge theory spectrum:			
Chiral multiplets	$U(M)$ Representation	$U(1)_R$ charge	real masses
ϕ_i, $i = 1, \ldots, N$	$\square_{+1}$	0	0

Abelianised gauge theory spectrum:			
Chiral multiplets	$U(1)^M$ charge	$U(1)_R$ charge	real masses
$\phi_i^{(1)}$, $i = 1, \ldots, N$	$(+1, 0, \cdots, 0)$	0	0
$\phi_i^{(2)}$, $i = 1, \ldots, N$	$(0, +1, \cdots, 0)$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\phi_i^{(M)}$, $i = 1, \ldots, N$	$(0, \cdots, 0, +1)$	0	0
W_{12} (W_{21})	$(+1, -1, 0, \cdots, 0)$ (with $+ \leftrightarrow -$)	2	0
W_{23} (W_{32})	$(0, +1, -1, \cdots, 0)$ (with $+ \leftrightarrow -$)	2	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
W_{1N} (W_{N1})	$(+1, 0, \cdots, 0, -1)$ (with $+ \leftrightarrow -$)	2	0

Table 4.1: The top part of the table shows the chiral matter multiplets of the $U(M)$ gauged linear sigma model of the complex Grassmannian manifold $\mathrm{Gr}(M, N)$, where the $U(M)$ representation is specified in terms of the Young tableau of the non-Abelian subgroup $SU(M)$ together with the charge of the diagonal $U(1)$ subgroup as a subscript. We work with the case where all the real masses, i.e., the flavour charges, have been set to zero. The bottom part of the table lists the chiral spectrum in the Abelianised theory, which comprises the decomposition of the non-Abelian matter multiplets into representations of the unbroken Abelian subgroup $U(1)^M$ together with the $2 \cdot \binom{M}{2}$ W-bosons that are part of the Abelianised spectrum.

Before we discuss the specific form of the classical and one-loop contributions to the partition function, let us take note of the fact that the chiral multiplet contribution to the one-loop determinants simplifies to,

$$Z_{\text{chiral}} = \left(\prod_{j=1}^{M} e^{\left(-\frac{\log(q)}{24} - \frac{1}{4}\left(\frac{\vec{\rho}^{(j)}\cdot\vec{\sigma}}{\log(q)}\right)^2 + \frac{1}{4}\left(\frac{\vec{\rho}^{(j)}\cdot\vec{\sigma}}{\log(q)}\right)\right)} \frac{1}{\left(e^{\vec{\rho}^{(j)}\cdot\vec{\sigma}}, q\right)_\infty} \right)^N . \tag{4.13}$$

The q-Pochhammer symbol in the expression above acquires poles when,

$$e^{\vec{\rho}^{(j)}\cdot\vec{\sigma}} = q^{-(d_j - \epsilon_j)} = e^{-\log(q)(d_j-\epsilon_j)} =: e^{-\log(q)(\tilde{d}_j)} , \tag{4.14}$$

with $d_j \in \mathbb{Z}_{\geq 0}$ and $\tilde{d}_j = n_j - \epsilon_j$. Here the variables ϵ_j's have been introduced in order to redefine the dummy variable of integration from σ_a to ϵ_j. Since the path integral simplifies to a sum over the poles of the integrand, according to the residue theorem we evaluate all the expressions within the integrand at the pole locus. The individual expressions for the classical and one-loop contributions at the pole locus are stated and simplified as follows.

(i) The classical Fayet-Iliopolous term is given by,

$$Z_{\text{FI}} = e^{2\pi \frac{\zeta}{\log(q)} \text{tr}(\vec{\sigma})} =: \tilde{Q}^{\left(\sum_{i=1}^{M} \tilde{d}_i\right)} , \tag{4.15}$$

where $\tilde{Q} := e^{-2\pi\zeta}$.

(ii) The classical Chern-Simons term is given by,

$$Z_{\text{CS}} = q^{\frac{1}{2}\kappa_S \text{tr}_{SU(M)}(\tilde{d}^2) + \frac{1}{2}\kappa_A \text{tr}_{U(1)}(\tilde{d}^2) + \kappa_R \text{tr}_R(\tilde{d})} , \tag{4.16}$$

where the parameters κ_S and κ_A are the Chern-Simons levels for the SU(M) and U(1) subgroups of U(M) and the parameter κ_R is the Chern-Simons level for the mixed U(1)-U(1)$_R$ term. Here, for the M-dimensional vector $\tilde{d} = \text{diag}(\tilde{d}_1, \ldots, \tilde{d}_M)$ the trace symbols are defined as,

$$\begin{aligned} &\text{tr}_{U(M)}(\tilde{d}^2) = \sum_a \tilde{d}_a^2 \quad , \quad \text{tr}_{U(1)}(\tilde{d}^2) = \frac{1}{M}\left(\sum_a \tilde{d}_a\right)^2 , \\ &\text{tr}_{SU(M)}(\tilde{d}^2) = \text{tr}_{U(M)}(\tilde{d}^2) - \text{tr}_{U(1)}(\tilde{d}^2) \quad , \quad \text{tr}_R(\tilde{d}) = \sum_a \tilde{d}_a . \end{aligned}$$

(iii) The one-loop determinant from the chiral matter can be simplified by inserting (4.14) in (4.13),

$$\begin{aligned} Z_{\text{chiral}} &= \left(\prod_{j=1}^{M} q^{\left(-\frac{1}{24} - \frac{1}{4}(\tilde{d}_j)^2 - \frac{1}{4}(\tilde{d}_i)\right)} \frac{1}{\left(q^{-\tilde{d}_j}, q\right)_\infty} \right)^N , \\ &= q^{\left(-\frac{MN}{24} - \frac{N}{4}\sum_{j=1}^{M}(\tilde{d}_j^2 + \tilde{d}_i)\right)} \cdot \prod_{j=1}^{M} \frac{1}{\left(q^{-\tilde{d}_j}, q\right)_\infty^N} . \end{aligned} \tag{4.17}$$

Simplifying one q-Pochhammer term as follows,

$$\begin{aligned}
\frac{1}{\left(q^{-\tilde{d}_j},q\right)_\infty} &= \frac{1}{(q,q)_\infty}\cdot\frac{\Gamma_q(-\tilde{d}_j)}{(1-q)^{1+\tilde{d}_j}}\ , \\
&= \frac{1}{(q,q)_\infty}\cdot\frac{1}{(1-q)^{1+\tilde{d}_j}}\cdot(-)^{d_j}q^{\binom{d_j+1}{2}-d_j\epsilon_j}\cdot\frac{\Gamma_q(\epsilon_j)\Gamma_q(1-\epsilon_j)}{\Gamma_q(1+n_j-\epsilon_j)}\ , \\
&= \frac{(-)^{d_j}}{(q,q)_\infty}\cdot\frac{q^{\binom{d_j+1}{2}-d_j\epsilon_j}}{(1-q)^{1-\epsilon_j}}\cdot\Gamma_q(\epsilon_j)\cdot\left(\frac{1}{\prod_{r=1}^{d_j}(1-q^{r-\epsilon_j})}\right)\ .
\end{aligned} \tag{4.18}$$

In the first equality we used the identity relating the q-Gamma function[2] to the q-Pochhammer function,

$$\Gamma_q(x) = \frac{(q,q)_\infty}{(q^x,q)_\infty}(1-q)^{1-x}\ , \tag{4.19}$$

and in the second equality we used the identity,

$$\Gamma_q(x-k)\Gamma_q(1-x+k) = (-)^k q^{\frac{k(k+1)}{2}-kx}\ \Gamma_q(x)\Gamma_q(1-x)\ . \tag{4.20}$$

Plugging the modified expression for the single q-Pochhammer (4.18) into the chiral one-loop determinant (4.17) and simplifying the exponent of q we finally obtain,

$$Z_{\text{chiral}} = q^{\left(-\frac{MN}{24}+\frac{N}{4}\sum_{j=1}^{M}\left(\tilde{d}_j^2+\tilde{d}_j-2\binom{\epsilon_j}{2}\right)\right)}\prod_{j=1}^{M}\left(\frac{(-)^{d_j}}{(q,q)_\infty}\cdot\frac{\Gamma_q(\epsilon_j)}{(1-q)^{1-\epsilon_j}}\right)^N\cdot\left(\frac{1}{\prod_{r=1}^{d_j}(1-q^{r-\epsilon_j})^N}\right)\ .$$

(iv) The vector multiplet contribution to the one-loop determinant comes from the W-bosons in the abelianised theory such that starting from (4.11),

$$\begin{aligned}
Z_{\text{vector}} &= \prod_\alpha e^{-\frac{(\vec{\alpha}\cdot\vec{\sigma})^2}{4\log q}}\left(e^{\vec{\alpha}\cdot\vec{\sigma}},q\right)_\infty\ , \\
&= \prod_{1\le i<j\le M} q^{\frac{\tilde{d}_{ij}^2}{2}}\ (q^{\tilde{d}_{ij}},q)_\infty\ (q^{-\tilde{d}_{ij}},q)_\infty\ ,
\end{aligned} \tag{4.21}$$

where $\tilde{d}_{ij} := \tilde{d}_i - \tilde{d}_j$. Introducing $x := \epsilon_j - \epsilon_i$ and $k := d_j - d_i$ and using (4.20), the product of q-Pochhammer symbols in the above expression can be rewritten as,

$$\begin{aligned}
(q^{\tilde{d}_{ij}},q)_\infty\ (q^{-\tilde{d}_{ij}},q)_\infty &= \left((-)^k\frac{(q,q)_\infty^2(1-q)}{\Gamma_q(x)\Gamma_q(1-x)}\right)\cdot\left((1-q^{k-x}\cdot q^{kx-\frac{k}{2}(k+1)})\right)\ , \\
&= \left((-)^k\frac{(q,q)_\infty^2(1-q)}{\Gamma_q(x)\Gamma_q(1-x)}\right)\cdot q^{-\frac{(k-x)^2}{2}}\left(q^{-\frac{k-x}{2}}-q^{\frac{k-x}{2}}\right)q^{\binom{x}{2}}\ .
\end{aligned} \tag{4.22}$$

[2] The q-Gamma function is a q-extension of the Gamma function with $\lim_{q\to1}\Gamma_q(x) = \Gamma(x)$. For a formal definition, see [97, 112].

Substituting x and k in the last expression of the above equation and substituting this into the one-loop determinant for the vector multiplet (4.21), the final expression for Z_{vector} reads,

$$Z_{\text{vector}} = \prod_{1\le i<j\le M} (-)^{d_j - d_i} \frac{(q,q)_\infty^2 (1-q)}{\Gamma_q(-\epsilon_{ij})\Gamma_q(1-\epsilon_{ij})} q^{\binom{\epsilon_{ij}+1}{2}} q^{-\frac{\tilde{d}_{ij}^2}{2}} \cdot \left(q^{\frac{\tilde{d}_{ij}^2}{2}} (q^{\frac{\tilde{d}_{ij}}{2}} - q^{-\frac{\tilde{d}_{ij}}{2}}) \right) . \tag{4.23}$$

Collecting all the simplified expressions for the classical and the one-loop terms into the path integral, summing over the poles of the q-Pochhammer symbols and changing the variable of integration from $\vec{\sigma}$ to $\vec{\epsilon}$ we obtain,

$$\begin{aligned} Z_{D^2\times_q S^1} = \frac{(\log q)^M}{M!} \sum_{\vec{d}\in\mathbb{Z}^M_{\ge 0}} \oint \left(\prod_{a=1}^{M} \frac{d\epsilon_a}{2\pi i}\right) \tilde{Q}^{\left(\sum_{i=1}^{M}\tilde{d}_i\right)} \cdot q^{\frac{1}{2}\kappa_S \mathrm{tr}_{SU(M)}(\tilde{d}^2) + \frac{1}{2}\kappa_A \mathrm{tr}_{U(1)}(\tilde{d}^2) + \kappa_R \mathrm{tr}_R(\tilde{d})} \, . \\ \times q^{\left(-\frac{MN}{24} + \frac{N}{4}\sum_{j=1}^{M}\left(\tilde{d}_j^2 + \tilde{d}_i - 2\binom{\epsilon_j}{2}\right) - \sum_{i<j}\frac{\tilde{d}_{ij}^2}{2}\right)} . \\ \times \prod_{j=1}^{M} \left(\frac{(-)^{d_j}}{(q,q)_\infty} \cdot \frac{\Gamma_q(\epsilon_j)}{(1-q)^{1-\epsilon_j}} \right)^N \cdot \prod_{i<j} (-)^{d_j-d_i} \frac{(q,q)_\infty^2(1-q)}{\Gamma_q(-\epsilon_{ij})\Gamma_q(1-\epsilon_{ij})} q^{\binom{\epsilon_{ij}+1}{2}} \, . \\ \times \left(\frac{\prod_{1\le i<j\le M} \left(q^{\frac{\tilde{d}_{ij}^2}{2}} (q^{\frac{\tilde{d}_{ij}}{2}} - q^{-\frac{\tilde{d}_{ij}}{2}}) \right)}{\prod_{j=1}^{M}\prod_{r=1}^{d_j} (1-q^{r-\epsilon_j})^N} \right) . \end{aligned} \tag{4.24}$$

In the expression above the Chern-Simons levels κ_S, κ_A and κ_R can be redefined to incorporate the contributions in the form of the $q^{(\cdots)}$ terms from the one-loop determinants. We note that this correction of the bare Chern-Simons levels from the one-loop determinants is in fact caused by the massive fermions mentioned in Section 4.1. The new Chern-Simons levels are denoted by $\hat{\kappa}_S, \hat{\kappa}_A$ and $\hat{\kappa}_R$ are the *effective* Chern-Simons levels introduced in Section 4.1 and they fulfil the quantisation condition,

$$\hat{\kappa}_S \in \mathbb{Z} \quad , \quad \frac{\hat{\kappa}_S - \hat{\kappa}_A}{M} \in \mathbb{Z} \quad , \quad 2\hat{\kappa}_R \in \mathbb{Z} \, . \tag{4.25}$$

The quantisation condition that mixes $\hat{\kappa}_S$ and $\hat{\kappa}_S$ is a consequence of the global structure of the gauge group $\mathrm{U}(2) \simeq (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$.

Moreover the expression for the partition function can be split into a factor that is the summation index $\vec{d}$-dependent, known as Givental's permutation symmetric I^{SQK}-function for quantum K-theory on the Grassmannian $\mathrm{Gr}(M,N)$, and a $\vec{d}$-independent factor, known as the folding factor f_{D^2}. The consolidated expression reads,

$$Z_{D^2\times_q S^1} = \oint \left(\prod_{a=1}^{M}\frac{d\epsilon_a}{2\pi i}\right) f_{D^2}(q,\vec{\epsilon}) \cdot I^{SQK}_{\mathrm{Gr}(M,N)}(q,Q,\vec{\epsilon}) \, . \tag{4.26}$$

Givental's permutation symmetric I-function, $I^{SQK}_{\mathrm{Gr}(M,N)}$, is given by,

$$I^{SQK}_{\mathrm{Gr}(M,N)}(q,Q,\vec{\epsilon}) = c_0 \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} (-Q)^{\left(\sum\limits_{i=1}^{M}\tilde{d}_i\right)} q^{CS(\tilde{d})} \left(\frac{\prod\limits_{1\leq i<j\leq M} \left(q^{\frac{\tilde{d}^2_{ij}}{2}} \left(q^{\frac{\tilde{d}_{ij}}{2}} - q^{-\frac{\tilde{d}_{ij}}{2}}\right)\right)}{\prod\limits_{j=1}^{M}\prod\limits_{r=1}^{d_j}(1-q^{r-\epsilon_j})^N} \right) . \quad (4.27)$$

Here the normalisation factor c_0 is chosen such that $I^{\mathrm{SQK}}_{\mathrm{Gr}(M,N)}|_{Q=0} = (1-q)$ in order to compare with the formulae in [31] and is explicitly given as,

$$c_0 = \frac{1-q}{q^{CS(-\epsilon)} \prod\limits_{1<\leq i<j\leq M} q^{\epsilon^2_{ij}/2}\left(q^{-\epsilon_{ij}/2} - q^{\epsilon_{ij}/2}\right)} \, .$$

The factor $CS(-\epsilon)$ is such that $q^{CS(\tilde{d})}|_{(d=0)} =: q^{CS(-\epsilon)}$. In subsequent reference to the $I^{SQK}_{\mathrm{Gr}(M,N)}$-function we will not state the factor c_0 explicitly, but it will be implicit in all the calculations. As we will see, for the purpose of the derivation of the Wilson loop algebra for canonical Chern-Simons levels, it will not play an important role. The new parameter Q has been introduced in (4.27) such that $Q = (-)^{N+M}\tilde{Q}$. The Chern-Simons term $q^{CS(\tilde{d})}$ is given in terms of the effective Chern-Simons levels as,

$$q^{CS(\tilde{d})} = q^{\frac{1}{2}\hat{\kappa}_S \mathrm{tr}_{SU(M)}(\tilde{d}^2) + \frac{1}{2}\hat{\kappa}_A \mathrm{tr}_{U(1)}(\tilde{d}^2) + \hat{\kappa}_R \mathrm{tr}_R(\tilde{d})} \, , \quad (4.28)$$

where the effective levels $\hat{\kappa}$ are expressed in terms of the bare levels κ as follows,

$$\hat{\kappa}_S = \kappa_S - M + \frac{N}{2} \quad , \quad \hat{\kappa}_A = \kappa_A + \frac{N}{2} \qquad \hat{\kappa}_R = \kappa_R + \frac{N}{4} \, . \quad (4.29)$$

The folding factor $f_{D^2}(q,\vec{\epsilon})$ in (4.26) for $X := \mathrm{Gr}(M,N)$ is given by,

$$\begin{aligned} f_{D^2}(q,\vec{\epsilon}) =& (-)^{(1-M-N)\sum\limits_{i=1}^{M}\epsilon_i} \cdot \frac{(\log q)^M}{M!} (-\eta(q))^{-MN} (q,q)_\infty^{2\binom{M}{2}} \; q^{g(\epsilon)} \; (1-q)^{(-1+c_1(\epsilon))} \\ &\times \frac{\prod\limits_{i<j} (\log q)^2 \; \epsilon_{ij}\cdot\epsilon_{ji}}{\prod\limits_{i=1}^{M}(\log q \; \epsilon_i)^N} \cdot \Gamma_{X,q} \; \mathrm{td}_{\beta'}(X) \, , \end{aligned} \quad (4.30)$$

where $\eta(q) = q^{1/24}(q,q)_\infty$ is usual eta function, and $\beta' = \beta\hbar = -\log q$. The exponent of q in the folding factor is a characteristic class dependent quantity and is explicitly given by,

$$g(\epsilon) = -\frac{c_1}{2} - \mathrm{ch}_2(\epsilon) + CS(-\epsilon) \, ,$$

where $c_1(\epsilon)$ and $\mathrm{ch}_2(\epsilon)$ are the first Chern class and the second Chern character of the tangent bundle on $\mathrm{Gr}(M,N)$.

The characteristic classes $\Gamma_{X,q}$ and $\mathrm{td}_{\beta'}(X)$ of the target space $X = \mathrm{Gr}(M,N)$ appearing in the folding factor are known as the q-Gamma class and the β'-dependent Todd class, respectively.

They are defined as follows [112, 113] ,

$$
\begin{aligned}
\Gamma_{\mathrm{Gr}(M,N),q} &= \frac{\prod_{i=1}^{M} \Gamma_q(1+\epsilon_i)^N}{\prod_{i<j}^{M} \Gamma_q(1+\epsilon_{ij})\Gamma_q(1-\epsilon_{ij})} \ , \\
\mathrm{td}_{\beta'}(\mathrm{Gr}(M,N)) &= \left(\prod_{i=1}^{M} \frac{(\log q)\ \epsilon_i}{(1-q^{-\epsilon_i})}\right)^N \cdot \left(\prod_{i<j}^{M} \frac{(1-q^{\epsilon_{ij}})(1-q^{\epsilon_{-ij}})}{(\log q)^2\ \epsilon_{ij}\cdot\epsilon_{ji}}\right) \ .
\end{aligned}
\tag{4.31}
$$

Here the ϵ's play the role of the Chern roots of the universal subbundle on the Grassmannian. For more details on the Grassmannian and its characteristic classes we refer the reader to the Appendix A. We note in passing that the expression for the folding factor is sensitive to the boundary conditions of the fields, thus the expression in (4.30) is specific to the case when all fields have Neumann boundary conditions.

This concludes our analysis of the partition function on $D^2 \times_q S^1$ for Grassmannian target spaces.

4.3 Quantum K-theoretic I- and J-functions

In the previous section we precociously split the partition function on a $D^2 \times_q S^1$ for a $\mathrm{Gr}(M,N)$ target space into an I-function, precisely the Givental's permutation symmetric I-function which was denoted by $I^{SQK}_{\mathrm{Gr}(M,N)}$, and a folding factor that was determined by the boundary conditions of the gauge theory content and the characteristic classes of the target space $\mathrm{Gr}(M,N)$. We will now briefly review quantum K-theory and Givental's quantum K-theoretic I- and J-functions to the extent required to understand the concepts and new results. This will be far from a comprehensive review of these purely mathematical concepts and will stand in primarily as a physicist's interpretation owing to the author's background as a physicist. For an introduction to K-theory we refer the reader to [114, 115]. The definitions of Givental's quantum K-theoretic I- and J-functions [31] and statements about the 3d gauge theory/quantum K-theory correspondence follow the work in [97] closely.

In order to define the K-theory group $K(X)$ on a target space X one starts with the semigroup, i.e., a group without the inverse operation, of vector bundles on X where the composition of two elements is done with the Whitney sum of vector bundles. To augment this semi-group with the inverse elements, the following equivalence relation must be obeyed for three arbitrary vector bundles E, F and H on X,

$$(E, F) \sim (E \oplus H, F \oplus H) \ .$$

This implies that the inverse element for a pair of vector bundles (E, F) is given by the pair (F, E). As in the case of [97], in this work as well the K-theory group $K(X)$ denotes the free part topological K-theory group $K^0(X)$. The Chern character $\mathrm{ch}(\cdot)$, see Appendix A for details, defines a ring homomorphism from $K^0(X)$ to the even-cohomology ring on X over the field of rationals $\mathbb{Q}$,

$$\begin{aligned} \mathrm{ch} : K^0(X) &\to H^{\mathrm{ev}}(X,\mathbb{Q}) \\ (E,F) &\mapsto \mathrm{ch}(E) - \mathrm{ch}(F) \ , \end{aligned} \tag{4.32}$$

known as the *Chern homomorphism.* Atiyah and Hirzebruch established the ring isomorphism for $K^0(X) \otimes \mathbb{Q}$ [116],

$$\mathrm{ch} : K^0(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^{\mathrm{ev}}(X,\mathbb{Q}) \ . \tag{4.33}$$

In K-theory the analog of the classical intersection pairing between elements of cohomology is the K-theory pairing given by Euler characteristic, i.e., $\chi(X;E,F) =: (E,F)$ for the vector bundles $E, F \in K(X)$. The Euler characteristic can be computed using the Hirzebruch-Riemann-Roch theorem,

$$(E,F) := \chi(X;E,F) = \int_X \mathrm{td}(X) \wedge \mathrm{ch}(E) \wedge \mathrm{ch}(F) \ , \tag{4.34}$$

for the Todd class $\mathrm{td}(X)$ of the tangent bundle TX on X and the Chern characters $\mathrm{ch}(\cdot)$ of the vector bundles E and F.

Recall that in Section 3.5 of Chapter 3 the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps φ from a Riemann surface $\Sigma_{g,n}$ of genus g with n marked points to a target space X was considered, where the map φ, with $\varphi : \Sigma_{g,n} \to X$, is represented by $\beta \in H_2(X,\mathbb{Z})$, s.t., $\varphi([\Sigma_{g,n}]) = \beta$. We also introduced n evaluation maps, $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$, such that $\mathrm{ev}_i(\Sigma_{g,n}, p_1, \ldots, p_n, \varphi) = \varphi(p_i)$. The cohomological Gromov-Witten invariants are then defined using pullback evaluation maps on elements of $H^*(X)$ and appear as the coefficients of the Kähler parameters in the A-model correlators.

Analogous to the augmentation of the classical cohomology ring on X to the quantum cohomology ring by the cohomological Gromov-Witten invariants encoded in correlators, the classical K-theory ring defined by pairing (4.34) can be augmented to a quantum K-theory ring with the quantum corrections encoded in certain correlators. As in case of quantum cohomology the Riemann surface of interest to us has genus zero. The correlators that are identified with quantum K-theoretic invariants at genus zero are computed for inputs $t_i(q) \in K(X)[q,q^{-1}]$, i.e., Laurent polynomials in q with coefficients in $K(X)$. They are given by the holomorphic Euler characteristic of their corresponding pulled back counterparts on the on the moduli space of stable maps, i.e.,

$$\langle t_1(q), t_2(q), \ldots, t_m(q)\rangle_{0,m,\beta} = \chi(\overline{\mathcal{M}}_{0,m}(X,\beta) \ ; \mathrm{ev}_1^* t_1(L_1) \otimes \ldots \otimes \mathrm{ev}_m^* t_m(L_m) \otimes \mathcal{O}^{\mathrm{vir}}) \ , \tag{4.35}$$

where the L_i's denote the universal cotangent line bundles on $\overline{\mathcal{M}}_{0,m}(X,\beta)$ at the i^{th} marked point and $\mathcal{O}^{\mathrm{vir}}$ is the so-called virtual structure sheaf on $\overline{\mathcal{M}}_{0,m}(X,\beta)$ [117] needed to define the correlator as an Euler characteristic. The notation $\mathrm{ev}_i^* t_i(L_i)$ denotes that for $t_i(q) = \mathcal{E} \cdot q^k$ with $\mathcal{E} \in K(X)$, $\mathrm{ev}_i^* t_i(L_i) = \mathrm{ev}_i^*(\mathcal{E}) \otimes (L_i)^{\otimes k}$.

Now, as in the case of the quantum cohomology, the quantum K-theoretic invariants (4.35) are encoded in the Givental's J-function for quantum K-theory. The J-function for *ordinary quantum K-theory* with the input $t(q)$ is given by the expansion,

$$J_K(t) = (1-q) + t(q) + \sum_{\beta \geq 0} \sum_{n \geq 0} \sum_i \frac{\Phi^i}{n!} \left\langle \frac{\Phi_i}{1-qL_1}, t(L_2), \ldots t(L_{n+1}) \right\rangle_{0,n+1,\beta} Q^\beta \ . \tag{4.36}$$

The first term is known as the dilaton shift and the second term spells out the input of the J-function. The $\beta \in H_2(X, \mathbb{Z})$ indexes the instanton sector Q^β where Q is the Kähler parameter as usual and is to be identified with the identically denoted Fayet-Iliopolous parameter in the gauge theory. The Φ_i is the basis of ring $K(X)$ and using the pairing (4.34),

$$\chi_{ij} := (\Phi_i, \Phi_j) = \int_X \mathrm{td}(X) \wedge \Phi_i \wedge \Phi_j \ , \tag{4.37}$$

a dual basis Φ^i can be defined by the relation $\Phi_i = \chi_{ij}\Phi^j$. The J-function (4.36) takes values in the formal ring $\mathcal{K} := K(X) \otimes \mathbb{C}(q, q^{-1}) \otimes \mathbb{C}[[Q]]$. Here $\mathbb{C}(q, q^{-1})$ denotes the field of fractions of the ring $\mathbb{C}[q, q^{-1}]$ which itself is simply the ring of polynomials in q and q^{-1} over the $\mathbb{C}$-field. The formal ring $\mathcal{K}$ can split as $\mathcal{K}_+ \oplus \mathcal{K}_-$ [118, 119] where,

$$\begin{aligned} \mathcal{K}_+ &= K(X) \otimes \mathbb{C}[q, q^{-1}] \otimes \mathbb{C}[[Q]] \ , \\ \mathcal{K}_- &= K(X) \otimes \{r(q) \in R(q) | r(0) \neq \infty \text{ and } r(\infty) = 0\} \otimes \mathbb{C}[[Q]] \ , \end{aligned} \tag{4.38}$$

where $R(q)$ denotes the field of rational functions in q. In particular, the input $t \in \mathcal{K}_+$ and the quantum K-theory invariants are in $\mathcal{K}_-$. This differentiation is elemental in separating the input from the invariants in the J-function expansion.

In the works [31] in addition to the ordinary quantum K-theory J-function $J_K(t)$, J-functions $J_K^{\mathrm{eq}}(t)$ and $J_K^{\mathrm{sym}}(t)$ were introduced for the permutation equivariant quantum K-theory and the permutation symmetric quantum K-theory, respectively. The permutation equivariant quantum K-theory invariants are refinements of the ordinary quantum K-theory invariants (4.35). They are defined such that they are equivariant with respect to the action of the symmetric group S_n on the last n marked points as permutations thereof. The permutation symmetric quantum K-theory invariants are a special case of the equivariant invariants where only the one-dimensional fully symmetric representation of S_n is considered.

As for the cohomological I- and J-functions, in quantum K-theory as well the I- and the J-functions are connected by a 3d version of the *mirror map*. However as we will note shortly, the gauge theories of interest yield a I-function with vanishing input, and at vanishing input,

$$J_K(0) = J_K^{\mathrm{eq}}(0) = J_K^{\mathrm{sym}}(0) = I(0) \ .$$

For the course of this thesis we will deal with geometries that yield unperturbed I-functions thus a need to distinguish between I- and J-function is not merited.

In [97] it was noted that the $D^2 \times_q S^1$ index yields the permutation symmetric I-function of quantum K-theory which can have a non-zero input in the generic case. For projective surfaces $\mathbb{P}^N$ it was noted that the I-function that the gauge theory yields is in fact with zero input. A similar phenomenon occurs for the I-function (4.27) obtained from gauge theory with target space Grassmannian $\mathrm{Gr}(M, N)$ and canonical Chern-Simons levels considered in [32]. For the quintic target space $\mathbb{P}^4[5]$, however, the gauge theory yields the permutation symmetric I-function with a non-zero input.

4.3.1 Wilson Loop Algebra

As we noted in Chapters 2 and 3, the 2d partition function of a supersymmetric gauge theory with four supercharges encodes enumerative information of the target space related to its quantum cohomology ring. The correspondence in [97] establishes an analogous correspondence between the 3d partition function and the quantum K-theory ring on the target spaces. In particular this correspondence equates the vortex sum in the 3d gauge theory partition function to the I-function for the permutation symmetric quantum K-theory. Moreover the quantum K-theoretic chiral ring is generated by Wilson line operators on the S^1 of the $D^2 \times_q S^1$ at the level of the gauge theory. This can be viewed as a parallel to the statement that the scalar σ of the twisted chiral field Σ encountered in the 2d $\mathcal{N} = (2,2)$ generates the quantum cohomology ring on the target space. In this section we again zoom in on gauge theories with gauge group $G = \mathrm{U}(M)$ and the Grassmannian manifold $\mathrm{Gr}(M,N)$ as a target space and compute the Wilson loop algebra thereof. The chiral ring of Wilson loops thus obtained is shown to match with the quantum product of the Schubert structure sheaves on the $\mathrm{Gr}(M,N)$ computed in [120] for canonical Chern-Simons levels.

The Wilson line on the S^1 was defined in (4.5) and from the expression of the partition function (4.6) it becomes evident that in the abelianised theory an insertion of a Wilson loop of charge w_a under the a^{th} U(1) factor of the maximal torus of G will contribute a factor of $q^{-w_a \tilde{d}_a}$. Explicitly, the pole position defined by (4.14) for the a^{th} U(1) factor is,

$$e^{\sigma_a} = q^{-(d_a - \epsilon_a)} \ . \tag{4.39}$$

We define an Abelianised I-function I_{ab} by introducing auxiliary Fayet-Iliopolous parameters corresponding to those U(1) factors in the maximal torus of G that arise from semi-simple subgroups in G, i.e.,

$$I_{\text{ab}} = \sum_{\vec{d} \in \mathbb{Z}_{\geq 0}} c_{\vec{d}} \left(\prod_{i=1}^{\text{rk}(G)} (-Q_i)^{\tilde{d}_i} \right) \ , \tag{4.40}$$

and when the auxiliary Fayet-Iliopolous parameters are set to one, we get the usual I-function, i.e.,$I_{\text{ab}}|_{\vec{Q}_{\text{aux}}=1} = I^{SQK}(q, Q, \vec{\epsilon})$.

The insertion of a factor $e^{w_a \sigma_a}$ corresponding to a Wilson line of charge w_a under the a^{th} U(1) factor in the path integral will shift the abelianised I-function I_{ab} as,

$$I_{\text{ab}} = \sum_{\vec{d} \in \mathbb{Z}_{\geq 0}} c_{\vec{d}} \left(\prod_{i=1}^{\text{rk}(G)} (-Q_i)^{\tilde{d}_i} \right) \longrightarrow \sum_{\vec{d} \in \mathbb{Z}_{\geq 0}} c_{\vec{d}} \left(\prod_{i=1}^{\text{rk}(G)} (-Q_i)^{\tilde{d}_i} \right) \cdot q^{-w_a \tilde{d}_a} \ =: \ I'_{\text{ab}} \ . \tag{4.41}$$

The insertion of such a Wilson line in the I-function can be emulated by the action of a shift operator on the same and is given by,

$$q^{-w_a \theta_a} \cdot I_{\text{ab}} = \sum_{\vec{d} \in \mathbb{Z}_{\geq 0}} c_{\vec{d}} \left(\prod_{i=1}^{\text{rk}(G)} (-Q_i)^{\tilde{d}_i} \right) \cdot q^{-w_a \tilde{d}_a} \ = \ I'_{\text{ab}} \ , \tag{4.42}$$

where $\theta_a = Q_a \partial_{Q_a}$ is the usual logarithmic derivative with respect to the a^{th} Fayet-Iliopolous parameter Q_a.

Evaluating the algebra of Wilson line operators boils down to evaluating the expectation value of compositions of such Wilson operators and expressing the result as a linear combination of insertions of individual Wilson line insertions. Given two Wilson line operators $W_{\vec{a}}$ and $W_{\vec{b}}$ with charge vectors $\vec{w}_{\vec{a}}$ and $\vec{w}_{\vec{b}}$ under the maximal torus group $\mathrm{U}(1)^M$ of $\mathrm{U}(M)$, an insertion of the product of these operators will alter the Abelianised I-function as,

$$I_{\text{ab}} \longrightarrow q^{-(\vec{w}_{\vec{a}}+\vec{w}_{\vec{b}})\cdot\vec{\theta}} \cdot I_{\text{ab}} \ , \tag{4.43}$$

where $\vec{\theta}$ is the M-dimensional vector of logarithmic derivatives. Thus classically the Wilson loop algebra of the Abelianised theory is given by,

$$W_{\vec{a}} \cdot W_{\vec{b}} = W_{\vec{a}+\vec{b}} \ . \tag{4.44}$$

Before we analyse the quantum corrections to the classical algebra of Wilson line operators stated in the equation above, we note that for the gauge group $\mathrm{U}(M)$ with the maximal torus $\mathrm{U}(1)^M$ the Wilson lines must be expressible as permutation symmetric combinations of the Wilson lines charges under each $\mathrm{U}(1)$ factor therein. This can be explained on a heuristic level by observing that for the Abelianised spectrum of this gauge theory, detailed in Table 4.1, the charge vectors of the chiral fields as well as the W-bosons under the Abelianised group are completely symmetric with respect to permutations of the $\mathrm{U}(1)$ factors. This in turn can be understood to be directly correlated with the fact the even cohomology ring of the Grassmannian target space, $H^{\text{ev}}(\mathrm{Gr}(M,N))$, is generated by Schubert cycles that can be represented by Schur polynomials of the Chern roots of the universal subbundle of $\mathrm{Gr}(M,N)$. A Schur polynomial σ_μ is labelled by a Young tableau μ and is a permutation symmetric polynomial in its arguments, see Appendix A for details.

Our focus is the algebra of Wilson line operators W_μ labelled by the Young tableaus μ that lie in the $M \times (N-M)$ box. The vertical limit on the Young tableau comes the number of variables to be symmetrise corresponding to the dimension of the universal subbundle and the horizontal limit comes from the dimension of the universal quotient bundle, as is detailed in Appendix A. Young tableaus outside the box either vanish identically or can be rewritten as Young tableaus inside the box using the ideal of difference operators to be derived shortly. The horizontal dimension, and hence the number of operators in the chiral ring, is susceptible to an increase if the Chern-Simons level do not lie in a certain window which is defined in the original work [32]. We will stay in the regime where the acceptable operators lie in the $M \times (N-M)$ box. For the correspondence between permutation equivariant quantum K-theory and 3d gauge theory at a non-trivial Chern-Simons level and explicit connection between the K-theory correlators and Verlinde numbers see [121, 122].

The insertion of the Wilson line W_μ is akin to the action of an operator $\mathfrak{D}_\mu := \sigma_\mu(q^{\vec{\theta}})$, where σ_μ is the Schur polynomial with the shift operators q^{θ_i}, $i = 1, \ldots, M$, as arguments. For instance,

$$\sigma_\square(q^{\vec{\theta}}) = q^{\theta_1} + q^{\theta_2} + \ldots + q^{\theta_M} \ .$$

The aim is to compute the structure constants $C^\lambda_{\mu\nu}$ in the algebra,

$$\mathfrak{D}_\mu * \mathfrak{D}_\nu = \sum_\lambda C^\lambda_{\mu\nu} \, \mathfrak{D}_\lambda \ , \tag{4.45}$$

with the $*$ denoting the quantum product of the line operators.

Derivation of the Wilson loop algebra

To compute the algebra of Wilson loops we start with the permutation symmetric I-function extracted from the partition function which was given in (4.27). The effective Chern-Simons level term $q^{CS(\tilde{d})}$ was given in (4.28) and (4.29). We now outline a methodology to compute quantum products between Wilson operators.

(i) The Abelianised I-function (4.40) corresponding to the physical I-function (4.27) can be rewritten as,

$$\begin{aligned} I^{SQK}_{\mathrm{Gr}(M,N),\mathrm{ab}}(q,Q,\vec{\epsilon}) &= \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} \left(\prod_{i=1}^{M} Q_i^{\tilde{d}_i}\right) q^{CS(\tilde{d})} \left(\frac{\prod\limits_{1\leq i<j\leq M} \left(q^{\frac{\tilde{d}_{ij}^2}{2} - \left(\frac{\tilde{d}_i+\tilde{d}_j}{2}\right)} (q^{\tilde{d}_i} - q^{\tilde{d}_j}) \right)}{\prod\limits_{j=1}^{M}\prod\limits_{r=1}^{d_j} (1-q^{r-\epsilon_j})^N} \right) \\ &= \Delta \cdot \tilde{I} \ , \end{aligned} \tag{4.46}$$

where $\Delta = \prod\limits_{i<j}(p_i - p_j)$, with $p_i = q^{\theta_i}$ and,

$$\tilde{I} = \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} \left(\prod_{i=1}^{M} Q_i^{\tilde{d}_i}\right) q^{CS(\tilde{d})} \left(\frac{\prod\limits_{1\leq i<j\leq M} \left(q^{\frac{\tilde{d}_{ij}^2}{2} - \left(\frac{\tilde{d}_i+\tilde{d}_j}{2}\right)} \right)}{\prod\limits_{j=1}^{M}\prod\limits_{r=1}^{d_j} (1-q^{r-\epsilon_j})^N} \right) . \tag{4.47}$$

(ii) The function $\tilde{I}$ in (4.47) can be rewritten using the definitions (4.28) and (4.29) as,

$$\tilde{I} = \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} q^{\gamma \sum\limits_{i<j} \tilde{d}_i \tilde{d}_j} \prod_i I^{\mathbb{P}^{N-1}}_{\tilde{d}_i,\alpha,\beta}(Q_i) \ , \tag{4.48}$$

where the constants α, β and γ are given for $\Delta_\kappa = \frac{\kappa_A - \kappa_S}{M}$ by,

$$\begin{aligned} \alpha &= \hat{\kappa}_S + \Delta_\kappa + M && \xrightarrow{\hat{\vec{\kappa}}=0} \quad M-1 \ , \\ \beta &= \hat{\kappa_R} - \frac{1}{2}(M-1) && \xrightarrow{\hat{\vec{\kappa}}=0} \quad -\frac{1}{2}(M-1) \ , \\ \gamma &= \frac{\hat{\kappa}_A - \hat{\kappa}_S}{M} - 1 && \xrightarrow{\hat{\vec{\kappa}}=0} \quad -1 \ . \end{aligned} \tag{4.49}$$

Here the limit $\hat{\vec{\kappa}} = 0$ corresponds to the canonical Chern-Simons levels, which is the regime we work in to derive the Wilson loop algebra. The function $I^{\mathbb{P}^{N-1}}_{\tilde{d}_i,\alpha,\beta}$ is given by,

$$I^{\mathbb{P}^{N-1}}_{\tilde{d}_i,\alpha,\beta}(Q_i) = \frac{Q_i^{\tilde{d}_i} q^{\frac{\alpha}{2}\tilde{d}_i^2} + \beta\tilde{d}_i}{\prod_{r=1}^{d_i}(1-q^{r-\epsilon_i})^N} \ , \tag{4.50}$$

is denoted in this way as it be interpreted as a generalised summand of the projective space $\mathbb{P}^{N-1}$ quantum K-theoretic I-function. The complete I-function will involve a sum over the vortex sectors by the integers $d_i \geq 0$. The target space $\mathbb{P}^{N-1}$ is a special case of the Grassmannian target space with $M = 1$, i.e., $G = U(1)$ and $\mathrm{Gr}(1, M) \simeq \mathbb{P}^{N-1}$. The indices α and β can be exactly identified with the bare Chern-Simons levels corresponding to the gauge group U(1) mixing with itself and with the $\mathrm{U}(1)_R$ symmetry, respectively. Furthermore, it is evident from (4.48) that the generically non-vanishing constant γ hinders a factorisation of the function $\tilde{I}$ into a product of I-functions of projective spaces. In particular, for canonical Chern-Simons levels $\gamma = -1$ from (4.49).

(iii) The identity (4.46), schematically denoted as $I_{\mathrm{ab}} = \Delta\tilde{I}$, implies that the Wilson loop operator $\mathfrak{D}_u$ acts as,

$$\mathfrak{D}_u \cdot I_{\mathrm{ab}} = \Delta\ \mathfrak{D}_u \cdot \tilde{I} \ , \qquad \because\ [\Delta, \mathfrak{D}_\mu] = 0 \ . \tag{4.51}$$

The ideal of difference equations, i.e., equations in the shift operator $p_i = q^{\theta_i}$, can be obtained by studying the form of (4.47). For the values pf α, β and γ for the canonical Chern-Simons levels in (4.49) it is given by the relation,

$$\delta_i^N \tilde{I} = (-Q_i)\frac{p_i^M}{\prod_{j=1}^{M} p_j}\tilde{I} + \mathcal{O}(\epsilon_i^N) \ , \qquad i = 1, \ldots, M \ , \tag{4.52}$$

for $\delta_i = 1 - p_i$. The correction term $\mathcal{O}(\epsilon_i^N)$ corresponds to a trivial element in the cohomology ring of the $\mathrm{Gr}(M, N)$ as it does not correspond to a Schur polynomial inside the $M \times (N - M)$ box. In light of this ideal, we define shifted Wilson line operators $\widehat{W}_n = 1 - W_n$ such that the shift operator that emulates their insertion in the path integral corresponds to an action of δ_{-n} instead of p_{-n}. Consequentially, it is useful to compute a Wilson loop algebra modified from (4.45) to,

$$\hat{\mathfrak{D}}_\mu * \hat{\mathfrak{D}}_\nu = \sum_\lambda \hat{C}^\lambda_{\mu\nu}\ \hat{\mathfrak{D}}_\lambda \ , \tag{4.53}$$

where $\hat{\mathfrak{D}}_\mu := \sigma_\mu(1 - q^{\vec{\theta}})$, with σ_μ being the Schur polynomial labelled by the Young tableau μ as before.

(iv) The algebra of the modified Wilson line operators must be reduced using the ideal of relations (4.52). In other words, the quantum product between two Wilson line operators is expressed the action of the modified shift operators $\hat{\mathfrak{D}}_\mu$ modulo the difference ideal. That is, any time a δ_i^M is encountered, it is replaced by the R.H.S of (4.52). In doing to inverse powers of the p_i will be encountered and they must be simplified in the following way. Since the ideal of difference equations is a polynomial in the basic shift operators p_i

and the inverses thereof, we can re-express this ideal as,

$$\begin{aligned}(1-p_i)^N &= 1+\sum_{k=1}^{N}\binom{N}{k}(-p_i)^k = (-Q_i)\frac{p_i^{M-1}}{\prod\limits_{i\neq j} p_j}\ ,\\ \Rightarrow \frac{1}{p_i} &= (-Q_i)\frac{p_i^{M-2}}{\prod\limits_{i\neq j} p_j}+\sum_{k=1}^{N}\binom{N}{k}(-p_i)^{k-1}\ ,\end{aligned} \tag{4.54}$$

for $i=1,\dots,M$. The final expression can be used as a replacement for the inverse powers of the shift operator p_i encountered when substituting the ideal of relations. This is a deceptively tautological step, however, the usefulness of this replacement lies in the fact that it treats the operators p_i as variables of a polynomial relation that generates an ideal. This replacement will need to be done recursively in order to eliminate all inverse powers of the shift operators.

(v) Finally, we note that the reduced algebra of Wilson line operators $\widehat{W}_\mu$ contains extra terms of the order $\mathcal{O}(1-q)$. This is because when the Wilson loops in (4.53) are such that the difference operator $(\delta_i)^L$, with $L>N$, is encountered, then δ_i^{L-N} acts non-trivially on the $\tilde{Q}_i$ after the replacement (4.52). This becomes quickly evident on noticing the modified product rule for difference operators,

$$\begin{aligned}q^\theta(f_1\cdot f_2) &= \sum_{k\geq 0}^{\infty}\frac{(\log q\ \theta)^k}{k!}(f_1\cdot f_2) = \sum_{k\geq 0}^{\infty}\frac{(\log q)^k}{k!}\sum_{n=0}^{k}\binom{k}{n}\theta^{k-n}f_1\cdot\theta^n f_2\\ &= q^\theta f_1\cdot q^\theta f_2\ .\end{aligned}$$

This implies that for difference operator $\delta = 1-p$ and a function f that is independent of Q,

$$\delta(Q\ f) = (1-q^\theta)(Q\ f) = Q\ f-(qQ)(q^\theta f) = Q(\delta f)+(1-q)Q(q^\theta f)\ . \tag{4.55}$$

The last term is irrelevant for the Wilson loop algebra as it is a result for the non-trivial action of the shift operator on the Q term in the ideal relations (4.52). This subtle point can be understood by realising that the correspondence of the Wilson line insertions to the action of the shift operator was only applicable to the Q-terms in the I-function and not the extra Q-dependences that arise in the ideal. For more general Q-dependent terms in the ideal relations one can similarly separate a spurious $\mathcal{O}(1-q)$ term or simply set q to 1 to eliminate such terms. Interestingly in the 2d limit, that we will discuss in subsection 4.3.3, $q\to 1$ and this term vanishes. Hence, a quick way to obtain the physical algebra from the reduced algebra is to set q to one.

Incorporating this limit on q and returning to the fully non-Abelian theory, i.e, where $\vec{Q}_{\text{aux}}\to 1$, the algebra of Wilson line operators $\widehat{W}_\mu$ for canonical Chern-Simons levels is given by,

$$\widehat{W}_\mu * \widehat{W}_\nu = \sum_{\lambda\in\mathcal{B}_D}\hat{C}^\lambda_{\mu\nu}(Q)\ \widehat{W}_\lambda\ , \tag{4.56}$$

where $\mathcal{B}_D$ denote the Young tableaux inside the $M\times(N-M)$ box.

This concludes our discussion of the method to compute Wilson line operator products using the ideal of relations.

Wilson Loop Algebra and Quantum Cohomology

Before we proceed to discuss the Wilson loop algebra with canonical Chern-Simons levels for an explicit example, we discuss another important implication of the ideal of relations generating the Wilson loop algebra. In the aforementioned window of Chern-Simons levels, the dimension of the Wilson line algebra equals $\dim K(\mathrm{Gr}(M,N)) = \binom{N}{M}$. Furthermore, the algebra of Wilson lines is isomorphic to the quantum cohomology ring of the Grassmannian $\mathrm{Gr}(M,N)$. This can be seen explicitly for the special case of the Chern-Simons levels such that $\alpha = \beta = \gamma = 0$ in (4.48). The effective Chern-Simons levels for this case are given by,

$$\hat{\kappa}_S = -M \quad , \quad \hat{\kappa}_A = 0, \quad , \quad \hat{\kappa}_R = \frac{M-1}{2} \ .$$

For this choice, the ideal of difference relations can be calculated analogously and is given by,

$$\delta_i^N = -Q_i \ , \tag{4.57}$$

for $i = 1, \ldots, M$ and where $\delta_i = (1 - p_i)$ as usual. We can now interpret this ideal as the one generates the algebra of quantum cohomology ring of the Grassmannian. This ring is generated by by Schubert cycles labelled by Young tableaus, see Appendix A for details. Solving for the algebra of cohomology classes using this ideal and projecting onto to the full non-Abelian group $\mathrm{U}(M)$ results in algebra that is isomorphic to the quantum cohomology ring of the $\mathrm{Gr}(M,N)$ [66, 123, 124].

The work of Witten in [125] establishes an isomorphism between the quantum cohomology ring of the Grassmannian and the Verlinde algebra [126] of the gauged Wess-Zumino-Witten model $\mathrm{U}(M)/\mathrm{U}(M)$ at level $N - M$. Thus this certain K-theory algebra which is isomorphic to quantum cohomology is interpreted as being isomorphic also to a special Verlinde algebra.

Example: Grassmannian $\mathrm{Gr}(2,4)$

In this section we want to study the Wilson loop algebra of $\mathcal{N} = 2$ gauge theory on the $D^2 \times_q S^1$ with the simplest non-trivial Grassmannian $\mathrm{Gr}(2,4)$ as its target space. Specifically, the gauge group is $\mathrm{U}(2)$ and there are four chiral fields in the fundamental representation, c.f. Table 4.1.

The ideal (4.52) for this space becomes the system of equations,

$$\delta_1^4 \tilde{I} = (-Q_1)\frac{p_1}{p_2}\tilde{I} \qquad , \qquad \delta_2^4 \tilde{I} = (-Q_2)\frac{p_2}{p_1}\tilde{I} \ . \tag{4.58}$$

Correspondingly the inverse shift operators p_1^{-1} and p_2^{-1} are given by (4.54),

$$\frac{1}{p_{1/2}} = -\frac{Q_{1/2}}{p_{2/1}} + \sum_{k=1}^{4} \binom{4}{k} (-p_{1/2})^{k-1} \ . \tag{4.59}$$

Here the auxiliary parameters Q_i's are in the basis of the maximal torus group $\mathrm{U}(1)^2$ with respect to which the Abelianised fields are charged in Table 4.1 and in the non-Abelian theory $Q_1, Q_2 \to Q$. The algebra (4.53) or equivalently (4.56) can explicitly computed for Schur polynomials σ_μ, where μ labels the Young tableau. We perform the quantum multiplication of σ_1 with itself and σ_2 explicitly to illustrate the methodology.

$\underline{\boldsymbol{\sigma_1 * \sigma_1}}$:
The simplest Schur polynomial in two variables is given by,

$$\sigma_1(\delta_1, \delta_2) = \delta_1 + \delta_2 \ , \tag{4.60}$$

and the action of $\sigma_1 * \sigma_1$ on the Abelianised permutation symmetric I-function (4.47) is given by,

$$(\delta_1 + \delta_2)^2 \ I^{SQK}_{\text{ab}} = (\delta_1 + \delta_2)^2 \Delta \tilde{I} = (\sigma_2 + \sigma_{1,1}) \Delta \tilde{I} = (\sigma_2 + \sigma_{1,1}) \ I^{SQK}_{\text{ab}} \ , \tag{4.61}$$

where,

$$\sigma_2(\delta_1, \delta_2) = \delta_1^2 + \delta_1\delta_2 + \delta_2^2 \qquad , \qquad \sigma_{1,1}(\delta_1, \delta_2) = \delta_1\delta_2 \ . \tag{4.62}$$

Thus,

$$\sigma_1 * \sigma_1 = \sigma_2 + \sigma_{1,1} \ .$$

Since neither δ_1^4 nor δ_2^4 was not encountered in this quantum product, the replacement by the ideal was not required. In other words, this quantum product receives no quantum corrections.

$\underline{\boldsymbol{\sigma_1 * \sigma_2}}$:
The action of this product on the Abelianised I-function is given by,

$$(\delta_1 + \delta_2)(\delta_1^2 + \delta_1\delta_2 + \delta_2^2) \ \Delta \tilde{I} = (\delta_2^4 - \delta_1^3\delta_2 + \delta_2^3\delta_1 - \delta_1^4)\tilde{I} \ , \tag{4.63}$$

where again $\Delta = p_1 - p_2 = \delta_2 - \delta_1$.

First of all, the classical terms are those that do not require the ideal to be employed, i.e., appear as powers of the shift operators lower than 4, can be simplified as,

$$(-\delta_1^3\delta_2 + \delta_2^3\delta_1)\tilde{I} = (\delta_1 + \delta_2)(\delta_1\delta_2)(\Delta\tilde{I}) = (\sigma_{2,1}) \ I^{SQK}_{\text{ab}} \ . \tag{4.64}$$

For the remaining terms we use (4.58) and (4.59) to write,

$$(\delta_1)^4(\tilde{I}) = (-Q_1)(-Q_2 + p_1 \left(4 - 6p_2 + 4p_2^2 - p_2^3\right)) \ , \tag{4.65}$$

and similarly for $(\delta_2)^4(\tilde{I})$ with $1 \leftrightarrow 2$ in the above equation. The consolidated Q-dependent term in the quantum product becomes after setting $Q_i = Q$ and replacing $p_i = 1 - \delta_i$ is,

$$\begin{aligned}(\delta_2^4 - \delta_1^4)(\tilde{I}) &= Q(2 + (\delta_1 + \delta_2) + (\delta_2^2 + \delta_1^2) - (\delta_2\delta_1^2 + \delta_2^2\delta_1))(\Delta\tilde{I}) \\ &= Q(2 + \sigma_1 + \sigma_2 - \sigma_{1,1} - \sigma_{2,1}) \ I^{SQK}_{\text{ab}} \ .\end{aligned} \tag{4.66}$$

The total quantum product is given by,

$$\sigma_1 * \sigma_2 = \sigma_{2,1} + Q(2 + \sigma_1 + \sigma_2 - \sigma_{1,1} - \sigma_{2,1}) \ . \tag{4.67}$$

The Multiplication Table In a similar fashion the quantum products corresponding to all permissible Wilson line operators can be computed. Up to order $O(1-q)$ terms the multiplication table is given by,

$$\begin{array}{llllll}
\sigma_1 * \sigma_1 & = & \sigma_2 + \sigma_{1,1} \,, & \sigma_2 * \sigma_{2,2} & = & Q(\sigma_{1,1} + \sigma_{2,1} - \sigma_{2,2}) + Q^2 \,, \\
\sigma_1 * \sigma_2 & = & \sigma_{2,1} + Q(2+\rho_1) \,, & \sigma_{1,1} * \sigma_{1,1} & = & \sigma_{2,2} \,, \\
\sigma_1 * \sigma_{1,1} & = & \sigma_{2,1} \,, & \sigma_{1,1} * \sigma_{2,1} & = & Q\rho_1 \,, \\
\sigma_1 * \sigma_{2,1} & = & \sigma_{2,2} + Q(1+\rho_1) \,, & \sigma_{1,1} * \sigma_{2,2} & = & Q(\sigma_2 - \sigma_{2,1}) \,, \\
\sigma_1 * \sigma_{2,2} & = & Q\rho_1 \,, & \sigma_{2,1} * \sigma_{2,1} & = & Q(\sigma_2 + \sigma_{1,1} - \sigma_{2,2}) + Q^2 \,, \\
\sigma_2 * \sigma_2 & = & \sigma_{2,2} + Q(\sigma_1 + \rho_2) + Q^2 \,, & \sigma_{2,1} * \sigma_{2,2} & = & Q(\sigma_{2,1} - \sigma_{2,2}) + Q^2 \,, \\
\sigma_2 * \sigma_{1,1} & = & Q(1+\rho_1) \,, & \sigma_{2,2} * \sigma_{2,2} & = & Q^2 \,. \\
\sigma_2 * \sigma_{2,1} & = & Q\rho_2 + Q^2 \,, & & &
\end{array} \tag{4.68}$$

Here σ_μ stands for either the Wilson line $\widehat{W}_\mu$ of the algebra in (4.56) or the corresponding shift operator $\hat{\mathfrak{D}}_\mu(\delta)$ in (4.53). We used the abbreviations $\rho_1 = \sigma_1 + \sigma_2 - \sigma_{1,1} - \sigma_{2,1}$, $\rho_2 = \sigma_1 + \sigma_2 + \sigma_{1,1} - \sigma_{2,2}$. The classical K-theory ring can be obtained by setting $Q \to 0$ and agrees with the mathematical result on K-theoretic products [127].

Basis of Grothendieck Polynomials The Schur polynomials $\sigma_\mu(x)$ are related by a linear transformation of determinant one to the Grothendieck polynomials, denoted by $\mathcal{O}_\mu(x)$. These were originally defined in [128] and an explicit formula to compute them is summarised in the Appendix A. As with the Schur polynomials, they too are labelled by Young tableau however a Grothendieck polynomial might be linear combination of Schur polynomials labelled by tableaux of varying degrees.

The two basis are related for the Gr(2,4) as,

$$\begin{array}{lll}
\mathcal{O}_1 & = & \sigma_1 - \sigma_{1,1} \,, \qquad \mathcal{O}_2 = \sigma_2 - \sigma_{2,1} \,, \qquad \mathcal{O}_{1,1} = \sigma_{1,1} \,, \\
\mathcal{O}_{2,1} & = & \sigma_{2,1} - \sigma_{2,2} \,, \qquad \mathcal{O}_{2,2} = \sigma_{2,2} \,.
\end{array} \tag{4.69}$$

Using the Chern isomorphism the $\mathcal{O}_\mu$ can equivalently be seen as the Chern characters of the the structure sheaves of the Schubert cycles. After the basis change, we obtain for the quantum multiplication of the structure sheaves,

$$\begin{array}{c|ccccc}
* & \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_{1,1} & \mathcal{O}_{2,1} & \mathcal{O}_{2,2} \\ \hline
\mathcal{O}_1 & \mathcal{O}_2 + \mathcal{O}_{1,1} - \mathcal{O}_{2,1} & - & - & - & - \\
\mathcal{O}_2 & \mathcal{O}_{2,1} & \mathcal{O}_{2,2} & - & - & - \\
\mathcal{O}_{1,1} & \mathcal{O}_{2,1} & Q & \mathcal{O}_{2,2} & - & - \\
\mathcal{O}_{2,1} & \mathcal{O}_{2,2} + Q\,(1 - \mathcal{O}_1) & Q\,\mathcal{O}_1 & Q\,\mathcal{O}_1 & Q\,(\mathcal{O}_2 + \mathcal{O}_{1,1} - \mathcal{O}_{2,1}) & - \\
\mathcal{O}_{2,2} & Q\,\mathcal{O}_1 & Q\,\mathcal{O}_{1,1} & Q\,\mathcal{O}_2 & Q\,\mathcal{O}_{2,1} & Q^2
\end{array} \tag{4.70}$$

Here the $-$ represent the symmetry of the quantum product. These multiplications agree with the result of ref. [120], which has been obtained by quite different methods. We mention here that the well known geometric duality between Grassmaniianns $\mathrm{Gr}(M,N)$ and $\mathrm{Gr}(N-M,N)$ persists at the level of quantum K-theory. In the case of the Gr(2,4), which is self-dual, it manifests itself as the symmetry under the exchange of $\mathcal{O}_2$ and $\mathcal{O}_{1,1}$. A more non-trivial example of the Gr(2,5) and Gr(3,5) has been stated in the work [32].

The Inner Product Recall that the analog to the intersection pairing in quantum K-theory is given by the pairing (4.37) between a basis of the quantum K-theoretic ring. We compute the

Todd class of the tangent bundle on the Gr(2,4) and the Chern characters of the K-theoretic basis in question in the Appendix A. Using these results the inner product in either the Schur or the Grothendieck basis can be computed by performing the integral in (4.37) which amount to selecting the coefficient of the top-form as the integral is over Gr(2,4).

The inner product on the Schur basis, indexed in the order,

$$\{\sigma_0, \sigma_1, \sigma_2, \sigma_{1,1}, \sigma_{2,1}, \sigma_{2,2}\} \ ,$$

is given by the matrix,

$$\chi(\sigma_\mu, \sigma_\nu) = \begin{pmatrix} 1 & 2 & 3 & 1 & 2 & 1 \\ 2 & 4 & 2 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{4.71}$$

The inner product on the Grothendieck basis, indexed in the order,

$$\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_{1,1}, \mathcal{O}_{2,1}, \mathcal{O}_{2,2}\} \ ,$$

is given by the matrix,

$$\chi(\mathcal{O}_\mu, \mathcal{O}_\nu) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{4.72}$$

Here both σ_0 and $\mathcal{O}_0$ correspond to the identity element in the quantum K-theory ring. Although the determinants of both the inner product matrices is 1, the matrix is non-minimal in the Schur basis as it does not consist of only 0's and 1's.

4.3.2 q-Difference Equations

Recall that in Section 3.5 of Chapter 3 we introduced and discussed Givental's cohomological I-function corresponding to a target space X which, as the name suggests, is a cohomology-valued function. We derived relations among the correlators of twisted chiral scalars in a 2d gauged linear sigma model and imparted them an interpretation as operators annihilating the Fayet-Iliopolous parameter-dependent ground state. We further associated to these operators an ideal in the ring of south pole correlators and gave the scalars of the twisted chiral field a differential operator representation, $\mathrm{tr}(\sigma) \sim Q\partial_Q$, resulting in the correlator relations being represented as a differential ideal. The corresponding differential operator was shown to annihilate the cohomological I-function for explicit examples of target spaces. The pre-existing literature presents techniques to compute differential operators that annihilate the I-function however the derivation of correlator relations provides purely gauge theoretic approach to computing these operators.

In this section we study Givental's quantum K-theoretic permutation equivariant I-function, which can be extracted from the partition function calculation, in a similar light. It becomes evident from the form of the I-function that it is annihilated by difference operators, $p \sim q^{Q\partial_Q}$. Our focus will continue to be the gauge theories with Grassmannian target spaces in the footsteps of the analysis of difference operators for K-theoretic I-functions of toric varieties and complete intersections therein done in [97, 129].

q-difference Equations for Projective Spaces

To illustrate the emergence of q-difference operator explicitly we start with the example computation of projective space $\mathbb{P}^{N-1}$ as target space. The 3d theory is gauged by $U(1)$ and charged matter spectrum as listed in Table 4.2.

Chiral multiplets	$U(1)$ charge	$U(1)_R$ charge	real masses
ϕ_i, $i = 1, \ldots, N$	$+1$	0	0

Table 4.2: Matter spectrum of the $U(1)$ gauge theory of the projective space $\mathbb{P}^{N-1}$.

The permutation symmetric I-function for the canonical Chern-Simons terms is the unperturbed I-function, i.e, has a vanishing input and is explicitly given by,

$$I^{SQK}_{\mathbb{P}^{N-1}}(q, Q, \epsilon) = \sum_{d=0}^{\infty} \frac{Q^{d-\epsilon}}{\prod_{r=1}^{d} (1 - q^{r-\epsilon})^N} \ . \tag{4.73}$$

For generic Chern-Simons levels the projective space I-function was stated in (4.50). The parameter Q in the I-function above has been scaled down by a power of $(1-q)^{c_1(\mathbb{P}^{N-1})}$, where c_1 stands for the first Chern class of the tangent bundle which for this space is equal to N. The recursion relation for this I-function as power series in Q implies,

$$(1 - q^{\theta})^N I^{SQK}_{\mathbb{P}^{N-1}} = Q \cdot I^{SQK}_{\mathbb{P}^{N-1}} \ , \tag{4.74}$$

where $\theta = Q\partial_Q$ as usual. This is the q-difference equation annihilating the I-function of the projective space. In the forthcoming we will discuss the 2d limits of these difference equations and see that becomes precisely the differential equation annihilating the cohomological I-function.

q-difference Equations for Grassmannians

The technique of studying the recursion relation in order to derive the q-difference equation annihilating the permutation symmetric I-function is not as straightforward for Grassmannian target spaces as it was for the projective space. This becomes evident upon studying the appearance of the quantum K-theory ring generators of the I-function stated in (4.27). Recall that we defined the generators of the quantum K-theory on $\mathrm{Gr}(M, N)$ as the shifted Wilson line operators $\widehat{W}_\mu$, stated before (4.53), labelled by Young tableaux μ. The correspondence of Wilson loops and q-difference operators,

$$\widehat{W}_\mu(w_1, \ldots, w_M) = \sigma_\mu(w_1, \ldots, w_M) \quad \longrightarrow \quad \hat{\mathfrak{D}}_\mu(\delta_1, \ldots, \delta_M) = \sigma_\mu(\delta_1, \ldots, \delta_M) \ , \tag{4.75}$$

where $w_i = 1 - q^{-\epsilon_i}$ and $\delta_i = 1 - q^{\theta_i}$, can only be consistently done in the Abelianised theory. While the arguments of the Schur polynomials appears in expression for the I-function (4.27), the explicit combinations of these arguments as Schur polynomials does not. This suggests that it might be sensible to work in the Abelianised theory. However, this in turn must be done such that upon returning to the physical theory the q-difference operators of the Abelianised theory combine in a gauge covariant way in the physical theory.

It seems more expeditious to study the I-function as the J-function as at vanishing input,$I(0) = J(0)$. One can then utilise the relation of the J-function to a $T \in \mathrm{End}(K(Gr))$ from [130] given by,

$$J(t) = (1-q)T\Phi_0 = (1-q)\sum_t (T\Phi_i, \Phi_0)\Phi^i \ , \tag{4.76}$$

where $(\cdot,\cdot)$ is the pairing of the K-theory generators given by the Euler characterstic (4.34). T is an endomorphism in the K-group that satisfies the so-called 'flatness equations' [131]. This relation implies that difference equations that annihilate the J-function can be associated to relations among K-theory generators and certain covariant derivatives thereof.

At vanishing input, $t = 0$, this correspondence can be spelled out as follows. For a difference equation,

$$\sum_k a_k \ p^k J = 0 \ , \tag{4.77}$$

with $p = q^\theta$, exists a relation,

$$\sum_k a_k \ \nabla^k \Phi_0 = 0 \ , \tag{4.78}$$

where the covariant derivative is given by $\nabla(\cdot) = (p - \Phi_1 * p)(\cdot)$. For the $\mathrm{Gr}(M,N)$ at hand $k \leq \binom{N}{M}$ as that is the number of independent Young tableaux in an $N \times (N-M)$ box and hence the number of Schur/Grothendieck polynomials that generate the quantum K-theory ring. Any higher covariant derivatives can necessarily be expressed as linear combination of lower derivatives.

Thus a q-difference operator that annihilates the I-function at vanishing input can be computed by evaluating successive covariant derivatives of the identity element in the quantum K-ring and solving for a minimal set with maximal rank.

Illustrative Example: $\mathrm{Gr}(2,4)$

We will now illustrate the aforementioned technique utilising the quantum K-theoretic ring structure to compute the q-difference operators. We work in the basis of Grothendieck polynomials defined in the Appendix A. The maximal rank of the matrix of all the covariant derivatives is 5, i.e., one lower that the dimension 6. For instance,

$$\begin{aligned} \nabla(1) \ &= (p - \mathcal{O}_1 * p)(1) = 1 - \mathcal{O}_1 \ , \\ \nabla^2(1) &= (p - \mathcal{O}_1 * p)(1-\mathcal{O}_1) = (1-\mathcal{O}_1) - (\mathcal{O}_1 - (\mathcal{O}_2 + \mathcal{O}_{1,1} - \mathcal{O}_{2,1})) \\ &= 1 - 2\mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_{1,1} - \mathcal{O}_{2,1} \ . \end{aligned} \tag{4.79}$$

The list of covariant derivatives up to maximal rank is given by,

$$\begin{aligned} \nabla^3(1) = \ & (1+Q) + (-Q-3)\mathcal{O}_1 + 3\mathcal{O}_2 + 3\mathcal{O}_{1,1} - 5\mathcal{O}_{2,1} + \mathcal{O}_{2,2} \ , \\ \nabla^4(1) = \ & (1+5Q+qQ) + (-4-6Q-2qQ)\mathcal{O}_1 + (6+qQ)\mathcal{O}_2 \\ & + (6+qQ)\mathcal{O}_{1,1} + (-14-qQ)\mathcal{O}_{2,1} + 6\mathcal{O}_{2,2} \ , \\ \nabla^5(1) = \ & (14Q + 5qQ + q^2Q + q^2Q^2) + \left(-5 - 20Q - 11qQ - 3q^2Q - q^2Q^2\right)\mathcal{O}_1 \\ & + \left(10 + 6qQ + 3q^2Q\right)\mathcal{O}_2 + \left(10 + 6qQ + 3q^2Q\right)\mathcal{O}_{1,1} \\ & + \left(-30 - 6qQ - 5q^2Q\right)\mathcal{O}_{2,1} + \left(20 + q^2Q\right)\mathcal{O}_{2,2} \ . \end{aligned} \tag{4.80}$$

Solving for a_k's in (4.78) and substituting the same in (4.77) yields the difference operator,

$$(\delta^5 + Q(pq+1)(p^2q-1))I_{\mathrm{Gr}(2,4)} = 0 \ , \tag{4.81}$$

with the identification of the I- and J-function at vanishing input. In the original work [32] the difference operators for a few other Grassmannians as well as their 2d limits have been listed. We find agreement with the differential operators listed in [70] that annihilate the cohomological I-function.

4.3.3 The 2d Limit

Having studied both 2d and 3d gauge theories with four supercharges and having encountered similar structures vis-à-vis the target space geometry, it is natural to compare these theories in the limit where the length scale of the third dimension goes to zero. This is the primary aim of this section. We will schematically compare the partition functions and the cohomological/K-theoretic I-functions that can be extracted therefrom. Furthermore, the differential ideal and difference ideal defining the quantum product in cohomology and K-theory, respectively, can also be compared. A similar comparison follows for the differential and difference operators annihilating the I-functions in either theory.

Partition Function

In Chapter 3 we studied gauge theories in 2d with four supercharges on a two-sphere S^2 and we summarised the computation of the partition function using localisation techniques in Chapter 2 following the work of [34, 35]. The authors of [34] made a comparison of the partition function on the S^2 to the partition function of an $\mathcal{N}=2$ gauge theory on $S^2 \times S^1$ of [45, 132] by taking the radius of the S^1 to zero and found agreement up to an overall normalisation factor.

Our focus in this chapter has been on theories defined on $D^2 \times S^1$, and thus, to make an appropriate comparison we refer to the works [46, 47] in 2d. Amongst other results, they spell out the the partition function of $\mathcal{N}=(2,2)$ gauge theories on the disc D^2. To interest of us are Neumann boundary conditions for the vector and the chiral multiplets. For a matter content consisting of N chiral multiplets with weights $\vec{\rho}_i$ and and a gauge group G with roots $\vec{\alpha}$, the partition function on the D^2 reads,

$$Z_{D^2} = \frac{1}{|\mathcal{W}_G|} \oint \left(\prod_{a=1}^{\mathrm{rk}(G)} \frac{d\sigma_a}{2\pi i} \right) Z_{\mathrm{cl}}(\zeta, \vec{\sigma}) \; Z_{\text{1-loop}}(\vec{\sigma}) \; . \tag{4.82}$$

The 1-loop determinant contribution reads,

$$\begin{gathered} Z_{\text{1-loop}}(\vec{\sigma}) = Z^{2d}_{\mathrm{chiral}} \cdot Z^{2d}_{\mathrm{vector}} \; , \\ Z^{2d}_{\mathrm{chiral}} = \prod_{i=1}^{N} \Gamma\left(\vec{\rho}_i \cdot \vec{\sigma} - \frac{\Delta_i}{2} \right) \qquad ; \qquad Z^{2d}_{\mathrm{vector}} = \prod_{\alpha>0} (-\vec{\alpha}\cdot\vec{\sigma}) \frac{\sin(\pi \vec{\alpha}\cdot\vec{\sigma})}{\pi} \; . \end{gathered} \tag{4.83}$$

For simplicity we focus on the case with no non-trivial flavour symmetry which is akin to vanishing twisted masses. The twisted chiral scalar σ of 2d is distinct from that used to define the Wilson line operator e^{σ} in 3d. In order to compare the expressions in 2d and 3d we must perform a sum over residues and substitute the variable of integration accordingly. We will not state this explicitly here as our aim is to make a schematic comparison of the expressions inside the 1-loop determinant.

Recall that the corresponding 1-loop determinants in 3d are given by (4.9) and (4.11). The circumference of the S^1 is βr, introduced after (4.5), and $q = e^{-\beta\hbar}$, i.e., in the 2d limit $q \to 1$.

This has the consequence that all Chern-Simons-like terms of the form $q^{(\dots)}$ have a trivial contribution in the 2d limit. The identity (4.19) allows for the $(q^x,q)_\infty$ to be expressed as,

$$(q^x,q)_\infty \sim \Gamma_q(x)^{-1} \ ,$$

where the $\sim$ camouflages overall q-dependent normalisation terms, similar to the analysis of [34]. Consequently, (4.9) becomes,

$$\lim_{\beta\to 0} Z^{3d}_{\text{chiral}} = \lim_{\beta\to 0} \prod_{i=1}^{N} q^{(\dots)} \frac{1}{(q^v,q)_\infty} = \prod_{i=1}^{N} \Gamma(v) \ , \tag{4.84}$$

where v denotes the appropriate linear combination of the weights and R-charges after taking the sum over residues and we use that $\lim_{q\to 1}\Gamma_q(x) = \Gamma(x)$. This matches the structure of Z^{2d}_{chiral}. Similarly (4.11) becomes,

$$\lim_{\beta\to 0} Z^{3d}_{\text{vector}} = \lim_{\beta\to 0} \prod_{\alpha} q^{(\dots)} \left(q^{\vec{\alpha}\cdot\vec{\sigma}},q\right)_\infty = \prod_{\alpha>0} \frac{1}{\Gamma(\vec{\alpha}\cdot\vec{\sigma})\Gamma(-\vec{\alpha}\cdot\vec{\sigma})} \ , \tag{4.85}$$

which using the Γ-function identity, $\Gamma(z)\Gamma(-z) = -\frac{\pi}{x\sin(\pi x)}$, becomes structurally equivalent to Z^{2d}_{vector}.

I-Function

For the comparison of the cohomological and K-theoretic I-function we hone in on the Grassmannian target space $\mathrm{Gr}(M,N)$. In [85] the cohomological I-function is stated up to a normalisation factor as,

$$I^{\text{Coh.}}_{\mathrm{Gr}(M,N)} = \sum_{\vec{d}\in\mathbb{Z}^M_{\geq 0}} Q^{\left(\sum\limits_{i=1}^{M}\tilde{d}_i\right)} \frac{\prod\limits_{1\leq i<j\leq M}(\tilde{d}_i-\tilde{d}_j)}{\prod\limits_{j=1}^{M}\prod\limits_{r=1}^{d_j}(r-\epsilon_j)^N} \ . \tag{4.86}$$

The quantum K-theoretic I-function for $\mathrm{Gr}(M,N)$ given in (4.27) simplifies to the cohomological I-function in the 2d limit up to a normalisation factor and with an appropriate identification of the Kähler parameters. This becomes evident upon noting,

$$\lim_{\beta\to 0} \prod_{1\leq i<j\leq M} \left(q^{\frac{\tilde{d}^2_{ij}}{2}}\left(q^{\frac{\tilde{d}_{ij}}{2}} - q^{-\frac{\tilde{d}_{ij}}{2}}\right)\right) = \lim_{\beta\to 0} \prod_{1\leq i<j\leq M} (-)q^{\frac{\tilde{d}^2_{ij}}{2}-\frac{\tilde{d}_{ij}}{2}}\left(1-q^{\tilde{d}_{ij}}\right) \ ,$$

and using the limit of vanishing S^1 radius,

$$\lim_{\beta\to 0}\frac{1-q^n}{1-q} = n \ . \tag{4.87}$$

Quantum Ring Structure

To compare the cohomological and K-theoretic ring structures we also utilise the identity (4.87) for the difference operator, i.e., $\lim\limits_{\beta\to 0}\frac{1-q^\theta}{1-q} = \theta$. The ideal of difference equations (4.52) defining the quantum K-theory ring on target space $\mathrm{Gr}(M,N)$,

$$(\beta\theta_a)^N = Q_a \ , \tag{4.88}$$

and after a renormalisation of the Fayet-Iliopolous parameter given by $Q' = Q\beta^{-N}$, this becomes the generating ideal for the quantum cohomology ring of the Grassmannian. The multiplication table of the generators of quantum cohomology, given by Schur polynomials, can now be computed using this differential ideal to yield results that agree with literature [123–125].

The q-difference equations that annihilate the quantum K-theoretic I-function can computed in the 2d limit in a similar fashion. Explicitly, making the substitution,

$$q = e^{-\beta\hbar} \quad , \quad Q_a = Q'_a\beta^N \quad , \quad p = e^{-\beta\hbar\theta} \ , \tag{4.89}$$

in the q-difference operator and taking the limit $\beta \to 0$ yields the differential operator that annihilates the cohomological I-function. For the difference operator (4.81) for the Gr(2,4), this limit yields,

$$\theta^5 - 2(2\theta + 1)Q' \ , \tag{4.90}$$

and it matches exactly with our computation of the operator in 2d done in Chapter 3, see (3.85) for $\epsilon \to 1$. The difference operators for several other Grassmannians and their respective 2d limits have been listed in the work [32].

CHAPTER 5

Summary and Outlook

This thesis deals with gauge theories with four supercharges in two and three dimensions. The spacetime geometries that arise in string theory can be difficult to probe using direct methods due to their novel 'stringy' nature. The supersymmetric gauge theories that are central to this thesis can be modelled to play the role of string worldsheet theories and prove to be powerful tools in studying these geometries. Several techniques available in the gauge theory setup can be used to yield results that shed light on the stringy quantum geometry we are interested in. Furthermore, the low dimensional gauge theories studied in this thesis encode a wealth of interesting structures from the perspective of mathematics. We will expand upon these multi-disciplinary axes with which to approach such theories shortly.

Supersymmetric gauge theories, including but not limited to those that are central to this thesis, are attractive for another reason: solvability. This characteristic manifests itself cleanly in the technique of localisation, discussed in Chapter 2. This technique allows for the partition function and certain correlators of the gauge theory to be computed exactly. The crucial idea behind the localisation toolset is to exploit the nilpotency of the BRST symmetry which can be constructed using the underlying supersymmetry [44]. This nilpotency allows for the path integral, needed to compute partition functions and correlators, to localise to a finite dimensional fixed point locus of the BRST symmetry. This finite dimensional fixed point locus can be made conveniently tangible using the nature of the BRST symmetry at hand. Localisation was originally conceived in a seminal work of Witten [23, 33] in the context of the supersymmetric theory on the worldsheet, known as the non-linear sigma model, of string theory. The requisite nilpotent BRST symmetry for localisation was fashioned in this case by performing the so-called topological twist. Twisting of the non-linear sigma model might yield an A- or a B-model. Once such a twist is performed, the computation of physical quantities like the Yukawa couplings on the target space becomes easier. However, the degree of difficulty by the way of the A-model is far higher than that of the B-model, which finds relevance in the field of mirror symmetry which establishes an equivalence between the A- and B-models manifolds that are mirror to one another.

Having motivated the pertinence of these gauge theories to compactification spaces in string theory as well to solve supersymmetric quantum field theories exactly, we outline their conception in 2d. The narrative started with the work of Witten on $\mathcal{N} = (2, 2)$ gauge theories in 2d, called gauged linear sigma models [22]. This work establishes a clean correspondence of these 2d supersymmetric gauge theories to target space geometries of string theory by studying how the former flow to the latter in the infrared given the right starting ingredients, and is summarised in Chapter 3. The target space attained in the flow to the infrared is in fact a special case of a superconformal field theory, various (not necessarily geometrical) facets of which can be encountered in different phases of the gauge theory. The phase structure is provided to the gauge theory due to the parameter space of a Fayet-Iliopolous coupling corresponding to each U(1) factor in the gauge group. For the course of the work presented in this thesis, the focus remains on the so-called geometric phase, i.e., one which corresponds to a string-theoretic target space geometry.

Our original work [30] delves into gauged linear sigma models on the two-sphere S^2, using as fodder the work of [29] which provided a localisation-motivated technique to compute certain correlators of gauged linear sigma models. In particular, we study the structure of these correlators and find non-trivial relations between them. These relations assume a deeper importance when given an interpretation as the generators of an ideal of the non-commutative ring of correlators. A specific representation of this ideal associates differential operators annihilating a moduli-dependent ground state to these correlator relations. This moduli-dependent ground state can be perceived to lie in a Hilbert space of states labelled by the Fayet-Iliopolous moduli and other equivariant parameters in the gauge theory. The differential operators thus obtained are found to coincide with the GKZ-system of differential operators in mathematics [133]. This is confirmed by noting that these operators annihilate Givental's cohomological I-function, named as such because it takes values in the cohomology of the target space. The cohomological I-function is the generating function of certain enumerative invariants of the target space that correspond to non-perturbative objects in string theory known as worldsheet instantons. The work of [84, 85] found an explicit correspondence between the A-twisted correlators of gauged linear sigma models and the cohomological I-function, in that a bilinear pairing of the latter was shown to be a generating function of the former. This ties in with the central idea that correlators of these supersymmetric gauge theories in 2d yield the quantum cohomology ring of the target space. The quantum cohomology ring encodes all the operator product expansions between the operators in the so-called (a, c)-ring of an $\mathcal{N} = (2, 2)$ superconformal algebra, where a stands for anti-chiral and c for chiral [134].

In conclusion, the original work presented provides a method to compute the defining differential operators corresponding to geometries from a purely gauge theory toolset. The geometries for which this is done explicitly are toric varieties, complete intersections therein, Grassmannian varieties and complete intersection spaces therein, too. The examples relevant from a string theory target space perspective correspond to Calabi-Yau manifolds, which from a mathematical perspective are Ricci flat complex Kähler manifolds and from a gauge theoretic perspective arise in the infrared limits of gauged linear sigma models where the axial anomaly of the $\mathrm{U}(1)_R$ symmetry cancels. We highlighted the main idea of this work and its ramifications in Chapter 3. Furthermore, specific examples illustrating the computation of the aforementioned correlator relations and their differential operator representation was also done in that chapter. Specifically for Calabi-Yau manifolds with one and two parameters, the original work also computed an explicit formula yielding their respective Picard-Fuchs operators in terms of correlators alone using a different perspective on the problem.

Supersymmetric gauge theories in 3d with 4 supercharges, i.e., those labelled as $\mathcal{N} = 2$ theories in 3d, are the focus of study in Chapter 4. At first sight these theories do not seem to be directly related to worldsheet theories of geometries relevant for string theory the way the 2d gauged theories are. However, they find relevance in several different ways to target space geometries of string theory. The results of [97] have established a correspondence between 3d $\mathcal{N} = 2$ gauge theories and quantum K-theory. This statement is parallel to the correspondence of the 2d $\mathcal{N} = (2,2)$ gauge theories to quantum cohomology. Our original work [32] which was also presented in this thesis deals primarily with gauge theories defined on the twisted solid torus $D^2 \times_q S^1$, where q denotes the twisting parameter of the S^1 fibered over the D^2. This 3d geometry is such that in the limit where the radius of the S^1 vanishes, various results on the 3d theory collapse to those on the 2d theory and consequently the 3d theory can be viewed as a lift of the 2d theory. The parameter q quantifying the non-trivial fibration of the S^1 over the D^2 proves elemental in generalising various mathematical structures that were encountered in the 2d theory.

In the work [32] we utilise the 3d gauge theory/quantum K-theory correspondence for gauge theories with Grassmannians as target spaces. Grassmannian target spaces can be achieved when the gauge group in the 2d theory is $\mathrm{U}(M)$. This is a generalisation of the work in [97] which focused on Abelian gauge groups. We study the algebra of Wilson line operators wrapping the S^1 and find that it gives rise to the quantum K-theory ring on the target space. This algebra is dependent on a new ingredient specific to the 3d ecosystem known as the supersymmetric Chern-Simons terms. For certain values of the Chern-Simons levels, known as the canonical choice, this Wilson loop algebra is isomorphic to the quantum product between the K-theory generators of the ring of Schubert structure sheaves and the result is shown to match with the mathematical result of [120]. For a different value of the Chern-Simons levels the Wilson loop algebra generates the Verlinde algebra [126] of the unitary group, which itself has been proven to be isomorphic to the quantum cohomology algebra of the Grassmannian [123–125]. The limit to 2d can be achieved by contracting the S^1 to zero and we find that, irrespective of the value of the Chern-Simons levels, all inequivalent Wilson loop algebras collapse to the quantum cohomology algebra on the target space. Finally, we compute the limit to 2d of the partition function on the $D^2 \times_q S^1$ and find schematic agreement with the 2d partition function on D^2 computed in [46,47]. The 3d partition function encodes a generating function of quantum K-theoretic enumerative invariants, known as the quantum K-theoretic I-function, which we also extract from our computation [31]. Lastly, we derive q-difference operators that annihilate the K-theoretic I-function and find that they reduce to the differential operators computed in Chapter 3 using correlator relations in the 2d limit.

Outlook

The field of exploration of supersymmetric gauge theories using localisation techniques is relatively nascent and has many promising avenues to offer, having been set off by Pestun's computation of the partition function on the four-sphere S^4 [24]. Supersymmetric gauge theories in 2d have a variety of hitherto unscaled concepts and questions, one of which the author is currently involved in the process of concluding. This has to do with the subtleties that come with 2d Riemann surfaces with boundaries on which the gauge theory is defined. It is a well known result in string theory that D-branes on Calabi-Yau manifolds form a category of coherent sheaves on this manifold when the corresponding non-linear sigma model is B-twisted. Gauged linear sigma models with non-Abelian gauge groups and suitable matter representations exhibit dualities which relate distinct ultraviolet gauge theories with the same infrared physics [49]. Such dualities can be checked by comparing partition functions of dual gauge theories on S^2 or D^2. In our work we test dualities by matching pairs of boundary conditions, which implies an equivalence among brane spectra arising from dual gauge theories. Such equivalences among boundary conditions are realised in terms of suitable duality defects, which separate pairs of dual gauge theories along particular one-dimensional interfaces. In a geometric setting such equivalences can be understood mathematically as derived equivalences between the bounded derived category of coherent sheaves of the respective target space geometries. In particular, we study the duality which relates the theory with gauge group $\mathrm{U}(k)$ and with N flavours in the fundamental representation to the theory with gauge group $\mathrm{U}(N-k)$ and with N flavours in the fundamental representation of this dual gauge group. This duality geometrically realises the duality between the complex Grassmannians $\mathrm{Gr}(k, N)$ and $\mathrm{Gr}(N-k, N)$. We construct the identity defects and the duality defects for such Grassmannian target spaces and prove them to be quasi-isomorphic, thus accomplishing our goal. Furthermore, we generalise the identity defect to hypersurfaces and complete intersections in Grassmannians. It would be very interesting to extend these results to the wider array of Seiberg-like dualities [25] for $\mathrm{SO}(k)$ and $\mathrm{USp}(k)$ gauge groups proposed in [49]. In particular, a certain duality involving $\mathrm{USp}(k)$ groups relates two Calabi-Yau manifolds and is also known as the Grassmannian-Pfaffian duality. This relates closely to the homological projective duality program of [58, 59] and thus would be a fruitful endeavour to explore in detail.

The topic of supersymmetric gauge theories in 3d, in particular the mathematical connection to quantum K-theory, is fledgling and proves to be fertile with unexplored and interesting questions. While the 2d gauge theories can be directly associated to the worldsheet of a string, the 3d gauge theory can be speculated to be the worldsheet of the 11-dimensional M-theory [19]. M-theory is difficult to model or probe directly as a formalism to quantise the membranes that give rise to M-theory does not exist. It is also known that the 3d gauge theories with four supercharges arise as low energy theories of compactifications of M-theory on Calabi-Yau manifolds with four complex dimensions. This is analogous to the connection between 2d gauge theories as worldsheet theories of the type II string compactified on Calabi-Yau threefolds. Furthermore, the fact that the 3d theory can be interpreted as a lift of the 2d theory is reminiscent of the lift of the type II string theory to M-theory. All these results provide hints at the interpretation of the 3d theory as a membrane theory in some limit. Exploring the 3d gauge theory setup further to concretise a connection to M-theory is an attractive, if ambitious, problem.

We now shed light upon a technical aspect of our computation that can be generalised. We brushed upon the topic of supersymmetric Chern-Simons levels and derived a majority of the results for the canonical values of these levels. An added simplicity that accompanies this choice is that the input of the quantum K-theoretic J-function resulting from the partition function computation has a vanishing input. Givental's reconstruction theorem states that that a non-trivial input can be built into the J-function with vanishing input by the action of a reconstruction operator. This exercise was illustrated for projective spaces and complete intersection therein in the work of [97] and in an ongoing project the author is involved on extending this technique to Grassmannian target spaces [111].

Another important direction which is being explored in the upcoming work is that of the superfields in the 3d gauge theory on the $D^2 \times_q S^1$ being subject to different permutations of boundary conditions. In the presented work of the author [32], we adhered to Neumann boundary conditions for all chiral fields as well as the vector field. Dirichlet boundary conditions on the vector field break the gauge group to a global symmetry group and lead to several nuances in the structure of the partition function [102]. It must be mentioned that the generators of the quantum K-theory ring of the Grassmannian are Schur polynomials, thus making most computations more expensive compared to the case of when the gauge group is simply $\mathrm{U}(1)$ and the target space is a projective space. Tackling such complexity and finding ways to minimise it remains an important aspect of this research. More broadly, in two dimensional gauge theories with boundary conditions there exists a robust understanding of the mathematical features of D-branes [135–143] in the infrared. A similar understanding of the corresponding branes, sometimes called membranes, appearing in the infrared of three dimensional gauge theories with boundaries evades us [54, 97, 102] and unfolds several new directions of research.

APPENDIX **A**

The Grassmannian

A.1 Basics of the Grassmannian $\mathrm{Gr}(M,N)$

In this appendix we review some basic notions of complex Grassmannians, and we collect some properties of the canonical vector bundles on Grassmannians that are relevant in defining the quantum cohomology and quantum K-theory rings. Furthermore, we discuss certain characteristic classes on the Grassmannian that made an appearance in the computation of separating the 3d partition function into the K-theoretic I-function and a folding factor in Chapter 4. For a concise reference, we refer the reader to [65], from which many fundamental concepts presented here are inspired.

A.1.1 Introductory Definitions

(i) A complex Grassmannian, generally denoted by $\mathrm{Gr}(M,V)$, is the space of M-dimensional linear subspaces of a finite-dimensional complex vector space V. For $V \simeq \mathbb{C}^N$, it is usually denoted by $\mathrm{Gr}(M,N)$. This is the notation to which we adhere to in the entirely of this thesis.

Let $\mathrm{U}_{M,N}$ be the space of complex matrices of maximal rank with dimension $M \times N$, where $0 < M < N$. Then, an arbitrary element $\Lambda \in \mathrm{U}_{M,N}$ which is explicitly given by,

$$\Lambda = \begin{pmatrix} v_{11} & v_{12} & \dots & v_{1N} \\ \vdots & \vdots & & \vdots \\ v_{M1} & v_{k2} & \dots & v_{MN} \end{pmatrix} \ , \quad v_{ij} \in \mathbb{C} \ , \quad \mathrm{rk}(\Lambda) = M \ , \tag{A.1}$$

describes an M-plane in V and therefore a point on the Grassmannian $\mathrm{Gr}(M,N)$. This is not a unique representation of an M-plane in V in terms of a matrix $\Lambda \in \mathrm{U}_{M,N}$ and this can be understood by noting that for two elements $\Lambda, \Lambda' \in \mathrm{U}_{M,N}$,

$$\Lambda \sim \Lambda' \quad \text{iff} \quad \Lambda' = g \cdot \Lambda \ , \quad \text{for some} \quad g \in \mathrm{GL}(M,\mathbb{C}) \ ,$$

where $\mathrm{GL}(M,\mathbb{C})$ is the general linear group of M-dimensional complex matrices. Thus, the Grassmannian $\mathrm{Gr}(M,N)$ can be viewed as the quotient space $\mathrm{U}_{M,N}/\mathrm{GL}(M,\mathbb{C})$.

In particular, for $M = 1$ we have $\mathrm{GL}(1,\mathbb{C}) \simeq \mathbb{C}^*$ and $\mathrm{Gr}(1,N) = (\mathbb{C}^N \setminus \{0\})/\mathbb{C}^* \simeq \mathbb{P}^{N-1}$, i.e., the complex projective space of dimension $N-1$.

(ii) On any complex Grassmannian $\mathrm{Gr}(M, V)$ three standard complex vector bundles can be defined: the *trivial bundle* $\mathcal{V}$, the *universal subbundle* $\mathcal{S}$ and the *universal quotient bundle* $\mathcal{Q}$. For over-arching definitions of fibre bundles we refer the physics reader to [144].

The trivial bundle $\mathcal{V}$ of rank N corresponds to the vector space V trivially fibered over the Grassmannian $\mathrm{Gr}(M, V)$, i.e.,

$$\begin{aligned} \mathcal{V} \;:=\; \mathrm{Gr}(M,N) \times V &\to \mathrm{Gr}(M,N) \\ (l,v) &\mapsto l \; . \end{aligned}$$

The universal subbundle $\mathcal{S}$ is a subbundle of the trivial bundle $\mathcal{V}$. It is of rank M and is defined as,

$$\mathcal{S} := \{(l,v) \in \mathrm{Gr}(M,N) \times V \mid v \in l\} \; .$$

The dual of $\mathcal{S}$ is commonly known as the universal hyperplane bundle $\mathcal{S}^*$.

A determinant line bundle $\mathcal{L}$ on the $\mathrm{Gr}(M, N)$ can be constructed corresponding to the the dual universal subbundle by associating to each fibre in $\mathcal{S}^*$, which is an M-plane, the one-dimensional fibre of the M-dimensional exterior power of the plane. Owing to its trivial transformation under a $\mathrm{U}(M)$ transformation, a section of this determinant line bundle makes an appearance in constructing the gauge-invariant superpotential for a gauged linear sigma model with a Grassmannian target space. For instance, for the Rødland model discussed Chapter 3, the gauge group is $\mathrm{U}(2)$ and $\mathcal{L} = \epsilon_{\alpha\beta}\phi_i^\alpha\phi_j^\beta \in \wedge^2(\mathcal{S}^*)$ where i, j are the flavour indices and α, β are the gauge indices.

The universal quotient bundle $\mathcal{Q}$, or simply the quotient bundle, is defined as a quotient of the trivial bundle and the universal subbundle, i.e., $\mathcal{Q} = \mathcal{V}/\mathcal{S}$. The rank of the quotient bundle is $\mathrm{rk}(\mathcal{V}) - \mathrm{rk}(\mathcal{S}) = N - M$.

Equivalently, the quotient bundle $\mathcal{Q}$ is defined via the short exact sequence of vector bundles,

$$0 \to \mathcal{S} \to \mathcal{V} \to \mathcal{Q} \to 0 \; . \tag{A.2}$$

Note that $\mathcal{Q} \otimes \mathcal{S}^* \simeq \mathrm{T}\mathrm{Gr}(k, N)$ is the tangent bundle on the grassmannian $\mathrm{Gr}(M, N)$. In particular, for $\mathbb{P}^{N-1} = \mathrm{Gr}(1, N)$, $\mathcal{S} \simeq \mathcal{O}(-1)$, $\mathcal{V} \simeq \mathcal{O}^{\oplus N}$, $\mathcal{Q} \simeq \mathrm{T}\mathbb{P}^{N-1}(-1)$ and the short exact sequence (A.2) becomes the well-known *Euler sequence* of the complex projective space $\mathbb{P}^{N-1}$,

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus N} \to \mathrm{T}\mathbb{P}^{N-1}(-1) \to 0 \; .$$

(iii) A Grassmannian can be realised as a subvariety of a projective space using the Plücker embedding. The essential idea is to map an M-plane of the $\mathrm{Gr}(M, N)$ to an exterior power of the M (N-dimensional) vectors defining the M-plane, i.e.,

$$\begin{aligned} \iota : \mathrm{Gr}(M,N) &\to \mathbb{P}(\wedge^M V) = \mathbb{P}^{\binom{N}{M}-1} \\ \Lambda &\mapsto v_1 \wedge \ldots \wedge v_M \; , \end{aligned}$$

where Λ was defined in (A.1). The subvariety of the projective space $\mathbb{P}^{\binom{N}{M}-1}$ that is isomorphic to the Grassmannian $\mathrm{Gr}(M, N)$ is given by certain quadratic relations known as the Plücker relations. For the simplest non-trivial Grassmannian $\mathrm{Gr}(2, 4)$, there is a single quadratic Plücker relation leading to the isomorphism,

$$\mathrm{Gr}(2,4) \simeq \mathbb{P}^5[2] \ ,$$

mentioned in Chapter 3.

This concludes a lightning-fast review of Grassmannian manifolds, important vector bundles thereon and the Plücker embedding of Grassmannians into projective varieties.

A.1.2 Characteristic Classes on $\mathrm{Gr}(M, N)$

Characteristic classes of vector bundles on a base space associate certain cohomology classes on the space to the said vector bundles. There are a few recurring characteristic classes of the tangent bundle $\mathrm{TGr}(k, N)$ on the Grassmannian $\mathrm{Gr}(M, N)$ in Chapters 3 and 4 that we will compute in this subsection. For a review on the definitions of these and a wider range of characteristic classes we refer the reader of a physics background to [144].

Chern Class

The Chern class, as most characteristic classes, help identify whether two seemingly identical vector bundles on a base space are in fact identical by associating to the vector bundles a topological invariant known as the Chern class. The total Chern class corresponding to a vector bundle $\mathcal{E}$ can be expanded as,

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + c_2(\mathcal{E}) + \ldots + c_n(\mathcal{E}) \ ,$$

where $c_i(\mathcal{E}) \in H^{2i}(\mathcal{E})$. For an m-dimensional base space, the j^{th} Chern class c_j with $2j > m$ vanishes identically. Regardless of this constraint if the complex dimension of the fibre is k, then c_j with $j > k$ vanishes as well. For any short exact sequence, say (A.2), the Chern class has a property,

$$c(\mathcal{S}) \cdot c(\mathcal{Q}) = c(\mathcal{V}) \ .$$

Using the splitting principle of vector bundles and the additive property of the Chern class under the Whitney sum, the Chern class over any vector bundle can be written as a product over Chern roots a_i as,

$$c(\mathcal{E}) = \prod_{i=1}^{n} (1 + a_i) \ . \tag{A.3}$$

The Chern classes can be solved for to yield $c_j = s_j(a_1, \ldots, a_n)$, where s_j are the elementary symmetric polynomials of order j.

For the specific case of the Grassmannian $\mathrm{Gr}(M, N)$ we note the following points.

(i) There are M Chern roots of the M-dimensional dual universal subbundle $\mathcal{S}^*$ and they are given by $\epsilon_1, \epsilon_2, \ldots \epsilon_M \in H^2(\mathrm{Gr}(M, N))$. This terminology is to be compared with the appearance of ϵ's in the permutation equivariant I-function of quantum K-theory in Chapter 4.

(ii) For the trivial bundle $\mathcal{V}$, $c(\mathcal{V}) = 1$ since it is a trivially fibered bundle. Thus the Chern class of the quotient bundle is the inverse of that of the universal subbundle by the short exact sequence (A.2).

(iii) The Chern class of the tangent bundle $\mathrm{TGr}(k, N) \simeq \mathcal{Q} \otimes \mathcal{S}^*$ can be computed using the Chern roots of $\mathcal{Q}$, denoted by q_i, and the Chern roots of $\mathcal{S}^*$ using the splitting principle. It is given by,

$$\begin{aligned} c(\mathrm{Gr}(M,N)) \ &:= \ c(\mathcal{Q} \otimes \mathcal{S}^*) = c\left(\left(\bigoplus_i^{N-M} \mathcal{Q}_i\right) \otimes \left(\bigoplus_i^{M} \mathcal{S}_i^*\right)\right) \\ &= \prod_{i=1}^{M} \prod_{j=1}^{N-M} (1 + \epsilon_i + q_j) \ , \end{aligned} \tag{A.4}$$

where $\mathcal{Q}_i$ and $\mathcal{S}_i^*$ are the line bundles obtained upon the splitting of $\mathcal{Q}$ and $\mathcal{S}^*$, respectively. Consequently, they have as Chern roots q_i and ϵ_i, respectively. In the last equality we have used the property of the Chern class for a Whitney sum of vector bundles $c(A \oplus B) = c(A) \wedge c(B)$ and .

(iv) Finally, we note that the Chern class of the tangent bundle can alternatively be computed due the result of [113]. We now state this result in the language of gauge theory, which is more relevant for us. If the Grassmannian is given by the symplectic quotient space $X /\!\!/ G$ with $G = \mathrm{U}(M)$ then we define the corresponding symplectic quotient space $X /\!\!/ T$, where T is the maximal torus of G, i.e., $T = \mathrm{U}(1)^M$ and the bundles $E = \oplus_{\alpha>0} L_\alpha$, $\widetilde{E} = \oplus_{\alpha<0} L_\alpha$ where L_α is a line bundle associated to the the root α of G. Then,

$$c(X /\!\!/ G) = \frac{c(X /\!\!/ T)}{c(E) \wedge c(\widetilde{E})} \ .$$

Explicitly it is given by,

$$c(\mathrm{Gr}(M,N)) = \frac{\prod_{i=1}^{M} (1+\epsilon_i)^N}{\prod_{1 \le i < j \le M} (1+\epsilon_{ij})(1-\epsilon_{ij})} \ , \tag{A.5}$$

where $\epsilon_{ij} = \epsilon_i - \epsilon_j$. The expression (A.4) can be shown to match with the formula above order by order.

This concludes our discussion of the Chern class of the Grassmannian.

Chern Character

The Chern character $\mathrm{ch}(\cdot)$ is another characteristic class that makes an appearance in the discussion of the inner product between generators of K-theory. In particular the formula for the holomorphic Euler characteristic of vector bundles $\mathcal{E}, \mathcal{F} \in K(X)$ given by the Hirzebruch-Riemann-Roch theorem takes as an input the Chern characters of these bundles and the Todd class of X. In terms of the Chern roots defined in (A.3), the Chern character of a vector bundle $\mathcal{E}$ can simply be written as,

$$\mathrm{ch}(\mathcal{E}) = e^{a_1} + e^{a_2} + \ldots + e^{a_n} \ . \tag{A.6}$$

Using the expression in terms of the Chern roots the Chern character can be written in term of Chern classes as,

$$\mathrm{ch}(\mathcal{E}) = \mathrm{rank}(\mathcal{E}) + \mathrm{ch}_1 + \mathrm{ch}_2 + \ldots = \mathrm{rank}(\mathcal{E}) + c_1(\mathcal{E}) + \frac{1}{2}\left(c_1^2 - 2c_2\right) + \ldots \ ,$$

where $\mathrm{ch}_i(\mathcal{E}) \in H^{2i}(\mathcal{E})$. The properties of the Chern character under a tensor product, $\mathrm{ch}(\mathcal{E} \otimes \mathcal{F}) = \mathrm{ch}(\mathcal{E}) \wedge \mathrm{ch}(\mathcal{F})$, and Whitney sum, $\mathrm{ch}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{ch}(\mathcal{E}) \oplus \mathrm{ch}(\mathcal{F})$, of vector bundles can help compute the characters of the modified Wilson loop algebra generators of Chapter 4.

Todd Class

The final ingredient to compute the holomorphic Euler characteristic is the characteristic class known as the Todd class. The Todd class of the Grassmannian also makes an appearance in the expression of the folding factor inside the partition function on the $D^2 \times_q S^1$. For a vector bundle $\mathcal{E}$ with n Chern roots a_i as in (A.3), the Todd class $\mathrm{td}(\mathcal{E})$ given by,

$$\mathrm{td}(\mathcal{E}) = \prod_{i=1}^{n} \left(\frac{a_i}{1 - e^{-a_i}} \right) \ . \tag{A.7}$$

Using the formula above the Todd class can be written in terms of the Chern class at each level. For the tangent bundle on the Grassmannian, one can either use the Chern roots given in (A.4) to compute the Todd class, or alternatively one can also use the technique of [113] to write,

$$\mathrm{td}(\mathrm{Gr}(M, N)) = \prod_{i=1}^{M} \left(\frac{\epsilon_i}{1 - e^{-\epsilon_i}} \right)^N \cdot \prod_{i<j}^{M} \left(\frac{(1 - e^{-\epsilon_{ij}})(1 - e^{-\epsilon_{ji}})}{(\epsilon_{ij})(\epsilon_{ji})} \right) \ . \tag{A.8}$$

This is can be compared with the expression for the β'-dependent Todd class inside the folding factor of the partition function on the $D^2 \times_q S^1$.

Gamma class

The Gamma class associated to a vector bundles is a certain square root of the Todd class, we refer the reader to [112] for a comprehensive definition. The q-Gamma class was introduced in [97] as a q-deformation of the Gamma class. For the case of the Grassmannian Gr(M, N) we deduce the q-Gamma class from the folding factor. It takes the form,

$$\Gamma_{\mathrm{Gr}(M,N),q} = \frac{\prod\limits_{i=1}^{M} \Gamma_q(1 + \epsilon_i)^N}{\prod\limits_{i<j}^{M} \Gamma_q(1 + \epsilon_{ij})\Gamma_q(1 - \epsilon_{ij})} \ .$$

The structure of the q-Gamma class, as a ratio of factors corresponding to the symplectic quotient space with respect to the maximal torus of G and the line bundles corresponding to the roots of the group G, falls in line with the argument of [113].

A.2 Quantum Cohomology Ring on $\mathrm{Gr}(M,N)$

The cohomology elements of the Grassmannian $\mathrm{Gr}(M,N)$ are generated by Schubert cycles Σ_ν of Young tableaux with ν with at most M rows and $N-M$ columns. There is a surjective ring homomorphism from the Schubert cycles Σ_ν to the Schur polynomials σ_ν such that to each Schubert cycle can be associated a Schur polynomial labelled by the same Young tableau in variables $x_1,\ldots,x_M$.

First of all, we will explicitly define Schur as well Grothendieck polynomials, that made an appearance in Chapter 4, for completion.

Schur Polynomials

Schur polynomials are symmetric polynomials that are indexed by integer partitions, which for the purpose of the cohomology of the Grassmannian $\mathrm{Gr}(M,N)$ are Young tableaus that lie in an $M\times(N-M)$ box. For a Young tableau labelled by the non-increasing integer partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_M)$, the Schur polynomial $\sigma_\lambda(x_1,\ldots,x_M)$ is given by,

$$\sigma_\lambda(x_1,\ldots,x_M)=\frac{\det(A_\lambda)}{\prod\limits_{1\leq i<j\leq M}(x_i-x_j)}\ ,\quad \text{with}\quad A_\lambda=\begin{bmatrix} x_1^{\lambda_1+M-1} & x_2^{\lambda_1+M-1} & \ldots & x_M^{\lambda_1+M-1}\\ x_1^{\lambda_2+M-2} & x_2^{\lambda_2+M-2} & \ldots & x_M^{\lambda_2+M-2}\\ \vdots & \vdots & & \vdots\\ x_1^{\lambda_M} & x_2^{\lambda_M} & \ldots & x_M^{\lambda_M}\end{bmatrix} \tag{A.9}$$

Grothendieck Polynomials

There exists another basis of symmetric polynomials which are related to the Schur polynomials by a linear transformation of determinant one, known as the Grothendieck polynomials. These become more relevant in defining the generators of the quantum K-theory ring as they appear frequently in the mathematical literature in this context [128]. A Grothendieck polynomial $\mathcal{O}_\lambda$ in variables $x_1,\ldots,x_M$ is labelled by a Young tableau λ and is defined as,

$$\mathcal{O}_\lambda(x_1,\ldots,x_M)=\frac{\det(B_\lambda)}{\prod\limits_{1\leq i<j\leq M}(x_i-x_j)}\ ,\qquad \text{with}\quad (B_\lambda)_{ij}=(A_\lambda)_{ij}\ \cdot(1-x_j)^{i-1}\ , \tag{A.10}$$

where $(A_\lambda)_{ij}=(x_j)^{\lambda_i+M-i}$ is borrowed from the definition of Schur polynomials (A.9). For the sample case of the Gr(2,4) the basis change between Schur and Grothendieck polynomials was spelled out in (4.69).

For complex Grassmannian varieties the deformation of the classical cohomology ring to the quantum cohomology ring is established in refs. [123–125]. More generally, for Fano varieties Siebert and Tian show that if the ordinary cohomology ring is a polynomial ring with relations as in formula (A.13), then the quantum cohomology is captured by a Q-dependent deformation of these relations [69].

The classical product of Schubert cycles generating the quantum cohomology ring can be obtained by the *Pieri* and *Giambelli* rules, for a definition of which we refer the reader to [65]. When represented as Schur polynomials, the classical product coincides with the classical product of the corresponding Young tableaus. The rules to perform the quantum product of the even cohomology elements on the Grassmannian were given in [145].

Here we approach the problem of computing the quantum cohomology ring over the Grassmannian using the gauged linear sigma model correlator technique. To this end, we look at the specific example of the Grassmannian Gr(2,4) to highlight the correspondence between 2d gauge theory and quantum cohomology.

Example: $\mathbf{Gr(2,4)}$

For the Grassmannian $\mathrm{Gr}(2,4)$ the generators of the cohomology ring are the Schur polynomials in two variables $\epsilon_1, \epsilon_2 \in H^2(\mathrm{Gr}(2,4))$. Note that we use for them identical terminology as for the Chern roots of the dual universal subbundle. Classically, they generate the symmetric polynomial ring $\mathbb{C}[\epsilon_1,\epsilon_2]^{S_2}$ and obeying,

$$\sigma_\nu \cdot \sigma_\mu \;=\; \sigma_{\nu\otimes\mu} \;, \tag{A.11}$$

in terms of the tensor product $\otimes$ of Young tableaux of the permutation group S_2. The kernel of the ring homomorphism from Schubert cycles to corresponding Schur polynomials is given by the two relations,

$$\sigma_3(\epsilon_1,\epsilon_2) \;=\; \epsilon_1^3+\epsilon_1^2\epsilon_2+\epsilon_1\epsilon_2^2+\epsilon_2^3 \;=\; 0 \;, \qquad \sigma_4(\epsilon_1,\epsilon_2) \;=\; \epsilon_1^4+\epsilon_1^3\epsilon_2+\epsilon_1^2\epsilon_2^2+\epsilon_1\epsilon_2^3+\epsilon_2^4 \;=\; 0 \;, \tag{A.12}$$

Here the subscripts of Schur polynomials σ refer to the integer partition of a Young diagram. The cohomology ring thus becomes,

$$H^*(\mathrm{Gr}(2,4),\mathbb{C}) \;\simeq\; \mathbb{C}[\epsilon_1,\epsilon_2]^{S_2}\Big/\langle\sigma_3,\sigma_4\rangle \;\simeq\; \mathbb{C}[N_1,N_2]\Big/\Big\langle N_1N_2, -\tfrac{N_1^4}{4}+N_1^2N_2+\tfrac{N_2^2}{4}\Big\rangle \;. \tag{A.13}$$

In the last step the ideal $\langle\sigma_3,\sigma_4\rangle$ of the kernel of symmetric polynomials is expressed in terms of the *Newton polynomials* $N_\ell = x_1^\ell + x_2^\ell$, $\ell = 1,2$, which in turn generate the symmetric polynomial ring $\mathbb{C}[\epsilon_1,\epsilon_2]^{S_2}$.

When the quantum deformation of is applied to the Grassmannian $\mathrm{Gr}(2,4)$ it yields the quantum cohomology ring [69],

$$H^*_\star(\mathrm{Gr}(2,4),\mathbb{C}) \;\simeq\; \mathbb{C}[N_1,N_2][[Q]]\Big/\Big\langle N_1N_2, -\tfrac{N_1^4}{4}+N_1^2N_2+\tfrac{N_2^2}{4}+Q\Big\rangle \;, \tag{A.14}$$

i.e., the latter relation generating the classical cohomology relations is deformed by a factor of Q. This is reminiscent of the deformation of the classical to the quantum cohomology ring for projective spaces. The presented formulation relates directly to the gauge theory correlators and its correlator relations, as discussed in Chapter 3. We first note that the gauge invariant insertions are canonically identified with the Newton polynomials $N_r = x_1^r + x_2^r$ according to,

$$\mathrm{tr}(\sigma^r) \;\longleftrightarrow\; N_r \;. \tag{A.15}$$

The Schubert cycles are multiplied with the Q-dependent quantum product and then integrated over the Grassmannian $\mathrm{Gr}(2,4)$. Here, we have used the following relations among Newton and Schur polynomials,

$$N_1 \;=\; \sigma_1 \;, \qquad N_2 \;=\; \sigma_2 - \sigma_{1,1} \;. \tag{A.16}$$

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