ON THE THEORY OF HIGHER SEGAL SPACES

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to Shanti and Timo

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Summary

This thesis contains three chapters, each dealing with one particular aspect of the theory of higher Segal spaces introduced by Dyckerhoff and Kapranov:

1. By exhibiting the simplex category as an ∞-categorical localization of the dendrex category of Moerdijk and Weiss, we identify the homotopy theory of 2-Segal spaces with that of invertible ∞-operads.
2. Inspired by a heuristic analogy with the manifold calculus of Goodwillie and Weiss, we characterize the various higher Segal conditions in terms of purely categorical conditions of higher weak excision on the simplex category and on Connes' cyclic category.
3. We establish a large class of ∞-categorical Morita-equivalences of Dold–Kan type. As an application we describe higher Segal simplicial objects in the additive context as truncated coherent chain complexes; in the stable context, we identify higher Segal Γ-objects with polynomial functors in the sense of Goodwillie.

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Chapter 0

Introduction

> A well-known scientist (some say it was Bertrand Russell) once gave a public lecture on astronomy. He described how the earth orbits around the sun and how the sun, in turn, orbits around the center of a vast collection of stars called our galaxy. At the end of the lecture, a little old lady at the back of the room got up and said: “What you have told us is rubbish. The world is really a flat plate supported on the back of a giant tortoise.” The scientist gave a superior smile before replying, “What is the tortoise standing on?” “You’re very clever, young man, very clever,” said the old lady. “But it’s turtles all the way down!”
>
> “A Brief History of Time”, Chapter 1
> S. W. Hawking

This thesis begins with a short story about algebraic structures, homotopy types and the nature of equality. The goal is not yet to delve into the actual mathematics that the author has produced in the last three years and a half, but rather to transmit a particular way of thinking that he has acquired while doing so. Of course, none of these ideas are original in any way; they are just the author’s personal glimpse into a paradigm that is still unfolding in modern mathematics.

The impatient reader who immediately wants to know what this thesis is actually about is welcome to skip ahead to Section 0.3.

0.1 Higher algebraic structures—a very informal introduction

Mathematical objects come in many shapes and forms and—barring tautological answers like “that which mathematicians study”—it is probably impossible to give a precise and complete general definition of what a mathematical object is. In many areas of mathematics such as Geometry, Topology, Algebra or Representation Theory a central role is often played, however, by objects which can roughly be described as follows:

- An underlying thingamajig
- equipped with some *structure*
- satisfying certain *properties.*

For example, an abelian monoid is

- a set M
- equipped with a special element $0 \in M$, and a binary operation $+\colon M \times M \to M$ which to each pair (a, b) of elements of M associates a new element $a + b$,
- satisfying the familiar axioms of
  - unitality: $0 + a = a = a + 0$,
  - associativity: $(a + b) + c = a + (b + c)$,
  - commutativity: $a + b = b + a$

  for all $a, b, c \in M$.

This example shows a pattern very common for mathematical objects which are of "algebraic" nature: the underlying thingamajig is a *set*, the structure consists of a bunch of *operations*, and the axioms postulate certain *equalities* between various ways of applying these operations.

The notion of equality is so basic that most mathematicians rarely stop and give it a second thought. One reason for this is that we are used to dealing with sets, where equality is very easy, very black-and-white: either two elements of a set are equal, or they are not. Furthermore, given two random$^{1)}$ mathematical objects—say the real number π and the abelian group $\mathbb{Z}$—few mathematicians would ever consider asking whether they are equal; in fact, many$^{2)}$ would argue that the statement "$\pi = \mathbb{Z}$" (or "$\pi \neq \mathbb{Z}$" for that matter) is intrinsically ill formed and that one should only ever ask about equality between objects which are known *a priori* to be elements of a common set. From this perspective, the question of equality is at the very core of the concept of a set, which we might thus introduce via the slogan:

> A **set** is a collection of objects with a well behaved notion of equality.

What happens then, when the notion of equality in the underlying thingamajig gets more complicated? For example, think about the case of a (suitably well behaved) geometric/topological object like a CW complex, which will henceforth just call a space. It is a perfectly valid question to ask whether two points in a space are equal or not, but usually a topologist is uneasy about any situation where this question of equality is—or seems to be—an essential feature. One reason for this uneasiness is that not all non-equalities are equal: if two points are different but connected by a path one might say that they are "less non-equal" than if they lie in different path components entirely.

Of maybe bigger interest to the algebraically minded reader, we might consider the related question of equality between algebraic objects; let's say vector spaces (over some fixed field $\Bbbk$) for concreteness. Assume that we wanted to consider the set$^{3)}$ of all (finite dimensional) vector spaces as an algebraic structure, for instance by equipping it with the (external) direct sum ($\oplus$) or the tensor product ($\otimes$). Equality in the set of all vector spaces is just as easy as in any other set: either two vector spaces are the same—they consist of the same elements and the same addition/scalar multiplication—or they are not. There are many trivial vector spaces, *e.g.*, $\{0\}$, $\{1\}$ and $\{(0, 1)\} = \{0\} \oplus \{1\}$, which are pairwise non-equal; yet even the most pedantic mathematician will often just denote "the" zero vector space by 0 and happily write the "equation" $0 \oplus 0 = 0$. The situation for the tensor product is even worse: asked to explicitly define "the" vector space $V \otimes W$, different mathematicians might even write down non-equal definitions. Does this mean that the tensor product is an ill defined concept? Of course not: in practice, nobody$^{4)}$ is confused about what $V \otimes W$ is, even though it is not so easy to say what "is" really means in this context. So where did we go wrong?

An easy answer would be that we were not considering the correct notion of equality on the underlying thingamajig to begin with: maybe we should define points in a space to be equal if

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$^{1)}$ in the colloquial sense of the word

$^{2)}$ including the author

$^{3)}$ don't worry about size issues

$^{4)}$ except every student who learns about tensor products for the first time there is a path connecting them and maybe we should define two vector spaces to be equal if there is an isomorphism between them.

The examples outlined above highlight two different problems with this simple-minded approach:

- If we declare path connected points in a space X to be equal then we are no better off than studying the set of path components of X. This would mean that the theory of spaces just collapses to the theory of sets.
- Let V be a vector space equipped with some additional structure, let's say an inner product. If we now have some other vector space W which is (abstractly) isomorphic to V, we might still not be able to write down an inner product on W. Even if we are somehow able to choose isomorphisms along which to transfer the additional structure, this procedure could be ambiguous and we might end up (after many such transfers) with a new inner product on the original vector space V that is different from the original one.

More abstractly:

- Declaring objects to be equal might make an interesting theory collapse and lose its richness.
- Equal objects should behave the same; moreover it should be possible to transfer properties and structure across equality without getting into trouble.

The problem is not just that we gave an answer that is too naive, but that we were implicitly trying to answer a subtly flawed question: When are two things to be considered equal?
The hidden—and potentially pernicious—assumption in this question lies in the unassuming word "when"; it encapsulates the fundamental dogma that equality is a *property* which two objects might or might not have; a property which is either true or false. This does often not reflect mathematical practice: when a mathematician writes an "equation" $V = W$ between vector spaces, they typically have a specific isomorphism in mind; this isomorphism is then implicitly used whenever some structure is transferred from one side to the other. In other words, they are not just keeping track *when* two objects are equal (a property), but *how* they are equal (a structure). This shift of perspective is the core of homotopy theory, which, in the words of Barwick, should be thought of "as an enrichment of the notion of equality, dedicated to the primacy of *structure* over *properties*" [Bar17] or, slightly catchier:

> Equality is not a property, but a structure.

Whenever we would formerly say that two objects are equal, we should now have to explicitly pick and remember an equality witnessing this fact. These equalities should then be treated on the same footing as any other structure; in particular it should make sense to ask whether two equalities are themselves equal5), or rather—keeping the fundamental slogan in mind—how they are equal. These equalities should also allow to perform basic deductive steps. For instance, there should be

- for each object x, a special equality $x = x$ from x to itself (reflexivity),
- a way to compose two equalities $x = y$ and $y = z$ to an equality $x = z$ (transitivity),
- a procedure for reversing an equality $x = y$ into an equality $y = x$ (symmetry).

So what sort of higher structures are we supposed to be studying? What sort of object is formed by such infinite hierarchies of highly structured higher equalities? Giving a precise mathematical answer is unfortunately a difficult question which is outside the scope of this informal introduction so we will have to make do with a slogan:

> A **homotopy type**6) is a collection of objects with a well behaved notion of equality.

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5) Of course it only makes sense to ask about equality for pairs of equalities between the same two objects.

6) This perspective on the word "homotopy type" and its accompanying discussion about the nature of equality is heavily inspired by the ideas of homotopy type theory [HoTT13].

The attentive reader will not have missed that this slogan is the same that was supposed to characterize sets; the difference, of course, is the new interpretation of the crucial word "equality": in the case of sets, equality was a mere property; now it is a structure. Between two objects of a homotopy type, there is a collection of equalities which themselves need to have a well behaved notion of equality, *i.e.*, form a homotopy type. We can view every set as a homotopy type where the homotopy type of equalities between any two objects is either the singleton type $\{\star\}$ (which is the set with exactly one element $\star$) if they are equal, or the empty type $\varnothing$ (which is the set with no objects) if they are not. A homotopy type whose homotopy types of equalities are actual sets is precisely a groupoid, *i.e.*, a category with only invertible morphisms; from this perspective it is natural to use the synonym "**∞-groupoid**" instead of "homotopy type". Between two homotopy types we may consider *functors*, which—generalizing functions between sets and functors between groupoids—are maps that send objects to objects and equalities to equalities.

For vector spaces—as for many basic algebraic objects—one usually considers the identity as the only possible equality between isomorphisms; in other words, equality between isomorphisms is a property. Hence the hierarchy of equalities stops after three steps:

(0) vector spaces
(1) isomorphisms between vector spaces
(2) (actual) equality of isomorphisms.

This state of affairs makes the situation seem deceptively simple, since it gives the impression that higher equalities do not play a role. However, even in such a simple situation, higher equalities have a tendency to creep into the picture as soon as one wants to perform any sort of universal construction. Let us illustrate with an example: The set of finite dimensional vector spaces up to isomorphism is an abelian monoid under direct sum $\oplus$; freely adding inverses to this monoid—a process called *group completion*—gives rise to the abelian group $\mathrm{K}_0(\Bbbk)$, isomorphic to $\mathbb{Z}$[7]. If one wants to perform the analogous procedure while taking into account the full homotopy type $\mathbf{vect}_\Bbbk$ of vector spaces (which has non-trivial levels 0 and 1), one should consider $(\mathbf{vect}_\Bbbk, \oplus)$ as a "higher abelian monoid"[8] and then group complete it with respect to $\oplus$. When this group completion is performed in the correct homotopy theoretic sense[9], it gives rise to a "higher abelian *group*"[10] $\mathrm{K}(\Bbbk)$—called the *connective* K*-theory spectrum of* $\Bbbk$—whose underlying homotopy type contains non-trivial information in all degrees.

The concept of a homotopy type first arose in algebraic topology were historically it was roughly synonymous with "space up to homotopy equivalence". Every space X does indeed give rise to a homotopy type—called the *fundamental* ∞*-groupoid of* X—whose objects are the points of X and whose homotopy types of equalities are, recursively, the homotopy types associated to the spaces of paths between pairs of points. Grothendieck's[11] homotopy hypothesis[12] states that this procedure should provide an equivalence between the homotopy theory of spaces and that of homotopy types. In this context, a homotopy theory—also called $(\infty, 1)$-category, or ∞-category for short—is the homotopy theoretic version of a category, which we can summarize in the following slogan:

An **∞-category** is a collection of objects equipped with homotopy types of compos-

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[7] $\mathrm{K}_0(\Bbbk)$ becomes more interesting when $\Bbbk$ is no longer a field, but an arbitrary ring or scheme.

[8] The usual name for this structure is *symmetric monoidal* ∞*-groupoid*, or, since $\mathbf{vect}_\Bbbk$ has no higher equalities, symmetric monoidal (1-)groupoid.

[9] *i.e.*, it satisfies the correct universal property in the world of symmetric monoidal ∞-groupoids

[10] From the perspective of algebraic topology, these "higher abelian groups" are called *grouplike* E*-algebras*, or—in view of the recognition principle of Boardman and Vogt [BV73] and May [May72]—*infinite loop spaces* or *connective spectra*.

[11] Grothendieck explained these ideas in a letter to Quillen which appears as the beginning of the manuscript "Pursuing stacks". Scans of the original manuscript are hosted at Maltsiniotis's web-page [Gro]; see also `https://thescrivener.github.io/PursuingStacks/ps-online.pdf` for a version retyped in LaTeX.

[12] The name "homotopy hypothesis" was popularized by Baez [Bae07]

able arrows between them.

Note how this slogan would describe an ordinary category if we were to replace the expression "homotopy type" with the word "set"; and since every set is a homotopy type, each ordinary category is an example of an ∞-category. Moreover just like sets and functions form a category, one can define an ∞-category whose objects are the homotopy types themselves: just like the functions between two given sets S and T assemble naturally into a set T^S, the functors between two given homotopy types S and T assemble canonically into a homotopy type T^S. The homotopy theory of spaces has spaces as objects and between two spaces X and Y the homotopy type of the mapping space $\mathrm{Map}(X, Y)$. Nowadays there are various ways to make the notion of homotopy types and homotopy theories precise; the first test each of these frameworks always needs to pass is that the homotopy hypothesis must hold, either by definition or by theorem. The fact that homotopy types were historically first modeled via spaces and homotopies is the reason why in many mathematical writings (including this one) the word "**space**" is used synonymously with "homotopy type" and the higher equalities are typically referred to as (higher) *homotopies*. Echoing Barwick again we would like to stress, however, that there is nothing intrinsically topological in the notion of a homotopy type and that the theory of spaces just "so happens to be one way (and historically the first way) to model homotopical thinking" [Bar17].

The correct notion of equality between homotopy types is called *equivalence* and is the analog of the notion of isomorphism but now in the ∞-category of homotopy types rather than in the ordinary category of sets. Hence, just like the—slightly pretentious—answer to the question "What is a vector?" goes "An element of a vector space!", the question "What is a homotopy type?" can only be answered conclusively once one understands what it means to be "an object of the ∞-category of homotopy types".

Making rigorous and useful the notion of an ∞-category is a hard problem which in recent decades has generated a variety of different solutions and frameworks, each suited for different situations and needs. It would go far beyond the scope of this introduction to go deeper into this issue; the interested reader is referred to Bergner's survey book [Ber18].

At this point the reader is hopefully convinced that the homotopy-theoretic ideal of equality-as-structure—and the resulting extension of the theory of sets and categories to that of homotopy types and ∞-categories—is a natural and ubiquitous extension of the equality-as-property paradigm. Let us then go back to the study of basic algebraic objects, but this time through the lens of homotopy theory. We will attempt to define the homotopy theoretic analog of an abelian monoid with an underlying homotopy type (instead of set) $\mathcal{M}$. By turning equality from a property to a structure at every opportunity, we can then start to define an abelian monoid with underlying homotopy type $\mathcal{M}$ to consist of

- a binary operation $+\colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and a special object $0 \in \mathcal{M}$.
- for each $a, b, c \in \mathcal{M}$, equalities[13)]

$$\begin{aligned} \mathfrak{l}_a\colon &\quad 0 + a = a \\ \mathfrak{r}_a\colon &\quad a + 0 = a \\ \mathfrak{ass}_{a,b,c}\colon &\quad (a+b)+c = a+(b+c) \\ \mathfrak{sym}_{a,b}\colon &\quad a+b = b+a \end{aligned}$$

If we were to stop our definition here, we would not do justice to many common situations including our running example of vectors spaces and the operation $\oplus$. In this case, for example, we wouldn't want to choose any old isomorphism $\mathfrak{sym}_{V,W}\colon V \oplus W \xrightarrow{\cong} W \oplus V$. Instead, we probably have the specific choice $(v, w) \mapsto (w, v)$ in mind which has many special properties; for instance, it satisfies that the composition $V \oplus W \xrightarrow{\cong} W \oplus V \xrightarrow{\cong} V \oplus W$ is equal to the identity

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[13)] We should of course also require the families $\mathfrak{l}$, $\mathfrak{r}$, $\mathfrak{ass}$ and $\mathfrak{sym}$ to be equipped with suitable naturality equalities.

on $V \oplus W$. Hence it is reasonable to continue the definition with various equalities involving $\mathfrak{sym}$ (and also $\mathfrak{l}$, $\mathfrak{r}$, $\mathfrak{ass}$). For example,

- for each $a, b \in \mathcal{M}$, an equality $\mathfrak{ff}_{a,b}$ between the composition $a + b \overset{\mathfrak{sym}}{=} b + a \overset{\mathfrak{sym}}{=} a + b$ and the identity $\mathrm{Id}_{a+b} \colon a + b = a + b$.

In the case of vector spaces, whose homotopy type is just a groupoid, we are done now, because there are no non-trivial higher equalities whose compatibility could be questioned; we have thus defined what it means for the homotopy type of vector spaces with the operation $\oplus$ to be a so-called *symmetric monoidal groupoid*[14)]. For a general homotopy type we would have to keep going by adding suitable equalities—involving $\mathfrak{ff}$, for example—and then equalities between those equalities, and so on, adding more structure at every level to witness the coherence of the level before it. In other words, this enhanced version of an abelian monoid—typically called a *symmetric monoidal* ∞*-groupoid*[15)]—is ~~turtles~~ structure all the way ~~down~~ up.

0.2 Segal presheaves

This begs the question: *How can one efficiently write down homotopy coherent algebraic structures?* One possible strategy to answer this question goes back to the following brilliant insights of Segal [Seg74] which explain how to define the homotopy theoretic analog of an abelian monoid:

(1) There is a certain category Γ such that an abelian monoid can be encoded as a Γ-set (*i.e.*, a functor $\Gamma^{\mathrm{op}} \to \mathbf{Set}$ from the opposite of Γ to the category of sets) satisfying certain special conditions.

(2) The aforementioned special conditions still make sense for functors $\Gamma^{\mathrm{op}} \to \mathcal{S}$ which now take values in the ∞-category $\mathcal{S}$ of spaces rather than sets; furthermore it is a good idea to *define* a **symmetric monoidal ∞-groupoid** to be such a special presheaf.

Before we explain in more detail what the category Γ is and how Γ-sets encode abelian monoids, let us first abstract Segal's ideas to obtain the following general recipe for extending the definition of an algebraic object $\langle X \rangle$ to the homotopy theoretic world:

(1) Find a suitable category Z and identify the category of $\langle X \rangle$es with the category of functors $Z^{\mathrm{op}} \to \mathbf{Set}$ satisfying certain conditions $\langle P \rangle$.

(2) Define the ∞-category of ∞-$\langle X \rangle$es to be the ∞-category of space-valued functors $Z^{\mathrm{op}} \to \mathcal{S}$ satisfying the correct homotopy theoretic version of $\langle P \rangle$.

How useful such a definition of an ∞-$\langle X \rangle$ turns out to be does of course depend crucially on the specific choice of the category Z and the conditions $\langle P \rangle$. Segal's notion of a special Γ-space, for instance, became a convenient framework for studying the theory of algebra and modules over "higher rings" [BF78; Lyd99; Sch99] and inspired Lurie's definitions of (symmetric) ∞-operads and symmetric monoidal ∞-categories [Lur17].

There is one such category Z which towers high over all others in terms of historical and mathematical significance: The simplex category Δ—whose detailed introduction we hereby add to the queue of explanations owed to the reader—was introduced by Eilenberg and Zilber [EZ50] who without ever using the words "category" or "functor"[16)] developed a theory of homology for what are nowadays called *simplicial sets*[17)], namely set-valued presheaves $\Delta^{\mathrm{op}} \to \mathbf{Set}$ on Δ. The goal of studying simplicial sets, just like that of simplicial complexes before them, was to make algebraic topology more combinatorial. The main idea was that one should consider topological spaces built from simplices—points, lines, triangles, tetrahedra, and so on—by specifying only a discrete set of data which determines how these simplices are glued together along their faces. A

[14)] See [Mac98, VII and XI] for a full list of axioms, including the famous pentagon and hexagon equations.

[15)] or, from the point of view of spaces, E_∞-algebra

[16)] The language of category theory had in fact been introduced by Eilenberg himself and MacLane [EM45] only a couple of years prior.

[17)] At the time Eilenberg and Zilber called them "complete semi-simplicial complexes".

topological 2-sphere could be built, for instance, by gluing all faces of a 2-simplex (a.k.a. triangle) to a single 0-simplex (a.k.a. point) but also by gluing two 2-simplices to each other along their boundary. The hope was that by expressing spaces in terms of combinatorial data, one could easier compute—maybe even algorithmically with the help of computers—fundamental invariants like homology or homotopy groups. From this perspective, what ultimately matters about a simplicial set is the homotopy type it encodes, called its *geometric realization.* Kan [Kan58a], for instance, defined the homotopy groups of a simplicial set—*i.e.*, those of its geometric realization—using only its combinatorial structure$^{18)}$. It is a famous result due to Quillen [Qui67] that geometric realization induces an equivalence between the homotopy theory of certain simplicial sets—called *Kan complexes*—and the homotopy theory of spaces. Since then it has become very common in algebraic topology and homotopy theory to work with simplicial sets rather than topological spaces, since they are typically easier to manipulate and reason about.

Before we come to the promised definitions of the simplex category Δ and Segal's category Γ, let us explain one way—the most important way—in which Δ fits into Segal's recipe for defining algebraic structures: Rezk's model for ∞-categories [Rez01].

(1) The data of a category C can be encoded into a simplicial set $\mathrm{N}(C)\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ called its nerve; the simplicial sets arising this way are precisely those which satisfy what Rezk calls the *Segal conditions.*

(2) Rezk then defines an ∞-category to be a simplicial space $\Delta^{\mathrm{op}} \to \mathcal{S}$ satisfying the correct analog of the Segal conditions.

As an aside, let us address an apparent circularity that appears here: it seems that in order to talk about simplicial spaces we already need to have a good notion of ∞-categories or, at the very least, the ∞-category of spaces. Rezk—just like many mathematicians before and after him—solved this issue by using the language of model categories which was introduced by Quillen [Qui67] long before there was any usable framework to work with ∞-categories directly. Model categories—which Baez calls "a trick for getting $(\infty, 1)$-categories" [Bae07]—make it possible to reason about the homotopy theory of spaces (and many others ∞-categories) in an indirect way, without ever having to leave the world of ordinary categories.

So then, without further ado, let us answer the question which is surely burning in the reader's mind by now: What are Γ and Δ and how do Γ-sets and simplicial sets encode abelian groups and categories, respectively?

- The category $\Gamma \coloneqq \mathbf{Fin}_\star^{\mathrm{op}}$ is the opposite of the category $\mathbf{Fin}_\star$ of finite pointed sets $\langle n\rangle \coloneqq \{\star, 1, \dots, n\}$ (with basepoint $\star$) and pointed (a.k.a. basepoint-preserving) maps between them. Each abelian monoid M gives rise to a functor $\mathrm{N}(M)\colon \Gamma^{\mathrm{op}} = \mathbf{Fin}_\star \to \mathbf{Set}$ which maps the object $\langle n\rangle$ to the set $\mathrm{N}(M)_{\langle n\rangle} \coloneqq M^n$ and each pointed map $f\colon \langle n\rangle \to \langle m\rangle$ to the function $\mathrm{N}(M)_f\colon M^n \to M^m$ given by the formula $(a_i)_{i=1}^n \mapsto (\sum_{i\in f^{-1}\{j\}} a_i)_{j=1}^m$. What is more, one can show that a Γ-set $\mathcal{X}\colon \Gamma^{\mathrm{op}} \to \mathbf{Set}$ is isomorphic to $\mathrm{N}(M)$ for some abelian monoid M if and only if satisfies the following special conditions:
  - for each natural number $n \geq 0$, the canonical projections $\delta_i\colon \langle n\rangle \to \langle 1\rangle$, which send i to 1 and all other elements to the basepoint, induce an equivalence (*i.e.*, bijection) $(\mathcal{X}_{\delta_i})_{i=1}^n\colon \mathcal{X}_{\langle n\rangle} \xrightarrow{\cong} \prod_{i=1}^n \mathcal{X}_{\langle 1\rangle}$.

  The abelian monoid associated to a special Γ-set $\mathcal{X}$ has underlying set $\mathcal{X}_{\langle 1\rangle}$; its addition is the map $\mathcal{X}_{\langle 1\rangle} \times \mathcal{X}_{\langle 1\rangle} \cong \mathcal{X}_{\langle 2\rangle} \to \mathcal{X}_{\langle 1\rangle}$ encoded in the value of $\mathcal{X}$ at the pointed map $1, 2 \mapsto 1$ from $\langle 2\rangle$ to $\langle 1\rangle$. The procedure $M \mapsto \mathrm{N}(M)$ extends to a functor from the category of abelian monoids to the category of special Γ-sets; it is easy to show that it is actually equivalence of categories.

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18) Kan gives a direct combinatorial formula only for those simplicial sets—nowadays called Kan complexes—which satisfy what he calles the *extension condition.* He also introduces a combinatorial procedure which replaces an arbitrary simplicial set K by a Kan complex $\mathrm{Ex}^\infty K$ that represents the same homotopy type.

Since homotopy types, just like sets, admit a well-behaved notion of Cartesian products, one can translate the definition of a special Γ-set *verbatim* to obtain the definition of a special Γ-space.

- The **simplex category** Δ is the category of finite non-empty linearly ordered sets and weakly monotone maps between them. A more fancy way of saying this is that Δ is the full subcategory of the category **Cat** of categories spanned by the categories which are isomorphic to those of the form $[n] := \{0 \to 1 \to \cdots \to n\}$. The nerve of a category C is the simplicial set defined by the composition

$$\mathrm{N}(C)\colon \Delta^{\mathrm{op}} \hookrightarrow \mathbf{Cat}^{\mathrm{op}} \xrightarrow{\mathbf{Cat}(-,C)} \mathbf{Set}.$$

Explicitly, the first few values of $\mathrm{N}(C)$ are at $[0]$ the set of objects of C, at $[1]$ the set of morphisms of C, and at $[2]$ and at $[3]$ the set of commutative triangles and tetrahedra

$$\begin{tikzcd}
 & \bullet_1 \arrow[dr, red] & \\
\bullet_0 \arrow[ur, red] \arrow[rr] & & \bullet_2
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \bullet_1 \arrow[dr, red] \arrow[dd] & \\
\bullet_0 \arrow[ur, red] \arrow[rr] \arrow[dr] & & \bullet_2 \arrow[dl, red] \\
 & \bullet_3 &
\end{tikzcd}$$

in C, respectively. The Segal conditions encode the following fundamental property of $\mathcal{X} = \mathrm{N}(C)$: Specifying a commutative simplex (*e.g.*, a triangle or tetrahedron) in C is the same as specifying just its *spine* consisting of the arrows $\bullet_{i-1} \to \bullet_i$ drawn in red above: every other arrow in the simplex is then uniquely determined as a composition of arrows in the spine. Formally, this can be phrased by saying that the canonical map

$$\mathcal{X}_{[n]} \xrightarrow{\cong} \mathcal{X}_{\{0,1\}} \times_{\mathcal{X}_{\{1\}}} \mathcal{X}_{\{1,2\}} \times_{\mathcal{X}_{\{2\}}} \cdots \times_{\mathcal{X}_{\{n-1\}}} \mathcal{X}_{\{n-1,n\}}$$

(induced by the inclusions $\{(i-1) \to i\} \hookrightarrow [n]$) is an equivalence for all $n \in \mathbb{N}$. As in the case of abelian monoids and Γ-sets, it is not hard to show that the nerve construction induces an equivalence between the category of categories and the category of Segal simplicial sets. The category associated to a Segal simplicial set $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ has $\mathcal{X}_{[0]}$ and $\mathcal{X}_{[1]}$ as its sets of objects and arrows, respectively; the composition of composable arrows is determined by the span

$$\mathcal{X}_{\{0,1\}} \times_{\mathcal{X}_{\{1\}}} \mathcal{X}_{\{1,2\}} \xleftarrow{\cong} \mathcal{X}_{[2]} \longrightarrow \mathcal{X}_{\{0,2\}},$$

where the first map is invertible by the Segal condition.

By replacing fiber products of sets with the correct analog[19] for homotopy types, the Segal conditions for simplicial sets translate again *verbatim* to those for simplicial spaces. These so-called Segal spaces[20] form Rezk's famous model for ∞-categories.

0.3 Higher Segal spaces, or: What is this thesis about?

After having illustrated what higher algebraic structures are and how they can sometimes be encoded as special presheaves on suitably chosen categories, we come to a two-paragraph overview

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[19] The fiber product of homotopy types does *not* correspond to the ordinary pullback in the category of CW complexes (which would not be invariant under homotopy equivalence) but rather to the so-called *homotopy pullback* which has the correct universal property in the ∞-category of spaces.

[20] For the sake of simplicity we are sweeping one additional condition—called completeness—under the rug here: roughly speaking it states that in an ∞-category described by a Segal space $\mathcal{X}$, the equivalences agree with the equalities already present in the homotopy type $\mathcal{X}_{[0]}$ of objects.

summarizing what this thesis is actually about, which shall then be expanded and explained in much greater detail.

In 2012 Dyckerhoff and Kapranov introduced a new type of simplicial object satisfying a weakening of Rezk's Segal conditions and gave many examples, including a construction of Waldhausen famous for its homotopical meaning in algebraic K-theory. One of their main insights was that these 2-*Segal spaces* carry not just homotopical but also higher *algebraic* information as witnessed by the fact that they give rise to a certain class of algebras—called *Hall algebras*—of great representation-theoretic interest. But this was just the beginning: they also observed that Segal and 2-Segal spaces are just the start of a fascinating hierarchy of so-called *higher Segal spaces* whose basic properties and fundamental examples—generalizing the constructions of Waldhausen and Segal—were then established by Poguntke. The fundamental question guiding this thesis is:

> What are higher Segal spaces?

And, more specifically:

> What higher algebraic structure is encoded in a higher Segal space?

This thesis consists of three chapters, each dealing with a different aspect of the theory of higher Segal spaces:

(1) The first chapter exclusively discusses 2-Segal spaces and gives a complete explanation of their algebraic structure by relating them to the ∞-operads of Cisinski and Moerdijk.
(2) The second chapter contains an intrinsic characterization of higher Segal spaces in terms of purely categorical notions of higher excision.
(3) In the third and last chapter we establish a generalized and homotopy coherent version of the Dold–Kan correspondence which we then apply to study higher Segal objects in the additive context.

For a graphical overview of the results in this thesis, see Figure 1 below.

0.3.1 Previous publications

Most of the results of this thesis were previously made available in separate publications/preprints:

- Section 0.4 and Chapter 1 cover the material of [Wal17].
- Section 0.5 and Chapter 2 cover the material of [Wal19a].
- Section 0.6 and Sections 3.1–3.6 of Chapter 3 cover the material of [Wal19b].
- Section 3.7.1 contains a slightly stronger version of results first obtained with G. Jasso and T. Dyckerhoff [DJW19].

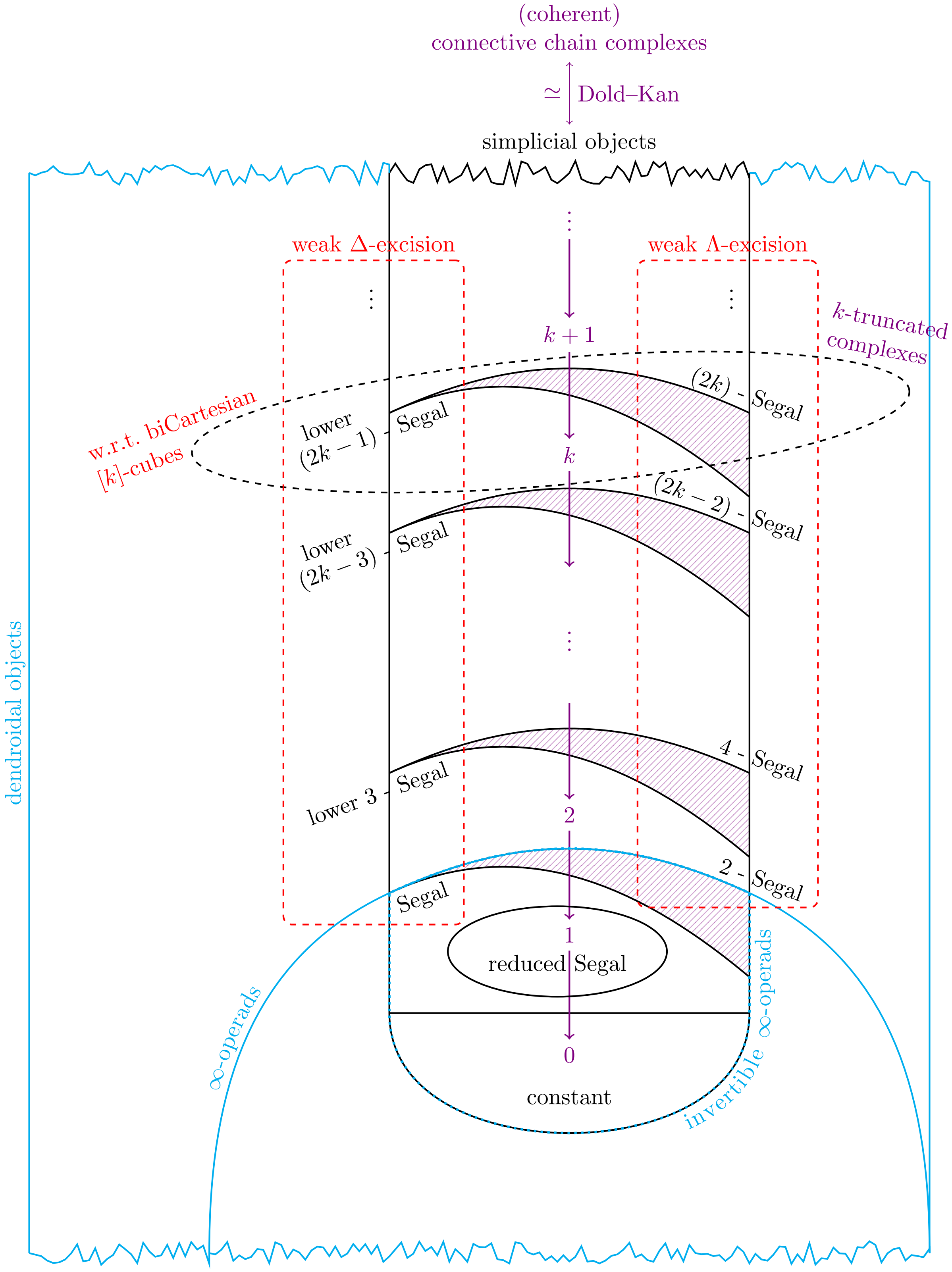

Figure 1: An artistic depiction of the hierarchy of higher Segal spaces: Chapter 1 (cyan) explains how simplicial spaces and ∞-operads intersect precisely in the 2-Segal spaces/invertible ∞-operads. In Chapter 2 (red), lower odd Segal and even Segal objects are characterized separately via weak excision on Δ and Λ, respectively. In the additive situation we can use the ∞-categorical Dold–Kan correspondence discussed in Chapter 3 (violet) to translate between higher Segal conditions and truncation conditions on chain complexes; the shaded discrepancy between lower $(2k-1)$-Segal objects and $2k$-Segal objects disappears in this case.

0.4 2-Segal spaces as invertible ∞-operads (Chapter 1)

We have seen how a simplicial set $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ encodes a category if and only if it satisfies Rezk's *Segal conditions*. Recall that the category corresponding to $\mathcal{X}$ has $\mathcal{X}_{[0]}$ as its set of objects and $\mathcal{X}_{[1]}$ as its set of morphisms; composition of morphisms is defined by the span

$$\mu\colon \mathcal{X}_{[1]} \times_{\mathcal{X}_{[0]}} \mathcal{X}_{[1]} \xleftarrow{\cong} \mathcal{X}_{[2]} \longrightarrow \mathcal{X}_{[1]}, \tag{0.4.1}$$

where the left pointing map is guaranteed to be a bijection by the first of the Segal conditions. It was Rezk's fundamental insight [Rez01] that one can model ∞-categories as simplicial *spaces* which satisfy the correct homotopy coherent analog of the Segal conditions, obtained by replacing bijections of sets by weak equivalences of spaces and fiber products by their homotopy coherent counterparts; the contractible homotopy fibers of the left pointing map in (0.4.1) parameterize the choices of composition.

Dyckerhoff and Kapranov [DK12] study the case where the first map in the span (0.4.1) is not an equivalence anymore. In this case one can still interpret μ as a "multi-valued composition law", where the space of possible results of a composition is parameterized by the possibly non-contractible or even empty fibers of the first map in the span (0.4.1). This multi-valued composition law is unital and associative (up to coherent homotopies) precisely if the simplicial object $\mathcal{X}$ satisfies the 2-*Segal conditions*, a weakening of Rezk's Segal conditions. 2-Segal spaces were also introduced independently by Gálvez-Carrillo, Kock and Tonks [GKT18a; GKT18b; GKT18c] under the name *decomposition spaces* for applications in combinatorics.

The main source of examples of 2-Segal spaces—apart from all ordinary Segal spaces—is *Waldhausen's* S-*construction* [Wal85], which assigns to a suitable (∞-)category $\mathcal{C}$ a 2-Segal simplicial space $\mathcal{S}(\mathcal{C})$ (see Example 1.3.3.7). While Waldhausen was originally interested in the *homotopical* meaning of the S-construction—the homotopy groups of $\mathcal{S}(\mathcal{C})$ compute the algebraic K-theory of $\mathcal{C}$—, it turns out that the S-construction also carries interesting *algebraic* information: under suitable finiteness assumptions, one can turn the simplicial space $\mathcal{S}(\mathcal{C})$ into the so called *Hall algebra* of $\mathcal{C}$ by an appropriate linearization procedure. In this context, the 2-Segal property enjoyed by $\mathcal{S}(\mathcal{C})$ can be seen to be directly responsible for the unitality and associativity of the multiplication in the Hall algebra. Variants of Hall algebras, such as the cohomological Hall algebra of Kontsevich and Soibelmann [KS11] or the derived Hall algebra of Toën [Toë06], can be obtained by considering variants of this construction; see [Dyc18] for a survey on this perspective. Dyckerhoff and Kapranov also recover classical convolution algebras such as the Iwahori and Hecke algebra as linearizations of certain 2-Segal spaces. Hall and Hecke algebras play an important role in representation theory, for instance due to their close connection to quantum groups.

When constructing (strictly) associative algebras out of 2-Segal spaces, one really only needs the 3-skeleton of these simplicial spaces and the corresponding truncated version of the 2-Segal conditions. It is thus natural to ask: What precisely is the higher algebraic structure encoded in a 2-Segal space? In Chapter 1 we establish the following theorem (see Corollary 1.3.4.2) which provides the first complete answer to this question.

Theorem 1. There is a canonical equivalence between

- the ∞-category of 2-Segal spaces and
- the ∞-category of *invertible* ∞*-operads*21). ◊

The theory of ∞-operads, originally introduced in the setting of algebraic topology by May [May72] and Boardman–Vogt [BV73] to study the algebraic structure of iterated loop spaces, has since become a fundamental organizational tool in the study of higher algebraic

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21) colored, non-symmetric

structures. Roughly speaking, an operad is a generalized category which admits not just morphisms $x \to y$ from one object to another, but also “many-to-one” morphisms $(x_1, \dots, x_n) \to y$, called *operations*, together with suitably associative composition laws (see Definition 1.1.1.1).

An operad is called **invertible** (see Definition 1.3.3.1) if each operation can uniquely be decomposed into other operations, as long as the shape of this decomposition is specified in advance; more precisely, we require that each 1-ary operation is the identity and that, after fixing $0 \leq i \leq j \leq n$, each n-ary operation $(x_1, \dots, x_n) \to z$ can be written uniquely as a composition of two operations $(x_{i+1}, \dots, x_j) \to y$ and $(x_1, \dots, x_i, y, x_{j+1}, \dots, x_n) \to z$. A trivial example of an invertible operad is the commutative operad which has a unique operation of each arity. More interestingly, there is, for each abelian category A, an invertible operad $\mathrm{S}(A)$—corresponding to the aforementioned Waldhausen S-construction under the equivalence of Theorem 1—whose colors and 1-ary operations are the objects of A and whose 2-ary operations are short exact sequences (see Example 1.3.3.7).

The passage from operads to ∞-operads is analogous to the passage from categories to ∞-categories and arises by replacing strict composition of operations by composition laws which are only well-defined and associative up to a coherent system of higher homotopies. To study ∞-operads we use the convenient framework of dendroidal spaces introduced by Moerdijk and Weiss [MW07] and later developed further by Cisinski and Moerdijk [CM11; CM13]. In this framework the simplex category Δ is replaced by a bigger category Ω_π of plane rooted trees whose definition we recall in Section 1.1.1. Generalizing Rezk's ideas from the simplicial case, Cisinski and Moerdijk observe that operads are identified via a dendroidal version of the nerve functor with *dendroidal sets* $\Omega_\pi^{\mathrm{op}} \to \mathbf{Set}$ satisfying the dendroidal analog of the Segal conditions (see Definition 1.3.1.1). More generally, they show that ∞-operads are modeled by (complete[22)]) Segal dendroidal spaces.

The equivalence in Theorem 1 is constructed by pulling back along an explicit functor

$$\mathcal{L}_\pi \colon \Omega_\pi \longrightarrow \Delta$$

(see Section 1.1.2) of ordinary categories, which we prove to be an ∞-categorical localization in the following sense: There is an explicit class S of maps in Ω_π which are sent by $\mathcal{L}_\pi$ to equivalences in Δ and, moreover, $\mathcal{L}_\pi$ is universal with this property among all functors of ∞-categories. More precisely, we have the following result (see Theorem 1.2.0.1).

Theorem 2. Let $\mathcal{C}$ be an ∞-category. The functor

$$\mathcal{L}_\pi^\star \colon \mathrm{Fun}(\Delta, \mathcal{C}) \longrightarrow \mathrm{Fun}(\Omega_\pi, \mathcal{C})$$

induced by $\mathcal{L}_\pi$ is fully faithful; the essential image is spanned by those functors $\Omega_\pi \to \mathcal{C}$ which send all maps in S to equivalences in $\mathcal{C}$. ◊

Theorem 1 follows from Theorem 2 (after passing to opposite categories) by observing that $\mathcal{L}_\pi^\star$ identifies 2-Segal simplicial objects in its domain with (complete) Segal dendroidal objects in its essential image.

It is often worthwhile to enhance simplicial objects with “additional symmetries”. We consider the following two main examples:

(1) Segal's *special* Γ*-spaces* [Seg74]—used to model the homotopy theory of connective spectra—can be seen as Segal simplicial spaces $\mathcal{X}$ enhanced by compatible actions

$$\mathfrak{S}_n \curvearrowright \mathcal{X}_n$$

of the symmetric groups. In terms of the algebraic structures described in Section 0.2, these additional symmetries account for the difference between monoids and *abelian* monoids.

[22)] Completeness is an additional technical condition which will be vacuous in the cases we consider.

(2) Cyclic symmetries on $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ are encoded by lifts of $\mathcal{X}$ to *Connes' cyclic category* $\Lambda \supset \Delta$ and described informally by a compatible system of actions

$$\mathrm{C}_{n+1} \curvearrowright \mathcal{X}_n$$

by cyclic groups. 2-Segal cyclic objects play a central role in Dyckerhoff–Kapanov's construction [DK18] of topological Fukaya categories of surfaces.

One important feature of our proof of Theorem 1 is that it can be generalized to clarify how cyclic (resp. symmetric) enhancements of 2-Segal spaces correspond precisely to cyclic (resp. symmetric) structures on the corresponding invertible ∞-operads. To do this we consider two variants of the category Ω_π of plane rooted trees:

(1) The category Ω_{sym} is precisely the category Ω of Moerdijk and Weiss. The objects of Ω_{sym} are rooted trees (without a chosen plane embedding); by the work of Cisinski and Moerdijk [CM13], (complete) Segal presheaves on Ω_{sym} are known to model *symmetric* ∞-operads.

(2) By slightly modifying a construction of Joyal and Kock [JK09], we introduce the category Ω_{cyc} of plane rootable trees (see Section 1.4); it is expected[23] that (complete) Segal presheaves on Ω_{cyc} are a model for *cyclic* ∞-operads.

These categories of trees come equipped with canonical functors

$$\mathcal{L}_{\mathrm{sym}}\colon \Omega_{\mathrm{sym}} \longrightarrow \Gamma \qquad \text{and} \qquad \mathcal{L}_{\mathrm{cyc}}\colon \Omega_{\mathrm{cyc}} \longrightarrow \Lambda$$

(see Section 1.1.3 and Section 1.4). Our methods directly generalize to obtain the following version of Theorem 2 and Theorem 1 (see Theorem 1.1.3.2, Theorem 1.4.0.18 and Remark 1.3.4.13).

Theorem 3. The functors $\mathcal{L}_{\mathrm{sym}}$ and $\mathcal{L}_{\mathrm{cyc}}$ are ∞-categorical localizations. Moreover the functor $\mathcal{L}_{\mathrm{sym}}$ induces an equivalence of ∞-categories between:

- 2-Segal Γ-spaces and
- invertible symmetric ∞-operads. ◊

Since the localization functor $\mathcal{L}_{\mathrm{cyc}}$ identifies 2-Segal cyclic objects with invertible Segal dendroidal objects, Theorem 3 also implies the following conjecture if we assume the conjectural existence of a complete Segal cyclic dendroidal model for cyclic ∞-operads (see Remark 1.4.0.20).

Conjecture 1. The functor $\mathcal{L}_{\mathrm{cyc}}$ induces an equivalence between 2-Segal cyclic spaces and invertible cyclic ∞-operads. ◊

Remark 0.4.0.1. The functor $\mathcal{L}_{\mathrm{sym}}\colon \Omega_{\mathrm{sym}} \to \Gamma$ was already considered by Boavida de Brito and Moerdijk [BM17, Theorem 1.1]; their main theorem states that this functor induces an equivalence between the ∞-category of special Γ-spaces and the ∞-category of what they call *covariantly fibrant* complete Segal dendroidal spaces. We obtain their equivalence—as well as the obvious variants for Λ and Δ—by restricting our equivalences to the appropriate full subcategories (see Corollary 1.3.2.2). ◊

Remark 0.4.0.2. Throughout Chapter 1 we write "2-Segal" to denote what Dyckerhoff and Kapranov originally called "unital 2-Segal". This is justified by the recent observation of Feller, Garner, Kock, Proulx and Weber [FGK+19] that unitality follows automatically from the 2-Segal conditions. In Section 2.5 we generalize this result to higher Segal spaces of all dimensions. ◊

Remark 0.4.0.3. Theorem 2 makes it possible to construct homotopy-coherent simplicial objects by specifying (possibly strict) dendroidal objects which send certain maps to weak equivalences. While this is easier *a priori*, the author does not know of any new simplicial objects that arise this way. When it comes to 2-Segal spaces, one should probably not expect new examples to

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[23] For instance, see [DH18, Remark 6.9] for a precise conjecture.

arise from our result: first, because most operads appearing in the literature are not invertible and second, because every 2-Segal space can already be constructed by a generalized version of Waldhausen's S-construction [BOO+18]. Therefore, the results of Chapter 1 should not be seen as a way to construct new 2-Segal spaces but rather as a new way of repackaging the higher algebraic structure encoded in such an object. This operadic perspective makes available tools and generalizations that were not evident in the original theory: While it is, for instance, not immediately obvious how to define 2-Segal objects with values in a general (not necessarily Cartesian) symmetric monoidal (∞-)category, the definition of invertible (∞-)operads directly generalizes to this setting; moreover, one can now hope to obtain new information about a 2-Segal space by studying algebras over the associated ∞-operad. ♢

Remark 0.4.0.4. Recently, a different algebraic interpretation of 2-Segal spaces was given by Stern [Ste19], who identified the ∞-category of 2-Segal objects in $\mathcal{C}$ with an ∞-category of algebras in correspondences in $\mathcal{C}$. Similarly, Stern shows that 2-Segal cyclic objects are identified with Calabi-Yau algebras in correspondences. ♢

0.5 Higher Segal spaces via higher excision (Chapter 2)

The starting point of Chapter 2 is the easy but little-known observation that Rezk's Segal objects can be characterized by a condition which is purely categorical, in the sense that it can be defined without having to know anything about the inner workings of Δ.

Observation. A simplicial object $\Delta^{\mathrm{op}} \to \mathcal{C}$ is Segal if and only if it sends biCartesian squares in Δ to Cartesian squares in $\mathcal{C}$. ♢

In 2012, Dyckerhoff and Kapranov generalized Rezk's Segal condition and introduced what they call *higher Segal spaces*[24]. Their definition is very geometric in nature: They consider the so called *cyclic polytopes* $\mathrm{C}(n,d)$, defined as the convex hull of $n+1$ points on the d-dimensional moment curve $t \mapsto (t, t^2, \dots, t^d)$. The main feature of these polytopes in this context is that they have two canonical triangulations, called the *lower triangulation* and the *upper triangulation*, respectively. Each of these triangulations defines a simplicial subcomplex $\mathcal{T}$ of the standard n-simplex Δ^n; Dyckerhoff and Kapranov then impose conditions on simplicial objects by requiring that the value[25] on the inclusion $\mathcal{T} \hookrightarrow \Delta^n$ is an equivalence: a simplicial object is called **lower** (resp. **upper**) **d-Segal** if this is true for the lower (resp. upper) triangulation of $\mathrm{C}(n,d)$ and **d-Segal** if this is true for all triangulations of $\mathrm{C}(n,d)$.

The purpose of Chapter 2 is to characterize the various flavors of higher Segal conditions in terms of purely categorical notions of higher excision. We first do this for lower $(2k-1)$-Segal spaces, since they are the most fundamental[26] amongst all versions of higher Segal spaces. The following is the first main result of this chapter:

Theorem 4 (Theorem 2.6.2.2)**.** Let $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object in an ∞-category $\mathcal{C}$ with finite limits. The following are equivalent:

(1) the simplicial object $\mathcal{X}$ is lower $(2k-1)$-Segal;

(2) the functor $\mathcal{X}$ sends every strongly biCartesian[27] $(k+1)$-dimensional cube in Δ to a limit diagram in $\mathcal{C}$. ♢

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[24] not to be confused with Barwick's n-fold Segal spaces [Bar05]

[25] Every simplicial object can be canonically evaluated on simplicial sets by Kan extension along the Yoneda embedding; see Section 2.4.1.

[26] This vague assertion is made precise by the path space criterion [Pog17, Proposition 2.7] which expresses all higher Segal conditions in terms of lower $(2k-1)$-Segal conditions.

[27] A cube is strongly biCartesian if each of its 2-dimensional faces is biCartesian; see Definition 2.2.3.4.

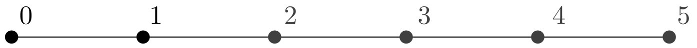

Figure 2: The lower triangulation of the cyclic polytope $\mathrm{C}(n,1)$, here depicted with $n=5$.

We call a functor $\mathcal{Z}^{\mathrm{op}} \to \mathcal{C}$ satisfying condition (2) of Theorem 4 **weakly k-excisive**; compare this with Goodwillie's calculus of functors [Goo92], where a (covariant) functor $\mathcal{Z} \to \mathcal{C}$ is called k-excisive if it sends strongly *co*Cartesian $(k+1)$-dimensional cubes in $\mathcal{Z}$ to limit diagrams in $\mathcal{C}$.

We illustrate Theorem 4 with some examples.

- The cyclic polytope $\mathrm{C}(n,1)$ is just the interval $\Delta^{\{0,n\}}$; its lower triangulation (see Figure 2) yields the simplicial complex

  $$\mathrm{Sp}[n] \coloneqq \Delta^{\{0,1\}} \cup \cdots \cup \Delta^{\{n-1,n\}} \subset \Delta^n.$$

  Rezk's Segal condition for a simplicial object says precisely that the inclusion $\mathrm{Sp}[n] \hookrightarrow \Delta^n$ needs to be sent to an equivalence; this is what Dyckerhoff and Kapranov call the lower 1-Segal condition. For $n=1$, this condition says precisely that the biCartesian square

  $$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 12 \arrow[d] \\ 01 \arrow[r] & 012 \end{tikzcd} \tag{0.5.1}$$

  in Δ needs to be sent to a limit diagram. More generally, every square of the form

  $$\begin{tikzcd} \{i\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \{i,\dots,n\} \arrow[d] \\ \{0,\dots,i\} \arrow[r] & \{0,\dots,n\} \end{tikzcd}$$

  (for $0<i<n$) is biCartesian in Δ; it is in fact an often used characterization of Segal objects to require these squares to be sent to pullbacks.
- The cyclic polytope $\mathrm{C}(4,3)$ is a double triangular pyramid; its lower triangulation (see Figure 3) induces the simplicial complex

  $$\mathcal{T} = \Delta^{\{1,2,3,4\}} \cup \Delta^{\{0,1,3,4\}} \cup \Delta^{\{0,1,2,3\}} \subset \Delta^4.$$

  By definition, a simplicial object satisfies the first lower 3-Segal condition if it sends the canonical inclusion $\mathcal{T} \hookrightarrow \Delta^4$ to an equivalence; this is equivalent to sending the cube

  $$\begin{tikzcd} 13 \arrow[rr] \arrow[dr] \arrow[dd] & & 134 \arrow[dr] \arrow[dd] & \\ & 123 \arrow[rr] \arrow[dd] & & 1234 \arrow[dd] \\ 013 \arrow[rr] \arrow[dr] & & 0134 \arrow[dr] & \\ & 0123 \arrow[rr] & & 01234 \end{tikzcd}$$

  which is strongly biCartesian in Δ, to a limit diagram.

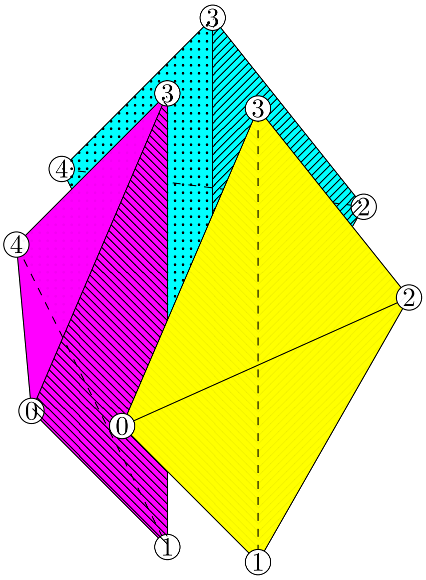

Figure 3: The three 3-simplices $\Delta^{\{1234\}}$, $\Delta^{\{0134\}}$ and $\Delta^{\{0123\}}$ (depicted in cyan, magenta and yellow, respectively) assemble into the lower triangulation of the double triangular pyramid $\mathrm{C}(4,3)$.

In general, the first non-trivial lower $(2k-1)$-Segal condition (*i.e.*, the one for $n = 2k$) can always be expressed in terms of a strongly biCartesian cube in Δ of dimension $k+1$ and this cube is the unique such cube which is in a certain sense “basic”. However, for bigger n both the number of simplices in the lower triangulation of $\mathrm{C}(n, 2k-1)$ and the number of basic strongly biCartesian cubes grows very rapidly so that, *a priori*, the behavior of weakly k-excisive simplicial objects and lower $(2k-1)$-Segal objects diverges dramatically.

In the work of Dyckerhoff and Kapranov [DK18] on topological Fukaya categories and in the work of Stern [Ste19] related to Calabi-Yau algebras and 2-dimensional quantum field theories, a special role is played by *cyclic* 2-Segal spaces. The next results show that this is no coincidence and that the 2-Segal conditions—and more generally the $2k$-Segal conditions—are most naturally expressed in terms of higher weak excision relative to Connes’ cyclic category Λ.

Theorem 5 (Theorem 2.6.2.2)**.** Let $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object in an ∞-category $\mathcal{C}$ with finite limits. The following are equivalent:

(1) the simplicial object $\mathcal{X}$ is $2k$-Segal;

(2) the functor $\mathcal{X}$ sends to Cartesian cubes in $\mathcal{C}$ those $(k+1)$-dimensional cubes in Δ which become strongly biCartesian in Λ (under the canonical functor $\Delta \to \Lambda$). ◊

Corollary 1 (Corollary 2.6.2.3)**.** Let $\mathcal{C}$ be an ∞-category with finite limits. The cyclic $2k$-Segal objects in $\mathcal{C}$ are precisely the weakly k-excisive functors $\Lambda^{\mathrm{op}} \to \mathcal{C}$. ◊

We again illustrate the theorem with some examples:

- The square (0.5.1) encoding the first Segal condition is typically not sent to a Cartesian square by 2-Segal objects. This is explained by Theorem 5: while the square (0.5.1) is biCartesian in Δ, it is no longer a pushout square in Λ.
- The 2-dimensional cyclic polytope $\mathrm{C}(4,2)$ is a square. It has the two triangulations (see Figure 4) whose corresponding Segal condition expresses that the two squares

$$\begin{tikzcd} 13 \arrow[r] \arrow[d] & 123 \arrow[d] \\ 013 \arrow[r] & 0123 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 02 \arrow[r] \arrow[d] & 012 \arrow[d] \\ 023 \arrow[r] & 0123 \end{tikzcd} \tag{0.5.2}$$

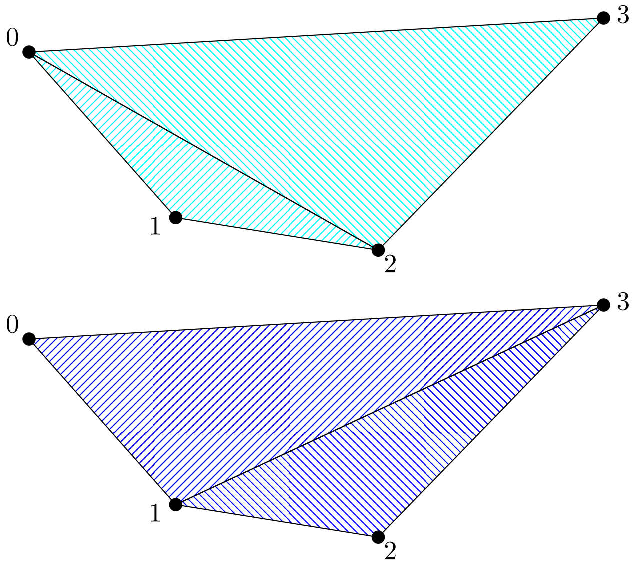

Figure 4: The lower and the upper triangulations of the cyclic polytope $\mathrm{C}(4,2)$.

in Δ are sent to a limit diagram. Both of the squares (0.5.2) are biCartesian in Λ.

- The squares

$$\begin{tikzcd} 11' \arrow[r, "\mathrm{d}^0"] \arrow[d, "\mathrm{s}^0"'] & 011' \arrow[d, "\mathrm{s}^1"] \\ 1 \arrow[r, "\mathrm{d}^0"'] & 01 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 0'0 \arrow[r, "\mathrm{d}^2"] \arrow[d, "\mathrm{s}^0"'] & 0'01 \arrow[d, "\mathrm{s}^0"] \\ 0 \arrow[r, "\mathrm{d}^1"'] & 01 \end{tikzcd}$$

  are biCartesian both in Δ and in Λ. Hence they need to be sent to pullback squares by every Segal object (by Theorem 4) and by every 2-Segal object (by Theorem 5). While the first of these facts is easy, the second is non-trivial; it is precisely the statement that 2-Segal spaces are automatically unital, which was discovered only very recently by Feller, Garner, Kock, Proulx and Weber [FGK+19].

Finally, we remark that our main theorem implies a non-trivial bound (Proposition 2.6.3.1) on how many values of a higher Segal object can be trivial without the whole object collapsing. Whether this bound is sharp is still unknown (at least to the author) and remains to be investigated in future research.

0.6 Homotopy coherent theorems of Dold–Kan type (Chapter 3)

The classical Dold–Kan correspondence [Dol58; Kan58b] is a remarkable equivalence of categories

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, A) \overset{\simeq}{\longleftrightarrow} \mathrm{Ch}_{\geq 0}(A) \tag{0.6.1}$$

between simplicial objects in A and connective chain complexes in A, where A is the category of abelian groups or, more generally, any abelian category [DP61]. In the past decades, many related equivalences have been constructed [Pir00; Sło04; Sło11; Hel14; CEF15; LS15] where the simplex category Δ is replaced by other categories which are of similar “combinatorial nature”.

The first goal of Chapter 3 is to simultaneously generalize these equivalences in the homotopy coherent context of ∞-categories. To this end we study categories B equipped with the structure $\mathbb{B} = (B, E, E^\vee)$ of a so-called *DK-triple* (see Definition 3.2.1.1); to each such DK-triple $\mathbb{B}$ we associate a pointed category $N_0 = N_0(\mathbb{B})$[28] and prove the following homotopy coherent correspondence of Dold–Kan type:

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[28] unrelated to the nerve $\mathrm{N}(C)$ of a category C despite the typographic similarity

Theorem 6 (Corollary 3.2.3.4)**.** For each weakly idempotent complete[29] additive[30] ∞-category $\mathcal{A}$, the DK-triple $\mathbb{B}$ induces a natural[31] equivalence

$$\operatorname{Fun}(B, \mathcal{A}) \xleftrightarrow{\simeq} \operatorname{Fun}^0(N_0, \mathcal{A}) \tag{0.6.2}$$

between the ∞-categories of diagrams $B \to \mathcal{A}$ and of pointed diagrams $N_0 \to \mathcal{A}$. ♢

Before going into more details about DK-triples, we explain how Theorem 6 subsumes and generalizes previous results in the literature.

(1) In the case where A is an abelian category, we recover the classical Dold–Kan correspondence (0.6.1) by applying Theorem 6 to $\mathcal{A} = A^{\mathrm{op}}$ and to a suitable DK-triple $\mathbb{B}_\Delta = (\Delta, E_\Delta, E_\Delta^\vee)$ whose associated pointed category $N_0(\mathbb{B}_\Delta) = \mathrm{Ch}_{\geq 0}$ is the shape of connective chain complexes; see Section 3.3.1 for more details.

(2) More generally, Theorem 6 specializes to the ∞-categorical Dold–Kan correspondence originally sketched by Joyal [Joy08, Section 35]

$$\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \xleftrightarrow{\simeq} \mathrm{Ch}_{\geq 0}(\mathcal{A}) \tag{0.6.3}$$

between simplicial objects and coherent connective chain complexes in any weakly idempotent complete additive ∞-category $\mathcal{A}$.

(3) Denote by $\mathbf{Fin}_\star$ the category of finite pointed sets and by **Surj** the category of (possibly empty) finite sets and surjections between them. Pirashvili[Pir00] constructed an equivalence

$$\operatorname{Fun}(\mathbf{Fin}_\star, \mathbf{Ab}) \xleftrightarrow{\simeq} \operatorname{Fun}(\mathbf{Surj}, \mathbf{Ab}) \tag{0.6.4}$$

between $\mathbf{Fin}_\star$-shaped and **Surj**-shaped diagrams[32] of abelian groups. We recover this equivalence from Theorem 6 which more generally yields a natural equivalence

$$\operatorname{Fun}(\mathbf{Fin}_\star, \mathcal{A}) \xleftrightarrow{\simeq} \operatorname{Fun}(\mathbf{Surj}, \mathcal{A}), \tag{0.6.5}$$

between Γ-objects and **Surj**-shaped diagrams in any weakly idempotent complete preadditive[33] ∞-category $\mathcal{A}$; see Section 3.3.2 for more details.

(4) Denote by FI$\sharp$ the category of finite sets and partial injections; let $\mathbf{Fin}^\simeq$ be the groupoid of finite sets and bijections. For each commutative ground ring R, [CEF15, Theorem 4.1.5] (which is a special case of [Sło04, Theorem 1.5]) describes an equivalence

$$\operatorname{Fun}(\mathrm{FI}\sharp, \mathbf{Mod}{-}R) \xleftrightarrow{\simeq} \operatorname{Fun}(\mathbf{Fin}^\simeq, \mathbf{Mod}{-}R) \simeq \prod_{n \in \mathbb{N}} (\mathfrak{S}_n{-}\mathbf{Rep}_R) \tag{0.6.6}$$

between the categories of FI$\sharp$-modules and of tuples of representations of all symmetric groups $\mathfrak{S}_n$. Again, our main result generalizes this equivalence to coherent diagrams/representations with values in arbitrary weakly idempotent complete preadditive ∞-categories.

(5) When $\mathcal{A}$ is an idempotent complete additive *ordinary* category, Theorem 6 recovers the general Dold–Kan type equivalence of Lack and Street [LS15, Theorem 6.8] which includes as special cases (0.6.1), (0.6.4), (0.6.6) and many more. See Section 3.5.1 for a detailed comparison.

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[29] weakly idempotent complete = closed under direct complements

[30] additive = has direct sums and is enriched in abelian groups

[31] natural in $\mathcal{A}$ with respect to additive functors

[32] To be precise, Pirashvili only considers diagrams whose value on $\star \in \mathbf{Fin}_\star$ and on $\varnothing \in \mathbf{Surj}$ is zero; these diagrams correspond to each other under the equivalence (0.6.4)

[33] preadditive = has direct sums

(6) Some of the equivalences of Theorem 6—including the one for $\Gamma = \mathbf{Fin}_\star^{\mathrm{op}}$ but not the one for Δ—were already established by Helmstutler [Hel14] in the language of model categories; see Remark 3.3.2.3 for more details. Note that unlike Theorem 6, Helmstutler's result cannot be dualized so easily to yield, for instance, a model categorical version of the equivalence (0.6.5).

(7) In a *stable* ∞-category $\mathcal{D}$, coherent connective chain complexes can be encoded more conveniently as filtered objects, *i.e.*, as diagrams $\mathbb{N} \to \mathcal{D}$; an explicit equivalence

$$\mathrm{Fun}(\mathbb{N}, \mathcal{D}) \simeq \mathrm{Ch}_{\geq 0}(\mathcal{D}) \tag{0.6.7}$$

is part of Stefano Ariotta's Ph.D. thesis [Ari]. In this stable context, Lurie proved an ∞-categorical Dold–Kan correspondence [Lur17, Theorem 1.2.4.1] in the form of an equivalence

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}(\mathbb{N}, \mathcal{D}), \tag{0.6.8}$$

of ∞-categories; we expect this equivalence to agree with (0.6.1) under the identification (0.6.7). Note that while both equivalences (0.6.7) and (0.6.8) need the stability of $\mathcal{D}$ to work, Theorem 6—just like the ordinary Dold–Kan correspondence—only needs that $\mathcal{A}$ is weakly idempotent complete additive. See Section 3.5.2 for a more detailled discussion.

We now introduce the notion of a DK-triple $\mathbb{B} = (B, E, E^\vee)$ on which Theorem 6 is based. It consists of a three-fold factorization system of type

$$\bullet \xrightarrow{E} \bullet \longrightarrow \bullet \xrightarrow{E^\vee} \bullet,$$

where the unnamed middle piece together with suitably encoded zero relations gives rise to the pointed category $N_0(\mathbb{B})$ appearing on the right side of the equivalence (0.6.2). This notion is inspired by similar concepts appearing in [Sło04; Hel14; LS15]. We give an illustration in the examples of Γ and Δ, which are discussed in greater detail in Section 3.3.2 and Section 3.3.1.

- Every map $f\colon I \leftarrow J$ in $\Gamma = \mathbf{Fin}_\star^{\mathrm{op}}$ can be written uniquely as the composition

  $$I \longleftarrow \operatorname{Im} f \longleftarrow \frac{J}{\operatorname{Ker} f} \longleftarrow J, \tag{0.6.9}$$

  where

  - the leftmost map is a bijection onto its image,
  - the middle map is surjective and sends only the basepoint to the basepoint; in other words it just amounts to a surjection between the (possibly empty) sets obtained by omitting the basepoints,
  - the rightmost map is bijective outside of its kernel (such maps are often called *inert*).

  The category of those arrows which appear as the middle piece of (0.6.9) is precisely (the opposite of) the category **Surj**; there are no zero relations in this case.

- Every arrow in Δ can be written uniquely as the composition

  $$\bullet \xrightarrow{\mathrm{s}^{\geq 0}} \bullet \xrightarrow{(\mathrm{d}^0)} \bullet \xrightarrow{\mathrm{d}^{>0}} \bullet$$

  where

  - the left arrow $\mathrm{s}^{\geq 0}$ is a (possibly empty) composition of codegeneracy maps,
  - the middle arrow is either the identity or a 0-th coface map,
  - the right arrow is a (possibly empty) composition of i-th coface maps d^i for $i > 0$.

  If one focuses only on the arrows of the second type, one obtains a category $\mathrm{Ch}_{\geq 0}$

  $$\begin{tikzcd} \underline{0} \arrow[r, "\mathrm{d}^0", ""{name=A, below}] & \underline{1} \arrow[r, "\mathrm{d}^0", ""{name=B, below}] & \underline{2} \arrow[r, "\mathrm{d}^0", ""{name=C, below}] & \cdots \arrow[from=A, to=B, dotted, bend right, no head] \arrow[from=B, to=C, dotted, bend right, no head] \end{tikzcd}$$

with zero relations by declaring the composite of two 0-th coface maps to vanish (because it is not again a 0-face map). Connective chain complexes are then exactly zero-preserving presheaves on $\mathrm{Ch}_{\geq 0}$. In order to properly encode the coherent zero relations in the ∞-categorical context, we actually consider $\mathrm{Ch}_{\geq 0}$ as a pointed category by adding an additional zero object through which all zero morphisms factor; for a more detailed explanation of this issue, see Section 3.1.2.

0.6.1 Applications: additive higher Segal objects and stable Goodwillie calculus

The Dold–Kan type theorems are a very useful tool in the study of higher Segal objects in the additive context. For example, we can use the Dold–Kan correspondence to compute a large class of *membrane spaces* as introduced in Section 2.5 (see Proposition 3.7.1.3). It is then easy to deduce, for instance, the following characterization of higher Segal objects.

Theorem 7 (Proposition 3.7.1.9 and Proposition 3.6.1.6)**.** Let $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{A}$ be a simplicial object in a weakly idempotent complete additive ∞-category and denote by $\overline{\mathcal{X}} \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ the coherent chain complex corresponding to $\mathcal{X}$ under the ∞-categorical Dold–Kan correspondence (0.6.3). The following are equivalent:

- the simplicial object $\mathcal{X}$ is weakly k-excisive (*i.e.*, lower $(2k-1)$-Segal);
- the simplicial object $\mathcal{X}$ is lower weakly k-Λ-excisive (*i.e.*, lower $2k$-Segal);
- the simplicial object $\mathcal{X}$ is upper weakly k-Λ-excisive (*i.e.*, upper $2k$-Segal);
- the functor $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{A}$ is a left Kan extension of its restriction to $\Delta^{\mathrm{op}}_{\leq k}$;
- the chain complex $\overline{\mathcal{X}}$ is k-truncated, *i.e.*, $\overline{\mathcal{X}}_n \simeq 0$ for all $n > k$. ◊

Remark 0.6.1.1. Many of these ideas were already present in [DJW19, Section 4] where the case of lower/upper $2k$-Segal objects is covered in abelian and stable (∞-)categories using the classical Dold–Kan correspondence (0.6.1) and Lurie's stable Dold–Kan correspondence (0.6.8), respectively. ◊

Another application is the following result which in the stable context identifies higher Segal Γ-spaces with higher excisive functors in the sense of Goodwillie [Goo92].

Theorem 0.6.1.2 (Theorem 3.7.3.1)**.** Let $\mathcal{D}$ be a presentable stable ∞-category and fix $k \in \mathbb{N}$. Restriction along $\Gamma^{\mathrm{op}} = \mathbf{Fin}_\star \hookrightarrow \mathcal{S}_\star$ induces an equivalence between the ∞-categories of

- Γ-objects $\Gamma^{\mathrm{op}} \to \mathcal{D}$ whose underlying simplicial object is weakly k-excisive and
- k-excisive functors $\mathcal{S}_\star \to \mathcal{D}$ which preserve filtered colimits □

In a similar spirit we can use the ∞-categorical version (0.6.5) of Pirashvili's equivalence to better describe how (filtered-colimit-preserving) k-excisive functors $\mathcal{S}_\star \to \mathcal{D}$ in some presentable stable ∞-category $\mathcal{D}$ can be assembled from representations (in $\mathcal{D}$) of the various symmetric groups $\mathfrak{S}_n$ for $n \leq k$; see Section 3.7.2.

0.7 Basic tools of ∞-categories

Most statements in this thesis are intrinsically homotopy coherent and are most naturally formulated and proven by employing a language of $(\infty, 1)$-categories which extends that of ordinary category theory. Since homotopy coherent thinking requires one to keep in mind infinite hierarchies of higher structures at once, it is not obvious that humans are even capable of it. It is the author's hope that one day the foundations of mathematics will have reached a state where it is possible for mathematicians to grow up as native speakers of such a homotopy theoretic language, just like today many of us take for granted the basic language of category theory.

Until that day, however, we have to rely on ever more sophisticated crutches and tools to make our intuitions precise and to make sure that our proofs stay correct.

In this thesis we employ the framework of quasi-categories which are simplicial sets satisfying the *restricted Kan condition* of Boardman and Vogt [BV73]; their homotopy theory is described by Joyal's model structure[34) on simplicial sets. The theory of quasi-categories was developed extensively by Joyal [Joy02; Joy08] and Lurie [Lur09; Lur17]. Our main references are Lurie's books—whose notation and terminology we mostly adopt—as well as Cisinski's [Cis19]. For a non-technical introduction, see Groth's survey [Gro10]. The reason that the theory of quasi-categories—which, following Lurie, we just call ∞-categories—is so useful for us is that it allows many basic ∞-categorical arguments to be performed on a very high level while the delicate combinatorics of simplicial sets remains in the background and makes the machine work. In particular, most statements and constructions from category theory have an analog for quasi-categories.

For the convenience of the reader we collect here some of the basic notation, conventions and results which throughout this thesis are used without further mention. Readers comfortable with the language of category theory are encouraged to skip this section and proceed directly to Chapter 1.

0.7.1 The basic language

We view the theory of ∞-categories as an extension of ordinary category theory by identifying an ordinary category C with its nerve $\mathrm{N}(C)$. Typically, we use ordinary capital letters (*e.g.*, C, Z, P, A) for 1-categories and the corresponding Euler Script letters (*e.g.*, $\mathcal{C}, \mathcal{Z}, \mathcal{P}, \mathcal{A}$) for ∞-categories.

- We denote by $\mathcal{S}$ the ∞-category of spaces/∞-groupoids [Lur09, Section 1.2.16]; it contains the category **Set** of sets as a full subcategory.
- The **opposite** of an ∞-category $\mathcal{C}$ is denoted $\mathcal{C}^{\mathrm{op}}$.
- Each ∞-category $\mathcal{C}$ is equipped with a functor $\mathrm{Map}_{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ which assigns to each pair of objects the homotopy type of arrows (called **mapping space**) between them [Cis19, Section 5.8]. The mapping space functor takes values in **Set** if and only if $\mathcal{C}$ is (equivalent to) an ordinary category; in this case $\mathrm{Map}(-,-)$ is just the ordinary Hom-functor.
- We write $\mathrm{Fun}(\mathcal{Z}, \mathcal{C})$ for the ∞-category of functors $\mathcal{Z} \to \mathcal{C}$ [Lur09, Section 1.2.7]. The arrows in $\mathrm{Fun}(\mathcal{Z}, \mathcal{C})$ are called natural transformations. Given ∞-categories $\mathcal{Z}$,$\mathcal{Z}'$ and $\mathcal{C}$, there is a canonical "currying" equivalence $\mathrm{Fun}(\mathcal{Z}' \times \mathcal{Z}, \mathcal{C}) \simeq \mathrm{Fun}(\mathcal{Z}', \mathrm{Fun}(\mathcal{Z}, \mathcal{C}))$.
- A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is called **fully faithful** if it induces, for each $c, d \in \mathcal{C}$, an equivalence $\mathrm{Map}_{\mathcal{C}}(c, d) \simeq \mathrm{Map}_{\mathcal{C}'}(Fc, Fd)$ on mapping spaces. The **essential image** of F consists of the objects which are up to equivalence of the form $F(c)$ for $c \in \mathcal{C}$.

  Given an ∞-category $\mathcal{C}$, we can talk about the **full subcategory**[35) of $\mathcal{C}$ spanned by some collection of objects; its mapping spaces are inherited from $\mathcal{C}$ and the inclusion is—by definition—fully faithful.
- A functor $\mathcal{Z}^{\mathrm{op}} \to \mathcal{C}$ is sometimes called a $\mathcal{C}$-valued **presheaf** on $\mathcal{Z}$ or just presheaf in the case $\mathcal{C} = \mathcal{S}$. Each small category $\mathcal{Z}$ admits the fully faithful **Yoneda embedding** $\mathcal{Z} \hookrightarrow \mathcal{P}(\mathcal{Z}^{\mathrm{op}}) := \mathrm{Fun}(\mathcal{Z}^{\mathrm{op}}, \mathcal{S})$ which shares most formal properties of its 1-categorical counterpart [Lur09, Section 5.1] .
- An **adjunction** [Cis19, Section 6.1][Lur09, Section 5.2] between ∞-categories $\mathcal{C}'$ and $\mathcal{C}$ consists of two functors $L\colon \mathcal{C} \rightleftarrows \mathcal{C}' :R$ together with an equivalence $\mathrm{Map}(L(-),-) \simeq$

[34) Unfortunately, Joyal's original paper containing its construction is not publicly available; instead, see for instance his lecture notes [Joy].

[35) Here "subcategory" is synonymous with "sub-∞-category"; we use the former out of convenience.

Map$(-, R(-))$ of functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}' \to \mathcal{S}$. Each adjunction has a unit $\eta\colon \mathrm{Id}_{\mathcal{C}} \to RL$ and a counit $\varepsilon\colon LR \to \mathrm{Id}_{\mathcal{C}'}$.

- The inclusion $\mathbf{Set} \hookrightarrow \mathcal{S}$ has a left adjoint $\pi_0\colon \mathcal{S} \to \mathbf{Set}$ which sends a space to its set of path components (or, in terms of homotopy types, identifies equal objects).[36]
- Given an ∞-category $\mathcal{Z}$ and an object $x \in \mathcal{Z}$ one can form the **under-category** $\mathcal{Z}_{x/}$ and the **over-category** $\mathcal{Z}_{/x}$ [Lur09, Section 1.2.9]. In this thesis we only really need it when $\mathcal{Z} = Z$ is an ordinary category, in which case its objects are given by arrows $x \to \bullet$ (resp. $\bullet \to x$) in Z and its arrows are given by commutative triangles under (resp. over) x. If we are also given a functor $f\colon \mathcal{Z}' \to \mathcal{Z}$ then we abuse notation and write ${\mathcal{Z}'}_{x/} \coloneqq \mathcal{Z}_{x/} \times_{\mathcal{Z}} \mathcal{Z}'$ and ${\mathcal{Z}'}_{/x} \coloneqq \mathcal{Z}' \times_{\mathcal{Z}} {\mathcal{Z}'}_{/x}$ where $\times_{\mathcal{Z}}$ denotes the fiber product of ∞-categories with the functor f implicit. When $\mathcal{Z}' = Z'$ and $\mathcal{Z} = Z$ are ordinary categories, the categories ${Z'}_{x/}$ and ${Z'}_{/x}$ have objects given by pairs $(\bullet \in Z', x \to f(\bullet))$ and $(\bullet \in Z', f(\bullet) \to x)$, respectively; morphisms are given between the bullets in Z' with the obvious commutativity requirements in Z.
- Given two ∞-categories $\mathcal{Z}'$ and $\mathcal{Z}$, we can form their **join** $\mathcal{Z}' \star \mathcal{Z}$ [Lur09, Section 1.2.8]. We only really need the special cases of the **right cone** $\mathcal{Z}^{\triangleright} \coloneqq \mathcal{Z} \star \{+\infty\}$ and the **left cone** $\mathcal{Z}^{\triangleleft} \coloneqq \{-\infty\} \star \mathcal{Z}$ obtained from $\mathcal{Z}$ by adjoining a new terminal object $+\infty$ or initial object $-\infty$, respectively.
- The **classifying space** $\mathrm{B}Z \coloneqq |\mathrm{N}(Z)|$ of a category Z is the geometric realization of its nerve. A category is called **weakly contractible** if $\mathrm{B}Z$ has the homotopy type of a point.
- The inclusion $\mathcal{S} \hookrightarrow \mathbf{Cat}_\infty$ of ∞-groupoids into ∞-categories has a right adjoint which takes an ∞-category $\mathcal{C}$ to its **groupoid core** $\mathcal{C}^{\simeq}$ obtained by throwing away all non-invertible arrows. It also has a left adjoint which sends each 1-category to its classifying space and is given in general by localizing at all arrows (see Section 0.7.3 below).
- We denote by $\mathbf{Cat}_\infty$ the ∞-category of (small) ∞-categories whose mapping spaces are the groupoid core $\mathrm{Map}(-,-) = \mathrm{Fun}(-,-)^{\simeq}$ of the functor categories. As a full subcategory, $\mathbf{Cat}_\infty$ contains the $(2,1)$-category of (small) categories, functors and natural isomorphisms; the inclusion has a left adjoint which sends an ∞-category $\mathcal{C}$ to its **homotopy category** $\mathrm{h}\mathcal{C}$ obtained by applying π_0 to each mapping space.
- We treat each partially ordered set (poset) as a category with objects given by its elements and a unique arrow $x \to y$ if and only if $x \leq y$. In the spirit of invariance, we use the word "poset" also for categories which are only equivalent to partially ordered sets, *i.e.*, have at most one arrow between any two objects[37].
- Throughout this thesis we mostly avoid set theoretic issues of size. Implicitly, we work with respect to three nested Grothendieck universes containing "small", "large" and "very large" objects, respectively; when necessary, we may pass to an even larger selection of universes where formerly large objects are small. By default, sets and spaces are small, while categories and ∞-categories are large.

0.7.2 Coherent diagrams, (co)limits and Kan extension

A commutative square

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g"'] & \bullet \arrow[d, "g'"] \\ \bullet \arrow[r, "f'"'] & \bullet \end{tikzcd} \tag{0.7.1}$$

in an ordinary category consists of arrows f, g, f', g' such that the equality $g' \circ f = f' \circ g$ holds. In an ∞-category, this equality is an additional structure which also must be specified. To make

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[36] The inclusion $\mathbf{Set} \hookrightarrow \mathcal{S}$ does *not* have a right adjoint because it does not preserve colimits.

[37] From the perspective of order theory such a category amounts to what is called a *preorder*, *i.e.*, a set with a reflexive and transitive relation.

this precise, we define a commutative square in an ∞-category $\mathcal{C}$ to be a functor $\alpha\colon \square \to \mathcal{C}$, where $\square := \{0 \to 1\} \times \{0 \to 1\}$ is the walking commutative square (a category). Such a functor sends the equality $(00 \to 01 \to 00) = (00 \to 10 \to 00)$ (which is unique, since $\square$ is an ordinary category) to a homotopy $\alpha(00 \to 01 \to 00) \simeq \alpha(00 \to 10 \to 00)$ (for which there might be many non-equivalent choices) in the ∞-category $\mathcal{C}$. More complicated commutative diagrams in $\mathcal{C}$ can be similarly described as functors from suitably chosen categories or ∞-categories. The upshot is that we typically don't have to explicitly worry about coherence when specifying diagrams indexed by a category we can write down: the formalism of functors automatically encodes all of the necessary higher data. We will often be a bit sloppy and just say, for instance, that (0.7.1) "is a commutative square", leaving the rest of the implicit structure unspoken.

Commutative diagrams in ∞-categories admit a well behaved calculus of limits, colimits and Kan extensions; see [Lur09, Chapter 4] and [Cis19, Section 6.2]:

- Objects $\varnothing$ and $\star$ of $\mathcal{C}$ are called initial and terminal, respectively, if the homotopy types $\mathrm{Map}(\varnothing, c)$ and $\mathrm{Map}(c, \star)$ are contractible for all $c \in \mathcal{C}$. Initial/terminal objects of $\mathcal{C}$ are essentially unique in the sense that they form an ∞-groupoid which is either contractible or empty.
- A diagram $\alpha\colon \mathcal{Z}^{\triangleright} \to \mathcal{C}$ is a **colimit cone** if it is initial amongst all cones with the base $\alpha\big|_{\mathcal{Z}}\colon \mathcal{Z} \subset \mathcal{Z}^{\triangleright} \to \mathcal{C}$. A diagram $\alpha\colon \mathcal{Z}^{\triangleleft} \to \mathcal{C}$ is a **limit cone** if it is terminal amongst all cones with the base $\alpha\big|_{\mathcal{Z}}\colon \mathcal{Z} \subset \mathcal{Z}^{\triangleleft} \to \mathcal{C}$. In this case we also say that α exhibits $\alpha(+\infty)$ (resp. $\alpha(-\infty)$) as the colimit (resp. limit) of the diagram $\alpha\big|_{\mathcal{Z}}$.
- More generally let $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ be a fully faithful functor. We say that a functor $\alpha\colon \mathcal{Z}' \to \mathcal{C}$ is a **left/right Kan extension**[38] of its restriction $\alpha\big|_{\mathcal{Z}}$ if it is initial/terminal amongst all functors extending $\alpha\big|_{\mathcal{Z}}$.
- [Lur09, Definition 4.3.2.2] Right/left Kan extension along a fully faithful functor $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ can be computed and characterized pointwise at each $x \in \mathcal{Z}'$ by the induced limit/colimit of shape $\mathcal{Z}_{x/}$ and $\mathcal{Z}_{/x}$, respectively.
- [Lur09, Proposition 4.3.2.15] Restriction along a fully faithful functor $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ induces an equivalence of ∞-categories between the full subcategories of $\mathrm{Fun}(\mathcal{Z}', \mathcal{C})$ and $\mathrm{Fun}(\mathcal{Z}, \mathcal{C})$ consisting of those functors which *are* a right/left Kan extension and those functors which *have* a right/left Kan extension, respectively.
- [Lur09, Corollary 4.3.2.16, Proposition 4.3.2.17] If every functor $\mathcal{Z} \to \mathcal{C}$ admits a right/left Kan extension along the fully faithful functor $\mathcal{Z} \hookrightarrow \mathcal{Z}'$ then there is a unique fully faithful right/left Kan extension functor $\mathrm{Fun}(\mathcal{Z}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{Z}', \mathcal{C})$ which is right/left adjoint to the restriction functor; its essential image is spanned by those functors $\mathcal{Z}' \to \mathcal{C}$ which are a right/left Kan extension along $\mathcal{Z} \hookrightarrow \mathcal{Z}'$.
- [Lur09, Proposition 4.1.3.1] A functor $Z' \to Z$ between ordinary categories is **homotopy terminal**[39] if and only if each under-category ${Z'}_{x/}$ (for each $x \in Z$) is weakly contractible. Dually $Z' \to Z$ is **homotopy initial** if and only if each over-category ${Z'}_{/x}$ is weakly contractible. It follows in particular that left/right adjoint functors are homotopy initial/terminal since in this case the relevant over/under-categories have a terminal/initial object.
- [Lur09, Proposition 4.1.1.8] The limit/colimit of a Z-shaped diagram $Z \to \mathcal{C}$ can be computed after precomposing with any homotopy initial/terminal functor $Z' \to Z$.

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[38] It does also make sense to define Kan extensions along functors which are not fully faithful [Lur09, Section 4.3.3]; such Kan extensions play no role in this thesis.

[39] Joyal and Lurie would say *cofinal* which, confusingly, is the word Cisinski uses for the dual concept (what we call homotopy initial). We avoid this potential confusion by using the hopefully unambiguous terminology of Dugger [Dug].

0.7.3 Localization

Categorical localization is the procedure of formally adding inverses to arrows which might previously not have them. A **localization**[40)] [Cis19, Definition 7.1.2] of an ∞-category $\mathcal{Z}$ at a class $W \subset \mathcal{Z}$ of arrows is a functor $\mathcal{Z} \to \mathcal{Z}[W^{-1}]$ which is universal amongst all functors that send the arrows in W to equivalences. More precisely for each ∞-category $\mathcal{C}$ the restriction functor

$$\mathrm{Fun}(\mathcal{Z}[W^{-1}], \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{Z}, \mathcal{C})$$

is fully faithful with essential image consisting of those functors $\mathcal{Z} \to \mathcal{C}$ that send all arrows in W to equivalences in $\mathcal{C}$ Such ∞-categorical localizations always exist and are essentially unique, see [Cis19, Proposition 7.1.3].

An important special case occurs when we want to invert *all* arrows of the ∞-category $\mathcal{Z}$. In this case the result $\mathcal{Z}[\mathcal{Z}^{-1}]$ is an ∞-groupoid; the assignment $\mathcal{Z} \mapsto \mathcal{Z}[\mathcal{Z}^{-1}]$ assembles to a *groupoidification* functor $\mathbf{Cat}_\infty \to \mathcal{S}$ which is left adjoint to the inclusion. If we think of $\mathcal{Z}$ as a simplicial set (which we rarely do) then the space $\mathcal{Z}[\mathcal{Z}^{-1}]$ can be computed explicitly as its geometric realization; in particular, localizing an ordinary category Z at all arrows yields precisely its classifying space $\mathrm{B}Z = |\mathrm{N}(Z)|$.

Let the reader be warned that only under very special circumstances[41)] is the localization of a category an (ordinary) category again; in general the result will be an honest ∞-category with non-discrete mapping spaces. Strikingly, *every* ∞-category is the localization of a category at some collection of arrows[42)] and every ∞-groupoid can even be obtained by localizing a suitable *poset.*

[40)] Here our terminology differs from Lurie's who uses the word "localization" to refer to a special kind of localization functor which admits a fully faithful right adjoint (see [Lur09, Definition 5.2.7.2 and Warning 5.2.7.3]).

[41)] One example of such a special situation is discussed in Lemma 1.2.1.1.

[42)] In fact, there is a homotopy theory of "categories with arrows to be inverted" which is equivalent to that of ∞-categories [BK12]

Chapter 1

2-Segal spaces as invertible ∞-operads

1.1 The localization functors

Recall that the simplex category Δ is the category of finite non-empty linearly ordered sets and weakly monotone maps between them; when convenient we identify Δ with its skeleton consisting of the standard ordinals $[n] = \{0 < \cdots < n\}$.

1.1.1 The category Ω_π of plane rooted trees

We recall some basic facts about (colored, non-symmetric) operads and the category Ω_π of plane rooted trees as introduced by Moerdijk and Weiss [MW07].

Definition 1.1.1.1. A **colored, non-symmetric operad** (or **operad** for short) $\mathcal{O} = (\mathcal{O}, O, \circ)$ consists of

- a collection O of **objects** (or **colors**),
- given colors $x_1, \ldots, x_n, y \in O$, a set $\mathcal{O}(x_1, \ldots, x_n; y)$ of **n-ary operations** from $(x_1, \ldots, x_n)$ to y and
- for each $k, n_1, \ldots, n_k \in \mathbb{N}$ and colors $x^i_{j_i}, z \in O$ (for $0 \leq j_i \leq n_i$, $0 \leq i \leq k$), a **composition map**

$$\left(\coprod_{y_1,\ldots,y_k \in O} \left(\mathcal{O}(x^1_1, \ldots, x^1_{n_1}; y_1) \times \cdots \times \mathcal{O}(x^k_1, \ldots, x^k_{n_k}; y_k)\right) \times \mathcal{O}(y_1, \ldots, y_k; z) \right) \tag{1.1.1}$$
$$\xrightarrow{\circ} \mathcal{O}(x^1_1, \ldots, x^1_{n_1}, \ldots, x^k_1, \ldots, x^k_{n_k}; z)$$

- a **unit map**

$$1 \colon O \longrightarrow \coprod_{x,y \in O} \mathcal{O}(x; y) \tag{1.1.2}$$

which assigns to each color $x \in O$ the 1-ary **identity operation** $1_x \in \mathcal{O}(x; x)$

such that the obvious associativity and unitality conditions are satisfied. There is an obvious notion of a morphism of operads, we denote the resulting category of operads by **Op**. ♢

Remark 1.1.1.2. By plugging suitable identity operations into the general composition law (1.1.1) one can define the special compositions

$$\circ_{i+1} \colon \left(\coprod_{y \in O} \mathcal{O}(x_{i+1}, \ldots, x_j; y) \times \mathcal{O}(x_1, \ldots, x_i, y, x_{j+1}, \ldots, x_n; z) \right) \longrightarrow \mathcal{O}(x_1, \ldots, x_n; z) \tag{1.1.3}$$

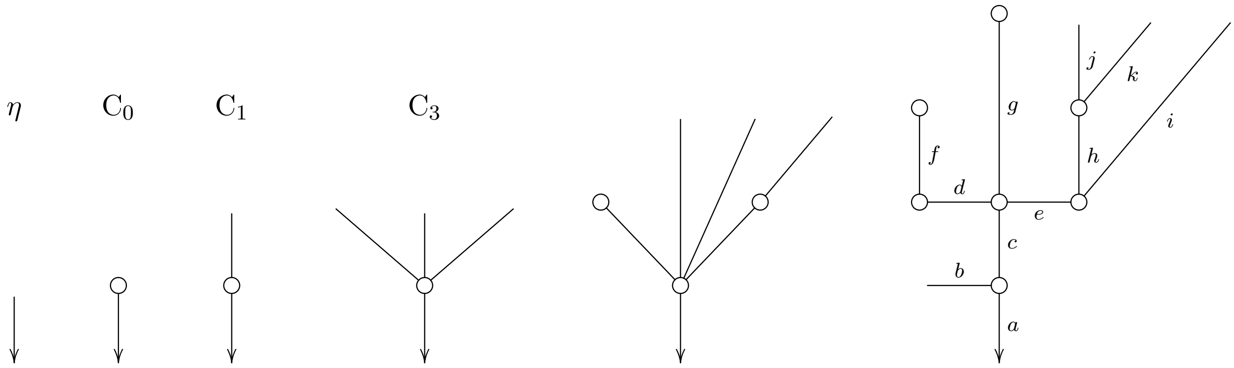
for all $0 \leq i \leq j \leq n$ and all $x_1, \ldots, x_n, z \in O$. It is called $\circ_{i+1}$ (the j is left implicit) because the output of the first operation is inserted at position $i+1$ into the second. It is a easy to verify that every general composition map (1.1.1) can be assembled as a suitable composition of such $\circ_{i+1}$-compositions (for varying i and j). $\Diamond$

Remark 1.1.1.3. As originally introduced by Boardman–Vogt and May, an "operad" would be assumed to be mono-colored. Since there is no reason for us to single out this special case we will instead take operads to be colored by default. Moreover it is most convenient for us to reserve the word "operad" for the least structured situation and add further adjectives (e.g. symmetric or cyclic) whenever we equip our operad with extra structure (see also Section 1.4 and Section 1.1.3). We warn the reader that this is a rather uncommon convention: most authors (including Moerdijk and Weiss and Cisinski and Moerdijk) will define operads to be symmetric by default. $\Diamond$

Remark 1.1.1.4. Each operad $(\mathcal{O}, O, \circ)$ has an underlying category with objects $x \in O$ and morphism sets $\mathcal{O}(x; y)$. Conversely, each category can be viewed as an operad which has only 1-ary operations. More precisely, we have an adjunction $\mathbf{Cat} \rightleftarrows \mathbf{Op}$ with fully faithful left adjoint. $\Diamond$

An object of Ω_π is called a **plane rooted tree** and consist of a finite plane rooted trees in the usual graph-theoretic sense together with a marking of some degree 1 vertices including the root-vertex. An edge between unmarked vertices is called internal, the other edges are called external. The unique external edge connected to the root-vertex is called the **root** (or output edge); an external edge attached to a marked non-root vertex is called a **leaf** (or input edge).

Example 1.1.1.5. We depict some trees in Ω_π, including the special tree η, some corollas (C_0, C_1, C_3) and two typical trees (of arity 3 and 4, respectively).



The root is marked with a little arrow and drawn towards the bottom. $\Diamond$

Remark 1.1.1.6. From now on we completely ignore the marked vertices of a tree and never speak of them again. Thus "vertex" always means "unmarked vertex". When drawing trees, we omit the marked vertices and instead draw the external edges "towards infinity". $\Diamond$

The number of leaves of a tree is its **arity**. Each vertex of a tree has some number (the **arity** of that vertex) of input edges and a unique output edge (which is the one that points in the direction of the root). The input edges of a vertex are linearly ordered left-to-right by the plane embedding. We denote by η or $[0]$ the tree with only a single edge (which is both the root and a leaf); we denote by $\mathrm{C}_{[n]}$ or C_n the **n-corolla**, i.e. the unique n-ary tree with a single vertex. Given two edges e, e' in a plane tree T, we say that e is a **predecessor** of e' and that e' is a **successor** of e, if the unique path in T going from e to the root of T goes through e'; note that every edge is a predecessor of the root. Given two edges d, e in T, we say that d **lies to the left** of e and that e **lies to the right** of d, if there are successors d' of d and e' of e which are input edges at a common vertex v and such that e' lies (strictly) to the left of e with respect to the left-to-right linear order at v. Observe that for any two edges e, d we have the following two mutually exclusive cases:

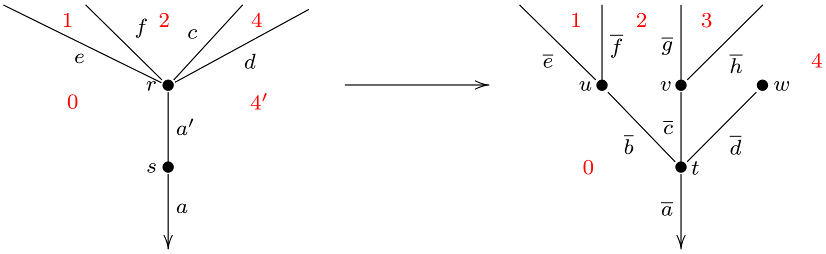
- d is a successor or a predecessor of e (this includes the case $d = e$) or

- d lies to the left or to the right of e.

Example 1.1.1.7. In the last tree of Example 1.1.1.5: The predecessors of the edge e are e itself, h, j, k and i; the successors of e are e itself, c and the root a. To the left of e lie the edges d, f, g and b; no edge lies to the right of e. ♢

Each plane rooted tree T gives rise to a free operad (also denoted by T): it has a color for each edge of T and its operations are freely generated by the vertices of T (an n-ary vertex is seen as an n-ary operation from its input edges to its output edge). A morphism in Ω_π between two trees is defined to be a morphism of the corresponding operads.

Example 1.1.1.8. Consider the following two plane rooted trees. The operad associated to the left tree has colors $\{a', a, c, d, e, f\}$ and three non-unit operations $s\colon a' \to a$ and $r\colon (e, f, c, d) \to a'$ and $r \circ s\colon (e, f, c, d) \to a$. The other one has colors $\{\overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}, \overline{f}, \overline{g}, \overline{h}\}$ and eleven non-unit operations (t, u, v, w and all their composites).



The depicted morphism is described on colors by $a' \mapsto \overline{a}$, $a \mapsto \overline{a}$, $c \mapsto \overline{c}$ etc. and on generating operations by $s \mapsto 1_{\overline{a}}$ and $r \mapsto (u, 1_{\overline{c}}, 1_{\overline{d}}) \circ t$. (The red numbers are for later reference.) ♢

A **(planar) dendroidal object** in an ∞-category $\mathcal{C}$ is functor $\Omega_\pi^{\mathrm{op}} \to \mathcal{C}$. We denote by $\mathbf{d}_\pi\mathbf{Set} := [\Omega_\pi^{\mathrm{op}}, \mathbf{Set}]$ the category of (planar) dendroidal sets, i.e. dendroidal objects in **Set**. Given a plane rooted tree T, we denote by $\Omega_\pi[T]$ the dendroidal set represented by T. There is a canonical fully faithful embedding $\Delta \hookrightarrow \Omega_\pi$ of the simplex category Δ by interpreting every linearly ordered set as a linear tree. This embedding gives rise to an adjunction $\mathbf{sSet} \rightleftarrows \mathbf{d}_\pi\mathbf{Set}$ with fully faithful left adjoint. The inclusion $\Omega_\pi \hookrightarrow \mathbf{Op}$ (which is full by construction) gives rise to a realization/nerve adjunction

$$\mathbf{d}_\pi\mathbf{Set} \rightleftarrows \mathbf{Op} : \mathrm{N}_\mathrm{d}$$

by the formula $\mathrm{N}_\mathrm{d}(\mathcal{O})\colon T \mapsto \mathrm{Hom}_{\mathbf{Op}}(T, \mathcal{O})$, which extends the usual adjunction

$$\mathbf{sSet} \rightleftarrows \mathbf{Cat} : \mathrm{N}.$$

1.1.2 The localization functor $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$

Let us introduce the main player in our game.

Construction 1.1.2.1 (Covariant description of $\mathcal{L}_\pi$)**.** Each plane rooted tree $T \in \Omega_\pi$ (which we visualize with its external edges going towards infinity) partitions the plane into a set $\mathcal{L}_\pi T$ of "areas" which is linearly ordered clockwise starting from the root. It is straightforward to extend this assignment to a functor $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$. ♢

We give an alternative, more formal, construction of the functor $\mathcal{L}_\pi$ at the end of this section, see Construction 1.1.2.10 below.

Example 1.1.2.2. The functor $\mathcal{L}_\pi$ sends the morphism depicted in Example 1.1.1.8 to the map $\{0, 1, 2, 4, 4'\} \to \{0, 1, 2, 3, 4\}$ in Δ which sends $i', i \mapsto i$. ♢

Remark 1.1.2.3. Specifying two adjacent "areas" of a plane rooted tree $T \in \Omega_\pi$ uniquely determines an external edge of T that separates them. If we write $[n] := \mathcal{L}_\pi T$ (where n is the arity of T) then

- each minimal edge $\{i-1, i\} \hookrightarrow [n]$ (for $1 \leq i \leq n$) corresponds precisely to a leaf of T and
- the maximal edge $\{0, n\} \hookrightarrow [n]$ corresponds to the root of T. $\lozenge$

Remark 1.1.2.4. Usually the category of trees is related to the simplex category by the inclusion $\Delta \hookrightarrow \Omega_\pi$ of the linear trees. The composition $\Delta \hookrightarrow \Omega_\pi \xrightarrow{\mathcal{L}_\pi} \Delta$ is constant with value $[1] \in \Delta$. The two occurrences of the category Δ in relation to the category Ω_π are in some sense "orthogonal": the first is sensitive to the "height" of a tree, the second measures the "width". $\lozenge$

Definition 1.1.2.5. A map of plane rooted trees is called **boundary preserving** if it maps the root to the root and each leaf to a leaf. $\lozenge$

Definition 1.1.2.6. A **collapse map** in Ω_π is a boundary preserving map $\mathrm{C}_{[n]} \to T$ out of a corolla (where n is the arity of T). A dendroidal object $\mathcal{X}\colon \Omega_\pi^{\mathrm{op}} \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is called **invertible** if $\mathcal{X}$ maps all collapse maps to equivalences in $\mathcal{C}$. $\lozenge$

Remark 1.1.2.7. A boundary preserving map $\alpha\colon T \to S$ of plane rooted trees induces a bijection between the leaves of T and the leaves of S. Hence the functor $\mathcal{L}_\pi$ maps boundary preserving maps to isomorphisms. $\lozenge$

Remark 1.1.2.8. The motivation for the word "invertible" in Definition 1.1.2.6 will become apparent in Section 1.3.3 when we discuss invertible operads (in the sense of Dyckehoff and Kapranov [DK12]) and show that an operad is invertible if and only if its nerve is an invertible dendroidal set (Lemma 1.3.3.5). $\lozenge$

Here is one version of our main result which we explain and prove in Section 1.2 below:

Theorem 1.1.2.9. The functor $\mathcal{L}_\pi$ exhibits Δ as an ∞-categorical localization of Ω_π at the set of collapse maps. $\square$

Before going forward, we give a "contravariant" description of the functor $\mathcal{L}_\pi$. This description is useful because unlike the covariant one it can easily be adapted to the case of symmetric trees (see Section 1.1.3). Denote by Δ_{b} the following category: objects are (possibly empty) linearly ordered sets; a morphism $N \to M$ is a weakly monotone map

$$\{-\infty\} \,\dot{\cup}\, N \,\dot{\cup}\, \{+\infty\} \to \{-\infty\} \,\dot{\cup}\, M \,\dot{\cup}\, \{+\infty\}$$

which preserves $-\infty$ and $+\infty$ (where $-\infty$ and $+\infty$ are a new minimal and maximal element, respectively). It is an easy fact (see also Lemma 2.5.2.2) that the category Δ is isomorphic to $\Delta_{\mathrm{b}}^{\mathrm{op}}$ via the assignment (described here only on objects)

$$\Delta \ni N \longmapsto \{\text{non-empty proper initial segments of } N\} \in \Delta_{\mathrm{b}}^{\mathrm{op}}.$$

Using the identification $\Delta \simeq \Delta_{\mathrm{b}}^{\mathrm{op}}$ we can give the following description of the functor $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta_{\mathrm{b}}^{\mathrm{op}}$, which is easily seen to be equivalent to Construction 1.1.2.1.

Construction 1.1.2.10 (Contravariant description of $\mathcal{L}_\pi$)**.** To each plane rooted tree $T \in \Omega_\pi$ we associate the (possibly empty) linearly ordered set $\mathcal{L}_\pi T \in \Delta_{\mathrm{b}}$ of its leaves. This association extends to maps in the following way: Given a map $\alpha\colon S \to T$ of trees, we need to define a map $\{-\infty\} \,\dot{\cup}\, \mathcal{L}_\pi T \,\dot{\cup}\, \{+\infty\} \to \{-\infty\} \,\dot{\cup}\, \mathcal{L}_\pi S \,\dot{\cup}\, \{+\infty\}$. We have no choice but to send $-\infty$ and $+\infty$ to $-\infty$ and $+\infty$, respectively. Denote by r_S the root of S and let $a \in \mathcal{L}_\pi T$; there are three cases:

- If a is a predecessor of $\alpha(r_S)$ then there is a unique leaf b of S such that $\alpha(b)$ is a successor of a; in this case we define $(\mathcal{L}_\pi \alpha)(a) := b$ to be this unique leaf.
- If a lies to the left of $\alpha(r_S)$ then we define $(\mathcal{L}_\pi \alpha)(a) := -\infty$.

- If a lies to the right of $\alpha(r_S)$ then we define $(\mathcal{L}_\pi \alpha)(a) \coloneqq +\infty$.

It is straightforward to verify that this assignment defines a functor $\mathcal{L}_\pi \colon \Omega_\pi \to \Delta_{\mathrm{b}}^{\mathrm{op}}$. $\Diamond$

Example 1.1.2.11. The map of trees from Example 1.1.1.8 gets sent by $\mathcal{L}_\pi$ to the map

$$\left\{-\infty, \overline{e}, \overline{f}, \overline{g}, \overline{h}, +\infty\right\} \to \{-\infty, e, f, c, d, +\infty\}$$

in Δ_{b} given by $\overline{e} \mapsto e$, by $\overline{f} \mapsto f$ and by $\overline{g}, \overline{h} \mapsto c$. $\Diamond$

1.1.3 Symmetric operads and Segal's category Γ

Before moving on with the proof of our main localization theorem, we briefly describe the analog construction in the world of symmetric operads, *i.e.*, operads $(\mathcal{O}, O, \circ)$ equipped with permutation isomorphisms

$$\mathcal{O}(x_1, \ldots, x_n; y) \xrightarrow{\cong} \mathcal{O}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; y),$$

(for all $n \in \mathbb{N}$, $x_1, \ldots, x_n, y \in O$ and $\sigma \in \mathfrak{S}_n$), which form actions of the symmetric groups compatible with composition of operations.

Denote by Ω_{sym} the category of symmetric rooted trees (i.e. trees without a plane embedding), defined as a suitable full subcategory of the category **symOp** of symmetric operads; this is the category of trees which Moerdijk and Weiss [MW07, Section 3] simply call Ω. Boundary preserving maps and collapse maps in **symOp** are defined in the same way as for plane trees.

The symmetric analog of the simplex category is Segal's category $\Gamma \coloneqq \mathbf{Fin}_\star^{\mathrm{op}}$, the opposite of the category of finite pointed sets. We define a functor $\mathcal{L}_{\mathrm{sym}} \colon \Omega_{\mathrm{sym}} \to \Gamma$, which is analogous to $\mathcal{L}_\pi$ by adapting Construction 1.1.2.10:

Construction 1.1.3.1 (The functor $\mathcal{L}_{\mathrm{sym}}$)**.** We define the functor $\mathcal{L}_{\mathrm{sym}} \colon \Omega_{\mathrm{sym}} \to \mathbf{Fin}_\star^{\mathrm{op}} = \Gamma$ as follows: To each tree T we assign the set of external edges which is pointed at the root. Given a morphism $\alpha \colon S \to T$ of rooted trees and a leaf a of T there is at most one leaf b of S such that $\alpha(b)$ is a successor of a; we define $(\mathcal{L}_{\mathrm{sym}}\alpha)(a) \coloneqq b$ if such a b exists and $(\mathcal{L}_{\mathrm{sym}}\alpha)(a) \coloneqq \star$ otherwise. $\Diamond$

It is straightforward to show that $\mathcal{L}_{\mathrm{sym}} \colon \Omega_{\mathrm{sym}} \to \mathbf{Fin}_\star^{\mathrm{op}}$ is well defined and extends the functor $\mathcal{L}_\pi$ in the sense that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{Op} \arrow[d, "\mathrm{sym}"] & \Omega_\pi \arrow[l, hook'] \arrow[r, "\mathcal{L}_\pi"] \arrow[d] & \Delta & \Delta_{\mathrm{b}}^{\mathrm{op}} \arrow[l, hook', "\simeq"'] \arrow[dl] \\
\mathbf{symOp} & \Omega_{\mathrm{sym}} \arrow[l, hook'] \arrow[r, "\mathcal{L}_{\mathrm{sym}}"] & \mathbf{Fin}_\star^{\mathrm{op}} &
\end{tikzcd}$$

where the leftmost arrow is the symmetrization functor and the rightmost diagonal arrow forgets the linear ordering and adds a basepoint.

We have the following localization result (see Section 1.2):

Theorem 1.1.3.2. The functor $\mathcal{L}_{\mathrm{sym}} \colon \Omega_{\mathrm{sym}} \to \Gamma$ exhibits Γ as an ∞-categorical localization of Ω_{sym} at the set of collapse maps. $\square$

Remark 1.1.3.3. The functor $\mathcal{L}_{\mathrm{sym}} \colon \Omega_{\mathrm{sym}} \to \mathbf{Fin}_\star^{\mathrm{op}}$ can be described as $\mathcal{L}_{\mathrm{sym}} \colon T \mapsto \lambda(T) \mathbin{\dot\cup} \{\star\}$, where $\lambda(T)$ is the set of leaves of a tree T. In this guise, it was introduced by Boavida de Brito and Moerdijk [BM17]. $\Diamond$

1.2 The localization theorem

The following theorem expresses that the functor $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$ (and its symmetric sibling $\mathcal{L}_{\mathrm{sym}}$) is universal (in the ∞-categorical sense) with the property of inverting the collapse maps in $\mathcal{L}_\pi$.

Theorem 1.2.0.1. For every ∞-category $\mathcal{C}$, the functor $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$ induces a fully faithful functor

$$\mathcal{L}_\pi^\star\colon \operatorname{Fun}(\Delta, \mathcal{C}) \longrightarrow \operatorname{Fun}(\Omega_\pi, \mathcal{C})$$

of ∞-categories with essential image spanned by those functors $\Omega_\pi \to \mathcal{C}$ which map collapse maps $\mathrm{C} \to T$ to equivalences. The analogous statement holds for the functor $\mathcal{L}_{\mathrm{sym}}\colon \Omega_{\mathrm{sym}} \to \Gamma$. □

Corollary 1.2.0.2. The categories Ω_π and Ω_{sym} are weakly contractible □

Proof. Clearly the categories Δ and Γ are contractible because they have a terminal object and a zero object, respectively. Since the localization functors of Theorem 1.2.0.1 induce weak equivalences on classifying spaces, the result follows. ■

Remark 1.2.0.3. The weak contractibility of Ω_{sym} (and implicitly of Ω_π) was proved with a different method by Ara, Cisinski and Moerdijk [ACM19]. ◊

1.2.1 The general situation

Our strategy to prove Theorem 1.2.0.1 is to apply the following general lemma which we will prove separately in Section 1.2.2 below.

Lemma 1.2.1.1. Let $L\colon W \to D$ be a functor of (ordinary) categories and for each $n \in D$ let $B_n \subset W_n$ be a subcategory of the weak fiber W_n of L such that (with the notation of Remark 1.2.1.2 below)

- B_n has an initial object c_n and
- the inclusion $\mathrm{N}(B_n) \hookrightarrow \mathrm{N}(W)_{/n}$ is homotopy terminal.

Then for every ∞-category $\mathcal{C}$, composition with L induces a fully faithful functor

$$L^\star\colon \operatorname{Fun}(\mathrm{N}(D), \mathcal{C}) \longrightarrow \operatorname{Fun}(\mathrm{N}(W), \mathcal{C})$$

of ∞-categories with the essential image spanned by those functors $\mathrm{N}(W) \to \mathcal{C}$ which send all the edges of the form $c_n \to t$ in $\mathrm{N}(B_n)$ (for $n \in D$) to equivalences. □

Remark 1.2.1.2. Recall that the weak fiber W_n of $L\colon W \to D$ is the category whose objects consist of an object $t \in W$ and an isomorphism $t \xrightarrow{\cong} n$ in D. The left fiber $W_{/n} \supset W_n$ has objects $(t, f\colon t \to n)$ where f is not required to be an isomorphism. ◊

Let Ω be any one of the categories Ω_π and Ω_{sym}; let $\mathcal{L}$ be the corresponding functor (among $\mathcal{L}_\pi$ and $\mathcal{L}_{\mathrm{sym}}$) and denote its target (which is either Δ or Γ) by $\mathcal{D}$. For every object $[n] \in \mathcal{D}$ we denote by $\Omega_{/[n]}$ the left fiber, by $\Omega_{[n]}$ the weak fiber and by $\mathrm{bp}_{[n]} \subset \Omega_n$ the subcategory of $\Omega_{[n]}$ with the same objects but only boundary preserving morphisms. We shall now show that the functors $\mathcal{L}$ satisfy the requirements for Lemma 1.2.1.1, thus concluding the proof of Theorem 1.2.0.1.

Proposition 1.2.1.3. Fix an object $[n] \in \mathcal{D}$.

(1) The n-corolla $\mathrm{C}_{[n]}$ (together with any identification $\mathcal{L}\mathrm{C}_{[n]} \xrightarrow{\cong} [n]$) is an initial object in the category $\mathrm{bp}_{[n]}$.

(2) The inclusion $\mathrm{bp}_{[n]} \subset \Omega_{[n]} \hookrightarrow \Omega_{/[n]}$ has a left adjoint; in particular it is homotopy terminal. □

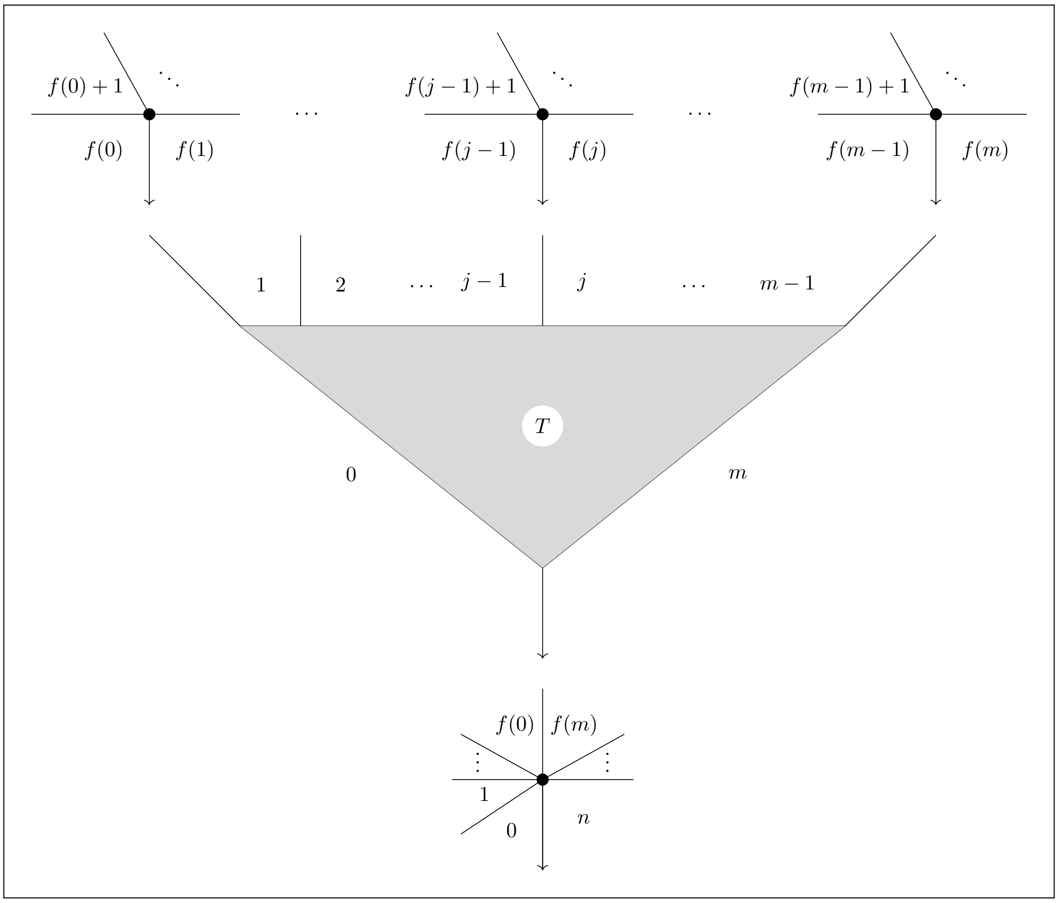


Figure 1.1: The construction of the tree T_f in the case $\mathcal{L} = \mathcal{L}_\pi$. The little arrows decorate the roots of the various trees. Forgetting the root and/or the plane embedding describes the analogous construction in the cases $\mathcal{L} = \mathcal{L}_{\mathrm{cyc}}, \mathcal{L}_{\mathrm{sym}}, \mathcal{L}_{\mathrm{abs}}$

Proof (of Proposition 1.2.1.3)**.** The first statement is obvious.
The functor $\Omega_{/[n]} \to \mathrm{bp}_{[n]}$ is constructed as follows: Given an object $(T, f \colon \mathcal{L}T = [m] \to [n])$ we define the tree T_f by glueing some corollas to T along its outer edges (see also Figure 1.1). We only describe this process explicitly for $\mathcal{L} = \mathcal{L}_\pi$; the construction is analogous for $\mathcal{L}_{\mathrm{sym}}$.

- To a leaf of T corresponding to the minimal edge $\{j-1, j\} \hookrightarrow [m]$ we glue a corolla $\mathrm{C}^f_{j-1,j}$ (of arity $f(j) - f(j-1)$) with leaves $\{i-1, i\}$ for $f(j-1) < i \leq f(j)$ (this might be a 0-corolla if $f(j-1) = f(j)$).
- To the root (corresponding to the maximal edge $\{0, m\} \hookrightarrow [m]$) we glue a corolla $\mathrm{C}^f_{\max}$ with leaves

$$\{0,1\}, \{1,2\}, \ldots, \{f(0)-1, f(0)\}, \{f(0), f(m)\}, \{f(m), f(m)+1\}, \ldots, \{n-1, n\}$$

  along the special leaf $\{f(0), f(m)\}$ of $\mathrm{C}^f_{\max}$.

The adjunction unit at (T, f) is the inclusion $T \hookrightarrow T_f$ which we denote by f_T. We need to prove that given a morphism of trees $\alpha \colon T \to S$ over $f \colon [m] \to [n]$ there is a unique factorization $T \xrightarrow{f_T} T_f \xrightarrow{\alpha^{\mathrm{bp}}} S$ with α^{bp} in $\mathrm{bp}_{[n]}$. We have no other choice than to define α^{bp} as α on the subtree $T \hookrightarrow T_f$ and to make it the identity on the boundary; hence uniqueness is clear. It is straightforward to verify that this map of trees is indeed well defined. ■

1.2.2 Proof of the key lemma

This section is devoted to the proof of Lemma 1.2.1.1

Let M be defined as the Grothendieck construction of the functor $\Delta^1 \to \mathbf{Cat}$ which parameterizes the functor $L\colon W \to D$. Explicitly, an object in M is either an object $t \in W$ or an object $n \in D$; for $s, t \in W$ and $m, n \in D$ we put $M(t,s) = W(t,s)$ and $M(n,m) = D(n,m)$ and $M(t,n) = D(Lt,n)$ and $M(n,t) = \varnothing$. We have a factorization $L\colon W \hookrightarrow M \xrightarrow{\overline{L}} D$ where the first arrow is the obvious fully faithful inclusion and the second arrow has a fully faithful right adjoint $D \hookrightarrow M$. We identify D with its image in M and we denote by $\eta\colon \mathrm{Id}_M \to \overline{L}$ the unit of the adjunction $\overline{L}\colon M \rightleftarrows D$; it is an isomorphism (in fact the identity) at exactly those objects in M that belong to D.[1] We deal with the two components of $L\colon W \hookrightarrow M \rightleftarrows D$ individually by using standard techniques from Higher Topos Theory [Lur09]. Lemma 1.2.1.1 is a direct consequence of Corollary 1.2.2.4 and Corollary 1.2.2.9 below.

Remark 1.2.2.1. For each $n \in D$ the forgetful functor $B_n \subset W_n \to W$ extends to a functor ${B_n}^{\triangleright} \hookrightarrow M$ by sending the new vertex v to n and the new arrow $(t,f) \to v$ (for $(t,f) \in B_n$) to the arrow $f\colon t \to n$ of M. ♢

Fix an ∞-category $\mathcal{C}$. We recall the following result.

Lemma 1.2.2.2. [Lur09, Proposition 5.2.7.12] Let $\overline{L}\colon \mathcal{M} \to \mathcal{D}$ be a reflective localization functor of ∞-categories (i.e. $\overline{L}$ has a fully faithful right adjoint) and let $\mathcal{C}$ be another ∞-category. Then composition with $\overline{L}$ induces a fully faithful functor

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathcal{M}, \mathcal{C})$$

with essential image consisting of those functors that map an edge f in $\mathcal{M}$ to an equivalence in $\mathcal{C}$ provided that $\overline{L}f$ is an equivalence in $\mathcal{D}$. ☐

Lemma 1.2.2.3. Let $F\colon \mathrm{N}(M) \to \mathcal{C}$ be a functor of ∞-categories. The following are equivalent:

1. For every edge f in $\mathrm{N}(M)$, if $\overline{L}f$ is an equivalence in D then Ff is an equivalence in $\mathcal{C}$.
2. For every $n \in D$, the functor F maps all edges in $\mathrm{N}(B_n)^{\triangleright}$ to equivalences in $\mathcal{C}$.
3. F sends every component $\eta_t\colon t \to \overline{L}t$ of the unit to an equivalence in $\mathcal{C}$.

We denote by K^+ the full subcategory of $\mathrm{Fun}(\mathrm{N}(M), \mathcal{C})$ spanned by such functors. ☐

Proof. Clearly (1) implies (2) (because $\overline{L}(f)$ is the identity for each edge f of $\mathrm{N}(B_n)^{\triangleright}$) and (2) trivially implies (3).
Observe that if $f\colon t \to s$ is a morphism in M then we have a commutative naturality square

$$\begin{tikzcd} t \arrow[r, "\eta_t"] \arrow[d, "f"] & \overline{L}t \arrow[d, "\overline{L}f"] \\ s \arrow[r, "\eta_s"] & \overline{L}s \end{tikzcd}$$

Hence (3) implies (1) by the two-out-of-three property for equivalences in $\mathcal{C}$. ■

Corollary 1.2.2.4. Composition with the functor $\overline{L}\colon M \to D$ induces a fully faithful functor $\mathrm{Fun}(\mathrm{N}(D), \mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathrm{N}(M), \mathcal{C})$ with essential image K^+. ■

Let us recall the following result.

Lemma 1.2.2.5. [Lur09, Proposition 4.3.1.12] Let $\mathcal{C}$ be an ∞-category and let $\overline{F}\colon B^{\triangleright} \to \mathcal{C}$ be a diagram where B is a weakly contractible simplicial set and $\overline{F}$ carries each edge of B to an equivalence in $\mathcal{C}$. Then $\overline{F}$ is a colimit diagram in $\mathcal{C}$ if and only if it carries every edge in $B^{\triangleright}$ to an equivalence in $\mathcal{C}$. ☐

Lemma 1.2.2.6. Let $F\colon \mathrm{N}(W) \to \mathcal{C}$ be a functor. The following are equivalent:

---

[1] The components $\eta_t\colon t \to \overline{L}t$ of the adjunction are precisely the coCartesian morphisms of the coCartesian fibration $M \to \Delta^1$.

(1) The functor F admits a left Kan extension along $W \hookrightarrow M$ and the resulting functor $\mathrm{N}(M) \to \mathcal{C}$ lies in K^+.

(2) For every $n \in D$ the functor F maps every edge of $\mathrm{N}(B_n)$ to an equivalence in $\mathcal{C}$.

(3) For every $n \in D$ and every $t \in B_n$ the functor F maps the unique edge $c_n \to t$ in $\mathrm{N}(B_n)$ to an equivalence in $\mathcal{C}$.

We denote by K the full subcategory of $\mathrm{Fun}(\mathrm{N}(W), \mathcal{C})$ spanned by such functors. □

Proof. The equivalence between (2) and (3) is obvious because c_n is an initial element in B_n. Using description (2) of Lemma 1.2.2.3 it is clear that (1) implies (2).

Let us prove the converse: By the pointwise construction of Kan extensions [Lur09, Lemma 4.3.2.13], a left Kan extension of F along $W \hookrightarrow M$ can be assembled from colimit cones for the diagrams $\mathrm{N}(W)_{/n} \to \mathrm{N}(W) \xrightarrow{F} \mathcal{C}$ (for $n \in D$). Recall that $B_n \hookrightarrow W_{/n}$ is homotopy terminal, hence we can reduce to finding colimits for the diagrams $\mathrm{N}(B_n) \hookrightarrow \mathrm{N}(W_{/n}) \to \mathrm{N}(W) \xrightarrow{F} \mathcal{C}$. All edges of these diagrams are equivalences by condition (2) and $\mathrm{N}(B_n)$ is contractible (because B_n has an initial element). Therefore by Lemma 1.2.2.5 these colimits exists and the corresponding colimit cones $\mathrm{N}(B_n)^{\triangleright} \to \mathcal{C}$ map all edges to equivalences in $\mathcal{C}$, thus verifying condition (2) of Lemma 1.2.2.3. ■

Fix the following notation:

- Denote by H^+ the full subcategory of $\mathrm{Fun}(\mathrm{N}(M), \mathcal{C})$ spanned by those functors which are the left Kan extension of their restriction to $W \subset M$.
- Denote by H the full subcategory of $\mathrm{Fun}(\mathrm{N}(W), \mathcal{C})$ spanned by those functors which admit a left Kan extension along $W \hookrightarrow M$.

Recall the following result.

Lemma 1.2.2.7. [Lur09, Proposition 4.3.2.15] The restriction functor along $\mathrm{N}(W) \hookrightarrow \mathrm{N}(M)$ is a trivial fibration $H^+ \to H$ of simplicial sets. □

Lemma 1.2.2.8. We have inclusions $K^+ \subset H^+$ and $K \subset H$ and a pullback square

$$\begin{tikzcd}
K^+ \arrow[r, hook] \arrow[d] & H^+ \arrow[d] \\
K \arrow[r, hook] & H
\end{tikzcd}$$

of simplicial sets with vertical arrows given by restriction along $W \hookrightarrow M$. □

Proof. This follows directly from Lemma 1.2.2.3 and Lemma 1.2.2.6 ■

Since trivial fibrations of simplicial sets are stable under pullbacks we obtain:

Corollary 1.2.2.9. The restriction functor along the inclusion $W \hookrightarrow M$ is a trivial fibration $K^+ \to K$ of simplicial sets. ■

This concludes the proof of Lemma 1.2.1.1 and therefore of Theorem 1.2.0.1.

1.3 Applications

Consider the category $\mathbf{sSet} := [\Delta^{\mathrm{op}}, \mathbf{Set}]$ of simplicial sets equipped with the Kan–Quillen left proper combinatorial simplicial model structure [Qui67]. Denote by $\mathcal{S} := \mathrm{N}_\Delta(\mathbf{sSet}^\circ)$ the corresponding **∞-category of spaces** obtained as the simplicial nerve of the subcategory of fibrant-cofibrant objects [Lur09, Definition 1.2.16.1]. A dendroidal (resp. simplicial) object in $\mathcal{S}$ is called a dendroidal (resp. simplicial) space.

1.3.1 2-Segal simplicial objects and Segal dendroidal objects

In this section we compare the dendroidal Segal conditions due to Cisinski and Moerdijk [CM13] and the simplicial 2-Segal conditions due to Dyckerhoff and Kapranov [DK12].

Definition 1.3.1.1. [CM13, Definition 2.2] The **Segal core** of a tree $\eta \neq T \in \Omega_{\mathrm{sym}}$ is the union

$$\mathrm{Sc}[T] \coloneqq \bigcup_v \Omega_{\mathrm{sym}}[\mathrm{C}_{n(v)}]$$

where v runs over all vertices of T and $\mathrm{C}_{n(v)} \hookrightarrow T$ denotes the subtree with vertex v. We use the convention $\mathrm{Sc}[\eta] \coloneqq \Omega_{\mathrm{sym}}[\eta]$ for the trivial tree.

A symmetric dendroidal space $\mathcal{X} \colon \Omega_{\mathrm{sym}}^{\mathrm{op}} \to \mathcal{S}$ is **Segal** if for any tree $T \in \Omega_{\mathrm{sym}}$ the map

$$\mathcal{X}_T = \mathrm{Hom}(\Omega_{\mathrm{sym}}(T), \mathcal{X}) \longrightarrow \mathrm{Hom}(\mathrm{Sc}[T], \mathcal{X})$$

is a trivial fibration. ♢

We adapt this definition as follows.

Definition 1.3.1.2. A dendroidal object $\mathcal{X} \colon \Omega_{\pi}^{\mathrm{op}} \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is called **Segal** if $\mathcal{X}$ sends the diagram

$$\begin{tikzcd} T & T_2 \arrow[l, hook'] \\ T_1 \arrow[u, hook] & e \arrow[l, hook'] \arrow[u, hook] \end{tikzcd} \tag{1.3.1}$$

to a pullback square in $\mathcal{C}$ whenever the tree $T \in \Omega_\pi$ arises by grafting two trees T_1 and T_2 along a common edge e. ♢

Remark 1.3.1.3. Clearly Definition 1.3.1.1 and Definition 1.3.1.2 make sense, mutatis mutandis, for symmetric dendroidal objects. Another way of saying this is that a symmetric dendroidal object is Segal if and only if the underlying dendroidal object is Segal. ♢

Remark 1.3.1.4. If a tree T arises by grafting two trees T_1 and T_2 along a common edge e then clearly $\mathrm{Sc}[T] = \mathrm{Sc}[T_1] \sqcup_e \mathrm{Sc}[T_2]$. By successively decomposing a tree along its inner edges we therefore see that Definition 1.3.1.1 and Definition 1.3.1.2 agree for dendroidal objects in the ∞-category $\mathcal{S}$ of spaces. ♢

The importance of the dendroidal Segal conditions is highlighted by the following result, which has an obvious analog for non-symmetric operads and dendroidal sets.

Proposition 1.3.1.5. [CM13, Corollary 2.6] The symmetric dendroidal nerve functor

$$\mathrm{N_d} \colon \mathbf{symOp} \longrightarrow \mathbf{dSet}$$

is fully faithful and the essential image consists precisely of the Segal symmetric dendroidal sets. ☐

Definition 1.3.1.6. [DK12, Proposition 2.3.2] A simplicial object $\mathcal{X} \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is called 2**-Segal** (or **unital** 2**-Segal**) if for each $0 \leq i \leq j \leq m$ it maps the square

$$\begin{tikzcd} \{0, \ldots, m\} & \{i, \ldots, j\} \arrow[l, hook'] \\ \{0, \ldots, i, j, \ldots m\} \arrow[u] & \{i, j\} \arrow[l, hook'] \arrow[u] \end{tikzcd} \tag{1.3.2}$$

in Δ to a pullback square square in $\mathcal{C}$. ♢

Remark 1.3.1.7. We always interpret the elements i and j in the lower row of Diagram 1.3.2 as distinct; thus in the case $i = j$ the vertical arrows are codegeneracy maps. ♢

Remark 1.3.1.8. The original definition of 2-Segal objects only includes the case $i \neq j$ of (1.3.2); the condition for $i = j$ is called *unitality.* Since unitality is now known to be redundant [FGK+19], we drop that adjective entirely. ♢

Lemma 1.3.1.9. A simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is 2-Segal if and only if the composition $\mathcal{L}_\pi^\star \mathcal{X}\colon \Omega_\pi^{\mathrm{op}} \xrightarrow{\mathcal{L}_\pi} \Delta^{\mathrm{op}} \xrightarrow{\mathcal{X}} \mathcal{C}$ is a Segal dendroidal object. ☐

Proof. Let $T = T_1 \cup_e T_2$ be a grafting of trees where e is the root of T_2 and a leaf of T_1. Put $[m] := \mathcal{L}_\pi T$. Applying $\mathcal{L}_\pi$ to the inclusion $e \hookrightarrow T$ defines a map $[1] = \mathcal{L}_\pi e \xrightarrow{f} [m]$, so we can define $i := f(0)$ and $j := f(1)$. It is easy to see that with this notation $\mathcal{L}_\pi$ sends Diagram (1.3.1) to Diagram (1.3.2) and that every instance of Diagram (1.3.2) arises this way. ■

1.3.2 Segal simplicial objects and covariantly fibrant dendroidal objects

Recall that a simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is called **reduced Segal** if $\mathcal{X}_{[n]} \xrightarrow{\simeq} \mathcal{X}_{[1]}^n$ via the inert maps $\{i-1, i\} \hookrightarrow [n]$ in Δ (in particular $\mathcal{X}_{[0]}$ is a terminal object in $\mathcal{C}$). A similar condition makes sense when replacing Δ by $\Gamma := \mathbf{Fin}_\star^{\mathrm{op}}$; such functors $\mathcal{X}\colon \Gamma^{\mathrm{op}} \to \mathcal{C}$ were introduced (in the case $\mathcal{C} := \mathcal{S}$) by Segal [Seg74] under the name *special* Γ*-spaces.*

Definition 1.3.2.1. [BM17] A dendroidal object $\mathcal{X}\colon \Omega_\pi^{\mathrm{op}} \to \mathcal{C}$ (or $\mathcal{X}\colon \Omega_{\mathrm{sym}}^{\mathrm{op}} \to \mathcal{C}$) is **covariantly fibrant** if for each n-ary tree T the inclusion of its leaves $l_1, \ldots, l_n$, induces an equivalence $\mathcal{X}_T \xrightarrow{\simeq} \prod_{i=1}^n \mathcal{X}_{l_i}$. ♢

It is clear from the definitions that

- a simplicial object $\mathcal{X}$ in $\mathcal{C}$ is reduced Segal if and only if $\mathcal{L}_\pi^\star \mathcal{X}$ is covariantly fibrant,
- every covariantly fibrant $\mathcal{X}\colon \Omega_\pi^{\mathrm{op}} \to \mathcal{C}$ maps collapse maps to equivalences.

(And similarly for the symmetric case.) Therefore Theorem 1.2.0.1 immediately implies the following result, proved by Boavida de Brito and Moerdijk [BM17, Theorem 1.1] for $\mathcal{C} = \mathcal{S}$ in the language of model categories.

Corollary 1.3.2.2. For every ∞-category $\mathcal{C}$, the functor $\mathcal{L}_\pi$ (resp. $\mathcal{L}_{\mathrm{sym}}$) induces an equivalence of ∞-categories between

- reduced Segal simplicial (resp. Γ-) objects in $\mathcal{C}$
- covariantly fibrant plane (resp. symmetric) dendroidal objects in $\mathcal{C}$. ■

1.3.3 2-Segal simplicial sets and invertible operads

Definition 1.3.3.1. [DK12, Definition 3.6.7] An operad $\mathcal{O}$ is called **invertible** if the unit map (1.1.2) and all the composition maps (1.1.1) are invertible. ♢

Remark 1.3.3.2. It follows from Remark 1.1.1.2 that an operad is invertible if and only if the unit map (1.1.2) and all $\circ_{i+1}$-compositions (1.1.3) are invertible. ♢

Remark 1.3.3.3. It follows from the condition on the unit map that if an operad is invertible then its underlying category is discrete, *i.e.*, has only identity arrows. ♢

Proposition 1.3.3.4. [DK12, Theorem 3.6.8] Fix a set B of colors. Then there is an equivalence of categories between invertible B-colored operads and 2-Segal simplicial sets $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ with $\mathcal{X}_{[1]} = B$. ☐

We can characterize invertibility of an operad in terms of its dendroidal nerve.

Lemma 1.3.3.5. Let $\mathcal{O}$ be an operad and let $\mathrm{N_d}(\mathcal{O})\colon \Omega_\pi^{\mathrm{op}} \to \mathbf{Set}$ be its dendroidal nerve. The following are equivalent:

(1) The dendroidal set $\mathrm{N_d}(\mathcal{O})$ maps all boundary preserving maps to isomorphisms.

(2) The dendroidal set $\mathrm{N_d}(\mathcal{O})$ is invertible, i.e. it inverts all collapse maps.

(3) The operad $\mathcal{O}$ is invertible. □

Proof. If $\alpha\colon T \to S$ is boundary preserving, then clearly the collapse map for S factors through the collapse map for T as $\mathrm{C} \to T \xrightarrow{\alpha} S$. Hence (1) and (2) are equivalent by the 2-out-of-3-property for isomorphisms.

The unit map (1.1.2) in Definition 1.1.1.1 is precisely the image under $\mathrm{N_d}(\mathcal{O})$ of the collapse map $\mathrm{C}_1 \to \eta$. Taking the coproduct over all the composition maps for fixed $k, n_1, \ldots, n_k \in \mathbb{N}$ yields (putting $n := \sum_{i=1}^{k} n_i$) precisely the image of the collapse map $\mathrm{C}_n \to T_k^{n_1,\ldots,n_k}$, where $T_k^{n_1,\ldots,n_k}$ is tree obtained by glueing (for all $0 \le i \le k$) the corolla C_{n_i} to the i-th leaf of the corolla C_k. Hence (2) implies (3). The converse holds because every "generalized composition map" represented by a collapse map $\mathrm{C} \to T$ can be written as the composition of unit and composition maps as in Definition 1.1.1.1. ■

Using

- the characterization of operads as Segal dendroidal sets (the non-symmetric analogue of Proposition 1.3.1.5),
- the characterization of invertible operads (Lemma 1.3.3.5),
- our main result (Theorem 1.1.2.9) in the case $\mathcal{C} = \mathbf{Set}$ and
- the corresponcence between Segal dendroidal objects and 2-Segal simplicial objects (Lemma 1.3.1.9)

we recover the following more elegant version of Proposition 1.3.3.4.

Corollary 1.3.3.6. The composition $\mathbf{sSet} \xrightarrow{\mathcal{L}_\pi^\star} \mathbf{d}_\pi\mathbf{Set} \longrightarrow \mathbf{Op}$ restricts to an equivalence of categories between the full subcategories of 2-Segal simplicial sets on one side and invertible operads on the other. ■

Before moving on, we discuss some examples of invertible operads.

Example 1.3.3.7 (Waldhausen's S-construction [Wal85]). Let A be an abelian category[2]. Consider the following operad $\mathrm{S}(A)$:

- The colors of $\mathrm{S}(A)$ are the objects of A (up to isomorphism).
- The 2-ary operations of $\mathrm{S}(A)$ are short exact sequences

$$\begin{tikzcd} {}_0x_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {}_1x_2 \arrow[d] \\ 0 \arrow[r] & {}_0x_2 \end{tikzcd}$$

(up to isomorphism) each of which is viewed as a 2-ary operation $({}_0x_1, {}_1x_2) \to {}_0x_2$.

---

[2] Waldhausen's S-construction applies in much greater generality; we restrict to abelian categories for simplicity.

- More generally, the n-ary operations $({}_0x_1, {}_1x_2, \dots, {}_{n-1}x_n) \longrightarrow {}_0x_n$ of $\mathrm{S}(A)$ are diagrams

$$\begin{tikzcd}
{}_0x_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {}_0x_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_0x_{n-2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {}_0x_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {}_0x_n \arrow[d] \\
0 \arrow[r] & {}_1x_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & {}_0x_{n-2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {}_0x_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {}_0x_n \arrow[d] \\
 & 0 \arrow[r] & \cdots \arrow[r] & {}_0x_{n-2} \arrow[r] \arrow[d] & {}_0x_{n-1} \arrow[r] \arrow[d] & {}_0x_n \arrow[d] \\
 & & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & & 0 \arrow[r] & {}_{n-2}x_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {}_{n-2}x_n \arrow[d] \\
 & & & & 0 \arrow[r] & {}_{n-1}x_n
\end{tikzcd} \tag{1.3.3}$$

  in A (up to isomorphism), where each square is required to be biCartesian, *i.e.*, both a pushout and a pullback.
- The $\circ_{i+1}$ composition of an operation

$$f\colon ({}_ix_{i+1}, \dots, {}_{j-1}x_j) \longrightarrow {}_ix_j$$

  with an operation

$$g\colon ({}_0x_1, \dots, {}_ix_j, \dots, {}_{n-1}x_n) \longrightarrow {}_0x_n$$

  (for $0 \leq i \leq j \leq n$) is the operation

$$(g \circ_{i+1} f)\colon ({}_0x_1, {}_1x_2, \dots, {}_{n-1}x_n) \longrightarrow {}_0x_n$$

  whose associated diagram (1.3.3) is uniquely characterized by the fact that it extends the corresponding diagrams for f and g.

It is not hard to verify that $\mathrm{S}(A)$ is a well defined operad; it is invertible because, for each fixed $0 \leq i \leq j \leq n$, each operation (1.3.3) arises as the composition $g \circ_{i+1} f$ for a unique pair of operations (f, g) as above. Under the equivalence of Theorem 3 this operad corresponds to Waldhausen's S-construction which is the 2-Segal simplicial set $\mathrm{S}(A)\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ that maps $[n] \in \Delta$ to the set of isomorphism classes of diagrams (1.3.3) with face/degeneracy maps given by simultaneously omitting/duplicating rows and columns. If instead of working up to isomorphism we keep track of those isomorphisms, we get an invertible operad/2-Segal object in groupoids rather than sets. ◊

Remark 1.3.3.8. Let $\mathcal{X}$ be an invertible Segal dendroidal object. Let T be the closed n-corolla (i.e. the grafting of n many 0-corollas on top of a n-corolla). We have two maps

$$\mathcal{X}(\mathrm{C}_0) \xleftarrow{\simeq} \mathcal{X}(T) \xrightarrow{\simeq} \mathcal{X}(\mathrm{C}_n) \times_{\mathcal{X}(\eta)^n} \mathcal{X}(\mathrm{C}_0)^n$$

which are equivalences by invertibility and the Segal conditions respectively. In the example where $\mathcal{X} = \mathrm{S}(A)$ is the Waldhausen S-construction of an abelian category A, the groupoid $\mathcal{X}(\mathrm{C}_0) \simeq \{0\}$ is trivial, hence this condition says precisely that a flag (1.3.3) of length n with trivial subquotients is trivial. Note, however, that in general a flag is not determined by its subquotients, which would be the Segal condition $\mathcal{X}(\mathrm{C}_n) \xrightarrow{\simeq} \mathcal{X}(\eta)^n$. ◊

Example 1.3.3.9 (E_k-operads). The commutative operad E_∞ (viewed as a symmetric operad) has a contractible space of operations in each degree, hence is invertible for trivial reasons; it corresponds to the constant Γ-space on a point. Its underlying non-symmetric operad is the associative operad which is invertible and corresponds to the constant simplicial space on a point. For all other $1 \leq k < \infty$, the operad E_k of little k-cubes is easily seen to *not* be invertible. ◊

Example 1.3.3.10. Each monoid M (multiplicatively written) gives rise to an invertible operad $\mathrm{N}(M)$ as follows: The set of colors is M. The set of n-ary operations is M^n, where each tuple $({}_0m_1, \ldots, {}_{n-1}m_n) \in M^n$ is viewed as an operation

$$({}_0m_1, \ldots, {}_{n-1}m_n) \longrightarrow {}_0m_1 \cdots {}_{n-1}m_n =: {}_0m_n$$

and is, for each $0 \leq i \leq j \leq n$, the $\circ_{i+1}$-composition of

$$({}_im_{i+1}, \ldots, {}_{j-1}m_j) \longrightarrow {}_im_j$$

and

$$({}_0m_1, \ldots, {}_{i-1}m_i, {}_im_j, {}_jm_{j+1}, \ldots, {}_{n-1}m_n) \longrightarrow {}_0m_n.$$

If M is abelian then the operad $\mathrm{N}(M)$ can be canonically enhanced to a symmetric operad. Under the equivalence of Theorem 3, the operad $\mathrm{N}(M)$ corresponds to the nerve $\mathrm{N}(M)\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ which is not just 2-Segal but Segal.

This example can be categorified to interpret each monoidal ∞-groupoid as an invertible ∞-operad; see Example 1.3.4.3 and Remark 1.3.4.4. ◊

1.3.4 2-Segal simplicial spaces and invertible ∞-operads

As a direct consequence of Theorem 1.1.2.9 and Lemma 1.3.1.9 we obtain the following comparison result.

Corollary 1.3.4.1. Composition with $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$ induces an equivalence between the ∞-category of 2-Segal simplicial spaces and the ∞-category of invertible Segal dendroidal spaces. ■

The goal of this Section 1.3.4 is to give an interpretation of this result by identifying the ∞-category of invertible Segal dendroidal spaces as a full subcategory of the ∞-category of complete Segal dendroidal spaces. We treat the latter as a model for (non-symmetric) ∞-operads (in analogy to results due to Cisinski and Moerdijk [CM13] in the symmetric case) so that we can rephrase Corollary 1.3.4.2 as follows:

Corollary 1.3.4.2. Composition with $\mathcal{L}_\pi\colon \Omega_\pi \to \Delta$ induces an equivalence between the ∞-category of 2-Segal simplicial spaces and the ∞-category of invertible (non-symmetric) ∞-operads. ■

Example 1.3.4.3. Every monoidal category $(\mathcal{M}, \otimes)$ gives rise to an operad $\mathcal{O}_\mathcal{M}$ in groupoids: Its groupoid of colors $\mathcal{O}_\mathcal{M}(\eta) := \mathcal{M}^\simeq$ is the groupoid core of $\mathcal{M}$ and its groupoid of 1-ary operations is the groupoid $\mathcal{O}_\mathcal{M}(1) := \mathrm{Fun}(\Delta^1, \mathcal{M})^\simeq$ of arrows in $\mathcal{M}$. The groupoid $\mathcal{O}_\mathcal{M}(n)$ of n-ary operations is the groupoid of arrows $\bullet_1 \otimes \cdots \otimes \bullet_n \to \bullet$, *i.e.*, the pullback

$$\begin{tikzcd}
\mathcal{O}_\mathcal{M}(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}_\mathcal{M}(1) \arrow[d, "s"] \\
\mathcal{O}_\mathcal{M}(\eta)^n \arrow[r, "\otimes"'] & \mathcal{O}_\mathcal{M}(\eta)
\end{tikzcd} \tag{1.3.4}$$

Composition in the operad $\mathcal{O}_\mathcal{M}$ is induced by composition of arrows in $\mathcal{M}$. The operad $\mathcal{O}_\mathcal{M}$ is invertible if and only if all arrows in the underlying category $\mathcal{M}$ are invertible, *i.e.*, if and only if $\mathcal{M}$ is a monoidal *groupoid*. In this case, the right vertical map in (1.3.4)—which sends each arrow to its source—is an equivalence; hence the same is true for the left vertical map. This amounts to saying that, viewed as a Segal dendroidal groupoid, $\mathcal{O}_{\mathcal{M}}$ is covariantly fibrant.

Under the equivalence of Corollary 1.3.2.2, the operad $\mathcal{O}_{\mathcal{M}}$ corresponds to the complete Segal simplicial space obtained by interpreting $\mathcal{M}$ as an ∞-category with a single object, $\mathcal{M}$ as its space of arrows and composition given by $\otimes$. This generalizes Example 1.3.3.10, where the monoidal groupoid $\mathcal{M}$ is discrete. ♢

Remark 1.3.4.4. In view of Example 1.3.4.3 and considering that complete reduced Segal simplicial spaces are a model for monoidal ∞-groupoids[3], Corollary 1.3.2.2 allows us to interpret "being covariantly fibrant" as the property which characterizes those ∞-operads which are monoidal ∞-groupoids. ♢

The theory of complete Segal dendroidal spaces was developed by Cisinski and Moerdijk [CM13] and spelled out in detail for *symmetric* dendroidal spaces. They prove that complete Segal symmetric dendroidal spaces are a model for symmetric ∞-operads (see Theorem 1.3.4.6 below). We briefly retrace their main definitions in the world of non-symmetric operads. We will use the resulting model category of complete Segal *planar* dendroidal spaces (or rather, its underlying ∞-category) as a model for (non-symmetric) ∞-operads.

Construction 1.3.4.5. [CM13, Sections 5 and 6] We build the simplicial model category $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{cS}}$ of **complete Segal dendroidal spaces** (also called **dendroidal Rezk model category**) as constructed by Cisinski and Moerdijk in the symmetric case:

Take the Reedy model structure[4] on the functor category $\mathbf{dsSet} := [\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]$ and then Bousfield-localize [Lur09, Proposition A.3.7.3] two times:

(1) at the Segal core inclusions $\mathrm{Sc}[T] \longrightarrow \Omega_\pi[T]$ and

(2) at the maps $\Omega_\pi[T] \otimes J_d \longrightarrow \Omega_\pi[T]$, where J_d is the dendroidal nerve of the category $\bullet \overset{\cong}{\longrightarrow} \bullet$ with two objects and a single isomorphism between them. ♢

The Reedy model category $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{Reedy}}$ has a canonical simplicial enrichment [RV14, Theorem 10.3] which is maintained by the Bousfield localization processes [Lur09, Proposition A.3.7.3]. Therefore we can construct what we call the **∞-category of ∞-operads** as the simplicial nerve of the fibrant-cofibrant objects:

$$\mathcal{O}\mathrm{p} := \mathrm{N}_\Delta([\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{cS}}^\circ)$$

The name is justified by the following result.

Theorem 1.3.4.6. [CM13, Corollary 6.8] The inclusion $\mathbf{dSet} \hookrightarrow [\Omega_{\mathrm{sym}}, \mathbf{sSet}]_{\mathrm{cS}}$ is a left Quillen equivalence between the model category of symmetric ∞-operads as defined by Cisinski and Moerdijk [CM11] and the model category of complete Segal symmetric dendroidal spaces. □

Definition 1.3.4.7. We denote by $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{iS}}$ the Bousfield localization of $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{cS}}$ at the collapse maps

$$\Omega_\pi[\mathrm{C}_n] \longrightarrow \Omega_\pi[T]$$

for each n-ary tree T; we call it the **model category of invertible Segal dendroidal spaces**. We denote by

$$\mathrm{i}\mathcal{O}\mathrm{p} := \mathrm{N}_\Delta([\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{iS}}^\circ)$$

the corresponding **∞-category of invertible ∞-operads** ♢

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[3] For instance, Lurie [Lur17, Definition 4.1.3.6] defines (non-symmetric) monoidal ∞-categories as those co-Cartesian fibrations over Δ^{op} which under the straightening/unstraightening equivalence correspond to reduced Segal simplicial ∞-categories; monoidal ∞-groupoids are then precisely those that take values in ∞-groupoids rather than ∞-categories.

[4] Cisinski and Moerdijk actually use a generalized version of the Reedy model structure since the category Ω_{sym} of symmetric rooted trees is not a Reedy category (unlike Ω_π, which is).

Remark 1.3.4.8. It is immediate from the characterization of Bousfield localization that $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]^\circ_{\mathrm{iS}}$ is a full simplicial subcategory of $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]^\circ_{\mathrm{cS}}$. Hence the ∞-category $\mathrm{i}\mathcal{O}\mathrm{p}$ of invertible ∞-operads is a full subcategory of the ∞-category $\mathcal{O}\mathrm{p}$ of (all) ∞-operads. ♢

Lemma 1.3.4.9. The ∞-category $\mathrm{i}\mathcal{O}\mathrm{p}$ of invertible ∞-operads is equivalent to the full subcategory of $\mathrm{Fun}(\Omega_\pi^{\mathrm{op}}, \mathcal{S})$ consisting of those dendroidal spaces $\mathcal{X}\colon \Omega_\pi^{\mathrm{op}} \to \mathcal{S}$ which are invertible Segal and satisfy the following **completeness** condition:

- For each tree T, the map $\Omega_\pi[T] \otimes J_d \to \Omega_\pi[T]$ from Construction 1.3.4.5 induces an equivalence
$$\mathrm{Hom}(\Omega_\pi[T] \otimes J_d, \mathcal{X}) \xrightarrow{\simeq} \mathcal{X}_T.$$
in $\mathcal{S}$. ☐

To prove Lemma 1.3.4.9 we use the following result.

Proposition 1.3.4.10. [Lur09, Proposition 4.2.4.4.] Let $\mathbb{A}$ be a combinatorial simplicial model category, D a small simplicial category and S a simplicial set equipped with an equivalence $\mathfrak{C}[S] \xrightarrow{\simeq} D$. Then the induced map

$$\mathrm{N}_\Delta([D, \mathbb{A}]^\circ) \longrightarrow \mathrm{Fun}(S, \mathrm{N}_\Delta(\mathbb{A}^\circ))$$

is a categorical equivalence of simplicial sets. ☐

Remark 1.3.4.11. In Proposition 1.3.4.10 it does not matter whether we equip $[D, \mathbb{A}]$ with the injective, projective or (if D is a Reedy category) with the Reedy model structure, since they are all Quillen equivalent [Lur09, Remark A.2.9.23]. ♢

Proof (of Lemma 1.3.4.9)**.** We specialize Proposition 1.3.4.10 to $\mathbb{A} := \mathbf{sSet}$ and $D := \Omega_\pi^{\mathrm{op}}$ (seen as a discrete simplicial category); we put $S := \mathrm{N}(\Omega_\pi^{\mathrm{op}}) = \mathrm{N}_\Delta(\Omega_\pi^{\mathrm{op}})$ equipped with the adjunction counit $\mathfrak{C}[\mathrm{N}_\Delta(\Omega_\pi^{\mathrm{op}})] \xrightarrow{\simeq} \Omega_\pi$. We obtain an equivalence

$$\mathrm{N}_\Delta([\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]^\circ_{\mathrm{Reedy}}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{N}(\Omega_\pi^{\mathrm{op}}), \mathcal{S}) \tag{1.3.5}$$

of ∞-categories. Passing to Bousfield localizations replaces the simplicial category $[\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]^\circ_{\mathrm{Reedy}}$ by the full subcategory of the new fibrant-cofibrant objects. Therefore the equivalence (1.3.5) restricts to an equivalence between $\mathrm{i}\mathcal{O}\mathrm{p} := \mathrm{N}_\Delta([\Omega_\pi^{\mathrm{op}}, \mathbf{sSet}]^\circ_{\mathrm{iS}})$ and some full subcategory of $\mathrm{Fun}(\mathrm{N}(\Omega_\pi^{\mathrm{op}}), \mathcal{S})$ whose objects are determined by the fibrancy conditions in the three localization steps. Each of these steps corresponds precisely to one of the three conditions (invertibility, Segal, completeness) in Lemma 1.3.4.9. ■

We will now see that the completeness condition in Lemma 1.3.4.9 is redundant.

Lemma 1.3.4.12. An invertible Segal dendroidal space is automatically complete. ☐

Proof. A dendroidal Segal space $\mathcal{X}\colon \Omega_\pi^{\mathrm{op}} \to \mathcal{S}$ is complete if and only the underlying simplicial Segal space $\mathcal{X}\big|_{\Delta^{\mathrm{op}}}\colon \Delta^{\mathrm{op}} \subset \Omega_\pi^{\mathrm{op}} \to \mathcal{S}$ (obtained by restricting to linear trees) is complete. If $\mathcal{X}$ is invertible then $\mathcal{X}\big|_{\Delta^{\mathrm{op}}}$ is constant, hence trivially complete. ■

Lemma 1.3.4.12 motivates the name "invertible Segal" (rather than "invertible complete Segal") in Definition 1.3.4.7 and completes the transition from Corollary 1.3.4.1 to Corollary 1.3.4.2.

Remark 1.3.4.13. The story of Section 1.3.4 can be retold, mutatis mutandis, in the world of symmetric ∞-operads, symmetric dentroidal spaces and Γ-spaces; hence we obtain an equivalence between the ∞-categories of

- 2-Segal Γ-spaces and
- invertible symmetric ∞-operads. ♢

Remark 1.3.4.14. Example 1.3.4.3 and Remark 1.3.4.4 have obvious analogs in the world of symmetric ∞-operads and reduced Segal (a.k.a. special) Γ-spaces. ♢

1.4 Variant: Cyclic operads and cyclic objects

We recall the definition of Connes' cyclic category Λ.

Definition 1.4.0.1. [Con83] To each natural number $n \in \mathbb{N}$ corresponds an object $[n] \in \Lambda$ which we interpret as the unit circle S^1 in the complex plane with $n+1$ many equidistant marked points. The morphisms are homotopy classes of weakly monotone maps $S^1 \to S^1$ of degree 1 that send marked points to marked points. ◊

Remark 1.4.0.2. We fix the inclusion $\Delta \hookrightarrow \Lambda$ which arranges the $n+1$ many elements of an object $[n] \in \Delta$ as marked points on a circle. This inclusion is dense and faithful but not full. ◊

We define the category Ω_{cyc} of **plane *rootable* trees**. In analogy to how Ω_π is a full subcategory of the category **Op** of operads, we define Ω_{cyc} as a full subcategory of the category of cyclic operads whose definition due to Getzler and Kapranov[5] [GK95] we now recall briefly.

Definition 1.4.0.3. A **cyclic structure** on an operad $(\mathcal{O}, O, \circ)$ consists of

- an involution $(-)^\vee \colon O \to O$ on colors (called **duality**) and
- a system of **rotation isomorphisms**

$$\mathcal{O}(x_1, \ldots, x_n; y) \xrightarrow{\cong} \mathcal{O}(y^\vee, x_1, \ldots, x_{n-1}; x_n^\vee)$$

  which is compatible with the composition of operations;

such that for each $n \in \mathbb{N}$ the $(n+1)$-fold composition

$$\begin{aligned}\mathcal{O}(x_1, \ldots, x_n; y) \xrightarrow{\cong} \mathcal{O}(y^\vee, x_1, \ldots, x_{n-1}; x_n^\vee) \xrightarrow{\cong} \mathcal{O}(x_n^\vee, y^\vee, x_1, \ldots, x_{n-2}; x_{n-1}^\vee)\\ \xrightarrow{\cong} \cdots \xrightarrow{\cong} \mathcal{O}(x_2, \ldots, x_n, y^\vee; x_1^\vee) \xrightarrow{\cong} \mathcal{O}(x_1, \ldots, x_n; y)\end{aligned}$$

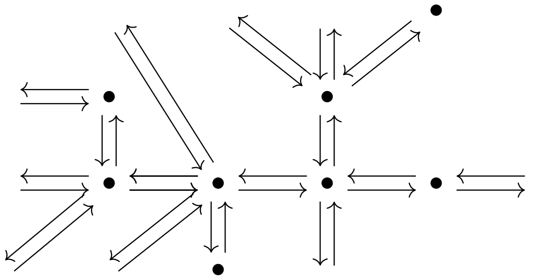
of rotation isomorphisms is equal to the identity.

A **cyclic operad** is an operad together with a cyclic structure. The cyclic operads are assembled into a category **cycOp** where the morphisms are required to be compatible with the additional structure in the obvious way. ◊

Remark 1.4.0.4. We have an adjunction $\mathbf{Op} \rightleftarrows \mathbf{cycOp}$ where the right adjoint forgets the cyclic structure and the left adjoint adds a cyclic structure freely. ◊

Definition 1.4.0.5. A **plane rootable tree** consists of vertices and (unoriented) edges arranged in the plane, where an edge can connect two vertices or go to infinity in one or (in the case of the unique tree η with no vertices) both directions. We require our trees to have at least one external edge (this is what we mean by "rootable"). We think of each unoriented edge as a pair of anti-parallel arrows. ◊

Example 1.4.0.6. A typical example of a plane rootable tree looks as follows:



◊

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[5] Getzler and Kapranov introduced cyclic operads in their mono-colored and symmetric version.

We call an arrow a **leaf** if comes from infinity and a **root** if it goes to infinity. An arrow a is called a **direct predecessor** of an arrow b (and b is then a **direct successor** of a) if there is a vertex which is both the target $t(a)$ of a and the source $s(b)$ of b. We say a is a **predecessor** of b (or b is a successor of a), if a is an iterated direct predecessor of b (this includes the case $a = b$). The **arity** of a tree (resp. a vertex) is n, where $n + 1$ is the number of arrows leaving (or, equivalently, entering) the tree (resp. the vertex).

Remark 1.4.0.7. For every arrow b in a tree T, the set of predecessors of b in T forms a plane rooted tree (the root is b itself). In particular there is a preferred linear order (clockwise along the boundary) on the set of those leaves a of T which are predecessors of b. ♢

Construction 1.4.0.8. Each plane tree T gives rise to a cyclic operad (also denoted T) as follows:

- Each arrow is a color.
- Each pair (v, a) consisting of an n-ary vertex $v \in T$ and an arrow a starting in v gives rise to an n-ary operation
$$v_a \colon (a_1, \ldots, a_n) \longrightarrow a$$
where the a_i's are the direct predecessors of a (hence $t(a_i) = v$) in clockwise order. All other operations are freely generated by these v_a's.
- The involution on the colors exchanges the two anti-parallel arrows associated to a single edge.
- The rotation isomorphisms are given on generators by $v_a \mapsto v_{a_n^\vee}$. ♢

Definition 1.4.0.9. We define the category $\Omega_{\text{cyc}} \subset$ **cycOp** of plane rootable trees to be the full subcategory spanned by the cyclic operads T constructed as above. A cyclic dendroidal object in an ∞-category $\mathcal{C}$ is a functor $\Omega_{\text{cyc}}^{\text{op}} \to \mathcal{C}$. ♢

Remark 1.4.0.10. Our category Ω_{cyc} is very close to the category of plane unrooted trees introduced by Joyal and Kock [JK09]; the only difference is that we require our trees to have at least one external edge. For instance, we do not allow the tree $\bullet$ which consists only of a single vertex, since this tree can not be interpreted as a cyclic operad in a meaningful way. ♢

Remark 1.4.0.11. The free-cyclic-structure functor **Op** → **cycOp** induces an inclusion $\Omega_\pi \to \Omega_{\text{cyc}}$ which replaces each edge with two anti-parallel arrows and forgets the root. ♢

Remark 1.4.0.12. The cyclic operad corresponding to the tree η (which has no vertices and exactly two mutually anti-parallel arrows) consists of two colors which are dual to each other and no non-identity operations. This cyclic operad η has an involution given by exchanging the two colors, i.e. the two arrows. A morphism $\eta \to \mathcal{O}$ to some cyclic operad $\mathcal{O}$ corresponds to a color of $\mathcal{O}$; the involution on the colors of $\mathcal{O}$ is induced by the involution on η. ♢

Remark 1.4.0.13. It is easy to check that an operation in the cyclic operad $T \in \Omega_{\text{cyc}}$ is uniquely determined by its input and output colors. Hence a map $S \to T$ between such operads is uniquely determined by the value at each arrow. Such a map would not, however, be determined by its values on unoriented edges; for instance, every unoriented edge e of a tree T gives rise to two different maps $\eta \to T$ in Ω_{cyc} corresponding to the two mutually dual colors described by e.

If one were only interested in mono-colored cyclic operads or, more generally, cyclic operads with trivial duality (i.e. every color is self-dual), then it would be enough to consider unoriented edges. This point of view is taken by Hackney-Robertson-Yau [HRY19]. ♢

Definition 1.4.0.14. A map of plane rootable trees is called **boundary preserving** if it maps leaves to leaves and roots to roots. A **collapse map** in Ω_{cyc} is a boundary preserving map $\mathrm{C} \to T$ out of a corolla. A cyclic dendroidal object $\Omega_{\text{cyc}}^{\text{op}} \to \mathcal{C}$ in some ∞-category $\mathcal{C}$ is called **invertible** if it maps all collapse maps to equivalences in $\mathcal{C}$. ♢

As the notation suggests, the category Ω_{cyc} of plane rootable trees has a close relationship to the cyclic category: the latter is a localization of the former as we will see next.

Construction 1.4.0.15 (Covariant description of $\mathcal{L}_{\text{cyc}}$)**.** Analogously to the case of plane rooted trees, a plane rootable tree partitions the plane into "areas" which are arranged clockwise around a circle. This assignment is a functor $\mathcal{L}_{\text{cyc}} \colon \Omega_{\text{cyc}} \to \Lambda$ which extends the functor $\mathcal{L}_{\pi} \colon \Omega_{\pi} \to \Delta$. $\Diamond$

Construction 1.4.0.16 (Contravariant description of $\mathcal{L}_{\text{cyc}}$)**.** Using the self-duality $\Lambda \cong \Lambda^{\text{op}}$ (which interchanges marked points and intervals on a circle) we can define the functor $L \colon \Omega_{\text{cyc}} \to \Lambda^{\text{op}}$ instead:

A tree T gets mapped to its set of leaves which are naturally arranged around a circle. The image of a morphism $\alpha \colon S \to T$ sends each leaf a of T to the unique leaf b of S such that $\alpha(b)$ is a successor of a. This assignment does not yet uniquely determine $L\alpha$ as a morphism in Λ; we still need to specify a linear order on the pre-images $(L\alpha)^{-1}(b)$ (for every leaf b of S) but this is taken care of by Remark 1.4.0.7. $\Diamond$

Remark 1.4.0.17. By combining the ideas from Section 1.4 and Section 1.1.3 we can construct a category of (non-plane) rootable trees as a full subcategory of cyclic symmetric operads$^{6)}$. The corresponding functor $\mathcal{L}_{\text{abs}} \colon \Omega_{\text{abs}} \to \mathbf{Fin}^{\text{op}}_{\neq\varnothing}$ maps a tree to its non-empty set of leaves (i.e. incoming arrows). $\Diamond$

Proposition 1.2.1.3 still holds for $\mathcal{L} \in \{\mathcal{L}_{\text{cyc}}, \mathcal{L}_{\text{abs}}\}$ with essentially the same proof, hence Lemma 1.2.1.1 yields the following cyclic version of Theorem 1.2.0.1:

Theorem 1.4.0.18. The functor $\mathcal{L}_{\text{cyc}} \colon \Omega_{\text{cyc}} \to \Lambda$ (resp. $\mathcal{L}_{\text{abs}} \colon \Omega_{\text{abs}} \to \mathbf{Fin}^{\text{op}}_{\neq\varnothing}$) exhibits Λ (resp. $\mathbf{Fin}^{\text{op}}_{\neq\varnothing}$) as an ∞-categorical localization of Ω_{cyc} (resp. Ω_{abs}) at the set of collapse maps. ■

Corollary 1.4.0.19. The classifying space of Ω_{cyc} is $\mathrm{B}S^1$. □

Proof. Follows immediately from Theorem 1.4.0.18 because the classifying space of the cyclic category Λ is known to be $\mathrm{B}S^1$ [Con83, Theorem 10]. ■

Remark 1.4.0.20. Analogously to Corollary 1.3.4.1 one can show that the functor $\mathcal{L}_{\text{cyc}}$ induces an equivalence between the ∞-categories of

- 2-Segal cyclic objects and
- invertible cyclic Segal dendroidal objects

in any ∞-category $\mathcal{C}$, where 2-Segal/Segal are defined either as the obvious analogs of Definition 1.3.1.2 and Definition 1.3.1.6 or, alternatively, by referring to the underlying simplicial/dendroidal object.

Unfortunately, there is currently no result in the literature exhibiting (complete) Segal cyclic dendroidal spaces as a model for cyclic ∞-operads. One promising approach to resolve this issue is proposed by Drummond-Cole and Hackney who construct [DH18, Theorem 6.5] a Dwyer–Kan type model structure on the category of simplicially enriched cyclic operads$^{7)}$ and conjecture[DH18, Remark 6.9] that it should be Quillen equivalent to a "complete Segal space"-type model structure on cyclic dendroidal simplicial sets lifted from the complete Segal model structure of Cisinski and Moerdijk. Conditional on their conjecture, we can then say that 2-Segal cyclic spaces are equivalent to invertible cyclic ∞-operads. $\Diamond$

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6) Such operads have both a cyclic and a symmetric structure which are compatible when regarding the symmetric group $\mathfrak{S}_n$ and the cyclic group $\mathbb{Z} / (n+1)$ as a subgroup of $\mathfrak{S}_{n+1}$.

7) Drummond-Cole and Hackney call *non-Σ positive cyclic operads* what we simply call cyclic operads.

Chapter 2

Higher Segal spaces via higher excision

The main conceptual framework which informs our approach in this chapter is a version for the simplex category of the Goodwillie–Weiss [Wei99; GW99] manifold calculus. In Section 2.1 we explain a system of heuristic analogies between manifold calculus (in its version described by Boavida de Brito and Weiss [BW13]) and a "manifold calculus" on Δ. While the mathematics in the rest of the chapter stands on its own, it is the author's opinion that these informal analogies to manifold calculus can be very helpful when digesting the definitions and building intuition. Interestingly, they also explain how one might have guessed the definition of higher Segal spaces without knowing about cyclic polytopes. One practical upshot of the analogy to manifold calculus is that it inspires the definition of *polynomial* simplicial objects, a notion which is implied by higher weak excision (while being, *a priori*, weaker) and which can be compared more easily to the higher Segal conditions.

In Section 2.2 we recall basic definitions and facts about the categories Δ and Λ, (co)Cartesian and strongly (co)Cartesian cubes, as well as general notions of excision, weak excision and descent. In Section 2.3, we explicitly classify strongly Cartesian and biCartesian cubes in Δ and in Λ. In Section 2.4 we explain a descent theory on Δ and study polynomial simplicial objects in this framework. In Section 2.5 we show that polynomial simplicial objects agree with weakly excisive ones; our key arguments here are a version of the ones in [FGK+19] repackaged in a way which directly generalizes to arbitrary dimensions. The main theorem (Theorem 2.6.2.2)—comparing higher Segal conditions with weak excision—is proved in the last section (Section 2.6) by considering a series of descent conditions which interpolate between the higher Segal conditions and the conditions of being polynomial.

2.1 A "manifold calculus" for the simplex category

A contravariant functor $\mathcal{X}$ defined on the topological (*i.e.*, ∞-) category **Man** of smooth d-manifolds and smooth embeddings is usually called *polynomial of degree* ≤ 1 if its value on a manifold M can be computed by cutting M up into smaller open pieces, evaluating $\mathcal{X}$ piece by piece and then reassembling the values. More precisely, for each pair of disjoint closed subsets subsets $A_0, A_1 \subset M$, one requires the canonical map

$$\mathcal{X}(M) \longrightarrow \mathcal{X}(M \setminus A_0) \times_{\mathcal{X}(M \setminus A_0 \cup A_1)} \mathcal{X}(M \setminus A_1)$$

to be an equivalence.

Boavida de Brito and Weiss [BW13] show that polynomial functors of degree ≤ 1 are precisely the (homotopy) sheaves on **Man** for the Grothendieck topology $\mathcal{J}_1$ of open covers. More generally, they consider a hierarchy $\mathcal{J}_k$ of Grothendieck topologies on **Man** (with $k \geq 1$), where $\mathcal{J}_k$ consists of those open covers (called *k-covers*) which have the property that every set of k (or fewer) points is contained in some open set of the cover. The manifold calculus of Boavida de

Brito and Weiss is concerned with the systematic study of sheaves on $(\mathbf{Man}, \mathcal{J}_k)$. They introduce the following classes of open covers:

(1) the class $\mathcal{J}_k^{\mathrm{h}}$ consists of open covers of the form

$$\{M \setminus A_i \hookrightarrow M \mid i = 0, \ldots, k\} \tag{2.1.1}$$

for pairwise disjoint closed subsets $A_0, \ldots, A_k \subset M$ of M.

(2) the class $\mathcal{J}_k^{\circ}$ consists of *good* k-covers, *i.e.*, k-covers with the property that every finite intersection of open sets is diffeomorphic to a disjoint union of at most k balls.

While the classes $\mathcal{J}_k^{\mathrm{h}}$ and $\mathcal{J}_k^{\circ}$ are not Grothendieck topologies anymore, they are so called *coverages*, hence they admit a well-behaved theory of descent and sheaves. Sheaves for the coverage $\mathcal{J}_k^{\mathrm{h}}$ are called *polynomial functors of degree* $\leq k$. One of the main results of Boavida de Brito and Weiss in this context is the following theorem:

Theorem 2.1.0.1. [BW13, Theorem 5.2 and Theorem 7.2] The coverages $\mathcal{J}_k$, $\mathcal{J}_k^{\mathrm{h}}$ and $\mathcal{J}_k^{\circ}$ define the same class of sheaves on **Man**. ☐

We shall now describe a similar theory for simplicial objects, *i.e.*, presheaves on the simplex category Δ (see Section 2.2.1 for the notation). It turns out that the following list of analogies is useful; we put terms coming from the language of manifold in quotes to emphasize that they should be thought of heuristically:

- We think of the object $[n] = \{0, \ldots, n\} \in \Delta$ as a "manifold" with "points" given by pairs $(x-1, x)$ with $x = 1, \ldots, n$.
- An "open subset" of $[n]$ is simply an ordinary subset $U \subseteq \{0, \ldots, n\}$; it contains the "points" $(x-1, x)$ such that $\{x-1, x\} \subseteq U$.
- We say that two "open subsets" U, U' of the "manifold" $[n]$ are "disjoint" if they are disjoint as subsets of $[n]$; note that this is a stronger condition than requiring U and U' to share no "point".
- A "closed set" A of $[n]$ is an ordinary subset of $A \subseteq [n]$; it contains all the points not contained in its complement $[n] \setminus A \subseteq [n]$ (viewed as an "open set"); explicitly, A contains all "points" $(x-1, x)$ with $x \in A$ or $x - 1 \in A$.
- We say that two "closed sets" $A, A' \subseteq [n]$ are "disjoint" if they share no "point"; note that this is stronger than being disjoint as subsets of $[n]$.
- Each "point" $p = (x-1, x)$ has a unique minimal "open neighborhood" given by the subset $U^p = \{x-1, x\} \subseteq [n]$, which we think of as a very small "open ball" around the "point" p.

Armed with this intuition, we can define analogs of the coverings $\mathcal{J}_k^{\mathrm{h}}$ and $\mathcal{J}_k^{\circ}$ in the simplex category:

(1) The open covers (2.1.1) can be translated to Δ by putting everything in quotation marks: For every collection $A_0, \ldots, A_k$ of "nonempty and pairwise disjoint closed subsets" of the "manifold" $[n]$, we can define the "open cover"

$$\{[n] \setminus A_i \hookrightarrow [n] \mid i = 0, \ldots, k\} \tag{2.1.2}$$

of $[n]$. See also Section 2.4.2.

(2) Heuristically[1)], one way to produce good k-covers of a manifold M is as follows: Fix a Riemannian metric on M and, for every tuple $p = (p_1, \ldots, p_k)$ of k points in M, choose very small (with respect to the geodesic distance between the points p_i) balls $U_i^p \ni p_i$. Then the collection $\left\{\dot{\bigcup}_{i=1}^{k} U_i^p \,\middle|\, p \in M^k\right\}$ is a k-good cover of M.

[1)] See Proposition 2.10 in [BW13] for an actual proof.

In our analogy, every "point" p of a "manifold" $[n] \in \Delta$ has a canonical/minimal "open ball" U^p surrounding it. Hence each $[n] \in \Delta$ has a canonical "good k-cover" containing all those "open subsets" of $[n] \in \Delta$ that can be written as union of the form

$$\dot{\bigcup}_{i=1}^{k} U^{p_i},$$

where $p_1, \ldots, p_k$ are "points" of the "manifold" $[n]$ with "pairwise disjoint neighborhoods" U^{p_i}. See also Section 2.6.1.

Inspired by the analogy, we call a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$ **polynomial of degree** $\leq k$ if it is a sheaf for the "open covers" of type (1) (see Definition 2.4.2.1).

The following easy observation was the author's original motivation for this line of inquiry because it shows on one hand that the canonical "good k-covers" are a meaningful concept and on the other hand that a "manifold calculus" of Δ can be a powerful organizational principle for higher Segal spaces.

Observation 2.1.0.2. Sheaves on Δ with respect to the canonical "good k-covers" of (2) are precisely the lower $(2k-1)$-Segal spaces of Dyckerhoff and Kapranov. □

The notion of polynomial simplicial objects might be a bit unsatisfying because its very definition relies on an informal analogy to manifold calculus; without this analogy, the "open covers" (2.1.1) might seem a bit mysterious and devoid of intrinsic meaning. We will clarify this issue by showing that a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$ is polynomial of degree $\leq k$ if and only if it is weakly k-excisive (see Theorem 2.5.1.1). In this light, our main result (Theorem 2.6.2.2) relating lower $(2k-1)$-Segal objects with weakly k-excisive functors should be seen as a discrete analog of Theorem 2.1.0.1 of Boavida de Brito and Weiss.

We will not spell out the whole story for $2k$-Segal objects since it is very similar. Let us just say that one should now consider a "manifold calculus" not on the simplex category Δ but on Connes' cyclic category Λ, where the "manifold" $[n] = \{0, \ldots, n\}$ now has an additional "point" given by $(n, 0)$.

2.2 Preliminaries

2.2.1 The simplex category

The **augmented simplex category** Δ_+ is the category of finite linearly ordered sets and order preserving (*i.e.*, weakly monotone) maps between them. The **simplex category** $\Delta \subset \Delta_+$ is the full subcategory spanned by the *nonempty* finite linearly ordered sets. Every object in Δ is isomorphic, by a unique isomorphism, to a standard ordinal of the form $[n] := \{0 < 1 < \cdots < n\}$ for some $n \in \mathbb{N}$; when convenient can we therefore identify Δ with its skeleton spanned by $\{[n] \mid n \in \mathbb{N}\}$.

Definition 2.2.1.1. A **simplicial object** in an $(\infty\text{-})$category $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. ◊

The augmented simplex category has a monoidal structure

$$\star\colon \Delta_+ \times \Delta_+ \longrightarrow \Delta_+,$$

given by left-to-right concatenation or **join** of linearly ordered sets. Explicitly we have

$$\{a_0 < \cdots < a_n\} \star \{b_0 < \cdots < b_m\} := \{a_0 < \cdots < a_n < b_0 < \cdots < b_m\};$$

the monoidal unit for $\star$ is the empty set $\varnothing \in \Delta_+$. We use the convention $[-1] := \varnothing \in \Delta_+$ and $[n \setminus i] := \{i+1 < \cdots < n\}$ for all $-1 \leq i \leq n$ so that we always have $[n] = [i] \star [n \setminus i]$. Given a simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$, the **left path object** $P^{\triangleleft}\mathcal{X}$ and the **right path object** $P^{\triangleright}\mathcal{X}$ are defined as the compositions

$$P^{\triangleleft}\mathcal{X}\colon \Delta^{\mathrm{op}} \xrightarrow{[0]\star -} \Delta^{\mathrm{op}} \xrightarrow{\mathcal{X}} \mathcal{C} \qquad \text{and} \qquad P^{\triangleright}\mathcal{X}\colon \Delta^{\mathrm{op}} \xrightarrow{-\star[0]} \Delta^{\mathrm{op}} \xrightarrow{\mathcal{X}} \mathcal{C},$$

respectively.

A morphism $f\colon [m] \to [n]$ in Δ is called **left active** if it preserves the minimal element (*i.e.*, $f(0) = 0$) and **right active** if it preserves the maximal element (*i.e.*, $f(m) = n$). We call f **active** if it is both left and right active. Denote by $\Delta^{\min}$, $\Delta^{\max}$ and $\Delta^{\mathrm{act}} := \Delta^{\min} \cap \Delta^{\max}$ the wide subcategories of Δ containing the left active, right active and active morphisms, respectively. Call a morphism $f\colon [m] \to [n]$ **left strict** (resp. **right strict**) if it satisfies $f^{-1}\{0\} = \{0\}$ (resp. $f^{-1}\{n\} = \{m\}$). For each $n \in \mathbb{N}$, we denote by $a_n\colon [1] \to [n]$ the unique active map; explicitly given as $a_n(0) = 0$ and $a_n(1) = n$.

2.2.2 The cyclic category

We have already introduced Connes' cyclic category Λ in Section 1.4. Since some of the results in this chapter require explicit computations of pullbacks and pushouts in Λ, a more detailed combinatorial definition is now in order.

A **finite cyclic set** is a pair (N, T) consisting of a finite set N together with an endomorphism $T\colon N \to N$ which is transitive, *i.e.*, for each $x, y \in N$ there is some $i \in \mathbb{N}$ such that $T^i x = y$. A **linearly ordered subset** $L = (L_0, \prec)$ of (N, T) is a subset L_0 of N (called the **underlying set** of L) equipped with a linear order $\prec$ such that the restriction of T to L agrees with the successor function induced by $\prec$. A morphism $(f, f^\star)\colon (N', T') \longrightarrow (N, T)$ of finite cyclic sets consists of

- a map of sets $N' \to N$ which we also denote by f and
- an assignment $f^\star$, which for each linearly ordered subset $L \subset N$ produces a linearly ordered subset $f^\star L \subset N'$ with underlying set $f^{-1}L$ such that $f^\star L = f^\star L' \star f^\star L''$ whenever the linerly ordered subset $L \subset N$ is decomposed as $L = L' \star L''$.

Composition of morphisms $N'' \xrightarrow{(f', f'^\star)} N' \xrightarrow{(f, f^\star)} N$ between finite cyclic set is given by the usual composition of underlying set maps and $(f \circ f')^\star = f'^\star \circ f^\star$.

Definition 2.2.2.1. [Con83] **Connes' cyclic category** Λ consists of nonempty finite cyclic sets and morphisms between them. A **cyclic object** in some (∞-)category $\mathcal{C}$ is a functor $\mathcal{X}\colon \Lambda^{\mathrm{op}} \to \mathcal{C}$. ♢

Remark 2.2.2.2. Following the usual naming convention, a cyclic object in the category of sets would also be called a cyclic set, hence produce a naming clash with the finite cyclic sets introduced above. This will not be an issue since cyclic objects in the category of sets never explicitly appear in this thesis. ♢

For each $n \in \mathbb{N}$, we have the standard finite cyclic set

$$\langle n \rangle := \left(\mathbb{Z}\big/ (n+1), +1 \right).$$

It is easy to see that every nonempty finite cyclic set is (non-canonically) isomorphic to exactly one such standard cyclic set. Motivated by this, we use the notation $+m := T^m$ and $-m := T^{-m}$ for arbitrary finite cyclic sets (N, T) and omit T from the notation entirely.

For every finite cyclic set $(N, +1)$, the automorphism group $\mathrm{Aut}_\Lambda(N)$ is cyclic of order $|N|$ and is generated by the structure morphism $+1\colon N \to N$ where $(+1)^\star := -1$ is given by

$$N \supset L \longmapsto L - 1 := \{x - 1 \,|\, x \in L\} \subset N.$$

Specifying a morphism $f\colon N \to \langle 0\rangle$ amounts to the choice of what we call a **linear order on the cyclic set** N, namely a linearly ordered subset $f^\star\{0\} \subset N$ with underlying set $f^{-1}\{0\} = N$. A commutative triangle

$$\begin{tikzcd} N' \arrow[rr] \arrow[dr, "f'"'] & & N \arrow[dl, "f"] \\ & \langle 0\rangle & \end{tikzcd}$$

corresponds precisely to an order preserving map $f'^\star\{0\} \to f^\star\{0\}$. We conclude that the assignment $f \mapsto f^\star\{0\}$ describes a functor

$$\Lambda_{/\langle 0\rangle} \xrightarrow{\cong} \Delta,$$

which is easily seen to be an isomorphism of categories between Δ and the slice of Λ over $\langle 0\rangle$. Under this identification, the object $[n] \in \Delta$ corresponds to $\langle n\rangle \in \Lambda$ which is equipped with the structure map $[n]\colon \langle n\rangle \to \langle 0\rangle$ induced by the standard linear order $0 < 1 < \cdots < n$ on $\mathbb{Z}\,/\,(n{+}1)$.

Composition in Λ induces a free and transitive right group action

$$\begin{aligned} \Lambda(N, \langle 0\rangle) \times \mathrm{Aut}_\Lambda(\langle n\rangle) &\longrightarrow \Lambda(N, \langle 0\rangle); \\ (f, +m) &\longmapsto f^{+m} \end{aligned}$$

which corresponds to cyclic rotation of linear orders: if $[n]\colon \langle n\rangle \to \langle 0\rangle$ is the structure map corresponding to the standard order $<$ on $[n]$, then $[n]^{+m}$ corresponds to the linear order $\prec$ on the set $\{0, 1 \ldots, n\}$ given by $n - m + 1 \prec \cdots \prec n \prec 0 \prec \cdots \prec n - m$.

2.2.3 Cartesian and coCartesian cubes

Fix a finite set S and denote by $\mathbf{P}(S)$ the powerset of S, partially ordered by inclusion.

Definition 2.2.3.1. An S**-cube** in some (∞-)category $\mathcal{C}$ is a functor $Q\colon \mathbf{P}(S) \to \mathcal{C}$. ◊

Remark 2.2.3.2. Since the poset $\mathbf{P}(S)$ is canonically isomorphic to its opposite (via the assignment $S \supseteq T \mapsto S \setminus T$), we will often write cubes in an (∞-)category $\mathcal{Z}$ as functors $\mathbf{P}^{\mathrm{op}}(S) \to \mathcal{Z}$. This is convenient when studying contravariant functors $\mathcal{X}\colon \mathcal{Z}^{\mathrm{op}} \to \mathcal{C}$, where we can then say that the cube $\mathbf{P}^{\mathrm{op}}(S) \to \mathcal{Z}$ in $\mathcal{Z}$ is sent by $\mathcal{X}$ to the composite $\mathbf{P}(S) \to \mathcal{Z}^{\mathrm{op}} \xrightarrow{\mathcal{X}} \mathcal{C}$; the main example in this thesis is of course the case where $\mathcal{Z} = \Delta$ and $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is a simplicial object in $\mathcal{C}$. ◊

Let $s \in S$ and write $S' := S \setminus \{s\}$. We have an isomorphism of posets

$$\Delta^1 \times \mathbf{P}(S') \xrightarrow{\cong} \mathbf{P}(S)$$

given by $(0, T) \mapsto T$ and $(1, T) \mapsto T \dot\cup \{s\}$. For every ∞-category $\mathcal{C}$ we get an induced equivalence

$$\mathrm{Fun}(\mathbf{P}(S), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathbf{P}(S'), \mathcal{C}))$$

of ∞-categories, which we denote by $Q \mapsto Q^s$. We say that a cube Q is the pasting in s-direction of two cubes Q' and Q'' if we have an identification $Q^s = Q'^s \circ Q''^s$.

Denote by $\mathbf{P}_*(S) := \mathbf{P}(S) \setminus \{\varnothing\}$ the poset of *nonempty* subsets of S.

Definition 2.2.3.3. An S-cube $Q\colon \mathbf{P}(S) \to \mathcal{C}$ is called

- **Cartesian** if it is a limit diagram in $\mathcal{C}$; *i.e.*, if Q is the right Kan extension of its restriction to $\mathbf{P}_*(S)$.
- **coCartesian** if it is a colimit diagram in $\mathcal{C}$; *i.e.*, if Q is the left Kan extension of its restriction to $\mathbf{P}(S) \setminus \{S\}$.

A cube is called **biCartesian** if it is both Cartesian and coCartesian. ◊

Definition 2.2.3.4. An S-cube $Q\colon \mathbf{P}^{\mathrm{op}}(S) \to \mathcal{Z}$ is called **strongly Cartesian** or **strongly coCartesian** if, for each $T \subset S$ and $s, s' \in S \setminus T$ with $s \neq s'$, the 2-dimensional face

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & T \mathbin{\dot{\cup}} \{s\} \arrow[d] \\
T \mathbin{\dot{\cup}} \{s'\} \arrow[r] & T \mathbin{\dot{\cup}} \{s, s'\}
\end{tikzcd}$$

is sent by Q to a pullback square or a pushout square in $\mathcal{Z}$, respectively. A cube is called **strongly biCartesian** if it is both strongly Cartesian and strongly coCartesian. ◊

Remark 2.2.3.5. Denote by $\mathbf{P}^{\mathrm{op}}_{\leq 1}(S)$ and by $\mathbf{P}^{\mathrm{op}}_{\geq |S|-1}(S)$ the subposet of $\mathbf{P}^{\mathrm{op}}(S)$ spanned by the subsets $T \subset S$ of cardinality $|T| \leq 1$ and $|T| \geq |S| - 1$, respectively. It is easy to see that a cube $Q\colon \mathbf{P}^{\mathrm{op}}(S) \to \mathcal{Z}$ is strongly Cartesian if and only if it is the right Kan extension of its restriction to $\mathbf{P}^{\mathrm{op}}_{\leq 1}(S)$; it is strongly coCartesian if and only if it is the left Kan extension of its restriction to $\mathbf{P}^{\mathrm{op}}_{\geq |S|-1}(S)$. ◊

Remark 2.2.3.6. If $|S| \geq 2$, then every strongly (co)Cartesian cube is also (co)Cartesian; thus justifying the terminology. Beware however, that for $|S| = 1$ an S-cube is just an arrow; it is always strongly biCartesian and is (co)Cartesian if and only if it is an equivalence. ◊

Lemma 2.2.3.7. Let $\mathcal{C}$ be an ∞-category. Let $s \in S$ and put $S' := S \setminus \{s\}$. The restriction functor

$$p\colon \operatorname{Fun}(\mathbf{P}(S'), \mathcal{C}) \longrightarrow \operatorname{Fun}(\mathbf{P}_*(S'), \mathcal{C})$$

is a coCartesian fibration which is Cartesian if and only if $\mathcal{C}$ admits limits of shape $\mathbf{P}_*(S)$. An S-cube $Q\colon \mathbf{P}(S) \to \mathcal{C}$ is Cartesian if and only if the corresponding edge $Q^s\colon \Delta^1 \to \operatorname{Fun}(\mathbf{P}(S'), \mathcal{C})$ is p-Cartesian. ☐

Proof. Lemma 2.2.3.7 is the higher dimensional analog of [Lur09, Lemma 6.1.1.1]; the proof is essentially the same. ■

We say that an S-cube Q is **degenerate in direction** $s \in S$ if the corresponding natural transformation Q^s of $S \setminus \{s\}$-cubes is an equivalence. It follows directly from Lemma 2.2.3.7 that **degenerate cubes**—cubes that are degenerate in at least one direction—are automatically Cartesian and coCartesian.

The following lemma is a standard argument which is useful to compare Cartesian cubes of different dimensions.

Lemma 2.2.3.8. Let $Q\colon \mathbf{P}(S) \to \mathcal{C}$ be an S-cube in an ∞-category $\mathcal{C}$ with finite limits. Fix $s \in S$ and write $S' := S \setminus \{s\}$. Assume that the S'-cube $Q^s(1)\colon T \mapsto Q(T \mathbin{\dot{\cup}} \{s\})$ is Cartesian. Then the canonical map

$$\lim Q\big|_{\mathbf{P}_*(S)} \longrightarrow \lim Q\big|_{\mathbf{P}_*(S')}$$

is an equivalence. In particular, the original S-cube Q is Cartesian if and only if the restricted S'-cube $Q\big|_{\mathbf{P}(S')} = Q^s(0)\colon T \mapsto Q(T)$ is Cartesian. ☐

Proof. Consider the following commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
Q(\varnothing) \arrow[r] \arrow[dr] & \lim Q\big|_{\mathbf{P}_*(S)} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Q(\{s\}) \arrow[d, "\simeq"] \\
& \lim Q\big|_{\mathbf{P}_*(S')} \arrow[r] & \lim Q^s(1)\big|_{\mathbf{P}_*(S')}
\end{tikzcd} \tag{2.2.1}$$

which is induced by the universal properties of the various limits. By a standard decomposition argument for limits, the rightmost square in the diagram (2.2.1) is Cartesian; moreover, the rightmost vertical map is an equivalence by assumption. It follows that the left vertical map is also an equivalence; the result follows. ■

2.2.4 Čech cubes, descent and weak excision

Let $\mathcal{Z}$ be an ∞-category.

Definition 2.2.4.1. Let S be a finite set. An **S-pronged claw** (or just **S-claw**, for short) $\mathcal{F}$ on an object N in $\mathcal{Z}$ is an S-indexed tuple $\mathcal{F} = (f_s \colon I_s \to N \mid s \in S)$ of maps f_s in $\mathcal{Z}$ with common codomain $N \in \mathcal{Z}$ or, equivalently, a diagram $\mathcal{F} \colon \mathbf{P}^{\mathrm{op}}_{\leq 1}(S) \to \mathcal{Z}$ with $\mathcal{F}(\varnothing) = N$. $\Diamond$

Example 2.2.4.2. A $[2]$-pronged claw $\mathcal{F} \models N$ looks as follows

$$\begin{tikzcd}
I_0 \arrow[dr, "f_0"'] & I_1 \arrow[d, "f_1"] & I_2 \arrow[dl, "f_2"] \\
 & N &
\end{tikzcd}$$

(recall that $[2] \in \Delta$ has three elements). $\Diamond$

Given an S-claw $\mathcal{F} = (f_s \colon I_s \to N \mid s \in S)$ on $N \in \mathcal{Z}$, we write $\mathcal{F} \models N$ to make the codomain N explicit in the notation while keeping the f_s, the I_s and sometimes even the S anonymous. In a similar spirit we will use the symbol $f \in \mathcal{F}$ to mean f_s for some s. With this convention f_s and $f_{s'}$ should be considered distinct if $s \neq s'$, even if they are the same map in $\mathcal{Z}$. Each subset $T \subset S$ induces a restricted T-claw of $\mathcal{F}$ given by $\mathcal{F}\big|_T := (f_t \mid t \in T) \models N$.

Definition 2.2.4.3. An S-claw $\mathcal{F} \models N$ in $\mathcal{Z}$ is called a **candidate S-covering** if it can be extended to a strongly Cartesian S-cube $\check{\mathrm{C}}\mathcal{F} \colon \mathbf{P}^{\mathrm{op}}(S) \to \mathcal{Z}$. In this case we call $\check{\mathrm{C}}\mathcal{F}$ the **Čech cube** associated to $\mathcal{F}$. $\Diamond$

If it exists, the Čech cube $\check{\mathrm{C}}\mathcal{F}$ is given by the formula

$$S \supseteq T \longmapsto \lim \mathcal{F}\big|_T. \tag{2.2.2}$$

We shall sometimes think of the prongs $f_s \colon I_s \to N$ as generalized subobjects of N; the values (2.2.2) of the Čech cube should then be thought of as generalized intersections. In this spirit it is sometimes convenient to use the notation $\bigcap_{t \in T} f_t := \check{\mathrm{C}}\mathcal{F}(T) = \lim \mathcal{F}\big|_T$ and denote, for instance, the Čech square of two maps $f \colon I \to N$ and $f' \colon I' \to N$ as follows:

$$\begin{tikzcd}
I \cap I' \arrow[r, "f \cap I'"] \arrow[d, "I \cap f'"'] \arrow[dr, "f \cap f'"] & I' \arrow[d, "f'"] \\
I \arrow[r, "f"] & N
\end{tikzcd}$$

Definition 2.2.4.4. Let $\mathcal{F}$ be a candidate covering in $\mathcal{Z}$. A functor $\mathcal{X} \colon \mathcal{Z}^{\mathrm{op}} \to \mathcal{C}$ is said to **satisfy descent with respect** $\mathcal{F}$ if it sends the Čech cube $\check{\mathrm{C}}\mathcal{F}$ to a Cartesian cube in $\mathcal{C}$; in this case we also say that $\mathcal{F}$ is **$\mathcal{X}$-local**. $\Diamond$

Following Boavida de Brito and Weiss we say that a **coverage** τ on $\mathcal{Z}$ is a collection of candidate coverings. If $\mathcal{F} \models N$ is an element of τ then we say that $\mathcal{F}$ is a **τ-covering**; if the coverage τ is implicit from the context then we say that $\mathcal{F}$ is a **covering** of N.

Definition 2.2.4.5. A $\mathcal{C}$-valued **sheaf** for the coverage τ is a functor $\mathcal{X} \colon \mathcal{Z}^{\mathrm{op}} \to \mathcal{C}$ which satisfies descent with respect to all τ-coverings. $\Diamond$

Remark 2.2.4.6. For each $k \geq 0$, there is a canonical coverage τ_k on $\mathcal{Z}$ which consists of *all* candidate $[k]$-coverings. A presheaf $\mathcal{Z}^{\mathrm{op}} \to \mathcal{C}$ is a sheaf for this coverage τ_k if and only if it is an k-excisive (covariant) functor in the sense of Goodwillie [Goo92], *i.e.*, if it sends strongly coCartesian $[k]$-cubes in $\mathcal{Z}^{\mathrm{op}}$ to Cartesian cubes in $\mathcal{C}$. $\Diamond$

We say that an S-claw is **strongly biCartesian** if it is a candidate covering and if its Čech cube is strongly coCartesian (hence strongly biCartesian).

Definition 2.2.4.7. A functor $\mathcal{Z}^{\mathrm{op}} \to \mathcal{C}$ is called **weakly S-excisive** if it is a sheaf for the coverage of strongly biCartesian S-claws, *i.e.*, if it sends all strongly biCartesian S-cubes to Cartesian cubes in $\mathcal{C}$. $\diamondsuit$

We will also need the following relative notion:

Definition 2.2.4.8. Let $\mathcal{Z} \to \mathcal{Z}'$ be a limit-preserving functor. We call a functor $\mathcal{X} \colon \mathcal{Z}^{\mathrm{op}} \to \mathcal{C}$ **weakly S-$\mathcal{Z}'$-excisive** (with the functor $\mathcal{Z} \to \mathcal{Z}'$ left implicit) if it is a sheaf with respect to those candidate S-coverings which become strongly biCartesian in $\mathcal{Z}'$. $\diamondsuit$

Clearly the property of being weakly S-excisive (both in the relative and in the absolute sense) only depends on the cardinality of S. For $k \in \mathbb{N}$, we say that $\mathcal{X} \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is **weakly k-excisive** if it is weakly $[k]$-excisive. We will stick to S-cubes instead of $[k]$-cubes whenever possible, because the latter might suggest a dependency on the linear order of the coordinates.

Remark 2.2.4.9. In the setting of Definition 2.2.4.8, if every candidate covering in $\mathcal{Z}'$ admits a lift to a candidate covering in $\mathcal{Z}$ then a functor $\mathcal{Z}'^{\mathrm{op}} \to \mathcal{C}$ is weakly S-excisive if and only if its restriction to $\mathcal{Z}$ is weakly S-$\mathcal{Z}'$-excisive. $\diamondsuit$

2.2.5 Sheafification

One of the most fundamental features of Goodwillie calculus is the existence of Taylor approximations [Goo03, 1.8 Theorem] [Lur17, Theorem 6.1.1.10]: given a functor $\mathcal{X} \colon \mathcal{Z} \to \mathcal{C}$ between suitable ∞-categories (for instance $\mathcal{S}$, $\mathcal{S}_\star$ or $\mathcal{S}p$), there exists a tower

$$\begin{tikzcd}
 & \cdots & & \mathcal{X} \arrow[dll] \arrow[dl] \arrow[d] \\
\cdots \arrow[r] & P_2\mathcal{X} \arrow[r] & P_1\mathcal{X} \arrow[r] & P_0\mathcal{X}
\end{tikzcd}$$

where $\mathcal{X} \to P_k\mathcal{X}$ is a *universal k-excisive approximation* of $\mathcal{X}$, *i.e.*, induces an equivalence $\mathrm{Map}(P_k\mathcal{X}, \mathcal{X}') \xrightarrow{\simeq} \mathrm{Map}(\mathcal{X}, \mathcal{X}')$ for each k-excisive functor $\mathcal{X}'$. Another way of saying this is that for each $k \in \mathbb{N}$, the inclusion

$$\mathrm{Exc}^k(\mathcal{Z}, \mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{Z}, \mathcal{C})$$

of the full subcategory spanned by the k-excisive functors admits a left adjoint P_k; the map $\mathcal{X} \to P_k\mathcal{X}$ is the adjunction unit.

From the sheaf-theoretic perspective discussed in Section 2.2.4, the k-excisive approximation $P_k\mathcal{X}$ of a functor $\mathcal{Z} \to \mathcal{C}$ is called the **sheafification** of $\mathcal{X}$ (viewed as a presheaf on $\mathcal{Z}^{\mathrm{op}}$) with respect to the coverage τ_k on $\mathcal{Z}^{\mathrm{op}}$ of all candidate $[k]$-coverings.

In this thesis, we are mostly interested in sheaves on *small* categories, like Δ of Λ. It turns out that in this case a sheafification/approximation always exists, at least when the target category is presentable (see [Lur09, Section 5.5]), *e.g.*, spaces or spectra or any ∞-category arising from a combinatorial simplicial model category.

Lemma 2.2.5.1. Let $\mathcal{Z}$ be a small ∞-category and fix a small set $\mathcal{R} = \{\alpha_i \colon K_i^{\triangleleft} \to \mathcal{Z} \mid i \in I\}$ of cones in $\mathcal{Z}$ (where each K_i is a simplicial set). Let $\mathcal{C}$ be a presentable ∞-category which admits limits of all shapes K_i. Then the full subcategory $\mathrm{Fun}^{\mathcal{R}}(\mathcal{Z}, \mathcal{C}) \subset \mathrm{Fun}(\mathcal{Z}, \mathcal{C})$ spanned by those functors which send all cones in $\mathcal{R}$ to limit diagrams in $\mathcal{C}$ is presentable and the inclusion

$$\mathrm{Fun}^{\mathcal{R}}(\mathcal{Z}, \mathcal{C}) := \{\forall \alpha \in \mathcal{R} \colon \alpha \mapsto \text{limit cone}\} \hookrightarrow \mathrm{Fun}(\mathcal{Z}, \mathcal{C})$$

admits a left adjoint. $\square$

Proof. The ∞-category $\mathrm{Fun}^{\mathcal{R}}(\mathcal{Z},\mathcal{C})$ fits into the following pullback square of ∞-categories

$$\begin{tikzcd}
\mathrm{Fun}^{\mathcal{R}}(\mathcal{Z},\mathcal{C}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathcal{Z},\mathcal{C}) \arrow[d, "\alpha_i^\star"] \\
\mathrm{Fun}(\coprod_{i\in I} K_i,\mathcal{C}) \arrow[r, hook] & \mathrm{Fun}(\coprod_{i\in I} K_i^{\triangleleft},\mathcal{C})
\end{tikzcd} \tag{2.2.3}$$

where:

- all other ∞-categories are $\mathcal{C}$-valued diagram categories, hence presentable because $\mathcal{C}$ is (see [Lur09, Proposition 5.5.3.6]);
- the lower horizontal arrow is given by right Kan extension along $\coprod_{i\in I} K_i \hookrightarrow \coprod_{i\in I} K_i^{\triangleleft}$ and has a left adjoint given by restriction;
- the right vertical arrow $\alpha_i^\star$ has a left adjoint given by left Kan extension.

The (very large) category $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ of presentable ∞-categories and right adjoint functors has all limits and the inclusion $\mathcal{P}\mathrm{r}^{\mathrm{R}} \hookrightarrow \mathbf{CAT}_\infty$ preserves them (see [Lur09, Theorem 5.5.3.18]). It follows that the ∞-category $\mathrm{Fun}^{\mathcal{R}}(\mathcal{Z},\mathcal{C})$ is presentable and that both structure maps in the pullback (2.2.3) have left adjoints. This concludes the proof. ■

Corollary 2.2.5.2. Let $\mathcal{Z}$ be a small ∞-category and let τ be coverage on $\mathcal{Z}$. Let $\mathcal{C}$ be a presentable ∞-category. Then the inclusion

$$\{\tau\text{-sheaves}\} \hookrightarrow \mathrm{Fun}(\mathcal{Z}^{\mathrm{op}},\mathcal{C})$$

admits a left adjoint; in other words, each $\mathcal{C}$-valued presheaf on $\mathcal{Z}$ can be τ-sheafified. □

Proof. Since $\mathcal{Z}$ is small, so is the set of τ-coverings. Hence Corollary 2.2.5.2 follows by applying Lemma 2.2.5.1 to the ∞-category $\mathcal{Z}^{\mathrm{op}}$ and to the small set $\mathcal{R} := \{\check{\mathrm{C}}\mathcal{F} \mid \mathcal{F} \in \tau\}$ of Čech cubes arising from τ-coverings. ■

2.3 Strongly biCartesian cubes in Δ and Λ

The goal of this section is to classify and explicitly describe the strongly biCartesian cubes in the simplex category and the cyclic category.

2.3.1 Strongly biCartesian cubes in the simplex category

Definition 2.3.1.1. An S-claw $\mathcal{F} = (f_s \mid s \in S)$ on $[n]$ in Δ_+ is called

- **backwards compatible** if for each $i \in [n]$ there is at most one $s \in S$ such that the preimage $f_s^{-1}\{i\}$ has more than one element;
- **compatible** if it satisfies the following two conditions:

(**BC1**) for each $i \in [n]$, there is at most one $s \in S$ such that the preimage $f_s^{-1}\{i\}$ is not a singleton;

(**BC2**) for each $0 < i \leq n$, there is at most one $s \in S$ such that the subset $\{i-1, i\} \subseteq [n]$ is not contained in the image of f_s. ◊

Remark 2.3.1.2. The S-claw $\mathcal{F}$ satisfies condition (**BC1**) if and only if it is backwards compatible and: if the preimage $f_s^{-1}\{i\}$ is empty for some $i \in [n]$ and $s \in S$ then the preimage $f_{s'}^{-1}\{i\}$ is a singleton for all $s' \in S \setminus s$. In the language of Section 2.1, condition (**BC2**) says precisely that the images of the maps f_s are of the form $[n] \setminus A_s$, where the $(A_s \mid s \in S)$ are "pairwise disjoint closed subsets" of the "manifold" $[n]$. ◊

We call a diagram in Δ_+ **left active** or **right active** if it takes values in the subcategory of Δ spanned by the left active or right active morphisms, respectively.

Remark 2.3.1.3. It will be useful to visualize S-claws $\mathcal{F} \models [n]$ in Δ_+ as arrays as in the following example (with $n = 9$ and $S = [3]$):

$$\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 0 & * & * & * & * & * & 3 & 2 & * & * & * \\ 1 & * & \varnothing & * & * & * & * & * & * & * & * \\ 2 & * & * & * & \varnothing & \varnothing & * & * & * & * & * \\ 3 & * & * & * & * & * & * & * & \varnothing & 2 & \varnothing \end{array} \tag{2.3.1}$$

There is one row for each prong $f_s \colon I_s \to [n]$ of $\mathcal{F}$ and one column for each $i \in [n]$; in the entry (s, i) we draw:

- a star $*$ if the preimage $f_s^{-1}\{i\}$ is a singleton,
- the symbol $\varnothing$ if the preimage $f_s^{-1}\{i\}$ is empty or
- a number l if the preimage $f_s^{-1}\{i\}$ has $l > 1$ many elements.

A claw is backwards compatible if and only if in each column there is at most one entry with a number $l > 1$. It is compatible if and only if it satisfies the following two conditions:

- in each column there is at most one “special” entry, *i.e.*, a cell which is not a star $*$;
- each pair of two empty cells is either in the same row or separated by a column with no empty cells.

The example (2.3.1) depicts the left active compatible claw

$$\begin{tikzcd} & \{0,2,3,4,5,6,7,8,9\} \arrow[dr, hook] & \{0,1,2,5,6,7,8,9\} \arrow[d, hook] & \{0,1,2,3,4,5,6,8,8'\} \arrow[dl] \\ \{0,1,2,3,4,5,5',5'',6,6',7,8,9\} \arrow[rr, two heads] & & {[9]} & \end{tikzcd}$$

defined by $i, i', i'' \mapsto i \in [9]$. ♢

Proposition 2.3.1.4. Let $\mathcal{F} \models [n]$ be an S-claw in Δ_+.

(a) The claw $\mathcal{F}$ is a candidate S-covering in Δ_+ if and only if $\mathcal{F}$ is backwards compatible. The Čech cube $\check{\mathrm{C}}\mathcal{F} \colon \mathbf{P}^{\mathrm{op}}(S) \to \Delta_+$ is given explicitly by the formula

$$\check{\mathrm{C}}\mathcal{F} \colon T \longmapsto \bigstar_{i \in [n]} \prod_{t \in T} f_t^{-1}\{i\}. \tag{2.3.2}$$

(b) The S-claw $\mathcal{F}$ is strongly biCartesian (*i.e.*, the Čech cube $\check{\mathrm{C}}\mathcal{F}$ of $\mathcal{F}$ is strongly biCartesian) if and only if $\mathcal{F}$ is compatible. □

Corollary 2.3.1.5. A claw in Δ is strongly biCartesian if and only if it is compatible. □

Proof. Corollary 2.3.1.5 follows directly from Proposition 2.3.1.4 and the easy observation that the whole Čech cube of a *compatible* claw $\mathcal{F} \models [n]$ in Δ_+ lies in Δ provided that $n \neq -1$. ■

Example 2.3.1.6. The [1]-claw

$$\begin{array}{ccc} 0 & 1 & 2 \\ \hline \varnothing & * & * \\ * & * & \varnothing \end{array} \tag{2.3.3}$$

is compatible and gives rise to the biCartesian square

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 12 \arrow[d] \\ 01 \arrow[r] & 012 \end{tikzcd} \tag{2.3.4}$$

in Δ which encodes the lowest instance of Rezk’s Segal conditions. ♢

Proof (of Proposition 2.3.1.4)**.** (a) *A priori*, the formula (2.3.2) describes a strongly Cartesian extension $\check{C}\mathcal{F}\colon \mathbf{P}^{\mathrm{op}}(S) \to \mathbf{Pos}$ of $\mathcal{F}$ in the category of posets. Since the canonical inclusion $\Delta_+ \hookrightarrow \mathbf{Pos}$ preserves limits, we conclude that $\check{C}\mathcal{F}$ is a strongly Cartesian extension of $\mathcal{F}$ in Δ_+ if and only if $\check{C}\mathcal{F}$ takes values in linearly ordered posets. This happens if and only if each product $\prod_{t\in T} f_t^{-1}\{i\}$ has at most one factor which is not empty or a singleton; this is precisely the backwards compatibility condition on $\mathcal{F}$.

(b) Assume that $\mathcal{F}$ is backwards compatible so that the Čech cube $\check{C}\mathcal{F} \coloneqq \mathbf{P}^{\mathrm{op}}(S) \to \Delta_+$ is well defined by part (a). We need to understand when $\check{C}\mathcal{F}$ is additionally strongly coCartesian. By definition, the cube $\check{C}\mathcal{F}$ is strongly coCartesian if and only it for every subset $T \subset S$ and every pair of distinct elements $s, s' \in S \setminus T$, the square

$$\begin{tikzcd}
\mathop{\bigstar}\limits_{i\in[n]} \left(f_s^{-1}\{i\} \times f_{s'}^{-1}\{i\} \times \prod_{t\in T} f_t^{-1}\{i\}\right) \arrow[r] \arrow[d] & \mathop{\bigstar}\limits_{i\in[n]} \left(f_{s'}^{-1}\{i\} \times \prod_{t\in T} f_t^{-1}\{i\}\right) \eqqcolon B' \arrow[d] \\
B \coloneqq \mathop{\bigstar}\limits_{i\in[n]} \left(f_s^{-1}\{i\} \times \prod_{t\in T} f_t^{-1}\{i\}\right) \arrow[r] & \mathop{\bigstar}\limits_{i\in[n]} \left(\prod_{t\in T} f_t^{-1}\{i\}\right) \eqqcolon N
\end{tikzcd} \tag{2.3.5}$$

is a pushout in Δ_+.

To show "if" in the claimed equivalence, assume that $\mathcal{F}$ is compatible; we will show that then each square (2.3.5) is a pushout in Δ_+. Condition (**BC1**) implies that, for every $i \in [n]$, if one amongst $f_s^{-1}\{i\}$ and $f_{s'}^{-1}\{i\}$ is empty then the other is a singleton; it follows that the square (2.3.5) is a pushout on the level of underlying sets. It remains to show that a map of sets $\beta\colon N \to M$ is weakly monotone if it is weakly monotone when composed with $B \to N$ and $B' \to N$; for this it is sufficient to show that each pair of adjacent elements in N is contained in the image of $B \to N$ or in the image of $B' \to N$. Let $x < x+1 \eqqcolon x'$ be two adjacent elements of N and denote by i and i' their respective images in $[n]$. It is enough to show that the subset $\{i, i'\} \subseteq [n]$ is contained in the image of f_s or in the image of $f_{s'}$. If $i = i'$ then this follows from condition (**BC1**); if $i' = i+1$ then this follows from condition (**BC2**). We may therefore assume $i < i+1 \leq i'-1 < i'$. For each $i < i'' < i'$ the product $\prod_{t\in T} f_t^{-1}\{i''\}$ must be empty by adjacency of x and x'. Hence there must be $t, t' \in T$ such that $f_t^{-1}\{i+1\}$ and $f_{t'}^{-1}\{i'-1\}$ are empty; in particular the subsets $\{i, i+1\}$ and $\{i'-1, i\}$ of $[n]$ are not contained in the image of f_t and $f_{t'}$, respectively. Condition (**BC2**) implies that the sets $\{i, i+1\}$, $\{i'-1, i\}$ and, *a fortiori*, $\{i, i'\}$ are contained in the image of both f_s and $f_{s'}$.

To show "only if", assume that the cube $\check{C}\mathcal{F}$ is strongly biCartesian. We show that conditions (**BC1**) and (**BC2**) hold, *i.e.*, that $\mathcal{F}$ is compatible.

(**BC1**) Let $i \in [n]$ and $s \in S$ be such that $f_s^{-1}\{i\}$ is empty. For each $s' \in S \setminus \{s\}$ consider the following commutative diagram, where the inner solid square is the pushout square (2.3.5) (for $T = \varnothing$):

$$\begin{tikzcd}
\mathop{\bigstar}\limits_{j\in[n]} f_s^{-1}\{j\} \times f_{s'}^{-1}\{j\} \arrow[r] \arrow[d] & \mathop{\bigstar}\limits_{j\in[n]} f_{s'}^{-1}\{j\} \arrow[d] \arrow[ddr] & \\
\mathop{\bigstar}\limits_{j\in[n]} f_s^{-1}\{j\} \arrow[r] \arrow[d, equal] & {[n]} \arrow[dr, dashed] & \\
\mathop{\bigstar}\limits_{j\in[n]\setminus\{i\}} f_s^{-1}\{j\} \arrow[r] & {[n] \setminus \{i\}} \arrow[r, hook] & {[i-1] \star f_{s'}^{-1}\{j\} \star \{i+1, \dots, n\}}
\end{tikzcd}$$

The dashed arrow—which exists by the pushout property—exhibits $f_{s'}^{-1}\{i\}$ as a retract of the singleton $\{i\}$, hence as a singleton itself.

(BC2) Fix $0 < i \leq n$ and distinct elements $s, s' \in S$. Consider the commutative diagram

$$\begin{tikzcd}
\underset{j\in[n]}{\bigstar} f_s^{-1}\{j\} \times f_{s'}^{-1}\{j\} \arrow[r] \arrow[d] & \underset{j\in[n]}{\bigstar} f_{s'}^{-1}\{j\} \arrow[d] \arrow[rdd, dashed] & \\
\underset{j\in[n]}{\bigstar} f_s^{-1}\{j\} \arrow[r] \arrow[rrd, dashed] & {[n]} \arrow[rd] & \\
 & & {[n]}
\end{tikzcd}$$

where $[n] \to [n]$ is the (not order preserving) map that exchanges $i-1$ and i. By the pushout property of the solid square, at least one of the dashed composites must be not order preserving; this can only happen if least one of the maps f_s and $f_{s'}$ contains the subset $\{i-1, i\} \subseteq [n]$ in its image. ■

Remark 2.3.1.7. An S-claw $\mathcal{F} = (f_s \mid s \in S)$ is backwards compatible if and only if for each pair of distinct elements $s, s' \in S$ the induced $\{s, s'\}$-subclaw is backwards compatible. Hence it follows from Proposition 2.3.1.4, that $\mathcal{F}$ admits a Čech cube in Δ_+ if and only if each pair $f_s, f_{s'}$ (for distinct $s, s' \in S$) admits pullback in Δ_+. Similarly, an S-claw admits a strongly biCartesian Čech cube if and only if each two-pronged subclaw is compatible. ◊

2.3.2 Strongly biCartesian cubes in the cyclic category

In this section, we characterize strongly biCartesian cubes in Λ. To this end, we introduce the cyclic analog of a compatible claw. Heuristically, this corresponds to adding the new "point" $(n, 0)$ to the "manifold" $[n] \in \Delta$.

Definition 2.3.2.1. An S-claw $\mathcal{F} \models [n]$ in Δ is called **cyclically compatible** if the claw $\mathcal{F}$ is compatible and all but at most one $f \in \mathcal{F}$ have the set $\{0, n\} \subseteq [n]$ in their image. ◊

Remark 2.3.2.2. Let $\iota\colon I'' \hookrightarrow I_0 \star I'' \star I_1 = I$ and $\alpha\colon I'' \to I'$ be an inert map and an active map in Δ, respectively. Define $[n] := I_0 \star I' \star I_1$. It is easy to see that the [1]-claw $(I' \hookrightarrow [n], \mathrm{Id} \star \alpha \star \mathrm{Id}\colon I \to [n])$ is cyclically compatible and that I'' is the associated pullback. By definition, the *decomposition spaces* of Gálvez-Carrillo, Kock and Tonks [GKT18a; GKT18b; GKT18c] are precisely those simplicial objects which send to Cartesian squares the biCartesian squares that arise this way. ◊

Example 2.3.2.3. The [1]-claws

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \hline \varnothing & * & * & * \\ * & * & \varnothing & * \end{array} \qquad \text{and} \qquad \begin{array}{cc} 0 & 1 \\ \hline * & 2 \\ \varnothing & * \end{array}$$

are cyclically compatible and arise as the pushouts of the inert map $\mathrm{d}^0\colon [1] \to [2]$ along the active maps $\mathrm{d}^1\colon [1] \to [2]$ and $\mathrm{s}^0\colon [1] \to [0]$, respectively. They encode the first upper 2-Segal condition and an instance of unitality. The [1]-claw (2.3.3) of Example 2.3.1.6 is not cyclically compatible because the "point" $(2, 0)$ of the "manifold" [2] is not covered by any prong; the corresponding Čech square (2.3.4) is not coCartesian in the cyclic category. ◊

The following is the main result of this section:

Proposition 2.3.2.4. An S-claw $\mathcal{F} \models [n]$ in Δ has a strongly biCartesian image in Λ if and only if it is cyclically compatible. □

Corollary 2.3.2.5. The following three classes of S-cubes in Λ agree:

- strongly biCartesian S-cubes in Λ
- images of left active strongly biCartesian S-cubes in Δ
- images of right active strongly biCartesian S-cubes in Δ. ☐

Before we can prove Proposition 2.3.2.4 and Corollary 2.3.2.5 we need a couple of lemmas.

Lemma 2.3.2.6. Let $\mathcal{F} = (f_s\colon I_s \to [n] \mid s \in S)$ be an S-claw in Δ. If $\mathcal{F}$ is compatible and either left active or right active then $\mathcal{F}$ is cyclically compatible. Moreover, the following are equivalent:

(1) the claw $\mathcal{F}$ is cyclically compatible;
(2) for every $m \in [n]$, the cyclic rotation $\mathcal{F}^{+m} := \big({f_s}^{+m}\colon {I_s}^{+m} \to [n]^{+m} \bigm| s \in S\big)$ of the claw $\mathcal{F}$ is compatible;
(3) there is an $m \in [n]$ such that the cyclic rotation $\mathcal{F}^{+m}$ of the claw $\mathcal{F}$ is left active and compatible;
(4) there is an $m \in [n]$ such that the cyclic rotation $\mathcal{F}^{+m}$ of the claw $\mathcal{F}$ is right active and compatible. ☐

Proof. The first statement follows directly from the definitions. It is clear from the definition that the property of being cyclically compatible is preserved under cyclic rotation; hence we have the implications ((1) $\Longrightarrow$ (2)), ((3) $\Longrightarrow$ (1)) and ((4) $\Longrightarrow$ (1)). Given a compatible S-claw $\mathcal{F} = (f_s \mid s \in S)$ on $[n]$ in Δ, there is an element $m \in [n]$ which is in the image of all the f_s. Then for any such m, the rotated claws $\mathcal{F}^{-m}$ and $\mathcal{F}^{-m-1}$ are left active and right active, respectively. We thus obtain the implications ((2) $\Longrightarrow$ (3)) and ((2) $\Longrightarrow$ (4)). ■

Lemma 2.3.2.7. Let $Q\colon \mathbf{P}^{\mathrm{op}}(S) \to \Lambda$ be an S-cube in the cyclic category. The following are equivalent:

(1) the cube Q is strongly Cartesian;
(2) there is a strongly Cartesian S-cube in Δ which is mapped to Q under the canonical functor $\Delta \to \Lambda$;
(3) every S-cube Q' in Δ which maps to Q is strongly Cartesian. ☐

Proof. The implications (2) $\Longrightarrow$ (1) $\Longrightarrow$ (3) follow from the general fact about slice categories that the projection $\Delta \cong \Lambda_{/\langle 0\rangle} \to \Lambda$ preserves and reflects pullbacks. The implication (3) $\Longrightarrow$ (2) holds because the cube Q lifts to a cube in $\Delta \cong \Lambda_{/\langle 0\rangle}$ by choosing any map $Q(\varnothing) \to \langle 0\rangle$. ■

Lemma 2.3.2.8. Let

$$\begin{tikzcd}
I \cap I' \arrow[r, "f \cap I'"] \arrow[d, "I \cap f'"'] \arrow[dr, phantom, "\square"] & I' \arrow[d, "f'"] \\
I \arrow[r, "f"'] & {[n]}
\end{tikzcd} \tag{2.3.6}$$

be the left active biCartesian Čech square associated to a left active compatible claw $(f, f') \models [n]$ in Δ. Then the image in Λ of the square (2.3.6) is a pushout. ☐

Proof. Consider a solid commutative diagram

$$\begin{tikzcd}
I \cap I' \arrow[r, "f \cap I'"] \arrow[d, "I \cap f'"'] & I' \arrow[d, "f'"'] \arrow[rdd, "p'"] & \\
I \arrow[r, "f"] \arrow[rrd, "p"'] & \langle n\rangle \arrow[rd, dashed] & \\
 & & N
\end{tikzcd} \tag{2.3.7}$$

in Λ, where the top left square is the image of the square (2.3.6). We need to show that there is a unique dashed morphism $p\colon \langle n\rangle \to N$ of cyclic sets making the diagram (2.3.7) commute.

- First, we treat the case $N = \langle 0\rangle$. In this case the maps $p\colon I \to \langle 0\rangle$, $p'\colon I' \to \langle 0\rangle$ and $p''\colon I\cap I' \to \langle 0\rangle$ correspond to cyclic rotations $\prec$ of the linear order on I, I' and $I'' := I\cap I'$, respectively; we have to show that there is a unique linear order $\prec$ on the cyclic set $\langle n\rangle$ such that both f and f' are order preserving with respect to $\prec$. Uniqueness is clear, because by compatibility of (f, f') each set $\{i-1, i\}$ (for $i \in [n]$) is in the image of f or of f'.

  To construct the linear order $\prec$ on $[n]$, denote by x and x' the maximal elements in the linearly ordered sets $(I, \prec)$ and $(I', \prec)$, respectively, *i.e.*, the unique elements with $x+1 \prec x$ and $x'+1 \prec x'$. Without loss of generality, assume $i' := f(x') \leq f(x) =: i$. Define $\prec$ to be the unique linear order on the cyclic set $\langle n\rangle$ which has i as its maximum. We need to show that f and f' preserve the orders $\prec$; for this it is enough to verify that $i < f(x+1)$ and $i < f'(x'+1)$ (because $f(x) \leq i$ and $f'(x') \leq i$).

  Denote by z'', z' and z the $<$-minimal elements of I'', I' and I, respectively; they satisfy $(f\cap I')(z'') = z'$, $(I\cap f')(z'') = z$ and $f(z) = 0 = f'(z')$ because the square (2.3.6) was assumed to be left active.

  - Assume that $i = f(x) = f(x+1)$. Then by backwards compatibility of (f, f') we must have a unique $y' \in I'$ with $f'(y') = i$. By the explicit formula for Čech cubes we deduce that the order preserving map (with respect to both $\prec$ and $<$) $I\cap f'\colon I'' \to I$ restricts to a bijection $I''\cap\{i\} \xrightarrow{\cong} I\cap\{i\}$ which is therefore an isomorphism (with respect to $\prec$ and $<$). Denote by $\overline{x}, \overline{x+1} \in I''$ the (unique) preimages under $I\cap f'$ of x and $x+1$, respectively; they satisfy $\overline{x}+1 = \overline{x+1} \prec \overline{x}$ by the isomorphism property, which means they are the maximal and minimal element of the linearly ordered set $(I'', \prec)$, respectively. Since both $\overline{x}$ and $\overline{x+1}$ are mapped to y' by $f\cap I'$ we deduce that $f\cap I'\colon I'' \to I'$ is constant. This can only happen if f was already constant and f' was an equivalence. Hence the square (2.3.6) is degenerate and therefore trivially a pushout in Λ.
  - The case $i' = f'(x') = f(x'+1)$ is analogous.

  We may therefore assume that x and x' are the maximal elements (with respect to both $<$ and $\prec$) of their corresponding preimages $f^{-1}\{i\}$ and $f'^{-1}\{i'\}$. It follows directly that $f(x+1) > i$ and $f'(x'+1) > i'$; it remains to show $f'(x'+1) > i$ and we may assume that $i' < i$. Next, we show that there is no $j \in [n]$ with $i' < j \leq i$ which is in the image of $f'' := f\cap f'\colon I'' \to [n]$:

  - Otherwise, choose $w'' \in I''$ with $f''(w'') = j$. Set $w' := (f\cap I')(w'') \in I'$ and $w := (I\cap f')(w'') \in I$. We have $z < w$ and $z' \leq x' < w'$ by construction and $w \leq x$ because x is maximal for $<$ in the preimage $f^{-1}\{i\}$. Hence we have (after cyclic rotation and using that x and x' are $\prec$-maximal) $z \prec w \preceq x$ and $w' \prec z' \preceq x'$, which implies $z'' \prec w''$ and $w'' \prec z''$, respectively. Contradiction.

  Since i is in the image of f (by definition) and each j with $i' < j \leq i$ is not in the image of f'', it follows from the compatibility of (f, f') that each such j is not in the image of f'. Since we already know $f'(x'+1) > i'$ we obtain $f'(x'+1) > i$, as desired; this concludes the case $N = \langle 0\rangle$.

- We prove the case of a general N. To see the existence of the dashed map in the diagram (2.3.7), choose any map $N \to \langle 0\rangle$. By the case $N = \langle 0\rangle$ which we have just shown, we can fill the dotted morphism $\langle n\rangle \to \langle 0\rangle$ of cyclic sets in the following commutative diagram

$$\begin{tikzcd}
I \cap I' \arrow[r, "f \cap I'"] \arrow[d, "I \cap f'"'] & I' \arrow[d, "f"'] \arrow[ddr, "p'"] & & \\
I \arrow[r, "f"] \arrow[drr, "p"'] & \langle n \rangle \arrow[dr, dashed] \arrow[drr, dotted] & & \\
& & N \arrow[r] & \langle 0 \rangle
\end{tikzcd} \tag{2.3.8}$$

Thus we have constructed a diagram in the overcategory $\Lambda_{/\langle 0 \rangle}$. Under the canonical identification $\Delta \cong \Lambda_{/\langle 0 \rangle}$, the top left square of the diagram (2.3.8) gets identified with a cyclic rotation of the original diagram (2.3.6). Since any cyclic rotation of a left active compatible claw is compatible, we deduce from Corollary 2.3.1.5 that the corresponding Čech square is a pushout in $\Delta \cong \Lambda_{/\langle 0 \rangle}$. We conclude by the pushout property that the desired dashed map $\langle n \rangle \to N$ in (2.3.8) and *a fortiori* in (2.3.7) exists.

To prove uniqueness, recall that the square (2.3.6) is a pushout on the level of underlying sets, so that the dashed map is unique as a function of underlying sets. If $\langle n \rangle \to N$ is constant then it factors uniquely as $\langle n \rangle \to \langle 0 \rangle \to N$, hence is unique by the case $N = \langle 0 \rangle$. If $\langle n \rangle \to N$ is not constant then it is uniquely determined by its underlying function of sets. ■

Proof (of Proposition 2.3.2.4)**.** If $\mathcal{F}$ is cyclically compatible then by Lemma 2.3.2.6 there is a cyclic rotation $\mathcal{F}^{-m}$ of $\mathcal{F}$ which is left active and compatible. Since $\mathcal{F}$ and $\mathcal{F}^{-m}$ have isomorphic images in Λ, it is enough to show that the latter image is strongly biCartesian. Since the Čech cube $\check{C}\mathcal{F}^{-m}$ is left active and strongly biCartesian, it follows from Lemma 2.3.2.7 and Lemma 2.3.2.8 (applied to each 2-dimensional face of the cube) that its image in Λ is still strongly biCartesian.

Conversely, let Q be a strongly biCartesian cube in Λ extending $\mathcal{F}$. Then every choice of $m \in [n]$ yields a structure map $[n]^{+m} \colon Q(\varnothing) = \langle n \rangle \to \langle 0 \rangle$ which gives rise to a cube Q_m in $\Lambda_{/\langle 0 \rangle} \cong \Delta$ that maps to Q and extends the claw $\mathcal{F}^{+m}$. Since the slice projection $\Delta \to \Lambda$ reflects pullbacks and pushouts, we deduce that each of these cubes Q_m is strongly biCartesian. Hence by Corollary 2.3.1.5 the corresponding claw $\mathcal{F}^{+m}$ is compatible. We conclude by Lemma 2.3.2.6 that the original claw $\mathcal{F}$ is cyclically compatible. ■

Proof (of Corollary 2.3.2.5)**.** Recall from Corollary 2.3.1.5 that strongly biCartesian S-cubes in Δ are precisely the Čech cubes of compatible S-claws. Hence Corollary 2.3.2.5 follows directly from Proposition 2.3.2.4 and Lemma 2.3.2.6. ■

2.3.3 Primitive decomposition of biCartesian cubes

In this section we show how a strongly biCartesian cube in Δ can be decomposed into simpler building blocks.

Definition 2.3.3.1. A map $f \colon I \to [n]$ in Δ is called **primitive** if there is exactly one $i \in [n]$ such that $f^{-1}\{i\}$ is not a singleton; the map f is called **preprimitive** if it is primitive or an isomorphism. A candidate covering $\mathcal{F}$ in Δ_+ (and the corresponding Čech cube $\check{C}\mathcal{F}$) is called **(pre)primitive** if the claw $\mathcal{F}$ consists only of (pre)primitive maps. ◊

Construction 2.3.3.2. Let $f \colon I \to [n]$ be a map in Δ. For each $i \in \{-1, 0, \ldots, n\}$, we define objects

$$I_i := f^{-1}[i] \star [n \setminus i]$$

in Δ. Then f admits a canonical factorization

$$f \colon I = I_n \xrightarrow{\overline{f}_n} \ldots \xrightarrow{\overline{f}_{i+1}} I_i \xrightarrow{\overline{f}_i} \ldots \xrightarrow{\overline{f}_1} I_0 \xrightarrow{\overline{f}_0} I_{-1} = [n] \tag{2.3.9}$$

where each map $\overline{f}_i \colon I_i \to I_{i-1}$ is given as

$$\overline{f}_i \coloneqq \mathrm{Id}_{f^{-1}[i-1]} \star \big(f \cap \{i\} : f^{-1}\{i\} \to \{i\}\big) \star \mathrm{Id}_{[n\setminus i]}.$$

Observe that each map $\overline{f}_i$ is preprimitive. ◊

Lemma 2.3.3.3. Let $(f \colon I \to [n], f' \colon I' \to [n])$ be backwards compatible and factorize f as in Construction 2.3.3.2.

(1) For each $i \in [n]$, the composition $I_i \to [n]$ in (2.3.9) is backwards compatible with f' so that by Proposition 2.3.1.4 we can form the pullbacks

$$\begin{tikzcd}
I \cap I' \arrow[r] \arrow[d, "I\cap f'"'] & I_{n-1} \cap I' \arrow[r] \arrow[d, "I_{n-1}\cap f'"'] & \dots \arrow[r] & I_1 \cap I' \arrow[r] \arrow[d, "I_1\cap f'"] & I_0 \cap I' \arrow[r] \arrow[d, "I_0\cap f'"] & I' \arrow[d, "f'"] \\
I \arrow[r, "\overline{f}_n"] \arrow[rrrrr, bend right, "f"'] & I_{n-1} \arrow[r, "\overline{f}_{n-1}"] & \dots \arrow[r, "\overline{f}_2"] & I_1 \arrow[r, "\overline{f}_1"] & I_0 \arrow[r, "\overline{f}_0"] & {[n]}
\end{tikzcd}$$

which factorize the Čech square of f and f' into smaller Čech squares.

(2) The original claw (f, f') is compatible if and only if the claw $(\overline{f}_i, I_{i-1} \cap f') \models I_{i-1}$ is compatible for each $i \in [n]$.

(3) The original claw (f, f') is cyclically compatible if and only if the claw $(\overline{f}_i, I_{i-1} \cap f') \models I_{i-1}$ is cyclically compatible for each $i \in [n]$. ☐

Proof. Follows by direct inspection of the explicit constructions. ■

Lemma 2.3.3.4. (1) Every strongly biCartesian cube Q in Δ can be decomposed into a pasting of preprimitive strongly biCartesian cubes. If Q was left active then each of these cubes can be chosen to be left active. If Q was right active then each of these cubes can be chosen to be right active.

(2) Every cube in Q in Δ which becomes strongly biCartesian in Λ can be decomposed into a pasting of preprimitive strongly biCartesian cubes, each of which is left active or right active.

(3) If the original cube Q in (1) or (2) is non-degenerate then the pastings can be chosen to consist of primitive cubes. ☐

Proof. By Corollary 2.3.1.5, each strongly biCartesian cube in Δ is the Čech cube $\check{C}\mathcal{F}$ of some compatible S-claw $\mathcal{F} = (f_s \,|\, s \in S)$. By Proposition 2.3.2.4, each cube in Δ which becomes strongly biCartesian in Λ is of this form $\check{C}\mathcal{F}$ where $\mathcal{F}$ is cyclically compatible. For each $s \in S$, consider the factorization of f_s into preprimitive maps from Construction 2.3.3.2. By a repeated application of Lemma 2.3.3.3, we can decompose the cube $\check{C}\mathcal{F}$ into a pasting of Čech cubes of compatible claws which are cyclically compatible if $\mathcal{F}$ was. Parts (1) and (2) of Lemma 2.3.3.4 now follow by applying Corollary 2.3.1.5, Proposition 2.3.2.4 and by the observing that preprimitive cyclically compatible claws are automatically either left active or right active. Part (3) follows with the same procedure by dropping all identities appearing in the factorizations produced by Construction 2.3.3.2. ■

2.4 Precovers and intersection cubes

Let $\mathcal{F} \models [n]$ be a S-claw on $[n]$ in Δ. If all of the maps in the claw $\mathcal{F}$ are injective then we call $\mathcal{F}$ an **(S-)precover** on $[n]$. Since precovers are trivially backwards compatible, Proposition 2.3.1.4 guarantees the existence of the Čech cube $\check{C}\mathcal{F}$; we call it the **intersection cube** of $\mathcal{F}$. If we view the injective maps $\mathcal{F} \ni f_s \colon I_s \hookrightarrow [n]$ as subsets $I_s \subseteq [n]$ of $[n]$ then the intersection cube of $\mathcal{F}$ is given explicitly by the intersections

$$T \longmapsto \bigcap_{t \in T} I_t,$$

(where the empty intersection is $[n]$ by convention); thus the terminology "intersection cube" is justified. A **cover** (not to be confused with cover*ing* as in Section 2.2.4) is a precover $\mathcal{F}$ whose prongs are jointly surjective, *i.e.*, $\bigcup \mathcal{F} = [n]$ when we identify the prongs of $\mathcal{F}$ with subsets of $[n]$.

2.4.1 Membrane spaces and refinements

By right Kan extension along the Yoneda embedding $\Delta \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$, we can extend any simplicial object $\mathcal{X} \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ to a functor

$$\mathcal{X} \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})^{\mathrm{op}} \longrightarrow \mathcal{C},$$

which we still denote by $\mathcal{X}$. Given any simplicial set K, we can calculate the value of $\mathcal{X}$ at K—which Dyckerhoff and Kapranov call the **object of K-membranes in $\mathcal{X}$**—by the pointwise formula for Kan extensions:

$$\mathcal{X}_K \simeq \lim \left(\left(\Delta_{/K} \right)^{\mathrm{op}} \to \Delta^{\mathrm{op}} \xrightarrow{\mathcal{X}} \mathcal{C} \right)$$

The inclusion $\Delta \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$ factors as $\Delta \hookrightarrow \Delta_+ \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$, where the second map sends the initial object $\varnothing$ to the initial presheaf. We can therefore evaluate any simplicial objet $\mathcal{X} \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ at $\varnothing$ and the value will be a terminal object in $\mathcal{C}$.

Given a candidate covering $\mathcal{F} = (f_s \colon I_s \to [n] \,|\, s \in S)$ in Δ, we obtain a simplicial set $\widetilde{\mathcal{F}}$ as the colimit

$$\widetilde{\mathcal{F}} \coloneqq \mathrm{colim} \left(\mathbf{P}_*^{\mathrm{op}}(S) \xrightarrow{\check{\mathrm{C}}\mathcal{F}} \Delta \longhookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set}) \right)$$

which comes equipped with a canonical map $\widetilde{\mathcal{F}} \to \Delta^n$. It is easy to see that if $\mathcal{F}$ is a precover (*i.e.*, if all maps f_s are injective) then $\widetilde{\mathcal{F}} \subseteq \Delta^n$ can be identified with the simplicial subset $\widetilde{\mathcal{F}} \coloneqq \bigcup_{I_s \in S} \Delta^{I_s}$ of the n-simplex. We say that a precover $\mathcal{F}' \models [n]$ is a **refinement** of $\mathcal{F} \models [n]$—written $\mathcal{F}' \preceq \mathcal{F}$—if and only if $\widetilde{\mathcal{F}'}$ is a simplicial subset of $\widetilde{\mathcal{F}}$; explicitly, this means that for every $I' \in \mathcal{F}'$ there is at least one $I \in \mathcal{F}$ such that $I' \subseteq I$ (as subobjects of $[n]$). We say the refinement $\mathcal{F}' \preceq \mathcal{F}$ is **degenerate** if $\widetilde{\mathcal{F}'} = \widetilde{\mathcal{F}}$. For each $[n] \in \Delta$ the assignment $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ describes an equivalence of categories between the category (which is just a preorder) of precovers and refinements on $[n]$ and the full subcategory of the overcategory $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})_{/\Delta^n}$ spanned by the simplicial subsets of Δ^n. An explicit inverse is given by identifying each simplicial subset $K \subseteq \Delta^n$ with the precover given by the maximal simplices of K. We will implicitly use this identification and write

$$\widetilde{\mathcal{F}} \coloneqq \left(I \,\middle|\, \Delta^I \hookrightarrow \widetilde{\mathcal{F}} \text{ maximal} \right) \models [n]$$

for the precover obtained from a precover $\mathcal{F}$ by "removing redundant subsets".

Remark 2.4.1.1. For every precover $\mathcal{F}$, the restriction $\check{\mathrm{C}}\mathcal{F}|_{\mathbf{P}_*^{\mathrm{op}}(S)} \colon \mathbf{P}_*^{\mathrm{op}}(S) \to \Delta_{+/\widetilde{\mathcal{F}}}$ of the Čech cube of $\mathcal{F}$ has a left adjoint given by

$$([m], \alpha \colon \Delta^m \to \widetilde{\mathcal{F}}) \longmapsto \left\{ s \in S \,\middle|\, \alpha(\Delta^m) \subseteq \Delta^{I_s} \right\}$$

which becomes a right adjoint after passing to opposite categories. Since left adjoints are homotopy initial, the canonical map

$$\mathcal{X}_{\widetilde{\mathcal{F}}} \simeq \lim \mathcal{X}|_{\left(\Delta_{+/\widetilde{\mathcal{F}}}\right)^{\mathrm{op}}} \xrightarrow{\simeq} \lim \mathcal{X} \circ \check{\mathrm{C}}\mathcal{F}|_{\mathbf{P}_*(S)}$$

is an equivalence. In particular, $\mathcal{X}$ satisfies descent with respect to $\mathcal{F}$ if and only if $\mathcal{X}$ sends the inclusion $\widetilde{\mathcal{F}} \hookrightarrow \Delta^n$ to an equivalence. ♢

Definition 2.4.1.2. We say that a refinement $\mathcal{F}' \preceq \mathcal{F}$ of precovers $[n]$ is $\mathcal{X}$-local if the induced morphism $\widetilde{\mathcal{F}'} \to \widetilde{\mathcal{F}}$ of simplicial sets is sent by $\mathcal{X}$ to an equivalence in $\mathcal{C}$. ◊

The following lemma (which is essentially [DJW19, Corollary 3.16]) is the main tool to compare to one another descent conditions with respect to various precovers.

Lemma 2.4.1.3. Let $\mathcal{F} \models [n]$ be a precover in Δ and $I \subset [n]$ a subset. Let $\mathcal{X} \colon \Delta_+^{\mathrm{op}} \to \mathcal{C}$ be an augmented simplicial object and assume that the restricted precover

$$\mathcal{F} \cap I := \big(I' \cap I \,\big|\, I' \in \mathcal{F}\big) \models I$$

on I is $\mathcal{X}$-local. Then the refinement $\mathcal{F} \preceq \widetilde{\mathcal{F} \cup \{I\}}$ is $\mathcal{X}$-local. In particular, the original precover $\mathcal{F}$ is $\mathcal{X}$-local if and only if the extended precover $\widetilde{\mathcal{F} \cup \{I\}}$ is $\mathcal{X}$-local. ☐

Proof. The refinement $\mathcal{F} \preceq \widetilde{\mathcal{F} \cup \{I\}}$ can be written as the composition of refinements

$$\mathcal{F} \preceq \mathcal{F} \cup \{I\} \preceq \widetilde{\mathcal{F} \cup \{I\}}. \tag{2.4.1}$$

The first refinement in the composition (2.4.1) is $\mathcal{X}$-local by Lemma 2.2.3.8 (due to the assumption of the Lemma 2.4.1.3 and using the identification of Remark 2.4.1.1); the second refinement is degenerate, hence always local. The claim follows. ■

2.4.2 Polynomial simplicial objects

Recalling the analogy to manifold calculus described in Section 2.1, we observe that compatible precovers can be identified precisely with the "open covers" of the form (2.1.2). Indeed, an S-precover $\mathcal{F}$ on $[n] \in \Delta$ is compatible if and only if every "point" $(x-1, x)$ of the "manifold" $[n]$ is contained in all but at most one of the elements of $\mathcal{F}$, which we think of as "open subsets" of $[n]$; in other words, $\mathcal{F}$ consists precisely of "open subsets" with "pairwise disjoint closed complements". The analogy thus motivates the following definition:

Definition 2.4.2.1. We call a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$ **polynomial of degree** $\leq |S|$ (or S-**polynomial**, for short) if $\mathcal{X}$ satisfies descent with respect to all compatible S-covers in Δ. ◊

Example 2.4.2.2. We depict, for $k = 1, 2, 3$, the unique non-degenerate compatible $[k]$-cover on $[2k]$:

0	1	2
∅	∗	∗
∗	∗	∅

0	1	2	3	4
∅	∗	∗	∗	∗
∗	∗	∅	∗	∗
∗	∗	∗	∗	∅

0	1	2	3	4	5	6
∅	∗	∗	∗	∗	∗	∗
∗	∗	∅	∗	∗	∗	∗
∗	∗	∗	∗	∅	∗	∗
∗	∗	∗	∗	∗	∗	∅

Note that for $n < 2k$, there are no non-degenerate compatible $[k]$-covers on $[n]$. ◊

The number of compatible S-covers on $[n] \in \Delta$ grows quite rapidly in n. Thus *a priori* to determine that a simplicial object is S-polynomial, there is an increasing number of conditions to check in each dimension. We show now that it suffices to check *any one* non-trivial condition in each dimension.

Proposition 2.4.2.3. Let $\mathcal{X} \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object in some ∞-category with finite limits. Assume that for each $n \geq 2k$ there exists a non-degenerate compatible $[k]$-cover $\mathcal{F} \models [n]$ in Δ which is $\mathcal{X}$-local. Then all compatible $[k]$-covers are $\mathcal{X}$-local. ☐

Proof. Assume the assumption of Proposition 2.4.2.3. Recall that degenerate covers are automatically local. Hence there is nothing to show for $n < 2k$ because in this case there are no non-degenerate compatible $[k]$-covers on $[n]$. We prove by induction on $n \geq 2k$ that all non-degenerate compatible $[k]$-covers are $\mathcal{X}$-local. The inductions start is the case $n = 2k$, which is trivial because there is a unique non-degenerate compatible $[k]$-cover on $[2k]$. For the induction step consider the following directed graph:

- Vertices are non-degenerate compatible $[k]$-covers on $[n]$.
- Let $\mathcal{F}$ be a non-degenerate compatible $[k]$-cover and let $I \in \mathcal{F}$ and $x \in [n] \setminus I$ such that $I' := I \cup \{x\} \neq [n]$. Then the cover $\mathcal{F}' := \widetilde{\mathcal{F} \cup \{I'\}}$ is easily seen to be again $[k]$-pronged, compatible and non-degenerate. We add the refinement

$$\mathcal{F} \preceq \widetilde{\mathcal{F} \cup \{I'\}}$$

  to the graph as an arrow $\mathcal{F} \to \mathcal{F}'$. Observe that in the language of Remark 2.3.1.3, the cover $\mathcal{F}'$ arises from the cover $\mathcal{F}$ by choosing a row with at least two $\varnothing$'s and replacing one of them by $*$.

With the notation above it is easy to see that the restricted $[k]$-cover $\mathcal{F} \cap I' \models I'$ is still compatible, hence $\mathcal{X}$-local by the induction hypothesis (since $I' \subsetneq [n]$). It follows from Lemma 2.4.1.3 that every arrow in the graph corresponds to an $\mathcal{X}$-local refinement. The proof of Proposition 2.4.2.3 is concluded by the easy combinatorial observation that the graph is connected as an undirected graph, *i.e.*, one can connect every pair of non-degenerate compatible $[k]$-covers by a zigzag of $\mathcal{X}$-local refinements as above. ■

Remark 2.4.2.4. The directed graph constructed in the proof of Proposition 2.4.2.3 is just the Hasse diagram of the poset of non-degenerate compatible $[k]$-covers under refinement. Our proof therefore shows that if there is an $n \geq 2k$ such that $\mathcal{X}$ satisfies descent with respect to all compatible $[k]$-covers in $\Delta_{<n}$ then all refinements between non-degenerate compatible $[k]$-covers on $[n]$ are $\mathcal{X}$-local. ◊

2.5 Weakly excisive and weakly Λ-excisive simplicial objects

Fix an ∞-category $\mathcal{C}$ with finite limits. Recall from Section 2.2.4 that a simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is

- weakly S-excisive if it sends strongly biCartesian S-cubes in Δ to Cartesian cubes in $\mathcal{C}$.
- weakly S-Λ-excisive if it sends to Cartesian cubes in $\mathcal{C}$ those S-cubes in Δ which become strongly biCartesian in Λ after applying the canonical functor $\Delta \to \Lambda$.

Remark 2.5.0.1. It follows from Remark 2.2.4.9 that a cyclic object $\Lambda^{\mathrm{op}} \to \mathcal{C}$ is weakly S-excisive if and only if its restriction to Δ is weakly S-Λ-excisive. ◊

We can refine the notion of weak Λ-excision as follows:

Definition 2.5.0.2. A simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in $\mathcal{C}$ is called

- **lower weakly S-Λ-excisive** if $\mathcal{X}$ sends every left active strongly biCartesian S-cube in Δ to a Cartesian cube in $\mathcal{C}$;
- **upper weakly S-Λ-excisive** if $\mathcal{X}$ sends every right active strongly biCartesian S-cube in Δ to a Cartesian cube in $\mathcal{C}$. ◊

The terminology is justified by the following easy lemma.

Lemma 2.5.0.3. A simplicial object is weakly S-Λ-excisive if and only if it is both lower weakly S-Λ-excisive and upper weakly S-Λ-excisive. □

Proof. By Lemma 2.3.3.4, every S-cube in Δ with strongly biCartesian image in Λ can be decomposed into a pasting of strongly biCartesian cubes each of which is left active or right active; thus we have "if". The converse "only if" follows from the fact (Corollary 2.3.2.5) that every strongly biCartesian in Δ which is left active or right active has a strongly biCartesian image in Λ. ■

2.5.1 Weakly excisive = polynomial

As explained in Section 2.4.2, a polynomial functor of degree $\geq k$ is a simplicial object $\Delta^{\mathrm{op}} \to \mathcal{C}$ which sends all strongly biCartesian intersection $[k]$-cubes to Cartesian cubes in $\mathcal{C}$. *A priori*, this does not agree with weak k-excision, because it only takes into account strongly biCartesian cubes which consist of *injective* maps. The next theorem states that this discrepancy is illusory both for weak (Δ-)excision and for (lower and/or upper) weak Λ-excision.

Theorem 2.5.1.1. Let $\mathcal{C}$ be an ∞-category with all finite limits. A simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is

(a) weakly S-excisive if and only if it sends primitive strongly biCartesian intersection S-cubes in Δ to Cartesian cubes in $\mathcal{C}$;

(b) lower weakly S-Λ-excisive if and only if it sends primitive strongly biCartesian left active intersection S-cubes in Δ to Cartesian cubes in $\mathcal{C}$;

(c) upper weakly S-Λ-excisive if and only if it sends primitive strongly biCartesian right active intersection S-cubes in Δ to Cartesian cubes in $\mathcal{C}$. ☐

Before we prove Theorem 2.5.1.1, we deduce the following criterion for detecting weak Λ-excision of a simplical object in terms of weak (Δ-)excision of its path objects.

Corollary 2.5.1.2 (Path space criterion)**.** A simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in an ∞-category with all finite limits is

- lower weakly S-Λ-excisive if and only if the left path object $P^{\triangleleft}\mathcal{X} := \mathcal{X} \circ ([0] \star -)$ is weakly S-excisive;
- upper weakly S-Λ-excisive if and only if the right path object $P^{\triangleright}\mathcal{X} := \mathcal{X} \circ (- \star [0])$ is weakly S-excisive. ☐

Proof. Observe that composition with the functor $[0] \star -\colon \Delta \to \Delta$ identifies compatible S-covers in Δ with left active compatible S-covers in Δ; hence by Corollary 2.3.1.5 it identifies strongly biCartesian intersection S-cubes in Δ with left active strongly biCartesian intersection S-cubes Δ. The first statement of Corollary 2.5.1.2 now follows directly from Theorem 2.5.1.1; the proof of the second statement is analogous. ■

Remark 2.5.1.3. The proof of Corollary 2.5.1.2 makes crucial use of Theorem 2.5.1.1 because in general a left active diagram in Δ need not factor through the functor $[0] \star -\colon \Delta \to \Delta$. It is the fact that we can reduce to diagrams of *injective* maps that makes this argument work. ♢

To prove Theorem 2.5.1.1 we isolate the following key lemma which we prove separately below. Recall that, for each $m \geq 0$, we denote the unique active maps $[1] \to [m]$ in Δ by a_m.

Lemma 2.5.1.4 (Key lemma)**.** Let $p\colon \mathcal{C} \to \mathcal{B}$ be a Cartesian fibration of ∞-categories. Let $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object. Assume that, for all $m \geq 1$, the edge $\mathcal{X}(a_m)$ of $\mathcal{C}$ is p-Cartesian. Then the edge $\mathcal{X}(\alpha)$ is also p-Cartesian for every active morphism α in Δ. ☐

Proof (of Theorem 2.5.1.1)**.** We will prove part (a); the proof for (b) or (c) is the same, word by word, by only considering cubes which are left or right active, respectively. The direction "only if" is trivial.

To prove "if" let $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object which sends primitive strongly biCartesian intersection S-cubes in Δ to Cartesian cubes in $\mathcal{C}$. Assume that there is a counterexample to Theorem 2.5.1.1, *i.e.*, a compatible S-claw $\mathcal{F} = (f_s \,|\, s \in S)$ on $[n] \in \Delta$ such that the corresponding Čech cube $\check{\mathrm{C}}\mathcal{F}$ is not sent by $\mathcal{X}$ to a Cartesian cube in $\mathcal{C}$. By Lemma 2.3.3.4 we may choose $\mathcal{F}$ to be preprimitive. We may assume that $\mathcal{F}$ is primitive because otherwise it would be degenerate; and degenerate cubes are always sent to Cartesian cubes. By induction we may additionally assume that the number

$$d\mathcal{F} := |\{s \in S \,|\, f_s \text{ is not injective}\}|$$

is minimal amongst all counterexamples. The number $d\mathcal{F}$ has to be at least one, because otherwise $\check{\mathrm{C}}\mathcal{F}$ would be an intersection S-cube which is not a counterexample by assumption. Choose an $s \in S$ such that f_s is not injective and write $S' \coloneqq S \setminus \{s\}$. Since f_s is primitive, it is of the form

$$f_s = \mathrm{Id}_{[i-1]} \star \big(f_s^{-1}\{i\} \to \{i\}\big) \star \mathrm{Id}_{[n\setminus i]}.$$

for some $i \in [n]$. Denote by L, A and R the S-claws obtained by restricting the S-claw $\mathcal{F}$ to $[i-1]$, $\{i\}$ and $[n \setminus i]$, respectively. Hence we have $\mathcal{F} = L \star A \star R$. Denote by L' and R' the S'-claws induced from L and R, respectively. Since the restriction of f_s to both $[i-1]$ and $[n \setminus i]$ is the identity, the edges

$$\check{\mathrm{C}}^s L \colon \Delta^1 \longrightarrow \mathrm{Fun}(\mathbf{P}^{\mathrm{op}}(S'), \Delta) \qquad \text{and} \qquad \check{\mathrm{C}}^s R \colon \Delta^1 \longrightarrow \mathrm{Fun}(\mathbf{P}^{\mathrm{op}}(S'), \Delta),$$

corresponding to the Čech cubes $\check{\mathrm{C}}L$ and $\check{\mathrm{C}}R$, are the identity on the objects $\check{\mathrm{C}}L'$ and $\check{\mathrm{C}}R'$ of $\mathrm{Fun}(\mathbf{P}^{\mathrm{op}}(S'), \Delta)$, respectively. Denote by $\mathrm{const}\colon \Delta \to \mathrm{Fun}(\mathbf{P}^{\mathrm{op}}(S'), \Delta)$ the constant-diagram functor and define a cosimplicial object Y in $\mathrm{Fun}(\mathbf{P}^{\mathrm{op}}(S'), \Delta)$ by

$$Y \colon \Delta \xrightarrow{\mathrm{const}} \mathrm{Fun}(\mathbf{P}^{\mathrm{op}}(S'), \Delta) \xrightarrow{\check{\mathrm{C}}L' \star (-) \star \check{\mathrm{C}}R'} \mathrm{Fun}(\mathbf{P}^{\mathrm{op}}(S'), \Delta)$$

Denote by $\mathcal{Y}$ the simplicial object

$$\mathcal{Y} \colon \Delta^{\mathrm{op}} \xrightarrow{Y^{\mathrm{op}}} \mathrm{Fun}(\mathbf{P}^{\mathrm{op}}(S'), \Delta)^{\mathrm{op}} = \mathrm{Fun}(\mathbf{P}(S'), \Delta^{\mathrm{op}}) \xrightarrow{\mathcal{X} \circ -} \mathrm{Fun}(\mathbf{P}(S'), \mathcal{C})$$

and by

$$p\colon\ \mathrm{Fun}(\mathbf{P}(S'), \mathcal{C}) \longrightarrow \mathrm{Fun}(\mathbf{P}_*(S'), \mathcal{C})$$

the Cartesian fibration of Lemma 2.2.3.7. Observe, that the value of Y at the (active) edge $f_s \cap \{i\} \colon (f_s^{-1}\{i\} \to \{i\})$ is precisely the edge $\check{\mathrm{C}}^s\mathcal{F}$ in $\mathrm{Fun}(\mathbf{P}(S'), \Delta)$ associated to the Čech cube $\check{\mathrm{C}}\mathcal{F}$. By Lemma 2.2.3.7, the simplicial object $\mathcal{X}$ sends the cube $\check{\mathrm{C}}\mathcal{F}$ to a Cartesian cube if and only if the edge $\mathcal{Y}(f_s \cap \{i\})$ is p-Cartesian.

To complete the proof we set up an application of the key lemma (Lemma 2.5.1.4) to show that this edge $\mathcal{Y}(f_s \cap \{i\})$ is p-Cartesian, so that the cube $\check{\mathrm{C}}\mathcal{F}$ was not a counterexample after all. Let $m \geq 1$ and consider the S-claw $\mathcal{F}^m = \big(f_{s'}^m \mid s' \in S\big)$ on $[i-1] \star [m] \star [n \setminus i]$ given by

$$f_{s'}^m \coloneqq (f_{s'} \cap [i-1]) \star \mathrm{Id}_{[m]} \star (f_{s'} \cap [n \setminus i])$$

for all $s' \neq s$ and by

$$f_s^m \coloneqq \mathrm{Id}_{[i-1]} \star (a_m \colon [1] \to [m]) \star \mathrm{Id}_{[n\setminus i]}.$$

It is clear that the S-claw $\mathcal{F}^m$ inherits compatibility from $\mathcal{F}$ and that the Čech cube $\check{\mathrm{C}}\mathcal{F}^m$ corresponds precisely to the edge

$$Y(a_m) \colon \Delta^1 \xrightarrow{a_m} \Delta \xrightarrow{Y} \mathrm{Fun}(\mathbf{P}^{\mathrm{op}}(S'), \Delta).$$

For every $s' \in S \setminus \{s\}$, the map $f_{s'}^m$ is injective if and only if $f_{s'}$ is injective. Furthermore, the map f_s^m is injective (this is where we use the condition $m \neq 0$); hence the number $d\mathcal{F}^m$ is smaller than $d\mathcal{F}$. By the minimality assumption on the counterexample $\mathcal{F}$, we conclude that the simplicial object $\mathcal{X}$ sends the Čech cube $\check{\mathrm{C}}\mathcal{F}^m$ to a Cartesian cube. By Lemma 2.2.3.7 this translates to the fact that the corresponding edge $\mathcal{X} \circ \check{\mathrm{C}}^s\mathcal{F}^m = \mathcal{Y}(a_m)$ in $\mathrm{Fun}(\mathbf{P}(S'), \mathcal{C})$ is p-Cartesian. Finally, we apply the key lemma (Lemma 2.5.1.4) to the Cartesian fibration p and the simplicial object $\mathcal{Y}$ to deduce that $\mathcal{Y}$ sends all active maps in Δ to p-cartesian edges; in particular this is true for the active map $f_s \cap \{i\} \colon f_s^{-1}\{i\} \to \{i\}$. This completes the proof. ■

2.5.2 Proof of the key lemma

Construction 2.5.2.1. Via the functor

$$J \longmapsto J \mathbin{\dot\cup} \{\infty\}$$

we identify the augmented simplex category Δ_+ with the wide subcategory $\Delta^{\mathrm{rstr}} \subset \Delta^{\max}$ spanned by the right strict morphisms. For every right active morphism $f\colon [m] \to [n]$ in Δ we define a left active morphism $f^-\colon [n] \to [m]$ by the formula

$$f^- : j \longmapsto \min f^{-1}\{j, \dots, n\}.$$

For every left active morphism $g\colon [n] \to [m]$ in Δ we define a left active morphism $g^+\colon [m] \to [n]$ by the formula

$$g^+ : i \longmapsto \max g^{-1}\{0, \dots, i\}. \qquad \lozenge$$

Lemma 2.5.2.2 (Joyal duality)**.** The assignments $f \mapsto f^-$ and $g \mapsto g^+$ of Construction 2.5.2.1 are mutually inverse and assemble to an isomorphism of categories

$$\Delta^{\max} \overset{\cong}{\longleftrightarrow} \Delta^{\min,\mathrm{op}}$$

(given by the identity on objects) which restricts to an isomorphism

$$\Delta_+ \cong \Delta^{\mathrm{rstr}} \overset{\cong}{\longleftrightarrow} \Delta^{\mathrm{act,op}}. \qquad \square$$

Proof. This is a straightforward calculation. ■

The category Δ^{act} has an initial object $[1]$ and a terminal object $[0]$ which, under the identification $\Delta_+ \cong \Delta^{\mathrm{act,op}}$ of Lemma 2.5.2.2 correspond to the objects $[0]$ and $\varnothing$ of Δ_+, respectively.

Lemma 2.5.2.3. Let $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a simplicial object in any ∞-category $\mathcal{C}$. Then the restriction of $\mathcal{X}$ to the subcategory $\Delta^{\mathrm{act,op}} \subset \Delta^{\mathrm{op}}$ is a limit cone. ☐

Proof. [Lur09, Lemma 6.1.3.16] states (after passing to opposite categories) that every augmented cosimplicial object $\Delta_+ \cong \Delta^{\mathrm{rstr}} \to \mathcal{C}$ which extends to a diagram $\Delta^{\max} \to \mathcal{C}$ is automatically a limit diagram. Hence by Lemma 2.5.2.2 every diagram $\Delta^{\min,\mathrm{op}} \to \mathcal{C}$ and, *a fortiori*, every simplicial object $\Delta^{\mathrm{op}} \to \mathcal{C}$ restricts to a limit diagram $\Delta^{\mathrm{act,op}} \to \mathcal{C}$. ■

Proof (of they key lemma, Lemma 2.5.1.4)**.** Denote by $\mathcal{X}^{\mathrm{act}}$ the restriction of $\mathcal{X}$ to Δ^{act}. Denote by $\Delta^{\mathrm{act}}_{\geq 1}$ the full subcategory of Δ^{act} spanned by the objects $[m]$ with $m \geq 1$. Applying Lemma 2.5.2.3 twice we deduce that $\mathcal{X}^{\mathrm{act}}$ and $p \circ \mathcal{X}^{\mathrm{act}}$ are limit cones; it follows from [Lur09, Proposition 4.3.1.5] that $\mathcal{X}^{\mathrm{act}}$ is also a p-limit cone, *i.e.*, a right p-Kan extensions of its restriction to $\Delta^{\mathrm{act,op}}_{\geq 1}$. Since the object $[1] \in \Delta^{\mathrm{act}}$ is initial, the assumption of Lemma 2.5.1.4 expresses precisely that the restriction of $\mathcal{X}^{\mathrm{act}}$ to $\Delta^{\mathrm{act,op}}_{\geq 1}$ is the right p-Kan extension of its restriction to $\{[1]\} \subset \Delta^{\mathrm{act}}$. We conclude by transitivity of p-Kan extensions [Lur09, Proposition 4.3.2.8] that $\mathcal{X}^{\mathrm{act}}$ is a right p-Kan extension of its restriction to $\{[1]\}$, which implies by the pointwise formula at $[0] \in \Delta^{\mathrm{act}}$ that the edge $\mathcal{X}(a_0\colon [1] \to [0])$ is p-Cartesian. For every active map $\alpha\colon [m] \to [n]$ in Δ we have $a_n \circ \alpha = a_m$ and we already know that the edges $\mathcal{X}(a_n)$ and $\mathcal{X}(a_m)$ are p-Cartesian; it follows by the left cancellation property of p-Cartesian edges [Lur09, Proposition 2.4.1.7] that the edge $\mathcal{X}(\alpha)$ is also p-Cartesian. ■

2.6 Higher Segal conditions

In this last section, we explain the relationship between the higher Segal spaces of Dyckerhoff and Kapranov and the notions of higher weak excision studied in 2.5.

2.6.1 Higher Segal covers

Fix a positive natural number $k \geq 1$. Given a subset $I \subseteq [n]$, a **gap** of I (with $[n]$ implicit) is an element $x \in [n]$ with $x \notin I$. A gap x of $I \subseteq [n]$ is called **even** if the cardinaity $|\{y \in I \mid x < y\}|$ is even. A subset $I \subseteq [n]$ is called **even** if all its gaps are even. Note that even subsets $I \subseteq [n]$ of cardinality $2k$ are precisely those which can be written as a disjoint union of the form

$$I = \dot{\bigcup}_{i=1}^{k} \{x_i - 1, x_i\},$$

with $0 \leq x_1 - 1 < x_1 < x_2 - 1 < \cdots < x_{k-1} < x_k - 1 < x_k \leq n$.

Definition 2.6.1.1. For each $n \geq 2k$, the **lower $(2k-1)$-Segal cover** on $[n] \in \Delta$ is defined as follows:

$$\mathrm{lSeg}^k_n := \{I \subset [n] \,|\, I \text{ even with of cardinality } |I| = 2k\} \models [n] \qquad \lozenge$$

Observe that the lower $(2k-1)$-Segal covers are precisely the canonical "good k-covers" described in Section 2.1. The first lower $(2k-1)$-Segal cover lSeg^k_n, *i.e.*, the one for $n = 2k$, is the unique non-degenerate compatible $[k]$-cover on $[n]$. As n grows bigger, the behavior of lower $(2k-1)$-Segal covers on $[n]$ and non-degenerate compatible $[k]$-covers on $[n]$ diverges dramatically: In the first case the number of prongs increasingly rapidly with $[n]$, but each subset of $[n]$ remains of constant size $2k$; in the second case it is the number of prongs $(k+1)$ that stays constant, while most of the subsets appearing in a compatible $[k]$-cover are large. This dichotomy should remind the reader of the analogous behavior of $\mathcal{J}^{\circ}_k$ and $\mathcal{J}^{\mathrm{h}}_k$ described in Section 2.1:

- Good k-covers of a manifold typically consist of a large number of open subsets; however, each of these subsets is simple and small (just a disjoint union of at most k balls)
- The open covers in $\mathcal{J}^{\mathrm{h}}_k$ always contain exactly $k+1$ open subsets $M \setminus A_i$; however, each of these open subsets is usually big and complicated.

Example 2.6.1.2. The following is a depiction of the first two lower 3-Segal covers:

0	1	2	3	4
∅	∗	∗	∗	∗
∗	∗	∅	∗	∗
∗	∗	∗	∗	∅

and

0	1	2	3	4	5
∅	∅	∗	∗	∗	∗
∅	∗	∗	∅	∗	∗
∗	∗	∅	∅	∗	∗
∅	∗	∗	∗	∗	∅
∗	∗	∅	∗	∗	∅
∗	∗	∗	∗	∅	∅

Observe that the left cover is the unique non-degenerate compatible [2]-cover on $[4] = [2k]$. $\lozenge$

We now come to the definition of higher Segal objects. The definition we will use is not the original one, but rather a reformulation called the *path space criterion* [Pog17, Proposition 2.7].

Definition 2.6.1.3. A simplicial object $\mathcal{X} \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is called

- **lower $(2k-1)$-Segal** if, for each $n \geq 2k$, it satisfies descent with respect to the lower $(2k-1)$-Segal cover lSeg^k_n;
- **lower $2k$-Segal** if the left path object $P^{\lhd}\mathcal{X}$ is lower $(2k-1)$-Segal;
- **upper $2k$-Segal** if the right path object $P^{\rhd}\mathcal{X}$ is lower $(2k-1)$-Segal;
- **$2k$-Segal** if $\mathcal{X}$ is both lower and upper $2k$-Segal. $\lozenge$

2.6.2 Segal = polynomial = weakly excisive

We come now to the main result of this chapter, the comparison of higher Segal conditions and weak excision. The key ingredient is the following theorem, which identifies the hierarchy of lower odd Segal objects with the hierarchy of polynomial functors.

Theorem 2.6.2.1. Let $\mathcal{C}$ be an ∞-category with finite limits. The lower $(2k-1)$-Segal objects in $\mathcal{C}$ are precisely the polynomial functors $\Delta^{\mathrm{op}} \to \mathcal{C}$ of degree $\leq k$. ☐

Before we prove Theorem 2.6.2.1, we use it to deduce our main theorem.

Theorem 2.6.2.2. A simplicial object in an ∞-category with finite limits is

1. lower $(2k-1)$-Segal if and only if it is weakly k-excisive.
2. lower $2k$-Segal if and only if it is lower weakly k-Λ-excisive.
3. upper $2k$-Segal if and only if it is upper weakly k-Λ-excisive.
4. $2k$-Segal if and only if it is weakly k-Λ-excisive. ☐

Proof (of Theorem 2.6.2.2)**.** In Theorem 2.5.1.1 we have seen that a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$ is polynomial of degree $\leq k$ if and only if it is weakly k-excisive; thus part (1) is an immediate consequence of Theorem 2.6.2.1. The rest of Theorem 2.6.2.2 then follows immediately from the path space criterion for weak Λ-excision (Corollary 2.5.1.2). ■

Recall that a cyclic object $\Lambda^{\mathrm{op}} \to \mathcal{C}$ is defined to be $2k$-Segal if the underlying simplicial object $\Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}} \to \mathcal{C}$ is $2k$-Segal.

Corollary 2.6.2.3. A cyclic object in an ∞-category with finite limits is $2k$-Segal if and only if it is weakly k-excisive. ☐

Proof. Corollary 2.6.2.3 follows directly from Theorem 2.6.2.2 and Remark 2.5.0.1. ■

We now give the proof of Theorem 2.6.2.1.

Proof (of Theorem 2.6.2.1)**.** Fix a simplicial object $\mathcal{X} \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ with finite limits. By the characterization of strongly biCartesian intersection cubes in Δ (Corollary 2.3.1.5) we only need to show that $\mathcal{X}$ satisfies descent with respect to all lower $(2k-1)$-Segal covers if and only if $\mathcal{X}$ satisfies descent with respect to all compatible $[k]$-covers. In view of Proposition 2.4.2.3, we only have to relate, for each $n \geq 2k$, the lower $(2k-1)$-Segal cover to *one* non-degenerate compatible k-cover. For each $n \geq 2k$ and each $j \in \{-1, 0, \ldots, k\}$, we define a cover $\mathcal{F}^n_j \models [n]$ (with the k left implicit since it is fixed throughout the proof) to consist of the following subsets of $[n]$:

- $I^n_i \coloneqq [n] \setminus \{2i\}$ for $i = 0, \ldots, j$
- those $I \in \mathrm{lSeg}^n_k$ that satisfy $[2j] = \{0, 1, \ldots, 2j\} \subset I$.

Clearly $\mathcal{F}^n_{-1}$ is nothing but the lower $(2k-1)$-Segal cover $\mathrm{lSeg}^n_k \models [n]$. Moreover, we have a chain of refinements

$$\mathrm{lSeg}^n_k = \mathcal{F}^n_{-1} \preceq \mathcal{F}^n_0 \preceq \ldots \preceq \mathcal{F}^n_k \tag{2.6.1}$$

because every $I \in \mathrm{lSeg}^n_k$ with $[2(j-1)] \subset I$ must either satisfy $[2j] \subset I$ or $2j \notin I$. The last cover $\mathcal{F}^n_k = (I^n_i \mid i \in k)$ in the refinement (2.6.1) is a non-degenerate compatible $[k]$-claw; in this sense, the chain (2.6.1) is an interpolation between the Segal condition and the descent condition with respect to the family $\{\mathcal{F}^n_k \mid n \geq 2k\}$ of non-degenerate compatible $[k]$-covers in Δ.

We establish the following two facts:

1. If $n = 2k$ then the chain (2.6.1) of refinements collapses, *i.e.*, we have
$$\mathrm{lSeg}^{2k}_k = \mathcal{F}^{2k}_{-1} = \mathcal{F}^{2k}_0 = \cdots = \mathcal{F}^{2k}_k.$$
2. For every $n > 2k$ and every $j = 0, \ldots, n$ the refinement $\mathcal{F}^n_{j-1} \preceq \mathcal{F}^n_j$ is $\mathcal{X}$-local provided that the cover $\mathcal{F}^{n-1}_{j-1} \models [n-1]$ is $\mathcal{X}$-local.

Fact (1) is immediate from the definition. For each $j = 0, \ldots, k$ we have $\mathcal{F}^n_j = \mathcal{F}^n_{j-1} \mathbin{\widetilde{\cup}} \left\{ I^n_j \right\}$ and the cover $\mathcal{F}^n_{j-1} \cap I^n_j \models I^n_j$ is easily seen to be isomorphic (under the unique isomorphism $I^n_j \cong [n-1]$) to the cover $\mathcal{F}^{n-1}_{j-1} \models [n-1]$; hence fact (2) follows from Lemma 2.4.1.3.

By a straightforward inductive argument, facts (1) and (2) imply that the following three conditions are equivalent:

- For all $n \geq 2k$, the cover $\mathrm{lSeg}_k^n = \mathcal{F}_{-1}^n \models [n]$ is $\mathcal{X}$-local.
- For all $n \geq 2k$ and all $j = -1, \dots, k$, the cover $\mathrm{lSeg}_k^n = \mathcal{F}_j^n \models [n]$ is $\mathcal{X}$-local.
- For all $n \geq 2k$, the (nondegenerate, compatible, $[k]$-pronged) cover $\mathcal{F}_k^n \models [n]$ is $\mathcal{X}$-local.

We have therefore related the Segal conditions to one hierarchy of descent conditions with respect to non-degenerate compatible $[k]$-covers; Proposition 2.4.2.3 precisely states that this is enough, hence the proof is concluded. ■

2.6.3 Triviality bounds for higher Segal objects

Let $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a lower or upper d-Segal object in $\mathcal{C}$. Since for each $m > d$ the d-Segal conditions express the value $\mathcal{X}_m$ as a cubical limit of the values $\mathcal{X}_n$ with $n \leq d$, it is obvious that $\mathcal{X}$ is trivial (*i.e.*, $\mathcal{X}_n$ is a terminal object in $\mathcal{C}$ for each $[n] \in \Delta$) as soon as $\mathcal{X}$ is trivial when restricted to $\Delta_{\leq d}$. From the comparison with weak excision we can deduce the following sharper bounds:

Proposition 2.6.3.1. Fix $d \geq 2$ and let $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ be a lower or upper d-Segal object in an ∞-category $\mathcal{C}$ with finite limits. If $\mathcal{X}$ is trivial when restricted to $\Delta_{<d}$ then $\mathcal{X}$ is trivial. □

Remark 2.6.3.2. Since not every monoid is trivial, it is not true that a lower 1-Segal object (*i.e.*, a Segal object in the sense of Rezk) is trivial as soon as its restriction to $\Delta_{\leq 0}$ is trivial. Hence the assumption $d \geq 2$ in Proposition 2.6.3.1 is necessary. ◊

Proof (of Proposition 2.6.3.1)**.** First, we prove the case of lower odd Segal objects. Let $k \geq 2$ and assume that $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ is lower $(2k-1)$-Segal and trivial on $\Delta_{<2k-1}$. It suffices to show that $\mathcal{X}_{[2k-1]}$ is trivial. Consider the following compatible $[k]$-claw $\mathcal{F}$ on $[2k-2]$:

$$\begin{array}{c|ccccccccc}
 & 0 & 1 & 2 & 3 & 4 & \cdots & 2k-4 & 2k-3 & 2k-2 \\
\hline
0 & * & 2 & * & * & * & \cdots & * & * & * \\
1 & \varnothing & * & * & * & * & \cdots & * & * & * \\
2 & * & * & \varnothing & * & * & \cdots & * & * & * \\
3 & * & * & * & * & \varnothing & \cdots & * & * & * \\
\vdots & & & & & & & & & \\
k-1 & * & * & * & * & * & \cdots & \varnothing & * & * \\
k & * & * & * & * & * & \cdots & * & * & \varnothing
\end{array} \tag{2.6.2}$$

The corresponding biCartesian Čech cube $\check{\mathrm{C}}\mathcal{F}\colon \mathbf{P}^{\mathrm{op}}([k]) \to \Delta$ satisfies $\check{\mathrm{C}}\mathcal{F}(\{0\}) \cong [2k-1]$ and $\check{\mathrm{C}}\mathcal{F}(T) \in \Delta_{<2k-1}$ for all $T \neq \{0\}$. It follows that the $[k]$-cube $\mathcal{X} \circ \check{\mathrm{C}}\mathcal{F}$ sends every $T \subseteq [k]$, except possibly $T = \{0\}$, to a terminal object in $\mathcal{C}$. Since $\mathcal{X}$ is weakly k-excisive by Theorem 2.6.2.2, this cube in $\mathcal{C}$ is Cartesian. It then follows that we have a Cartesian square

$$\begin{tikzcd}
(\mathcal{X} \circ \check{\mathrm{C}}\mathcal{F})(\varnothing) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \lim\limits_{\substack{0 \notin T \subseteq [k] \\ \varnothing \neq T}} (\mathcal{X} \circ \check{\mathrm{C}}\mathcal{F})(T) \arrow[d] \\
(\mathcal{X} \circ \check{\mathrm{C}}\mathcal{F})(\{0\}) \arrow[r] & \lim\limits_{\substack{0 \in T \subseteq [k] \\ \{0\} \neq T}} (\mathcal{X} \circ \check{\mathrm{C}}\mathcal{F})(T)
\end{tikzcd}$$

in $\mathcal{C}$, where all but the lower left corner are trivial; we conclude that $(\mathcal{X} \circ \check{\mathrm{C}}\mathcal{F})(\{0\}) \simeq \mathcal{X}_{[2k-1]}$ is also trivial.

If $d = 2k$ is even with $k \geq 1$ then the same proof works for lower or upper $2k$-Segal objects by considering instead of (2.6.2) the left active compatible k-claw

	0	1	2	3	$\cdots$	$2k-3$	$2k-2$	$2k-1$
0	2	$*$	$*$	$*$	$\cdots$	$*$	$*$	$*$
1	$*$	$\varnothing$	$*$	$*$	$\cdots$	$*$	$*$	$*$
2	$*$	$*$	$*$	$\varnothing$	$\cdots$	$*$	$*$	$*$
$\vdots$								
$k-1$	$*$	$*$	$*$	$*$	$\cdots$	$\varnothing$	$*$	$*$
k	$*$	$*$	$*$	$*$	$\cdots$	$*$	$*$	$\varnothing$

on $[2k-1]$ or its obvious right active analog.

Recall from [Pog17, Proposition 2.7] that a simplicial object is upper $(2k+1)$-Segal if and only if its left path object is upper $2k$-Segal (or, equivalently, if its right path object is lower $2k$-Segal); the result for upper odd Segal objects thus follows immediately from the one for (lower or upper) even Segal objects. ■

It is not known to the author if the bounds in Proposition 2.6.3.1 are sharp. More precisely, the author does not know the answer to the following question, which remains to be investigated in future work:

Question 2.6.3.3. Let $k \geq 1$ and let $\mathcal{X}$ be a simplicial object which is lower $(2k-1)$-Segal, or upper $2k$-Segal or lower $2k$-Segal. If $\mathcal{X}$ is trivial when restricted to $\Delta_{\leq k}$, does it follow that $\mathcal{X}$ is trivial? ◊

Chapter 3

Homotopy coherent theorems of Dold–Kan type

3.1 Preliminaries

3.1.1 Pointed ∞-categories

Recall that an ∞-category $\mathcal{P}$ is called **pointed** if it has a zero object, *i.e.*, an object $0 \in \mathcal{P}$ which is both initial and terminal in $\mathcal{P}$. A functor $\mathcal{P}' \to \mathcal{P}$ between pointed ∞-categories is called **pointed** if it sends one (equivalently, each) zero object of $\mathcal{P}'$ to a zero object of $\mathcal{P}$. We denote by $\mathbf{Cat}^{\mathbf{0}}_{\infty}$ the ∞-category of (small) pointed ∞-categories and pointed functors between them; it comes equipped with a canonical forgetful functor

$$\mathbf{Cat}^{\mathbf{0}}_{\infty} \longrightarrow \mathbf{Cat}_{\infty}.$$

Given two pointed ∞-categories $\mathcal{P}'$ and $\mathcal{P}$, we denote by $\mathrm{Fun}^0(\mathcal{P}', \mathcal{P}) \subset \mathrm{Fun}(\mathcal{P}', \mathcal{P})$ the full subcategory spanned by the pointed functors.

Construction 3.1.1.1 (Free pointed category)**.** Let Z be an ordinary category. We define a pointed category Z_+ by freely adjoining a zero object to Z. Explicitly, it is described as follows:

- The objects of Z_+ are the objects of Z plus an additional object 0.
- For every object $x \in Z_+$ we put

  $$Z_+(x, 0) = \{0\} \qquad \text{and} \qquad Z_+(0, x) = \{0\}$$

  (in other words, $0 \in Z_+$ is a zero object as the notation suggests). Given objects $x, y \in Z$, we set

  $$Z_+(x, y) \coloneqq Z(x, y) \mathbin{\dot\cup} \{0\}$$

  where here 0 denotes the composite map $x \to 0 \to y$.
- The composition in Z_+ is inherited from the composition in Z. ◊

The pointed category Z_+ comes equipped with the canonical (non-full) inclusion functor $Z \to Z_+$.

Construction 3.1.1.2 (Free pointed ∞-category)**.** Let $\mathcal{Z}$ be an ∞-category. Denote by

$$\mathcal{Z}^{\triangleleft\triangleright} \coloneqq \{-\infty\} \star \mathcal{Z} \star \{+\infty\}$$

the ∞-category obtained from $\mathcal{Z}$ by freely adjoining an initial object $-\infty$ and a terminal object $+\infty$. We define $\mathcal{Z}_+$ to be the localization of $\mathcal{Z}^{\triangleleft\triangleright}$ at the (essentially unique) edge $-\infty \to +\infty$ connecting the initial to the terminal object. The ∞-category $\mathcal{Z}_+$ is pointed (since localizations preserve both initial and terminal objects$^{1)}$) and comes equipped with the defining functor $\mathcal{Z} \hookrightarrow \mathcal{Z}^{\triangleleft\triangleright} \to \mathcal{Z}_+$. ◊

$^{1)}$ This follows, for instance, from [Cis19, Proposition 7.1.10]

If the category $\mathcal{Z}$ in Construction 3.1.1.2 happens to be an ordinary category, then $\mathcal{Z}^{\triangleleft\triangleright}$ is again an ordinary category. It is however not clear *a priori* that the the same is true for $\mathcal{Z}_+$, because the localization procedure has the potential to turn an ordinary category into one that isn't. The following lemma addresses this issue.

Lemma 3.1.1.3. Let Z be an ordinary category. Then the functor $Z \to Z_+$ from Construction 3.1.1.2 agrees with the one from Construction 3.1.1.1. In particular, Z_+ is an ordinary category again. ☐

Proof. Let Z_+ be as in Construction 3.1.1.1 and consider the canonical functor

$$\gamma\colon Z^{\triangleleft\triangleright} = \{-\infty\} \star Z \star \{+\infty\} \longrightarrow Z_+$$

given by the canonical inclusion of Z and by $-\infty, +\infty \mapsto 0$. We need to show that γ exhibits Z_+ as the ∞-categorical localization of $\{-\infty\} \star Z \star \{+\infty\}$ at the unique map $-\infty \to +\infty$. Denote by $\langle -\infty, +\infty \rangle$ the full subcategory of $Z^{\triangleleft\triangleright}$ spanned by $-\infty$ and $+\infty$. Since $\langle -\infty, +\infty \rangle \cong \Delta^1$ is weakly contractible, it follows by comparing universal properties that the desired localization can be computed as the pushout $Z^{\triangleleft\triangleright} \sqcup_{\langle -\infty, +\infty \rangle} \{0\}$ (of ∞-categories). To conclude the proof, it therefore suffices to show that—after passing to nerves—the canonical square

$$\begin{tikzcd}
\langle -\infty, +\infty \rangle \arrow[r] \arrow[d, hook] & \{0\} \arrow[d] \\
Z^{\triangleleft\triangleright} \arrow[r] & Z_+
\end{tikzcd}$$

of categories becomes a (Joyal) homotopy pushout of simplicial sets. Since the left vertical map is a monomorphism, it suffices to show that the map

$$\mathrm{N}(\{0\}) \sqcup_{\mathrm{N}(\langle -\infty, +\infty \rangle)} \mathrm{N}(Z^{\triangleleft\triangleright}) \longrightarrow \mathrm{N}(Z_+) \tag{3.1.1}$$

from the (strict) pushout of simplicial sets is a (Joyal) weak equivalence; we will now show that it is in fact an inner anodyne extension.

The simplices of $\mathrm{N}(Z_+)$ can be described explicitly as follows: Each m-simplex of $\mathrm{N}(Z_+)$ is of the form

$$\sigma(k, x, t)\colon 0^{t(0)} \to x^1 \to 0^{t(1)} \to x^2 \to 0^{t(2)} \to \cdots \to 0^{t(k-1)} \to x^k \to 0^{t(k)}, \tag{3.1.2}$$

where

- k is a natural number
- each $x^i\colon x^i_0 \to \cdots \to x^i_{n(i)}$ (for $1 \le i \le k$) is an $n(i)$-simplex of $\mathrm{N}(Z)$.
- $t(0), \ldots, t(k)$ are natural numbers of which all but $t(0)$ and $t(k)$ are required to be positive.
- $0^{t(i)}$ denotes a chain $0 \to \cdots \to 0$ with $t(i)$ many zeros.
- the dimension $m := t(0) - 1 + \sum_{i=1}^{k} (n(i) + 1 + t(i))$ is non-negative.

Denote by $\mathrm{N}(Z_+)^{\le d} \subset \mathrm{N}(Z_+)$ the simplicial subset containing those simplices $\sigma(k, x, t)$ with $k \le d$. The following are straightforward to verify:

1. The map (3.1.1) induces an isomorphism $\mathrm{N}(\{0\}) \sqcup_{\mathrm{N}(\langle -\infty, +\infty \rangle)} \mathrm{N}(Z^{\triangleleft\triangleright}) \xrightarrow{\cong} \mathrm{N}(Z_+)^{\le 1}$.
2. For each $d \ge 1$, we have a pushout of simplicial sets

$$\begin{tikzcd}
\coprod\limits_{k,x,t} \Delta^{t(0)+n(1)+t(1)} \sqcup_{\Delta^{t(1)-1}} \Delta^{m'} \arrow[r] \arrow[d, hook] & \mathrm{N}(Z_+)^{\le d-1} \arrow[d, hook] \\
\coprod\limits_{k,x,t} \Delta^{m} \arrow[r, "(\sigma(k,x,t))"] & \mathrm{N}(Z_+)^{\le d} \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} \tag{3.1.3}$$

which corresponds to the decomposition of each chain (3.1.2) into the two overlapping chains

$$0^{t(0)} \to x^1 \to 0^{t(1)} \qquad \text{and} \qquad 0^{t(1)} \to x^2 \to 0^{t(2)} \to \cdots \to 0^{t(k-1)} \to x^k \to 0^{t(k)}$$

of dimensions $t(0)+n(1)+t(1)$ and $m' := -1+t(1)+\sum_{i=2}^{k}\left(n(i)+1+t(i)\right)$, respectively.

(3) The simplicial set $\mathrm{N}(Z_+)$ is the union of the ascending chain $\mathrm{N}(Z_+)^{\leq 1} \subset \mathrm{N}(Z_+)^{\leq 2} \subset \cdots$ of simplicial subsets.

The left vertical map in the square (3.1.3) is an inner anodyne extension; it follows from (1), (2) and (3) that the same is true for the map (3.1.1); this concludes the proof. ■

Remark 3.1.1.4. In view of Lemma 3.1.1.3, we are justified in tacitly assuming that the free pointed ∞-category Z_+ on an ordinary category Z is given by the explicit description of Construction 3.1.1.1. ◊

The following lemma establishes the universal property of the free pointed ∞-category construction.

Proposition 3.1.1.5. Let $\mathcal{Z}$ be a (small) ∞-category. For every pointed ∞-category $\mathcal{P}$, restriction along the functor $\mathcal{Z} \to \mathcal{Z}_+$ induces an equivalence

$$\mathrm{Fun}^0(\mathcal{Z}_+, \mathcal{P}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{Z}, \mathcal{P}).$$

of ∞-categories. In particular, the construction $\mathcal{Z} \mapsto \mathcal{Z}_+$ yields a left adjoint to the forgetful functor $\mathbf{Cat}_\infty^{\mathbf{0}} \to \mathbf{Cat}_\infty$ ☐

Proof. The functors $\mathcal{Z} \hookrightarrow \mathcal{Z}^{\triangleleft} \hookrightarrow \mathcal{Z}^{\triangleleft\triangleright} \longrightarrow \mathcal{Z}_+$ induce the following commutative diagram of functor ∞-categories and their various subcategories defined as indicated:

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{Z}_+, \mathcal{P}) \arrow[dd] \arrow[dr, "\simeq", "1"'] & \mathrm{Fun}^0(\mathcal{Z}_+, \mathcal{P}) \arrow[l, hook'] \arrow[dr, "\simeq", "2"'] & \\
& \{\text{inverts } -\infty \to +\infty\} \arrow[dl, hook'] & \{\text{inverts } -\infty \to +\infty \text{ and } -\infty, +\infty \mapsto 0\} \arrow[l, hook'] \\
\mathrm{Fun}(\mathcal{Z}^{\triangleleft\triangleright}, \mathcal{P}) \arrow[dd] & & \\
& \{+\infty \mapsto 0\} \arrow[ul, hook'] \arrow[dl, "\simeq", "3"'] & \{-\infty, +\infty \mapsto 0\} \arrow[l, hook'] \arrow[uu, hook, "="', "6"] \arrow[dl, "\simeq", "4"'] \\
\mathrm{Fun}(\mathcal{Z}^{\triangleleft}, \mathcal{P}) \arrow[d] & \{-\infty \mapsto 0\} \arrow[l, hook'] \arrow[dl, "\simeq", "5"'] & \\
\mathrm{Fun}(\mathcal{Z}, \mathcal{P}) & &
\end{tikzcd}$$

Restriction along $\mathcal{Z}^{\triangleleft\triangleright} \to \mathcal{Z}_+$ induces the equivalence 1 by the universal property of the localization. The functors labeled by 3 and 5 are equivalences because they have an inverse given by right Kan extension and left Kan extension, respectively (using that $0 \in \mathcal{P}$ is a terminal and an initial object, respectively). The equivalences 2 and 4 are induced by restricting to appropriate full subcategories. Since $\mathrm{Map}_{\mathcal{P}}(0,0) \simeq \mathrm{pt}$, every functor $\mathcal{Z}^{\triangleleft\triangleright} \to \mathcal{P}$ which sends $-\infty$ and $+\infty$ to zero objects must invert the edge $-\infty \to +\infty$; thus the inclusion labeled 6 is an equality of full subcategories. The result follows. ■

Remark 3.1.1.6. If $\mathcal{C}$ is an ∞-category which is not necessarily pointed but still has a terminal object $\star$, then the proof of Proposition 3.1.1.5 still shows that restriction along $\mathcal{Z}^{\triangleleft} \to \mathcal{Z}_+$ induces an equivalence between the ∞-categories $\mathrm{Fun}^{\star}(\mathcal{Z}_+, \mathcal{C})$ and $\mathrm{Fun}^{\star}(\mathcal{Z}^{\triangleleft}, \mathcal{C})$ consisting of those functors that send the zero object $0 \in \mathcal{Z}_+$ and the cone point $-\infty \in \mathcal{Z}^{\triangleleft}$ to a terminal object $\star \in \mathcal{C}$. The category $\mathrm{Fun}^{\star}(\mathcal{Z}^{\triangleleft}, \mathcal{C})$ is canonically equivalent to the ∞-category $\mathrm{Fun}(\mathcal{Z}, \mathcal{C}_\star)$ of $\mathcal{Z}$-shaped diagrams in the pointed ∞-category $\mathcal{C}_\star := \mathcal{C}_{\star/}$ of *pointed objects in* $\mathcal{C}$. Hence restriction along $\mathcal{Z} \to \mathcal{Z}_+$ induces an equivalence

$$\mathrm{Fun}^{\star}(\mathcal{Z}_+, \mathcal{C}) \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{Z}, \mathcal{C}_\star)$$

which explicitly sends the diagram $\mathcal{X} \in \mathrm{Fun}^{\star}(\mathcal{Z}_+, \mathcal{C})$ to the diagram $x \mapsto (\star = \mathcal{X}(0) \to \mathcal{X}(x))$. ◊

Lemma 3.1.1.7. Let $\mathcal{P}$ be a pointed ∞-category and let $\{f_i\colon x_i \to y_i \,|\, i \in I\}$ be a finite set of morphisms in $\mathcal{P}$. Assume that the product

$$\prod_{i\in I} f_i\colon \prod_{i\in I} x_i \longrightarrow \prod_{i\in I} y_i$$

exists in $\mathcal{P}$ and is an equivalence. Then for each $i \in I$, the morphism $f_i\colon x_i \to y_i$ is an equivalence.

□

Proof. Given an inverse $g\colon \prod_i y_i \to \prod_i x_i$ to $\prod f_i$, it is easy to see that for each $j \in I$ the composition

$$y_j \xrightarrow{\iota^j} \prod_i y_i \xrightarrow{g} \prod_i x_i \xrightarrow{\pi_j} x_j, \qquad \text{where} \qquad \pi_i \iota^j := \begin{cases} \mathrm{Id}\colon y_j \to y_j & \text{if } i = j \\ 0\colon y_j \to y_i & \text{if } i \neq j \end{cases}$$

is an inverse of f_j. ■

In a pointed ∞-category it makes sense to talk about fibers and cofibers which are the ∞-categorical analog of kernels and cokernels. The **fiber** and **cofiber** of an arrow $f\colon x \to y$ are the pullback and pushout of the diagrams

$$\begin{tikzcd} & 0 \arrow[d] \\ x \arrow[r, "f"] & y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \\ 0 & \end{tikzcd}$$

respectively. More generally, we define the **total cofiber** tot-fib D of a conical diagram $D\colon K^{\triangleright} \to \mathcal{P}$ as the cofiber of the canonical map

$$\mathrm{colim}(K \subset K^{\triangleright} \xrightarrow{D} \mathcal{P}) \to D(+\infty)$$

and the **total fiber** tot-cof D of a conical diagram $D\colon K^{\triangleleft} \to \mathcal{P}$ as the fiber of the canonical map

$$D(-\infty) \to \lim(K \subset K^{\triangleleft} \xrightarrow{D} \mathcal{P}).$$

To recover the case of the ordinary fiber/cofiber set $K = \Delta^0$, hence $K^{\triangleright} \cong \Delta^1 \cong K^{\triangleleft}$.

Another way of computing the total cofiber (resp. total fiber) of a $K^{\triangleright}$-shaped (resp. $K^{\triangleleft}$-shaped) diagram D is to first pass to its right (resp. left) Kan extension along the first inclusion $K^{\triangleright} \hookrightarrow K^{\triangleright} \sqcup_K K^{\triangleright}$ (resp. $K^{\triangleleft} \hookrightarrow K^{\triangleleft} \sqcup_K K^{\triangleleft}$)—which is given explicitly by setting the value on the cone point of the second copy of $K^{\triangleright}$ (resp. $K^{\triangleleft}$) to $0 \in \mathcal{P}$—and then taking the colimit (resp. limit) of this diagram. The advantage of this description is that it is well defined even if the colimit (resp. limit) of $D\big|_K$ does not exist in $\mathcal{P}$.

3.1.2 Quotient categories and coherent chain complexes

A chain complex in an ordinary pointed category P is a diagram $\mathbb{Z}^{\mathrm{op}} \to P$, which we might depict as

$$\cdots \xleftarrow{d} \bullet \xleftarrow{d} \bullet \xleftarrow{d} \bullet \xleftarrow{d} \cdots$$

such that any composite of more than one d is sent to the zero morphism in P. In other words, the category of chain complexes in P is a full subcategory of the category $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, P)$ of P-valued presheaves on $\mathbb{Z}$. In the ∞-categorical world, this naive definition would no longer be satisfactory because

- for a map in an ∞-category, being zero is no longer a *property* but the *structure* of an explicit null-homotopy and
- there should be be higher coherence data exhibiting all the trivializations $d \circ \cdots \circ d \simeq 0$ as compatible

Let Z be a category equipped with an ideal $S \subseteq Z$ (*i.e.*, a set of arrows satisfying $Z \circ S \circ Z \subseteq S$), we would like to say what it means to equip a diagram $Z \to \mathcal{P}$ with a coherent trivialization of all arrows in S.

Construction 3.1.2.1. We define a pointed category $\frac{Z}{S}$ as follows:

- The objects of $\frac{Z}{S}$ are the objects $x \in Z$ plus an additional zero object 0.
- The morphisms of $\frac{Z}{S}$ are determined by setting

  $$\frac{Z}{S}(x,y) \coloneqq \frac{Z(x,y)}{S} \cong \{f \in Z(x,y) \mid f \notin S\} \,\dot{\cup}\, \{x \to 0 \to y\}$$

  for $x, y \in Z$, with composition induced by the one in Z.

The category $\frac{Z}{S}$ comes equipped with the canonical functor $Z \to \frac{Z}{S}$ which is the identity on objects and sends precisely the arrows in S to zero. ♢

Remark 3.1.2.2. If $x \in Z$ is an object with $\mathrm{Id}_x \in S$ then the unique morphisms $x \to 0$ and $0 \to x$ are mutually inverse isomorphisms in $\frac{Z}{S}$. ♢

Definition 3.1.2.3. Let $Z \to \mathcal{P}$ be a Z-shaped diagram in a pointed ∞-category $\mathcal{P}$. We say that a **trivialization** of all arrows in S is an extension of $Z \to \mathcal{P}$ along $Z \to \frac{Z}{S}$ to a pointed functor $\frac{Z}{S} \to \mathcal{P}$. ♢

Example 3.1.2.4.

- The quotient $\frac{Z}{\varnothing}$ of Z by the empty ideal is the free pointed category Z_+. Hence Proposition 3.1.1.5 can be read as saying that the empty set of arrows can always be trivialized in a unique way.
- If the category Z is already pointed and $S = (0)$ consists of all zero maps $\bullet \to 0 \to \bullet$ then $\frac{Z}{(0)} \cong Z$.
- Every category Z has an ideal consisting of all non-isomorphisms; the corresponding quotient $\frac{Z}{\not\cong}$ is the free pointed category $Z_+^{\simeq}$ on the groupoid core $Z^{\simeq}$ of Z. ♢

Definition 3.1.2.5. We denote by $\mathrm{Ch} \coloneqq \frac{\mathbb{Z}}{(\to\to)}$ the quotient of the poset $\mathbb{Z}$ by the ideal $(\to\to)$ of all maps $\underline{n} \to \underline{m}$ with $m - n \geq 2$. A **coherent chain complex** in $\mathcal{P}$ is a pointed presheaf $\mathrm{Ch}^{\mathrm{op}} \to \mathcal{P}$; we denote by $\mathrm{Ch}(\mathcal{P}) \coloneqq \mathrm{Fun}^0(\mathrm{Ch}^{\mathrm{op}}, \mathcal{P})$ the ∞-category of coherent chain complexes in $\mathcal{P}$. Similarly, we set $\mathrm{Ch}_{\geq 0} \coloneqq \frac{\mathbb{N}}{(\to\to)}$ and define the ∞-categories of **connective chain complexes** in $\mathcal{P}$ as $\mathrm{Ch}_{\geq 0}(\mathcal{P}) \coloneqq \mathrm{Fun}^0\left(\mathrm{Ch}_{\geq 0}^{\mathrm{op}}, \mathcal{P}\right)$. ♢

Remark 3.1.2.6. If P is a pointed 1-category then it is straightforward to check that the restriction functor

$$\mathrm{Fun}^0\left(\frac{Z}{S}, P\right) \hookrightarrow \mathrm{Fun}(Z, P)$$

is fully faithful and that the essential image consists of those functors $Z \to P$ which send arrows in S to zero maps in P. This means that "sending arrows in S to zero" is a *property* which a diagram $Z \to P$ might or might not have. If $\mathcal{P}$ is an ∞-category, this is no longer true in general: specifying a lift of a diagram $Z \to \mathcal{P}$ to a pointed diagram $\frac{Z}{S} \to \mathcal{P}$ might require an infinite amount of additional *structure*. ♢

Remark 3.1.2.7. Another way to make the notion of trivialization of arrows in S precise would have been to work with ∞-categories enriched in *pointed* spaces or even in pairs of spaces. Then we could study pairs-enriched diagrams $Z \to \mathcal{P}$, where Z is pairs-enriched via S and where $\mathcal{P}$ is pairs-enriched (even $\mathcal{S}_\star$-enriched) via the zero maps. From this perspective one can see in a different way how the additional structure encoded in such trivializations comes in: unlike the forgetful functor $\mathbf{Set}_\star \to \mathbf{Set}$ from pointed sets to sets, the "forgetful" functor $\mathcal{S}_\star \to \mathcal{S}$ from the ∞-category of pointed spaces to the ∞-category of spaces is not faithful and in fact not even injective on π_0 of mapping spaces. ♢

3.1.3 Additive and preadditive ∞-categories

Definition 3.1.3.1. [GGN15, Definitions 2.1 and 2.6] An ∞-category $\mathcal{A}$ with finite products and coproducts is called **preadditive** if

- it is pointed, *i.e.*, the canonical map $\varnothing \xrightarrow{\simeq} \star$ from the initial objects to the terminal object is an equivalence.
- for any two objects $X, X' \in \mathcal{A}$, the canonical morphism

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} : X \sqcup X' \xrightarrow{\simeq} X \times X'$$

  (which exists, since $\mathcal{A}$ is pointed) is an equivalence.

The ∞-category $\mathcal{A}$ is called **additive** if additionally

- for each object $X \in \mathcal{A}$, the *shear map*

$$\begin{pmatrix} \mathrm{Id} & \mathrm{Id} \\ 0 & \mathrm{Id} \end{pmatrix} : X \sqcup X \xrightarrow{\simeq} X \times X$$

  is an equivalence.

A functor between preadditive ∞-categories is called **additive** if it is pointed and preserves finite products (or, equivalently, finite coproducts). ♢

Remark 3.1.3.2. Since products and coproducts in a preadditive ∞-category are canonically identified, it is customary to call them **direct sums** and denote them by the same symbol $\oplus$. ♢

Remark 3.1.3.3. When specializing to the case where $\mathcal{A}$ is an ordinary category, Definition 3.1.3.1 recovers the classical notion of an additive category (as defined, for instance, in [Mac98, Chapter VIII]). However, we warn the reader that our use of the word "preadditive" (which is taken from [GGN15]) might be confusing, since many authors write "preadditive category" to mean a category enriched in abelian groups. ♢

Lemma 3.1.3.4. [Lur18, Definition C.1.5.1] Let $\mathcal{A}$ be an ∞-category with finite products and coproducts. Then $\mathcal{A}$ is preadditive/additive if and only if its homotopy category $\mathrm{h}\mathcal{A}$ is preadditive/additive. □

Proof. The three maps defining the preadditivity/additivity of $\mathcal{A}$ in Definition 3.1.3.1 are sent by the functor $\mathcal{A} \to \mathrm{h}\mathcal{A}$ to the corresponding three maps defining the preadditivity/additivity of $\mathrm{h}\mathcal{A}$. Since a map is an equivalence in $\mathcal{A}$ if and only if it is an equivalence (*i.e.*, isomorphism) in $\mathrm{h}\mathcal{A}$, the result follows. ■

Example 3.1.3.5. Every stable ∞-category $\mathcal{D}$ (see [Lur17, Chapter 1]) is additive. More generally, every full subcategory $\mathcal{A} \subset \mathcal{D}$ closed under direct sums is additive. ♢

The following lemma states that every additive ∞-category $\mathcal{A}$ arises this way; at least if one is willing to pass to a larger universe where $\mathcal{A}$ is small.

Lemma 3.1.3.6. For every (small) additive ∞-category $\mathcal{A}$ there exists a stable ∞-category $\mathcal{D}$ and a fully faithful, additive functor $\mathcal{A} \hookrightarrow \mathcal{D}$. ☐

Proof. Let $\mathcal{D}$ stable be the ∞-category of additive spectral presheaves $\mathcal{A}^{\mathrm{op}} \to \mathcal{S}p$. It follows from Proposition C.1.5.7 and Remark C.1.5.9 in [Lur18] that there is a fully faithful, additive Yoneda embedding $\mathcal{A} \hookrightarrow \mathcal{D}$. ■

Finally, we need the following easy lemma.

Lemma 3.1.3.7. Let $\mathcal{A}$ be an additive ∞-category. Consider two n-tuples $(X_i)_{i=1}^n$, $(Y_i)_{i=1}^n$ of objects of $\mathcal{A}$ and a matrix $F = (f_{i,j} \colon X_i \to Y_j)_{i,j=1}^n$ of maps between them. Assume that

- all diagonal entries $f_{i,i} \colon X_i \to Y_i$ are equivalences
- all entries below the diagonal (*i.e.*, $f_{i,j}$ with $i > j$) factor through a zero object $0 \in \mathcal{A}$.

Then F, viewed as a map

$$F \colon \coprod_{i=1}^n X_i \xrightarrow{\simeq} \prod_{i=1}^n Y_i,$$

is an equivalence. ☐

Proof. By passing to the homotopy category, we may reduce to the case of ordinary additive categories; Lemma 3.1.3.7 is standard in this case. ■

3.1.4 Weakly idempotent complete ∞-categories

Recall that an additive 1-category A is called idempotent complete (or Karoubian) if every idempotent endomorphisms $p \colon X \to X$ induces a direct sum decomposition $X \cong \operatorname{Im} p \oplus \operatorname{Ker} p$. If A is embedded as a full additive subcategory of some abelian category, this amounts to saying that A is closed under summands; in particular, every abelian category is idempotent complete.

In the ∞-categorical world, the situation is a bit less favorable; for instance, even stable ∞-categories are not idempotent complete in general[2]. Fortunately the weaker condition of *weak* idempotent completeness will suffice for our purposes. While idempotent completeness is a way to say that the category is "closed under summands", weakly idempotent completeness should be read as "closed under direct complements"; in other words A is weakly idempotent complete additive if for each $X \in A$ and each direct sum decomposition $X \cong X' \oplus X''$ (in some ambient abelian category) we have $X' \in A$ if and only if $X'' \in A$. One way to intrinsically make this definition without reference to any ambient category is to say that an additive category A is weakly idempotent complete if each retraction (a.k.a. split epimorphism) has a kernel and each section (a.k.a. split monomorphism) has a cokernel (see for instance [TT90, A.5.1] and [Büh10, Definition 7.2]).

Next, we define weak idempotent completeness in the ∞-categorical setting. Let $\mathcal{P}$ be a pointed ∞-category. A **section-retraction pair** in $\mathcal{P}$, is a composable pair (r, s) of maps in $\mathcal{P}$ whose composite $r \circ s$ is an equivalence. We say that two section-retraction pairs (r, s) and (r', s') are **complementary**, if they fit in a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \bullet \arrow[r] \arrow[d, "s'"] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
\bullet \arrow[r, "s"] \arrow[d] \arrow[dr, phantom, "\square"] & \bullet \arrow[r, "r"] \arrow[d, "r'"] \arrow[dr, phantom, "\square"] & \bullet \arrow[d] \\
0 \arrow[r] & \bullet \arrow[r] & 0
\end{tikzcd}$$

(3.1.4)

---

[2] Splitting a 1-categorical idempotent p amounts to computing the kernels of p and of $\mathrm{Id} - p$ which exist in any abelian category. In contrast, the splitting a (coherent) ∞-categorical idempotent must be computed as an *infinite* limit which is not always possible.

where all squares are biCartesian (*i.e.*, both a pushout and a pullback); more precisely, we say that the diagram (3.1.4) exhibits (r', s') as the complement of (r, s), and vice versa.

Remark 3.1.4.1. It is not hard to show using Kan extensions that the evident forgetful functor

$$\{\text{diagrams (3.1.4) in } \mathcal{P}\} \xrightarrow{\simeq} \{\text{section-retraction pairs in } \mathcal{P} \text{ which admit a complement}\}$$

is an equivalence ∞-categories. This is the sense in which the complement of a section-retraction pair (together with the data exhibiting it as complementary) is essentially unique (if it exists). ◊

Definition 3.1.4.2. A pointed ∞-category $\mathcal{P}$ is called **weakly idempotent complete** if every section-retraction pair has a complement. ◊

Remark 3.1.4.3. When $\mathcal{P} = A$ is an additive 1-category, specifying a diagram (3.1.4) amounts to exhibiting s' as the kernel of r and r' as the cokernel of s. Hence in this case Definition 3.1.4.2 agrees with the classical notion of weak idempotent completeness. ◊

Example 3.1.4.4. Every stable ∞-category is weakly idempotent complete. More generally, each stable ∞-category gives rise to many examples by passing to subcategories which are closed under direct complements. ◊

3.2 Dold–Kan type theorems

3.2.1 DK-triples

In this section we describe the axiomatic framework of DK-triples which encompasses—and is essentially equivalent—to the setting of Lack and Street [LS15]; see Section 3.5.1 for a detailed comparison. Similar ideas were already present in prior work of Słomińska [Sło04; Sło11] and of Helmstutler [Hel14] (cf. Remark 3.3.2.3).

Let B be a category equipped with two subcategories $E, E^{\vee} \subset B$, each of which contains all isomorphism (in particular all objects). Arrows in E and $E^{\vee}$ are called E**pis** and **dual** E**pis**, respectively; we depict them with the symbols $\twoheadrightarrow$ (a two-headed arrow) and $\rightarrowtail$ (a tailed arrow), respectively. For each $b \in B$ we denote by $E(b)$ the category of Epis under b3); similarly, we denote by $E^{\vee}(b)$ the category of dual Epis over b.

We make the following auxiliary definitions:

- We call an arrow in B **singular** if it lies in the right ideal $\mathrm{Sing} := E^{\vee}_{\not\simeq} \circ B$ generated by the non-invertible dual Epis.
- An arrow which is not singular is called **regular**; we denote by $\mathrm{Reg} := B \setminus \mathrm{Sing}$ the set of regular arrows.
- We call an arrow a M**ono** if it does not lie in the left ideal generated by the non-invertible Epis. We denote by $M := B \setminus (B \circ E_{\not\simeq})$ the set of Monos.
- For each $b \in B$ we have a pairing $- \circ -\colon E(b) \times E^{\vee}(b) \to \mathrm{Ar}\, B$ given by composition (where $\mathrm{Ar}\, B$ denotes the category of arrows in B). We denote by

  $$\langle -; - \rangle_b \colon \pi_0 E(b) \times \pi_0 E^{\vee}(b) \longrightarrow \pi_0 \mathrm{Ar}\, B$$

  the induced pairing on isomorphism classes.

Definition 3.2.1.1. The datum $\mathbb{B} := (B, E, E^{\vee})$ is called

- A **DK-triple**4) if it satisfies the following properties (using the auxiliary notation introduced above):

---

3) The category $E(b)$ is nothing but the undercategory $E_{b/}$ (where b is viewed as an object of E). We do not use the latter notation because it can unfortunately be confused with the undercategory $E_{b/} = B_{b/} \times_B E$ (where b is viewed as an object of B).

4) Unsurprisingly, DK stands for Dold–Kan.

  - **(T1)** Every arrow f of B can be written uniquely (up to unique isomorphism) as a composition of the form

$$\begin{tikzcd} \bullet \arrow[r, two heads, "e'\in E"] & \bullet \arrow[r, "\overline{f}\in(M\cap\mathrm{Reg})"] & \bullet \arrow[r, tail, "e^\vee\in E^\vee"] & \bullet \end{tikzcd} \tag{3.2.1}$$

  - **(T2)** For each $b \in B$, the pairing $\langle -;-\rangle_b$ can be described by a finite square matrix which is "unipotent upper triangular modulo non-isomorphisms",
    i.e., there is a number $n \geq 1$ and bijections $\pi_0 E(b) \cong \{1,\dots,n\} \cong \pi_0 E^\vee(b)$, such that the pairing $\langle -;-\rangle_b$ induces an $n \times n$-matrix

$$\begin{pmatrix} \simeq & ? & \cdots & ? \\ \not\simeq & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & ? \\ \not\simeq & \cdots & \not\simeq & \simeq \end{pmatrix}$$

    with values in $\pi_0 \operatorname{Ar} B$ which has invertible arrows on the diagonal and non-invertible arrows below the diagonal (there is no condition on the arrows above the diagonal).
  - **(T3)** The set $\overline{B} \coloneqq E^\vee \circ E$ is closed under composition.
  - **(T4)** The composition of two regular Monos is a (not necessarily regular) Mono,
    i.e., $(M \cap \mathrm{Reg}) \circ (M \cap \mathrm{Reg}) \subset M$
  - **(T5)** The singular arrows form a left module over M, *i.e.*, we have $M \circ \mathrm{Sing} \subseteq \mathrm{Sing}$.
- a **diagonalizable DK-triple** if it if satisfies all axioms (T1)–(T5) above and the matrix in (T2) can even be made *diagonal* modulo non-isomorphisms.
- **reduced** if $B = E^\vee \circ E$. ◊

The following observations follow immediately from Definition 3.2.1.1.

Lemma 3.2.1.2. Let $\mathbb{B} = (B, E, E^\vee)$ be a DK-triple.

1. For each b there is a unique bijection $(-)^\vee\colon \pi_0 E(b) \longleftrightarrow \pi_0 E^\vee(b)$ such that for each $e \in E(b)$ the composition $e \circ e^\vee$ is an isomorphism in B.
2. Every Epi is a split epimorphism and every dual Epi is a split monomorphism in B.
3. For each $b \in B$, the categories $E(b)$ and $E^\vee(b)$ are both (equivalent to) posets.
4. Both Reg and M contain all isomorphisms of B.
5. An arrow B decomposed as in (3.2.1) is regular if and only if the component $e^\vee \in E^\vee$ is invertible and it is a Mono if and only if the component $e' \in E$ is invertible.
6. We have $M = (M \cap \mathrm{Reg}) \circ E$ and $\mathrm{Reg} = E^\vee \circ (M \cap \mathrm{Reg})$.
7. The datum $\overline{\mathbb{B}} \coloneqq (\overline{B}, E, E^\vee)$ is again a DK-triple which is automatically reduced.
8. If $\mathbb{B}$ is reduced then we have $M = E^\vee$ and $\mathrm{Reg} = E$ and $M \cap \mathrm{Reg} = B^\simeq$.
9. If $\mathbb{B}$ is reduced then the dual datum $\mathbb{B}^{\mathrm{op}} \coloneqq (B^{\mathrm{op}}, (E^\vee)^{\mathrm{op}}, E^{\mathrm{op}})$ is again a (reduced) DK-triple. □

Proof. Straightforward and left to the reader. ■

Each DK-triple $\mathbb{B} = (B, E, E^\vee)$ induces a canonical partial order $\leq$ on the set $\pi_0 B$ by declaring $b' \leq b$ if there exists a dual Epi $b' \rightarrowtail b$ or equivalently (by (1)) an Epi $b \twoheadrightarrow b'$. To see that $\leq$ is antisymmetric (*i.e.*, $b \leq b' \leq b$ implies $b \cong b'$) choose an Epi $e\colon b' \twoheadrightarrow b$ and an Epi $b \twoheadrightarrow b'$: the induced maps $- \circ e\colon \pi_0 E(b) \hookrightarrow \pi_0 E(b')$ and $\pi_0 E(b') \hookrightarrow \pi_0 E(b)$ are injective because Epis are (split) epimorphisms. Since the sets $\pi_0 E(b)$ and $\pi_0 E(b')$ are finite by (T2), this implies that $e \circ -$ is a bijection; hence e is a split monomorphism because $\mathrm{Id}_{b'}$ lies in the image of $e \circ -$; hence e is an isomorphism.

For each $b \in B$ the set $\{b' \in \pi_0 B \,|\, b' \leq b\}$ of predecessors of b is finite by (T2), hence the poset $(\pi_0 B, \leq)$ is suited for inductive arguments.

3.2.2 Key constructions

Construction 3.2.2.1. Assume that $\mathbb{B}$ is a DK-triple. We define a pointed category $N_0 = N_0(\mathbb{B})$ as the quotient

$$N_0 \coloneqq \frac{M}{M \cap \mathrm{Sing}}$$

of M by the two-sided ideal $M \cap \mathrm{Sing}$. Explicitly:

- The pointed category N_0 has a zero object 0 and for each object $b \in B$ an object $\underline{b} \in N$
- For every pair of objects $\underline{b'}, \underline{b} \in N$, we have the hom-set

$$N_0(\underline{b'}, \underline{b}) \coloneqq \frac{M(b', b)}{(M \cap \mathrm{Sing})} = (M \cap \mathrm{Reg})(b', b) \mathbin{\dot\cup} \left\{\underline{b'} \to 0 \to \underline{b}\right\}.$$

- Composition in N_0 is induced by composition in B; it is well defined because of (T4) and (T5).

For convenience we write N for the full subcategory of N_0 spanned by all objects except the zero object 0. ♢

Remark 3.2.2.2. A particularly simple case of Construction 3.2.2.1 occurs when the set $M \cap \mathrm{Reg}$ of regular Monos is closed under composition. In this case, $M \cap \mathrm{Reg}$ is a subcategory of B and the quotient $N_0 \coloneqq \frac{M}{M \cap \mathrm{Sing}} \cong \frac{M \cap \mathrm{Reg}}{\varnothing} \cong \frac{N}{(0)}$ is simply the free pointed category on the category $M \cap \mathrm{Reg}$. ♢

Notation 3.2.2.3. To minimize the potential confusion, we adopt the following conventions: Objects in N are denoted by n, n', n''. Objects in B are denoted by b, b', b''. Given an object $n \in N$, we denote by $[n]$ the corresponding object in B. ♢

We now come to the key construction of this chapter.

Construction 3.2.2.4. Let $\mathbb{B}$ be a DK-triple. We define the pointed category $V = V(\mathbb{B})$ as the "upper triangular" category

$$V \coloneqq \begin{pmatrix} N_0 & R_0 \\ 0 & B_+ \end{pmatrix} \coloneqq \begin{pmatrix} \frac{M}{\mathrm{Sing}} & \mathrm{Sing}\backslash B \\ 0 & B_+ \end{pmatrix}$$

associated to the N_0-B_+-bimodule $R_0 \coloneqq \mathrm{Sing}\backslash B$. More precisely, the category V is given explicitly as follows:

- The objects of V are given by the objects $n \in N$, the objects $b \in B$ and a zero object 0; in other words we have $\mathrm{Ob}\, V \coloneqq \mathrm{Ob}\, N_0 \sqcup_{\{0\}} \mathrm{Ob}\, B_+$.
- The hom-sets in V between two objects of N_0 or between two objects of B_+ are inherited from N_0 or from B_+, respectively.
- The only arrow in V from an object $n \in N_0$ to an objects $b \in B_+$ is the zero arrow $n \to 0 \to b$
- The set of arrows in V from $b \in B$ to $n \in N$ is defined to be

$$V(b, n) \coloneqq R_0(b, n) \coloneqq {}_{\mathrm{Sing}}\Big\backslash^{B(b, [n])} = \mathrm{Reg}(b, [n]) \mathbin{\dot\cup} \{b \to 0 \to n\}$$

- Composition in V is induced by the composition in N_0 and in B_+; the composition

$$N_0(n, n') \times R_0(b, n) \times B_+(b', b) \longrightarrow R_0(b', n')$$

  is well defined because $M \circ \mathrm{Sing} \circ B \subseteq \mathrm{Sing}$.

The pointed category V comes equipped with the two fully faithful embeddings

$$B_+ \hookrightarrow V \hookleftarrow N_0;$$

for convenience we identify B_+ and N_0 with their images in V. ♢

Notation 3.2.2.5. We denote by $!\colon [n] \to n$ the arrow corresponding to the identity $\mathrm{Id}_{[n]} \in \mathrm{Reg}([n],[n])$. For every non-zero arrow $u\colon b \to n$ in V we denote by $[u] \in \mathrm{Reg}(b,[n])$ the corresponding regular arrow in b; in other words, $[u]\colon b \to [n]$ is the unique arrow satisfying $![u] = u$. ◊

Remark 3.2.2.6. Assumptions (T4) and (T5) are needed to guarantee that Construction 3.2.2.1 and Construction 3.2.2.4 are well defined. In many examples M is actually a subcategory of B; in this case $M \cap \mathrm{Sing}$ is a two-sided ideal in M in the usual sense and Construction 3.2.2.1 becomes an instance of Construction 3.1.2.1. The notation in Construction 3.2.2.1 and Construction 3.2.2.4 should be understood with this more special (but still very general) case in mind. ◊

3.2.3 Statement

We now state the main theorem of this chapter.

Theorem 3.2.3.1 (Homotopy coherent correspondences of Dold–Kan type)**.** Let $\mathbb{B} = (B, E, E^\vee)$ be a DK-triple with associated pointed category $N_0 = N_0(\mathbb{B})$.

(a) For any weakly idempotent complete additive ∞-category $\mathcal{A}$, the restriction functors

$$\mathrm{Res}\colon \mathrm{Fun}^0(V,\mathcal{A}) \longrightarrow \mathrm{Fun}^0(B_+,\mathcal{A}) \qquad \text{and} \qquad \mathrm{Res}\colon \mathrm{Fun}^0(V,\mathcal{A}) \longrightarrow \mathrm{Fun}^0(N_0,\mathcal{A})$$

from Construction 3.2.2.4 admit a left adjoint LKE (left Kan extension) and a right adjoint RKE (right Kan extension), respectively.

(b) The composite adjunction

$$\begin{tikzcd} \mathrm{Fun}^0(B_+,\mathcal{A}) \arrow[r, shift left, "\mathrm{LKE}"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}^0(V,\mathcal{A}) \arrow[l, shift left, "\mathrm{Res}"] \arrow[r, shift left, "\mathrm{Res}"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}^0(N_0,\mathcal{A}) \arrow[l, shift left, "\mathrm{RKE}"] \end{tikzcd} \tag{3.2.2}$$

is an adjoint equivalence of ∞-categories.

(c) The adjoint equivalence (3.2.2) is natural in $\mathcal{A}$ with respect to additive functors.

(d) Consider a pointed functor $\mathcal{X}\colon B_+ \to \mathcal{A}$ and denote by $\overline{\mathcal{X}}\colon N_0 \to \mathcal{A}$ the pointed functor corresponding to $\mathcal{X}$ under the equivalence (3.2.2). Then for each $n \in N$ the canonical maps

$$\operatorname*{colim}_{b\in E^\vee_{\not\simeq}([n])} \mathcal{X}_b \longrightarrow \mathcal{X}_{[n]} \longrightarrow \lim_{b\in E_{\not\simeq}([n])} \mathcal{X}_b \tag{3.2.3}$$

form a section-retraction pair with complement equivalent to $\overline{\mathcal{X}}_n$. □

Remark 3.2.3.2. The notions of (pre)additivity and weak idempotent completeness are manifestly self-dual. Therefore in Theorem 3.2.3.1 (and all of the results below) we can replace the target ∞-category by its opposite, or, equivalently, B_+ by $(B_+)^{\mathrm{op}}$ and N_0 by $(N_0)^{\mathrm{op}}$. ◊

Remark 3.2.3.3. Since we are not assuming that our target category $\mathcal{A}$ has finite limits or colimits, it is not clear *a priori* that the limits/colimits indicated in (3.2.3) even exist; part of the statement of Theorem 3.2.3.1 (d) is that they do. Similarly, (a) is not automatic; in fact, the heart of the proof of Theorem 3.2.3.1 is an explicit inductive pointwise construction of the Kan extensions (3.2.2) in the case where $\mathbb{B}$ is reduced (see Proposition 3.4.2.1). ◊

Corollary 3.2.3.4. In the situation of Theorem 3.2.3.1, the span $B \subset B_+ \hookrightarrow V \hookleftarrow N_0$ induces a natural equivalence

$$\mathrm{Fun}(B,\mathcal{A}) \overset{\simeq}{\longleftrightarrow} \mathrm{Fun}^0(N_0,\mathcal{A}) \tag{3.2.4}$$

of ∞-categories for each weakly idempotent complete additive ∞-category $\mathcal{A}$. □

Proof. Compose the equivalence of Theorem 3.2.3.1 with the natural equivalence

$$\mathrm{Fun}(B,\mathcal{A}) \xleftarrow{\simeq} \mathrm{Fun}^0(B_+,\mathcal{A})$$

produced by the universal property of the free pointed category $B \to B_+$. ∎

Remark 3.2.3.5. In the situation of Remark 3.2.2.2, where $N_0 = (M \cap \mathrm{Reg})_+$ is a *free* pointed category, we can simplify the statement of Corollary 3.2.3.4 even more and obtain a natural equivalence

$$\mathrm{Fun}(B,\mathcal{A}) \xleftrightarrow{\simeq} \mathrm{Fun}(M \cap \mathrm{Reg},\mathcal{A})$$

between ordinary (*i.e.*, non-pointed) ∞-categories of diagrams. All equivalences discussed in Section 3.3.2 are of this form. ♢

Specializing Corollary 3.2.3.4 to the 1-categorical case, we recover the main theorem of Lack and Street.

Corollary 3.2.3.6. [LS15, Theorem 6.8] Each DK-triple $\mathbb{B} = (B, E, E^\vee)$ induces a natural equivalence

$$\mathrm{Fun}(B,A) \xleftrightarrow{\simeq} \mathrm{Fun}^0(N_0(\mathbb{B}),A)$$

of categories for each weakly[5] idempotent complete additive category A. ∎

Remark 3.2.3.7. Since the functor $\mathcal{A} \to \mathrm{h}\mathcal{A}$ to the homotopy category is additive, the naturality of equivalence (3.2.4) implies the existence of a commutative square

$$\begin{tikzcd}
\mathrm{Fun}(B,\mathcal{A}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Fun}^0(N_0,\mathcal{A}) \arrow[d] \\
\mathrm{Fun}(B,\mathrm{h}\mathcal{A}) \arrow[r, "\simeq"] & \mathrm{Fun}^0(N_0,\mathrm{h}\mathcal{A})
\end{tikzcd}$$

where the lower equivalence is an instance of Corollary 3.2.3.6. ♢

Remark 3.2.3.8. If the DK-triple $\mathbb{B}$ is diagonalizable, then in all of the results above one can weaken the assumption on $\mathcal{A}$ and only require it to be weakly idempotent complete and *pre*additive. Indeed, the additivity of $\mathcal{A}$ is only used once (in the proof of Proposition 3.4.2.1) to invert certain upper triangular matrices in $\mathcal{A}$ obtained from the matrices $\langle -; - \rangle_b$ defined in (T2); if $\mathbb{B}$ is diagonalizable then these matrices in $\mathcal{A}$ are diagonal, hence inverting them only requires preadditivity. See also Remark 3.4.2.4. ♢

Remark 3.2.3.9. Theorem 3.2.3.1 (d) implies that one can compute the value $\overline{\mathcal{X}}_n$ of the diagram $\overline{\mathcal{X}}\colon N_0 \to \mathcal{A}$ at an object $n \in N$ in two seemingly unrelated ways: as a total fiber

$$\overline{\mathcal{X}}_n \simeq \text{tot-fib}\left(E([n]) \to B \xrightarrow{\mathcal{X}} \mathcal{A}\right) \tag{3.2.5}$$

along the Epis, or as a total cofiber

$$\overline{\mathcal{X}}_n \simeq \text{tot-cof}\left(E^\vee([n]) \to B \xrightarrow{\mathcal{X}} \mathcal{A}\right).$$

along the dual Epis. ♢

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[5] To be precise, Lack and Street assume A to be idempotent complete.

3.3 Examples

3.3.1 The ∞-categorical Dold–Kan correspondence

We explain how to equip the simplex category $B = \Delta$ with the structure of a DK-triple; see also [LS15, Example 3.2] for a similar discussion. Recall that Δ is the category of finite non-empty linearly ordered sets and weakly monotone maps between them. We denote by $[n]$ the standard ordinal $\{0 < 1 < \cdots < n\}$; every object of Δ is of this form up to unique isomorphism. Let $E \subset \Delta$ be the wide subcategory of surjective maps and let $E^\vee_{\mathrm{min}} \subset \Delta$ be the wide subcategory of those injectives maps that preserve minimal elements. The following observations are straightforward to verify and imply that $\mathbb{B}^\Delta_{\mathrm{min}} = (\Delta, E, E^\vee_{\mathrm{min}})$ is a DK-triple:

- A map $f\colon [n] \to [m]$ is singular if and only if there is a non-minimal element of $[m]$ which is not in the image of f.
- The set M of *M*onos consists precisely of the injective maps in Δ. Since M is closed under composition, (T4) is satisfied.
- The set $M \cap \mathrm{Reg}$ of regular monos consists of the identities and the 0-th coface maps

  $$\mathrm{d}^0\colon [n-1] \cong \{1 < \cdots < n\} \hookrightarrow [n].$$

  Note that $M \cap \mathrm{Reg}$ is not closed under composition.

(T1) Each map $[n] \to [m]$ in Δ admits a unique (up to unique isomorphism) factorization of type $E^\vee_{\mathrm{min}} \circ (M \cap \mathrm{Reg}) \circ E$, namely

$$[n] \twoheadrightarrow (\mathrm{Im}\, f) \hookrightarrow (\{0\} \cup \mathrm{Im}\, f) \rightarrowtail [n].$$

(T3) The set $E^\vee_{\mathrm{min}} \circ E$ consists of the minimum-perserving arrows in Δ, hence $E^\vee_{\mathrm{min}} \circ E$ is closed under composition.

(T5) If $0 \neq i \in [m]$ is a non-minimal element which is not in the image of $f\colon [n] \to [m]$ then, for each injective map $g\colon [m] \to [m']$, the element $0 \neq g(i) \in [m']$ is not minimal and not contained in the image of $g \circ f$.

(T2) For each $[n], [m] \in \Delta$ we have a bijection

$$(-)^\vee\colon E([n],[m]) \xrightarrow{\cong} E^\vee_{\mathrm{min}}([m],[n])$$

which sends a surjection $e\colon [n] \to [m]$ to its minimal section $e^\vee\colon [n] \to [n]$ given by the formula $i \mapsto \min e^{-1}\{i\}$. A composition $e' \circ e^\vee$ of an *E*pi $e'\colon [n] \twoheadrightarrow [m]$ with a dual *E*pi $e^\vee\colon [n] \rightarrowtail [m]$ is

- an isomorphism if $e' = e$
- not an isomorphism if $e^\vee \ngeq e'^\vee$ poinwise as maps $[m] \to [n]$

(note that we make no claim when $e'^\vee < e^\vee$). Hence, for each $[n] \in \Delta$, the lexicographic ordering on $\pi_0 E^\vee_{\mathrm{min}}([n])$ makes the matrix

$$\pi_0 E^\vee_{\mathrm{min}}([n]) \times \pi_0 E^\vee_{\mathrm{min}}([n]) \xleftarrow[(-)^\vee]{\cong} \pi_0 E([n]) \times \pi_0 E^\vee_{\mathrm{min}}([n]) \xrightarrow{\langle -;- \rangle_{[n]}} \pi_0 \operatorname{Ar} \Delta$$

into upper triangular shape modulo non-isomorphisms.

Example 3.3.1.1. The matrix $\langle -;- \rangle_{[2]}$, can be depicted as follows

	0	01	02	012
(012)	**0**	(01)	(12)	(012)
0(12)	0	**01**	**02**	0(12)
(01)2	0	(01)	**02**	(01)2
012	0	01	02	**012**

where the rows are labeled by equivalence classes of Epis $[2] \twoheadrightarrow [m]$ (written by grouping elements with the same image); dually, the rows are labeled by equivalence classes of dual Epis $[m] \twoheadrightarrow [2]$ (written by listing the elements in the image). The isomorphisms are highlighted in red and bold, showing that the matrix is—modulo non-isomorphisms—unipotent upper triangular but not diagonal. In particular, this example shows that the DK-triple $\mathbb{B}^{\Delta}_{\mathrm{min}}$ is not diagonalizable. ◊

There is an equivalence $\mathrm{Ch}_{\geq 0} := \frac{\mathbb{N}}{(\to\to)} \xrightarrow{\simeq} N_0 := \frac{M}{M \cap \mathrm{Reg}}$ of pointed categories which is given on objects by $\underline{n} \mapsto [n]$ and is determined on morphisms by sending the arrow $\underline{n} \to \underline{n+1}$ to the 0-th coface map $\mathrm{d}^0 \colon [n] \to [n+1]$. Applying Corollary 3.2.3.4 to the DK-triple $\mathbb{B}^{\Delta}_{\mathrm{min}}$ thus establishes a natural equivalence

$$\mathrm{Fun}(\Delta, \mathcal{A}) \xleftrightarrow{\simeq} \mathrm{Fun}^0(\mathrm{Ch}_{\geq 0}, \mathcal{A})$$

for each weakly idempotent complete additive ∞-category $\mathcal{A}$. Replacing $\mathcal{A}$ by its opposite (which is again weakly idempotent complete additive) yields the more familiar form of the following **∞-categorical Dold–Kan correspondence**:

Corollary 3.3.1.2. The DK-triple $\mathbb{B}^{\Delta}_{\mathrm{min}} = (\Delta, E, E^{\vee}_{\mathrm{min}})$ induces a natural equivalence of ∞-categories

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \xleftrightarrow{\simeq} \mathrm{Fun}^0\Big(\mathrm{Ch}^{\mathrm{op}}_{\geq 0}, \mathcal{A}\Big) =: \mathrm{Ch}_{\geq 0}(\mathcal{A}) \tag{3.3.1}$$

between simplicial objects and connective chain complexes in any weakly idempotent complete additive ∞-category $\mathcal{A}$. ■

The simplex category Δ is part of a second DK-triple $\mathbb{B}^{\Delta}_{\mathrm{max}}$, where E is again the set of surjections and $E^{\vee}_{\mathrm{max}}$ is the set of maximum-preserving injections in Δ. The DK-triples $\mathbb{B}^{\Delta}_{\mathrm{min}}$ and $\mathbb{B}^{\Delta}_{\mathrm{max}}$ have canonically isomorphic quotient categories $N_0(\mathbb{B}^{\Delta}_{\mathrm{min}}) \cong \mathrm{Ch}_{\geq 0} \cong N_0(\mathbb{B}^{\Delta}_{\mathrm{max}})$ and correspond to each other under the canonical involution $\updownarrow \colon \Delta \overset{\cong}{\leftrightarrows} \Delta$ which reverses the linear order on each object of Δ. Hence we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \arrow[rr, leftrightarrow, "\simeq", "\updownarrow"'] \arrow[dr, leftrightarrow, "\simeq", "\mathrm{DK}^{\mathrm{min}}"'] & & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \arrow[dl, leftrightarrow, "\simeq"', "\mathrm{DK}^{\mathrm{max}}"] \\
& \mathrm{Ch}_{\geq 0}(\mathcal{A}) &
\end{tikzcd}$$

which intertwines the corresponding two versions of the Dold–Kan functor (3.3.1).

Lemma 3.3.1.3. For every additive ∞-category $\mathcal{A}$, the autoequivalence $\updownarrow \colon \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \overset{\simeq}{\leftrightarrows} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A})$ is equivalent to the identity; in other words, the two Dold–Kan functors $\mathrm{DK}^{\mathrm{min}}$ and $\mathrm{DK}^{\mathrm{max}}$ agree (up to equivalence). □

In the 1-categorical context, one can check explicitly that $\mathrm{DK}^{\mathrm{min}}$ and $\mathrm{DK}^{\mathrm{max}}$ both agree (up to natural isomorphism) with the normalized chain functor, hence with each other. For ∞-categories we provide the following alternative argument:

Proof. Choose a stable ∞-category $\mathcal{A} \subseteq \mathcal{D}$ into which $\mathcal{A}$ is embedded as a full additive subcategory (see Lemma 3.1.3.6). Since the involution $\updownarrow \colon \Delta \overset{\cong}{\leftrightarrows} \Delta$ preserves the filtration

$$\Delta_{\leq 0} \subset \Delta_{\leq 1} \subset \cdots \subset \Delta_{\leq n} \subset \cdots \subset \Delta,$$

the functor $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D}) \to \mathrm{Fun}(\mathbb{N}, \mathcal{D})$, which sends a simplicial objects to its sequence of partial colimits, is $\updownarrow$-invariant. Since $\mathcal{D}$ is stable, Lurie's stable Dold–Kan correspondence (which we also discuss in Section 3.5.2) states that this functor is an equivalence; hence the ($\updownarrow$-invariant) composition

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{A}) \hookrightarrow \mathrm{Fun}(\mathbb{N}, \mathcal{D}),$$

is fully faithful. The result follows. ■

3.3.2 Categories of partial maps

An important class of examples of diagonalizable DK-triples arises by considering partial maps with respect to certain factorization systems; we sketch here the corresponding discussion by Lack and Street [LS15, Example 3.1]. Let $(\mathscr{E}, \mathscr{M})$ be a factorization system on a category $\mathscr{A}$, *i.e.*,

- $\mathscr{E}$ and $\mathscr{M}$ are subcategories of $\mathscr{A}$ containing all isomorphisms and
- arrows in $\mathscr{A}$ factor, uniquely up to unique isomorphism, as compositions of the type $\mathscr{M} \circ \mathscr{E}$;

assume furthermore that

- the arrows in $\mathscr{M}$ are monomorphisms,
- each object of $\mathscr{A}$ has only finitely many $\mathscr{M}$-subobjects and
- the pullback of an arrow in $\mathscr{M}$ along an arbitrary map in $\mathscr{A}$ exists and lies again in $\mathscr{M}$.

The category $\mathrm{Par}^{\mathrm{op}}\mathscr{A} = \mathrm{Par}^{\mathrm{op}}_{\mathscr{M}}\mathscr{A}$ of **co-$\mathscr{M}$-partial maps** is defined to have

- the same objects as $\mathscr{A}$;
- morphism in $\mathrm{Par}^{\mathrm{op}}\mathscr{A}$ are equivalence classes of spans in $\mathscr{A}$ of the type $\bullet \xleftarrow{\mathscr{A}} \bullet \xrightarrow{\mathscr{M}} \bullet$, *i.e.*, where the second leg is required to lie in $\mathscr{M}$;
- composition in $\mathrm{Par}^{\mathrm{op}}\mathscr{A}$ is that of spans, *i.e.*, by pullback.

We define two wide subcategories $E, E^{\vee} \subset \mathrm{Par}^{\mathrm{op}}\mathscr{A}$ to consist of the spans of the type

$$\bullet \xleftarrow{\mathscr{M}} \bullet \xrightarrow{\cong} \bullet \qquad \text{and} \qquad \bullet \xleftarrow{\cong} \bullet \xrightarrow{\mathscr{M}} \bullet,$$

respectively. With the notation of Section 3.2.1, the Monos are the spans of type $\bullet \xleftarrow{\mathscr{E}} \bullet \xrightarrow{\mathscr{M}} \bullet$. The set Reg of regular morphisms consists of those morphisms in $\mathrm{Par}^{\mathrm{op}}\mathscr{A}$ which are totally defined, *i.e.*, the spans of type $\bullet \xleftarrow{\mathscr{A}} \bullet \xrightarrow{\cong} \bullet$; hence we have $\mathrm{Reg} \cong \mathscr{A}^{\mathrm{op}}$.

Lemma 3.3.2.1. The datum $\mathbb{B} := (\mathrm{Par}^{\mathrm{op}}\mathscr{A}, E, E^{\vee})$ is a diagonalizable DK-triple. □

Proof. The proof is straightforward and left to the reader. ■

The regular Monos in $\mathrm{Par}^{\mathrm{op}}\mathscr{A}$ are the spans of the type $\bullet \xleftarrow{\mathscr{E}} \bullet \xrightarrow{\cong} \bullet$; they form a subcategory equivalent to $\mathscr{E}^{\mathrm{op}}$. Hence Remark 3.2.2.2 says that the pointed category $N_0(\mathbb{B})$ constructed in Construction 3.2.2.1 is just the free pointed category $\mathscr{E}^{\mathrm{op}}_{+}$ on $M \cap \mathrm{Reg} \cong \mathscr{E}^{\mathrm{op}}$.

The upshot of this discussion is the following corollary of Corollary 3.2.3.4, taking into account that $N_0 = \mathscr{E}^{\mathrm{op}}_{+}$ is a free pointed category (see Remark 3.2.3.5) and that the DK-triple $\mathbb{B}$ is diagonalizable (see Remark 3.2.3.8).

Corollary 3.3.2.2. Let $\mathscr{A}$ and $(\mathscr{E}, \mathscr{M})$ be as above. The DK-triple $(\mathrm{Par}^{\mathrm{op}}_{\mathscr{M}}\mathscr{A}, E, E^{\vee})$ induces a natural equivalence

$$\mathrm{Fun}(\mathrm{Par}^{\mathrm{op}}_{\mathscr{M}}\mathscr{A}, \mathcal{A}) \xleftrightarrow{\simeq} \mathrm{Fun}(\mathscr{E}^{\mathrm{op}}, \mathcal{A})$$

for each weakly idempotent complete preadditive ∞-category $\mathcal{A}$. □

The prototypical example of Corollary 3.3.2.2 comes from the category **Fin** of finite sets, equipped with its canonical surjective-injective factorization system $(\mathbf{Surj}, \mathbf{Inj})$; in this case $\mathrm{Par}^{\mathrm{op}}\mathbf{Fin}$ is precisely Segal's category $\Gamma = \mathbf{Fin}^{\mathrm{op}}_{\star}$ [Seg74], hence we get a natural equivalence

$$\mathrm{Fun}(\Gamma, \mathcal{A}) \xleftrightarrow{\simeq} \mathrm{Fun}(\mathbf{Surj}^{\mathrm{op}}, \mathcal{A})$$

or, after dualizing (see Remark 3.2.3.2),

$$\mathrm{Fun}(\Gamma^{\mathrm{op}}, \mathcal{A}) \xleftrightarrow{\simeq} \mathrm{Fun}(\mathbf{Surj}, \mathcal{A}), \tag{3.3.2}$$

for all weakly idempotent complete preadditive ∞-categories $\mathcal{A}$.

We refer the reader to [LS15, Section 7] for many more examples in this spirit.

Remark 3.3.2.3. Up to minor differences[6)], the pairs $(\mathrm{Par}_{\mathscr{M}}\mathscr{A}, \mathscr{E})$ arising from a factorization system $(\mathscr{E}, \mathscr{M})$ as above are the *conjugate pairs* $(\mathcal{B}, \mathcal{A})$ of Helmstutler [Hel14]. For the convenience of the reader we provide a table translating Helmstutler's notation to the one of Lack and Street (which we are using in this section):

Lack and Street	$\mathscr{A}$	$\mathscr{M}$	$\mathscr{E}$	$\mathrm{Par}_{\mathscr{M}}\mathscr{A}$
Helmstutler	$\mathcal{U}$	$\mathcal{I}$	$\mathcal{A}$	$\mathcal{B}$

Helmstutler calls the arrows in $\mathscr{A} \subset \mathrm{Par}_{\mathscr{M}}\mathscr{A}$ *regular* (because they are totally, and not just $\mathscr{M}$-partially, defined) and the other arrows in $\mathrm{Par}_{\mathscr{M}}\mathscr{A}$ *singular*; this matches our use of those words. Moreover, he constructs a bimodule $\mathcal{U}_{+}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \longrightarrow \mathbf{Set}_{\star}$ (which is precisely our bimodule R_0 from Construction 3.2.2.4) and proves [Hel14, Theorem 6.2] that it induces, for each left proper stable model category $\mathcal{C}$, a Quillen equivalence $[\mathcal{B}^{\mathrm{op}}, \mathcal{C}] \rightleftarrows [\mathcal{A}^{\mathrm{op}}, \mathcal{C}]$ (left adjoint on top). Corollary 3.3.2.2 is our version of this result, where we replace a Quillen equivalence of model categories by an equivalence of ∞-categories. Note that the self-duality inherent to our ∞-categorical approach (see Remark 3.2.3.2) fixes the asymmetry problem addressed by Helmstutler in the note at the end of Section 6 in [Hel14]. ♢

3.4 Proof of the Dold–Kan type theorems

3.4.1 Cofinality lemmas

In order to get a better understanding of the Kan extensions appearing in Theorem 3.2.3.1 we use cofinality arguments to simplify the relevant pointwise formulas.

Construction 3.4.1.1. Fix an element $n \in N$. Consider the category

$$X_n := E^{\vee}([n]) \sqcup_{E^{\vee}_{\ncong}([n])} E^{\vee}_{\ncong}([n])^{\triangleright},$$

equipped with the functor $X_n \to B_{+/n}$ given by sending each $b \in E^{\vee}([n])$ to the composition $b \rightarrowtail [n] \to n$ (which is the zero map for all $b \in E^{\vee}_{\ncong}([n])$) and the cone point of $E^{\vee}_{\ncong}([n])^{\triangleright}$ to $0 \to n$. Since $E^{\vee}_{\ncong}([n])$ is (equivalent to) a poset, the same is true for X_n; the latter poset arises from $E^{\vee}([n])$ by adding one new element which is bigger than all elements of $E^{\vee}([n])$ except its terminal object $\mathrm{Id}\colon [n] \rightarrowtail [n]$.
Fix an element $b \in B$. Denote by $Y_b \subset (N_0)_{b/}$ the (non-full) subcategory defined as follows:

- objects are the maps $b \twoheadrightarrow \underline{b'}$ corresponding to Epis $b \twoheadrightarrow b'$ in B (recall that $\underline{b'} \in N$ denotes the object corresponding to $b' \in B$) and the unique map $b \to 0$.
- the only morphisms are isomorphisms under b and the zero morphisms $\underline{b'} \to 0$.

Observe that Y_b is equivalent to the right cone

$$\{b \twoheadrightarrow b'\}^{\triangleright}$$

on the discrete set $\{b \twoheadrightarrow b'\}$ containing some choice of representatives for the isomorphism classes of Epis out of b; the cone point corresponds to the object $b \to 0$ of Y_b. ♢

Lemma 3.4.1.2. For each $n \in N$, the inclusion $X_n \hookrightarrow B_{+/n}$ is homotopy terminal. □

Before we go into the rather technical proof of Lemma 3.4.1.2, we state the direct following corollary which is what we will use going forward.

[6)] For instance, Helmstutler's $\mathscr{M}$ is not required to contain all isomorphisms. Unlike Lack and Street (hence us) he does however require the pullback of an arrow in $\mathscr{E}$ along an arrow in $\mathscr{M}$ to lie in $\mathscr{E}$ again; this amounts to saying that the set M of Monos is closed under composition.

Corollary 3.4.1.3. Let $\mathcal{P}$ be a pointed ∞-category and $\mathcal{X}\colon B_+ \to \mathcal{P}$ a pointed diagram. Then a left Kan extension $\mathcal{X}^1$ of $\mathcal{X}$ along the inclusion $B_+ \hookrightarrow V$ exists if and only if for each $n \in N$ the total cofiber of the diagram

$$E^\vee([n]) \longrightarrow B_+ \xrightarrow{\mathcal{X}} \mathcal{A}$$

exists in $\mathcal{A}$. If it exists, this left Kan extension $\mathcal{X}^1$ is characterized pointwise at $n \in N$ by the fact that it extends the diagram

$$X_a \longrightarrow B_{+/n} \longrightarrow B_+ \xrightarrow{\mathcal{X}} \mathcal{A}$$

to a colimit cone in $\mathcal{A}$ with colimit $\mathcal{X}^1(n)$. ■

Remark 3.4.1.4. If the colimit of the diagram

$$E^\vee_{\not\simeq}([n]) \longrightarrow B_+ \xrightarrow{\mathcal{X}} \mathcal{A}$$

exists for each $n \in N$ then we can characterize the left Kan extension as in Corollary 3.4.1.3 by the fact that it induces cofiber sequences

$$\begin{tikzcd}
\operatorname*{colim}_{b \in E^\vee_{\not\simeq}([n])} \mathcal{X}_b \arrow[r] \arrow[d] & \mathcal{X}_0 \simeq 0 \arrow[d] \\
\mathcal{X}_{[n]} \arrow[r] & \mathcal{X}^1_n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

We will show in Proposition 3.4.2.1 that this colimit always exists when $\mathcal{A}$ is weakly idempotent complete additive. ♢

Proof (of Lemma 3.4.1.2). Fix $n \in N$ and let us abbreviate $X \coloneqq X_n$ to avoid proliferating subscripts. For each $b \in B$ and each arrow $u\colon b \to n$ in V, the undercategory $X_{u/}$ can be described explicitly as follows:

- objects are factorizations

$$\begin{tikzcd}
& b' \arrow[dr, "x"] & \\
b \arrow[ur, "f"] \arrow[rr, "u"'] & & n
\end{tikzcd}$$

  of u in V, where the arrow $x\colon b' \to n$ is required to lie in X;
- a morphism $(x, f) \to (x', f')$ between such factorizations is simply an arrow $x \to x'$ in X compatible with f and f'.

Observe that $X_{u/}$ is a poset (because X is). To prove that $X \hookrightarrow B_{+/n}$ is homotopy initial we have to show that all these categories of factorizations are weakly contractible. We distinguish two cases:

- Assume that $u\colon b \to n$ is a non-zero. Then the only factorization of u through an object of X is the tautological factorization $u\colon b \xrightarrow{[u]} [n] \xrightarrow{!} n$ hence the category $X_{u/}$ is a singleton.
- Assume that $u\colon b \xrightarrow{0} u$ is the zero map. In this case there are three types of factorizations:
  (1) given a non-invertible dual Epi $[x]\colon b' \overset{\not\simeq}{\rightarrowtail} [n]$ and given *any* map $f\colon b \to b'$ in B_+, there is a factorization $0\colon b \xrightarrow{f} b' \xrightarrow{x} n$;
  (2) for each *singular* map $s\colon b \to [n]$ in B_+, there is a factorization $0\colon b \xrightarrow{s} [n] \xrightarrow{!} n$;
  (3) there is the zero factorization $0\colon b \to 0 \to n$.

Denote by (x,f), $(!,s)$ and 0 the objects of $X_{u/}$ corresponding to the factorizations of type (1), (2) and (3), respectively. Denote by $Z \subset X_{u/}$ the subposet consisting of the objects (x,f). For each singular map $s\colon b \to [n]$, denote by $Z_s \subset Z$ the subposet consisting of those (x,f), where the composite $b \xrightarrow{f} b' \xrightarrow{[x]} [n]$ is equal to $s\colon b \to [n]$. We now describe the morphisms in the category $X_{u/}$.

- For each factorization (x,f) with $s := [x] \circ f$ (as maps $b \to [n]$), we have a unique map $(x,f) \to (!,s)$. There are no other maps between factorizations of types (1) and (2). In other words, the subposet $Z_s \cup \{(!,s)\} \subset X_{u/}$ is a (right) cone on Z_s with maximum $(!,s)$
- There are no maps between factorizations of types (2) and (3) (because there are no maps between $[n] \to n$ and $0 \to n$ in X)
- For each factorization (x,f), we have a unique map $(x,f) \to 0$. There are no other maps between factorizations of types (1) and (3). In other words, the subposet $Z \cup \{0\} \subset X_{u/}$ is a (right) cone on Z with maximum 0.

It follows that we have the following pushout of simplicial sets:

$$\begin{tikzcd}
\coprod\limits_{s\in\mathrm{Sing}(b,[n])} \mathrm{N}(Z_s) \arrow[r, hook] \arrow[d] & \coprod\limits_{s\in\mathrm{Sing}(b,[n])} \mathrm{N}(Z_s)^{\triangleright} \arrow[d] \\
\mathrm{N}(Z)^{\triangleright} \arrow[r, hook] & \mathrm{N}(X_{u/})
\end{tikzcd} \tag{3.4.1}$$

which is a (Kan) homotopy pushout because the top horizontal map is a monomorphism. By (T1), each singular arrow $s\colon b \to [n]$ admits a unique (up to unique isomorphism) factorization $s\colon b \to b' \overset{\not\simeq}{\rightarrowtail} [n]$, where $b \to b'$ is regular and $b' \overset{\not\simeq}{\rightarrowtail} [n]$ is a non-invertible dual Epi; viewed as a factorization of $0\colon b \to n$ it is an initial object of the category Z_s, which is hence contractible. Therefore the top horizontal map in the square (3.4.1) is a (Kan) weak equivalence, hence also the bottom horizontal map; this concludes the proof because $\mathrm{N}(Z)^{\triangleright}$ is contractible. ■

Lemma 3.4.1.5. For each $b \in B$, the inclusion $Y_b \hookrightarrow (N_0)_{b/}$ is homotopy initial. □

Proof. Fix $b \in B$ and abbreviate $Y := Y_b$. Similarly to the proof of Lemma 3.4.1.2, we have to show that, for each $n \in N_0$ and each map $u\colon b \to n$, the groupoid $Y_{/u}$ of factorizations

$$\begin{tikzcd}
& \underline{b'} \arrow[dr, "f"] & \\
b \arrow[ur, "y"] \arrow[rr, "u"'] & & n
\end{tikzcd} \tag{3.4.2}$$

(with $y \in Y$) is weakly contractible. Again, we distinguish two cases:

- If the arrow $u\colon b \to n$ is non-zero then factorizations (3.4.2) are the same as $(M \cap \mathrm{Reg}) \circ E$-factorizations of the corresponding regular map $[u]\colon b \to [n]$. By (T1), the groupoid of such factorizations is (equivalent to) a point.
- If the arrow u is the zero then the zero factorization $u\colon b \to 0 \to n$ is a terminal object of the category $Y_{/u}$, which is hence contractible. ■

Corollary 3.4.1.6. Let $\mathcal{A}$ be a pointed ∞-category with finite products. Every pointed diagram $\overline{\mathcal{X}}\colon N_0 \to \mathcal{A}$ admits a right Kan extension $\mathcal{X}\colon V \to \mathcal{A}$ along the inclusion $N_0 \hookrightarrow V$. Moreover, this right Kan extension is characterized pointwise by the product cones

$$\mathcal{X}_b \xrightarrow{\simeq} \prod_{b \twoheadrightarrow n} \overline{\mathcal{X}}_n \tag{3.4.3}$$

indexed by equivalence classes of Epis out of b. □

Proof. Let $\overline{\mathcal{X}}\colon N_0 \to \mathcal{A}$ be a pointed diagram and fix an objects $b \in B$. By Lemma 3.4.1.5 we can compute the pointwise right Kan extension $\mathcal{X}$ of $\overline{\mathcal{X}}$ along $N_0 \hookrightarrow V$ at b as the limit

$$\mathcal{X}_b \xrightarrow{\simeq} \lim\Big((N_0)_{b/} \to N_0 \xrightarrow{\overline{\mathcal{X}}} \mathcal{A}\Big) \xrightarrow{\simeq} \lim\Big(Y_b \to N_0 \xrightarrow{\overline{\mathcal{X}}} \mathcal{A}\Big) \simeq \lim\Big(\{b \twoheadrightarrow n\}^{\triangleright} \to N_0 \xrightarrow{\overline{\mathcal{X}}} \mathcal{A}\Big).$$

This limit formula is the same as the product formula (3.4.3) because the value of $\overline{\mathcal{X}}$ on the cone point of $\{b \twoheadrightarrow n\}^{\triangleright}$ is $\overline{\mathcal{X}}_0 \simeq 0$. ■

3.4.2 Inductive construction in the reduced case

Throughout this section we assume that the DK-triple $\mathbb{B} = (B, E, E^{\vee})$ is reduced, *i.e.*, that $B = E^{\vee} \circ E$ and hence $N_0 = B_{+}^{\simeq}$. By applying Construction 3.2.2.4 to the reduced DK-triple $\mathbb{B}$ and to its dual $\mathbb{B}^{\mathrm{op}}$, we obtain two categories

$$V = V(\mathbb{B}) := \begin{pmatrix} N_0^1 & R_0 \\ 0 & B_+ \end{pmatrix} \qquad \text{and} \qquad V^{\vee} := V(\mathbb{B}^{\mathrm{op}})^{\mathrm{op}} = \begin{pmatrix} B_+ & M_0 \\ 0 & N_0^0 \end{pmatrix}$$

where N_0^0 and N_0^1 are both just (a copy of) N_0, decorated with superscripts 0 and 1 to avoid confusing them. For every $n \in N$ we denote by n_0 its copy in $N^0 \subset V^{\vee}$ and by n_1 its copy in $N^1 \subset V$. Furthemore, we denote by $V_{\leq n} \subset V$ the full subcategories spanned by B_+ and by all the objects n'_1 with $n' \leq n$; similarly, $V^{\vee}_{\leq n} \subset V^{\vee}$ is the full subcategory which contains B_+ and all the objects n'_0 with $n' \leq n$.

Proposition 3.4.2.1. Let $\mathcal{A}$ be a weakly idempotent complete additive ∞-category $\mathcal{A}$ and let $\mathcal{X}\colon B_+ \to \mathcal{A}$ be a pointed functor. Then there exist functors

$$\mathcal{X}^0\colon V^{\vee} \to \mathcal{A} \qquad \text{and} \qquad \mathcal{X}^1\colon V \to \mathcal{A}$$

which are right and left Kan extension of $\mathcal{X}$, respectively. Moreover the functors $\mathcal{X}^0$ and $\mathcal{X}^1$ are a left Kan extension and a right Kan extension of their restriction to N_0^0 and N_0^1, respectively. ☐

Remark 3.4.2.2. By Corollary 3.4.1.6, the "moreover" part of Proposition 3.4.2.1 is saying that for each $b \in B$ the diagrams $\mathcal{X}^0$ and $\mathcal{X}^1$ induce direct sum decompositions

$$\coprod_{n \rightarrowtail b} \mathcal{X}^0(n) \xrightarrow{\simeq} \mathcal{X}^0(b) = \mathcal{X}_b \qquad \text{and} \qquad \mathcal{X}_b = \mathcal{X}^1(b) \xrightarrow{\simeq} \prod_{b \twoheadrightarrow n} \mathcal{X}^1(n) \tag{3.4.4}$$

where the coproduct/product is indexed over equivalence classes of dual Epis into b and Epis out of b, respectively. ♢

Remark 3.4.2.3. It follows from the universal property of the coproduct that each dual Epi $b' \rightarrowtail b$ induces a commutative square

$$\begin{tikzcd} \coprod\limits_{n \rightarrowtail b'} \mathcal{X}^0(n) \arrow[r, "\simeq"] \arrow[d] & \mathcal{X}_{b'} \arrow[d] \\ \coprod\limits_{n \rightarrowtail b} \mathcal{X}^0(n) \arrow[r, "\simeq"] & \mathcal{X}_b \end{tikzcd}$$

where the left vertical map is the inclusion of those summands that are labeled by a dual Epi which factors through $b' \rightarrowtail b$. Similarly each Epi $b \twoheadrightarrow b'$ induces projection onto those factors of the decomposition $\mathcal{X}_b \simeq \prod_{b \twoheadrightarrow n} \mathcal{X}^1(n)$ that are indexed by Epis which factor through $b \twoheadrightarrow b'$. ♢

Proof. For each $n \in N$ we prove:

1. A right Kan extension $\mathcal{X}^0_{\leq n}$ of $\mathcal{X}$ along $B_+ \hookrightarrow V^{\vee}_{\leq n}$ exists.
2. A left Kan extension $\mathcal{X}^1_{\leq n}$ of $\mathcal{X}$ along $B_+ \hookrightarrow V_{\leq n}$ exists.

(3) Each choice of such Kan extensions $\mathcal{X}^0_{\leq n}$ and $\mathcal{X}^1_{\leq n}$ induces, for each $b \leq [n]$, direct sum decompositions as in (3.4.4); moreover, the composition

$$\mathcal{X}^0_{\leq n}(n_0) \longrightarrow \mathcal{X}_{[n]} \longrightarrow \mathcal{X}^1_{\leq n}(n_1)$$

is an equivalence in $\mathcal{A}$.

By induction on the number $|\pi_0 E^\vee(n)| = |\pi_0 E(n)|$ we may assume that we have proved (1), (2) and (3) for all objects of N which are strictly smaller than n. Choose a right Kan extension $\mathcal{X}^0_{<n} \colon V^\vee_{<n} \to \mathcal{A}$ and a left Kan extension $\mathcal{X}^1_{<n} \colon V_{<n} \to \mathcal{A}$ of $\mathcal{X} \colon B_+ \to \mathcal{A}$ (they exist pointwise by assumption). By assumption, $\mathcal{X}^0_{<n}$ induces coproduct decompositions $\coprod_{n' \rightarrowtail b} \mathcal{X}^0_{<n}(n') \xrightarrow{\simeq} \mathcal{X}_b$ for all $b < [n]$. Since all dual Epis induce compatible inclusions of summands (see Remark 3.4.2.3), the diagram $\mathcal{X}^0$ provides an identification

$$\coprod_{n' \overset{\not\simeq}{\rightarrowtail} [n]} \mathcal{X}^0_{<n}(n') \simeq \operatorname*{colim}_{b \in E^\vee_{\not\simeq}([n])} \mathcal{X}_b \tag{3.4.5}$$

where the coproduct is indexed over equivalence classes of *non-invertible* dual Epis; moreover, this identification (3.4.5) is compatible with the respective structure maps to $\mathcal{X}_{[n]}$. By applying the dual argument to $\mathcal{X}^1_{<n} \colon V_{<n} \to \mathcal{A}$ we obtain an identification

$$\lim_{b \in E_{\not\simeq}([n])} \mathcal{X}_b \simeq \prod_{[n] \overset{\not\simeq}{\twoheadrightarrow} n'} \mathcal{X}^1_{<n}(n'),$$

again compatible with the structure maps from $\mathcal{X}_{[n]}$. We analyze the two composable maps

$$\coprod_{n' \overset{\not\simeq}{\rightarrowtail} [n]} \mathcal{X}^0_{<n}(n') \longrightarrow \mathcal{X}_{[n]} \longrightarrow \prod_{[n] \overset{\not\simeq}{\twoheadrightarrow} n''} \mathcal{X}^1_{<n}(n'') \tag{3.4.6}$$

and their composite Φ in terms of the components $\Phi_{n'',n'} \colon \mathcal{X}^0_{<n}(n') \to \mathcal{X}_{[n]} \to \mathcal{X}^1_{<n}(n'')$. We have the commutative diagram in $V^\vee_{<n} \sqcup_{B_+} V_{<n}$

$$\begin{tikzcd}
 & {[n'']} \arrow[dr, "!"] & \\
n'_0 \arrow[r] \arrow[dr, "!"'] & {[n]} \arrow[u, two heads] \arrow[r] & n''_1 \\
 & {[n']} \arrow[u, tail] &
\end{tikzcd}$$

where the vertical morphisms are the dual Epi $[n'] \rightarrowtail [n]$ and the Epi $[n] \twoheadrightarrow [n']$; their composition is—by definition—the map $\langle [n'']; [n'] \rangle_{[n]}$. Therefore, the map $\Phi_{n'',n'}$ is equivalent to the composition

$$\Phi_{n'',n'} \colon \mathcal{X}^0_{<n}(n'_0) \longrightarrow \mathcal{X}_{[n']} \xrightarrow{\mathcal{X}(\langle [n'']; [n'] \rangle_{[n]})} \mathcal{X}_{[n'']} \longrightarrow \mathcal{X}^1_{<n}(n''_1).$$

It follows that:

- If $\langle [n'']; [n'] \rangle_{[n]}$ is an isomorphism in B_+ (without loss of generality, the identity) then $\Phi_{n'',n'}$ is an equivalence by the induction hypothesis (3);
- If $\langle [n'']; [n'] \rangle_{[n]}$ is not an isomorphism in B_+ then it must be either singular or cosingular. If it is singular then the composition $[n'] \to [n''] \to n''_1$ factors through $0 \in V$; if it is cosingular then the composition $n'_0 \to [n'] \to [n'']$ factors through $0 \in V^\vee$; in either case $\Phi_{n'',n'}$ factors through $\mathcal{X}_0 \simeq 0$.

Therefore it follows from (T2) that $(\Phi_{n'',n'})$ is an upper triangular matrix with invertible diagonal entries; hence Φ is invertible because $\mathcal{A}$ is additive (see Lemma 3.1.3.7). This means that the two composable maps (3.4.6) are a section-retraction pair. Since $\mathcal{A}$ is weakly idempotent complete, this section-retraction pair admits a complement, *i.e.*, there is an essentially unique diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
\coprod_{n' \overset{\not\simeq}{\rightarrowtail} [n]} \mathcal{X}^0_{<n}(n') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_{[n]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_{[n] \overset{\not\simeq}{\twoheadrightarrow} n''} \mathcal{X}^1_{<n}(n'') \arrow[d] \\
0 \arrow[r] & Q \arrow[r] & 0
\end{tikzcd} \tag{3.4.7}$$

where all squares are biCartesian. By 3.4.1.3 (or, more precisely, by 3.4.1.4) and the identification (3.4.5), we conclude that the pointwise left Kan extension $\mathcal{X}^1(n)$ of $\mathcal{X}$ at n_1 exists and that its value on the structure map $!\colon [n] \to n_1$ is equivalent to the projection $\mathcal{X}_{[n]} \to Q$. By the dual argument, we conclude that the pointwise right Kan extension $\mathcal{X}^0(n)$ of $\mathcal{X}$ at n_0 exists and that its value on the structure map $!\colon n_0 \to [n]$ is equivalent to the inclusion $K \to \mathcal{X}_{[n]}$. To establish the inductive step for property (3), note that the diagram (3.4.7) encodes the required coproduct decompositions

$$\coprod_{n' \rightarrowtail [n]} \mathcal{X}^0(n') = K \sqcup \coprod_{n' \overset{\not\simeq}{\rightarrowtail} [n]} \mathcal{X}^0_{<n}(n') \xrightarrow{\simeq} \mathcal{X}_{[n]}$$

(and similarly the required product decomposition) and the fact that the composition

$$K \simeq \mathcal{X}^0(n) \xrightarrow{\mathcal{X}^0(!)} \mathcal{X}_{[n]} \xrightarrow{\mathcal{X}^1(!)} \mathcal{X}^1(n) \simeq Q$$

is an equivalence. ■

Remark 3.4.2.4. If the DK-triple $\mathbb{B}$ is diagonalizable then the matrix $(\Phi_{n'',n'})_{n'',n'}$ is actually a diagonal matrix. Hence to invert it, we do not need $\mathcal{A}$ to be additive but only preadditive. ♢

Remark 3.4.2.5. From the proof of Proposition 3.4.2.1 we can extract more detailed information. For each $n \in N$, the extensions $\mathcal{X}^0$ and $\mathcal{X}^1$ encode two complementary section-retraction pairs

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname*{colim}_{b \in E^\vee_{\not\simeq}([n])} \mathcal{X}_b \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & 0 \arrow[d] \\
\mathcal{X}^0(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}_{[n]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X}^1(n) \arrow[d] \\
0 \arrow[r] & \lim_{b \in E_{\not\simeq}([n])} \mathcal{X}_b \arrow[r] & 0
\end{tikzcd}$$

(in particular, the indicated limits/colimits exist). ♢

3.4.3 The general case

Proof (of Theorem 3.2.3.1)**.** We first prove (a), (b), and (d) in the case where the DK-triple $(B, E, E^\vee)$ is reduced. In this case, we have the following ingredients:

- Corollary 3.4.1.6 guarantees that the right Kan extension functor RKE: $\mathrm{Fun}^0(N_0, \mathcal{A}) \to \mathrm{Fun}^0(V, \mathcal{A})$ exists. Moreover, the explicit formula (3.4.3) implies that for any natural transformation $\alpha\colon \overline{\mathcal{X}}' \to \overline{\mathcal{X}}$ of pointed diagrams $N_0 \to \mathcal{A}$, the component $\alpha_n\colon \overline{\mathcal{X}}'_n \to \overline{\mathcal{X}}_n$

at $n \in N$ is a factor of the corresponding right Kan extended transformation (with the notation as in 3.4.1.6)

$$\begin{tikzcd}
\mathcal{X}'_{[n]} \arrow[r, "\mathrm{RKE}(\alpha)_{[n]}"] \arrow[d, "\simeq"'] & \mathcal{X}_{[n]} \arrow[d, "\simeq"] \\
\prod\limits_{[n]\twoheadrightarrow n'} \overline{\mathcal{X}}'_{n'} \arrow[r, "\prod \alpha_{n'}"'] & \prod\limits_{[n]\twoheadrightarrow n'} \overline{\mathcal{X}}_{n'}
\end{tikzcd}$$

at $[n] \in B$; hence it follows from Lemma 3.1.1.7 that the composition

$$\mathrm{Fun}^0(N_0, \mathcal{A}) \xrightarrow{\mathrm{RKE}} \mathrm{Fun}^0(V, \mathcal{A}) \xrightarrow{\mathrm{Res}} \mathrm{Fun}^0(B_+, \mathcal{A})$$

is conservative, *i.e.*, reflects equivalences.

- Proposition 3.4.2.1 states in particular—if we focus only on the statements about V and not about $V^\vee$—that
  - the left Kan extension functor LKE: $\mathrm{Fun}^0(B_+, \mathcal{A}) \to \mathrm{Fun}^0(V, \mathcal{A})$ exists and
  - on the image of this functor LKE, the unit $\mathrm{Id}_{\mathrm{Fun}^0(V,\mathcal{A})} \to \mathrm{RKE} \circ \mathrm{Res}$ of the adjunction

    $$\mathrm{Res} \colon \mathrm{Fun}^0(V, \mathcal{A}) \rightleftarrows \mathrm{Fun}^0(N_0, \mathcal{A}) : \mathrm{RKE}$$

    is an equivalence.

  Since left Kan extension along the fully faithful functor $B_+ \hookrightarrow V$ is fully faithful, the unit $\mathrm{Id}_{\mathrm{Fun}^0(B_+,\mathcal{A})} \to \mathrm{Res} \circ \mathrm{LKE}$ is an equivalence. We conclude that the unit

  $$\mathrm{Id}_{\mathrm{Fun}^0(B_+,\mathcal{A})} \longrightarrow \mathrm{Res} \circ \mathrm{LKE} = \mathrm{Res} \circ \mathrm{Id}_{\mathrm{Fun}^0(V,\mathcal{A})} \circ \mathrm{LKE} \longrightarrow \mathrm{Res} \circ \mathrm{RKE} \circ \mathrm{Res} \circ \mathrm{LKE}$$

  of the composite adjunction 3.2.2 is also an equivalence.

This already proves (a); assertion (b) follows from the general fact about adjunctions that if the right adjoint is conservative and the unit is an equivalence then the whole adjunction is an adjoint equivalence. Assertion (d) is spelled out in Remark 3.4.2.5 since $\overline{\mathcal{X}}_n$ is by definition equivalent to $\mathcal{X}^1(n)$.

To prove (a), (b) and (d) when $\mathbb{B}$ is not necessarily reduced, we make the following key observation:

- the criterion for constructing and detecting left Kan extension along $B_+ \hookrightarrow V$ (Corollary 3.4.1.3) and the criterion for constructing and detecting right Kan extension along $N_0 \hookrightarrow V$ (Corollary 3.4.1.6) both only depend on the values of a diagram on the dual Epis $E^\vee$ and on the Epis E.

Therefore we can reduce to the reduced case (no pun intended) by replacing the original DK-triple with the reduced DK-triple

$$\overline{\mathbb{B}} := (E^\vee \circ E, E, E^\vee).$$

To prove (c), note that the right Kan extension RKE: $\mathrm{Fun}^0(N_0, \mathcal{A}) \longrightarrow \mathrm{Fun}^0(V, \mathcal{A})$ is natural in $\mathcal{A}$ with respect to all functors which preserve the relevant pointwise limits; since all these pointwise limits are just products, this is true for every additive functor. ■

3.5 Comparison with...

3.5.1 ...the setting of Lack and Street

We provide a short dictionary/comparison between our setup described in Section 3.2.1 and Section 3.2.2 and the setting of Lack and Street [LS15, Section 2]. Unless stated otherwise, references in this section refer to their revised arXiv paper[LS14], *not* the published one [LS15] (see also the corrigendum [LS20]); we freely use the notation of [LS14, Section 2].

Their category $\mathscr{P}$ is the *dual* of our category B. Under this duality we have the following table of correspondence:

Lack and Street	$\mathscr{P}$	$\mathscr{M}$	$\mathscr{M}^*$	$\mathscr{R}$	$\mathscr{D}$	$\mathscr{S}$	$\{u \mid s_u \in \mathscr{R}\}$
our setup	B	E	$E^\vee$	$M \cap \mathrm{Reg}$	N	M	Reg

Lack and Street take as part of the data an isomorphism $(-)^* \colon \mathscr{M}^{\mathrm{op}} \cong \mathscr{M}^*$ (which in our language would be written as $(-)^\vee \colon E^{\mathrm{op}} \cong E^\vee$) which is the identity on objects and satisfies $m^* \circ m = \mathrm{Id}$ for all arrows m in $\mathscr{M}$. Their Assumption 2.5 translates to the fact that the set $\pi_0 E(b)$ is finite for each $b \in B$; Assumption 2.6 is saying that for each $b \in B$ there exists a linear order on $\pi_0 E(b)$ such that the matrix $\langle -; (-)^\vee \rangle_b \colon \pi_0 E(b) \times \pi_0 E(b) \to \pi_0 \operatorname{Ar} B$ has only singular entries below the diagonal. In our setup, (T2) replaces all these ingredients and repackages them as a property which more directly reflects the final use: what we ultimately want to exploit is that certain unipotent upper triangular matrices (3.4.6) induced from the matrices $\langle -; - \rangle_b$ can be inverted in any additive ∞-category. Note that while Lack and Street require the matrix entries below the diagonal to be singular, it suffices for our purposes if they are non-invertible. Furthermore:

- Their Assumption 2.1 and Assumption 2.4 correspond precisely to our axioms (T1) and (T3), respectively.
- Their Assumption 2.2 translates to our axiom (T4).
- Their Assumption 2.3 translates to $(M \cap \mathrm{Reg}) \circ E^\vee_{\not\simeq} \subset B \setminus (M \cap \mathrm{Reg})$ and is, *a priori*, weaker than our axiom (T5). However, they use Assumption 2.3 (in the presence of the other assumptions) to prove Proposition 2.10(b) which states that if two composable arrows v, u satisfy $s_v \notin \mathscr{R}$ and $u \in \mathscr{S}$, then also $s_{vu} \notin \mathscr{R}$. This statement translates to $M \circ \mathrm{Sing} \subseteq \mathrm{Sing}$, which is precisely (T5).

The preceding discussion proves:

Corollary 3.5.1.1. Let $\mathscr{P}$, $\mathscr{M}$, $\mathscr{M}^*$ and $\mathscr{D}$ be as in [LS14, Section 2]. Then $\mathbb{B} = (\mathscr{P}^{\mathrm{op}}, \mathscr{M}^{\mathrm{op}}, (\mathscr{M}^*)^{\mathrm{op}})$ is a DK-triple with associated pointed category $N_0(\mathbb{B}) = \frac{\mathscr{D}^{\mathrm{op}}}{(0)}$. ■

The main tool in the proof by Lack and Street is what they call the *kernel module* [LS15, Section 4]

$$M \colon \mathscr{D}^{\mathrm{op}} \times \mathscr{P} \longrightarrow 1/\mathrm{Set}$$

(where 1/Set is their notation for the category of pointed sets); it corresponds to our N_0-B_+-bimodule

$$R_0 \colon B_+^{\mathrm{op}} \times N_0 \longrightarrow \mathbf{Set}_\star$$

which we encode in its upper triangular category V. Their main theorem [LS14, Theorem 6.7] [LS15, Theorem 6.8] states that for each idempotent complete additive 1-category $\mathscr{X}$, the kernel module M induces an equivalence

$$\mathrm{Fun}(\mathscr{P}, \mathscr{X}) \simeq \mathrm{Fun}_{\mathbf{Set}_\star}(\mathscr{D}, \mathscr{X})$$

where $\mathrm{Fun}_{\mathbf{Set}_\star}$ denotes the category $\mathbf{Set}_\star$-enriched functors. Instead of using $\mathbf{Set}_\star$-enriched categories (or rather $\mathcal{S}_\star$-enriched ∞-categories; see also Remark 3.1.2.7) we chose to work with pointed categories and phrase our main result in terms of pointed functors on $N_0 = \frac{N}{(0)}$. Therefore Corollary 3.2.3.6 recovers their result because, for each pointed 1-category P and each $\mathbf{Set}_\star$-enriched category N, the inclusion $N \hookrightarrow N_0$ induces an equivalence of categories $\mathrm{Fun}^0(N_0, P) \xrightarrow{\simeq} \mathrm{Fun}_{\mathbf{Set}_\star}(N, P)$ (see Remark 3.1.2.6).

3.5.2 ...Lurie's stable Dold–Kan correspondence

Let $\mathcal{D}$ be an ∞-category with finite colimits and consider the functor

$$\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathcal{D}) \longrightarrow \operatorname{Fun}(\mathbb{N}, \mathcal{D}), \tag{3.5.1}$$

which sends a simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{D}$ to the filtered object

$$\hat{\mathcal{X}}\colon \operatorname{colim} \mathcal{X}_{\leq 0} \longrightarrow \cdots \longrightarrow \operatorname{colim} \mathcal{X}_{\leq n-1} \longrightarrow \operatorname{colim} \mathcal{X}_{\leq n} \longrightarrow \cdots$$

of its partial colimits $\hat{\mathcal{X}}_n \coloneqq \operatorname{colim} \mathcal{X}_{\leq n} = \operatorname{colim}(\mathcal{X}_{\leq n}\colon \Delta^{\mathrm{op}}_{\leq n} \hookrightarrow \Delta^{\mathrm{op}} \xrightarrow{\mathcal{X}} \mathcal{D})$. Lurie's stable Dold–Kan correspondence [Lur17, Theorem 1.2.4.1] states that the functor (3.5.1) is an equivalence when the target $\mathcal{D}$ is a *stable* ∞-category. The functor (3.5.1) lifts the ordinary Dold–Kan correspondence in the following sense: Each filtered object $\hat{\mathcal{X}}$ in a stable ∞-category $\mathcal{D}$ gives rise to a connective chain complex

$$\mathrm{h}\overline{\mathcal{X}}_0 \longleftarrow \cdots \longleftarrow \mathrm{h}\overline{\mathcal{X}}_{n-1} \longleftarrow \mathrm{h}\overline{\mathcal{X}}_n \longleftarrow \cdots \tag{3.5.2}$$

in the homotopy category $\mathrm{h}\mathcal{D}$, with $\overline{\mathcal{X}}_n \coloneqq \Omega^n \operatorname{cof}(\hat{\mathcal{X}}_{n-1} \to \hat{\mathcal{X}}_n)$. Moreover, there is a commutative diagram

$$\begin{tikzcd}
 & \operatorname{Fun}(\mathbb{N}, \mathcal{D}) \arrow[d, dotted] \arrow[dd, bend left] & & \hat{\mathcal{X}} \arrow[d, dotted, mapsto] \arrow[dd, bend left, mapsto] \\
\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathcal{D}) \arrow[ur, "\simeq"] \arrow[r, leftrightarrow, "\simeq"] \arrow[d] & \operatorname{Ch}_{\geq 0}(\mathcal{D}) \arrow[d] & & \overline{\mathcal{X}} \arrow[d, mapsto] \\
\operatorname{Fun}(\Delta^{\mathrm{op}}, \mathrm{h}\mathcal{D}) \arrow[r, leftrightarrow, "\simeq"] & \operatorname{Ch}_{\geq 0}(\mathrm{h}\mathcal{D}) & & \mathrm{h}\overline{\mathcal{X}}
\end{tikzcd} \tag{3.5.3}$$

where the top diagonal functor is (3.5.1) and the lower commutative square is the naturality square of Remark 3.2.3.7. In particular, the dotted equivalence $\hat{\mathcal{X}} \mapsto \overline{\mathcal{X}}$ exists and functorially lifts the incoherent chain complex (3.5.2) to a coherent one.

If we only assume that the target $\mathcal{D}$ is weakly idempotent complete additive but not necessarily stable then, even if sufficient colimits exist to define the functor (3.5.1), it need not be an equivalence anymore; similarly, the dotted functor $\hat{\mathcal{X}} \mapsto \overline{\mathcal{X}}$ (or even $\hat{\mathcal{X}} \mapsto \mathrm{h}\overline{\mathcal{X}}$) does not exist in this generality. For instance, in the ∞-category of *connective* spectra the filtered object $0 \to \mathbb{S} \to \mathbb{S} \to \mathbb{S} \to \ldots$ (which would correspond to the chain complex $0 \leftarrow \mathbb{S}[-1] \leftarrow 0 \leftarrow 0 \leftarrow$) does not arise from a simplicial object.

Remark 3.5.2.1. A systematic study of the relationship between coherent chain complexes and filtered objects in stable ∞-categories is part of Stefano Ariotta's Ph.D. thesis [Ari]. In particular, he directly constructs an equivalence $\operatorname{Fun}(\mathbb{N}, \mathcal{D}) \simeq \operatorname{Ch}_{\geq 0}(\mathcal{D})$ of ∞-categories which we expect to agree with the vertical dotted equivalence in (3.5.3) obtained by combining our result with Lurie's. ◊

3.6 Further tools

3.6.1 Measuring Kan extensions

Let $\mathbb{B} = (B, E, E^\vee)$ be a DK-triple with associated quotient $N_0 = N_0(\mathbb{B})$. Let $\mathcal{X}\colon B \to \mathcal{A}$ be a diagram in a weakly idempotent complete additive ∞-category $\mathcal{A}$ and let $\overline{\mathcal{X}}\colon N_0 \to \mathcal{A}$ be the pointed functor corresponding to $\mathcal{X}$ under the equivalence of Corollary 3.2.3.4.

In this section, we set out to answer the following question:

Question 3.6.1.1. What do the values of the diagram $\overline{\mathcal{X}}\colon N_0 \to \mathcal{A}$ tell us about the original diagram $\mathcal{X}\colon B \to \mathcal{A}$? ◊

The rough answer is that in favorable situations $\overline{\mathcal{X}}$ "measures" how far away $\mathcal{X}$ is from being a Kan extension of its restriction $\mathcal{X}_{<n}$. To make this precise, we make the following definition:

Definition 3.6.1.2. The DK-triple $\mathbb{B}$ is called **monotone** if all Monos make objects bigger, *i.e.*, if we have $b' \leq b$ whenever there exists a Mono $m\colon b' \to b$. We say that $\mathbb{B}$ is **partially monotone** if Monos at least do not make objects smaller, *i.e.*, there are no Monos $b' \to b$ if $b' > b$. ◊

Remark 3.6.1.3. If the partial order $\leq$ on $\pi_0 B$ is total, then the notions of monotone and partially monotone agree; in general being partially monotone is weaker than being monotone. ◊

Remark 3.6.1.4. Whether the DK-triple $(B, E, E^\vee)$ is (partially) monotone does not depend on $E^\vee$, since both the partial order $\leq$ and the class M of Monos are defined only in terms of the Epis. ◊

Example 3.6.1.5.
- In both the DK-triples $\mathbb{B}^\Delta_{\text{min}}$ and $\mathbb{B}^\Delta_{\text{max}}$ on Δ defined in Section 3.3.1 the partial order $\leq$ on the objects $[n] \in \Delta$ is just the usual comparison of cardinalities; the monos are the injective maps. Hence $\mathbb{B}^\Delta_{\text{min}}$ and $\mathbb{B}^\Delta_{\text{max}}$ are both monotone.
- Denote by $\mathbb{B}_\Gamma$ the DK-triple on Γ defined in Section 3.3.2. It is monotone since the Monos are opposite to the surjective maps in $\mathbf{Fin}_\star$ and the order $\leq$ is again just given by comparing cardinalities of finite pointed sets. ◊

Proposition 3.6.1.6. Let $\mathbb{B} = (B, E, M)$ be a partially monotone DK-triple with associated quotient $N_0 = N_0(\mathbb{B})$. Fix a diagram $\mathcal{X}\colon B \to \mathcal{C}$ in an arbitrary ∞-category $\mathcal{C}$ and an object $n \in N$.

1. The functor $\mathcal{X}$ is pointwise at $[n] \in B$ a right Kan extension of its restriction to $B_{<[n]}$ if and only if
$$(E_{\not\simeq}([n]))^{\triangleleft} \simeq E([n]) \hookrightarrow B \xrightarrow{\mathcal{X}} \mathcal{C}$$
is a limit cone in $\mathcal{C}$.
2. If the ∞-category $\mathcal{A} := \mathcal{C}$ is weakly idempotent complete additive (or preadditive if $\mathbb{B}$ is diagonalizable) then this happens if and only if the corresponding pointed diagram $\overline{\mathcal{X}}\colon N_0 \to \mathcal{A}$ vanishes at n, *i.e.*, if and only if $\overline{\mathcal{X}}_n$ is a zero object in $\mathcal{A}$.
3. Assume that $\mathbb{B}$ is monotone. Then $\mathcal{X}\big|_{E(n)}$ is a limit cone if and only if $\mathcal{X}$ is pointwise at $[n] \in B$ a right Kan extension of its restriction to $B_{\not\geq[n]}$. □

Proof. We only prove (1) and (2); the proof of (3) is analogous to (1). The pointwise right Kan extension of $\mathcal{X}_{<[n]}$ at $[n] \in B$ is computed as the limit of the diagram
$$\lim\left(\left(B_{<[n]}\right)_{[n]/} \longrightarrow B \xrightarrow{\mathcal{X}} \mathcal{A}\right)$$
We show that if $\mathbb{B}$ is partially monotone then the canonical inclusion $E_{\not\simeq}([n]) \to \left(B_{<[n]}\right)_{[n]/}$ is homotopy initial:

- This amounts to showing that for each object $b \in B$ with $b < [n]$ and each arrow $[n] \to b$, the poset of factorizations
$$\begin{tikzcd} & {[n']} \arrow[dr] & \\ {[n]} \arrow[ur, "E_{\not\simeq}"] \arrow[rr] & & b \end{tikzcd} \tag{3.6.1}$$
is weakly contractible. The first leg in the unique (E, M)-factorization
$$\begin{tikzcd} & {[n']} \arrow[dr, "M"] & \\ {[n]} \arrow[ur, "E"] \arrow[rr] & & b \end{tikzcd} \tag{3.6.2}$$

must be non-invertible because otherwise $n' \cong n > b$ would contradict the assumption that $\mathbb{B}$ is partially monotone. It follows that the unique factorization (3.6.2) is of type (3.6.1) and is therefore a terminal object in the the poset we wish to contract.

It follows that the desired pointwise right Kan extension is computed as the limit

$$\lim\Big(E_{\not\simeq}([n]) \longrightarrow B \xrightarrow{\mathcal{X}} \mathcal{A}\Big)$$

as required by (1). Statement (2) now follows from Theorem 3.2.3.1 (d) which states in particular that the canonical map

$$\mathcal{X}_{[n]} \longrightarrow \lim_{b \in E_{\not\simeq}([n])} \mathcal{X}_b.$$

is retraction with complement $\overline{\mathcal{X}}_n$. ∎

Fix a natural number $k \in \mathbb{N}$. Recall that $\Delta_{\leq k} \subset \Delta$ denotes the full subcategory spanned by the objects $[n]$ with $n \leq k$.

Corollary 3.6.1.7. A simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{A}$ in a weakly idempotent complete additive ∞-category is a left Kan extension of its restriction to $\Delta^{\mathrm{op}}_{\leq k}$ if and only if the corresponding connective chain complex $\overline{\mathcal{X}} \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ is k-truncated, *i.e.*, $\overline{\mathcal{X}}_n \simeq 0$ for all $n > k$. ∎

Proof. Apply Proposition 3.6.1.6 (2) to the DK-triple $\mathbb{B}^{\Delta}_{\mathrm{min}}$ (or, equivalently, to the DK-triple $\mathbb{B}^{\Delta}_{\mathrm{max}}$) and dualize. ∎

3.6.2 Functoriality

Definition 3.6.2.1. Let $\mathbb{B} = (B, E, E^{\vee})$ and $\mathbb{B}' = (B', E', E'^{\vee})$ be DK-triples. We say that a functor $F\colon B \to B'$ is a **DK-morphism** $\mathbb{B} \to \mathbb{B}'$ if, for each $b \in B$, it induces an equivalence $F\colon E(b) \xrightarrow{\simeq} E'(Fb)$ between the respective posets of Epis. ◊

Remark 3.6.2.2. Whether a functor $F\colon B \to B'$ is a DK-morphism only depends on the Epis of the DK-triples, not on the dual Epis. ◊

Example 3.6.2.3. Let $\Delta^{\mathrm{op}} \to \mathbf{Fin}_{\star}$ be the standard simplicial circle $\Delta^1 / \partial\Delta^1$. This functors sends $[n]$ to the pointed set $\langle n \rangle \cong E_{\Delta}([n],[1])_+$ and induces the map of posets

$$E_{\Delta}([n]) \longrightarrow E_{\Gamma}(\langle n \rangle) \cong \mathbf{P}(E_{\Delta}([n],[1])) \tag{3.6.3}$$

which sends an Epi $[n] \twoheadrightarrow [n']$ to the subset of $E_{\Delta}([n],[1])$ consisting of those Epis $[n] \twoheadrightarrow [1]$ that factor through $[n] \twoheadrightarrow [n']$. The map (3.6.3) has an inverse given by sending a set S of Epis $[n] \twoheadrightarrow [1]$ to the unique quotient of $[n]$ which identifies $i, i' \in [n]$ if and only if $e(i) = e(i')$ for all $e \in S$.

It follows that the circle functor $\Delta \to \Gamma$ defines DK-morphisms $\mathbb{B}^{\Delta}_{\mathrm{min}} \to \mathbb{B}_{\Gamma}$ and $\mathbb{B}^{\Delta}_{\mathrm{max}} \to \mathbb{B}_{\Gamma}$ ◊

Remark 3.6.2.4. Let $F\colon \mathbb{B} \to \mathbb{B}'$ be a DK-morphism as in Definition 3.6.2.1. Let $\mathcal{X}\colon B' \to \mathcal{A}$ be a diagram with values in an weakly idempotent complete additive ∞-category. Then for each $b \in B$, we have an equivalence

$$(\overline{\mathcal{X} \circ F})_{\underline{b}} \simeq \text{tot-fib}\Big((\mathcal{X} \circ F)\big|_{E(b)}\Big) \xrightarrow{\simeq} \text{tot-fib}\Big(\mathcal{X}\big|_{E'(b)}\Big) \simeq \overline{\mathcal{X}}_{\underline{Fb}},$$

where the outer two equivalences come from the formula (3.2.5) in Remark 3.2.3.9 and the middle arrow is an equivalence because the DK-morphism F identifies the posets $E(b)$ and $E'(b)$. ◊

Corollary 3.6.2.5. Let $F\colon (B, E, E^{\vee}) \to (B', E', E'^{\vee})$ be a DK-morphism between partially monotone DK-triples. Consider a diagram $\mathcal{X}\colon B' \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ and an object $b \in B$. The following are equivalent:

- Pointwise at b, the functor $\mathcal{X} \circ F$ is a right Kan extension of its restriction to $B_{<b}$.

- Pointwise at Fb, the functor $\mathcal{X}$ is a right Kan extension of its restriction to $B'_{<Fb}$. □

Proof. Corollary 3.6.2.5 follows directly from Proposition 3.6.1.6 (1), since the DK-morphism F identifies the posets $E(b)$ and $E'(Fb)$. ■

Fix a natural number $k \in \mathbb{N}$. Denote by $\mathbf{Fin}_\star^{\leq k} \subset \mathbf{Fin}_\star$ the full subcategory spanned by the pointed sets $\langle n \rangle := \{\star, 1, \dots, n\}$ with $n \leq k$,

Corollary 3.6.2.6. A Γ-object $\mathcal{X}\colon \mathbf{Fin}_\star \to \mathcal{C}$ in an ∞-category $\mathcal{C}$ is a left Kan extension of its restriction to $\mathbf{Fin}_\star^{\leq k}$ if and only if the underlying simplicial object $\mathcal{X}\big|_{\Delta^{\mathrm{op}}}\colon \Delta^{\mathrm{op}} \to \mathbf{Fin}_\star \to \mathcal{C}$ is a left Kan extension of its restriction to $\Delta^{\mathrm{op}}_{\leq k}$. □

Proof. Apply Corollary 3.6.2.5 to the circle functor $\Delta \to \Gamma$. ■

3.7 Higher Segal objects in the additive or stable context

3.7.1 Computing membrane spaces via the Dold–Kan correspondence

In this section we explain how to compute many membrane spaces of simplicial objects (see Section 2.4.1) in weakly idempotent complete additive ∞-categories as direct sums of the terms of the corresponding chain complex. This extends observation made in [DJW19].

By abuse of notation we identify a subset $J \subseteq [n]$ with the precover $\{J\} \models [n]$; for instance, given some precover $\mathcal{F} \models [n]$ which is refined by $\{J\}$ (*i.e.*, there is an $I \in \mathcal{F}$ with $J \subset I$) we write $J \preceq \mathcal{F}$ (instead of $\{J\} \preceq \mathcal{F}$) and say that J refines $\mathcal{F}$.

Definition 3.7.1.1. We say that a precover $\mathcal{F} \models [n] \in \Delta$ is at least k-fold if each subset $J \subseteq [n]$ with $|J| = k$ refines $\mathcal{F}$. We say that $\mathcal{F}$ is **exactly k-fold** if it is at least k-fold but not at least $(k+1)$-fold. ◊

Remark 3.7.1.2. Clearly, each precover $\mathcal{F} \models [n]$ is exactly k-fold for a unique number $k \in \mathbb{N}$, which is non-zero precisely if $\mathcal{F}$ is a cover. ◊

Proposition 3.7.1.3. Fix a simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{A}$ in a weakly idempotent complete additive ∞-category and denote by $\overline{\mathcal{X}} \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ the corresponding chain complex. Consider an object $[n] \in \Delta$ and an element $i \in [n]$. For all precovers $\mathcal{F} \models [n]$ with $i \in \bigcap \mathcal{F}$ there are equivalences

$$\mathcal{X}_{\mathcal{F}} \xrightarrow{\simeq} \bigoplus_{i \in J \preceq \mathcal{F}} \overline{\mathcal{X}}_{\underline{J}}$$

such that

(1) the Segal map $\mathcal{X}_n \simeq \mathcal{X}_{\{[n]\}} \longrightarrow \mathcal{X}_{\mathcal{F}}$ is identified with the factor projection

$$\bigoplus_{i \in J \subseteq [n]} \overline{\mathcal{X}}_{\underline{J}} \longrightarrow \bigoplus_{i \in J \preceq \mathcal{F}} \overline{\mathcal{X}}_{\underline{J}} \tag{3.7.1}$$

along $\{i \in J \preceq \mathcal{F}\} \subseteq \{i \in J \subset [n]\}$

(2) for each refinement $\mathcal{F} \preceq \mathcal{F}'$ of such precovers, the induced map $\mathcal{X}_{\mathcal{F}'} \to \mathcal{X}_{\mathcal{F}}$ is identified with the factor projection along $\{i \in J \preceq \mathcal{F}\} \subseteq \{i \in J \preceq \mathcal{F}'\}$. □

Remark 3.7.1.4. Proposition 3.7.1.3 is a unified and slightly more refined version of Propositions 4.6 and 4.23 in [DJW19], which deal with the case $i \in \{0, n\}$ in abelian and stable (∞-)categories, respectively. ◊

Consider the category $\Delta_\star := \Delta_{[0]/}$ of finite *pointed* linearly ordered sets. Denote by $E_\star$ and $E_\star^\vee$ the surjective and injective maps in $\Delta_\star$, respectively.

Lemma 3.7.1.5. The datum $\mathbb{B}^{\Delta_\star} = (\Delta_\star, E_\star, E_\star^\vee)$ is a reduced DK-triple. The associated pointed category $N_0(\mathbb{B}^{\Delta_\star}) \cong (\Delta_\star)_+^{\simeq}$ is free on the discrete category/set $\pi_0(\Delta_\star) = \{([n], i) \mid 0 \le i \le n\}$. The forgetful functor $\Delta_\star \to \Delta$ is a DK-morphism. ☐

Proof. Denote by Δ^{min} and Δ^{max} the wide subcategories of Δ consisting of all minimum-preserving and maximum-preserving maps, respectively. There are reduced DK-triples

$$\overline{\mathbb{B}^{\Delta}_{\mathrm{min}}} = (\Delta^{\mathrm{min}} = E^\vee_{\mathrm{min}} \circ E, E, E^\vee_{\mathrm{min}}) \qquad \text{and} \qquad \overline{\mathbb{B}^{\Delta}_{\mathrm{max}}} = (\Delta^{\mathrm{max}} = E^\vee_{\mathrm{max}} \circ E, E, E^\vee_{\mathrm{max}})$$

underlying the DK-triples $\mathbb{B}^{\Delta}_{\mathrm{min}} = (\Delta, E, E^\vee_{\mathrm{min}})$ and $\mathbb{B}^{\Delta}_{\mathrm{max}} = (\Delta, E, E^\vee_{\mathrm{max}})$ described in Section 3.3.1. There is a canonical equivalence

$$\Delta^{\mathrm{max}} \times \Delta^{\mathrm{min}} \xrightarrow{\simeq} \Delta_\star \qquad : \qquad (I, J) \longmapsto (I \vee J, \max I = \min J)$$

which identifies $\mathbb{B}^{\Delta_\star}$ with the datum $\overline{\mathbb{B}^{\Delta}_{\mathrm{min}}} \times \overline{\mathbb{B}^{\Delta}_{\mathrm{max}}}$. We conclude that $\mathbb{B}^{\Delta_\star}$ is the product of reduced DK-triples, hence itself a reduced DK-triple.

The rest of the claims are straightforward to check directly and left to the reader. ■

Proof (of Proposition 3.7.1.3)**.** We replace $\mathcal{A}$ by its dual so that we may work with diagrams $\mathcal{X}\colon \Delta \to \mathcal{A}$. By abuse of notation we also write $\mathcal{X}$ (instead of $\mathcal{X}\big|_{\Delta_\star}$) for the composition $\Delta_\star \to \Delta \xrightarrow{\mathcal{X}} \mathcal{A}$. Applied to the DK-triple $\mathbb{B}^{\Delta_\star}$ of Lemma 3.7.1.5, Theorem 3.2.3.1—or, more precisely, Corollary 3.2.3.4—produces an equivalence

$$\mathrm{Fun}(\Delta_\star, \mathcal{A}) \simeq \prod_{([n],i)} \mathcal{A}$$

under which our functor $\mathcal{X}\colon \Delta_\star \to \mathcal{A}$ corresponds to a tuple $\left(\overline{\mathcal{X}}_{\underline{([n],i)}}\right)_{n,i}$. Since $\Delta_\star \to \Delta$ is a DK-morphism, for each $i \in [n] \in \Delta$ we have $\overline{\mathcal{X}}_{\underline{([n],i)}} \simeq \overline{\mathcal{X}}_n$ (see Remark 3.6.2.4), *i.e.*, the value of $\overline{\mathcal{X}}$ does not depend on i. Proposition 3.4.2.1—or, more precisely, Remark 3.4.2.3—then states that there are direct sum decompositions

$$\bigoplus_{i \in J \subseteq [n]} \overline{\mathcal{X}}_{\underline{J}} \simeq \bigoplus_{i \in J \subseteq [n]} \overline{\mathcal{X}}_{\underline{(J,i)}} \xrightarrow{\simeq} \mathcal{X}_{([n],i)} = \mathcal{X}_n$$

with respect to which the value of $\mathcal{X}$ on a dual Epi $(I, i) \hookrightarrow ([n], i)$ is identified with the inclusion of those summands $\overline{\mathcal{X}}_{\underline{J}}$ which are indexed by a $i \in J \subseteq [n]$ which is contained in I. For each precover $\mathcal{F} \models [n]$ the membrane space $\mathcal{X}_{\mathcal{F}} \simeq \mathrm{colim}_{i \in I \preceq \mathcal{F}} \mathcal{X}_I$ is a colimit over a system of inclusions of subsums of (3.7.1); hence it can be explicitly computed to yield the statements of Proposition 3.7.1.3. ■

Corollary 3.7.1.6. Let $\mathcal{F} \models [n]$ be a precover on $[n] \in \Delta$ which is exactly k-fold and satisfies $\bigcap \mathcal{F} \neq \varnothing$. Then the simplicial object $\mathcal{X}\colon \Delta^{\mathrm{op}} \to \mathcal{A}$ in a weakly idempotent complete additive ∞-category $\mathcal{A}$ satisfies descent with respect to $\mathcal{F}$ if and only if the corresponding chain complex $\overline{\mathcal{X}}$ vanishes in the range $\{k+1, \dots, n\}$. ☐

Proof. Since the intersection $\bigcap \mathcal{F}$ is non-empty, we may pick an element $i \in \bigcap \mathcal{F}$ and apply Proposition 3.7.1.3 to identify the Segal map $\mathcal{X}_n \longrightarrow \mathcal{X}_{\mathcal{F}}$ with the projection (3.7.1), which is an equivalence if and only if $\overline{\mathcal{X}}_{\underline{J}} \simeq 0$ for all $i \in J \not\preceq \mathcal{F}$. Such a J must satisfy $J \setminus \{i\} \not\preceq \mathcal{F}$ (because $i \in \bigcap \mathcal{F}$), hence $\underline{J} = |J \setminus \{i\}| > k$ (because $\mathcal{F} \models [n]$ is at least k-fold). To show that the value $\underline{J} = k+1$ (hence every value in the interval $\{k+1, \dots, n\}$) is attained by such a J, choose $J' \subset [n]$ minimal such that $J' \not\preceq \mathcal{F}$; then $i \notin J'$ (otherwise $J' \setminus \{i\}$ would contradict minimality) and $|J'| = k+1$ (because $\mathcal{F}$ is exactly k-fold), hence $\underline{J' \dot\cup \{i\}} = |J'| = k+1$. ■

Definition 3.7.1.7. We call a set P of precovers k**-truncated** if

- every cover $\mathcal{F} \in P$ satisfies $\bigcap \mathcal{F} \neq \varnothing$,
- every cover $\mathcal{F} \in P$ is at least k-fold and
- for every $k < m \in \mathbb{N}$ there exists $n \geq m$ and a cover $\mathcal{F} \models [n]$ in P which is not at least m-fold. ◊

Example 3.7.1.8. Fix a natural number k.

- The **i-th n-dimensional horn**

  $$\Lambda^n_i := \{[n] \setminus \{j\} \mid j \neq i\} \models [n]$$

  is always exactly $(n-1)$-fold. Hence a subset of the set $\{\Lambda^n_i \models [n] \mid 0 \leq i \leq n > k\}$ of horns above dimension k is k-truncated if and only if it contains at least one horn of each dimension $n > k$. For example,

  - the set $\{\Lambda^n_0 \models [n] \mid n > k\}$ of left horns above dimension k and
  - the set $\{\Lambda^n_n \models [n] \mid n > k\}$ of right horns above dimension k

  are both k-truncated.
- A non-degenerate compatible $[k]$-cover (in the sense of Definition 2.3.1.1) is always exactly k-fold. Hence *any* infinite set of non-degenerate compatible $[k]$-covers is k-truncated. We highlight the following examples:
  - Lower weak $[k]$-Λ-excision (a.k.a. the lower $2k$-Segal-Segal condition) is equivalent to descent with respect to the infinite set of *left active* non-degenerate $[k]$-covers.
  - Upper weak $[k]$-Λ-excision (a.k.a. the upper $2k$-Segal-Segal condition) is equivalent to descent with respect to the infinite set of *right active* non-degenerate $[k]$-covers.
  - weak $[k]$-(Δ-)excision (a.k.a. the lower $(2k-1)$-Segal-Segal condition) is equivalent to descent with respect to the infinite set of *all* non-degenerate $[k]$-covers. ◊

Definition 3.7.1.7 is made precisely to make the following statement true:

Proposition 3.7.1.9. Consider a simplicial object $\mathcal{X} \colon \Delta^{\mathrm{op}} \to \mathcal{A}$ in a weakly idempotent complete additive ∞-category $\mathcal{A}$ and a number $k \in \mathbb{N}$. Let P be an k-truncated set of precovers. The following are equivalent:

- The simplicial object $\mathcal{X}$ satisfies descent with respect to all precovers in P.
- The chain complex $\overline{\mathcal{X}} \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ associated to $\mathcal{X}$ is k-truncated, *i.e.*, satisfies $\overline{\mathcal{X}}_n \simeq 0$ for all $n > k$. ☐

Proof. Follows directly from Definition 3.7.1.7 and Corollary 3.7.1.6. ■

Remark 3.7.1.10. Proposition 3.7.1.9 says in particular that, for a simplicial objects $\mathcal{X}$ in an additive ∞-category, all of the sets of descent conditions described in Example 3.7.1.8 are equivalent to the corresponding chain complex $\overline{\mathcal{X}}$ (potentially computed in some ambient weakly idempotent complete ∞-category) being k-truncated; in particular, they are all equivalent to each other. This subsumes, unifies and generalizes Theorems 4.12 and 4.27 in [DJW19], which cover left/right horns and lower/upper even Segal conditions in the case of abelian categories and stable ∞-categories. ◊

It follows from Proposition 3.7.1.9 that Question 2.6.3.3 has an affirmative answer if we assume the target ∞-category to be additive.

Corollary 3.7.1.11. Let $\mathcal{X} \colon \Delta^{\mathrm{op}} \to \mathcal{A}$ be a simplicial object in a additive ∞-category. Assume that $\mathcal{X}$ is lower $(2k-1)$-Segal (or, equivalently, lower/upper $2k$-Segal) and that $\mathcal{X}_k \simeq 0$. Then $\mathcal{X}$ is trivial, *i.e.*, $\mathcal{X}_n \simeq 0$ for all n. ☐

Proof. By suitably enlarging $\mathcal{A}$ inside some ambient stable ∞-category, we may assume that $\mathcal{A}$ is weakly idempotent complete. Then by assumption and Proposition 3.7.1.9, the chain complex $\overline{\mathcal{X}} \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ corresponding to $\mathcal{X}$ under the Dold–Kan correspondence vanishes in the range $\{k, k+1, \dots\}$. Moreover the value $\mathcal{X}_k \simeq \bigoplus_{[k] \twoheadrightarrow [n]} \overline{\mathcal{X}}_n$ (with the product decomposition of Corollary 3.4.1.6) can only vanish if $\overline{\mathcal{X}}$ vanishes in the range $\{0, \dots, k\}$. The result follows. ■

3.7.2 Goodwillie calculus in stable ∞-categories

Recall that in Goodwillie's calculus [Goo92; Goo03] a functor $F\colon \mathcal{Z} \to \mathcal{C}$ is called **k-excisive** if it sends strongly coCartesian $[k]$-cubes in $\mathcal{Z}$ to Cartesian $[k]$-cubes in $\mathcal{C}$. For example, F is 1-excisive (or just **excisive**) if it sends pushouts squares to pullback squares.

In favorable situations, the fully faithful inclusion

$$\mathrm{Exc}^k(\mathcal{Z}, \mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{Z}, \mathcal{C})$$

of the ∞-category of k-excisive functors into the ∞-category of all functors $\mathcal{Z} \to \mathcal{C}$ admits a left adjoint P_k called Taylor approximation. A functor $F\colon \mathcal{Z} \to \mathcal{C}$ is called **k-homogeneous** if it is k-excisive and its $(k-1)$-th Taylor approximation $P_{k-1}F$ is trivial (*i.e.*, equivalent to the terminal functor).

The goal of this section is to use the Dold–Kan type equivalence

$$\mathrm{Fun}(\mathbf{Fin}_\star, \mathcal{A}) \overset{\simeq}{\longleftrightarrow} \mathrm{Fun}(\mathbf{Surj}, \mathcal{A})$$

established in Section 3.3.2 to better describe excisive and homogeneous functors from the ∞-category $\mathcal{S}_\star$ of pointed spaces to some presentable stable ∞-category $\mathcal{D}$, *e.g.*, the ∞-category $\mathcal{S}p$ of spectra.

We start with a more general discussion. Let $\mathcal{Q}$ be a small pointed category. Let

$$\mathcal{Q} \hookrightarrow \mathcal{P}(\mathcal{Q}) := \mathrm{Fun}(\mathcal{Q}^{\mathrm{op}}, \mathcal{S})$$

be the fully faithful Yoneda embedding of $\mathcal{Q}$ into the ∞-category of space-valued presheaves on $\mathcal{Q}$. We identify $\mathcal{Q}$ with its image in $\mathcal{P}(\mathcal{Q})$, hence each object $q \in \mathcal{Q}$ with the corresponding representable presheaf $\mathrm{Map}_{\mathcal{Q}}(-, q)\colon \mathcal{Q}^{\mathrm{op}} \to \mathcal{S}$. Denote by

$$\mathcal{P}_\star(\mathcal{Q}) := \{0 \mapsto \star\} \subset \mathcal{P}(\mathcal{Q})$$

the full subcategory consisting of those presheaves $\mathcal{Q}^{\mathrm{op}} \to \mathcal{S}$ which send the zero object $0 \in \mathcal{Q}$ to the terminal object $\star \in \mathcal{S}$. The ∞-category $\mathcal{P}_\star(\mathcal{Q})$ contains all representables $q \in \mathcal{Q}$ because 0 is initial in $\mathcal{Q}$, *i.e.*, $\mathrm{Map}_{\mathcal{Q}}(0, q)$ is contractible. The ∞-category $\mathcal{P}_\star(\mathcal{Q})$ is presentable and the inclusion $i\colon \mathcal{P}_\star(\mathcal{Q}) \hookrightarrow \mathcal{P}(\mathcal{Q})$ has a left adjoint L (see Lemma 2.2.5.1) defined explicitly on $\mathcal{X} \in \mathcal{P}(\mathcal{Q})$ by the coCartesian square

$$\begin{tikzcd}
\mathrm{const}_{\mathcal{X}(0)} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\star \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & L\mathcal{X}
\end{tikzcd}$$

where the top map at q is the value of $\mathcal{X}$ at the unique map $q \to 0$ (using that $0 \in \mathcal{Q}$ is terminal). The right vertical arrows $\mathcal{X} \to L\mathcal{X}$ assemble to give rise to the unit of the adjunction $L \dashv i$.

Lemma 3.7.2.1. The inclusion $\mathcal{P}_\star(\mathcal{Q}) \hookrightarrow \mathcal{P}(\mathcal{Q})$ preserves colimits of weakly contractible shape, for example pushouts and filtered colimits. □

Proof. Colimits in $\mathcal{P}(\mathcal{Q})$ are computed pointwise and colimits in $\mathcal{S}$ of weakly contractible shape preserve the terminal object (in a way, that is precisely what "weakly contractible shape" means). Hence the weakly contractible colimit in question (computed in $\mathcal{P}(\mathcal{Q})$) of a diagram in $\mathcal{P}_\star(\mathcal{Q})$ lies again in $\mathcal{P}_\star(\mathcal{Q})$. The result follows. ■

Remark 3.7.2.2. Since the inclusion $\mathcal{P}_\star(\mathcal{Q}) \hookrightarrow \mathcal{P}(\mathcal{Q})$ does not preserve coproducts, one always needs to specify in which ambient category we are computing coproducts. To make this distinction clearer, we denote the coproduct in $\mathcal{P}_\star(\mathcal{Q})$ by $\bigvee$ (instead of $\coprod$) and it **wedge**. ◊

Given a small pointed category $\mathcal{Q}$ and a natural number $k \in \mathbb{N}$, we define the full subcategories

$$\mathcal{P}_\star^{\leq k}(\mathcal{Q}) := \left\{ \bigvee_{j \in I} q_j \;\middle|\; |I| \leq k, q_j \in \mathcal{Q} \right\} \subset \mathcal{P}_\star(\mathcal{Q}) \qquad \text{and} \qquad \mathcal{P}^{\leq k}(\mathcal{Q}) := \left\{ \coprod_{j \in I} q_j \;\middle|\; |I| \leq k, q_j \in \mathcal{Q} \right\} \subset \mathcal{P}(\mathcal{Q})$$

spanned by those objects which are the wedge/coproduct of at most k representables. We denote by $\mathcal{P}_\star^{<\infty}(\mathcal{Q}) := \bigcup_{k \in \mathbb{N}} \mathcal{P}_\star^{\leq k}(\mathcal{Q})$ the full subcategory of $\mathcal{P}_\star(\mathcal{Q})$ spanned by all finite wedges of representables.

Example 3.7.2.3. Consider the pointed category $\{1\}_+$ obtained by freely adding a basepoint to the terminal category $\{1\}$. Explicitly, $\{1\}_+$ is the category

$$\begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd}$$

where the composite $0 \to 1 \to 0$ is the identity Id_0. It follows from Remark 3.1.1.6 (using that $\{1\}_+$ is self-dual) that there is a canonical equivalence

$$\mathcal{P}_\star(\{1\}_+) \xrightarrow{\simeq} \mathcal{S}_\star \tag{3.7.2}$$

which sends a presheaf $\mathcal{X} \colon \{1\}_+^{\mathrm{op}} \to \mathcal{S}$ to the pointed space $\star = \mathcal{X}(0) \to \mathcal{X}(1)$, hence in particular the representables $0, 1 \in \{1\}_+$ to the pointed spaces $0 = \mathrm{pt}$ and S^0, respectively. It follows that the equivalence (3.7.2) restricts to equivalences

$$\mathcal{P}_\star^{<\infty}(\mathcal{Q}) \xrightarrow{\simeq} \mathbf{Fin}_\star \qquad \text{and} \qquad \mathcal{P}_\star^{\leq k}(\{1\}_+) \xrightarrow{\simeq} \mathbf{Fin}_\star^{\leq k}$$

because $\mathbf{Fin}_\star$ and $\mathbf{Fin}_\star^{\leq k}$ are precisely the full subcategories of $\mathcal{S}_\star$ spanned by wedges of finitely many/at most k many copies of S^0. ♢

Lemma 3.7.2.4. Let $\mathcal{Z}$ be an ∞-category and let $\mathcal{C}$ be a presentable ∞-category. For each object $x \in \mathcal{Z}$, denote by $x_! \colon \mathcal{C} \to \mathrm{Fun}(\mathcal{Z}, \mathcal{C})$ the left Kan extension functor along $x \colon \{\star\} \to \mathcal{Z}$.

(1) The functor $x_!$ is given explicitly on $X \in \mathcal{C}$ by

$$x_!(X) \simeq \mathrm{Map}_{\mathcal{Z}}(x, -) \otimes X.$$

where $\otimes \colon \mathcal{S} \times \mathcal{C} \to \mathcal{C}$ is the canonical tensoring of $\mathcal{C}$ over $\mathcal{S}$ (see [Lur09, Proposition 4.8.1.15]).

(2) The functor category $\mathrm{Fun}(\mathcal{Z}, \mathcal{C})$ is generated under colimits by the functors $x_!(X)$ for $x \in \mathcal{Z}$ and $X \in \mathcal{C}$. □

Proof. Lemma 3.7.2.4 is precisely the content of the first two paragraphs in the proof of [Lur17, Theorem 6.1.5.6], where it is stated for the specific ∞-category $\mathcal{Z} = \mathcal{P}^{\leq n}(\mathcal{C})$ but proved in a way that works for all $\mathcal{Z}$. ■

Recall that a functor is called finitary if it preserves filtered colimits. We will need the following theorem, which classifies finitary k-excisive functors in presentable stable ∞-categories.

Theorem 3.7.2.5. [Lur17, Theorem 6.1.5.6]. Let $\mathcal{Z}$ be a small ∞-category and $\mathcal{D}$ a presentable stable ∞-category. Fix a natural number $k \in \mathbb{N}$ and let $F \colon \mathcal{P}(\mathcal{Z}) \to \mathcal{D}$ be a functor. The following are equivalent:

(1) The functor F is a left Kan extension of its restriction to $\mathcal{P}_\star^{\leq k}(\mathcal{Z})$

(2) The functor F is k-excisive and preserves filtered colimits. □

More specifically, we need the following pointed version.

Corollary 3.7.2.6. Let $\mathcal{Q}$ be a small pointed ∞-category and $\mathcal{D}$ a presentable stable ∞-category. Fix a natural number $k \in \mathbb{N}$ and let $F \colon \mathcal{P}_\star(\mathcal{Q}) \to \mathcal{D}$ be a functor. The following are equivalent:

(1) The functor F is a left Kan extension of its restriction to $\mathcal{P}_\star^{\leq k}(\mathcal{Q})$

(2) The functor F is k-excisive and preserves filtered colimits. ☐

Proof. We first prove that (1) implies (2), mimicking the proof of [Lur17, Theorem 6.1.5.6]. It is enough to show (2) whenever F is a functor $q_!(X)\colon \mathcal{P}_\star(\mathcal{Q}) \to \mathcal{D}$ for some $q \in \mathcal{P}_\star^{\leq k}(\mathcal{Q})$ and $X \in \mathcal{D}$, because these are the left Kan extensions of the homonymous functors $q_!(X)\colon \mathcal{P}_\star^{\leq k}(\mathcal{Q}) \to \mathcal{D}$ which by Lemma 3.7.2.4 generate under colimits the ∞-category $\mathrm{Fun}(\mathcal{P}_\star^{\leq k}(\mathcal{Q}), \mathcal{D})$. If q is actually representable (*i.e.*, lies in $\mathcal{Q} \subset \mathcal{P}_\star^{\leq k}(\mathcal{Q})$) then it follows from Lemma 3.7.2.1 that $\mathrm{Map}_{\mathcal{P}_\star(\mathcal{Q})}(q, -)$ (which by the Yoneda lemma is just evaluation at q) preserves filtered colimits and pushouts. If $q = \bigvee q_j$ is the wedge of at most k representables then if follows that $\mathrm{Map}_{\mathcal{P}_\star(\mathcal{Q})}(q, -) \simeq \prod_j \mathrm{Map}_{\mathcal{P}_\star(\mathcal{Q})}(q_j, -)$ preserves filtered colimits (because in $\mathcal{S}$ filtered colimits commute with products) and sends strongly coCartesian $[k]$-cubes to coCartesian $[k]$-cubes (because in $\mathcal{S}$ the product of at most k strongly coCartesian $[k]$-cubes is coCartesian⁷⁾). It follows that $q_!(X)$ is finitary and k-excisive, because $- \otimes X\colon \mathcal{S} \to \mathcal{D}$ preserves all colimits and because coCartesian cubes in $\mathcal{D}$ are Cartesian by stability.

For the converse we must show that each finitary k-excisive functor $\mathcal{P}_\star(\mathcal{Q}) \to \mathcal{D}$ is the left Kan extension of some functor $\mathcal{P}_\star^{\leq k}(\mathcal{Q}) \to \mathcal{D}$. Since the localization functor L sends coproducts to wedges, it induces a commutative square

$$\begin{tikzcd}
\mathcal{P}^{\leq k}(\mathcal{Q}) \arrow[r, hook] \arrow[d, "L"'] & \mathcal{P}(\mathcal{Q}) \arrow[d, "L"] \\
\mathcal{P}_\star^{\leq k}(\mathcal{Q}) \arrow[r, hook] & \mathcal{P}_\star(\mathcal{Q})
\end{tikzcd}$$

to which we apply $\mathrm{Fun}(-, \mathcal{D})$ to obtain the following commutative square of adjoint pairs:

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{P}^{\leq k}(\mathcal{Q}), \mathcal{D}) \arrow[r, hook, shift left=1.5, "\mathrm{LKE}"] \arrow[r, phantom, "\perp"] \arrow[d, shift right=1.5, "L_!"'] \arrow[d, phantom, "\dashv"] & \mathrm{Fun}(\mathcal{P}(\mathcal{Q}), \mathcal{D}) \arrow[l, shift left=1.5, "\mathrm{Res}"] \arrow[d, shift right=1.5, "i^\star = L_!"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{Fun}(\mathcal{P}_\star^{\leq k}(\mathcal{Q}), \mathcal{D}) \arrow[r, hook, shift left=1.5, "\mathrm{LKE}"] \arrow[r, phantom, "\perp"] \arrow[u, shift right=1.5, "L^\star"'] & \mathrm{Fun}(\mathcal{P}_\star(\mathcal{Q}), \mathcal{D}) \arrow[l, shift left=1.5, "\mathrm{Res}"] \arrow[u, shift right=1.5, "L^\star = i_\star"']
\end{tikzcd} \tag{3.7.3}$$

For each finitary k-excisive functor $F\colon \mathcal{P}_\star(\mathcal{Q}) \to \mathcal{D}$, the functor $L^\star F\colon \mathcal{P}(\mathcal{Q}) \to \mathcal{D}$ is again finitary and k-excisive (because L preserves colimits); hence we can apply Theorem 3.7.2.5 to obtain a functor $g\colon \mathcal{P}_\star^{\leq k}(\mathcal{Q}) \to \mathcal{D}$ whose left Kan extension along $\mathcal{P}^{\leq k}(\mathcal{Q}) \hookrightarrow \mathcal{P}(\mathcal{Q})$ is $L^\star F$. Then by the commutativity of (3.7.3), the functor $F \simeq L_! L^\star F$ is the left Kan extension along $\mathcal{P}_\star^{\leq k}(\mathcal{Q}) \hookrightarrow \mathcal{P}_\star(\mathcal{Q})$ of $L_! g$. ■

Denote by

$$\widehat{\mathrm{Exc}}_{\mathrm{f}}(\mathcal{P}_\star(\mathcal{Q}), \mathcal{D}) \subset \mathrm{Fun}_{\mathrm{f}}(\mathcal{P}_\star(\mathcal{Q}), \mathcal{D})$$

the full subcategory generated under colimits by the finitary k-excisive functors for all $k \in \mathbb{N}$. We call the functors in $\widehat{\mathrm{Exc}}_{\mathrm{f}}(\mathcal{P}_\star(\mathcal{Q}), \mathcal{D})$ **coanalytic**.

Corollary 3.7.2.7. With $\mathcal{Q}$ and $\mathcal{D}$ as in Corollary 3.7.2.6, restriction and left Kan extension along $\mathcal{P}_\star^{<\infty}(\mathcal{Q}) \hookrightarrow \mathcal{P}_\star(\mathcal{Q})$ give rise to an equivalence

$$\widehat{\mathrm{Exc}}_{\mathrm{f}}(\mathcal{P}_\star(\mathcal{Q}), \mathcal{D}) \overset{\simeq}{\longleftrightarrow} \mathrm{Fun}(\mathcal{P}_\star^{<\infty}(\mathcal{Q}), \mathcal{D}).$$

which for each $k \in \mathbb{N}$ restricts to an equivalence

$$\mathrm{Exc}_{\mathrm{f}}^k(\mathcal{P}_\star(\mathcal{Q}), \mathcal{D}) \overset{\simeq}{\longleftrightarrow} \mathrm{Fun}(\mathcal{P}_\star^{\leq k}(\mathcal{Q}), \mathcal{D}). \tag{3.7.4}$$

---

7) This follows from [Lur17, Lemma 6.1.5.8] using the fact that the cartesian product in $\mathcal{S}$ commutes with colimits in each variable.

between the ∞-categories of finitary k-excisive functors $\mathcal{P}_\star(\mathcal{Q}) \to \mathcal{D}$ and (arbitrary) functors $\mathcal{P}_\star^{\leq k}(\mathcal{Q}) \to \mathcal{D}$. $\square$

Proof. Left Kan extension along $\mathcal{P}_\star^{<\infty}(\mathcal{Q}) \hookrightarrow \mathcal{P}_\star(\mathcal{Q})$ is fully faithful and induces equivalences (3.7.4) by Corollary 3.7.2.6. We need to show that its essential image agrees with $\widehat{\mathrm{Exc}}_{\mathrm{f}}(\mathcal{P}_\star(\mathcal{Q}), \mathcal{D})$ as full sucategories of $\mathrm{Fun}(\mathcal{P}_\star(\mathcal{Q}), \mathcal{D})$. The essential image is closed under colimits (because $\mathrm{Fun}(\mathcal{P}_\star^{<\infty}(\mathcal{Q}), \mathcal{D})$ has all colimits and left Kan extension preserves them), and contains all finitary k-excisive functors (for all k); hence the essential image contains $\widehat{\mathrm{Exc}}_{\mathrm{f}}$. Conversely, every functor F in the essential image can be written as the colimit $F \simeq \operatorname{colim}_{k \in \mathbb{N}} \mathrm{LKE}(F|_{\mathcal{P}_\star^{\leq k}(\mathcal{Q})})$ because the category $\mathcal{P}_\star^{<\infty}(\mathcal{Q})$ is the ascending union of the full subcategories $\mathcal{P}_\star^{\leq k}(\mathcal{Q})$. $\blacksquare$

Let us now focus on the special case $\mathcal{Q} = \{1\}_+$ described in Example 3.7.2.3. We have the following commutative diagram where the left half is described by Corollary 3.7.2.7 and the right half is induced from the Dold–Kan type equivalence (3.3.2).

$$\begin{tikzcd}
\widehat{\mathrm{Exc}}_{\mathrm{f}}(\mathcal{S}_\star, \mathcal{D}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, shift right=1.5ex, "P_k"'] \arrow[d, phantom, "\dashv"] \arrow[d, dashed, bend left=60, "\reflectbox{P}_k"] & \mathrm{Fun}(\mathbf{Fin}_\star, \mathcal{D}) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, shift left=1.5ex, "\mathrm{Res}"] \arrow[d, phantom, "\dashv"] & \mathrm{Fun}(\mathbf{Surj}, \mathcal{D}) \arrow[d, shift left=1.5ex, "\mathrm{Res}"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Exc}_{\mathrm{f}}^{k}(\mathcal{S}_\star, \mathcal{D}) \arrow[r, leftrightarrow, "\simeq"] \arrow[u, hook, shift right=1.5ex] \arrow[d, shift right=1.5ex, "P_{k-1}"'] \arrow[d, phantom, "\dashv"] \arrow[d, dashed, bend left=60, "\reflectbox{P}_{k-1}"] & \mathrm{Fun}(\mathbf{Fin}_\star^{\leq k}, \mathcal{D}) \arrow[r, leftrightarrow, "\simeq"] \arrow[u, hook, shift left=1.5ex, "\mathrm{LKE}"] \arrow[d, shift left=1.5ex, "\mathrm{Res}"] \arrow[d, phantom, "\dashv"] & \mathrm{Fun}(\mathbf{Surj}^{\leq k}, \mathcal{D}) \arrow[u, hook, shift left=1.5ex, "\mathrm{LKE}"] \arrow[d, shift left=1.5ex, "\mathrm{Res}"] \arrow[d, phantom, "\dashv"] \\
\mathrm{Exc}_{\mathrm{f}}^{k-1}(\mathcal{S}_\star, \mathcal{D}) \arrow[r, leftrightarrow, "\simeq"] \arrow[u, hook, shift right=1.5ex] & \mathrm{Fun}(\mathbf{Fin}_\star^{\leq k-1}, \mathcal{D}) \arrow[r, leftrightarrow, "\simeq"] \arrow[u, hook, shift left=1.5ex, "\mathrm{LKE}"] & \mathrm{Fun}(\mathbf{Surj}^{\leq k-1}, \mathcal{D}) \arrow[u, hook, shift left=1.5ex, "\mathrm{LKE}"]
\end{tikzcd} \tag{3.7.5}$$

We denote by $\reflectbox{$P$}_k \colon \widehat{\mathrm{Exc}}_{\mathrm{f}}(\mathcal{S}_\star, \mathcal{D}) \to \mathrm{Exc}_{\mathrm{f}}^{k}(\mathcal{S}_\star, \mathcal{D})$ the functor corresponding to restriction along $\mathbf{Fin}_\star^{\leq k} \hookrightarrow \mathbf{Fin}_\star$; it is right adjoint to the inclusion $\mathrm{Exc}_{\mathrm{f}}^{k}(\mathcal{S}_\star, \mathcal{D}) \to \widehat{\mathrm{Exc}}_{\mathrm{f}}(\mathcal{S}_\star, \mathcal{D})$ hence deserves the name **k-coTaylor approximation**. It follows from Corollary 3.7.2.7 that each coanalytic functor $F \colon \mathcal{S}_\star \to \mathcal{D}$ is the colimit of its coTaylor filtration

$$\reflectbox{P}_0(F) \longrightarrow \reflectbox{P}_1(F) \longrightarrow \cdots \longrightarrow \reflectbox{P}_k(F) \longrightarrow \cdots \longrightarrow F.$$

We say that a finitary functor $\mathcal{S} \to \mathcal{D}$ is **k-cohomogeneous** if it is k-excisive and has vanishing k-coTaylor approximation. Under the equivalence of (3.7.5), the k-cohomogeneous functors correspond precisely to those diagrams $\mathbf{Surj} \to \mathcal{D}$ which are non-zero only in degree k.

Now we can describe the adjunctions appearing in the rightmost columns of (3.7.5) more explicitly. Fix $k \in \mathbb{N}$. Consider the full subcategories $\mathbf{Surj}^{\leq k-1}$ and $\mathbf{Surj}^{=k} \simeq \mathrm{B}\mathfrak{S}_k$ of $\mathbf{Surj}^{\leq k}$ spanned by the objects $\langle n \rangle \in \mathbf{Surj}^{\leq k}$ with $n \leq k$ and $n = k$ respectively.

We have the following ladders of adjunctions given by Kan extension (left adjoints always on top).

$$\begin{tikzcd}[column sep=large]
\mathrm{Fun}(\mathbf{Surj}^{\leq k-1}, \mathcal{D}) \arrow[r, hook, shift left=1.5ex, "\mathrm{LKE}"] \arrow[r, phantom, "\perp"] \arrow[r, bend right=50, "\mathrm{RKE}"'] & \mathrm{Fun}(\mathbf{Surj}^{\leq k}, \mathcal{D}) \arrow[l, shift left=1.5ex, "\mathrm{Res}"] \arrow[l, dashed, bend right=50, "P_{k-1}"'] \arrow[r, shift left=1.5ex, "\mathrm{Res}"] \arrow[r, phantom, "\perp"] \arrow[r, dashed, bend right=50] & \mathrm{Fun}(\mathrm{B}\mathfrak{S}_k, \mathcal{D}) \arrow[l, hook, shift left=1.5ex, "\mathrm{RKE}"] \arrow[l, bend right=50, "\mathrm{LKE}"']
\end{tikzcd} \tag{3.7.6}$$

Observe that in $\mathbf{Surj}^{\leq k}$ there are no arrows going from $\mathbf{Surj}^{\leq k-1}$ to $\mathbf{Surj}^{=k}$; it follows that

- the essential image of left Kan extension along $\mathbf{Surj}^{\leq k-1} \hookrightarrow \mathbf{Surj}^{\leq k}$ is precisely the kernel of the restriction along $\mathrm{B}\mathfrak{S}_k \hookrightarrow \mathbf{Surj}^{\leq k}$ and
- the essential image of right Kan extension along $\mathrm{B}\mathfrak{S}_k \hookrightarrow \mathbf{Surj}^{\leq k}$ is precisely the kernel of the restriction along $\mathbf{Surj}^{\leq k-1} \hookrightarrow \mathbf{Surj}^{\leq k}$.

This implies that the ladder (3.7.6) can be completed with the dashed adjoints to a *ladder of recollements* in the sense of [BBD82; BGS88; AKLY17]. Note that under the correspondence (3.7.5) the left dashed functor corresponds precisely to the Taylor approximation functor P_{k-1}; its kernel—which corresponds to the ∞-category of finitary k-homogeneous functors—is precisely the essential image of left Kan extension along $\mathrm{B}\mathfrak{S}_k \hookrightarrow \mathbf{Surj}^{\leq k}$. In other words, restriction and left Kan extension give rise to an equivalence

$$\mathrm{Homog}^k_{\mathrm{f}}(\mathcal{S}_\star, \mathcal{D}) \overset{\simeq}{\longleftrightarrow} \mathrm{Fun}(\mathrm{B}\mathfrak{S}_k, \mathcal{D}) = \mathfrak{S}_k{-}\mathbf{rep}_{\mathcal{D}}$$

of ∞-categories between finitary k-homogeneous functors and coherent representations in $\mathcal{D}$ of the symmetric group $\mathfrak{S}_k$. Similarly, restriction and right Kan extension along $\mathrm{B}\mathfrak{S}_k \hookrightarrow \mathbf{Surj}^{\leq k}$ give rise to an equivalence

$$\mathrm{coHomog}^k_{\mathrm{f}}(\mathcal{S}_\star, \mathcal{D}) \overset{\simeq}{\longleftrightarrow} \mathrm{Fun}(\mathrm{B}\mathfrak{S}_k, \mathcal{D}) = \mathfrak{S}_k{-}\mathbf{rep}_{\mathcal{D}}$$

of ∞-categories between finitary k-cohomogeneous functors and coherent representations in $\mathcal{D}$ of the symmetric group $\mathfrak{S}_k$.

Warning 3.7.2.8. The ∞-categories of finitary k-homogeneous and k-cohomogeneous functors $\mathcal{S}_\star \to \mathcal{D}$ are both abstractly equivalent to $\mathfrak{S}_k{-}\mathbf{rep}_{\mathcal{D}}$, hence to each other. However, they do *not* form the same subcategory of $\widehat{\mathrm{Exc}_{\mathrm{f}}}(\mathcal{S}_\star, \mathcal{D})$ but are, in the language of semiorthogonal decompositions [BK89], *mutations* of each other. ♢

3.7.3 Higher Segal objects in stable Goodwillie calculus

We conclude this chapter by identifying finitary k-excisive functors in the sense of Goodwillie with higher Segal Γ-objects—at least when the target ∞-category is stable.

Theorem 3.7.3.1. Let $\mathcal{D}$ be a presentable stable ∞-category and let P be a k-truncated set of precovers in Δ. Then restriction along $\mathbf{Fin}_\star \hookrightarrow \mathcal{S}_\star$ induces an equivalence of ∞-categories between:

- Γ-objects $\mathbf{Fin}_\star \to \mathcal{D}$ whose underlying simplicial object satisfies P-descent and
- finitary k-excisive functors $\mathcal{S}_\star \to \mathcal{D}$. ☐

Proof. Follows directly by combining Corollary 3.7.2.7 (relating finitary k-excisive functors $\mathcal{S}_\star \to \mathcal{D}$ to Γ-objects which are left Kan extensions from $\Gamma^{\mathrm{op}}_{\leq k}$), Corollary 3.6.2.6 (relating left Kan extensions along $\Gamma^{\mathrm{op}}_{\leq k} \hookrightarrow \Gamma^{\mathrm{op}}$ to left Kan extensions along $\Delta^{\mathrm{op}}_{\leq k} \hookrightarrow \Delta^{\mathrm{op}}$), Proposition 3.6.1.6 (measuring left Kan extensions along $\Delta^{\mathrm{op}}_{\leq k} \hookrightarrow \Delta^{\mathrm{op}}$ in terms of the truncation of associated chain complexes) and Proposition 3.7.1.9 (which relates P-descent of a simplicial object to truncation of its associated chain complex). ■

Corollary 3.7.3.2. Let $\mathcal{D}$ be a presentable stable ∞-category and fix $k \in \mathbb{N}$. Restriction along $\mathbf{Fin}_\star \hookrightarrow \mathcal{S}_\star$ induces an equivalence between the ∞-categories of

- Γ-objects in $\mathcal{D}$ which are lower $(2k-1)$-Segal (equivalently, lower/upper $2k$-Segal) and
- finitary k-excisive functors $\mathcal{S}_\star \to \mathcal{D}$. ☐

Proof. Apply Theorem 3.7.3.1 to the k-truncated sets of Example 3.7.1.8. ■

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