The Mukai system of rank two and genus two

Dissertation

zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Isabell Hellmann (geb. Große-Brauckmann)
aus Bonn

Bonn 2020

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen
Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Dr. Daniel Huybrechts
2. Gutachter: Prof. Dr. Georg Oberdieck

Tag der mündlichen Prüfung: 10.11.2020
Erscheinungsjahr: 2020

THE MUKAI SYSTEM OF RANK TWO AND GENUS TWO

ISABELL HELLMANN

Abstract. We study the Mukai system $f\colon M \to \mathbb{P}^5$ of rank two and genus two from three different points of view. We compute the degrees and multiplicities of the irreducible components of the fibers over non-reduced curves. We provide examples of algebraically coisotropic and constant cycle subvarieties in M. We find all smooth birational models of M in a sequence of flops.

Introduction

Being asked what this thesis is about, the most precise answer would be

'Everything we know about the Mukai system of rank two and genus two'.

Let us explain, why this informal title, taken with a grain of salt, is very much to the point.

What is the Mukai system?

Let (S, H) be a polarized K3 surface of genus g (i.e. $H \in \operatorname{Pic}(S)$ is primitive and ample such that $H^2 = 2g - 2$) and assume that there is a smooth curve $C \in |H|$ (necessarily of genus $g \geq 2$). Fix two coprime integers $n \geq 1$ and s. We consider the moduli space $M_H(0, nH, s)$ of H-Gieseker stable coherent sheaves on S with Mukai vector $v = (0, nH, s)$. This is an irreducible holomorphic symplectic manifold, which is deformation equivalent to the Hilbert scheme of $n^2(g-1)+1$ points on S. A point in $M_H(0, nH, s)$ corresponds to a stable sheaf $\mathcal{E}$ on S such that $\mathcal{E}$ is pure of dimension one with support in the linear system $|nH|$ and such that $\chi(\mathcal{E}) = s$. Taking the (Fitting) support equips $M_H(0, nH, s)$ with a Lagrangian fibration

$$f\colon M_H(0, nH, s) \longrightarrow |nH| \cong \mathbb{P}^{n^2+1},\ \mathcal{E} \mapsto \operatorname{Supp}(\mathcal{E}),$$

known as the *Mukai system of rank n and genus g* [8], [44]. Over a point in $|nH|$ which corresponds to a smooth curve $D \subset S$, the fibers of f are abelian varieties isomorphic to $\operatorname{Pic}^{\delta}(D)$, where $\delta = s + n^2(g-1)$. This way $M_H(0, nH, s)$ can be viewed as a compactified relative Jacobian of the universal curve $\mathcal{C} \to |nH|$. The Mukai system has several beautiful features that we will take up below. For example, it specializes to the Hitchin system of rank n (associated to a smooth curve $C \in |H|$). Or if $s = \pm 1$, it is the birational model of $S^{[n^2(g-1)+1]}$ at the extreme

The author is supported by the SFB/TR 45 'Periods, Moduli Spaces and Arithmetic of Algebraic varieties' of the DFG (German Research Foundation) and the Bonn International Graduate School.

end of the movable cone.

Why rank two and genus two?
The Mukai system of rank one appears in the literature for various applications. One could say, that it is the prototypical example of a Lagrangian fibration in the Hyperkähler world. One usually assumes that $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$. Then the linear system $|H|$ has the crucial feature, that every curve $C \in |H|$ is integral. This implies, for example, that every fiber of $M_H(0, H, s) \to |H|$ is irreducible and generically smooth. Moreover, the Brill–Noether theory of a general curve $C \in |H|$ is well-understood [36] and has an interpretation for the birational geometry of $M_H(0, H, \pm 1)$ [40]. If $n \geq 2$, it is no longer true, that every curve in $|nH|$ is reduced. Hence, most of the above results fail. Therefore, the lowest dimensional case $n = 2$ and $g = 2$ deserves an in-depth study, to understand which phenomena can arise. This is the purpose of my thesis.

Summary.
We study the Mukai system of rank two and genus two from three different points of view. It turns out, that the isomorphism class of $M_H(0, 2H, s)$ for odd s is independent of the choice of s. Therefore, we specialize to the case

$$M \coloneqq M_H(0, 2H, -1).$$

In Part I, we see $f\colon M \to |2H|$ as a generalisation of the Hitchin system [20]. In this context, we want to understand the structure of the fibers over non-reduced curves. In analogy with the Hitchin system, such a fiber $N_C \coloneqq f^{-1}(2C)$ is called the *nilpotent cone* (associated to the curve $C \in |H|$). In contrast to the general fibers, N_C is reducible and its components can have higher multiplicities. Our main result is

Theorem (Thm 0.1)**.** *Let $C \in |H|$ be an irreducible curve. The nilpotent cone $N_C \coloneqq f^{-1}(2C)$ has two irreducible components*

$$(N_C)_{\mathrm{red}} = N_0 \cup N_1,$$

where $N_0 \cong M_C(2, 1)$ is isomorphic to the moduli space of stable vector bundles of rank two and degree one on C. The degrees of the two components are given by

$$\deg N_0 = 5 \cdot 2^9 \quad \textit{and} \quad \deg N_1 = 5^2 \cdot 2^{11}$$

and their multiplicities are

$$\mathrm{mult}_{N_C} N_0 = 2^3 \quad \textit{and} \quad \mathrm{mult}_{N_C} N_1 = 2.$$

Moreover, any fiber F of the Mukai system has degree $5 \cdot 3 \cdot 2^{13}$.

We also provide a description of the corresponding cohomology classes. The first step in the proof of the above theorem is to give a description of the component N_1, which we will also use in Part II.

In Part II, we study the Chow group of zero cycles in M. More precisely, the Mukai system is our playground for the hunt of constant cycle subvarieties. We combine results of Voisin [55] and Marian, Shen, Yin and Zhao [41, 50] in order to decide, whether a point $\mathcal{E} \in M$ is contained in a constant cycle subvariety of given dimension. This allows us to produce several examples of algebraically coisotropic and constant cycle subvarieties in M. Specifically, we find a series coming from singular curves and two examples from Brill–Noether theory.

Proposition (cf. Prop 7.3)**.** *Define $V_i := \{D \in |2H| \mid g(\tilde{D}) \leq i\} \subset |2H|$ for $i = 0, \ldots, 4$ and let $Z_i \subset V_i$ be an irreducible compoenent. Then*

$$M_{Z_i} := f^{-1}(Z_i) \subset M$$

is an algebraically coisotropic subvariety of codimension $5 - i$ and there is a $2i$-dimensional scheme T_i with a rational map

$$M_{Z_i} \dashrightarrow T_i$$

whose fibers are $5 - i$-dimensional constant cycle subvarieties in M.

Proposition (Prop 7.1)**.** *Let $B^\circ \subset B$ be the open subset of smooth curves and set $M^\circ := f^{-1}(B^\circ)$. We define*

$$Z_1 := \overline{\{\mathcal{E} \in M^\circ \mid h^0(\mathcal{E}) \geq 1\}} \subset M \quad \text{and} \quad Z_3 := \overline{\{\mathcal{E} \in M^\circ \mid h^0(\mathcal{E}) \geq 2\}} \subset M.$$

Then $Z_i \subset M, i = 1, 3$ is algebraically coisotropic of codimension $5 - i$ and there is a rational map

$$Z_i \dashrightarrow S^{[i]}$$

whose fibers are $5 - i$-dimensional, rational constant cycle subvarieties in M.

In fact, the subvarieties Z_1 and Z_3 are generically projective bundles and play an important role in Part III.

In Part III, we approach the Mukai system as a birational model of the Hilbert scheme $S^{[5]}$. Precisely, we will assume that $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$ and study the geometry of the essentially unique birational map $S^{[5]} \dashrightarrow M$, which leads to an analysis of the Brill–Noether loci in M. Moreover, we apply the techniques of Bayer and Macrì [6], to determine all birational models of M. Our main result is

Theorem (Thm 7.18)**.** *Let (S,H) be a polarized K3 surface with $\operatorname{Pic}(S)=\mathbb{Z}\cdot H$ and $H^2=2$. There are five smooth birational models of $S^{[5]}$ or $M\coloneqq M_H(0,2H,-1)$, respectively. They are connected by a chain of flopping contractions*

$$\begin{tikzcd}
& \mathrm{Bl}_{W_2} S^{[5]} \arrow[dl] \arrow[dr] & & \mathrm{Bl}_{\tilde{W}_3} X_1 \arrow[dl] \arrow[dr] & & \mathrm{Bl}_{\tilde{Z}_3} X_3 \arrow[dl] \arrow[dr] & & \mathrm{Bl}_{Z_1} M \arrow[dl] \arrow[dr] & \\
S^{[5]} \arrow[rr, dashed, "g_1"] & & X_1 \arrow[rr, dashed, "g_2"] & & X_2 & & X_3 \arrow[ll, dashed, "g_3"'] & & M \arrow[ll, dashed, "g_4"']
\end{tikzcd}$$

for some subvarieties $W_2\subset W_3\subset S^{[5]}$ such that

- *W_2 is a $\mathbb{P}^3$-bundle over $M_H(0,H,-6)$,*
- *$W_3\setminus W_2$ is a $\mathbb{P}^2$-bundle over an open subset of $M_H(0,H,-5)\times S$*

and subvarieties $Z_1\subset Z_3\subset M$ such that

- *Z_1 is a $\mathbb{P}^4$-bundle over S,*
- *$Z_3\setminus Z_1$ is a $\mathbb{P}^2$-bundle over an open subset of $S^{[3]}$.*

Here, $\tilde{W}_3$ (resp. $\tilde{Z}_3$) is the strict transform of W_3 (resp. Z_3) under g_1 (resp. g_4).

The three parts are rather self-contained and we refer to the respective introductions for a more precise statement of the results.

Thanks.
I'm extremely grateful to Daniel Huybrechts for all his advice and support over the years. Particular thanks go to Thorsten Beckmann for answering so many questions. It's a pleasure to acknowledge helpful discussions with Alberto Cattaneo, Hsueh-Yung Lin, Mirko Mauri, Denis Nesterov, Georg Oberdieck, Giulia Saccà, Johannes Schmitt and Andrey Soldatenkov. I thank them all.

Contents

I. The nilpotent cone in the Mukai system of rank two and genus two

Abstract. We study the nilpotent cone in the Mukai system of rank two and genus two. We compute the degrees and multiplicities of its irreducible components and describe their cohomology classes.

Introduction

Let (S, H) be a polarized K3 surface of genus g and fix two coprime integers $n \geq 1$ and s. The moduli space $M = M_H(v)$ of H-Gieseker stable coherent sheaves with Mukai vector $v = (0, nH, s)$ is a smooth Hyperkähler variety of dimension $2(n^2(g-1)+1)$. A point in M corresponds to a stable sheaf $\mathcal{E}$ on S such that $\mathcal{E}$ is pure of dimension one with support in the linear system $|nH|$. Taking the (Fitting) support defines a Lagrangian fibration

$$f\colon M \longrightarrow |nH| \cong \mathbb{P}^{n^2(g-1)+1},\ [\mathcal{E}] \mapsto \mathrm{Supp}(\mathcal{E})$$

known as the *Mukai system* [8], [44]. Over a general point in $|nH|$ which corresponds to a smooth curve $D \subset S$ the fibers of f are abelian varieties isomorphic to $\mathrm{Pic}^{\delta}(D)$, where $\delta = s - n^2(1-g)$. So, M can also be viewed as a compactified relative Jacobian associated to the universal curve $\mathcal{C} \to |nH|$.

The Mukai system is of special interest because of its relation to the classical and widely studied *Hitchin system*, see [28] for a survey. Let C be a smooth curve of genus g. A Higgs bundle on C is a pair $(\mathcal{E}, \phi)$ consisting of a vector bundle $\mathcal{E}$ on C and a morphism $\phi\colon \mathcal{E} \to \mathcal{E} \otimes \omega_C$, called Higgs field. The moduli space $M_{\mathrm{Higgs}}(n,d)$ of stable Higgs bundles of rank n and degree d is a smooth and quasi-projective symplectic variety. Sending $(\mathcal{E}, \phi)$ to the coefficients of its characteristic polynomial $\chi(\phi)$ defines a proper Lagrangian fibration

$$\chi\colon M_{\mathrm{Higgs}}(n,d) \longrightarrow \bigoplus_{i=1}^{n} H^0(C, \omega_C^i).$$

It is equivariant with respect to the $\mathbb{C}^*$-action that is given by scaling the Higgs field on $M_{\mathrm{Higgs}}(n,d)$ and by multiplication with t^i in the corresponding summand on the base. As a corollary the topology of $M_{\mathrm{Higgs}}(n,d)$ is controlled by the fiber over the origin. This fiber

$$N \coloneqq \chi^{-1}(0) = \{(\mathcal{E}, \phi) \in M_{\mathrm{Higgs}}(n,d) \mid \phi \text{ is nilpotent}\}$$

is called the *nilpotent cone*. In the late '80s Beauville, Narasimhan, and Ramanan discovered a beautiful interpretation of the space of Higgs bundles [7]. They showed that a Higgs bundle

$(\mathcal{E},\phi)$ with characteristic polynomial s corresponds to a pure sheaf of rank one on a so called spectral curve $C_s \subset |\omega_C|$ inside the total space of the canonical bundle. The curve C_s is defined in terms of $s = \chi(\phi)$ and is linearly equivalent to nC, the n-th order thickening of the zero section $C \subset |\omega_C|$. This idea was taken up by Donagi, Ein, and Lazarsfeld in [20]: The space $M_{\mathrm{Higgs}}(n,d)$ appears as a moduli space of stable sheaves on $|\omega_C|$ that are supported on curves in the linear system $|nC|$. Consequently, $M_{\mathrm{Higgs}}(n,d)$ has a natural compactification $\overline{M}_{\mathrm{Higgs}}(n,d)$ given by a moduli space of stable sheaves on the projective surface $S_0 = \mathbb{P}(\omega_C \oplus \mathcal{O}_C)$ with respect to the polarization $H_0 = \mathcal{O}_{S_0}(C)$. The Hitchin map extends to

$$\overline{M}_{\mathrm{Higgs}}(n,d) \to |nH_0| \cong \mathbb{P}(\oplus_{i=0}^{n} H^0(C,\omega_C^i))$$

and is nothing but the support map; the nilpotent cone is the fiber over the point $nC \in |nH_0|$. However, $\overline{M}_{\mathrm{Higgs}}(n,d)$ cannot admit a symplectic structure as it is covered by rational curves. At this point the Mukai system enters the picture. If S is a K3 surface that contains the curve C as a hyperplane section, one can degenerate (S,H) to (S_0,H_0) and consequently the Mukai system $M_H(v) \to |nH|$ with $v = (0, nH, d + n(1-g))$ degenerates to the compactified Hitchin system [20, §1]. From our perspective, this is a powerful approach to studying the Hitchin system. For instance, in a recent paper [14], de Cataldo, Maulik and Shen prove the P=W conjecture for $g = 2$ by means of the corresponding specialization map on cohomology.

In this note, we study the geometry of the nilpotent cone in the Mukai system, which is defined in parallel to the Hitchin system

$$N_C \coloneqq f^{-1}(nC),$$

for some curve $C \in |H|$. Alternatively, one could say that we study the most singular fiber type, see (2.2). We will fix the invariants $n = 2$ and $g = 2$ and the Mukai vector $v = (0, 2H, -1)$. In this case and if C is irreducible, the nilpotent cone has two irreducible components

$$(N_C)_{\mathrm{red}} = N_0 \cup N_1,$$

where the first component is isomorphic to the moduli space $M_C(2,1)$ of stable vector bundles of rank two and degree one on C and the second component is the closure of $N_C \setminus N_0$. Both components are Lagrangian subvarieties of $M = M_H(v)$. If C is smooth, then N_0 is smooth and the singularities of N_1 are contained in $N_0 \cap N_1$ (each understood with their reduced structure). However, both components occur with multiplicities.

Our first result is the computation of the multiplicities of the components as well as their degrees. Here, the degree is meant with respect to a naturally defined distinguished ample class $u_1 \in H^2(M,\mathbb{Z})$, see Definition 3.7.

Theorem 0.1. *Let $C \in |H|$ be an irreducible curve. The degrees of the two components of the nilpotent cone N_C are given by*

$$\deg_{u_1} N_0 = 5 \cdot 2^9 \quad \text{and} \quad \deg_{u_1} N_1 = 5^2 \cdot 2^{11}$$

and their multiplicities are

$$\mathrm{mult}_{N_C} N_0 = 2^3 \quad \text{and} \quad \mathrm{mult}_{N_C} N_1 = 2.$$

Moreover, any fiber F of the Mukai system has degree $5 \cdot 3 \cdot 2^{13}$.

As the smooth locus of every component with its reduced structure deforms from the Mukai to the Hitchin system, the multiplicities and degrees must coincide. Here, indeed, the same multiplicities can be found in [51, Propositions 34 and 35] and [29, Proposition 6], whereas, up to our knowledge, the degrees have not been determined in the literature. In our case, the degrees determine the multiplicities.

Our second result is a description of the cohomology classes $[N_0]$ and $[N_1] \in H^{10}(M, \mathbb{Z})$. The projective moduli spaces of stable sheaves on K3 surfaces are known to be deformation equivalent to Hilbert schemes of points. In our case, M is actually birational to $S^{[5]}$ (cf. Section 8). In particular, there is an isomorphism $H^*(M, \mathbb{Z}) \cong H^*(S^{[5]}, \mathbb{Z})$. The cohomology ring of $S^{[5]}$ is well understood, e.g. [37, §4] and the references therein. Recall that for any Hyperkähler variety X of dimension $2n$ there is an embedding $S^i H^2(X, \mathbb{Q}) \hookrightarrow H^{2i}(X, \mathbb{Q})$ for all $i \leq n$ [52, Theorem 1.7].

Theorem 0.2. *The classes $[N_0]$ and $[N_1] \in H^{10}(M, \mathbb{Q})$ are linearly independent and span a totally isotropic subspace of $H^{10}(M, \mathbb{Q})$ with respect to the intersection pairing. They are given by*

$$[N_0] = \frac{1}{48}[F] + \beta \text{ and } [N_1] = \frac{5}{12}[F] - 4\beta,$$

where $[F]$ is the class of a general fiber of the Mukai system and $0 \neq \beta \in (S^5 H^2(M, \mathbb{Q}))^\perp$ satisfies $\beta^2 = 0$. As $\deg_{u_1} \beta = 0$, the class β is not effective.

Outline. In Section 1 we introduce the Mukai system. In Section 2 we reduce the study of N_C to the case of a smooth curve C. We describe the irreducible components of the nilpotent cone following [20, §3], where it is shown that any point $[\mathcal{E}] \in N_C \setminus N_0$ fits into an extension of the form

$$0 \to \mathcal{L}(x) \otimes \omega_C^{-1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \to 0,$$

where $\mathcal{L} \in \mathrm{Pic}^1(C)$ is a line bundle and $x \in C$ a point. We specify a space $W \to \mathrm{Pic}^1(C) \times C$ parameterizing such extensions, and a compactification $\overline{W}$ of W that comes with a birational map $\nu \colon \overline{W} \to N_1$. In the Hitchin case, this idea originates from [51].

In Section 3 we prove Theorem 0.1. The proof relies on the functorial properties of the definition of u_1 via the determinant line bundle construction, see Section 3.1. It allows us to relate $u_1|_F$

and $u_1|_{N_0}$ with the (generalized) theta divisor on $F = f^{-1}(D) \cong \mathrm{Pic}^3(D)$ for $D \in |nC|$ smooth and $M_C(2,1)$, respectively, see Propositions 3.8 and 3.10. For $[N_1]$ the degree computation is achieved by determining $\nu^* u_1 \in H^2(\overline{W}, \mathbb{Z})$. Finally, the multiplicities are inferred from knowing the degrees. The last Section 4 is devoted to the proof of Theorem 0.2. It uses our previous results.

Notation. All schemes are of finite type over $k = \mathbb{C}$. In the entire paper, S is a K3 surface polarized by an ample class $H \in \mathrm{NS}(S)$ with $H^2 = 2g-2$.

1. Basics

In this section, we give a brief recollection on moduli spaces of sheaves on K3 surfaces and define the Mukai system. Recall that the Mukai vector induces an isomorphism

$$v\colon \mathrm{K}(S)_{\mathrm{num}} \xrightarrow{\sim} H^*_{\mathrm{alg}}(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus \mathrm{NS}(S) \oplus H^4(S,\mathbb{Z}).$$

It is given by

$$v(\mathcal{E}) \coloneqq \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{td}(S)} = (\mathrm{rk}(\mathcal{E}), c_1(\mathcal{E}), \chi(\mathcal{E}) - \mathrm{rk}(\mathcal{E})).$$

We write $M_H(v)$ for the moduli space of pure, H-Gieseker stable sheaves on S with Mukai vector v. If v is primitive and positive and H is v-generic then $M_H(v)$ is an irreducible holomorphic symplectic manifold of dimension $\langle v, v\rangle + 2$, which is deformation equivalent to the Hilbert scheme of $\frac{1}{2}\langle v, v\rangle + 1$ points on S [34, Theorem 10.3.1]. Here, $\langle\ ,\ \rangle$ is the Mukai pairing given by

$$\langle (r,c,s),(r',c',s')\rangle = cc' - rs' - r's.$$

Consider the Mukai vector

$$v \coloneqq (0, nH, s) \in H^*_{\mathrm{alg}}(S,\mathbb{Z}),$$

and assume that v is primitive. A pure sheaf $\mathcal{F}$ of Mukai vector v has one-dimensional support, first Chern class nH and Euler characteristic s. In particular, $\mathcal{F}$ admits a length one resolution $0 \to \mathcal{V} \xrightarrow{f} \tilde{\mathcal{V}} \longrightarrow \mathcal{F}$ by two vector bundles of the same rank r [31, §1.1]. We define the *(Fitting) support* of $\mathcal{F}$ to be

$$\mathrm{Supp}(\mathcal{F}) \coloneqq V(\det f) \subset S$$

the vanishing scheme of the induced morphism $\det f = \wedge^r f\colon \wedge^r \mathcal{V} \to \wedge^r \tilde{\mathcal{V}}$, for any resolution $0 \to \mathcal{V} \xrightarrow{f} \tilde{\mathcal{V}}$ of $\mathcal{F}$ as above. This definition is well-defined, i.e. independent of the chosen resolution [23, Definition 20.4].

Example 1.1. Let $i\colon C \hookrightarrow S$ be an integral curve and $\mathcal{E}$ a vector bundle of rank n on C. Then

$$\mathrm{Supp}(i_*\mathcal{E}) = nC$$

is the n-th order thickening of C in S.

By definition, $\mathrm{Supp}(\mathcal{F})$ is linearly equivalent to $c_1(\mathcal{F})$ and $\mathrm{Supp}(\mathcal{F})$ contains the usual support defined by the annihilator of $\mathcal{F}$. Moreover, the reduced locus $\mathrm{Supp}(\mathcal{F})_{\mathrm{red}}$ is the set-theoretic support of $\mathcal{F}$. The advantage of the above definition is, that it behaves well in families and thus induces a morphism [38, §2.2]

$$f\colon M_H(v) \longrightarrow |nH| \cong \mathbb{P}^{\tilde{g}}, \quad [\mathcal{E}] \mapsto \mathrm{Supp}(\mathcal{E}).$$

Here, $\tilde{g} = n^2(g-1)+1$. Moreover, $M_H(v)$ is irreducible holomorphic symplectic of dimension $n^2H^2+2 = 2\tilde{g}$ and hence, by Matsushita's result [42, Corollary 1] this morphism is a Lagrangian fibration (for an explicit proof see [20, Lemma 1.3]), called the *Mukai system (of rank n and genus g)*.

2. The nilpotent cone for $n = 2$ and $g = 2$

We now specialize to the case that $n = 2$ and $s = 3 - 2g$ with $g = 2$, i.e. we fix the Mukai vector

$$v = (0, 2H, -1).$$

In particular, a stable vector bundle of rank two and degree one on a smooth curve $C \in |H|$ defines a point in $M \coloneqq M_H(v)$. We have $\dim M = 8g - 6 = 10$ and M is birational to the Hilbert scheme $S^{[5]}$ of five points on S.

Taking (Fitting) supports defines a Lagrangian fibration

$$f\colon M \longrightarrow |2H| \cong \mathbb{P}^5.$$

We use the Segre map $m\colon |H| \times |H| \to |2H|$ to define the subloci

$$\Delta \coloneqq m(\Delta_{|H|}) \subset \Sigma \coloneqq \mathrm{im}(m) \subset |2H|. \tag{2.1}$$

Then $\Sigma \cong \mathrm{Sym}^2 |H|$ is four-dimensional and its generic member is reduced and has two smooth irreducible components in the linear system $|H|$ meeting transversally in two points. The subset $\Delta \cong |H| \cong \mathbb{P}^2$ is the locus of non-reduced curves. If $\rho(S) = 1$, then Σ is exactly the locus of non-integral curves. In this case, we can distinguish three fiber types following [13, Proposition 3.7.1]:

$$f^{-1}(x) \begin{cases} \text{is reduced and irreducible} & \text{if } x \in |2H| \setminus \Sigma \\ \text{is reduced and has two irreducible components} & \text{if } x \in \Sigma \setminus \Delta \\ \text{has two irreducible components with multiplicities} & \text{if } x \in \Delta. \end{cases} \tag{2.2}$$

If $\rho(S) \geq 1$, the list is still valid for the geometric generic point in the respective subvariety. However, over points that correspond to curves with more irreducible components, one also finds more irreducible components in the fiber [13, Proof of Lemma 3.3.2] (see also Section 5.2).

We will study fibers of the third type, namely

$$N_C \coloneqq f^{-1}(2C),$$

where $C \in |H|$ is irreducible. In analogy with the Hitchin system, we call N_C the *nilpotent cone*.

For the rest of this part, we fix a smooth curve $C \in |H|$ and write N instead of N_C. We will now identify the irreducible components of N following the ideas of [20].

2.1. Pointwise description of the nilpotent cone $N = N_C$. Let $[\mathcal{E}] \in N$ and consider its restriction $\mathcal{E}|_C$ to C. There are two cases, either $\mathcal{E}|_C$ is a stable rank two vector bundle on C or $\mathcal{E}|_C$ has rank one. By dimension reasons, the sheaves of the first kind contribute an irreducible component N_0 of N isomorphic to the moduli space $\mathrm{M}_C(2,1)$ of stable rank two and degree one vector bundles on C. In the second case, $\mathcal{E}|_C \cong \mathcal{L} \oplus \mathcal{O}_D$, where the first factor $\mathcal{L} \coloneqq \mathcal{E}|_C$ /torsion is a line bundle on C and $D \subset C$ is an effective divisor. We set

$$E_1 \coloneqq N \setminus N_0$$

with reduced structure.

Lemma 2.1. *Let $[\mathcal{E}] \in E_1$ and write $\mathcal{E}|_C = \mathcal{L} \oplus \mathcal{O}_D$. There is a short exact sequence of $\mathcal{O}_S$-modules*

$$0 \to i_*(\mathcal{L}(D) \otimes \omega_C^{-1}) \longrightarrow \mathcal{E} \longrightarrow i_*\mathcal{L} \to 0. \tag{2.3}$$

Moreover, $k \coloneqq \deg \mathcal{L} = 1$ and $d \coloneqq \deg D = 2g - 2k - 1 = 1$.

Proof. Noting that ω_C^{-1} is the conormal bundle of C in S, it is straightforward to obtain the sequence (2.3). Let us prove the numerical restrictions. From (2.3) we have

$$1 + 2(1-g) = \chi(\mathcal{E}) = \chi(\mathcal{L}(D) \otimes \omega_C^{-1}) + \chi(\mathcal{L}) = 2k + d - (2g-2) + 2(1-g).$$

Thus $d = 2g - 2k - 1$ and we find $k \leq g - 1$. On the other hand, $\mathcal{E}$ is stable, so the reduced Hilbert polynomials [31, Definition 1.2.3] of $\mathcal{E}$ and $\mathcal{L}$ satisfy $p(\mathcal{E}, t) < p(\mathcal{L}, t)$, which amounts to

$$\tfrac{1}{2}(1 + 2(1-g)) < k + 1 - g$$

or equivalently $k \geq 1$. $\square$

Remark 2.2. For $n = 2$ and arbitrary genus g, one has $\deg \mathcal{L} \in \{1, \ldots, g-1\}$ and a decomposition into locally closed subsets $N_{\mathrm{red}} = N_0 \sqcup E_1 \sqcup \ldots \sqcup E_{g-1}$ corresponding to the degree of $\mathcal{L}$. In fact, N_0 and the closures of E_k are the irreducible components of N.

We conclude that every point in E_1 defines a class in $\mathrm{Ext}^1_S(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1}))$ for some point $x \in C$ and some line bundle $\mathcal{L} \in \mathrm{Pic}^1(C)$. Conversely, an extension class in $\mathrm{Ext}^1_S(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1}))$ defines a point in E_1 if and only if its middle term is stable and has the point x as support of its torsion part when restricted to C, i.e. if it is not pushed forward from C. It turns out that all such extensions are stable.

Lemma 2.3. *Consider a coherent sheaf $\mathcal{E}$ on S that is given as an extension*

$$0 \to i_*\mathcal{L} \to \mathcal{E} \to i_*\mathcal{L}' \to 0,$$

where $\mathcal{L}'$ and $\mathcal{L}$ are line bundles on C of degree k and $1-k$, respectively, with $k \geq 1$. Moreover, assume that $\mathcal{E}$ itself does not admit the structure of an $\mathcal{O}_C$-module. Then $\mathcal{E}$ is H-Gieseker stable.

Proof. We have to prove $p(\mathcal{E}, t) < p(\mathcal{M}, t)$ or, equivalently, $\frac{\chi(\mathcal{E})}{c_1(\mathcal{E}).H} < \frac{\chi(\mathcal{M})}{c_1(\mathcal{M}).H}$ for every surjection $\mathcal{E} \twoheadrightarrow \mathcal{M}$. We can assume that $\mathrm{Supp}(\mathcal{M}) = C$ and $\mathcal{M} = i_*\mathcal{M}'$, where $\mathcal{M}'$ is a line bundle on C. Then because $\mathcal{E}|_C \cong \mathcal{L}' \oplus \mathcal{T}$ for some torsion sheaf $\mathcal{T}$, we find

$$\mathcal{H}om_{\mathcal{O}_S}(\mathcal{E}, i_*\mathcal{M}') \cong \mathcal{H}om_{\mathcal{O}_C}(\mathcal{E}|_C\ , \mathcal{M}') \cong \mathcal{H}om_{\mathcal{O}_C}(\mathcal{L}', \mathcal{M}')$$

and thus $i_*\mathcal{L}' \xrightarrow{\sim} \mathcal{M}$. ☐

Corollary 2.4. *The closed points of E_1 are in bijection with the following set*

$$\bigsqcup_{\substack{\mathcal{L}\in \mathrm{Pic}^1(C)\\ x\in C}} \mathbb{P}(\mathrm{Ext}^1_S(i_*\mathcal{L}, i_*(\mathcal{L}(x)\otimes \omega_C^{-1}))) \setminus \mathbb{P}(\mathrm{Ext}^1_C(\mathcal{L}, \mathcal{L}(x)\otimes\omega_C^{-1})),$$

i.e. with extension classes $[v] \in \mathbb{P}(\mathrm{Ext}^1_S(i_*\mathcal{L}, i_*(\mathcal{L}(x)\otimes\omega_C^{-1})))$ such that v is not pushed forward from C. Here, $\mathcal{L}$ varies over all line bundles on C with $\deg \mathcal{L} = 1$, and x varies over all points in C. The bijection is established by Lemma 2.1.

In Proposition 2.5 below, we will see that there is a short exact sequence

$$0 \to \mathrm{Ext}^1_C(\mathcal{L}, \mathcal{L}(x)\otimes\omega_C^{-1}) \to \mathrm{Ext}^1_S(i_*\mathcal{L}, i_*(\mathcal{L}(x)\otimes\omega_C^{-1})) \xrightarrow{\rho_{\mathcal{L},x}} H^0(C, \mathcal{O}_C(x)) \to 0,$$

where $\rho_{\mathcal{L},x}$ has the following interpretation modulo a scalar factor. If $\mathcal{E}$ is the middle term of a representing sequence of $v \in \mathrm{Ext}^1_S(i_*\mathcal{L}, i_*(\mathcal{L}(x)\otimes\omega_C^{-1}))$, then

$$\mathcal{E}|_C \cong \mathcal{L} \oplus \mathcal{O}_{V(\rho_{\mathcal{L},x}(v))}.$$

Hence, another way to phrase Corollary 2.4 is by fixing for every $x \in C$ a defining section $s_x \in H^0(C, \mathcal{O}_C(x))$ as follows. Let $\Delta \hookrightarrow C\times C$ be the diagonal, yielding a section $s_\Delta \in H^0(C\times C, \mathcal{O}(\Delta))$. For every $x \in C$, we set $s_x = s_\Delta\big|_{\{x\}\times C}$. Then we can write

$$\text{points of } E_1 \quad \overset{1:1}{\longleftrightarrow} \quad \bigsqcup_{\substack{\mathcal{L}\in \mathrm{Pic}^1(C)\\ x\in C}} \{v \in \mathrm{Ext}^1_S(i_*\mathcal{L}, i_*(\mathcal{L}(x)\otimes\omega_C^{-1})) \mid \rho_{\mathcal{L},x}(v) = s_x\}. \tag{2.4}$$

2.2. Extension spaces. So far, we have given a pointwise description of the nilpotent cone. Next, we will identify its irreducible components and their scheme structures. This subsection is a technical parenthesis in this direction. The reader may like to skip it.

Let S be a smooth projective surface and $i\colon C \hookrightarrow S$ a smooth curve with normal bundle $\mathcal{N}_{C/S} \cong \mathcal{O}_C(C)$. Let T be any scheme and let $\mathcal{F}$ and $\mathcal{F}'$ be two vector bundles on $T \times C$ considered as families of vector bundles on C. Denote by $\pi\colon T \times S \to T$ and $\pi_C\colon T \times C \to T$ the projections. For a morphism $f\colon X \to Y$, we write $\mathcal{E}xt_f$ instead of $Rf_* R\mathcal{H}om$.

Proposition 2.5. *There is a short exact sequence of $\mathcal{O}_T$-modules*

$$0 \to \mathcal{E}xt^1_{\pi_C}(\mathcal{F}', \mathcal{F}) \to \mathcal{E}xt^1_{\pi}((\mathrm{id} \times i)_*\mathcal{F}', (\mathrm{id} \times i)_*\mathcal{F}) \xrightarrow{\rho} \mathcal{E}xt^0_{\pi_C}(\mathcal{F}' \boxtimes \mathcal{O}_C(-C), \mathcal{F}) \to 0, \tag{2.5}$$

as well as for every $t \in T$ a short exact sequence of vector spaces

$$0 \to \mathrm{Ext}^1_C(\mathcal{F}'_t, \mathcal{F}_t) \xrightarrow{\xi} \mathrm{Ext}^1_S(i_*\mathcal{F}'_t, i_*\mathcal{F}_t) \xrightarrow{\rho_t} \mathrm{Ext}^0_C(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t) \to 0. \tag{2.6}$$

Note that the fibers of (2.5) must, in general, not coincide with (2.6), see Lemma 2.6.

Proof. Apply $R\pi_{C*}R\mathcal{H}om(\ \ , \mathcal{F})$ or $R\,\mathrm{Hom}(\ \ , \mathcal{F}_t)$, to the exact triangle

$$\mathcal{F}' \boxtimes \mathcal{O}_C(-C)[1] \to L(\mathrm{id} \times i)^*(\mathrm{id} \times i)_*\mathcal{F}' \to \mathcal{F}' \xrightarrow{[1]}$$

in $D^b(T \times C)$ (see [32, Corollary 11.4]) or its counterpart in $D^b(C)$, respectively, and consider the induced cohomology sequence. □

We can explicitly describe the morphism ρ_t in the sequence (2.6). Represent $v \in \mathrm{Ext}^1_S(i_*\mathcal{F}'_t, i_*\mathcal{F}_t)$ by $0 \to i_*\mathcal{F}_t \to \mathcal{E} \to i_*\mathcal{F}'_t \to 0$. Restriction to C yields

$$\ldots \to \mathcal{F}'_t \otimes \mathcal{O}_C(-C) \xrightarrow{\delta(v)} \mathcal{F}_t \to \mathcal{E}|_C \to \mathcal{F}'_t \to 0,$$

where we inserted $\mathcal{T}or_1^{\mathcal{O}_S}(i_*\mathcal{F}'_t, i_*\mathcal{O}_C) \cong \mathcal{F}'_t \otimes_{\mathcal{O}_C} \mathcal{T}or_1^{\mathcal{O}_S}(i_*\mathcal{O}_C, i_*\mathcal{O}_C) = \mathcal{F}'_t \otimes \mathcal{O}_C(-C)$. This gives a well-defined, linear map

$$\delta\colon\ \mathrm{Ext}^1_S(i_*\mathcal{F}'_t, i_*\mathcal{F}_t) \to \mathrm{Ext}^0_C(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t).$$

As $\mathrm{im}\,\xi = \ker\delta$, it follows by dimension reasons, that δ has to be surjective. So, $\rho_t = \delta$ up to post-composition with an isomorphism of $\mathrm{Ext}^0_C(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t)$.

Lemma 2.6. *For every $t \in T$ there is a commutative diagram of short exact sequences*

$$\begin{array}{ccccc}
\mathcal{E}xt^1_{\pi_C}(\mathcal{F}', \mathcal{F})(t) & \longrightarrow & \mathcal{E}xt^1_{\pi}((\mathrm{id} \times i)_*\mathcal{F}', (\mathrm{id} \times i)_*\mathcal{F})(t) & \xrightarrow{\rho(t)} & \mathcal{E}xt^0_{\pi_C}(\mathcal{F}' \boxtimes \mathcal{O}_C(-C), \mathcal{F})(t) \\
\downarrow \cong & & \downarrow & & \downarrow \\
\mathrm{Ext}^1_C(\mathcal{F}'_t, \mathcal{F}_t) & \longrightarrow & \mathrm{Ext}^1_S(i_*\mathcal{F}'_t, i_*\mathcal{F}_t) & \xrightarrow{\rho_t} & \mathrm{Ext}^0_C(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t),
\end{array}$$

where the first vertical arrow is an isomorphism. If $\mathrm{Ext}^0_C(\mathcal{F}'_t \otimes \mathcal{O}_C(-C), \mathcal{F}_t)$ has constant dimension for all $t \in T$ all vertical arrows are isomorphisms.

Proof. The vertical morphisms are the usual functorial base change morphisms. The lower line is (2.6) and hence also exact on the left. The first vertical arrow is an isomorphism because $\mathrm{Ext}^2_C(\mathcal{F}'_t, \mathcal{F}_t) = 0$. Consequently, also the upper line is exact on the left. $\square$

2.3. Irreducible components of N. In this section, we show that E_1 is irreducible and has the same dimension as N. Therefore its closure

$$N_1 \coloneqq \overline{E}_1 \subset N_{\mathrm{red}},$$

with reduced structure is an irreducible component of N. For the proof, we need some more notation. Let $\mathcal{P}_1$ be a Poincaré line bundle on $\mathrm{Pic}^1(C) \times C$ and $\Delta \subset C \times C$ the diagonal. Set $T \coloneqq \mathrm{Pic}^1(C) \times C$ and on T define the following sheaves

$$\begin{aligned}
\mathcal{V} &\coloneqq R^1 p_{12*}(p_{23}^* \mathcal{O}(\Delta) \otimes p_3^* \omega_C^{-1}),\\
\mathcal{W} &\coloneqq R^1 p_{12*} R\mathcal{H}om((\mathrm{id} \times i)_* p_{13}^* \mathcal{P}_1, (\mathrm{id} \times i)_*(p_{13}^* \mathcal{P}_1 \otimes p_{23}^* \mathcal{O}(\Delta) \otimes p_3^* \omega_C^{-1}) \text{ and}\\
\mathcal{U} &\coloneqq p_{12*} p_{23}^* \mathcal{O}(\Delta),
\end{aligned}$$

where p_{ij} are the appropriate projections from $\mathrm{Pic}^1(C) \times C \times C$. Considering the fiber dimensions, we see that $\mathcal{V}$ and $\mathcal{U}$ are vector bundles of rank 2 and 1, respectively. In fact, s_Δ induces an isomorphism $s_\Delta \colon \mathcal{O}_T \xrightarrow{\sim} \mathcal{U} = p_{12*} p_{23}^* \mathcal{O}(\Delta)$. Moreover, by Proposition 2.5 they fit into a short exact sequence

$$0 \to \mathcal{V} \longrightarrow \mathcal{W} \xrightarrow{\rho} \mathcal{O}_T \to 0. \tag{2.7}$$

Consequently, also $\mathcal{W}$ is a vector bundle and ρ induces a map of geometric vector bundles

$$|\rho| \colon |\mathcal{W}| = \underline{\mathrm{Spec}}_T(\mathrm{Sym}^\bullet \mathcal{W}^\vee) \longrightarrow T \times \mathbb{A}^1.$$

We set

$$W \coloneqq |\rho|^{-1}(T \times \{1\})$$

with the projection $\tau \colon W \to T$. We retain some immediate consequences of the construction.

(i) W is a principal homogeneous space under $|\mathcal{V}|$. In particular, it is an affine bundle over T.

(ii) Let $t = (\mathcal{L}, x) \in T$. Then by Lemma 2.6 we have

$$W_t = \tau^{-1}(t) \cong \mathbb{P}(\mathrm{Ext}^1_S(i_* \mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1}))) \setminus \mathbb{P}(\mathrm{Ext}^1_C(\mathcal{L}, \mathcal{L}(x) \otimes \omega_C^{-1})).$$

(iii) $\dim W = 5$.

(iv) W is compactified by the projective bundle $\overline{W} \coloneqq \mathbb{P}(\mathcal{W})$ with boundary isomorphic to $\mathbb{P}(\mathcal{V})$, i.e.

$$\overline{W} = W \cup \mathbb{P}(\mathcal{V}).$$

Remark 2.7. Actually, $\mathcal{V} \cong p_2^*(\omega_C \oplus \omega_C)$ and hence $\mathbb{P}(\mathcal{V}) \cong \mathbb{P}^1 \times \mathrm{Pic}^1(C) \times C$.

Next, we relate E_1 and $\overline{W}$. Recall that $N_1 \coloneqq \overline{E}_1 \subset N_{\mathrm{red}}$. We keep all the notations from the previous section, and

$$\begin{tikzcd}
\overline{W} \times C \arrow[r, hook] \arrow[d, "\tau_C"] & \overline{W} \times S \arrow[r, "\pi'"] \arrow[d, "\tau_S"] & \overline{W} \coloneqq \mathbb{P}(\mathcal{W}) \arrow[d, "\tau"] \\
T \times C \arrow[r, hook] & T \times S \arrow[r, "\pi"] & T.
\end{tikzcd}$$

Proposition 2.8. *There exists a 'universal' extension represented by*

$$0 \to \tau_S^*(\operatorname{id} \times i)_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1}) \boxtimes \mathcal{O}_\tau(1) \to \mathcal{G}_{\mathrm{univ}} \to \tau_S^*(\operatorname{id} \times i)_* p_{13}^* \mathcal{P}_1 \to 0,$$

such that $\mathcal{G}_{\mathrm{univ}} \in \operatorname{Coh}(\overline{W} \times S)$ *defines a birational morphism*

$$\nu \colon \overline{W} \longrightarrow N_1.$$

In particular, $N_{\mathrm{red}} = N_0 \cup N_1$ *is a decomposition into irreducible components.*

Proof. We set $\mathcal{F} \coloneqq \mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1}$ and $\mathcal{F}' \coloneqq p_{13}^* \mathcal{P}_1$. We are looking for a 'universal' extension, i.e. for

$$v_{\mathrm{univ}} \in \operatorname{Ext}^1_{\overline{W} \times S}(\tau_S^*(\operatorname{id} \times i)_* \mathcal{F}', \tau_S^*(\operatorname{id} \times i)_* \mathcal{F} \otimes \pi'^* \mathcal{O}_\tau(1)),$$

such that for $w \in W \subset \overline{W}$ the restriction of v_{univ} to $\{w\} \times S$ is the extension corresponding to $w \in W_{\tau(w)} \subset \operatorname{Ext}^1_S(i_* \mathcal{F}'_{\tau(w)}, i_* \mathcal{F}_{\tau(w)})$.
By definition, $\mathcal{W} = R^1 \pi_* R\mathcal{H}om((\operatorname{id} \times i)_* \mathcal{F}', (\operatorname{id} \times i)_* \mathcal{F})$. Hence, there is a base change map

$$\tau^* \mathcal{W} \to R^1 \pi'_* L\tau_S^* R\mathcal{H}om((\operatorname{id} \times i)_* \mathcal{F}', (\operatorname{id} \times i)_* \mathcal{F}).$$

We get

$$\begin{aligned}
H^0(\overline{W}, \tau^* \mathcal{W} \otimes \mathcal{O}_\tau(1)) &\to H^0(\overline{W}, R^1 \pi'_* L\tau_S^* R\mathcal{H}om((\operatorname{id} \times i)_* \mathcal{F}', (\operatorname{id} \times i)_* \mathcal{F}) \otimes \mathcal{O}_\tau(1)) \\
&\xleftarrow{\sim} H^1(\overline{W} \times S, L\tau_S^* R\mathcal{H}om((\operatorname{id} \times i)_* \mathcal{F}', (\operatorname{id} \times i)_* \mathcal{F}) \otimes \pi'^* \mathcal{O}_\tau(1))) \qquad (2.8) \\
&= \operatorname{Ext}^1_{\overline{W} \times S}(\tau_S^*(\operatorname{id} \times i)_* \mathcal{F}', \tau_S^*(\operatorname{id} \times i)_* \mathcal{F} \otimes \pi'^* \mathcal{O}_\tau(1)),
\end{aligned}$$

where the indicated isomorphism comes from the Leray spectral sequence. It is an isomorphism, because

$$R^0 \pi'_* L\tau_S^* R\mathcal{H}om((\operatorname{id} \times i)_* \mathcal{F}', (\operatorname{id} \times i)_* \mathcal{F}) \cong R^0 \pi'_{C*} L\tau_C^* R\mathcal{H}om(L(\operatorname{id} \times i)^* (\operatorname{id} \times i)_* \mathcal{F}', \mathcal{F}) = 0,$$

where $\pi_C \colon T \times C \to T$. The last equality follows from the long exact sequence

$$\begin{aligned}
\ldots \to 0 = R^0 \pi'_{C*} L\tau_C^* R\mathcal{H}om(\mathcal{F}', \mathcal{F}) &\to R^0 \pi'_{C*} L\tau_C^* R\mathcal{H}om(L(\operatorname{id} \times i)^* (\operatorname{id} \times i)_* \mathcal{F}', \mathcal{F}) \\
&\to 0 = R^0 \pi'_{C*} L\tau_C^* R\mathcal{H}om(\mathcal{F}' \boxtimes \mathcal{O}_C(-C)[1], \mathcal{F}) \to \ldots.
\end{aligned}$$

Finally, we consider the universal surjection as an element in $H^0(\overline{W}, \tau^* \mathcal{W} \otimes \mathcal{O}_\tau(1)))$ and take its image under (2.8). This produces the desired extension.

By construction, $\mathcal{G}_{\text{univ}} \in \mathrm{Coh}(\overline{W} \times S)$ defines a morphism $\nu\colon \overline{W} \to N_1 \subset M$ which restricts to a bijection $W \to E_1$ (see Corollary 2.4 and (2.4)). By degree reasons an extension on C of the form $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{L}' \to 0$, where $\deg \mathcal{L}' = 1$ and $\deg \mathcal{L} = 0$ is stable or split. However, the split extensions do not occur in $\mathbb{P}(\mathcal{V})$. Hence, ν is everywhere defined. Moreover, the boundary $\overline{W} \setminus W = \mathbb{P}(\mathcal{V})$ maps to $N_1 \setminus E_1 = N_0 \cap N_1$. □

Remark 2.9. One can show that $\nu\colon W \to E_1$ is actually an isomorphism of schemes. Moreover, $\nu\colon \overline{W} \to N_1$ is finite and hence a normalization map. Its tangent map is analyzed in [21, Proposition 7.5] and provides a characterization of the singularities of N_1.

3. Proof of Theorem 0.1

We will now prove Theorem 0.1.

Theorem. *Let $C \in |H|$ be an irreducible curve. The degrees of the two components of the nilpotent cone $N_C = N_0 \cup N_1$ are given by*

$$\deg_{u_1} N_0 = 5 \cdot 2^9 \quad \textit{and} \quad \deg_{u_1} N_1 = 5^2 \cdot 2^{11}$$

and their multiplicities are

$$\mathrm{mult}_{N_C} N_0 = 2^3 \quad \textit{and} \quad \mathrm{mult}_{N_C} N_1 = 2.$$

Moreover, any fiber F of the Mukai system has degree $5 \cdot 3 \cdot 2^{13}$.

All degrees will be computed with respect to a naturally defined distinguished ample class $u_1 \in H^2(M, \mathbb{Z})$, which we construct in Section 3.1. We set

$$d_i = \deg_{u_1}(N_i) := \int_M [N_i] u_1^5$$

for $i = 0, 1$, where by abuse of notation $[N_i] \in H^{10}(M, \mathbb{Z})$ is the Poincaré dual of the fundamental homology class $[N_i] \in H_{10}(M, \mathbb{Z})$.

The multiplicity is defined as follows. Let η_i be the generic point of N_i. Then

$$m_i = \mathrm{mult}_N N_i := \mathrm{lg}_{\mathcal{O}_{N,\eta_i}} \mathcal{O}_{N_i,\eta_i} = \mathrm{lg}_{\mathcal{O}_{N_i,\eta_i}} \mathcal{O}_{N_i,\eta_i}.$$

In particular, we have an equality $[F] = m_0[N_0] + m_1[N_1] \in H^{10}(M, \mathbb{Z})$ for any fiber F. Consequently, inserting $m_0 = 2^3$ and $m_1 = 2$, we find

$$\deg_{u_1}(F) = 5 \cdot 2^{12} + 5^2 \cdot 2^{12} = 5 \cdot 3 \cdot 2^{13}.$$

as stated in the theorem. We will see that the multiplicities are small in comparison with the degrees. Therefore, it is possible to determine the multiplicities from the knowledge of the degrees but not vice versa.

Proof of the multiplicities knowing all the degrees. Let $F \subset M$ be a smooth fiber. Then, we have $\deg F = m_0 d_0 + m_1 d_1$ and hence

$$5 \cdot 3 \cdot 2^{13} = m_0 \cdot 5 \cdot 2^9 + m_1 \cdot 5^2 \cdot 2^{11}.$$

The only possible solutions are $(m_0, m_1) = (28, 1)$ or $(m_0, m_1) = (8, 2)$. However, by [16, Proposition 4.11]

$$\dim T_{[\mathcal{E}]} N = \dim \operatorname{Ext}^1_{2C}(\mathcal{E}, \mathcal{E}) = \dim N + 1 \text{ for all } [\mathcal{E}] \in E_1.$$

So N_1 is not reduced and the first solution is ruled out. □

Remark 3.1. We will prove Theorem 0.1 for a fixed smooth curve $C \in |H|$, which implies the case of an irreducible and possibly singular curve by a deformation argument as follows. According to the careful analysis in [13, Section 3.7, in particular Propositions 3.7.23 & 3.7.19] the above description of the irreducible components of $f^{-1}(2C)$ is valid for every irreducible curve $C \in |H|$. Hence, if one deforms from a smooth to a singular, irreducible curve in $|H|$, the irreducible components of the fiber with their reduced structure deform as well. Consequently, degrees and multiplicities remain constant.

3.1. Construction of the ample class u_1. We use the determinant line bundle construction [31, Lemma 8.1.2] in order to produce an ample class on the moduli space M.

Let X and T be two projective varieties and assume that X is smooth. Let $p\colon T \times X \to T$ and $q\colon T \times X \to X$ denote the two projections. For any $\mathcal{W} \in \operatorname{Coh}(X \times T)$ flat over T, we define $\lambda_{\mathcal{W}}\colon \mathrm{K}(X)_{\mathrm{num}} \to H^2(T, \mathbb{Z})$ to be the following composition

$$\mathrm{K}(X)_{\mathrm{num}} \xrightarrow{q^*} \mathrm{K}^0(T \times X)_{\mathrm{num}} \xrightarrow{\cdot[\mathcal{W}]} \mathrm{K}^0(T \times X)_{\mathrm{num}} \xrightarrow{Rp_*} \mathrm{K}^0(T)_{\mathrm{num}} \xrightarrow{\det} \mathrm{NS}(T) \subset H^2(T, \mathbb{Z}).$$

We will take advantage of the functorial properties of this definition. These are

(i) $f^*\lambda_{\mathcal{W}} = \lambda_{(f \times \mathrm{id})^*\mathcal{W}}$ for any morphism $f\colon T' \to T$ and
(ii) $\lambda_{(\mathrm{id} \times i)_*\mathcal{W}}(x) = \lambda_{\mathcal{W}}(Li^*x)$ for all $x \in \mathrm{K}(X)_{\mathrm{num}}$ if $i\colon Y \hookrightarrow X$ is the inclusion of a closed, smooth subscheme and $\mathcal{W} \in \operatorname{Coh}(T \times Y)$.

The construction is especially interesting if $X = \mathrm{M}_T(c)$ is a fine moduli space, that parameterizes coherent sheaves on T of class $c \in \mathrm{K}(T)_{\mathrm{num}}$. Let $\mathcal{E}_{\mathrm{univ}}$ be a universal sheaf on $\mathrm{M}_T(c) \times T$, then

$$\lambda_{\mathcal{E}_{\mathrm{univ}} \otimes p^*\mathcal{M}}(x) = \lambda_{\mathcal{E}_{\mathrm{univ}}}(x) + \chi(c \cdot x) c_1(\mathcal{M})$$

for all $\mathcal{M} \in \operatorname{Pic}(\mathrm{M}_T(c))$. Hence,

$$\lambda_{\mathrm{M}_T(c)} \coloneqq \lambda_{\mathcal{E}_{\mathrm{univ}}}\colon c^{\perp,\chi} \longrightarrow \mathrm{NS}(\mathrm{M}_T(c))$$

is well-defined and does not depend on the choice of a universal sheaf. Here,

$$c^{\perp,\chi} = \{x \in \mathrm{K}(T)_{\mathrm{num}} \mid \chi(x \cdot c) = 0\}.$$

Example 3.2. Let C be a smooth curve of any genus $g \geq 0$. Then

$$(\mathrm{rk}, \deg)\colon \mathrm{K}(C)_{\mathrm{num}} \xrightarrow{\sim} \mathbb{Z} \oplus \mathbb{Z}.$$

Fix $n \geq 1$ and $d \in \mathbb{Z}$ coprime and let $c = (n, d) \in \mathrm{K}(C)_{\mathrm{num}}$. Then $\mathrm{M}_C(c) = \mathrm{M}_C(n, d)$ is the moduli space of stable vector bundles of rank n and degree d on C and we find

$$c^{\perp,\chi} = \langle (-n, d + n(1 - g)) \rangle.$$

The generalized Theta divisor can be defined by

$$\Theta_{\mathrm{M}_C(n,d)} \coloneqq \lambda_{\mathrm{M}_C(n,d)}(-n, d + n(1 - g)),$$

see [22, Théorème D]. A special case is $\mathrm{M}_C(1, k) = \mathrm{Pic}^k(C)$. In this case,

$$c^{\perp,\chi} = \langle (-1, k + 1 - g) \rangle \quad \text{and} \quad \Theta_k \coloneqq \lambda_{\mathrm{Pic}^k(C)}(-1, k + 1 - g)$$

is the class of the canonical Theta divisor in $\mathrm{Pic}^k(C)$.

Remark 3.3. Denote by $\mathrm{SM}_C(n, d)$ the moduli space of vector bundles with fixed determinant, i.e. a fiber of $\det\colon \mathrm{M}_C(n, d) \to \mathrm{Pic}(C)$ and by $\Theta_{\mathrm{SM}_C(n,d)}$ the restriction of $\Theta_{\mathrm{M}_C(n,d)}$ to $\mathrm{SM}_C(n, d)$. Taking the tensor product defines an étale map

$$h\colon \mathrm{SM}_C(n, d) \times \mathrm{Pic}^0(C) \longrightarrow \mathrm{M}_C(n, d)$$

of degree n^{2g}. Using [19, Corollary 6], we find the following relation if (n, d) are coprime

$$h^*\Theta_{\mathrm{M}_C(n,d)} = p_1^*\Theta_{\mathrm{SM}_C(n,d)} + n^2 p_2^*\Theta_0. \tag{3.1}$$

Lemma 3.4. *Let C be a smooth curve of genus g and $\mathcal{P}$ a Poincaré line bundle on* $\mathrm{Pic}^k(C) \times C$. *Then*

$$\lambda_{\mathcal{P}}\colon \mathrm{K}(C)_{\mathrm{num}} \to H^2(\mathrm{Pic}^k(C), \mathbb{Z})$$

is given by

$$(r, d) \mapsto (d + (k + 1 - g)r)\mu - r\Theta_k,$$

where $p_1^*\mu = c_1^{2,0}(\mathcal{P}) \in H^2(\mathrm{Pic}^k(C) \times C, \mathbb{Z})$ is the $(2, 0)$ Künneth component of $c_1(\mathcal{P})$.

By tensoring with a suitable line bundle on $\mathrm{Pic}^k(C)$, we can assume that $c_1^{2,0}(\mathcal{P}) = 0$.

Proof. Let us abbreviate $\mathrm{Pic}^k(C)$ to Pic^k. We decompose

$$c_1(\mathcal{P}) = c^{2,0} + c^{1,1} + c^{0,2}$$

into its Künneth components and write $c^{2,0} = p^*\mu$ for some $\mu \in H^2(\mathrm{Pic}^k, \mathbb{Z})$. Then by [2, VIII §2] the class $\gamma = c^{1,1}$ satisfies $\gamma^2 = -2\rho p^*\Theta_k$. Moreover, by definition, $c^{0,2} = k\rho$, where ρ is the pullback of the class of a point on C. Together, $c_1(\mathcal{P}) = p^*\mu + \gamma + k\rho$ and

$$\mathrm{ch}(\mathcal{P}) = 1 + p^*\mu + \gamma + k\rho + \rho p^*(k\mu - \Theta_k).$$

Now, let $x = (r, d) \in \mathrm{K}(C)_{\mathrm{num}}$. The Grothendieck–Riemann–Roch theorem gives

$$\begin{aligned}
\mathrm{ch}(Rp_*(\mathcal{P} \otimes q^*x)) &= p_*(\mathrm{ch}(\mathcal{P} \otimes q^*x)\,\mathrm{td}(\mathrm{Pic}^k \times C)) = p_*(\mathrm{ch}(\mathcal{P})\,\mathrm{ch}(q^*x) q^*\,\mathrm{td}(C)) \\
&= p_*(\mathrm{ch}(\mathcal{P})(r + ((1-g)r + d)\rho)) \\
&= kr + (1-g)r + d + (kr + (1-g)r) + d)\mu - r\Theta_k.
\end{aligned}$$

In particular, $\lambda_{\mathcal{P}}(x) = c_1(Rp_*(\mathcal{P} \otimes q^*x)) = ((k + 1 - g)r + d)\mu - r\Theta_k$. □

We come back to our original situation, i.e. (S, H) is a polarized K3 surface of genus 2 and $M = M_H(v)$ parameterizes H-stable sheaves with Mukai vector $v = (0, 2H, -1)$ or equivalently, with Chern character $v_{\mathrm{ch}} = (0, 2H, -1)$. In this setting λ_M induces an isomorphism, [31, Theorem 6.2.15]

$$\lambda_M \colon v_{\mathrm{ch}}^{\perp,\chi} \xrightarrow{\sim} \mathrm{NS}(M). \tag{3.2}$$

As v and v_{ch} coincide, we will notationally not distinguish between them anymore. We find

$$v^{\perp,\chi} = \{(2c.H, c, s) \mid c \in \mathrm{NS}(S), s \in \mathbb{Z}\}.$$

Warning 3.5. In this setting, one usually wants to consider the morphism λ_M in terms of the Mukai vector instead of the Chern character and the Mukai pairing instead of the intersection product, i.e. one considers the composition

$$v^{\perp,\langle,\rangle} \xrightarrow{\sim} v_{\mathrm{ch}}^{\perp,\chi} \xrightarrow{\lambda_M} \mathrm{NS}(M_H(v)),$$

which identifies the Mukai pairing on the left hand side with the Beauville–Bogomolov form on the right hand side. Here $v = v_{\mathrm{ch}} \cdot \sqrt{\mathrm{td}(S)}$. Explicitly, if $v = (r, c, s)$, then $v_{\mathrm{ch}} = (r, c, s - r)$ and

$$\langle (r', c', s'), (r, c, s) \rangle = \chi((-r', c', -s' - r') \cdot (r, c, s - r)).$$

Thus the first arrow is given by $(r', c', s') \mapsto (-r', c', -s' - r')$.

Definition 3.6. For all $s \in \mathbb{Z}$ we define

$$l_s \coloneqq \lambda_M((-4, -H, s)) \in H^2(M, \mathbb{Z}).$$

The value of s does not have any relevance for our computations. However, with the results of [5], it can be proven that l_s is ample for $s \gg 0$ and one can even compute the precise boundary of the ample cone (cf. 10.1).

Definition 3.7. For everything what follows, we fix $s_0 \gg 0$ such that l_{s_0} is ample and set

$$u_1 \coloneqq l_{s_0}.$$

3.2. Degree of a general fiber. We compute the degree of a general fiber.

Proposition 3.8. *Let $D \in |2H|$ be a smooth curve and let $F := f^{-1}(D)$ be the corresponding fiber. Let $u = \lambda_M(x)$ with $x = (2c.H, c, s) \in v^{\perp,\chi}$. Then*

$$u|_F = -2c.H \cdot \Theta_3,$$

where $\Theta_3 \in H^2(\mathrm{Pic}^3(D), \mathbb{Z})$ is the class of the Theta divisor. In particular, we have

$$\deg_{u_1} F = 5! \cdot 2^{10}.$$

Proof. Let $i\colon D \hookrightarrow S$ be the inclusion. The inclusion $\mathrm{Pic}^3(D) \cong F \hookrightarrow M$ is defined by $(\mathrm{id} \times i)_* \mathcal{P}_3$, where $\mathcal{P}_3$ is a Poincaré line bundle on $\mathrm{Pic}^3(D) \times D$. Hence,

$$u|_{\mathrm{Pic}^3(D)} = \lambda_{(\mathrm{id} \times i)_* \mathcal{P}_3}(x) = \lambda_{\mathcal{P}_3}(Li^* x).$$

Now, $Li^*\colon K(S)_{\mathrm{num}} \to K(D)_{\mathrm{num}} \cong \mathbb{Z}^{\oplus 2}$ maps (r, c, s) to $(r, c.D)$ and thus x to $2c.H \cdot (1, 1)$, whereas by definition $\theta_3 = \lambda_{\mathcal{P}_3}(-1, -1)$. Finally,

$$\deg_{u_1} F = \int_{\mathrm{Pic}^3(D)} (4\Theta_3)^5 = 2^{10} \cdot 5!.$$

□

Remark 3.9. One can also prove the above result using the Beauville–Bogomolov form $(\, , \,)_{BB}$ on $H^2(M, \mathbb{Z})$. Let $u_0 = f^* c_1(\mathcal{O}(1)) \in H^2(M, \mathbb{Z})$. Then $[F] = u_0^5 \in H^{10}(M, \mathbb{Z})$ and

$$\deg_{u_1}(F) = \int_M u_0^5 u_1^5 = 5! \cdot (u_0, u_1)_{BB}^5,$$

where we use that $(u_0, u_0)_{BB} = 0$ and that M is birational to $S^{[5]}$ in order to determine the correct Fujiki constant. One verifies that $u_0 = \lambda_M((0, 0, 1))$ [58, Lem 4.4] whereas, by definition, $u_1 = \lambda_M(-4, -H, s_0)$ with $s_0 \gg 0$. After correct identification (cf. Warning 3.5), one has

$$(\lambda_M(r, c, s), \lambda_M(r', c', s'))_{BB} = \langle (r, c, s), (r', c', s') \rangle + 2rr'.$$

This gives $(u_0, u_1)_{BB} = 4$.

3.3. Degree of the vector bundle component N_0. Next, we deal with the component N_0, which is isomorphic to $M_C(2, 1)$.

Proposition 3.10. *Let $x = \lambda_M(u)$ with $u = (2c.H, c, s) \in v^{\perp,\chi}$. Then*

$$x|_{N_0} = -c.H\Theta,$$

where $\Theta \in H^2(N_0, \mathbb{Z})$ is the the generalized Theta divisor. In particular,

$$u_1|_{N_0} = 2\Theta$$

and given $x_i = \lambda_M(2c_i.H, c_i, s_i)$ *for* $i = 1, \ldots, 5$, *we find*

$$\int_M x_1 \ldots x_5[N_0] = -\prod_{i=1}^{5} c_i.H \int_{N_0} \Theta^5 = -5 \cdot 2^4 \prod_{i=1}^{5} c_i.H.$$

Hence, $\deg_{u_1} N_0 = 5 \cdot 2^9$.

Proof. Let $i\colon C \hookrightarrow S$ be the inclusion. The inclusion $N_0 \hookrightarrow M$ is defined by $(\mathrm{id} \times i)_* \mathcal{E}_{\mathrm{univ}}$, where $\mathcal{E}_{\mathrm{univ}}$ is the universal vector bundle on $N_0 \times C$. Hence,

$$x|_{N_0} = \lambda_{(\mathrm{id} \times i)_* \mathcal{E}_{\mathrm{univ}}}(u) = \lambda_{N_0}(Li^* u).$$

Now, $Li^*\colon K(S)_{\mathrm{num}} \to K(C)_{\mathrm{num}} \cong \mathbb{Z}^{\oplus 2}$ maps (r, c, s) to $(r, c.H)$. In particular,

$$Li^* u = c.H(2, 1),$$

whereas by definition $\theta = \lambda_{N_0}(-2, -1)$.

Next, we compute $\int_{N_0} \Theta^5$ by pulling back along $h\colon \mathrm{SM}_C(2,1) \times \mathrm{Pic}^0(C) \to N_0$ from Remark 3.3.

$$\begin{aligned}\int_{\mathrm{M}_C(2,1)} \Theta^5 &\overset{(3.1)}{=} \frac{1}{2^4} \int_{\mathrm{SM}_C(2,1) \times \mathrm{Pic}^0(C)} (p_1^* \Theta_{\mathrm{SM}} + 4 p_2^* \Theta_0)^5 \\ &= \frac{1}{2^4} \binom{5}{3} \int_{\mathrm{SM}_C(2,1)} \Theta_{\mathrm{SM}}^3 \int_{\mathrm{Pic}^0(C)} (4\Theta_0)^2 = 5 \cdot 2^4.\end{aligned}$$

The value $\int_{\mathrm{SM}_C(2,1)} \Theta_{\mathrm{SM}}^3 = 4$ is given by the leading term of the Verlinde formula [60]. $\square$

Remark 3.11. The general formula is

$$\int_{\mathrm{M}_C(n,d)} \Theta^{\dim \mathrm{M}_C(n,d)} = \dim \mathrm{M}_C(n,d)!(2^{2g-2} - 2) \frac{(-1)^g 2^{2g-2} B_{2g-2}}{(2g-2)!},$$

where B_i is the i-th Bernoulli number. The second Bernoulli number is $B_2 = \frac{1}{6}$

Remark 3.12. In the general case, where $v = (0, nH, s)$ and $u_1 = \lambda_M(-n(2g-2), sH, *)$ with $s = n + d(1-g)$, we find $u_1|_F = n(2g-2)\Theta_\delta$ and $u_1|_{N_0} = (2g-2)\Theta$. Thus

$$\deg_{u_1} F = (n(2g-2))^{\dim N} \cdot \dim N! \quad \text{and} \quad \deg_{u_1} N_0 = (2g-2)^{\dim N} \int_{\mathrm{M}_C(n,d)} \Theta^{\dim \mathrm{M}_C(n,d)}.$$

Here, $\dim N = n^2(2g-2) + 2$.

3.4. Degree of the other component N_1. We complete the proof of Theorem 0.1 by dealing with the remaining component N_1. Recall from Proposition 2.8 that there is a birational map $\nu\colon \overline{W} \to N_1$, where $\tau\colon \overline{W} = \mathbb{P}(\mathcal{W}) \to T = \mathrm{Pic}^1(C) \times C$.

Proposition 3.13. *Let* $x_i = \lambda_M(u_i)$ *with* $u_i = (2c_i.H, c_i, s_i) \in v^{\perp,\chi}$ *for* $i = 1, \ldots, 5$. *Then*

$$\int_M x_1 \ldots x_5[N_1] = \int_{\overline{W}} \prod_{i=1}^{5} \nu^*(x_i|_{N_1}) = -5^2 \cdot 2^6 \prod_{i=1}^{5} c_i.H. \tag{3.3}$$

In particular, $\deg_{u_1} N_1 = 5^2 \cdot 2^{11}$.

Note that the first equality in (3.3) is immediate, because $\nu\colon \overline{W} \to N_1$ is birational. For the proof of the proposition, we need to introduce some more notation. We abbreviate $\mathrm{Pic}^1(C)$ to Pic^1 and in the following all cohomology groups have $\mathbb{Z}$ coefficients. We set

$$\zeta = c_1(\mathcal{O}_\tau(1)) \in H^2(\overline{W}) \quad \text{and write} \quad \rho = p_2^*[\mathrm{pt}] \in \mathrm{H}^2(\mathrm{Pic}^1 \times \mathrm{C})$$

for the pullback of the class of a point on C. If no confusion is likely, we suppress pullbacks from our notation, e.g. we will write $\Theta_1 \in H^2(\mathrm{Pic}^1 \times C)$ and also $\Theta_1 \in H^2(\mathbb{P}(\mathcal{W}))$ instead of $p_1^*\Theta_1$ and $\tau^* p_1^* \Theta_1$, respectively. Moreover, we define

$$\pi \coloneqq c_1(\mathcal{P}) - c_1^{2,0}(\mathcal{P}) \in H^2(\mathrm{Pic}^1 \times C),$$

where $\mathcal{P}$ is a Poincaré line bundle. Note that π is independent of the choice of $\mathcal{P}$.

Proof of Proposition 3.13. We will split the proof into the following three steps.

(i) Let $x = \lambda_M(2c.H, c, s)$. Then

$$\nu^*(x|_{N_1}) = \lambda_{\mathcal{G}_{\mathrm{univ}}}(x) = c.H(-4\Theta_1 + 2\pi - 7\rho - \zeta) \in H^2(\overline{W}).$$

(ii) We have

$$(-4\Theta_1 + 2\pi - 7\rho - \zeta)^5 = -5^2 2^5 \zeta^2 \rho \Theta_1^2 \in H^{10}(\overline{W}).$$

(iii) The top cohomology group $H^{10}(\overline{W})$ generated by $\frac{1}{2}\zeta^2\rho\Theta_1^2$ and we have

$$\int_{\mathbb{P}(\mathcal{W})} \zeta^2 \rho \Theta_1^2 = 2.$$

Proof of (i). In Proposition 2.8, we defined the morphism $\nu\colon \overline{W} \to N_1$ by means of $\mathcal{G}_{\mathrm{univ}} \in \mathrm{Coh}(\overline{W} \times S)$, which sits in the (universal) extension

$$0 \to \tau_S^*(\mathrm{id} \times i)_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1}) \boxtimes \mathcal{O}_\tau(1) \to \mathcal{G}_{\mathrm{univ}} \to \tau_S^*(\mathrm{id} \times i)_* p_{13}^* \mathcal{P}_1 \to 0,$$

where $\tau_S = \tau \times \mathrm{id}_S\colon \overline{W} \times S \to \mathrm{Pic}^1 \times C \times S$. So, by construction, we have

$$\begin{aligned}
\lambda_{\mathcal{G}_{\mathrm{univ}}}(x) &= \lambda_{\tau_S^*(\mathrm{id} \times i)_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1}) \boxtimes \mathcal{O}_\tau(1)}(x) + \lambda_{\tau_S^*(\mathrm{id} \times i)_* p_{13}^* \mathcal{P}_1}(x) \\
&= \lambda_{\tau_S^*(\mathrm{id} \times i)_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1})}(x) + k(Li^*x) \cdot \zeta + \tau^* p_1^* \lambda_{\mathcal{P}_1}(Li^*x) \\
&= \tau^*(\lambda_{\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta)}(Li^*x \cdot \omega^{-1}) + p_1^* \lambda_{\mathcal{P}_1}(Li^*x)) + k(Li^*x) \cdot \zeta,
\end{aligned}$$

where $\omega = c_1(\omega_C)$ and $k(Li^*x) = \mathrm{rk}\, Rp_*(\mathcal{P}_1 \boxtimes \mathcal{O}(\Delta) \boxtimes \omega_C^{-1} \boxtimes Li^*x) = \chi(Li^*x) = -c.H$.

The term $\lambda_{\mathcal{P}_1\boxtimes\mathcal{O}(\Delta)}(Li^*x\cdot\omega^{-1})+p_1^*\lambda_{\mathcal{P}_1}(Li^*x)$, is determined in Lemmas 3.14 and 3.4. Note that each summand depends on the choice of a Poincaré line bundle, whereas the sum does not. Together,

$$\begin{aligned}\nu^*(x|_{N_1}) &= \tau^*(\lambda_{\mathcal{P}_1\boxtimes\mathcal{O}(\Delta)}(Li^*x\cdot\omega^{-1})+p_1^*\lambda_{\mathcal{P}_1}(Li^*x)) - c.H\zeta \\ &= c.H(p_1^*(\lambda_{\mathcal{P}_1}(2,-3)+\lambda_{\mathcal{P}_1}(2,1))+2c_1(\mathcal{P}_1)-7\rho-\zeta) \\ &= c.H(-4\Theta_1+2\pi-7\rho-\zeta).\end{aligned}$$

□

Lemma 3.14. *Let $\mathcal{F}\in\mathrm{Coh}(X\times C)$. Then*

$$\lambda_{\mathcal{F}\boxtimes\mathcal{O}(\Delta)}(x) = {p_1}^*\lambda_{\mathcal{F}}(x)+rc_1(\mathcal{F})+c_0(\mathcal{F})(d-2r)\rho$$

for all $x=(r,d)\in\mathrm{K}(C)_{\mathrm{num}}$. In particular,

$$\lambda_{\mathcal{P}_1\boxtimes\mathcal{O}(\Delta)}(Li^*x\cdot\omega^{-1}) = c.H(p_1^*\lambda_{\mathcal{P}}(2,-3)+2c_1(\mathcal{P}_1)-7\rho).$$

Proof. We have

$$[\mathcal{F}\boxtimes\mathcal{O}(\Delta)] = [p_{13}^*\mathcal{F}]+[(\mathrm{id}\times i_\Delta)_*(p_2^*\omega_C^{-1}\otimes\mathcal{F})]\in\mathrm{K}(X\times C\times C),$$

where $i_\Delta\colon C\to C\times C$ is the diagonal and thus

$$\lambda_{\mathcal{F}\boxtimes\mathcal{O}(\Delta)}(x) = \lambda_{p_{13}^*\mathcal{W}}(x)+\lambda_{(\mathrm{id}\times i_\Delta)_*(p_2^*\omega_C^{-1}\otimes\mathcal{F})}(x)\in H^2(X\times C)$$

for all $x\in\mathrm{K}(C)_{\mathrm{num}}$. Now,

$$\begin{aligned}\lambda_{(\mathrm{id}\times i_\Delta)_*(p_2^*\omega^{-1}\cdot[\mathcal{F}])}(x) &= \det Rp_{12*}((\mathrm{id}\times i_\Delta)_*(p_2^*\omega^{-1}\cdot[\mathcal{F}])\cdot p_3^*x) \\ &= \det R(p_{12}\circ(\mathrm{id}\times i_\Delta))_*([\mathcal{F}]\cdot p_2^*(\omega^{-1}\cdot x)) \\ &= \det([\mathcal{F}]\cdot p_2^*(\omega^{-1}\cdot x)) = rc_1(\mathcal{F})+c_0(\mathcal{F})(d-r(2g-2))\rho.\end{aligned}$$

□

To prove the remaining steps, we need to understand the cohomology ring $H^*(\overline{W})$.

Lemma 3.15. *We have*

$$H^*(\overline{W})\cong H^*(\mathrm{Pic}^1\times C)[\zeta]/\zeta^3+4\rho\zeta^2.$$

In particular,

$$H^{10}(\overline{W}) = \zeta^2\cdot H^6(\mathrm{Pic}^1\times C).$$

Proof. By definition, $\overline{W}=\mathbb{P}(\mathcal{W})$. Hence,

$$H^*(\overline{W})\cong H^*(\mathrm{Pic}^1\times C)[\zeta]/\zeta^3+c_1(\mathcal{W})\zeta^2+c_2(\mathcal{W})\zeta+c_3(\mathcal{W}).$$

We use the short exact sequence $0 \to \mathcal{V} \to \mathcal{W} \to \mathcal{O}_T \to 0$ from (2.7) to compute the Chern classes of $\mathcal{W}$. Note that

$$\mathcal{V} = R^1 p_{12*}(p_{23}^* \mathcal{O}(\Delta) \otimes p_3^* \omega_C^{-1}) \cong p_2^* R^1 p_{1*}(\mathcal{O}(\Delta) \otimes p_2^* \omega_C^{-1}).$$

So the Chern classes of $\mathcal{V}$ can be computed by the push forward along the first projection of the following the short exact sequence

$$0 \to p_2^* \omega_C^{-1} \to \mathcal{O}(\Delta) \boxtimes \omega_C^{-1} \to p_2^* \omega_C^{-2}|_\Delta \to 0.$$

We find

$$0 \longrightarrow \omega_C^{-2} \to \mathcal{O}_C \otimes H^1(C, \omega_C^{-1}) \to R^1 p_{1*}(\mathcal{O}(\Delta) \otimes p_2^* \omega_C^{-1}) \longrightarrow 0.$$

Hence, $c_1(\mathcal{W}) = 4\rho$ and $c_i(\mathcal{W}) = 0$ if $i \geq 2$. □

Proof of (ii) and (iii). We want to show that

$$(-4\Theta_1 + 2\pi - 7\rho - \zeta)^5 = -5^2 2^5 \zeta^2 \rho \Theta_1^2 \in H^{10}(\overline{W}).$$

We compute

$$\begin{aligned}(-4\Theta_1 + 2\pi - 7\rho - \zeta)^5 &= \binom{5}{3}(-\zeta^3)(-4\Theta_1 + 2\pi - 7\rho)^2 + \binom{5}{2}\zeta^2(-4\Theta_1 + 2\pi - 7\rho)^3 \\ &= 10 \cdot \zeta^2((4\rho(-4\Theta_1 + 2\pi - 7\rho)^2 + (-4\Theta_1 + 2\pi - 7\rho)^3).\end{aligned}$$

The result is a combination of π, θ and ρ, which are classes of type $(1,1)+(0,2), (2,0)$ and $(0,2)$, respectively. Moreover, in the proof of Lemma 3.4 we computed $\pi = \rho + \gamma$ and $\pi^2 = \gamma^2 = -2\rho\Theta_1$. Hence, the only non-zero combinations are $\pi^2\Theta_1 = -2\rho\Theta_1^2 = -2\pi\Theta_1^2$. We find

$$\begin{aligned}&10 \cdot \zeta^2((4\rho(-4\Theta_1 + 2\pi - 7\rho)^2 + (-4\Theta_1 + 2\pi - 7\rho)^3) \\ &\quad = 10 \cdot \zeta^2(2^6\rho\Theta_1^2 + 3(-2^4\pi^2\Theta_1 + 2^5\pi\Theta_1^2 - 7 \cdot 2^4\rho\Theta_1^2) \\ &\quad\quad = 10(2^6 + 3(2^5 + 2^5 - 7 \cdot 2^4))\zeta^2\rho\Theta_1^2 = -5^2 2^5 \zeta^2 \rho \Theta_1^2.\end{aligned}$$

Finally, we want to show that $\int_{\overline{W}} \zeta^2 \rho \Theta^2 = 2$. Indeed,

$$\int_{\overline{W}} \zeta^2 \rho \Theta^2 = \tau_* \zeta^2 \int_{\mathrm{Pic}^1} \Theta^2 \int_C \rho = 2.$$

□

This concludes the proof of the proposition. □

4. Proof of Theorem 0.2

In this section we prove Theorem 0.2.

Theorem. *The classes $[N_0]$ and $[N_1] \in H^{10}(M,\mathbb{Q})$ are linearly independent and span a totally isotropic subspace of $H^{10}(M,\mathbb{Q})$ with respect to the intersection pairing. They are given by*

$$[N_0] = \frac{1}{48}[F] + \beta \text{ and } [N_1] = \frac{5}{12}[F] - 4\beta,$$

where $[F]$ is the class of a general fiber of the Mukai system and $0 \neq \beta \in (S^5H^2(M,\mathbb{Q}))^\perp$ satisfies $\beta^2 = 0$. As $\deg_{u_1}\beta = 0$, the class β is not effective.

From now on, all cohomology groups have $\mathbb{Q}$-coefficients.

Before coming to the proof, we want to point out, that the irreducible components over points in $\Sigma \setminus \Delta$ (see (2.2)) are of different cohomological nature. Let $D \in \Sigma \setminus \Delta$ be a reducible curve with two smooth components C_1 and C_2 meeting transversally. Then the two components N_1' and N_2' of $f^{-1}(D)$ contain an open sublocus parameterizing line bundles on D of bi-degree $(2,1)$ and $(1,2)$, respectively [13, Proposition 3.7.1 and Lemma 3.3.2]. The monodromy around $\Sigma\setminus\Delta$ exchanges C_1 and C_2 and consequently the classes of the irreducible components. We find

$$[N_1'] = [N_2'] = \tfrac{1}{2}[F].$$

In particular, the two components are linearly dependent. This is not true over Δ.

Proposition 4.1. *The classes $[N_0]$ and $[N_1] \in H^{10}(M)$ are linearly independent.*

The proof uses the following simple observation.

Lemma 4.2. *Let $M \to B$ be a Lagrangian fibration and F a smooth fiber. Then*

$$c_i(\mathcal{T}_M)|_F = 0 \text{ for all } i > 0.$$

Proof. We have a short exact sequence $0 \to \mathcal{T}_F \longrightarrow \mathcal{T}_M|_F \longrightarrow \mathcal{N}_{F/M} \to 0$. Now, $F \subset M$ is Lagrangian and hence $\mathcal{N}_{F/M} \cong \Omega_F$. Moreover, F is an abelian variety and hence all its Chern classes of degree greater than zero are trivial. □

Proof of Proposition 4.1. Assume that $[N_0]$ and $[N_1]$ are linearly dependent. Then there is some $\lambda \in \mathbb{Q}$ such that $[F] = \lambda[N_0]$, where $F \subset M$ is a smooth fiber. In particular, by the above lemma, any product of $[N_0]$ and the Chern classes of M vanishes. However, we will show that

$$\int_M c_2(\mathcal{T}_M)\cdot u_1^3 \cdot [N_0] \neq 0,$$

leading to the desired contradiction. We have $c(\mathcal{T}_M|_{N_0}) = c(\mathcal{T}_{N_0})c(\Omega_{N_0})$ and thus

$$c_2(\mathcal{T}_M)|_{N_0} = (2c_2 - c_1^2)(\mathcal{T}_{N_0}).$$

Moreover, our computation will use the following two inputs. Let $\alpha \in H^2(\mathrm{SM}_C(2,1))$ be the degree two Künneth component of $(c_1^2 - c_2)(\mathcal{V}_{\mathrm{univ}})$ with $\mathcal{V}_{\mathrm{univ}}$ being a universal bundle on $\mathrm{SM}_C(2,1) \times C$. It is known, e.g. [59, §5A], that

$$c_1(\mathcal{T}_{\mathrm{SM}_C(2,1)}) = 2\alpha, \quad c_2(\mathcal{T}_{\mathrm{SM}_C(2,1)}) = 3\alpha^2 \quad \text{and} \quad \int_{\mathrm{SM}_C(2,1)} \alpha^3 = 4.$$

Further, by [22, Théorème F] it is known that $\mathcal{O}_{\mathrm{SM}_C(2,1)}(-2\Theta) \cong \omega_{\mathrm{SM}_C(2,1)}$. Hence,

$$\Theta = -\frac{1}{2}c_1(\omega_{\mathrm{SM}_C(2,1)}) = \frac{1}{2}c_1(\mathcal{T}_{\mathrm{SM}_C(2,1)}).$$

This gives,

$$\begin{aligned}
\int c_2(\mathcal{T}_M)u_1^3[N_0] &\overset{3.10}{=} \int_{N_0} (2c_2 - c_1^2)(\mathcal{T}_{N_0}) \cdot (2\Theta)^3 \\
&= \frac{1}{2^4}\int_{\mathrm{SM}_C(2,1)\times \mathrm{Pic}^0} h^*((2c_2 - c_1^2)(\mathcal{T}_{N_0}) \cdot (2\Theta)^3) \\
&\overset{(3.1)}{=} \frac{2^3}{2^4}\int_{\mathrm{SM}_C(2,1)\times \mathrm{Pic}^0} p_1^*(2c_2 - c_1^2)(\mathcal{T}_{\mathrm{SM}}) \cdot (p_1^*\Theta_{\mathrm{SM}} + 4p_2^*\Theta_0)^3 \\
&= \frac{1}{2}\int_{\mathrm{SM}_C(2,1)\times \mathrm{Pic}^0} p_1^*(2c_2 - c_1^2)(\mathcal{T}_{\mathrm{SM}}) \cdot (3p_1^*\Theta_{\mathrm{SM}} \cdot 4^2 p_2^*\Theta_0^2) \\
&= 3 \cdot 2^3 \int_{\mathrm{SM}_C(2,1)} (2c_2 - c_1^2)(\mathcal{T}_{\mathrm{SM}}) \cdot \frac{1}{2}c_1(\mathcal{T}_{\mathrm{SM}}) \int_{\mathrm{Pic}^0} \Theta_0^2 \\
&= 3 \cdot 2^4 \int_{\mathrm{SM}_C(2,1)} (6\alpha^2 - 4\alpha^2)\alpha = 3 \cdot 2^7 \neq 0.
\end{aligned}$$

□

Proof of Theorem 0.2. We set $V \coloneqq S^5H^2(M) \subset H^{10}(M)$ so that we have an orthogonal decomposition with respect to the cup product $H^{10}(M) = V \oplus V^\perp$. Accordingly, we write $[N_i] = \alpha_i + \beta_i$ with $\alpha_i \in V$ and $0 \neq \beta_i \in V^\perp$ for $i = 1, 2$. We claim that

$$20[N_0] - [N_1] \in V^\perp. \tag{4.1}$$

To see this, we decompose the second cohomology group into its transcendental and algebraic part, i.e. $H^2(M) = T(M) \oplus \mathrm{NS}(M)$. Now, for $i = 1, 2$ consider

$$T(M) \to H^{12}(M), \quad \alpha \mapsto \alpha \cdot [N_i]. \tag{4.2}$$

As the symplectic form $\sigma \in T(M)$ vanishes on N_i, it follows by irreducibility of the Hodge structure $T(M)$ that the assignment (4.2) is trivial. Hence, it suffices to show that $20[N_0] - [N_1] \in (S^5\,\mathrm{NS}(M))^\perp$. By (3.2) any element in $S^5\,\mathrm{NS}(M)$ is of the form $x_1x_2\ldots x_5$, where $x_i = \lambda_M(2c_i.H, c_i, s_i)$. According to Propositions 3.10 and 3.13

$$\int [N_1]x_1x_2\ldots x_5 = -5^2 2^6 \prod_{i=1}^{5} c_i.H = 20\int [N_0]x_1x_2\ldots x_5.$$

This proves (4.1).

Next, we write $[N_1] - 20[N_0] = \alpha_1 - 20\alpha_0 + \beta_1 - 20\beta_0 \in V^\perp$ and conclude $\alpha_1 = 20\alpha_0$. We set $\alpha = \alpha_0$. On the one hand, we have by Theorem 0.1

$$2^3[N_0] + 2[N_1] = [F] = u_0^5 \in V,$$

but also

$$u_0^5 = 48\alpha + 8\beta_0 + 2\beta_1.$$

This gives $48\alpha = u_0^5$ and $\beta_1 = -4\beta_0$. Setting $\beta = \beta_0$ gives the desired expression.

The last assertion follows from $[N_0]^2 = (\frac{1}{48}u_0^5 + \beta)^2 = \beta^2$ and

$$[N_0]^2 = \int_{N_0} c_5(\mathcal{N}_{N_0/M}) = \int_{N_0} c_5(\Omega_{N_0}) = -e(N_0),$$

which is known to vanish, see [3, §9]. Hence also $[N_1]^2 = 0$ and $[N_i] \cdot \beta = 0$ for $i = 1, 2$. $\square$

II. Constant cycle subvarieties in the Mukai system of rank two and genus two

Abstract. Combining theorems of Voisin and Marian, Shen, Yin and Zhao, we compute the dimensions of the orbits under rational equivalence in the Mukai system of rank two and genus two. We produce several examples of algebraically coisotropic and constant cycle subvarieties.

Introduction

By a theorem of Beauville and Voisin [9], any point lying on a rational curve in a K3 surface S, determines the same zero cycle of degree one

$$c_S \in \mathrm{CH}_0(S),$$

called the *Beauville–Voisin class*. This class has the striking property that the image of the intersection product

$$\mathrm{Pic}(S) \otimes \mathrm{Pic}(S) \to \mathrm{CH}_0(S)$$

and $c_2(S)$ are contained in $\mathbb{Z}\cdot c_S$. It is expected that the Chow ring of an irreducible holomorphic symplectic manifold has a similar and particularly rich structure provided by the conjectural Bloch–Beilinson filtration and its conjectural splitting [10]. In this context, Voisin introduced in [55] the notion of an algebraically coisotropic subvariety, which is an generalization of Lagrangian subvariety.

The goal of this note is to investigate the Chow group of zero cycles for the Mukai system of rank two and genus two. Specifically, we produce several examples algebraically coisotropic subvarieties fibered into isotropic constant cycle subvarieties.

As before, let (S, H) be a polarized K3 surface of genus two, that is a double covering $\pi\colon S \to \mathbb{P}^2$ ramified over a smooth sextic curve $R \subset \mathbb{P}^2$ and $H = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is primitive. We consider the the *Mukai system of rank two and genus two*

$$f\colon M := M_H(0, 2H, s) \longrightarrow B := |2H| \cong \mathbb{P}^5,$$

where $s \equiv 1 \bmod 2$. This is an irreducible holomorphic symplectic variety of dimension 10, which is birational to $S^{[5]}$.

For any irreducible, holomorphic symplectic manifold X of dimension $2n$, a brute force approach to finding constant cycle subvarieties (see Section 6.1 for the definition) is to consider the orbit under rational equivalence of a point $x \in X$. This is the countable union of algebraic subvarieties defined by

$$O_x \coloneqq \{x' \in X \mid [x] = [x'] \in \mathrm{CH}_0(X)\} \subset X.$$

Then $\dim O_x$ is defined to be the supremum over the dimensions of the components of O_x. In [55], Voisin defines an increasing filtration $F_0X \subset F_1X \subset \ldots \subset F_nX = X$ on the points of X, where

$$F_iX \coloneqq \{x \in X \mid \dim O_x \geq n - i\}$$

is again a countable union of algebraic subvarieties. Our examples are based on the combination of two theorems. The first one is due to Voisin.

Theorem 4.3 ([55, Thm 1.3])**.** *We have $\dim F_iX \leq n + i$ and if $Z \subset F_iX$ is an irreducible component of dimension $n+i$. Then Z is algebraically coisotropic and the fibers of the isotropic fibration are constant cycle subvarieties of dimension $n - i$.*

The second theorem applies in the case that $X = M_\sigma(v)$ is a smooth projective moduli of Bridgeland stable objects in $D^b(S)$ and is due to Marian, Shen, Yin and Zhao. It establishes a link between rational equivalence in X and in S, which in particular results in a connection between Voisin's filtration $F_\bullet X$ and O'Grady's filtration $S_\bullet \mathrm{CH}_0(S)$ (See Section 6.1 for the definition).

Theorem 4.4 ([50],[41], Thms 6.2, 6.5)**.** (i) *Any two points $\mathcal{E}, \mathcal{E}' \in M_\sigma(v)$ are rational equivalent in $M_\sigma(v)$ if and only if $\mathrm{ch}_2(\mathcal{E}') = \mathrm{ch}_2(\mathcal{E}) \in \mathrm{CH}_0(S)$.*

(ii) *Let $\mathcal{E} \in M_\sigma(v)$ such that $\mathrm{ch}_2(\mathcal{E}) \in S_i \mathrm{CH}_0(S)$. Then $\mathcal{E} \in F_iM_\sigma(v)$. If $M_\sigma(v)$ is birational to the Hilbert scheme $S^{[n]}$, then also the converse implication holds true, i.e. in this case*

$$F_iM_\sigma(v) = \{\mathcal{E} \in M_\sigma(v) \mid \mathrm{ch}_2(\mathcal{E}) \in S_i \mathrm{CH}_0(S)\}.$$

We remark that both parts of the theorem can equally be formulated with c_2 instead of ch_2.

This opens the door to finding infinitely many examples of constant cycle or algebraically coisotropic subvarieties in $M = M_H(0, 2H, -1)$. For example, a first straightforward application yields.

Lemma 4.5 (Cor 6.9)**.** *The fiber $F = f^{-1}(D)$ is a constant cycle Lagrangian if and only if $D \in |2H|$ is a constant cycle curve in S.*

Or one can prove, that given $\mathcal{E} \in M$ such that $\mathrm{Supp}(\mathcal{E}) = D$. Then $\mathcal{E} \in F_{g(\tilde{D})}M$, where $g(\tilde{D})$ is the geometric genus of D. Here, the geometric genus of D is the genus of the normalization of D (resp. of D_{red}) and the sum over the genera of the normalizations of the irreducible components if D is reducible. This way, we find algebraically coisotropic subvarieties over singular curves. Precisely, for $i = 0, \dots, 4$ let

$$V_i \coloneqq \{D \in |2H| \mid g(\tilde{D}) \leq i\} \subset |2H|$$

and set $M_{V_i} \coloneqq f^{-1}(V_i)$.

Proposition 4.6 (Prop 7.3)**.** *The subvarieties M_{V_i} are equidimensional of codimension $n - i$ and satisfy*

$$M_{V_i} \subset F_i M.$$

In particular, they are algebraically coisotropic.

We will see that V_i is reducible due to reducible and non-reduced curves in the linear system $|2H|$. For every component we find the isotropic fibration and comment on the resulting constant cycle subvarieties. Most of them are rational. However, over the component of non-reduced curves $\Delta \subset V_2$, we find three-dimensional constant cycle subvarieties that are not rational (cf. Proposition 7.6).

Another series of examples comes from Brill–Noether theory. Let $B^\circ \subset B$ be the locus of smooth curves and $\mathcal{C}^\circ \to B^\circ$ the restricted universal curve. For any i, we have an isomorphism

$$M_H(0, 2H, i-4)^\circ \cong \mathrm{Pic}^i_{\mathcal{C}^\circ/B^\circ},$$

where $M_H(0, 2H, i-4)^\circ$ is the preimage of B° under the support map $M_H(0, 2H, i-4) \to B$. For $i = 1, \dots 4$, we define

$$\mathrm{BN}^0_i(B^\circ) \coloneqq \{\mathcal{L} \in M_H(0, 2H, i-4)^\circ \mid H^0(S, \mathcal{L}) \neq 0\} \subset M_H(0, 2H, i-4)^\circ.$$

We consider the closures for odd i. Namely,

$$Z_1 \coloneqq \overline{\mathrm{BN}^0_1(B^\circ)} \subset M_H(0, 2H, -3) \quad \text{and} \quad Z_3 \coloneqq \overline{\mathrm{BN}^0_3(B^\circ)} \subset M \coloneqq M_H(0, 2H, -1).$$

As $M_H(0, 2H, -3)$ and M are isomorphic (Lemma 5.1), Z_1 can also be seen as subvarieties in M.

Proposition 4.7 (Prop 7.1)**.** *The subvarieties $Z_i \subset M, i = 1, 3$ have codimension $5 - i$ and satisfy*

$$Z_i \subset F_i M.$$

In particular, they are algebraically coisotropic.

Outline. In Section 5, we collect general results on the Mukai system and describe the nature of its fibers. This requires an analysis of the singular curves in $|2H|$. In Section 6, we state Theorems 4.3 and 4.4 in more detail and apply them to $M = M_H(0, 2H, -1)$. Section 7 is devoted to present explicit examples. These include the examples from Brill–Noether theory (Section 7.1), the examples from singular curves together with their isotropic fibrations (Section 7.2) and a less conceptual mixture of examples of constant cycle Lagrangians and examples in $S^{[5]}$ (Section 7.3).

5. The Mukai system

Let (S, H) be a polarized K3 surface of genus 2 such that the linear system $|H|$ contains a smooth irreducible curve, i.e. S is a double covering $\pi\colon S \to \mathbb{P}^2$ ramified over a smooth sextic curve $R \subset \mathbb{P}^2$ and $H = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is primitive. We consider the moduli space $M = M_H(0, 2H, s)$ of H-Gieseker stable coherent sheaves on S with Mukai vector $v = (0, 2H, s)$ where $s \equiv 1 \mod 2$. This is an irreducible holomorphic symplectic variety of dimension 10, which is birational to $S^{[5]}$, cf. Section 8. A point in $M_H(0, 2H, s)$ corresponds to a stable sheaf $\mathcal{E}$ on S such that $\mathcal{E}$ is pure of dimension one with support in the linear system $|2H|$ and $\chi(\mathcal{E}) = s$. Taking the (Fitting) support defines a Lagrangian fibration

$$f\colon M_H(0, 2H, s) \longrightarrow B \coloneqq |2H| \cong \mathbb{P}^5$$

known as the *Mukai system of rank two and genus two* [8], [44].

As tensoring with $\mathcal{O}_S(H)$ induces an isomorphism

$$\tau_H\colon M_H(0, 2H, s) \xrightarrow{\sim} M_H(0, 2H, s+4),$$

it is immediate that the isomorphism class of $M_H(0, 2H, s)$ depends only on s modulo 4. The following Lemma shows that actually the isomorphism class is the same for all odd s. If $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$ one could also characterize $M_H(0, 2H, s)$ for odd s as the unique birational model of $S^{[5]}$ admitting a Lagrangian fibration (cf. Section 10).

Lemma 5.1. *There is an isomorphism*

$$M(0, 2H, 1) \longrightarrow M(0, 2H, -1),\ \mathcal{E} \mapsto \mathcal{E}^\vee \coloneqq \mathcal{E}xt^1_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_S).$$

In particular, all the moduli spaces $M_H(0, 2H, s)$ for odd s are isomorphic.

Proof. Every $\mathcal{E} \in M(0, 2H, 1)$ is pure of dimension one. Therefore, $\mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_S) = 0$ for $i \neq 1$ and the natural map

$$\mathcal{E} \xrightarrow{\sim} \mathcal{E}^{\vee\vee} = \mathcal{E}xt^1_{\mathcal{O}_S}(\mathcal{E}xt^1_{\mathcal{O}_S}(\mathcal{E}, \mathcal{O}_S), \mathcal{O}_S)$$

is an isomorphism, [31, Prop 1.1.10]. Finally, one easily sees that $\mathcal{E}^\vee$ is again H-Gieseker stable. $\square$

In the following, we usually choose $s = -1$ and set

$$M \coloneqq M_H(0, 2H, -1).$$

With this choice of s, a stable vector bundle of rank two and degree one on a smooth curve $C \in |H|$ defines a point in M.

5.1. The linear systems $|H|$ and $|2H|$. The geometry of the Mukai system is closely related to the structure of the curves in the linear systems $|H|$ and $|2H|$, which we want to analyze in this section. A curve in the linear system $|H|$ (resp. $|2H|$) has geometric genus 2 (resp. 5). In (2.1), we already introduced the subloci

$$\Delta \coloneqq m(\Delta_{|H|}) \subset \Sigma \coloneqq \mathrm{im}(m) \subset |2H|,$$

where $m\colon |H| \times |H| \to |2H|$ comes from the Segre map. We have $\Sigma \cong \mathrm{Sym}^2 |H|$ and its generic member is reduced and has two smooth irreducible components in the linear system $|H|$ meeting transversally in two points. The subset $\Delta \cong |H| \cong \mathbb{P}^2$ is the locus of non-reduced curves.

Recall that $\pi\colon S \to |H| \cong \mathbb{P}^2$ is a double covering, which is ramified along a smooth sextic curve $R \subset \mathbb{P}^2$. We have

$$H^0(S, \mathcal{O}_S(kH)) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) \oplus H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)),$$

and so in particular

$$H^0(S, \mathcal{O}_S(kH)) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k)) \text{ if } k = 1, 2.$$

We conclude that every curve in $|H|$ (resp. in $|2H|$) is the pullback of a line ℓ (resp. a quadric Q) in $\mathbb{P}^2$. In particular, every curve in $|H|$ (resp. in $|2H|$) has singularities depending on the intersection behavior of the ramification sextic R with ℓ (resp. Q) and has at most two (resp. four) irreducible components. For example, let $\ell \subset \mathbb{P}^2$ be a line and $C \coloneqq \pi^{-1}(\ell) \in |H|$. Assume that C is reducible. Then C consists of two irreducible components C_1 and C_2, each isomorphic to $\mathbb{P}^1$ with $C_1.C_2 = 3$. This is only possible if $\rho(S) \geq 2$. If S is general, then all curves $C \in |H|$ are irreducible. We give a complete list of the possible singularities in the following table.

$\#(\ell \cap R)_{\text{red}}$	intersection multiplicities	$g(\tilde{C})$	C irred. ?	singularities of C
5	$1,1,1,1,2$	1	$\surd$	1 node
4	$1,1,2,2$	0	$\surd$	2 nodes
4	$1,1,1,3$	1	$\surd$	1 cusp
3	$2,2,2$	0	$\times$	3 transversal intersection pts.
3	$1,2,3$	0	$\surd$	1 node &1 cusp
3	$1,1,4$	0	$\surd$	1 A_4-singularity
2	$3,3$	0	$\surd$	2 cusps
2	$2,4$	0	$\times$	2 intersection pts. with mult. 2 and 1 resp.
2	$1,5$	0	$\surd$	1 A_5-singularity
1	6	0	$\times$	1 intersection pt. with mult. 3

For an ample line bundle L on S and $0 \leq i \leq \frac{L^2}{2} + 1$, we can consider the closed subvariety

$$V(i, |L|) \coloneqq \{D \in |L| \mid g(\tilde{D}) \leq i\} \subset |L|, \tag{5.1}$$

which is called a (generalized) Severi variety. We have $\dim V(i, |L|) \leq i$ and there are various results about non-emptiness, irreducibility or smoothness of $V(i, |L|)$ in the literature, e.g. [17]. However, most of the results apply to a primitive linear system on a general K3 surface, but not to our situation, where we deduce a description of $V(i, |H|)$ from the above table.

Corollary 5.2. *The varieties $V(i, |H|)$ are non-empty of dimension i for $i = 0, 1$. Moreover, $V(1, |H|)$ is irreducible and the locus of nodal curves is dense in $V(1, |H|)$.*
If (S, H) is general. Then $V(1, |H|) \subset |H| \cong \mathbb{P}^2$ is a nodal curve of degree 30 and $V(0, |H|)$ consists of 324 points. Any curve in $V(1, |H|) \setminus V(0, |H|)$ is irreducible and has exactly one node or one cusp as singularities. Any curve in $V(0, |H|)$ has exactly two nodes as singularities.

Proof. It follows from the above table that $V(1, |H|)$ is parameterized by the tangents of R, i.e.

$$V(1, |H|) \cong R^\vee \subset (\mathbb{P}^2)^\vee = |\mathcal{O}_{\mathbb{P}^2}(1)|,$$

and the dual sextic $R^\vee$ has degree 30 if R is smooth. A curve in $V(1, |H|)$ is nodal if it corresponds to a tangent line that is tangent to R in exactly one point. Hence, this locus is dense. A general smooth sextic has exactly 324 bitangents [26, IV Ex. 2.3]. $\square$

Next, we study the linear system $|2H|$ and define

$$V_i \coloneqq V(i, |2H|) \tag{5.2}$$

for $i = 0, \ldots, 5$. Recall that $m\colon |H| \times |H| \to |2H|$ was the map coming from the Segre embedding. We set

$$\Sigma_{\{i,j\}} \coloneqq m(V(i, |H|) \times V(j, |H|)) \subset V_{i+j}$$

for $0 \leq i \leq j \leq 2$ and

$$\Delta_1 \coloneqq m(\Delta_{V(1,|H|)}) \subset V_1,$$

i.e. $\Sigma_{\{i,j\}} \subset \Sigma$ is the locus of reducible curves, whose components have geometric genus bounded by i and j, respectively and $\Delta_1 \subset \Delta$ is the locus of non-reduced curves, with underlying singular curve. We keep writing Σ for $\Sigma_{\{2,2\}}$.

Corollary 5.3. *We have*

$$\dim \Sigma_{\{i,j\}} = i + j \text{ and } \dim \Delta_1 = 1.$$

Moreover, $\Sigma_{\{i,j\}}$ and Δ_1 are irreducible if $i \neq 0$ and $\Sigma_{\{0,j\}}$ has 324 irreducible components.

Finally, we let

$$\Lambda_i \coloneqq \overline{\{D \in V_i \mid D \text{ is integral}\}} \subset V_i.$$

The same considerations leading to the above table in the case of quadrics instead of lines in $\mathbb{P}^2$ show that Λ_i is an irreducible subvariety of dimension i. Moreover, a general curve in Λ_i has exactly $5 - i$ nodes as its only singularities. We sum up our discussion in the following proposition.

Proposition 5.4. *The Severi varieties $V_i \subset |2H|$ are non-empty of pure dimension i. Their irreducible components correspond to integral, reducible and non-reduced curves, respectively. More precisely, we have*

$$\begin{aligned} V_4 &= \Lambda_4 \cup \Sigma \\ V_3 &= \Lambda_3 \cup \Sigma_{\{1,2\}} \\ V_2 &= \Lambda_2 \cup \Sigma_{\{0,2\}} \cup \Sigma_{\{1,1\}} \cup \Delta \\ V_1 &= \Lambda_1 \cup \Sigma_{\{0,1\}} \cup \Delta_1. \end{aligned}$$

Here, all varieties occurring on the right hand side but $\Sigma_{\{0,2\}}$ and $\Sigma_{\{0,1\}}$ are irreducible. $\square$

Note that $V_4 = \Lambda_4 \cup \Sigma \subset |2H| \cong \mathbb{P}^5$ is the discriminant divisor of f. We compute the degree of its components.

Lemma 5.5. *We have*

$$\deg[\Sigma] = 3 \quad \text{and} \quad \deg[\Lambda_4] = 42.$$

In particular, the discriminant divisor of f has degree 45.

Proof. The easiest way, to see that $\deg[\Sigma] = 3$ is a geometric argument. Choose 4 points $x_1, \ldots, x_4$ in general position and consider the line $\ell = \{D \in |2H| \mid x_i \in D \text{ for all } i = 1, \ldots 4\}$. There is a unique (resp. no) curve $C \in |H|$ passing through two (resp. three) points in general position. Hence, $\deg \Sigma = \#(\ell \cap \Sigma) = 3$, corresponding to the three possible partitions of

$x_1, \ldots, x_4$ into pairs of two points. Alternatively, after a choice of coordinates $\Sigma \cong \operatorname{Sym}^2 \mathbb{P}^2$ is embedded into $\mathbb{P}^5$ via the map induced by

$$\begin{aligned} \mathbb{P}^2 \times \mathbb{P}^2 &\to \mathbb{P}^5 \\ [x_0 : x_1 : x_2], [y_0 : y_1 : y_2] &\mapsto [x_0y_0 : x_1y_1 : x_2y_2 : x_0y_1 + x_1y_0 : x_0y_2 + x_2y_0 : x_1y_2 + x_2y_1]. \end{aligned} \tag{5.3}$$

One checks that the image is cut out by the equation,

$$f_0f_5^2 + f_1f_4^2 + f_2f_3^2 = 4f_0f_1f_2 + f_3f_4f_5,$$

where the coordinates f_i of $\mathbb{P}^5$ are ordered as in (5.3).
To prove $\deg[\Lambda_4] = 45$, we use the computation from [49, §5]. Let $\mathcal{C}' = \bigcup_{t\in\mathbb{P}^1} C_t$ be a general pencil of curves in the linear system $B = |2H|$, i.e. $\mathcal{C}' = \mathcal{C} \cap (S \times \mathbb{P}^1)$, where $\mathcal{C} \subset S \times B$ is the universal curve and $\mathbb{P}^1 \subset B$ a general line. Then $\mathcal{C}' \subset S \times \mathbb{P}^1$ is defined by a section $s \in H^0(S \times \mathbb{P}^1, \mathcal{O}_S(2H) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$ and

$$\mathcal{C}'_{\mathrm{sing}} \coloneqq V(s \oplus ds) = \bigcup_{t\in\mathbb{P}^1} (C_t)_{\mathrm{sing}}$$

is the union of the singular points of C_t, where $ds \in H^0(S \times \mathbb{P}^1, \Omega_S(2H) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))$. We compute

$$\deg c_3((\mathcal{O}_S(2H) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)) \oplus (\Omega_S(2H) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1))) = 48, \tag{5.4}$$

i.e. $\mathcal{C}'_{\mathrm{sing}}$ consists of 48 points. As a general pencil contains three curves in Σ which have two singular points and a generic integral singular curve has exactly one nodal singularity, we conclude $\deg \Lambda_4 = 42$. □

Remark 5.6. In [49, §5] the computation (5.4) serves as a demonstration for a formula of the degree of the discriminant locus of a Lagrangian fibration with 'good singular fibers'. An example of such a fibration is the Beauville–Mukai system over a primitive curve class and the discriminant divisor is irreducible of degree $6(n + 3)$, where n is the dimension of the base of the fibration. However, in our example the fibers over Δ are not 'good singular fibers' and we find a different result.

5.2. Fibers of the Mukai morphism and structure of M. In this section, we collect some information on the fibers of the Mukai morphism. Part of this has already appeared in Section 2.

The moduli space $M = M_H(0, 2H, -1)$ contains a dense open subset consisting of the sheaves that are line bundles on their support. The restriction of the Mukai morphism to this locus is smooth [38, Prop 2.8] and the image of the restricted morphism is $B \setminus \Delta$ [13, Lem 3.5.3]. In particular, $M_\Sigma \coloneqq f^{-1}(\Sigma)$ contains a dense open subset that parameterizes the push forwards of line bundles, but $M_\Delta \coloneqq f^{-1}(\Delta)$ does not.

Following [13, Proposition 3.7.1], we can give a description of the fibers of the Mukai morphism. To this end, first assume that $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$. The fibers of the Mukai morphism $f\colon M \to B$ show the following characteristics:

$$f^{-1}(x) = \begin{cases} \text{is reduced and irreducible} & \text{if } x \in B \setminus \Sigma \\ \text{is reduced and has two irreducible components} & \text{if } x \in \Sigma \setminus \Delta \\ \text{has two irreducible components with multiplicities} & \text{if } x \in \Delta. \end{cases} \tag{5.5}$$

Let us make this more precise for generic points:

- In the first case, let $x \in B \setminus \Sigma$ correspond to a smooth curve D, then $f^{-1}(x) \cong \mathrm{Pic}^3(D)$.
- In the second case, let $x \in \Sigma \setminus \Delta$ correspond to the union $D = D_1 \cup D_2$ of two smooth curves meeting transversally in two points. Then $f^{-1}(x)$ contains a dense open subset parameterizing line bundles on D. The two irreducible components of $f^{-1}(x)$ correspond to line bundles with partial degree $(2, 1)$ and $(1, 2)$.
- In the third case, let $x \in \Delta$ correspond to a non-reduced curve with smooth underlying curve $C \in |H|$. Then $f^{-1}(x)$ has two non-reduced irreducible components, which we denote as follows

$$M_{2C} \coloneqq f^{-1}(x)_{\mathrm{red}} = M^0_{2C} \cup M^1_{2C}. \tag{5.6}$$

  The first component M^0_{2C} consists of those sheaves, that are pushed forward from the reduced curve C. With its reduced structure it is isomorphic to the moduli space of stable vector bundles of rank two and degree one on C. The other component M^1_{2C} is the closure of those sheaves that can not be endowed with an $\mathcal{O}_C$-module structure. All these sheaves fit into a short exact sequence

$$0 \to i_*(\mathcal{L}(x) \otimes \omega_C^{-1}) \to \mathcal{E} \to i_*\mathcal{L} \to 0, \tag{5.7}$$

  where $i\colon C \hookrightarrow S$ is the inclusion, and $\mathcal{L} \in \mathrm{Pic}^1(C)$ is the torsionfree part of $\mathcal{E}|_C$ and $x \in C$ is the support of the torsion part of $\mathcal{E}|_C$. This extension is intrinsically associated to $\mathcal{E}$, see Part I.

In the case of a K3 surface of higher Picard rank, the general picture remains the same. But due to reducible curves in the linear system $|H|$ or $B \setminus \Sigma$, the fibers could exceptionally have more irreducible components. For example, if $x \in B \setminus \Sigma$ corresponds to a reducible curve with two smooth components, then $f^{-1}(x)$ still contains a dense open subset parameterizing line bundles. However, following [13, Lem 3.3.2] one finds, that the numerical restrictions imposed by the stability now allow partial degrees $(5, -1), (4, 0), (3, 1), (2, 2), (1, 3), (0, 4), (-1, 5)$. Thus, in this case $f^{-1}(x)$ has seven irreducible components.

The decomposition (5.6) also exists globally over the locus of curves $D = 2C$ with $C \in |H|$ smooth, which we denote by $\Delta^\circ \subset \Delta$. Here, we have

$$M_{\Delta^\circ} = f^{-1}(\Delta^\circ)_{\mathrm{red}} = M^0_{\Delta^\circ} \cup M^1_{\Delta^\circ},$$

where $M^0_{\Delta^\circ}$ is a relative moduli space of stable vector bundles and $M^1_{\Delta^\circ}$ the closure of its complement, [13, Proposition 3.7.23]. We set

$$M^0_{\Delta} = \overline{M^0_{\Delta^\circ}} \text{ and } M^1_{\Delta} = \overline{M^1_{\Delta^\circ}}. \tag{5.8}$$

6. Orbits under rational equivalence

Our strategy to find algebraically coisotropic subvarieties is to single out points whose orbit under rational equivalence has a high dimension. In this section, we explain how this can be done combining results of Voisin and Shen, Yin and Zhao.

6.1. Preliminaries. We start by recalling some general definitions. Let (X, σ) be an irreducible holomorphic symplectic manifold of dimension $2n$. For a smooth subvariety $Y \subset X$, we let

$$\mathcal{T}_Y^{\perp} \coloneqq \ker(\mathcal{T}_X \xrightarrow{\sim} \Omega_X \twoheadrightarrow \Omega_Y),$$

where the first arrow is given by σ.

(i) A subvariety $Y \subset X$ is a *constant cycle subvariety* [33] if all its points are rationally equivalent in X. Note that this is the case, if Y contains a dense open subset U, such that all points in U are rationally equivalent in X. Mumford's theorem [45] implies that a constant cycle subvariety Y is isotropic [55, Cor 1.2], i.e.

$$\mathcal{T}_{Z_{\mathrm{reg}}} \subset \mathcal{T}^{\perp}_{Z_{\mathrm{reg}}} \text{ or equivalently } \sigma|_{Y_{\mathrm{reg}}} = 0.$$

In particular, $\dim Y \leq n$ and if $\dim Y = n$, then Y is a Lagrangian subvariety.

(ii) A subvariety $Z \subset X$ is *algebraically coisotropic* [55, Def 0.6] if Z is coisotropic (i.e. $\mathcal{T}^{\perp}_{Z_{\mathrm{reg}}} \subset \mathcal{T}_{Z_{\mathrm{reg}}}$) and the corresponding foliation is algebraically integrable. For a subvariety of codimension i, this is equivalent to the existence of a $2n-2i$-dimensional variety T and a rational surjective map $\phi\colon Z \dashrightarrow T$ such that

$$\mathcal{T}^{\perp}_{Z_{\mathrm{reg}}} \cong \mathcal{T}_{Z/T} \text{ (where defined) or equivalently } \sigma|_Z = \phi^*\sigma_T \text{ for some } (2,0) \text{ form } \sigma_T \text{ on } T.$$

Actually, T and ϕ are unique up to birational equivalence. We call ϕ the associated isotropic fibration.

(iii) For a point $x \in X$, its *orbit under rational equivalence* is

$$O_x \coloneqq \{x' \in X \mid [x] = [x'] \in \mathrm{CH}_0(X)\} \subset X,$$

which is a countable union of closed algebraic subvarieties [53, Lem 10.7]. Its dimension is defined to be the supremum over the dimensions of its irreducible components.

Following [55, Def 0.2], we set

$$F_iX \coloneqq \{x \in X \mid \dim O_x \geq n - i\}^{1}.$$

for $i = 0, \ldots, n$. This is again a countable union of closed algebraic subvarieties and defines a filtration on the points of X

$$F_0X \subset F_1X \subset \cdots \subset F_nX = X. \tag{6.1}$$

From [55, Thm 1.3] it is known that

$$\dim F_iX \leq n + i \tag{6.2}$$

and conjecturally [55, Conj 0.4] equality holds true. The following theorem says that a component of maximal dimension is algebraically coisotropic.

Theorem 6.1 ([55, Thm 0.7])**.** *Let $Z \subset X$ be a subvariety of codimension $n - i$ such that $Z \subset F_iX$, then Z is algebraically coisotropic and the fibers of the associated isotropic fibration $\phi\colon Z \dashrightarrow T$ are constant cycle subvarieties of dimension $n - i$.*

Now, let $X = M_\sigma(v)$ be a smooth, projective moduli space of (Bridgeland-)stable objects in $D^b(S)$. In this situation, we have the following beautiful criterion for rational equivalence.

Theorem 6.2 ([41],[50, Conj 0.3])**.** *Two points $\mathcal{E}, \mathcal{E}' \in M_\sigma(v)$ satisfy*

$$[\mathcal{E}] = [\mathcal{E}'] \in \mathrm{CH}_0(M_\sigma(v))$$

if and only if

$$\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(\mathcal{E}') \in \mathrm{CH}_0(S).$$

Remark 6.3. As $\mathrm{ch}_2(\mathcal{E}) = \frac{1}{2}c_1(\mathcal{E})^2 - c_2(\mathcal{E})$, and $c_1(\mathcal{E})$ is fixed for all $\mathcal{E} \in M_\sigma(v)$, one could also phrase the theorem using c_2.

In particular, for $\mathcal{E} \in M_\sigma(v)$ we have

$$O_{\mathcal{E}} = \{\mathcal{E}' \in M_\sigma(v) \mid \mathrm{ch}_2(\mathcal{E}') = \mathrm{ch}_2(\mathcal{E}) \in \mathrm{CH}_0(S)\} \subset M_\sigma(v).$$

Using that the union of all constant cycle curves in S is Zariski dense and Theorem 6.2 allows one to prove.

Theorem 6.4 ([50, Thm 0.5(i)])**.** *For all $0 \leq i \leq n$ there is an algebraically coisotropic subvariety $Z \subset M_\sigma(v)$ of codimension i such that the isotropic fibration $Z \dashrightarrow T$ has generically constant cycle fibers of dimension i. In particular,*

$$\dim F_iM_\sigma(v) = n + i,$$

i.e. (6.2) *is actually an equality.*

[1] We reversed the numbering from [55].

Next, one could ask how the filtration $F_i M_\sigma(v)$ interferes with the second Chern classes. The answer is to consider O'Grady's filtration on $\mathrm{CH}_0(S)$. Let us recall some results about $\mathrm{CH}_0(S)$.

In [9], Beauville and Voisin prove that any point lying on a rational curve in S determines the same class

$$c_S \in \mathrm{CH}_0(S),$$

which has the property that the image of the intersection product $\mathrm{Pic}(S) \otimes \mathrm{Pic}(S) \to \mathrm{CH}_0(S)$ is contained in $\mathbb{Z} \cdot c_S$. In [48], building on this class, now called the *Beauville–Voisin class*, O'Grady introduces an increasing filtration $S_\bullet$ on $\mathrm{CH}_0(S)$,

$$S_0 \,\mathrm{CH}_0(S) \subset S_1 \,\mathrm{CH}_0(S) \subset \ldots \subset S_i \,\mathrm{CH}_0(S) \subset \ldots \subset \mathrm{CH}_0(S),$$

where $S_i \,\mathrm{CH}_0(S)$ is the union of cycles of the form $[z] + d \cdot c_S$ for some effective zero-cycle z of degree i and $d \in \mathbb{Z}$. In particular, $S_0 \,\mathrm{CH}_0(S) = \mathbb{Z} \cdot c_S$. O'Grady's filtration $S_\bullet \,\mathrm{CH}_0(S)$ has several useful properties, [48, Cor. 1.7 and Claim 0.2]:

(1) The filtration is compatible with addition, i.e. if $\alpha \in S_i \,\mathrm{CH}_0(S)$ and $\beta \in S_j \,\mathrm{CH}_0(S)$, then $\alpha + \beta \in S_{i+j} \,\mathrm{CH}_0(S)$.
(2) Each step of the filtration $S_i \,\mathrm{CH}_0(S)$ is closed under multiplication with $\mathbb{Z}$, i.e. if $\alpha \in S_i \,\mathrm{CH}_0(S)$ then $m \cdot \alpha \in S_i \,\mathrm{CH}_0(S)$ for every $m \in \mathbb{Z}$.
(3) If C is an irreducible, smooth projective curve and $f\colon C \to S$. Then

$$f_* \,\mathrm{CH}_0(C) \subset S_{g(C)} \,\mathrm{CH}_0(S).$$

Theorem 6.5 ([50, Thm 0.5(ii)])**.** *Let $M_\sigma(v)$ be a smooth projective moduli space of Bridgeland stable objects in $D^b(S)$ with $\dim M_\sigma(v) = 2n$. If $\mathcal{E} \in M_\sigma(v)$ satisfies $\mathrm{ch}_2(\mathcal{E}) \in S_i \,\mathrm{CH}_0(S)$, then $\mathcal{E} \in F_i M_\sigma(v)$. Moreover, if $M_\sigma(v)$ is birational to the Hilbert scheme $S^{[n]}$, then also the converse implication holds true, i.e. in this case*

$$F_i M_\sigma(v) = \{\mathcal{E} \in M_\sigma(v) \mid \mathrm{ch}_2(\mathcal{E}) \in S_i \,\mathrm{CH}_0(S)\}.$$

Proof. We sketch the proof along the lines of [50, Proof of Thm 0.5(ii)], where the first part of the theorem is proven. The case of $S^{[n]}$, i.e. that for all $\xi \in S^{[n]}$

$$\dim O_\xi \geq n - i \iff [\mathrm{Supp}(\xi)] \in S_i \,\mathrm{CH}_0(S)$$

is proven in [54, Thm 1.4]. Note that only the implication from left to right needs a proof. The other implication follows because any point representing the Beauville-Voisin lies on a rational curve and hence if $\mathrm{Supp}(\xi)$ contains $(n-i) \cdot c_S$, we have $\dim O_\xi \geq n - i$.
For the general case, let $\mathcal{E} \in M = M_\sigma(v)$. By [50, Thm 0.1], we can write

$$\mathrm{ch}_2(\mathcal{E}) = [\mathrm{Supp}(\xi)] + d \cdot [c_S] \in \mathrm{CH}_0(S)$$

for some $\xi \in S^{[n]}$ and $d \in \mathbb{Z}$ depending on the degree of $\mathrm{ch}_2(\mathcal{E})$, which is fixed. After knowing the result for $S^{[n]}$, the theorem translates into the statement

$$\dim O_\xi \geq n-i \Rightarrow \dim O_{\mathcal{E}} \geq n-i \quad (\text{resp. } \dim O_\xi \geq n-i \Leftrightarrow \dim O_{\mathcal{E}} \geq n-i, \text{ if } M \sim_{\mathrm{bir}} S^{[n]}).$$

The two orbits can be compared by means of the incidence variety

$$R = \{(\mathcal{E}, \xi) \in M \times S^{[n]} \mid \mathrm{ch}_2(\mathcal{E}) = [\mathrm{Supp}(\xi)] + d \cdot [c_S] \in \mathrm{CH}_0(S)\},$$

which is a countable union of Zariski closed subsets in $M \times S^{[n]}$. There exists an irreducible component $R_0 \subset R$ which projects generically finite and surjective to both factors, and hence yields a correspondence between the two orbits. However, in order to compare their dimensions, one needs to know that the components of maximal dimension in every orbit under rational equivalence are dense. This is known for the Hilbert scheme, whence the inclusion $S_i^{\mathrm{SYZ}} \mathrm{CH}_0(M) \subset S_i^{\mathrm{V}} \mathrm{CH}_0(M)$ always holds. The reverse inclusion is true if M is birational to $S^{[n]}$ but in general not known. □

6.2. Orbits under rational equivalence in M. We turn back to our favorite example $M = M_H(0, 2H, -1)$ with the goal in mind, to give explicit constructions of constant cycle subvarieties in M. The first step is to understand the orbits under rational equivalence in M and the filtration

$$F_0M \subset F_1M \subset \ldots \subset F_5M = M.$$

Recall that M is birational to $S^{[5]}$ and thus by Theorem 6.5, we know

$$F_iM = \{\mathcal{E} \in M \mid \dim O_{\mathcal{E}} \geq 5-i\} = \{\mathcal{E} \in M \mid \mathrm{ch}_2(\mathcal{E}) \in S_i \mathrm{CH}_0(S)\}$$

and

$$\dim F_iM = 5+i$$

for $0 \leq i \leq 5$. The following is a straightforward computation using the Grothendieck–Riemann–Roch theorem.

Lemma 6.6. *Let $i\colon D \hookrightarrow S$ be a reduced curve and let $\mathcal{F}$ be a vector bundle on D.*

(i) *Assume that D is irreducible and let $\nu\colon \tilde{D} \to D$ be its normalization. Then*

$$\mathrm{ch}_2(i_*\mathcal{F}) = i_*\nu_*c_1(\nu^*\mathcal{F}) - \mathrm{rk}(\mathcal{F})(\tfrac{1}{2}i_*\nu_*c_1(\omega_{\tilde{D}}) - \sum_{p \in D} m_p[p]) \in \mathrm{CH}_0(S), \tag{6.3}$$

where $m_p = \mathrm{lg}(\nu_*\mathcal{O}_{\tilde{D}}/\mathcal{O}_D)_p$. In particular,

$$\mathrm{ch}_2(i_*\mathcal{F}) \in \mathrm{im}(\mathrm{CH}_0(\tilde{D}) \xrightarrow{i_*\nu_*} \mathrm{CH}_0(S)) \subset S_{g(\tilde{D})} \mathrm{CH}_0(S).$$

(ii) *Assume that $D = D_1 \cup D_2$ has two irreducible components. Then*

$$\mathrm{ch}_2(i_*\mathcal{F}) = \mathrm{ch}_2(i_{1*}\mathcal{F}|_{D_1}) + \mathrm{ch}_2(i_{2*}\mathcal{F}|_{D_2}) - \mathrm{rk}(\mathcal{F})(D_1.D_2)c_S \in \mathrm{CH}_0(S), \tag{6.4}$$

were $i_k\colon D_k \hookrightarrow S, k = 1,2$ *are the inclusions of the components. In particular,*

$$\mathrm{ch}_2(i_*\mathcal{F}) \in S_{g(\tilde{D_1})+g(\tilde{D_2})}\,\mathrm{CH}_0(S).$$

$\square$

Example 6.7. Using Lemma 6.6 we compute $\mathrm{ch}_2(\mathcal{E})$ for some cases of stable sheaves $\mathcal{E}$ occuring in M:

(i) Let $\mathcal{E} \in M$ such that $D = \mathrm{Supp}(\mathcal{E})$ is smooth, then $\mathcal{E} = i_*\mathcal{L}$, where $i\colon D \hookrightarrow S$ is the inclusion and $\mathcal{L} \in \mathrm{Pic}^3(D)$. We find

$$\mathrm{ch}_2(\mathcal{E}) = -4c_S + i_*c_1(\mathcal{L}). \tag{6.5}$$

(ii) Let $\mathcal{E} \in M$ be the pushforward of a line bundle $\mathcal{L}$ on its support $D = \mathrm{Supp}\,\mathcal{E}$ and assume that $D = D_1 \cup D_2$ has two smooth and connected components. We write $\mathcal{E} = i_*\mathcal{L}$, then

$$\mathrm{ch}_2(\mathcal{E}) = -4c_S + i_{1*}c_1(\mathcal{L}|_{D_1}) + i_{2*}c_1(\mathcal{L}|_{D_2}), \tag{6.6}$$

where $i_k\colon D_k \hookrightarrow C, k = 1,2$ are the inclusions.

(iii) Let $\mathcal{E} \in M^0_{2C}$ for a smooth curve $C \in |H|$, i.e. $\mathrm{Supp}(\mathcal{E}) = 2C$ and $\mathcal{E} = i_*\mathcal{E}_0$, where $i\colon C \hookrightarrow S$ is the inclusion and $\mathcal{E}_0$ is a vector bundle of rank 2 and degree 1 on C. Then

$$\mathrm{ch}_2(\mathcal{E}) = -2c_S + i_*c_1(\mathcal{E}_0). \tag{6.7}$$

(iv) Let $\mathcal{E} \in M^1_{2C} \setminus M^0_{2C}$ for a smooth curve $C \in |H|$, i.e. $\mathrm{Supp}(\mathcal{E}) = 2C$ but $\mathcal{E}$ is not pushed forward along the inclusion $i\colon C \hookrightarrow S$. However, $\mathcal{E}$ fits into a short exact sequence

$$0 \to i_*(\mathcal{L}(x) \otimes \omega_C^{-1}) \to \mathcal{E} \to i_*\mathcal{L} \to 0,$$

for some $\mathcal{L} \in \mathrm{Pic}^1(C)$ and $x \in C$. Hence

$$\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(i_*(\mathcal{L}(x) \otimes \omega_C^{-1})) + \mathrm{ch}_2(i_*\mathcal{L}) = -4c_S + [i(x)] + 2i_*c_1(\mathcal{L}). \tag{6.8}$$

Corollary 6.8. *Let* $\mathcal{E} \in M$ *and let* $D = \mathrm{Supp}(\mathcal{E})$. *Then*

$$\mathrm{ch}_2(\mathcal{E}) \in F_{g(\tilde{D})}M,$$

where $g(\tilde{D})$ *is the geometric genus of* D. $\square$

The geometric genus of D is the genus the normalization of D (resp. of D_{red}) and the sum over the genera of the normalizations of the irreducible components if D is reducible.

Corollary 6.9. *The fiber* $F = f^{-1}(D)$ *over a curve* $D \in |2H|$ *is a constant cycle Lagrangian if and only if* D *is a constant cycle curve in* S. *If* $D \in \Delta$ *this means that the underlying reduced curve is constant cycle.*

Proof. It suffices to consider a dense open subset of F, in order to decide whether F is a constant cycle subvariety. First assume that $i\colon D \hookrightarrow S$ is reduced. Then F contains a dense open subset parameterizing line bundles of fixed degree. In Lemma 6.6 we saw that the class of $i_*\mathcal{L}$ in $\mathrm{CH}_0(M)$ depends on

$$\mathrm{Pic}^k(D) \to \mathrm{CH}_0(S),\ \mathcal{L} \mapsto i_*c_1(\mathcal{L}),$$

which is constant if D is a constant cycle curve. Conversely, assume that $F \subset M$ is a constant cycle subvariety. Then in particular,

$$i_*c_1(\mathcal{O}_D(kx)) = ki_*[x] \in S_0\,\mathrm{CH}_0(S)$$

and hence $[x] = c_S$ for all $x \in D$. (We use that $\mathrm{CH}_0(S)$ is torsionfree).
If $D = 2C$ is non-reduced, we apply the same argument to the explicit description (5.6) of the fiber F. □

7. Algebraically coisotropic subvarieties in M

We give several examples of algebraically coisotropic subvarieties in $M = M_H(0, 2H, -1)$.

7.1. Horizontal examples from Brill–Noether loci. Brill–Noether theory allows one to produce examples of constant cycle subvarieties.

Let $B^\circ \subset B$ be the locus of smooth curves and $\mathcal{C}^\circ \to B^\circ$ the restricted universal curve. For any k, we have an isomorphism

$$M_H(0, 2H, k-4)^\circ \cong \mathrm{Pic}^k_{\mathcal{C}^\circ/B^\circ},$$

where $M_H(0, 2H, k-4)^\circ$ is the preimage of B° under the support map $M_H(0, 2H, k-4) \to B$. For $k = 1, 3$, we define

$$\mathrm{BN}^0_k(B^\circ) \coloneqq \{\mathcal{L} \in M_H(0, 2H, k-4)^\circ \mid H^0(S, \mathcal{L}) \neq 0\} \subset M_H(0, 2H, k-4)^\circ$$

and consider the closures

$$Z_1 \coloneqq \overline{\mathrm{BN}^0_1(B^\circ)} \subset M_H(0, 2H, -3) \quad \text{and} \quad Z_3 \coloneqq \overline{\mathrm{BN}^0_3(B^\circ)} \subset M. \tag{7.1}$$

One can show that Z_3 is strictly contained in $\mathrm{BN}^0(M) \coloneqq \{\mathcal{E} \in M \mid H^0(S, \mathcal{E}) \neq 0\}$ as $\mathrm{BN}^0(M)$ has an additional component over Δ.

Proposition 7.1. *The subvarieties $Z_i \subset M_H(0, 2H, i-4)$ have codimension $5 - i$ for $i = 1, 3$ and satisfy*

$$Z_i \subset F_i M_H(0, 2H, i-4).$$

In particular, they are algebraically coisotropic.

Proof. A point in $\mathrm{BN}_i^0(B^\circ)$ is of the form $\mathcal{E} = i_*\mathcal{O}_D(\xi)$, where $D \in B^\circ$ and $\xi \subset D$ is an effective divisor of degree i. Hence

$$\mathrm{ch}_2(\mathcal{E}) \equiv [\mathrm{Supp}(\xi)] \bmod \mathbb{Z}\cdot c_S$$

in $\mathrm{CH}_0(S)$ and we conclude $\mathrm{ch}_2(\mathcal{E}) \in S_i \mathrm{CH}_0(S)$, which in turn gives $Z_i \subset F_i M_H(0, 2H, i-4)$. By Theorem 6.2, this implies $\dim Z_i \leq 5+i$, whereas the reverse inequality is known from Brill–Noether theory [2, IV Lem 3.3]. □

In Proposition 9.6, we prove that Z_1 is actually a projective bundle over S. Precisely, let $D \in |2H|$ and $\mathcal{L} \in Z_1 \cap f^{-1}(D)$, i.e. $\mathcal{L} \in \mathrm{Pic}^1(D)$ is effective and can uniquely be written as $\mathcal{O}_D(x)$ for some $x \in D$. This way, Z_1 is isomorphic the universal curve $\mathcal{C} \subset |2H| \times S$, which is a $\mathbb{P}^4$ bundle with respect to the second projection. With the same arguments, we also prove that Z_3 is generically a $\mathbb{P}^2$-bundle over $S^{[3]}$, which parameterizes the line bundles $\mathcal{O}_D(\xi)$ over $\xi \in S^{[3]}$.

In the following, we will consider Z_1 as a subvariety of M via the isomorphism

$$M_H(0, 2H, -3) \to M,\ \mathcal{E} \mapsto \mathcal{E}xt^1(\mathcal{E}, \mathcal{O}_S) \otimes \mathcal{O}_S(-H). \tag{7.2}$$

In particular, over a smooth curve $D \in |2H|$, we have

$$\mathrm{Pic}^1(D) \to \mathrm{Pic}^3(D),\ \mathcal{L} \mapsto \mathcal{L}^\vee \otimes \mathcal{O}_S(H)|_D.$$

Lemma 7.2. *We have*

$$Z_1 \subset \{\mathcal{E} \in M \mid h^0(\mathcal{E}) \geq 2\}.$$

In particular, there is an inclusion

$$Z_1 \subset Z_3.$$

Proof. It suffices to show the result over a smooth curve $D \in |2H|$. Let $\mathcal{L} \in \mathrm{Pic}^1(D)$ such that $H^0(D, \mathcal{L}) \neq 0$. We want to show that $\dim H^0(D, \mathcal{L}^\vee \otimes \mathcal{O}_S(H)|_D) \geq 2$. Write $\mathcal{L} = \mathcal{O}_D(x)$ for a point $x \in D$. On S, we have a short exact sequence

$$0 \to \mathcal{O}_S(-H) \to \mathcal{I}_x(H) \to \mathcal{O}_D(-x) \otimes \mathcal{O}_S(H)|_D \to 0$$

and the resulting long exact cohomology sequence proves the lemma. □

One can also define Z_1 directly as a subvariety of M. In Corollary 10.5, we prove that Z_1 is the closure of the Brill–Noether locus

$$\mathrm{BN}_3^1(B^\circ) \coloneqq \{\mathcal{L} \in M^\circ \mid h^0(S, \mathcal{L}) \geq 2\} \subset M.$$

Due to the non-primitivity of the linear system $|2H|$, unexpected things happen here. Namely, the smooth curves $D \in |2H|$ are hyperelliptic and we have $W_3^1(D) \neq \emptyset$ for all irreducible curves $D \in |2H|$, even though the Brill–Noether number $\rho(5,1,3) = 5 - 2(5-3+1)$ is negative (see also Remark 9.8).

7.2. Vertical examples from singular curves. In this section, we give examples of algebraically coisotropic subvarieties, that arise as preimages of subvarieties in B. In Corollary 6.9, we already treated the case of a fiber over a point $D \in B$. Namely, $f^{-1}(D)$ is a constant cycle Lagrangian, if and only if D is a constant cycle curve.

We set
$$M_{V_i} \coloneqq f^{-1}(V_i) \subset F_i M,$$
where $V_i \coloneqq \{D \in |2H| \mid g(\tilde{D}) \leq i\}$ for $i = 1, \ldots, 4$ was defined in (5.2).

Proposition 7.3. *The subvarieties M_{V_i} are equidimensional of codimension $5-i$ for $i = 1, \ldots, 4$ and satisfy*
$$M_{V_i} \coloneqq f^{-1}(V_i) \subset F_i M.$$
In particular, they are algebraically coisotropic.

Proof. We saw in Proposition 5.4 that $\dim V_i = i$ and in Corollary 6.8 that $g(\tilde{D}) \leq i$ implies that $f^{-1}(D) \subset F_i M$ for every $D \in |2H|$. □

In the following section, we find the isotropic fibrations for M_{V_i}.

7.2.1. Isotropic fibrations. In order to understand the constant cycle subvarieties resulting from the above examples, we write down the isotropic fibration for $M_\Sigma \subset M_{V_4}$ and $M_\Delta \subset M_{V_2}$ and $M_{\Lambda_i} \subset M_{V_i}$ for $i = 1, \ldots, 4$.

Proposition 7.4. *For every $i = 1, \ldots, 4$, there is a quasi-projective scheme T_i of dimension $2i$ fitting into a diagram*
$$\begin{tikzcd} M_{\Lambda_i} \arrow[rr, dashed, two heads, "\phi_i"] \arrow[dr, "f"'] & & T_i \arrow[dl] \\ & \Lambda_i. & \end{tikzcd}$$
The fibers of ϕ_i are rational constant cycle subvarieties of M of dimension $5-i$.

Proof. A general point in M_{Λ_i} is the pushforward of a line bundle on a singular curve in Λ_i. Its class in $\mathrm{CH}_0(S)$ however, depends only on the pullback of the line bundle to the normalization (cf. Lemma 6.6). This is what $M_{\Lambda_i} \dashrightarrow T_i$ encodes.
Consider the universal curve over $|2H|$ and let $\mathcal{C}_i \to \Lambda_i$ be its restriction to $\Lambda_i \subset |2H|$. By construction, the generic fiber of $\mathcal{C}_i$ is singular and so must be the total space $\mathcal{C}_i$. Hence, the normalization
$$\tilde{\mathcal{C}}_i \to \mathcal{C}_i$$
generically parameterizes the normalization of the curves in Λ_i. We set
$$T_i \coloneqq \mathrm{Pic}^3_{\tilde{\mathcal{C}}_i/U_i}.$$

Then pulling back along $\tilde{\mathcal{C}}_i \to \mathcal{C}_i$ defines

$$\phi_i \colon M_{\Lambda_i} \supset \mathrm{Pic}^3_{\mathcal{C}_i/\Lambda_i} \dashrightarrow T_i \coloneqq \mathrm{Pic}^3_{\tilde{\mathcal{C}}_i/U_i}$$

and by Lemma 6.6(i) the fibers are constant cycle subvarieties of M. Over the open dense subset of curves in Λ_i, that have exactly $5-i$ nodes as their only singularities, the fibers of ϕ_i are isomorphic to $\mathbb{G}_m^{5-i}$. □

Proposition 7.5. *There is an eight-dimensional quasi-projective scheme T_Σ fitting into a diagram*

$$\begin{tikzcd} M_\Sigma \arrow[rr, dashed, two heads, "\phi_\Sigma"] \arrow[dr, "f"'] & & T_\Sigma \arrow[dl] \\ & \Sigma. & \end{tikzcd}$$

The fibers of ϕ_Σ are rational constant cycle curves in M.

Proof. A general point in M_Σ is the pushforward of a line bundle on a reducible curve $i \colon D \hookrightarrow S$ and by Lemma 6.6 (ii) the class $[i_*\mathcal{L}] \in \mathrm{CH}_0(S)$ depends exactly on the restriction of $\mathcal{L}$ to each component. This is, what T_Σ shall parameterize.

Let $\mathcal{C}_{\Sigma\setminus\Delta} \to \Sigma\setminus\Delta$ be the universal curve over $\Sigma\setminus\Delta$. Even though every fiber has two irreducible components, the total space $\mathcal{C}_{\Sigma\setminus\Delta}$ is irreducible. However, after the base change

$$\begin{tikzcd} \tilde{\mathcal{C}}_{\Sigma\setminus\Delta} \arrow[r] \arrow[d] & \mathcal{C}_{\Sigma\setminus\Delta} \arrow[d] \\ \mathbb{P}^2 \times \mathbb{P}^2 \setminus \Delta \arrow[r] & \Sigma \setminus \Delta, \end{tikzcd}$$

we have a decomposition $\tilde{\mathcal{C}}_{\Sigma\setminus\Delta} = \tilde{\mathcal{C}}^1_{\Sigma\setminus\Delta} \cup \tilde{\mathcal{C}}^2_{\Sigma\setminus\Delta}$ into two irreducible components, which are identified under the natural $\mathbb{Z}/2\mathbb{Z}$-action. Note that the horizontal arrows are principal $\mathbb{Z}/2\mathbb{Z}$-bundles and the vertical arrows are $\mathbb{Z}/2\mathbb{Z}$-equivariant. On the level of Picard schemes, restricting to each component gives a $\mathbb{Z}/2\mathbb{Z}$-equivariant map

$$\mathrm{Pic}_{\tilde{\mathcal{C}}_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta} \longrightarrow \mathrm{Pic}_{\tilde{\mathcal{C}}^1_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta} \times_{\Sigma\setminus\Delta} \mathrm{Pic}_{\tilde{\mathcal{C}}^2_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta}. \tag{7.3}$$

Here, $\mathbb{Z}/2\mathbb{Z} = \langle\tau\rangle$ acts on the right hand side via

$$\tau \cdot (\mathcal{L}_1, \mathcal{L}_2) = (\tau^*\mathcal{L}_2, \tau^*\mathcal{L}_1).$$

(By a slight abuse of notation, τ also denotes the map identifying the isomorphic components $\tilde{\mathcal{C}}^1_{\Sigma\setminus\Delta}$ and $\tilde{\mathcal{C}}^2_{\Sigma\setminus\Delta}$). The quotient

$$\mathrm{Pic}_{\tilde{\mathcal{C}}_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta}/\tau \cong \mathrm{Pic}_{\mathcal{C}_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta} \longrightarrow (\mathrm{Pic}_{\tilde{\mathcal{C}}^1_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta} \times_{\Sigma\setminus\Delta} \mathrm{Pic}_{\tilde{\mathcal{C}}^2_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta})/\tau$$

of (7.3) by τ is what we are looking for, once the correct degree has been fixed. We set

$$T_\Sigma \coloneqq (\mathrm{Pic}^1_{\tilde{\mathcal{C}}^1_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta} \times_{\Sigma\setminus\Delta} \mathrm{Pic}^2_{\tilde{\mathcal{C}}^2_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta} \sqcup \mathrm{Pic}^2_{\tilde{\mathcal{C}}^1_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta} \times_{\Sigma\setminus\Delta} \mathrm{Pic}^1_{\tilde{\mathcal{C}}^2_{\Sigma\setminus\Delta}/\Sigma\setminus\Delta})/\tau$$

and take ϕ_Σ to be the above map, whose fibers are isomorphic to $\mathbb{G}_m$. $\square$

Proposition 7.6. *There is a four-dimensional quasi-projective scheme T_Δ fitting into a diagram*

$$\begin{tikzcd} M^i_\Delta \arrow[rr, dashed, two heads, "\phi^i_\Delta"] \arrow[dr, "f"'] & & T_\Delta \arrow[dl] \\ & \Delta & \end{tikzcd}$$

for $i = 0, 1$. The fibers of ϕ^0_Δ are three-dimensional rational constant cycle subvarieties in M. Over $2C \in \Delta$, the fibers of $(\phi^1_\Delta)_{2C}$ are birational to a $\mathbb{P}^2$-bundles over a curve of genus 17 that is étale of degree 16 over C. In particular, they yield examples of three-dimensional constant cycle subvarieties in M that are not rationally connected.

Proof. We consider the component M^0_Δ first. A general point in M^0_Δ is of the form $\mathcal{E} = i_*\mathcal{E}_0$, where $i\colon C \hookrightarrow S$ is the inclusion of a smooth curve $C \in |H| \cong \Delta$ and $\mathcal{E}_0$ is a vector bundle of rank 2 on C. The class $[i_*\mathcal{E}_0] \in \mathrm{CH}_0(S)$ is determined by $i_*c_1(\mathcal{E}_0)$. This suggests to set

$$T_\Delta \coloneqq \mathrm{Pic}^1_{\mathcal{C}_U/U},$$

where $U \subset |H|$ is the open subset of smooth curves and $\mathcal{C}_U \to U$ denotes the universal curve and then define ϕ^0_Δ as the determinant map. The fibers of ϕ^0_Δ are isomorphic to a moduli space of stable vector bundles of rank two with fixed determinant of degree one, which is rational [46], [47, Prop 2].

To deal with M^1_Δ, let $\mathcal{E} \in M^1_\Delta \setminus (M^0_\Delta \cap M^1_\Delta)$ such that $C \coloneqq \mathrm{Supp}(\mathcal{E})_{\mathrm{red}} \in U$. Then, by (5.7) and Lemma 6.6 the class $\mathrm{ch}_2(\mathcal{E})$ is determined by $i_*c_1(\mathcal{L}^{\otimes 2}(x) \otimes \omega_C^{-1})$, where $\mathcal{L} \coloneqq \mathcal{E}|_C/\mathcal{T} \in \mathrm{Pic}^1(C)$ and $x \coloneqq \mathrm{Supp}(\mathcal{T}) \in C$ with $\mathcal{T}$ being the torsion subsheaf of $\mathcal{E}|_C$ and $i\colon C \hookrightarrow S$ being the inclusion. Consequently, we define

$$\phi^1_\Delta\colon M^1_\Delta \dashrightarrow T_\Delta, \mathcal{E} \mapsto \mathcal{L}^{\otimes 2}(x) \otimes \omega_C^{-1}.$$

We want to compute the fibers of $(\phi^1_\Delta)_{2C}$. First, we forget the twist with ω_C^{-1}. Then we can factor $(\phi^1_\Delta)_{2C}$ as follows

$$M^1_{2C} \dashrightarrow \mathrm{Pic}^1(C) \times C \to \mathrm{Pic}^3(C),\ \mathcal{E} \mapsto (\mathcal{L}, x) \mapsto \mathcal{L}^{\otimes 2}(x).$$

The first arrow is defined outside the intersection $M^0_{2C} \cap M^1_{2C}$ and its fibers are a torsor under $\mathrm{Ext}^1_C(i_*\mathcal{L}, i_*(\mathcal{L}(x) \otimes \omega_C^{-1})) \cong \mathbb{C}^2$, cf. Corollary 2.4. Thus the fibers of $(\phi^1_\Delta)_{2C}$ are an $\mathbb{A}^2$-bundle over the fibers of the second arrow, which we factor as follows

$$\mathrm{Pic}^1(C) \times C \to \mathrm{Pic}^2(C) \times C \xrightarrow{\mu} \mathrm{Pic}^3(C),\ (\mathcal{L}, x) \mapsto (\mathcal{L}^{\otimes 2}, x) \mapsto \mathcal{L}^{\otimes 2}(x).$$

Here, the first map is étale of degree 16 and the fibers of the second map

$$\mu\colon \operatorname{Pic}^2(C)\times C\to \operatorname{Pic}^3(C),\ (\mathcal{L},x)\mapsto \mathcal{L}(x)$$

are isomorphic to C. To see this, let $\mathcal{M}\in \operatorname{Pic}^3(C)$ and consider $p_2\colon \mu^{-1}(\mathcal{M})\to C$. As $\mathcal{L}(x)\cong \mathcal{M}$ for fixed $x\in C$ determines $\mathcal{L}\in \operatorname{Pic}^2(C)$, this projection is an isomorphism and the claim follows. □

Remark 7.7. A combination of the proofs of Propositions 7.4, 7.5 and 7.6 allows one to find the isotropic fibrations for the remaining cases $M_{V_i}, i=1,2,3$.

7.3. More examples. We construct some more examples of algebraically coisotropic subvarieties.

7.3.1. Horizontal constant cycle Lagrangians. To start with, we produce a constant cycle Lagrangian that dominates B. For example, any section of $M\to B$ would work. Unfortunately, f does not admit a section [4]. Below, we produce a multisection of degree of 2^{10}. Recall that

$$M^\circ \cong \operatorname{Pic}^3_{\mathcal{C}^\circ/B^\circ}$$

and there is an exact sequence [24, (9.2.11.5)]

$$0\to \operatorname{Pic}(\mathcal{C}^\circ)/\operatorname{Pic}(B^\circ)\to \operatorname{Pic}_{\mathcal{C}^\circ/B^\circ}(B^\circ)\to \operatorname{Br}(B^\circ)\to \dots .$$

Moreover, one can show that

$$\operatorname{Pic}(\mathcal{C}^\circ)/\operatorname{Pic}(B^\circ)\cong \operatorname{Pic}(\mathcal{C})/\operatorname{Pic}(B)\cong \operatorname{Pic}(S),$$

where the last isomorphism holds because $\mathcal{C}\subset B\times S$ is a $\mathbb{P}^4$-bundle over S with $\mathcal{O}_{p_2}(1)=p_1^*\mathcal{O}_B(1)$. For $L\in \operatorname{Pic}(S)$ with $n=2H.L$, the corresponding section is given by

$$s_L\colon B^\circ\to \operatorname{Pic}^n_{\mathcal{C}^\circ/B^\circ},\ D\mapsto L.D.$$

If $\operatorname{Pic}(S)=\mathbb{Z}\cdot H$, for example, one gets sections for $n\equiv 0 \bmod 4$. There is always a section of $\operatorname{Pic}^2_{\mathcal{C}^\circ/B^\circ}$ that does not come from S.

Lemma 7.8. *There is a section*

$$g^1_2\colon B^\circ\to \operatorname{Pic}^2_{\mathcal{C}^\circ/B^\circ}$$

such that a curve $D\in B^\circ$ maps to the unique line bundle $g^1_2(D)\in \operatorname{Pic}^2(D)$ with $h^0(g^1_2(D))=2$. In particular,

$$(g^1_2)\otimes(g^1_2)=s_H\colon B^\circ\to \operatorname{Pic}^4_{\mathcal{C}^\circ/B^\circ}$$

and g^1_2 is not of the form s_L for $L\in \operatorname{Pic}(S)$.

Proof. This is a consequence of the same phenomenon occurring for the universal family of smooth quadrics in $\mathbb{P}^2$. We identify $B = |\mathcal{O}_{\mathbb{P}^2}(2)|$ and we will see that the lemma holds true for $B^\circ = B \setminus \Sigma$. Let $\mathcal{Q}^\circ \subset B^\circ \times \mathbb{P}^2$ be the universal quadric, which is an étale $\mathbb{P}^1$-fibration, but not a projective bundle. We claim that there is a section

$$s\colon B^\circ \to \operatorname{Pic}^1_{\mathcal{Q}^\circ/B^\circ}$$

that is not induced by a line bundle on $\mathcal{Q}^\circ$. Indeed, fix a line $\ell \subset \mathbb{P}^2$ and consider

$$\tilde{B^\circ} \coloneqq \mathcal{Q}^\circ \cap (B^\circ \times \ell) \to B^\circ.$$

This morphism is finite, flat of degree 2 and the base change $\tilde{\mathcal{Q}^\circ} \to \tilde{B^\circ}$ admits a section. Therefore we get

$$\tilde{s}\colon \tilde{B^\circ} \to \operatorname{Pic}^1_{\tilde{\mathcal{Q}^\circ}/\tilde{B^\circ}}.$$

As the two points in $\tilde{B^\circ}$ lying over a fixed point in B° define the same line bundle, $\tilde{s}$ descends to a section s. By definition, $s \otimes s$ is the section defined by $p_2^*\mathcal{O}_{\mathbb{P}^2}(1)$, which does not admit a square root. Pulling back s along $\mathcal{C}^\circ \to \mathcal{Q}^\circ$ defines g^1_2. ☐

Remark 7.9. In all our examples, it does not matter if we identify M and $M_H(0, 2H, -3)$ via the isomorphism (7.2) (given by tensorization and dualization) or the birational map induced by the section g^1_2. The composition of the one map with the inverse of the other is the rational involution on M that comes from the natural involution $\iota^{[5]}$ on $S^{[5]}$.

Now, we use the squaring map

$$\rho_2\colon \operatorname{Pic}^1_{\mathcal{C}^\circ/B^\circ} \longrightarrow \operatorname{Pic}^2_{\mathcal{C}^\circ/B^\circ},\ \mathcal{L} \mapsto \mathcal{L}^{\otimes 2},$$

to construct a constant cycle Lagrangian from $g^1_2(B^\circ)$. Specifically, we set

$$L^1_2 \coloneqq \overline{\rho_2^{-1}(g^1_2(B^\circ))} \subset M_H(0, 2H, -3) \xrightarrow{\sim} M.$$

Lemma 7.10. *The subvariety L^1_2 is a constant cycle Lagrangian in M, which is generically finite of degree 2^{10} over B.*

Proof. It is clear, that $\dim L^1_2 = 5$. We will show that $L^1_2 \subset F_0M_H(0, 2H, -3)$. Let $D \in B^\circ$ and $\mathcal{L} \in \operatorname{Pic}^1(D)$ such that $\mathcal{L}^{\otimes 2} = g^1_2(D)$. Then

$$4 \cdot i_*c_1(\mathcal{L}) = i_*c_1(\mathcal{O}_S(H)|_D) = 4c_S \in \mathrm{CH}_0(S).$$

This implies $i_*c_1(\mathcal{L}) = c_S$ because $\mathrm{CH}_0(S)$ is torsionfree. We conclude that a general point in L^1_2 is contained in $S_0\,\mathrm{CH}_0(S)$ as desired. Finally, ρ_2 is finite, étale of degree 2^{10}. ☐

Remark 7.11. Another example of a horizontal constant cycle Lagrangian is constructed more generally for any Lagrangian fibration by Lin [39].

7.3.2. Examples in M_Δ. Starting from $\phi^i_\Delta\colon M^i_\Delta \dashrightarrow T_\Delta$ (cf. Proposition 7.6), we construct two examples of constant cycle Lagrangians in M^0_Δ and M^1_Δ. Recall that $T_\Delta = \mathrm{Pic}^1_{\mathcal{C}_U/U}$, where $U \subset |H|$ is the open subset consisting of smooth curves and the fibers of ϕ^i_Δ are three-dimensional constant cycle subvarieties in M.

The idea of Example 7.12 is to find a constant cycle surface in T_Δ. Then the preimage under ϕ^i_Δ is a constant cycle Lagrangian in M contained in M^i_Δ. This idea is taken further in Example 7.13. Here, we find a surface in T_Δ, that consists of line bundles whose first Chern class is a multiple of the Beauville–Voisin class, when pushed forward to S. By Theorem 6.5, the preimage of this surface is also a constant cycle Lagrangian in M.
For simplicity, we assume from now on that $\mathrm{Pic}(S) = \mathbb{Z}\cdot H$. Then every curve in $|H|$ is integral and $\mathrm{Pic}^1_{\mathcal{C}_{|H|}/|H|}$ is representable by a smooth, quasi-projective scheme.

Example 7.12. We construct a constant cycle subvariety of $\mathrm{Pic}^1_{\mathcal{C}_{|H|}/|H|}$ applying the same trick as for the construction of L^1_2. Namely, let

$$\rho_2\colon\ \mathrm{Pic}^1_{\mathcal{C}_{|H|}/|H|} \to \mathrm{Pic}^2_{\mathcal{C}_{|H|}/|H|}$$

and consider the section s_H of $\mathrm{Pic}^2_{\mathcal{C}_{|H|}/|H|}$ defined by H. We set

$$Z_H \coloneqq \rho_2^{-1}(s_H(|H|)).$$

Since $\mathrm{Pic}^2_{\mathcal{C}_{|H|}/|H|}$ can be embedded as an open subset of $M_H(0,H,2)$, we can apply Theorem 6.5 to see that Z_H is a constant cycle subvariety, as in the proof of Lemma 7.10. Now, $Z_H \subset \mathrm{Pic}^2_{\mathcal{C}_{|H|}/|H|}$ is a smooth, quasi-projective surface and the morphism $Z_H \to |H|$ is finite, étale of degree 2^4, when restricted to the open subset of smooth curves $U \subset |H|$. By Corollary 5.2, we know that $U = |H| \setminus V(1,|H|)$ is the complement of a nodal curve of degree 30. Therefore $\pi_1(U) \cong \mathbb{Z}/30\mathbb{Z}$ [18, Prop 1.3 & Thm 1.13]. Consequently, Z_H must have 8 pairwise isomorphic connected components that restrict over U to the unique degree 2 cover of U. We replace Z_H by one of its irreducible components and define

$$L^i \coloneqq \overline{(\phi^i_\Delta)^{-1}(Z_H)} \subset M^i_\Delta \text{ for } i = 0,1.$$

By construction, these are constant cycle Lagrangians in M.

Example 7.13. The idea of this example is to consider the preimage of two-dimesional subvarieties in T_Δ that are not constant cycle subvarieties themselves, but consist of line bundles whose first Chern class is the Beauville–Voisin class when pushed forward to S.
To begin with, we have an embedding

$$\Theta\colon \mathcal{C}_{|H|} \hookrightarrow \mathrm{Pic}^1_{\mathcal{C}_{|H|}/|H|},\ C \ni x \mapsto \mathcal{O}_C(x).$$

Then, for example a vector bundle $\mathcal{E} \in M^0_\Delta$ lies over $\Theta(\mathcal{C}_{|H|})$ if and only if its determinant line bundle is effective (of degree one). Therefore,

$$(\phi^i_\Delta)^{-1}(\Theta(\mathcal{C}_{|H|})) \subset F_1M \quad \text{and} \quad \operatorname{codim}(\phi^i_\Delta)^{-1}(\Theta(\mathcal{C}_{|H|})) = 4.$$

In particular, $(\phi^i_\Delta)^{-1}(\Theta(\mathcal{C}_{|H|}))$ is algebraically coisotropic. The isotropic fibration is given by the composition of the projection $\mathcal{C} \to S$ with ϕ^i_Δ.
Refining this example yields constant cycle Lagrangians in M^i_Δ as follows. For example, let $C_{\mathrm{cc}} \subset S$ be a constant cycle curve and set

$$L_{C_{\mathrm{cc}}} \coloneqq (\phi^i_\Delta)^{-1}(\Theta(\mathcal{C}_{|H|} \cap C_{\mathrm{cc}} \times |H|)).$$

7.3.3. Examples in $S^{[5]}$. We can also produce easily examples of algebraically coisotropic subvarieties in $S^{[5]}$. As M and $S^{[5]}$ are birational, we have

$$\mathrm{CH}_0(S^{[5]}) \cong \mathrm{CH}_0(M),$$

[25, Expl 16.1.11] and algebraically coisotropic varieties that are not contained in the exceptional locus of a birational map can be transferred from $S^{[5]}$ to M and vice versa.

Example 7.14. This example can also be found in [55, §4 Exa 1]. For $i = 1, \ldots, 4$, define

$$E_i \coloneqq \{\xi \in S^{[5]} \mid \lg(\mathcal{O}_{\xi_{\mathrm{red}}}) \leq i\ \}.$$

Then $E_i \subset S^{[5]}$ is closed subvariety of codimension $5 - i$ [12]. For example, $E \coloneqq E_4$ is the exceptional divisor of the Hilbert–Chow morphism $s\colon S^{[5]} \to S^{(5)}$. The irreducible components $E_i^{\underline{n}}$ of E_i are indexed by ordered tuples of positive natural numbers $\underline{n} = (n_1 \geq n_2 \geq \ldots \geq n_i)$ such that $\sum_{k=1}^i n_k = 5$. In particular, E_4 and E_1 are irreducible, whereas E_3 and E_2 consists of two irreducible components. To sum up

$$E_1 \subset E_2 \subset E_3 \subset E_4 = E \subset S^{[5]}. \tag{7.4}$$

By definition of E_i and Theorem 6.2, we have

$$E_i \subset F_i S^{[5]}$$

for all $i = 1, \ldots, 4$ and hence E_i is algebraically coisotropic.

Example 7.15. We have $\mathbb{P}^2 \subset S^{[2]}$ given by $x \mapsto \pi^{-1}(x)$, where $\pi\colon S \to \mathbb{P}^2$. Consider the generically injective rational maps

$$g_3\colon \mathbb{P}^2 \times S^{[3]} \dashrightarrow S^{[5]} \quad \text{and} \quad g_1\colon \mathbb{P}^2 \times \mathbb{P}^2 \times S \dashrightarrow S^{[5]}$$

and set

$$P_i \coloneqq \overline{\mathrm{im}(g_i)} \subset S^{[5]} \text{ for } i = 1, 3.$$

Clearly, $P_i \subset F_i S^{[5]}$ and $\operatorname{codim} P_i = 5 - i$.

Example 7.16. This example is taken from [35]. As in (5.1), we consider the locus $V(j,|H|) \subset |H|$ of curves C with $g(\tilde{C}) \leq j$ for $j = 0,1,2$. Specifically, $V(2,|H|)$ is everything, $V(1,|H|) \subset |H|$ is irreducible and 1-dimensional and the generic curve in $V(1,|H|)$ has exactly one node, $V(0,|H|)$ is the discrete set of rational curves. We let $\mathcal{C}_j \to V(2-j,|H|)$ be the respective restriction of the universal curve $\mathcal{C}_{|H|} \to |H|$ and $\tilde{\mathcal{C}}_j$ its normalization. For $2-j \leq i \leq 4$, consider the diagram

$$\begin{tikzcd}
 & S^{[7-i-j]} \times S^{[i+j-2]} \arrow[r, dashed] \arrow[d, "s\times\mathrm{id}"] & S^{[5]} \\
\mathrm{Sym}^{7-i-j}_{V(2-j,|H|)}(\mathcal{C}_j) \times S^{[i+j-2]} \arrow[r] \arrow[rru, dashed, bend left, "f_i^j"] & S^{(7-i-j)} \times S^{[i+j-2]}, &
\end{tikzcd}$$

in which the lower horizontal map and hence f_i^j turns out to be generically injective [35, Thm 6.4]. We define

$$W_i^j \coloneqq \overline{\mathrm{im}(f_i^j)} \subset S^{[5]}.$$

This is a subvariety of codimension $5-i$, which is irreducible for $j \neq 2$. We have the following table of inclusions:

$$\begin{array}{ccccccccc}
W_2^0 & \subset & W_3^0 & \subset & W_4^0 & & & & \\
\cup & & \cup & & \cup & & & & \\
W_1^1 & \subset & W_3^1 & \subset & W_3^1 & \subset & W_4^1 & & \\
\cup & & \cup & & \cup & & \cup & & \\
W_0^2 & \subset & W_1^2 & \subset & W_2^2 & \subset & W_3^2 & \subset & W_4^2.
\end{array}$$

A generic point $\xi \in W_i^j$ corresponds to a subscheme in S that contains exactly $7-i-j$ points, which lie on a curve in $V(2-j,|H|)$ and the other $i+j-2$ points can move freely outside C. Hence, $[\xi] \in \mathrm{CH}_0(S)$ is contained in the $(2-j)+(i+j-2)$-th step of O'Grady's filtration. In other words,

$$W_i^j \subset F_i S^{[5]}$$

and W_i^j is algebraically coisotropic. The isotropic fibration on W_i^j is given by the Abel map

$$\mathrm{Sym}^{7-i-j}_{V(2-j,|H|)}(\mathcal{C}_j) \to \mathrm{Pic}^{7-i-j}_{\tilde{\mathcal{C}}_j/V(2-j,|H|)}$$

and endows W_i^j generically with the structure of a $\mathbb{P}^{5-i}$-bundle. By [35, Thm 6.4] the class of a line in the fibers is $H-(8-i+2j)\delta^\vee$.

In Proposition 9.4, we show directly that W_2^0 is the $\mathbb{P}^3$-bundle over $M_H(0,H,-6)$ parameterizing extensions $\mathrm{Ext}^1_S(\mathcal{E},\mathcal{O}_S(-H))$. Moreover, $W_3^0 \setminus W_2^0$ has the structure of a $\mathbb{P}^2$-bundle over a dense open subset of $S \times M_H(0, H, -5)$. This bundle parameterizes ideal sheaves $\mathcal{I} \in M_H(1, 0, -4) = S^{[5]}$ that fit into an extension

$$0 \to \mathcal{I}_x(-H) \to \mathcal{I} \to \mathcal{E} \to 0.$$

III. Birational geometry of the Mukai system of rank two and genus two

Abstract. Using the techniques of Bayer–Macrì, we determine the walls in the movable cone of the Mukai system of rank two for a general K3 surface S of genus two. We study the birational map to $S^{[5]}$ and decompose it into a sequence of flops. We give an interpretation of the exceptional loci in terms of Brill–Noether loci.

Introduction

As before, let (S,H) be a polarized K3 surface of genus two, that is a double covering $\pi\colon S\to\mathbb{P}^2$ ramified over a smooth sextic curve and $H=\mathcal{O}_S(1)=\pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. We assume in this part that $\mathrm{Pic}(S)=\mathbb{Z}\cdot H$.

For any n, we consider the Mukai system of genus two

$$f\colon M_H(0,n,-1)\longrightarrow |nH|\cong\mathbb{P}^{n^2+1},\ \mathcal{E}\mapsto\mathrm{Supp}(\mathcal{E}).$$

This is a particular nice example of an irreducible holomorphic symplectic manifold with a Lagrangian fibration and comes with a rich geometrical structure. An instance of this, shall be demonstrated below, where we determine all birational models in the case $n=2$.

It is easy to see that $M\coloneqq M_H(0,n,-1)$ is birational to $S^{[n^2+1]}$. Namely, let $\xi\in S^{[n^2+1]}$ such that $\mathrm{Supp}(\xi)$ consists of n^2+1 points in general position. Then there is a unique smooth curve $D\in|nH|$ such that $\xi\subset D$ and this allows to define a rational map

$$T\colon S^{[n^2+1]}\dashrightarrow M_H(0,n,-1),\ \xi\mapsto\mathcal{O}_D(-\xi)\otimes\mathcal{O}_S(n)|_D.$$

Conversely, a general point in M is given by $\mathcal{L}\in\mathrm{Pic}^{n^2-1}(D)$, for a smooth curve $D\in|nH|$ and thus generically $\dim H^0(S,\mathcal{L}^\vee\otimes\mathcal{O}_S(n)|_D)=1$. Hence, T is birational.
The morphism T can be defined more conceptually via the spherical twist

$$T_{\mathcal{O}_S(-n)}\colon D^b(S)\xrightarrow{\sim}D^b(S)$$

[32, §8.1]. Let $\mathcal{I}_\xi\in M_H(1,0,-n^2)$ be the ideal sheaf of a point $\xi\in S^{[n^2+1]}$, which is contained in the open subset, where $h^0(\mathcal{I}_\xi(n))=1$. By definition, $T_{\mathcal{O}_S(-n)}(\mathcal{I}_\xi)$ fits into a short exact sequence

$$0\to\mathcal{O}_S(-n)\to\mathcal{I}_\xi\to T_{\mathcal{O}_S(-n)}(\mathcal{I}_\xi)\to 0.$$

We conclude $T(\mathcal{I}_\xi)=T_{\mathcal{O}_S(-n)}(\mathcal{I}_\xi)(n)$. In other words, T is the composition

$$S^{[n^2+1]}=M_H(1,0,-n^2)\xrightarrow{T_{\mathcal{O}_S(-n)}}M_\sigma(0,n,-2n^2-1)\xrightarrow{-\otimes\mathcal{O}_S(n)}M_\sigma(0,n,-1)\dashrightarrow M_H(0,n,-1),$$

where the first two arrows are isomorphisms and σ is a suitable stability condition. The last arrow is the birational transformation coming from wall-crossing along a path from σ into the Gieseker chamber.

If $n = 1$, then all curves in $|H|$ are irreducible and therefore $T_{\mathcal{O}_S(-1)}(\mathcal{I}_\xi)$ is a stable sheaf provided that $h^0(\mathcal{I}_\xi(1)) = 1$. The indeterminancy of T is exactly the closed subset of $\xi \in S^{[2]}$ through which passes a pencil of curves in $|H|$, which is identified with

$$\mathbb{P}^2 \subset S^{[2]},\ x \mapsto \pi^{-1}(x).$$

A resolution of T is the original example of a Mukai flop [44]:

$$\begin{tikzcd}[column sep=small] & \mathrm{Bl}_{\mathbb{P}^2} S^{[2]} \cong \mathrm{Hilb}^2(\mathcal{C}/|H|) \arrow[dl] \arrow[dr] & & & (\xi \subset C) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ S^{[2]} \arrow[rr, dashed] & & M_H(0,1,-1), & \xi & & \mathcal{O}_C(-\xi) \otimes \mathcal{O}_S(1)|_C. \end{tikzcd}$$

For $n = 2$, we show

Proposition 7.17 (Prop 8.1)**.** *The spherical twist at $\mathcal{O}_S(-2)$ induces an isomorphism*

$$T\colon S^{[5]} \setminus \{\xi \in S^{[5]} \mid h^0(\mathcal{I}_\xi(2)) \geq 2\} \xrightarrow{\sim} M \setminus (\{\mathcal{E} \in M \mid h^0(\mathcal{E}) \geq 1\} \cup M^0_\Delta).$$

Here, $\Delta \subset |2H|$ is the locus of non-reduced curves and $M^0_\Delta \subset M$ is the irreducible component of $f^{-1}(\Delta)$ that consists of vector bundles of rank two and degree one on the underlying reduced curve.

From the birational geometer's point of view, $M_H(0,n,-1)$ and $S^{[n^2+1]}$ are extremal in the following sense. If one considers the decomposition of the movable cone

$$\mathrm{Mov}(S^{[n^2+1]}) \subset \mathrm{NS}(S^{[n^2+1]})_{\mathbb{R}} \cong \mathbb{R}^2$$

into chambers corresponding to birational models, $S^{[n^2+1]}$ is at the one end, for it admits a divisorial contraction given by the Hilbert–Chow morphism and $M_H(0,n,-1)$ with the Lagrangian fibration is at the other end. For $n = 1$, the Mukai system $M_H(0,1,-1)$ is the only other smooth birational model of $S^{[2]}$. If $n > 1$, the presence of reducible and non-reduced curves in the linear system $|nH|$ makes the situation more complicated. The article in hand, deals with the case $n = 2$. We prove the following result.

Theorem 7.18 (Thm 10.2)**.** *Let (S,H) be a polarized K3 surface with $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$ and $H^2 = 2$. There are five (smooth) birational models of $S^{[5]}$ or $M := M_H(0,2,-1)$, respectively. They are connected by a chain of flopping contractions*

$$\begin{tikzcd}[column sep=small] & \mathrm{Bl}_{W_2} S^{[5]} \arrow[dl] \arrow[dr] & & \mathrm{Bl}_{\tilde{W}_3} X_1 \arrow[dl] \arrow[dr] & & \mathrm{Bl}_{\tilde{Z}_3} X_3 \arrow[dl] \arrow[dr] & & \mathrm{Bl}_{Z_1} M \arrow[dl] \arrow[dr] & \\ S^{[5]} \arrow[rr, dashed, "g_1"] & & X_1 \arrow[rr, dashed, "g_2"] & & X_2 & & X_3 \arrow[ll, dashed, "g_3"'] & & M \arrow[ll, dashed, "g_4"'] \end{tikzcd}$$

for some subvarieties $W_2 \subset W_3 \subset S^{[5]}$ such that

- *W_2 is a $\mathbb{P}^3$-bundle over $M_H(0,1,-6)$,*
- *$W_3 \setminus W_2$ is a $\mathbb{P}^2$-bundle over an open subset of $M_H(0,1,-5) \times S$*

and subvarieties $Z_1 \subset Z_3 \subset M$ such that

- *Z_1 is a $\mathbb{P}^4$-bundle over S,*
- *$Z_3 \setminus Z_1$ is a $\mathbb{P}^2$-bundle over an open subset of $S^{[3]}$.*

Here, $\tilde{W}_3$ (resp. $\tilde{Z}_3$) is the strict transform of W_3 (resp. Z_3) under g_1 (resp. g_4).

We prove Theorem 7.18 using the methods of Bayer–Macrì [6]. Their machinery gives a procedure to compute the walls in the movable cone and to identify the curves which are contracted at every step. The exceptional loci are components of the Brill–Noether loci

$$\mathrm{BN}^i(S^{[5]}) \coloneqq \{\xi \in S^{[5]} \mid h^0(\mathcal{I}_\xi(2)) \geq i+1\} \subset S^{[5]},\ i = 1,2$$

and

$$\mathrm{BN}^i(M) \coloneqq \{\mathcal{E} \in M \mid h^0(\mathcal{E}) \geq i+1\} \subset M,\ i = 0,1.$$

More precisely, W_2 and W_3 (resp. Z_1 and Z_3) are the algebraically coisotropic subvarieties, that were defined in Example 7.16 (resp. in (7.1)).

Proposition 7.19. (i) *We have*

$$W_2 = \{\xi \in S^{[5]} \mid \textit{there is } C \in |H| \textit{ such that } \xi \subset C\} \subset \mathrm{BN}^2(S^{[5]}),\ \textit{and}$$

$$W_3 = \{\xi \in S^{[5]} \mid \textit{there is } x \in \xi \textit{ and } C \in |H| \textit{ such that } \xi \setminus \{x\} \subset C\} \subset \mathrm{BN}^1(S^{[5]}).$$

(ii) *Z_1 (resp. Z_3) is the component of $\mathrm{BN}^1(M)$ (resp. $\mathrm{BN}^0(M)$) that dominates the locus of smooth curves in $|2H|$.*

Outline. The core of this part is the application of the results of [6] to the Mukai system of rank two and genus two in Section 10. In particular, we compute the walls in $\mathrm{Mov}(S^{[5]})$ and at each wall, we get a numerical characterization of the projective bundles that get contracted. The preceding sections can be seen as the foundation for the geometrical interpretation of these computations. Precisely, in Section 8, we prove Proposition 7.17 by explicit considerations and in Section 9, we study likewise explicitly components of the appearing Brill–Noether loci leading to Proposition 7.19. These components will later be identified with the exceptional loci of the transformations in Theorem 7.18.

Notation. In this part, we assume throughout $\mathrm{Pic}(S) = \mathbb{Z} \cdot H$ and therefore suppress H from the notation. Moreover, we identify $H^*_{\mathrm{alg}}(S, \mathbb{Z}) \cong \mathbb{Z}^3$ and (other than in Part I) always equip it with the Mukai pairing denoted by $(-,-)$. As before,

$$f\colon M \coloneqq M_H(0, 2, -1) \longrightarrow B \coloneqq |2H| \cong \mathbb{P}^5$$

is the Mukai system of rank two and genus two and we have the subloci $\Delta \subset \Sigma \subset B$ of non-reduced and non-integral curves as in (2.1). In Lemma 5.1, we saw that $M \cong M_H(0, 2, k)$ for every $k \equiv 0 \bmod 2$.

8. The birational map $T\colon S^{[5]} \dashrightarrow M$

In this section, we study the birational map $T\colon S^{[5]} \dashrightarrow M$ from the introduction, which is induced by the spherical twist at $\mathcal{O}_S(-2)$. For the definition of the spherical twist, we refer to [32, §8.1].

Proposition 8.1. *The spherical twist at $\mathcal{O}_S(-2)$ defines a birational map*

$$T\colon S^{[5]} \dashrightarrow M,\ \xi \mapsto T_{\mathcal{O}_S(-2)}(\mathcal{I}_\xi) \otimes \mathcal{O}_S(2)$$

which induces an isomorphism

$$S^{[5]} \setminus \{\xi \in S^{[5]} \mid h^0(\mathcal{I}_\xi(2)) \geq 2\} \xrightarrow{\sim} M \setminus (\{\mathcal{E} \in M \mid h^0(\mathcal{E}) \geq 1\} \cup M^0_\Delta).$$

In particular, T is defined in $\xi \in S^{[5]}$ if there is a unique curve $D \in |2H|$ such that $\xi \subset D$. In this case,

$$T(\xi) \cong \ker(\mathcal{O}_S(2)|_D \to \mathcal{O}_\xi).$$

We want to point out that, due to $\rho(S) = 1$, there are actually no proper birational automorphisms of $S^{[5]}$. Precisely, we have

$$\mathrm{Aut}(S^{[5]}) = \mathrm{Bir}(S^{[5]}) = \langle \mathrm{id}, \iota^{[5]} \rangle,$$

where $\iota^{[5]}$ is the automorphism induced by the involution ι on S [11, Thm 1.1]. Hence, T is the only birational morphism $S^{[5]} \dashrightarrow M$, up to precomposition with $\iota^{[5]}$. Also note that the subvariety $\mathrm{BN}^1(S^{[5]}) = \{\xi \in S^{[5]} \mid h^0(\mathcal{I}_\xi(2)) \geq 2\} \subset S^{[5]}$ is left invariant under $\iota^{[5]}$.

Proof of Proposition 8.1. By definition of the spherical twist, there is an exact triangle in $D^b(S)$

$$R\Gamma(\mathcal{I}_\xi(2)) \otimes \mathcal{O}_S(-2) \to \mathcal{I}_\xi \to T_{\mathcal{O}_S(-2)}(\mathcal{I}_\xi) \xrightarrow{[1]} .$$

So, $T_{\mathcal{O}_S(-2)}(\mathcal{I}_\xi)$ is a complex in degrees -1 and 0, which is concentrated in degree 0 if and only if $h^0(\mathcal{I}_\xi(2)) = 1$, as $\chi(\mathcal{I}_\xi(2)) = 1$. In this case, $T_{\mathcal{O}_S(-2)}(\mathcal{I}_\xi)$ is as stated.
Let $\xi \in S^{[5]}$ and $s \in H^0(\mathcal{I}_\xi(2))$. We claim that if

$$\mathcal{E} \coloneqq T(\xi) \cong \ker(\mathcal{O}_S(2)|_D \to \mathcal{O}_\xi)$$

is unstable, then $h^0(\mathcal{I}_\xi(2)) \geq 2$. Here, D is the curve defined by the composition of s with the inclusion $\mathcal{I}_\xi(2) \hookrightarrow \mathcal{O}_S(2)$.
First, assume $D \in D \setminus \Sigma$. Then $\mathcal{E}$ is a rank one sheaf on the integral curve D_s and necessarily stable.
Next, if $D \in \Sigma \setminus \Delta$ write $D = D_1 \cup D_2$. Then $\mathcal{E}$ is stable, if and only if

$$\chi(\mathcal{E} \otimes \mathcal{O}_S(-1)|_{D_i}) < \tfrac{\chi(\mathcal{E})}{2} = -\tfrac{1}{2} < \chi(\mathcal{E}|_{D_i}) \text{ for } i = 1, 2.$$

Otherwise, the inclusion $\mathcal{E} \otimes \mathcal{O}_S(-1)|_{D_i} \hookrightarrow \mathcal{E}$ or the restriction $\mathcal{E} \twoheadrightarrow \mathcal{E}|_{D_i}$ to one component is destabilizing. Conversely, every destabilizing subbundle or surjection factors through the above. We find

$$\chi(\mathcal{E}|_{D_i}) = -\operatorname{lg}(\mathcal{O}_\xi \otimes \mathcal{O}_{D_i}) + \chi(\mathcal{O}_S(2)|_{D_i}) + \operatorname{lg}(\mathcal{T}or_1^{\mathcal{O}_D}(\mathcal{O}_\xi, \mathcal{O}_{D_i})) = \operatorname{lg}(\mathcal{O}_\xi \otimes \mathcal{O}_{D_{3-i}}) - 2,$$

where we used $\operatorname{lg}(\mathcal{T}or_1^{\mathcal{O}_D}(\mathcal{O}_{D_i}, \mathcal{O}_\xi)) = 5 - \operatorname{lg}(\mathcal{O}_\xi \otimes \mathcal{O}_{D_1}) - \operatorname{lg}(\mathcal{O}_\xi \otimes \mathcal{O}_{D_2})$. Similarly,

$$\chi(\mathcal{E} \otimes \mathcal{O}_S(-1)|_{D_i}) = \operatorname{lg}(\mathcal{O}_\xi \otimes \mathcal{O}_{D_{3-i}}) - 4.$$

Hence, $\mathcal{E}$ is unstable if and only if $\operatorname{lg}(\mathcal{O}_\xi \otimes \mathcal{O}_{D_i}) \geq 4$ for one $i = 1, 2$. Without loss of generality assume that $\operatorname{lg}(\mathcal{O}_\xi \otimes \mathcal{O}_{D_1}) \geq 4$. There are two cases. Either there is a reduced point $x \in \xi$ such that $\xi \setminus \{x\} \subset D_1$. Otherwise, $\xi_{\text{red}} \subset D_1$ and there is a point $x \in \xi$ whose multiplicity drops by one, when restricting to D_1. In both cases, D_2 can move in the pencil of curves in $|H|$ passing through x and thus $h^0(\mathcal{I}_\xi(2)) \geq 2$, cf. Lemma 8.2 below.
Finally, if $D = 2C \in \Delta$, the above arguments remain valid with $D_1 = D_2 = C$. This is, if $\mathcal{E}$ is unstable, then either $\operatorname{lg}(\mathcal{O}_\xi \otimes \mathcal{O}_C) = 4$ and Lemma 8.2 applies or ξ is completely contained in C. But then $\xi \subset C \cup C'$ for every curve $C' \in |H|$.
So far, we have proven that T is well-defined for all $\xi \in S^{[5]}$ such that $h^0(\mathcal{I}_\xi(2)) = 1$. A birational morphism between projective irreducible holomorphic symplectic manifolds is an isomorphism on the regular locus [30, 2.2]. Therefore, it is left to see that

$$T(S^{[5]} \setminus \{\xi \in S^{[5]} \mid h^0(\mathcal{I}_\xi(2)) \geq 2\}) = M \setminus (\{\mathcal{E} \in M \mid h^0(\mathcal{E}) \geq 1\} \cup M_\Delta^0).$$

For sure, we have an inclusion from left to right, since $H^0(T(\xi)) = 0$, whenever T is defined and over Δ, the sheaf $T(\xi)$ always has rank one on the reduced curve. More precisely, let $\xi \in S^{[5]}$ such that $2C$ is the only curve in $|2H|$ containing ξ. Then $\mathcal{E} \coloneqq T(\xi)$ fits into an extension on $2C$

$$0 \to \mathcal{L}(x) \otimes \omega_C^{-1} \to \mathcal{E} \to \mathcal{L} \to 0,$$

where x is the support of $\mathcal{T}or_1^{\mathcal{O}_{2C}}(\mathcal{O}_C, \mathcal{O}_\xi)$ and $\mathcal{L} = \omega_C^{\otimes 2}(-\xi \cap C) \in \operatorname{Pic}^1(C)$.
The converse inclusion is clear over $B \setminus \Sigma$. Let $D = D_1 \cup D_2 \in \Sigma \setminus \Delta$ and $\mathcal{E} \in f^{-1}(D)$ such that $h^0(\mathcal{E}) = 0$. We have to find $s \colon \mathcal{E} \to \mathcal{O}_S(2)|_D$. Then $\xi \coloneqq \operatorname{Supp}(\operatorname{coker}(s)) \in S^{[5]}$ and $\mathcal{E} = T(\xi)$.

Assume first that $\mathcal{L}_i \coloneqq \mathcal{E}|_{D_i}$ is torsionfree and without loss of generality that $\chi(\mathcal{L}_i) = i-1$, i.e. for smooth D_i, we have $\mathcal{L}_i \in \mathrm{Pic}^i(D_i)$. Now, we have an exact sequence

$$0 \to \mathrm{Hom}(\mathcal{L}_1, \mathcal{O}_S(1)|_{D_1}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{O}_S(2)|_D) \to \mathrm{Hom}(\mathcal{L}_2, \mathcal{O}_S(2)|_{D_2}) \\ \to \mathrm{Ext}^1(\mathcal{L}_1, \mathcal{O}_S(1)|_{D_1}) \to \dots . \quad (8.1)$$

If $\hom(\mathcal{L}_1, \mathcal{O}_S(1)|_{D_1}) = 1$, then everything is clear and if $\hom(\mathcal{L}_1, \mathcal{O}_S(1)|_{D_1}) = 0$, then also $\mathrm{ext}^1(\mathcal{L}_1, \mathcal{O}_S(1)|_{D_1}) = 0$. Thus in this case $\mathrm{Hom}(\mathcal{E}, \mathcal{O}(2)|_D) \cong \mathrm{Hom}(\mathcal{L}_2, \mathcal{O}(2)|_{D_2}) \neq 0$. (Actually, we must have $\hom(\mathcal{E}, \mathcal{O}(2)|_D) = 1$ because we assumed $h^0(\mathcal{E}) = 0$). Next, if $\mathcal{E}|_{D_i}$ has torsion, then $\mathcal{E}|_{D_i} \cong \mathcal{L}_i \oplus \mathcal{T}$, where $\mathcal{L}_i$ is torsionfree with $\chi(\mathcal{L}_i) = 0$ and $\mathcal{T}$ is supported on the intersection $D_1 \cap D_2$ with $\lg(\mathcal{T}) = 1$. In particular, also in this case the sequence (8.1) proves that $\mathrm{Hom}(\mathcal{E}, \mathcal{O}(2)|_D) \neq 0$.
Over Δ the argument is the same. Let $D = 2C \in \Delta$ and assume that we are given $\mathcal{E} \in M^1_{2C} \setminus M^0_{2C}$ such that $h^0(\mathcal{E}) = 0$. Again, $\mathcal{E} = T(\xi)$ if and only if $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_S(2)|_{2C}) \neq 0$. This time, we have $\mathcal{E}|_C = \mathcal{L} \oplus \mathcal{O}_x$ for some $\mathcal{L} \in \mathrm{Pic}^1(C)$ and $x \in C$ and the sequence (8.1) with $C = D_1 = D_2$ and $\mathcal{L} = \mathcal{L}_1 = \mathcal{L}_2$ proves what we need. □

We will see in Proposition 10.2, how the indeterminancy of T can be resolved by a sequence of blow-ups and blow-downs.

Lemma 8.2. *Let $\xi \subset S$ be a zero-dimensional subscheme of length n supported in a point $p \in S$. Assume there is an integral curve $C_1 \subset S$ such that $\lg(\mathcal{O}_\xi \otimes \mathcal{O}_{C_1}) = n-1$. Then*

$$\xi \subset C_1 \cup C_2$$

for every curve C_2 passing through p.

Proof. We can assume that $S = \mathrm{Spec}\, A$, where A is a local ring with maximal ideal $\mathfrak{m}$. Moreover, $\xi = V(I)$, and $C_1 = V(f)$ for some $f \in A$. By assumption, $\lg(A/I) = n$ and $\lg(A/(I,f)) = n-1$, hence $\lg((I,f)/I) = 1$. We we want to show that $f \cdot \mathfrak{m} \subset I$ or equivalently $(I, f \cdot \mathfrak{m}) = I$. We have a short exact sequence of $\mathbb{C}$-vector spaces

$$0 \to (I, f \cdot \mathfrak{m})/I \to (I,f)/I \to (I,f)/(I, f \cdot \mathfrak{m}) \to 0,$$

where the middle term is of dimension one. Hence, $(f \cdot \mathfrak{m}, I) = I$ is true if and only if the right outer term is non-zero. Assume $(I,f) = (I, f \cdot \mathfrak{m})$, then we can write $f = af + b$ for some $a \in \mathfrak{m}$ and $b \in I$. This implies $(1-a)f \in I$ and thus $f \in I$, which is a contradiction to our assumption. □

9. Brill–Noether loci in M and $S^{[5]}$

In Proposition 8.1, we established the isomorphism

$$T \colon S^{[5]} \setminus \{\xi \in S^{[5]} \mid h^0(\mathcal{I}_\xi(2)) \geq 2\} \xrightarrow{\sim} M \setminus (\{\mathcal{E} \in M \mid h^0(\mathcal{E}) \geq 1\} \cup M^0_\Delta).$$

In this section, we undertake a hands-on analysis of certain components of the Brill–Noether loci appearing here. Namely,

$$\mathrm{BN}^i(S^{[5]}) \coloneqq \{\xi \in S^{[5]} \mid h^0(\mathcal{I}_\xi(2)) \geq i+1\} \subset S^{[5]}$$

and

$$\mathrm{BN}^i(M) \coloneqq \{\mathcal{E} \in M \mid h^0(\mathcal{E}) \geq i+1\} \subset M$$

for $i \geq 0$. The first is also an actual Brill–Noether locus after the identification

$$S^{[5]} \cong M_H(1,2,0),\ \mathcal{I}_\xi \mapsto \mathcal{I}_\xi(2).$$

All these Brill–Noether loci generically have the structure of a projective bundle, which we explicitly state for certain components in Propositions 9.4 and 9.6.

9.1. Brill–Noether loci in $M_H(1,2,0) \cong S^{[5]}$. We study the Brill–Noether locus in $S^{[5]}$ or rather $M_H(1,2,0)$ first. Our first result shows that the only non-trivial cases are $i = 1, 2$. For $\xi \in S^{[5]}$, we introduce the linear subspace

$$B(\xi) \coloneqq \mathbb{P}(H^0(S, \mathcal{I}_\xi(2))) = \{D \in |2H| \mid \xi \subset D\} \subset |2H|.$$

Lemma 9.1. (i) *We have the inequalities*

$$0 \leq h^0(S, \mathcal{I}_\xi(1)) \leq 1 \leq h^0(S, \mathcal{I}_\xi(2)) \leq 3.$$

(ii) *If $h^0(S, \mathcal{I}_\xi(1)) = 1$, then $h^0(\mathcal{I}_\xi(2)) = 3$ and*

$$B(\xi) = m(C \times |H|) \subset \Sigma \subset |2H|,$$

where $C \in |H|$ is the unique curve containing ξ.

Proof. From the short exact sequence

$$0 \to \mathcal{I}_\xi(2) \to \mathcal{O}_S(2) \to \mathcal{O}_\xi \to 0,$$

it follows that $h^0(S, \mathcal{I}_\xi(2)) \geq 1$ for all $\xi \in S^{[5]}$.
First, assume that $B(\xi) \subset \Sigma$ then $\dim B(\xi) \leq 2$, as there is no three-dimensional linear subspace of $\mathbb{P}^5$ that is contained in $\Sigma = \mathrm{Sym}^2 \mathbb{P}^2$. So, in order to show $h^0(S, \mathcal{I}_\xi(2)) \leq 3$, we can assume that $B(\xi) \cap B \setminus \Sigma \neq \emptyset$, and we can even assume that there is a smooth curve $D \in |2H|$ such that $\xi \subset D$. We have compatible long exact sequences

$$\begin{tikzcd}
0 \arrow[r] & 0 = H^0(\mathcal{I}_\xi) \arrow[r] \arrow[d, hook] & H^0(\mathcal{I}_\xi(2)) \arrow[r, hook] \arrow[d, hook] \arrow[dr, "\alpha"] & H^0(\mathcal{I}_\xi(2)|_D) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & H^0(\mathcal{O}_S) \arrow[r] & H^0(\mathcal{O}_S(2)) \arrow[r] & H^0(\omega_D) \arrow[r] & 0,
\end{tikzcd}$$

where we inserted $\mathcal{O}_S(2)|_D \cong \omega_D$. Moreover,

$$\mathcal{I}_\xi(2)|_D \cong \omega_D(-\xi) \oplus \mathcal{O}_\xi.$$

Thus $H^0(\mathcal{I}_\xi(2)|_D) = H^0(\mathcal{O}_\xi) \oplus H^0(\omega_D(-\xi))$ and the first summand is the kernel of the third vertical map. Hence, $\dim \operatorname{im}(\alpha) \leq h^0(D, \omega_D(-\xi))$. Together, this gives

$$h^0(\mathcal{I}_\xi(2)) \leq \dim \operatorname{im}(\alpha) + h^0(\mathcal{O}_S) \leq h^0(D, \omega_D(-\xi)) + 1 = h^0(D, \mathcal{O}_D(\xi)) \leq 3,$$

where the last inequality uses Clifford's theorem [26, IV Thm 5.4].
Next, assume that $\xi \subset C$ for a curve $C \in |H|$. Then the analogous considerations yield

$$h^0(\mathcal{I}_\xi(1)) \leq \dim \ker(H^0(\mathcal{O}_S(1)) \to H^0(\mathcal{O}_S(1)|_C)) + h^0(\mathcal{I}_\xi(1)|_C) - 5 = 1 + 5 - 5 = 1.$$

This finishes the proof of (i).

Next, we prove (ii). Any non-zero section $s \in H^0(\mathcal{I}_\xi(1))$ induces a short exact sequence

$$0 \to \mathcal{O}_S(1) \xrightarrow{s} \mathcal{I}_\xi(2) \to \ker(\mathcal{O}_S(2)|_C \to \mathcal{O}_\xi) \to 0,$$

which gives the isomorphism $H^0(\mathcal{O}_S(1)) \cong H^0(\mathcal{I}_\xi(2))$ that translates into the statement for $B(\xi)$. $\square$

Next, we have two strategies to find explicit components of $\mathrm{BN}^i(S^{[5]})$. The first relies on the observation that, given a curve $D \in |2H|$ and $x \in D$, then also $\iota(x) \in D$, where $\iota\colon S \to S$ is the covering involution of $\pi\colon S \to |H| \cong \mathbb{P}^2$. Hence, the subvarieties in $S^{[5]}$ which parameterize subschemes that are partly invariant under ι are candidates to provide a component of the Brill–Noether locus. The second is based on Lemma 9.1(ii). Namely, we parameterize subschemes that are already or almost contained in a curve of the primitive linear system $|H|$.

Example 9.2 (cf. Example 7.15)**.** As mentioned in the introduction, we have an embedding

$$\mathbb{P}^2 \subset S^{[2]},\ x \mapsto \pi^{-1}(x).$$

We get generically injective rational maps

$$g_3\colon \mathbb{P}^2 \times S^{[3]} \dashrightarrow S^{[5]} \quad \text{and} \quad g_1\colon \mathbb{P}^2 \times \mathbb{P}^2 \times S \dashrightarrow S^{[5]}$$

and set

$$P_i := \overline{\operatorname{im}(g_i)} \subset S^{[5]} \text{ for } i = 1, 3.$$

Clearly, $P_3 \subset \mathrm{BN}^1(S^{[5]})$ and $P_1 \subset \mathrm{BN}^2(S^{[5]})$. Moreover, we note $\operatorname{codim} P_i = 5 - i$ and P_i is generically a $\mathbb{P}^{5-i}$-bundle over $S^{[i]}$.

Example 9.3 (cf. Example 7.16)**.** We define

$$W_2 := \{\xi \in S^{[5]} \mid H^0(\mathcal{I}_\xi(1)) \neq 0\} = \{\xi \in S^{[5]} \mid \text{there is } C \in |H| \text{ such that } \xi \subset C\}.$$

Then W_2 is the closure of the image of the generically injective rational map

$$\mathrm{Sym}^5_{\mathcal{C}_{|H|}/|H|}(\mathcal{C}_{|H|}) \dashrightarrow S^{[5]}$$

and therefore $\dim W_2 = 7$. Here, $\mathcal{C}_{|H|} \to |H|$ is the universal curve. We also define

$$W_3 \coloneqq \{\xi \in S^{[5]} \mid \text{there is } x \in \xi \text{ and } C \in |H| \text{ such that } \xi \setminus \{x\} \subset C\},$$

i.e. W_3 is the closure of the image of the generically injective rational map

$$\mathrm{Sym}^4_{\mathcal{C}_{|H|}/|H|}(\mathcal{C}_{|H|}) \times S \dashrightarrow S^{[5]}.$$

We conclude that $\dim W_3 = 8$.

Clearly, $W_2 \subset W_3$. By Lemma 9.1(i), $W_2 \subset \mathrm{BN}^2(S^{[5]})$. Similarly, one sees $W_3 \subset \mathrm{BN}^1(S^{[5]})$. Namely, if $\xi \setminus \{x\} \subset C$ as in the definition of W_3, then $\xi \subset C \cup C'$ for every $C' \in |H|$ such that $x \in C'$ and thus $\dim B(\xi) \geq 1$.

The subvarieties W_2 and W_3 have also appeared in Subsection 7.3.3 or [35, Thm 6.4] as examples of algebraically coisotropic subvarieties in $S^{[5]}$. We can give the precise structure of a projective bundle.

Proposition 9.4. (i) *The subvariety W_2 is a $\mathbb{P}^3$-bundle over $M_H(0,1,-6)$. More precisely, let $\mathcal{E}^{-6}_{\mathrm{univ}}$ be the universal bundle on $M_H(0,1,-6) \times S$ and define the sheaf*

$$\mathcal{E}_2 \coloneqq p_{1*} R\mathcal{H}om(\mathcal{E}^{-6}_{\mathrm{univ}}, p_2^* \mathcal{O}_S(-1)).$$

on $M_H(0,1,-6)$. Then $\mathcal{E}_2$ is a vector bundle and

$$W_2 \cong \mathbb{P}(\mathcal{E}_2).$$

In particular, W_2 is smooth.

(ii) *The subvariety $W_3 \setminus W_2$ is a $\mathbb{P}^2$-bundle over an open subset of $S \times M_H(0,1,-5)$. More precisely, let $\mathcal{E}^{-5}_{\mathrm{univ}}$ be the universal sheaf on $M_H(0,1,-5) \times S$ and $\mathcal{I}_\Delta$ the ideal sheaf of the diagonal $\Delta \subset S \times S$ and define the sheaf*

$$\mathcal{E}_3 \coloneqq p_{12*} R\mathcal{H}om(p_{23}^* \mathcal{E}^{-5}_{\mathrm{univ}}, p_3^* \mathcal{O}_S(-1) \otimes p_{13}^* \mathcal{I}_\Delta)$$

on $S \times M_H(0,1,-5)$. Then $\mathcal{E}_3$ a vector bundle on an open set $U \subset S \times M_H(0,1,-5)$ and

$$W_3 \setminus W_2 \cong \mathbb{P}(\mathcal{E}_3|_U).$$

Proof. (i) For every $\mathcal{E} \in M_H(0,1,-6)$ we have the base change map

$$\mathcal{E}_2(\mathcal{E}) \to H^0(S, R\mathcal{H}om(\mathcal{E}, \mathcal{O}_S(-1))) \cong \mathrm{Ext}^1_S(\mathcal{E}, \mathcal{O}_S(-1))$$

and $\mathrm{Ext}^i_S(\mathcal{E}, \mathcal{O}_S(-1)) = 0$ for $i \neq 1$ because $\mathcal{E}$ is stable of rank 0. Hence $\dim \mathrm{Ext}^1(\mathcal{E}, \mathcal{O}_S(-1)) = 4$ for all $\mathcal{E} \in M_H(0,1,-6)$ and $\mathbb{P}(\mathcal{E}_2) \to M_H(0,1,-6)$ is indeed a $\mathbb{P}^3$-bundle parameterizing extensions of $\mathcal{E} \in M_H(0,1,-6)$ by $\mathcal{O}_S(-1)$. Moreover, as $\mathcal{E}xt^1_S(\mathcal{E}, \mathcal{O}_S(-1))$ is torsion free, any non-split extension

$$0 \to \mathcal{O}_S(-1) \to \mathcal{I} \to \mathcal{E} \to 0 \tag{9.1}$$

does not admit a local splitting. Hence, the middle term $\mathcal{I}$ of such an extension is torsion free [31, Prop 1.1.10] and must be the ideal sheaf of a zero-dimensional subscheme [31, Expl 1.1.16]. In particular, $\mathcal{I}$ is H-Gieseker stable and the universal extension on $\mathbb{P}(\mathcal{E}_2) \times S$ defines a map

$$\psi_2 \colon \mathbb{P}(\mathcal{E}_2) \longrightarrow M_H(1,0,-4) = S^{[5]}$$

whose image is W_2. It is left to show that ψ_2 is an isomorphism onto its image. For injectivity, assume that there is $\xi \in S^{[5]}$ such that $\mathcal{I}_\xi$ fits into two different extensions of the form (9.1). But then $h^0(S, \mathcal{I}_\xi(1)) \geq 2$ which is absurd (cf. Lemma 9.1). Finally, we have $\mathrm{Hom}(\mathcal{O}_S(-1), \mathcal{E}) = 0$, which signifies that the extensions of the form (9.1) are rigid [24, Thm 6.4.5] and therefore ψ_2 is really an immersion of schemes.

(ii) In the case of W_3, we find

$$\mathcal{E}_3(x, \mathcal{E}) \to H^0(S, R\mathcal{H}om(\mathcal{E}, \mathcal{I}_x(-1)))$$

and the right hand side is isomorphic to $\mathrm{Ext}^1_S(\mathcal{E}, \mathcal{I}_x(-1))$ if $x \neq \mathrm{Supp}(\mathcal{E})$ and isomorphic to $\mathrm{Ext}^1_S(\mathcal{E}, \mathcal{I}_x(-1)) \oplus \mathbb{C}$ if $x \in \mathrm{Supp}(\mathcal{E})$. As above, we have $\dim \mathrm{Ext}^1(\mathcal{E}, \mathcal{I}_x(-1)) = 3$ for all $(x, \mathcal{E}) \in S \times M_H(0,1,-5)$. Hence, $\mathcal{E}_3$ is a vector bundle on the open subset $U \subset S \times M$ that is the inverse image of the complement of the universal curve $\mathcal{C}_{|H|} \subset S \times |H|$ under the product of the support morphism and the identity. Again, one checks that for $(x, \mathcal{E}) \in U$ the middle term of every non-split extension

$$0 \to \mathcal{I}_x(-1) \to \mathcal{I} \to \mathcal{E} \to 0, \tag{9.2}$$

is a pure, H-Gieseker stable sheaf with Mukai vector $(1,0,-4)$ and so the associated universal extension defines a map

$$\psi_3 \colon \mathbb{P}(\mathcal{E}_3|_U) \to S^{[5]}$$

whose image is clearly contained in W_3. We claim, that ψ_3 is an isomorphism onto $W_3 \setminus W_2$. Again, $\mathrm{Hom}(\mathcal{I}_x(-1), \mathcal{E}) = 0$ and so ψ_3 is a local isomorphism. If $\mathcal{I}_\xi$ can be written in two different extensions, say over $(x, \mathcal{E})$ and $(x', \mathcal{E}')$, it follows that $\xi \setminus \{x, x'\} \subset \mathrm{Supp}(\mathcal{E})$ as well as $\xi \setminus \{x, x'\} \subset \mathrm{Supp}(\mathcal{E}')$. However, a scheme of length 3 is at most contained in one curve $C \in |H|$. Hence $\mathrm{Supp}(\mathcal{E}) = \mathrm{Supp}(\mathcal{E}')$, which implies $x = x'$ and also identifies the first arrow up to a scalar. In other words, all the data match and ψ_3 is injective. Finally, $\xi \in \mathrm{im}(\psi_3)$ if and only if there is $x \in \xi$ such that $\xi \setminus \{x\}$ is contained in a unique curve $C \in |H|$ but $x \notin C$. Hence, $\mathrm{im}(\psi_3) = W_3 \setminus W_2$. □

In Proposition 10.2, we encounter the flop at W_2 and W_3, respectively.

9.2. Brill–Noether loci in M. In this section, we study the Brill–Noether loci in M. Recall that over a smooth curve $D \in |2H|$, the fiber $f^{-1}(D)$ is isomorphic to $\mathrm{Pic}^3(D)$ and therefore

$$\mathrm{BN}^i(M) \cap f^{-1}(D) = W^i_3(D) \subset \mathrm{Pic}^3(D)$$

is the classical Brill–Noether locus $W^1_3(D)$ [2]. It is known, that a general curve in a primitive linear system on a general K3 surface is Brill–Noether general [36]. This implies in particular that W^r_d (when non-empty) has the expected dimension

$$\dim W^r_d = \rho(g,r,d) = g-(r+1)(g-d+1).$$

However, for non-primitive linear systems unexpected things may happen, as we encounter below.

First, we deal with the structure of $\mathrm{BN}^i(M)$ over the locus of smooth curves $B^\circ \subset B$. Our construction uses the fact, that the moduli spaces $M_H(0,2,k)$ for odd k are all isomorphic (cf. Lemma 5.1). Let $\mathcal{C}^\circ \to B^\circ$ be the corresponding universal curve. For any k, we have an isomorphism

$$M_H(0,2,k-4)^\circ \cong \mathrm{Pic}^k_{\mathcal{C}^\circ/B^\circ},$$

where $M_H(0,2,k-4)^\circ$ is the preimage of B° under the support map $M_H(0,2,k-4)\to B$. We define

$$\mathrm{BN}^i_k(B^\circ) \coloneqq \{\mathcal{L}\in M_H(0,2,k-4)^\circ \mid h^0(S,\mathcal{L})\geq i+1\} \subset M_H(0,2,k-4)^\circ$$

and consider its closure in two particular cases

$$Z_1 \coloneqq \overline{\mathrm{BN}^0_1(B^\circ)} \subset M_H(0,2,-3) \quad \text{and} \quad Z_3 \coloneqq \overline{\mathrm{BN}^0_3(B^\circ)} \subset M = M_H(0,2-1).$$

We expect $Z_3 \subset BN^0(M)$ to be a strict inclusion, as the latter might have components over Σ or Δ.

In the following, we will consider Z_1 as a subvariety of M via the isomorphism

$$M_H(0,2,-3) \xrightarrow{\sim} M,\ \mathcal{E}\mapsto \mathcal{E}xt^1(\mathcal{E},\mathcal{O}_S)(-1).$$

In particular, over a smooth curve $D\in|2H|$, we have

$$\mathrm{Pic}^1(D)\to\mathrm{Pic}^3(D),\ \mathcal{L}\mapsto\mathcal{L}^\vee\otimes\mathcal{O}_S(1)|_D$$

and

$$Z_1\cap f^{-1}(D) = \{\mathcal{L}\in\mathrm{Pic}^3(D)\mid h^0(\mathcal{L}\otimes\mathcal{O}_S(1)|_D)\geq 4\} = \{\mathcal{L}\in\mathrm{Pic}^3(D)\mid h^1(\mathcal{L}\otimes\mathcal{O}_S(1)|_D)\neq 0\}.$$

Lemma 9.5 (= Lemma 7.2)**.** *We have*

$$Z_1\subset \mathrm{BN}^1(M).$$

In particular, there is an inclusion

$$Z_1\subset Z_3.$$

In Corollary 10.5, we prove that actually $Z_3\cap\mathrm{BN}^1(M) = Z_1$.

Proof. It suffices to show the result over a smooth curve $D \in |2H|$. Let $\mathcal{L} \in \mathrm{Pic}^1(D)$ such that $H^0(D, \mathcal{L}) \neq 0$. We want to show that $h^0(D, \mathcal{L}^\vee \otimes \mathcal{O}_S(1)|_D) \geq 2$. Write $\mathcal{L} = \mathcal{O}_D(x)$ for a point $x \in D$. On S, we have a short exact sequence

$$0 \to \mathcal{O}_S(-1) \to \mathcal{I}_x(1) \to \mathcal{O}_D(-x) \otimes \mathcal{O}_S(1)|_D \to 0$$

and the resulting long exact cohomology sequence proves the lemma. □

Proposition 9.6. (i) *There is an embedding*

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[dr] & M \arrow[d, "f"] \\ & B, \end{tikzcd}$$

whose image is Z_1. In particular, $\dim Z_1 = 6$ and Z_1 is a $\mathbb{P}^4$-bundle over S.

(ii) *Z_3 is generically isomorphic to a $\mathbb{P}^2$-bundle over $S^{[3]}$. In particular, $\dim Z_3 = 8$.*

Proof. The proof of (i) and (ii) works analogously. The idea is that, over $D \in B^\circ$ we want to parameterize the line bundles $\mathcal{O}_D(-\xi_i), i = 1, 3$ for a point $\xi_1 \in D$ and a divisor $\xi_3 \subset D$ of degree 3, respectively. These are the ideal schemes of $\xi_i \subset D$ and can be realized as the quotient of the respective ideal sheaves in S

$$0 \to \mathcal{I}_{D/S} = \mathcal{O}_S(-2) \to \mathcal{I}_{\xi_i/S} \to \mathcal{I}_{\xi_i/D} = \mathcal{O}_D(-\xi_i) \to 0.$$

Hence, our task is to parameterize $\xi_i \subset D$ for $\xi_i \in S^{[i]}$ and $D \in B$, define the above sequence universally and show that the cokernel defines a map to $M_H(0, 2, -i-4)$ for $i = 1, 3$.

The first step is straightforward. For $i = 1, 3$ define

$$\mathcal{X}_i := \mathbb{P}(p_{2*}(\mathcal{I}_{\mathcal{Z}_i} \otimes p_1^* \mathcal{O}_S(2))) \to S^{[i]},$$

where $\mathcal{Z}_i \subset S \times S^{[i]}$ is the universal subscheme and p_i are the projections from $S \times S^{[i]}$ for $i = 1, 2$. The inclusion

$$p_{2*}(\mathcal{I}_{\mathcal{Z}_i} \otimes p_1^* \mathcal{O}_S(2)) \hookrightarrow p_{2*}(\mathcal{O}_{S \times S^{[i]}} \otimes p_1^* \mathcal{O}_S(2)) \cong H^0(S, \mathcal{O}_S(2)) \otimes \mathcal{O}_{S^{[i]}}$$

defines an embedding $\mathcal{X}_i \subset S^{[i]} \times B$. We could also think of $\mathcal{X}_i$ as $\mathrm{Hilb}^i(\mathcal{C}/B)$, i.e.

$$\mathcal{X}_1 = \mathcal{C} = \{p \in D\} \subset S \times B \quad \text{and} \quad \mathcal{X}_3 = \{\xi \subset D\} \subset S^{[3]} \times B.$$

Note that $\mathcal{X}_1$ is a $\mathbb{P}^4$-bundle over S and $\mathcal{X}_3$ is generically a $\mathbb{P}^2$-bundle over $S^{[3]}$. On $S \times \mathcal{X}_i$, we have the sequence

$$0 \to (\mathrm{id} \times p_B)^* \mathcal{O}_{S \times B}(-\mathcal{C}) \to (p_S \times \mathrm{id})^* \mathcal{I}_{\mathcal{Z}_i} \to \mathcal{Q}_i \to 0.$$

Here, $\mathcal{Q}_i$ is defined to be the cokernel, which is flat over $\mathcal{X}_i$ and $v(\mathcal{Q}_i|_{S\times\{p\}}) = (0,2,-i-4)$ for all $p \in \mathcal{X}_i$. Consequently, $\mathcal{Q}_i$ gives a map

$$\begin{tikzcd} \mathcal{X}_i \arrow[rr, dashed, "\varphi_i"] \arrow[dr] & & M_H(0,2,-i-4), \arrow[dl] \\ & B & \end{tikzcd} \qquad p \mapsto \mathcal{Q}_i|_{S\times\{p\}}$$

which is defined in $p \in \mathcal{X}_i$, whenever $\mathcal{Q}_i|_{S\times\{p\}}$ is stable. By definition, we have $\operatorname{im}(\varphi_i) \subset Z_i$. For simplicity, we restrict to the case $i = 1$ in the rest of the proof. Actually, the more powerful methods from Proposition 10.2 allow us to conclude without explicit computation, that $Z_3 \setminus Z_1$ is a projective bundle.
We claim that φ_1 is everywhere defined and immersive. Clearly, φ_1 is defined and injective over $B \setminus \Sigma$ and with the same arguments as in the proof of Proposition 8.1 this also holds true over Δ. So $\operatorname{im}(\varphi_1) = Z_1$ and it is left to show that φ_1 is an immersion. We show that the induced map on tangent spaces is injective. To this end, let $S[\varepsilon] \coloneqq S \times_{\mathbb{C}} \mathbb{C}[\varepsilon] \xrightarrow{p} S$ and assume we are given a $\mathbb{C}[\varepsilon]$-valued point of $\mathcal{C}$ that maps to a trivial deformation of $\mathcal{E} \in M_H(0,2,-5)$, i.e. this point corresponds to a sequence

$$p^*\mathcal{O}_S(-2) \to \mathcal{S} \to p^*\mathcal{E} \to 0 \tag{9.3}$$

on $S[\varepsilon]$. Here, we use that the line bundle $\mathcal{O}_S(-2)$ is rigid. We want to see that $\mathcal{S} = p^*\mathcal{I}_x$, where $\mathcal{E}$, as before, sits in the sequence $0 \to \mathcal{O}_S(-2) \to \mathcal{I}_x \to \mathcal{E} \to 0$. By definition, (9.3) embeds into a diagram

$$\begin{tikzcd} p^*\mathcal{O}_S(-2) \arrow[r] \arrow[d, equal] & \mathcal{S} \arrow[r] \arrow[d] & p^*\mathcal{E} \arrow[r] \arrow[d] & 0 \\ p^*\mathcal{O}_S(-2) \arrow[r] & \mathcal{O}_{S[\varepsilon]} \arrow[r] \arrow[d] & \mathcal{O}_{\tilde{D}} \arrow[r] \arrow[d] & 0 \\ & \mathcal{O}_{\tilde{x}} \arrow[r, equal] \arrow[d] & \mathcal{O}_{\tilde{x}} \arrow[d] & \\ & 0 & 0, & \end{tikzcd}$$

where $\tilde{x} \subset \tilde{D} \subset S[\varepsilon]$ are deformations of x and D. As $\operatorname{Supp}(p^*\mathcal{E}) = D[\varepsilon]$, we must have $p^*\mathcal{E} = \mathcal{O}_{D[\varepsilon]}(-x[\varepsilon])$ and we can conclude that all deformations are trivial. Hence, $\mathcal{S} = p^*\mathcal{I}_x$. $\square$

Note that the smooth curves in $|2H|$ are hyperelliptic and so there is a unique line bundle $g_2^1(D) \in \operatorname{Pic}^2(D)$ such that $h^0(g_2^1) = 2$.

Corollary 9.7. *Let $D \in |2H|$ is a smooth curve and $\mathcal{L} \in f^{-1}(D)$. Then*

$$\mathcal{L} \in Z_1 \cap f^{-1}(D) \iff \mathcal{L} \cong \mathcal{O}_D(x) \otimes g_2^1 \text{ for some } x \in D.$$

Proof. By dimension reasons, it suffices to show one implication. Assume $\mathcal{L} \cong \mathcal{O}_D(x) \otimes g_2^1$ for some $x \in D$. We know that $\mathcal{O}_S(1)|_D \cong (g_2^1)^{\otimes 2}$ [26, Prop 5.3]. Thus

$$h^1(\mathcal{L} \otimes \mathcal{O}_S(1)|_D) = h^1(\mathcal{O}_D(x) \otimes (g_2^1)^{\otimes 3}) = h^0(\mathcal{O}_D(-x) \otimes g_2^1) \neq 0,$$

which proves the claim. □

Remark 9.8. Another consequence of Proposition 9.6 is that $W_3^0(D)$ has the expected dimension

$$\dim W_3^0(D) = \rho(5, 0, 3) = 3$$

and that

$$\dim W_3^1(D) \geq 1$$

for $D \in |2H|$ general, even though the Brill–Noether number $\rho(5, 1, 3) = 5 - 2(5 - 3 + 1)$ is negative. The latter also follows, since $g_2^1(D) \otimes \mathcal{O}_D(x) \in W_3^1(D)$ for all $x \in D$. In Corollary 10.5, we will see that every line bundle in $W_3^1(D)$ is of this form, i.e. $\dim W_3^1(D) = 1$. Finally, we know

$$W_3^2(D) = \emptyset$$

from Clifford's theorem [26, IV Thm 5.4].

10. Computation of the birational models

In this final section, we leave our hands-on methods behind and apply the techniques of Bayer and Macrì to get a full picture of the birational models of $S^{[5]}$ and the associated birational wall-crossing transformations. We find that there are five birational models of M (including M and $S^{[5]}$) and moreover match the exceptional loci of the flopping contractions with the subvarieties from the previous section. By a birational model, we mean a smooth projective variety with trivial canonical bundle that is birational to M.

10.1. Numerical characterization of the walls in $\mathrm{Mov}(M)$**.** We will compute the wall and chamber decomposition of the movable cone of $S^{[5]}$ (resp. M), whose chambers correspond to the birational models of $S^{[5]}$ (resp. M) using Bayer and Macrì's results [6]. We start by recalling the basic definitions and relevant statements in this context.

Let X be an irreducible holomorphic symplectic manifold. Recall that the *positive cone* $\mathrm{Pos}(X) \subset \mathrm{NS}(X)_{\mathbb{R}}$ is the connected component of $\{x \in \mathrm{NS}(X)_{\mathbb{R}} \mid (x, x) > 0\}$ containing a

Kähler class. The *movable cone* $\mathrm{Mov}(X) \subset \mathrm{NS}(X)_{\mathbb{R}}$ is the open cone generated by the classes of divisors D such that $|D|$ has no divisorial base locus. We have the inclusions

$$\overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X) \subset \overline{\mathrm{Mov}(X)} \subset \overline{\mathrm{Pos}(X)} \subset \mathrm{NS}(X)_{\mathbb{R}}.$$

The movable cone admits a locally polyhedral chamber decomposition, whose chambers correspond to smooth birational models of X. More precisely,

$$\overline{\mathrm{Mov}(X)} = \overline{\bigcup_g g^* \mathrm{Nef}(X')},$$

where the union is taken over all birational maps $g \colon X \dashrightarrow X'$ from X to another irreducible holomorphic symplectic manifold X' [27, Thm 7, Cor 19].

Assume that $X = M_\sigma(v)$ is a smooth projective moduli space of σ-stable objects in $D^b(S)$ with $v^2 > 0$. In this case, the Mukai morphism [6, Thm 3.6] gives the identification

$$\lambda_X \colon v^\perp \xrightarrow{\sim} \mathrm{NS}(X).$$

Here, $v^\perp \subset H^*_{\mathrm{alg}}(S, \mathbb{Z})$. By [6, Thm 12.1], the nef (resp. movable) cone of X is one of the chambers of the decomposition of the positive cone $\overline{\mathrm{Pos}(X)}$ whose walls are the orthogonal complement to linear subspaces

$$\lambda_X(\mathcal{H}^\perp),$$

where $\mathcal{H} \subset H^*_{\mathrm{alg}}(S, \mathbb{Z})$ is a primitive sublattice of signature $(1, 1)$ that contains v. If $\mathcal{H} = \langle v, a \rangle$, then $a \in H^*_{\mathrm{alg}}(S, \mathbb{Z})$ can be chosen such that $a^2 \geq -2$ and $0 \leq (v, a) \leq \frac{v^2}{2}$ (resp. ($a^2 = -2$ and $(v, a) = 0$) or ($a^2 = 0$ and $(a, v) \in \{1, 2\}$)). Following [6, Thm 5.7], the lattice $\mathcal{H}$ governs the geometry of the birational transformation at the respective wall. One can distinguish the following cases:

(a) The lattice $\mathcal{H}$ is isomorphic to one of the following: $\begin{pmatrix} -2 & 0 \\ 0 & v^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & v^2 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & v^2 \end{pmatrix}$. This case corresponds to a divisorial contraction.

(b) The lattice $\mathcal{H}$ is none of the above and there is either $s \in \mathcal{H}$ such that $s^2 = -2$ and $0 < (s, v) \leq \frac{v^2}{2}$ or v is the sum $v = a_1 + a_2$ of two positive classes $a_i \in \mathcal{H}$ (i.e. $a_i^2 \geq 0$ and $(a_i, v) > 0$ for $i = 1, 2$). This case corresponds to a flopping contraction.

(c) In all other cases, the birational transformation is actually an isomorphism.

The rough idea here is, that a wall of the ample cone is induced by a wall-crossing in the space of stability conditions and the associated contraction contracts precisely the curves of objects that are S-equivalent with respect to the stability condition on the wall [5, Thm 1.4(a)]. To a wall $\mathcal{W}$ (with respect to v) of the stability manifold, Bayer and Macrì associate a rank two sublattice [6, Prop 5.1]

$$\mathcal{H} := \{w \in H^*_{\mathrm{alg}}(S, \mathbb{Z}) \mid \phi_0(w) = \phi_0(v) \text{ for all } \sigma_0 \in \mathcal{W}\} \subset H^*_{\mathrm{alg}}(S, \mathbb{Z}).$$

Here, ϕ_0 is the phase associated to $\sigma_0 = (Z_0, \mathcal{A}_0)$. Then $\mathcal{H}$ has the property that if $\mathcal{E}$ is a σ-stable object and A_i is a factor in its Harder–Narasimhan filtration with respect to a stability condition σ_-, which lies sufficiently close on the other side of the wall, then $v(A_i) \in \mathcal{H}$. Now, let $\mathcal{E}_1$ and $\mathcal{E}_2 \in M_\sigma(v)$ have the same Harder–Narasimhan factors with respect to σ_-. As one can always find a Jordan–Hölder filtration that is a refinement of the Harder-Narasimhan filtration, this implies that $\mathcal{E}_1$ and $\mathcal{E}_2$ are S-equivalent and therefore contracted under the transformation induced from the wall-crossing. Consequently, in order to understand this transformation, one has to parameterize possible Harder–Narasimhan filtrations whose factors have Mukai vectors in $\mathcal{H}$. Unfortunately, it may also happen that the open subset $M^{\mathrm{st}}_{\sigma_0}(v) \subset M_{\sigma_0}(v)$ of stable objects is empty. In this case, the behavior at the wall is harder to control and we call $\mathcal{W}$ a *totally semistable wall*. By [6, Thm 5.7], $\mathcal{W}$ is totally semistable if and only if

(a') there is $w \in \mathcal{H}$ such that $w^2 = 0$ and $(v, w) = 1$ or
(b') there is $s \in \mathcal{H}$ such that $s^2 = -2, M_{\sigma_0}(s) \neq \emptyset$ and $(s, v) < 0$.

The content of Bayer and Macrì's article [6] is a detailed study of the possible lattices and the associated modifications of the moduli space, which in particular, yields the above lists.

For our computations, we fix the following notation: We set

$$v = (0, 2, -1) \quad \text{and} \quad v' = (1, 0, -4).$$

The respective Mukai morphisms fit in the commutative diagram

$$\begin{tikzcd}
H^*_{\mathrm{alg}}(S,\mathbb{Z}) \arrow[d, "T^*", "\cong"'] & v^\perp \arrow[l, hook'] \arrow[r, "\lambda_M"] \arrow[d, "\cong"'] & \mathrm{NS}(M) \arrow[d, "T^*", "\cong"'] \\
H^*_{\mathrm{alg}}(S,\mathbb{Z}) & v'^\perp \arrow[l, hook'] \arrow[r, "\lambda_{S^{[5]}}"] & \mathrm{NS}(S^{[5]}),
\end{tikzcd}$$

where, by abuse of notation, we also write $T^* \colon H^*_{\mathrm{alg}}(S,\mathbb{Z}) \to H^*_{\mathrm{alg}}(S,\mathbb{Z})$ for the isomorphism that makes the left square commute. It is defined as the composition

$$H^*_{\mathrm{alg}}(S,\mathbb{Z}) \xrightarrow{\cdot\,\mathrm{ch}(\mathcal{O}_S(-2))} H^*_{\mathrm{alg}}(S,\mathbb{Z}) \xrightarrow{\rho_{v(\mathcal{O}_S(-2))}} H^*_{\mathrm{alg}}(S,\mathbb{Z}),$$

where $\rho_{v(\mathcal{O}_S(-2))}$ is the reflection at the hyperplane orthogonal to $v(\mathcal{O}_S(-2)) = (1, -2, 5)$. In our usual basis, T^* is given by the matrix

$$\begin{pmatrix} -4 & -4 & -1 \\ 10 & 9 & 2 \\ -25 & -20 & -4 \end{pmatrix} \circ \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 2 \\ -1 & -4 & -4 \end{pmatrix}. \tag{10.1}$$

We have the following basis of $\mathrm{NS}(S^{[5]})$

$$\delta := \lambda_{S^{[5]}}(-1, 0, -4) \quad \text{and} \quad H := \lambda_{S^{[5]}}(0, -1, 0).$$

For the Hilbert scheme $S^{[n]}$, computing the walls in $\overline{\mathrm{Pos}(S^{[n]})}$ reduces to solving Pell's equation, cf. [6, Prop. 13.1] and also [15, Lem 2.5]. In our case, we get the following list of walls with respective intersection properties:

i	$a'_i \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$	(a'_i,a'_i)	(a'_i,v')	$D_i \in H^2(S^{[5]},\mathbb{Z})$	(D_i,D_i)	$R_i \in H_2(S^{[5]},\mathbb{Z})$	(R_i,R_i)
0	$(0,0,1)$	0	1	$-\delta$	-8	$-\delta^\vee$	$-\frac{1}{8}$
1	$(1,-1,2)$	-2	2	$4H-3\delta$	-40	$H-6\delta^\vee$	$-\frac{5}{2}$
2	$(1,-1,1)$	0	3	$8H-5\delta$	-72	$H-5\delta^\vee$	$-\frac{9}{8}$
3	$(-1,2,-5)$	-2	1	$-16H+9\delta$	-136	$-2H+9\delta^\vee$	$-\frac{17}{8}$
4	$(2,-3,5)$	-2	3	$24H-13\delta$	-200	$3H-13\delta^\vee$	$-\frac{25}{8}$
5	$(-1,2,-4)$	0	0	$-2H+\delta$	0	$-H+4\delta^\vee$	0.

Here, $\pm D_i \in \mathrm{NS}(S^{[5]})$ is the integral class defining the same wall as a'_i, i.e.

$$D_i^\perp = \lambda_{S^{[5]}}({a'}_i^\perp \cap {v'}^\perp)$$

or, in other words, D_i is a rational multiple of the orthogonal projection of a'_i to $({v'}^\perp)_{\mathbb{Q}}$. The sign here is chosen such that this multiple is positive, but it does not matter. In [43], the classes $\pm D_i$ are called wall divisors. Moreover, $R_i \in H_2(S^{[5]},\mathbb{Z})$ is the curve class corresponding to $D_i \in H^2(S^{[5]},\mathbb{Z})$, i.e. D_i is the smallest positive multiple of R_i contained in $H^2(S^{[5]},\mathbb{Z})$ under the embedding $H^2(S^{[5]},\mathbb{Z}) \subset H_2(S^{[5]},\mathbb{Z})$ coming from the intersection form. Below, we also give the list of walls in coordinates of M.

Corollary 10.1. *For a K3 surface S with $\mathrm{Pic}(S) = \mathbb{Z}\cdot H$ and $H^2 = 2$, there are five smooth birational models of $S^{[5]}$ (including $S^{[5]}$ itself).*

10.2. Geometrical characterization of the walls in $\mathrm{Mov}(M)$. So far, we know that the movable cone of M or $S^{[5]}$, respectively, is divided into five chambers. The outer ones correspond to M, with the Lagrangian fibration and to $S^{[5]}$ with the Hilbert–Chow morphism. Next, we want to understand exceptional loci (and their strict transforms in M and $S^{[5]}$, respectively) of the birational transformations between two models in adjacent chambers.

The geometry of the occuring contractions is studied in-depth in [6, §9], to which we refer for the precise results. As mentioned above, the rough idea is to parameterize objects with prescribed Harder-Narasimhan filtration with respect to a stability condition on the other side of the wall. This translates into finding decompositions $v = a_1 + \ldots + a_m$ into effective classes $a_i \in \mathcal{H}$, where $\mathcal{H} \subset H^*_{\mathrm{alg}}(S,\mathbb{Z})$ is the sublattice such that $\lambda_{M_\sigma(v)}(\mathcal{H}^\perp)$ cuts out the wall of the ample cone. Here, a class $a \in \mathcal{H}$ is called *effective* if $M_{\sigma_0}(a) \neq \emptyset$ [6, Prop 5.5], and a class $a \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$ is called *positive*, if $a^2 \geq 0$ and $(a,v) > 0$. All positive classes are effective.

Let $\mathcal{H}$ define a flopping wall for $M_\sigma(v)$. By [6, Prop 9.1] there are two cases: Either

(i) there is a decomposition $v = a + b$ into two positive classes and $\mathcal{H} = \langle v, a\rangle$. Or
(ii) there is a spherical class $s \in \mathcal{H}$ such that $0 < (s, v) \leq \frac{v^2}{2}$.

In case (i), assume moreover that the wall is not totally semistable with respect to a or b (e.g. if $\mathcal{H}$ does not contain any spherical or isotropic classes), and that $\phi_\sigma(a) < \phi_\sigma(b)$, where ϕ_σ is the phase of the stability condition σ. By [6, §9], the decomposition $v = a + b$ defines an irreducible component E of the exceptional locus of the contraction morphism associated to the wall defined by $\mathcal{H}$, such that a generic point $\mathcal{E} \in E$ is an extension

$$\mathcal{A} \to \mathcal{E} \to \mathcal{B} \xrightarrow{[1]},$$

where $\mathcal{A}$ and $\mathcal{B}$ are σ-stable objects of Mukai vector a and b, respectively. By assumption on the wall, $\mathcal{A}$ and $\mathcal{B}$ are generically also σ_0-stable, where σ_0 is a generic stability condition on the wall. And by definition of $\mathcal{H}$, we have $\phi_{\sigma_0}(a) = \phi_{\sigma_0}(b)$. Hence, $\mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0 = \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Finally, $\mathcal{E}$ defines a class in $\mathbb{P}^1(\mathrm{Ext}^1_{D^b(S)}(\mathcal{B}, \mathcal{A}))$ and we find

$$r := \dim \mathbb{P}^1(\mathrm{Ext}^1_{D^b(S)}(\mathcal{B}, \mathcal{A})) = (v - a, a) - 1.$$

Thus, E has generically the structure of a $\mathbb{P}^r$-bundle over $M_\sigma(a) \times M_\sigma(b)$. By assumption, $M_\sigma(a)$ and $M_\sigma(b)$ are both non-empty and

$$\begin{aligned}\operatorname{codim} E &= (v, v) + 2 - \dim M_\sigma(a) - \dim M_\sigma(b) - \dim \mathbb{P}^1(\mathrm{Ext}^1_{D^b(S)}(\mathcal{B}, \mathcal{A}))\\ &= (v, v) + 2 - ((a, a) + 2) - ((v - a, v - a) + 2) - ((v - a, a) - 1)\\ &= (v - a, a) - 1 = r.\end{aligned}$$

It may happen that case (ii) is a special case of (i). Otherwise $\mathcal{H} = \langle v, s\rangle$ and the above results also hold for the decomposition $v = s + (v - s)$ if s is effective and need extra care, if s is not effective [6, Proof of Prop 9.1].

The relevant data for our example is listed in the following table:

i	$a'_i \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$	$a_i \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$	(a_i, a_i)	(a_i, v)	r_i
0	$(0,0,1)$	$(-1,0,0)$	0	1	
1	$(1,-1,2)$	$(-2,1,-1)$	-2	2	3
2	$(1,-1,1)$	$(-1,1,-1)$	0	3	2
3	$(-1,2,-5)$	$(1,0,1)$	-2	1	2
4	$(2,-3,5)$	$(-1,1,-2)$	-2	3	4
5	$(-1,2,-4)$	$(0,0,1)$	0	0.	

Here, $T^* a_i = a'_i$ (cf. (10.1)), so that

$$\lambda_M(a_i^\perp \cap v^\perp) = \lambda_{S^{[5]}}({a'_i}^\perp \cap {v'}^\perp)$$

cuts out the walls in $\overline{\mathrm{Pos}(M)} \cong \overline{\mathrm{Pos}(S^{[5]})}$.

When following the birational transformations from $S^{[5]}$ to M, not all exceptional loci can naively be identified in $S^{[5]}$ as their codimension $r_i = (v - a_i, a_i) - 1$ is not strictly decreasing. However, if we approach from either side, we get a beautiful description.

Theorem 10.2. *The birational models of $S^{[5]}$ are connected by the following chain of flopping contractions*

$$\begin{tikzcd}[column sep=small]
& \mathrm{Bl}_{W_2} S^{[5]} \arrow[dl] \arrow[dr] & & \mathrm{Bl}_{\tilde{W}_3} X_1 \arrow[dl] \arrow[dr] & & \mathrm{Bl}_{\tilde{Z}_3} X_3 \arrow[dl] \arrow[dr] & & \mathrm{Bl}_{Z_1} M \arrow[dl] \arrow[dr] & \\
S^{[5]} \arrow[rr, dashed, "g_1"] & & X_1 \arrow[rr, dashed, "g_2"] & & X_2 & & X_3 \arrow[ll, dashed, "g_3"'] & & M. \arrow[ll, dashed, "g_4"']
\end{tikzcd}$$

Here, $\tilde{W}_3$ (resp. $\tilde{Z}_3$) is the strict transform of W_3 (resp. Z_3) under g_1 (resp. g_4).

Remark 10.3. In the primitive case, the birational transformation $S^{[g]} \dashrightarrow M_H(0,1,1)$, where $H^2 = 2g - 2$ can be resolved in one step as follows (e.g. [1, §3])

$$\begin{tikzcd}
& \mathrm{Hilb}^g(\mathcal{C}/|H|) \arrow[dl] \arrow[dr] & \\
S^{[g]} \arrow[rr, dashed] & & M_H(0,1,1),
\end{tikzcd}
\qquad
\begin{tikzcd}
& (\xi, D) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\xi & & \mathcal{O}_D(\xi),
\end{tikzcd}$$

where

$$\mathrm{Hilb}^g(\mathcal{C}/|H|) = \{(\xi, D) \mid \xi \subset D\} \subset S^{[g]} \times |H|.$$

If $g = 2$, the birational map $S^{[2]} \dashrightarrow M_H(0,1,1)$ is the original example of a Mukai flop [44, Expl 0.6].

More generally, the geometry of the birational map $S^{[g]} \dashrightarrow M_H(0,1,1)$, where $H^2 = 2g-2$ is studied in [40].

Proof of Theorem 10.2. We verify wall by wall that the contracted locus is as claimed in the Proposition. This means first of all, that we have to find decompositions $v = a + b$ inside $\mathcal{H}_i = \langle v, a_i \rangle$ for $i = 1, \ldots, 4$ corresponding to a flopping contraction. If we set $a = xa_i + yv$, solving for a positive decomposition reduces to solve

$$\begin{cases} a_i^2 x^2 + 2(a_i, v)xy + 8y^2 \geq 0 \\ 0 < (a_i, v)x + 8y \leq 4 \end{cases}$$

for integer solutions (x, y) and similar for decompositions, where a is a spherical class. Due to the small dimension of our example, we find by explicit computation that at every wall $v = a_i + (v - a_i)$ is the only suitable decomposition. We also compute that none of the walls is totally semistable with respect to v. Moreover, for all $i = 1, \ldots, 4$, the parallelogram inside $\mathcal{H}_i \otimes \mathbb{R}$ with vertices $0, a_i, v - a_i, v$ does not contain any other lattice points and so the decomposition does not admit a refinement. This is reflected by the fact that the exceptional locus in each step is irreducible and actually a projective bundle.

We attack the first two walls starting from $S^{[5]}$. The wall crossing can be realized by the following path in the stability manifold

$$\sigma'_t \coloneqq \sigma_{tH,-2H} = (Z_{tH,-2H}, \mathrm{Coh}^{-2}(S)),\ t \in (0, +\infty),$$

that actually hits every wall for v' [5, Thm 10.8]. (For the definitions, see [12, Prop 7.1] or the summary in [5, §6]). Explicitly, the walls arise when $Z_{tH,-2H}(v)$ and $Z_{tH,-2H}(a_i)$ are $\mathbb{R}$-linearly dependent. We find

$$+\infty > t_0 \coloneqq 2 > t_1 \coloneqq \sqrt{2} > t_2 \coloneqq \sqrt{\tfrac{5}{3}} > t_3 \coloneqq \sqrt{\tfrac{2}{3}} > 0$$

such that

$$M_{\sigma'_t}(v') \cong \begin{cases} S^{[5]} & \text{for } t > t_0 \\ X_i & \text{for } t_{i-1} > t > t_i \\ M & \text{for } t_3 > t > 0. \end{cases}$$

At the first wall, we have the sublattice $\mathcal{H}_1 = \langle a'_1, v' \rangle \cong (\mathbb{Z}^2, \left(\begin{smallmatrix} -2 & 2 \\ 2 & 8 \end{smallmatrix}\right))$. This lattice admits no decomposition of v' into positive classes. But we have

$$v' = a'_1 + b'_1 \text{ with } a'_1 = (1, -1, 2) = v(\mathcal{O}_S(-1)) \text{ and } b'_1 = (0, 1, -6)$$

and a'_1 is the only spherical class s with $0 < (s, v) \leq 4$ in $\mathcal{H}_1$. Hence an ideal sheaf $\mathcal{I}_\xi \in S^{[5]}$ is in the exceptional locus of $g_1 \colon S^{[5]} \dashrightarrow X_1$ if and only if it fits into a short exact sequence

$$0 \to \mathcal{O}_S(-1) \to \mathcal{I}_\xi \to \mathcal{Q} \to 0$$

with $\mathcal{Q} \in M_H(0, 1, -6)$. By Proposition 9.4, this is equivalent to $\xi \in W_2$. Hence g_1 is the flop at the projective bundle W_2 (see also [5, Expl 10.2]).

The second wall corresponds to the lattice $\mathcal{H}_2 = \langle a'_2, v' \rangle \cong (\mathbb{Z}^2, \left(\begin{smallmatrix} 0 & 3 \\ 3 & 8 \end{smallmatrix}\right))$, which contains no suitable spherical classes and admits exactly one decomposition into positive classes. Namely,

$$v' = a'_2 + b'_2 \text{ with } a'_2 = (1, -1, 1) \text{ and } b'_2 = (0, 1, -5).$$

Choose $t_1 < r < t_0$ such that $X_1 = M_{\sigma'_r}(v')$. Let $\xi \in S^{[5]} \setminus W_2$. Then $\mathcal{I}_\xi$ is not destabilized at the first wall and hence $\mathcal{I}_\xi \in X_1$. Now, $\mathcal{I}_\xi$ is in the exceptional locus of $g_2 \colon X_1 \dashrightarrow X_2$ if and only if there is an exact triangle

$$\mathcal{A} \to \mathcal{I}_\xi \to \mathcal{B} \xrightarrow{[1]},$$

where $\mathcal{A} \in M_{\sigma'_r}(a'_2)$ and $\mathcal{B} \in M_{\sigma'_r}(b'_2)$ are stable objects. We claim that $M_{\sigma'_r}(a'_2)$ is isomorphic to the original K3 surface S, via $S \ni x \mapsto \mathcal{I}_x(-1)$, where $\mathcal{I}_x \subset \mathcal{O}_S$ is the ideal sheaf of the point $x \in S$. Indeed, $\mathcal{I}_x(-1) \in \mathrm{Coh}^{-2}(S)$ for all $x \in S$. Moreover, $M_{\sigma'_r}(b'_2) \cong M_H(0, 1, -5)$, as there is only one wall for b'_2, which is defined by $v(\mathcal{O}_S(-1)) = (1, -1, 2)$ and hit by our path for $t = \sqrt{2} = t_1$ (cf. Remark 10.3). Consequently, $\mathcal{I}_\xi$ is contracted if and only if there is $x \in S$, $\mathcal{Q} \in M_H(0,1,-5)$ and an extension

$$0 \to \mathcal{I}_x(-1) \to \mathcal{I}_\xi \to \mathcal{Q} \to 0.$$

In other words, $\xi \in W_3$, see Proposition 9.4.

The remaining walls, we detect starting from M along the path $\sigma_t \coloneqq \sigma_{tH,0}$ for $t \in (1, +\infty)$. Then

$$M_{\sigma_t}(v) \cong \begin{cases} M & \text{for } t > \frac{\sqrt{6}}{2} < t \\ X_3 & \text{for } \frac{\sqrt{6}}{2} > t > 1, \end{cases}$$

i.e. this path only hits the first wall for $t = \frac{\sqrt{6}}{2}$ but it serves our purpose, in the sense that it provides us with a classical moduli description of the exceptional loci.

The contraction $g_4 \colon M \dashrightarrow X_3$ arises from the decomposition

$$v = a_4 + b_4 \text{ with } a_4 = (-1,1,-2) = -v(\mathcal{O}_S(-1)) \text{ and } b_4 = (1,1,1),$$

which is the only suitable decomposition in $\mathcal{H}_4 = \langle a_4, v\rangle \cong (\mathbb{Z}^2, \left(\begin{smallmatrix} -2 & 3 \\ 3 & 8 \end{smallmatrix}\right))$. Let $t > \frac{\sqrt{6}}{2}$. We note that $M_{\sigma_t}(a_4)$ consists of the point $\mathcal{O}_S(-1)[1]$ and $S \cong M_{\sigma_t}(b_4)$ via $x \mapsto \mathcal{I}_x(1)$. Moreover, $\phi_t(a_4) > \phi_t(b_4)$. Hence the exceptional locus of g_4 consists of those sheaves $\mathcal{E} \in M$ that arise as quotients

$$0 \to \mathcal{O}_S(-1) \to \mathcal{I}_x(1) \to \mathcal{E} \to 0$$

for some $x \in S$. This is the projective bundle Z_1, as defined in Proposition 9.6.

Finally, there is the wall defined by $\mathcal{H}_3 = \langle a_3, v\rangle \cong (\mathbb{Z}^2, \left(\begin{smallmatrix} -2 & 1 \\ 1 & 8 \end{smallmatrix}\right))$, which only admits the decomposition

$$v = a_3 + b_3 \text{ with } a_3 = (1,0,1) \text{ and } b_3 = (-1,2,-2).$$

Let $\mathcal{E} \in M \setminus Z_1$. Then $\mathcal{E}$ is not destabilized at the first wall and thus $\mathcal{E} \in X_3$. Now, $\mathcal{E}$ is in the exceptional locus of $g_3 \colon X_3 \dashrightarrow X_2$ if and only if there is an exact triangle

$$\mathcal{A} \to \mathcal{E} \to \mathcal{B} \xrightarrow{[1]},$$

where $\mathcal{A} \in M_{\sigma_t}(a_3)$ and $\mathcal{B} \in M_{\sigma_t}(b_3)$ are stable objects and $1 < t < \frac{\sqrt{6}}{2}$. The space $M_{\sigma_t}(a_3)$ consists of the point $\mathcal{O}_S$ and $S^{[3]} \cong M_{\sigma_t}(b_3)$, via $\mathcal{I}_\xi \mapsto R\mathcal{H}om(\mathcal{I}_\xi, \mathcal{O}_S)(-2)[1]$. Indeed, there are no walls for $S^{[3]}$ [11, Prop 5.6] and $R\mathcal{H}om(\mathcal{I}_\xi, \mathcal{O}_S)(-2) \in \mathrm{Coh}^0(S)$. Consequently, $\mathcal{E}$ is contracted if and only if there is $\xi \in S^{[3]}$ and an exact triangle

$$\mathcal{O}_S \to \mathcal{E} \to R\mathcal{H}om(\mathcal{I}_\xi, \mathcal{O}_S)(-2)[1] \xrightarrow{[1]}$$

or equivalently an exact sequence

$$0 \to \mathcal{O}_S(-2) \to \mathcal{I}_\xi \to \mathcal{E}xt^1(\mathcal{E}(2), \mathcal{O}_S) \to 0.$$

Here, $\mathcal{E}' \coloneqq \mathcal{E}xt^1(\mathcal{E}(2), \mathcal{O}_S) \in M_H(0,2,-7)$. By the proof of Proposition 9.6 we conclude that the exceptional locus of g_3 is $\tilde{Z}_3$. $\square$

Remark 10.4. For $v=(0,2,-1)$, the wall-crossing can not be realized entirely in the (H,H)-plane. Actually, only two walls intersect the (H,H)-plane.

Corollary 10.5. *We have that $Z_3 \setminus Z_1$ is isomorphic to a $\mathbb{P}^2$-bundle over an open subset of $S^{[3]}$. Moreover,*

$$Z_3 \cap \mathrm{BN}^1(M) = Z_1.$$

Proof. The proof of Theorem 10.2 implies that the map $\varphi_3\colon \mathcal{X}_3 \dashrightarrow M_H(0,2,-7) \cong M$ from Proposition 9.6 identifies the open subset of $\mathcal{X}_3$, where φ_3 is defined and injective with $Z_3 \setminus Z_1$. Let $D \in |2H|$ be a smooth curve and $\mathcal{L} \in \mathrm{Pic}^3(D)$ such that $h^0(\mathcal{L}) \geq 2$. Then φ_3 is defined but not injective in $\mathcal{L}^\vee$. Consequently, we must have $\mathcal{L} \in Z_1$. □

We could have also determined $(g_3)^{-1}\colon X_2 \dashrightarrow X_3$.

Proposition 10.6. *We have*

$$\mathrm{Bl}_{\tilde{P}_3} X_2 \cong \mathrm{Bl}_{\tilde{Z}_3} X_3,$$

i.e. $\tilde{P}_3 \subset X_2$ is the dual projective bundle of $\tilde{Z}_3 \subset X_3$.

Proof. We let $t_1 > r > t_2$ and $X_2 = M_{\sigma'_r}(v')$. We want to see that $\tilde{P}_3$ is the projective bundle parameterizing extensions of the form

$$\mathcal{A} \to \mathcal{I} \to \mathcal{B} \xrightarrow{[1]},$$

where $\mathcal{A} \in M_{\sigma'_r}(2,-2,1)$ and $\mathcal{B} \in M_{\sigma'_r}(-1,2,-5)$. Necessarily, $\mathcal{B} = \mathcal{O}_S(-2)[1]$ and

$$S^{[3]} \cong M_{\sigma'_r}(2,-2,1),\ \mathcal{I}_\xi \mapsto \mathcal{I}_\xi(-1) \oplus \mathcal{O}_S(-1).$$

Indeed, we verify that $\mathcal{E}[1] \coloneqq (\mathcal{I}_\xi(-1) \oplus \mathcal{O}_S(-1))[1] \in \mathrm{Coh}^{-2}(S)$. Assume that $\mathcal{F} \subset \mathcal{E}$ is a subbundle. Then either $\mathrm{rk}\,\mathcal{F} = 2$ and $c_1(\mathcal{F}) = c_1(\mathcal{E}) = -2H$. Or $\mathrm{rk}\,\mathcal{F} = 1$ and $\mathcal{F}$ embeds into $\mathcal{I}_\xi(-1)$ or into $\mathcal{O}_S(-1)$, which implies $c_1(\mathcal{F}).H \leq -2$. In both cases, we have $\frac{H.c_1(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}} + 2 \leq 0$ and hence $\mathcal{E}[1] \in \mathrm{Coh}^{-2}(S)$.
Let $\mathcal{I}_\xi$ be the ideal sheaf of a generic point in $P_3 \subset S^{[5]}$. Then $\xi = \zeta \cup \xi'$ for a subscheme $\zeta \in S^{[2]}$ such that $\mathrm{Supp}(\zeta) = \{x, \iota(x)\}$ and $\xi' \in S^{[3]}$ is disjoint from ζ (cf. Example 7.15). The assumption on ζ is equivalent to $\mathcal{I}_\zeta(1)$ being globally generated with $h^0(\mathcal{I}_\zeta(1)) = 2$. This gives a short exact sequence

$$0 \to \mathcal{O}_S(-2) \to \mathcal{O}_S(-1)^{\oplus 2} \to \mathcal{I}_\zeta \to 0. \tag{10.2}$$

Therefore, $\mathcal{I}_\xi = \mathcal{I}_\zeta \cdot \mathcal{I}_{\xi'}$ fits into a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{\xi'}(-2) \arrow[r] \arrow[d] & \mathcal{I}_{\xi'}(-1)^{\oplus 2} \arrow[r] \arrow[d] & \mathcal{I}_\xi \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_S(-2) \arrow[r] & \mathcal{O}_S(-1) \oplus \mathcal{I}_{\xi'}(-1) \arrow[r] & \mathcal{I}_\xi \arrow[r] & 0.
\end{tikzcd}$$

Here, the first line is (10.2) tensored with $\mathcal{I}_{\xi'}$ and the second line is the pushout along the left vertical arrow. This finishes the proof. □

Interestingly, the dual projective bundle of $\tilde{W}_3 \subset X_1$ does not yield a component of the Brill–Noether locus in M.

Proposition 10.7. *Let $\tilde{W}_3^\vee \subset X_2$ be the exceptional locus of $(g_2)^{-1}\colon X_2 \dashrightarrow X_1$ and $B_3 \subset M$ its strict transform then*

$$B_3 \subset M_\Sigma.$$

More precicesly, let $D = D_1 \cup D_2 \in \Sigma \setminus \Delta$. Then

$$B_3 \cap f^{-1}(D_1 \cup D_2) = \{\mathcal{E} \in f^{-1}(D_1 \cup D_2) \mid h^0(\mathcal{E}|_{D_i}) \neq 0 \text{ for both } i = 1, 2\}.$$

In particular, B_3 is not contained in $\mathrm{BN}^0(M)$ and T^{-1} is generically defined in B_3.

Proof. As above, we let $X_2 = M_{\sigma'_r}(v')$ with $t_1 > r > t_2$. We know that $\tilde{W}_3^\vee$ parameterizes extensions

$$\mathcal{A} \to \mathcal{F} \to \mathcal{B} \xrightarrow{[1]},$$

where $\mathcal{A} \in M_{\sigma'_r}(1, 0, -5)$ and $\mathcal{B} \in M_{\sigma'_r}(1, -1, 1)$. If $\mathcal{F} \in \tilde{W}_3^\vee$ is a generic point, then we saw in the proof of Theorem 10.2, that we can take $\mathcal{A} = \mathcal{Q} \in M_H(0, 1, -5)$ and $\mathcal{B} = \mathcal{I}_x(-1)$ for a point $x \in S$. This gives

$$\mathcal{Q} \to \mathcal{F} \to \mathcal{I}_x(-1) \xrightarrow{[1]} .$$

To find the strict transform in M, we have to apply the spherical twist $T = T_{\mathcal{O}_S(-2)}$ and tensor with $\mathcal{O}_S(2)$. We find an exact triangle in $D^b(S)$

$$T(\mathcal{Q}) \to T(\mathcal{F}) \to T(\mathcal{I}_x(-1)) \xrightarrow{[1]}, \tag{10.3}$$

where $T(\mathcal{Q})$ is concentrated in degree zero and fits into the sequence

$$0 \to \mathcal{Q} \to T(\mathcal{Q}) \to \mathcal{O}_S(-2) \to 0. \tag{10.4}$$

Moreover, we compute that

$$\mathcal{O}_S(-2)^{\oplus 2} \to \mathcal{I}_x(-1) \to T(\mathcal{I}_x(-1)) \xrightarrow{[1]}$$

and thus $\mathcal{H}^{-1}(T(\mathcal{I}_x(-1))) = \mathcal{O}_S(-3)$ and $\mathcal{H}^0(T(\mathcal{I}_x(-1))) = \mathcal{O}_{\iota(x)}$. Then the long exact cohomology sequence of (10.3) reads

$$0 \to \mathcal{H}^{-1}(T(\mathcal{F})) \to \mathcal{O}_S(-3) \to T(\mathcal{Q}) \to \mathcal{H}^0(T(\mathcal{F})) \to \mathcal{O}_{\iota(x)} \to 0. \tag{10.5}$$

Generically, $T(\mathcal{F})$ is a Gieseker stable sheaf and in this case $T(\mathcal{F})(2)$ is a generic point in B_3. Combining (10.4) and (10.5), we see that if $T(\mathcal{F})$ is a sheaf, then $\mathrm{Supp}(T(\mathcal{F})) = D_1 \cup D_2$, where $D_1 = \mathrm{Supp}(\mathcal{Q})$ and $D_2 \in |H|$ such that $x \in D_2$. Hence $B_3 \subset M_\Sigma$.

Now, if we assume that $x \notin D_1$, then $\mathcal{E} := T(\mathcal{F})(2)$ is an extension

$$0 \to \mathcal{Q} \otimes \mathcal{O}_S(2)|_{D_1} \to \mathcal{E} \to \mathcal{O}_{D_2}(\iota(x)) \to 0. \tag{10.6}$$

As $\mathcal{T}or_1^{\mathcal{O}_{D_1 \cup D_2}}(\mathcal{O}_{D_1}, \mathcal{O}_{D_2}(\iota(x))) = 0$, this implies $\mathcal{E}|_{D_1} \in \mathrm{Pic}^2(D_1)$, which is trivially effective. Moreover, restricting (10.6) to D_2 gives

$$0 \to \mathcal{T}or_1^{\mathcal{O}_{D_1 \cup D_2}}(\mathcal{O}_{D_2}, \mathcal{O}_{D_2}(\iota(x))) \xrightarrow{\sim} \mathcal{O}_{D_1 \cap D_2} \xrightarrow{0} \mathcal{E}|_{D_2} \xrightarrow{\sim} \mathcal{O}_{D_2}(\iota(x)) \to 0.$$

In particular, $\mathcal{E}|_{D_2}$ is an effective line bundle of degree one. Conversely,

$$\dim\{\mathcal{E} \in f^{-1}(D_1 \cup D_2) \mid h^0(\mathcal{E}|_{D_i}) \neq 0 \text{ for both } i = 1, 2\} = 4.$$

Hence, by dimension reasons, $B_3 \cap f^{-1}(D_1 \cup D_2)$ is as claimed for every $D = D_1 \cup D_2 \in \Sigma \setminus \Delta$.

It is left to show, that for a generic sheaf $\mathcal{E} \in B_3$, we have $h^0(\mathcal{E}) = 0$. To see this, we can assume $\mathcal{E} \in \mathrm{Pic}^{(1,2)}(D_1 \cup D_2)$ and that $h^0(\mathcal{E}|_{D_i}) = 1$ for $i = 1, 2$. If $h^0(\mathcal{E}) \neq 0$, necessarily $h^0(\mathcal{E}) = 1$ and the restriction to each component induces an isomorphism on global sections. However, this determines $\mathcal{E}$ completely. Indeed, we have $\mathcal{E}|_{D_1} = \mathcal{O}_{D_1}(x)$ and $\mathcal{E}|_{D_2} = \mathcal{O}_{D_2}(y+z)$, for unique points x, y and z. Then any non-zero section $\mathcal{O}_{D_1 \cup D_2} \to \mathcal{E}$ is necessarily injective with cokernel supported on $\xi = \{x, y, z\}$. In other words, $\mathcal{E}$ has a section if and only if $\mathcal{E}^\vee = \ker(\mathcal{O}_D \to \mathcal{O}_\xi)$. Finally, we saw in Proposition 8.1, that T^{-1} is defined for all $\mathcal{E} \in M_{\Sigma \setminus \Delta}$ such that $h^0(\mathcal{E}) = 0$. $\square$

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Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: `igb@math.uni-bonn.de`