# Extremes of the discrete Gaussian free field in dimension two

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

von
**Christian Joachim Maximilian Fels**
aus
Bonn

Bonn, 21.10.2020

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn.

| | |
|---|---|
| 1. Gutachter: | Prof. Dr. Anton Bovier |
| 2. Gutachter: | Prof. Dr. Nicola Kistler |

| | |
|---|---|
| Tag der Promotion: | 28.01.2021 |
| Erscheinungsjahr: | 2021 |

# Acknowledgments

I would like to thank a number of people that enabled me in writing this dissertation. The first to name here is my advisor Anton Bovier. I am extremely happy and grateful to have him as my advisor. His wonderful lectures sparked my interest in probability theory. I am particularly grateful for his personal address which is also why I wrote my Bachelor thesis on the maximum of Gaussian branching random walks, my first contact with extreme value theory. He showed real interest in me and my research. In particular, he always actively encouraged and supported me to continue my studies and research while creating and maintaining a wonderful working environment. But not only in this sense Anton is a great advisor, I really appreciate his frank and straightforward criticism which always was spot on and which helped me a lot to improve. Moreover, he has a wonderful sense of humor and it has always been a pleasure for me to come to the institute to work and spend time together.

The second person I would like to thank here is Lisa Hartung. I enjoyed her great hospitality in New York as well as in Mainz. It has always been a great pleasure for me to collaborate with her, discuss mathematics and go on hikes. She has also been very encouraging and supporting me, gave me numerous valuable remarks and has always been available for advice.

I would also like to thank Oren Louidor, whom I visited and worked with during my PhD studies. Apart from work, I enjoyed his great sense of humour and his great hospitality.

I would like to thank my colleagues from the Stochastics group who largely contributed to the welcoming, pleasant working environment that I enjoyed so much.

I would also like to thank Nicola Kistler for willing to be the second referee of my thesis, as well as Stefan Mueller and Corinna Kollath for being members of the committee.

I would also like to thank all the institutions that supported me not only financially but also provided wonderful working conditions, enabling me to conduct research, travel to summer schools and conferences, present and discuss my own research and get into touch with other mathematicians. To name here are the DFG which supported me through the CRC 1060 ‘The Mathematics of Emergent Effects’ and the Bonn International Graduate School in Mathematics (BIGS), the Insitute for Applied Mathematics in Bonn, the Courant Institute of Mathematical Sciences, the Technion-Israel Institute of Technology, the Weizmann Institute of Science and the Oberwolfach Research Institute for Mathematics.

Last but by far not least, I would like to thank my parents and family for their great and enduring support.

# Abstract


In recent years, there have been many advances towards an understanding of the extreme value theory of *log-correlated random fields*. Log-correlated random fields are conjectured to compose a universality class for the extremal values of strongly correlated fields. In the general context of extreme value statistics there are two natural basic questions to answer. Akin to the central limit theorem one may ask: Is there a deterministic recentring and rescaling such that the maximum value of the sequence converges to a non-trivial limit?

And second, if such a recentring and rescaling exists, how does the process look like when recentring and rescaling each random variable as done for the maximum value?

Both questions were answered in the context of independent identically distributed random variables during the first half of the past century. The theory developed in this context is commonly referred to as *classical extreme value theory*. We state the main results in the general case of independent identically distributed random variables and then turn to the case of Gaussian distributions.

To analyze the extreme value statistics of correlated models, it is natural to start with simple models that capture the essential details, which in our case are the *hierarchical* ones. We start with a rather classical model, the *generalized random energy model* (GREM), which can be realized as a branching random walk with Gaussian increments, and then discuss (variable-speed) *branching Brownian motion* (BBM), a model that has attracted a lot of interest in the last decade.

An important example of a log-correlated Gaussian random field is the *two-dimensional discrete Gaussian free field* (2d DGFF). It is a natural object of major interest both in mathematics and physics. Its extremal values have been investigated in the last 20 years.

We then introduce the model we studied, which is a generalization of the 2d DGFF, the so-called *scale-inhomogeneous two-dimensional discrete Gaussian free field*. Similarly to variable-speed BBM in the context of BBM, it allows for a richer class of correlation structures. It turns out that it is possible to classify its extremal values into three possible cases, one being the two-dimensional discrete Gaussian free field. In this thesis, we present our contributions in the study of the extremal values of the scale-inhomogeneous 2d DGFF. In any of the three possible cases and when there are only finitely many scales we determine the sub-leading order correction to the maximum value and prove tightness of the centred maximum. Moreover, in the *case of weak correlations* we provide a complete characterization of the extreme value theory of the scale-inhomogeneous 2d DGFF.

# Contents

# CHAPTER 1

# Introduction

## 1.1 Organization

First, let us explain the structure of the introduction. In Section 1.2, we make a few preliminary remarks on extreme value theory and log-correlated random fields. In Section 1.3, we start with historical remarks on extreme value theory and then give an overview of the relevant contents from classical extreme value theory. At the end of this section, we highlight the particular case of iid Gaussian random variables. In Section 1.4, we discuss the extreme value theory of Gaussian processes that can be indexed by trees such as the *random energy model* (REM), GREM as well as variable-speed BBM. In Section 1.5, we introduce the 2d DGFF, a non-hierarchical log-correlated Gaussian random field. In Section 1.6, we introduce the main model of this thesis, the scale-inhomogeneous 2d DGFF, put it into the context of Gaussian processes on trees and present the original contributions of this thesis. In particular, we give heuristic explanations for most of the results. Finally, we shortly discuss open problems for the models discussed in the introduction, which can be found in Section 1.7, and furthermore, in Section 1.8 we provide a glimpse of what is being done in related models.

## 1.2 Preliminaries

Extreme events are rare events, but as they can have major effects it is important and of natural interest to understand their behaviour. One fundamental example are floods. Due to the necessity of sufficient water resources, human settlements need to be in reasonable vicinity. As rivers also provide a convenient way of transportation and communication, proximity to these have always been preferential. The unwanted side effects of extraordinarily high floods that can potentially devastate entire cities has to be taken into consideration. It has been only at the end of the 19th century when mathematicians started to systematically develop the so-called classical theory of extremal values. The classical theory of extremal values deals with sequences of events that are independent and identically distributed. Extreme value statistics allows to quantify the behaviour of unusually large values whose occurrence is, of course, scarce. In particular, it allows to better estimate the tail area of the distribution of extremal values. One major limitation to the classical theory is the assumption of independence. Nevertheless, it turns out that the theory for independent and identically distributed events also applies to correlated models, provided correlations decay sufficiently fast. It is the case of *log-correlated fields* in which correlations start to affect the behaviour of the extremal values. A random field, $\{X_v\}_{v\in V}$, that belongs to this class can be indexed by the elements of a metric space, $(V, |\cdot|)$. The key properties of log-correlated fields is that their variances have a logarithmic singularity and that covariances decay approximately with the negative logarithm of the distance between index points, i.e. $\mathbb{E}[X_v X_w] \sim -\log|v-w|$, for $v, w \in V$. Important examples that fall into this class are e.g. branching Brownian motion (BBM), the branching random walk (BRW), the Gaussian free field in dimension two (2d DGFF), the field of hitting times of Brownian motion on the two-dimensional torus, the logarithm of the characteristic polynomial of random matrices or the randomized Riemann zeta function. Note that many models belong to the universality class of log-correlated fields and do not satisfy the previous properties for all their index points, e.g. the 2d DGFF. Log-correlated random fields, and in particular their extremal values, have a rich structure. Due to their common multi-scale nature their analysis is often interrelated. In the last three decades, and in particular in the last 20 years, there has been a huge push towards the understanding of the extreme value theory for (Gaussian) log-correlated fields. This is partly due to insightful conjectures in the physics literature concerning the extremal values of such fields, which sparked lots of interest and which are based on a statistical mechanics approach, see [32, 57, 58, 56].

## 1.3 Classical extreme value theory

Extreme events are part of nature and ever since of major importance to humankind. Prominent examples arise from observing sequences of events such as floods, earthquakes, volcanic eruptions or weather extremes. More recent applications can be found in astronomy, meteorology, oceanography, quality control, building code, mutations in DNA, polymerization or in the financial industry. We start with a short historical background of the mathematical theory, which is based on the one given in Emil Gumbel's classical standard reference [63]. Considering its relevance in real world applications, it is fairly recent that the statistical nature of extremal events was realized. The question of what the distribution of the maximum value of a growing number of observables is was already posed in Nicolas Bernoulli's *Specimina artis conjectandi, ad quaestiones juris applicatae* (1709) [15], in which he considered the lifetime of the last survivor among $n$ men if they are to die within $k$ time. He reduced this problem to finding the expected value of the maximum of $n$ independent and uniformly distributed variates. Extreme events are by nature rare events. The number of rare events can be described by the *Poisson distribution*. L. von Bortkiewicz [84] was the first to realize its statistical relevance for extreme value theory in his study of the number of soldiers in the Prussian army killed by horse-kicks over certain time periods. In 1922, L. von Bortkiewicz was also the first to study extremal values of normal random variables [85, 86], with subsequent contributions from R. von Mises [87], who discovered the Gumbel distribution as limiting distribution for independent standard Gaussians, and Tippett [83]. In light of the central limit theorem, with the basic statistical motivation stemming from repeated, independent measurements of the same quantity, and in which the Gaussian distribution emerges as universal limiting distribution of the properly normalized sum of those measurements for a large class of underlying distributions, studying the case of Gaussian distribution seemed to be natural. E.L. Dodd [48] was the first to study extremal values for independent random variables, different from Gaussian. In 1927, Fréchet [55] started a systematic study of the maximum value of a collection of random variables, not necessarily normally distributed, and laid the foundation for a classification of extremal distributions. In analogy to the notion of *sum-stability* in the context of the central limit theorem, he introduced the notion of *max-stability* of a distribution. The key idea is the following: If one samples independent random variables according to a max-stable distribution, then the maximum of all samples should have the same distribution as any of the samples up to an affine transformation, which itself should depend only on the number of samples. Fréchet conjectured max-stability to be a crucial property of a distribution function to be a candidate distribution describing the maximum value of a sequence of iid random variables. Shortly after and based on the concept of Fréchet's max-stability, Fisher and Tippett [54] identified the only two other possible non-trivial limit distributions. R. von Mises [88] identified conditions on the initial distributions to belong to the domain of attraction of one of the possible limit distributions. In 1943, Gnedenko [61] added to this by providing necessary and sufficient conditions. Emil Gumbel's monograph [63] is the first systematic overview of and reference for the theory of extremal values for collections of independent identically distributed random variables. Gumbel's book does not only provide an overview of the mathematical theory but also explains how to apply it in applications and discusses real world examples, making this monograph one of the most cited references in this field. However, there are several severe limitations of the classical theory, as Gumbel remarked:

*Another limitation of the theory is the condition that the observations from which the extremes are taken should be independent. This assumption, made in most statistical work, is hardly ever realized.*

In the 1970s, 1980s and 1990s the study of extremal values of (weakly) correlated sequences started.

Most of the theory can be found in the two monographs by Leadbetter, Lindgren and Rootzén [68] and by Resnick [78]. In the book of Leadbetter, Lindgren and Rootzén [68] from 1983 the authors treat the extreme value theory for Gaussian stationary sequences and stationary stochastic processes under a mixing condition. In particular, it is shown that under these assumptions the theory is identitical to the one in the independent case. In [78], Resnick studies the distributional convergence of extremes and upper order statistics using the elegant theory of weak convergence of point processes. [78] also provides a rigorous theory of extremal values for multivariate iid sequences.

The remainder of this section briefly covers the most important results in the general setting of iid random variables. We then shift our focus to the particular, and to us most relevant case when distributions are Gaussian. Most of what we discuss in this context and more can be found in [22, 65]. The basic motivation for studying the theory of extremal values naturally stems from statistics, when recording data corresponding to partial observations or a sequence of events. Let us call such a sequence of events $\{X_n\}_{n\in\mathbb{N}}$, where $X_n$ are random variables taking values in the real numbers. Thus, $\{X_n\}_{n\in\mathbb{N}}$ is a stochastic process in discrete time defined on some underlying probability space, $(\Omega, \mathcal{F}, \mathbb{P})$. In the context of extremal values, the most natural question to ask concerns the distribution of the *maximum value* up to time $N$, which we denote by

$$M_N := \max_{1\leq i\leq N} X_i. \tag{1.1}$$

The question then reads, what is $\mathbb{P}(M_N \leq x)$, for $N$ large and $x \in \mathbb{R}$? In the spirit of the central limit theorem for random variables, one can ask for a deterministic centring, $\{b_N\}_{N\in\mathbb{N}}$, and rescaling, $\{a_N\}_{N\in\mathbb{N}}$, such that

$$\mathbb{P}\left(\frac{M_N - b_N}{a_N} \leq x\right) \tag{1.2}$$

has a non-trivial limit as $N \to \infty$, for fixed $x \in \mathbb{R}$. In other words, does $\frac{M_N - b_N}{a_N}$ converge to a random variable with a non-trivial distribution function? Note that studying the *minimum value* is an equally well choice which, up to a possible deterministic shift of the mean, can be reduced to the study of the maximum value of $\{-X_i\}_{1\leq i\leq N}$.

A second natural question in the context of extreme value theory is to understand the *joint distribution of the reordered sequence*

$$X_1 \geq X_2 \geq \ldots . \tag{1.3}$$

Beyond these two basic questions, for fixed common distributions of the random variables, goes the more fundamental question in extreme value theory: Are there *universal laws* that describe the limiting processes? And if such universal laws exist, can we describe their *domain of attraction* depending only on their common initial distribution? All these questions have been answered in the affirmative in the case of independent and identically distributed random variables.

### 1.3.1 Independent identically distributed random variables

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{X_i\}_{i\in\mathbb{N}}$ be a collection of independent identically distributed random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and with common distribution function

$$F(x) = \mathbb{P}(X_1 \leq x). \tag{1.4}$$

Recall that we denote the *maximum value* up to $N$ by

$$M_N := \max_{1\leq i\leq N} X_i. \tag{1.5}$$

Note that $\{M_N\}_{N\in\mathbb{N}}$ is a stochastic process defined on the same probability space, $(\Omega, \mathcal{F}, \mathbb{P})$. To get started, let us first examine $\mathbb{P}(M_N \leq x)$, for $N$ large and arbitrary but fixed $x \in \mathbb{R}$. Using the fact that the random variables $X_i$ are iid allows for the following simple computation:

$$\mathbb{P}(M_N \leq x) = \mathbb{P}(\forall i \in \{1, \ldots, N\} : X_i \leq x) = \mathbb{P}(X_1 \leq x)^N = (F(x))^N. \tag{1.6}$$

Using $F(x) \in [0, 1]$ in (1.6), we observe that

$$\mathbb{P}(M_N \leq x) = (F(x))^N \to \begin{cases} 0, & \text{if } F(x) < 1 \\ 1, & \text{if } F(x) = 1, \end{cases} \tag{1.7}$$

as $N \to \infty$. Regardless of the common distribution of the random variables, $\{X_n\}_{n\in\mathbb{N}}$, (1.7) implies that, for any fixed $x \in \mathbb{R}$, we observe a trivial behaviour of the ordinary maximum value. Similarly as for the *central limit theorem*, one should ask the following question: Do deterministic centrings, $\{b_N\}_{N\in\mathbb{N}}$, scalings, $\{a_N\}_{N\in\mathbb{N}}$, and a non-trivial distribution function, $G$, exist such that

$$\mathbb{P}\left(\frac{M_N - b_N}{a_N} \leq x\right) \to G(x), \quad \text{as } N \to \infty\,? \tag{1.8}$$

Rewriting the left-hand side of (1.8) as in (1.6), we see that the tails of the underlying distribution $F$ play a crucial role, i.e.

$$\mathbb{P}\left(\frac{M_N - b_N}{a_N} \leq x\right) = \mathbb{P}(M_N \leq b_N + a_N x) = (F(b_N + a_N x))^N. \tag{1.9}$$

The question becomes: Do deterministic sequences $\{a_N\}_{N\in\mathbb{N}}$, $\{b_N\}_{N\in\mathbb{N}}$ and a non-trivial distribution function, $G$, exist, such that

$$(F(b_N + a_N x))^N \to G(x)\,? \tag{1.10}$$

And if the answer is positive, one may further ask:

*What are possible limiting distributions? What is their domain of attraction?*

In 1943, Gnedenko [61] established a complete classification of possible limiting distributions.

**Theorem 1.3.1.** *Let $\{X_i\}_{i\in\mathbb{N}}$ be independent identically distributed random variables. If there exist*

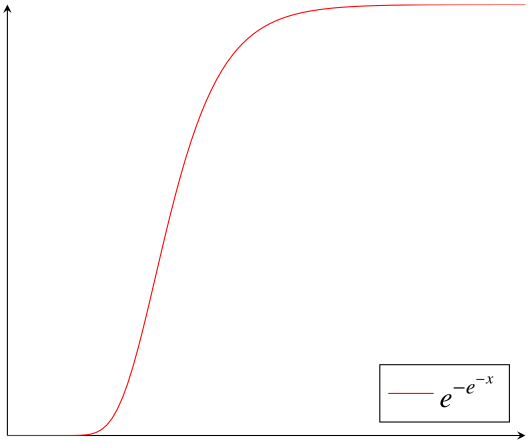



Figure 1.1: Gumbel.

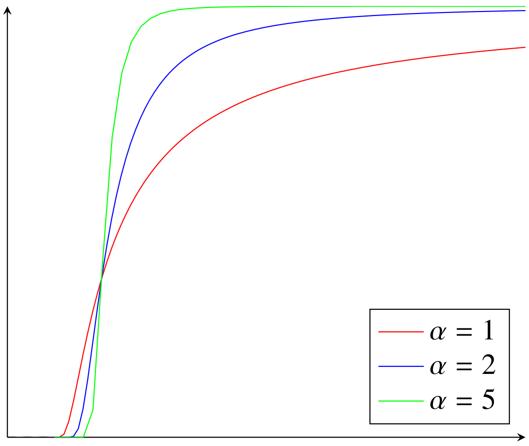



Figure 1.2: Fréchet.

*deterministic sequences* $\{a_N\}_{N\in\mathbb{N}}$, $\{b_N\}_{N\in\mathbb{N}}$ *and a non-degenerate distribution function, G, such that*

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{M_N - b_N}{a_N} \le x\right) = G(x), \tag{1.11}$$

*then, up to an affine transformation in x, G must be one of the following three types:*

1. ***Gumbel-distribution:*** $G(x) = e^{-e^{-x}}$, $\forall x \in \mathbb{R}$.

2. ***Fréchet-distribution:*** *For some* $\alpha > 0$, $G(x) = \begin{cases} 0, & \text{if } x \le 0, \\ e^{-x^{-\alpha}}, & \text{if } x > 0. \end{cases}$

3. ***Weibull-distribution:*** *For some* $\alpha > 0$, $G(x) = \begin{cases} e^{-(-x)^{\alpha}}, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$

Note that it is of course not true that for any sequence of iid random variables, $\{X_i\}_{i\in\mathbb{N}}$, one obtains a non-degenerate distribution as in Theorem 1.3.1. Think for example of random variables supported

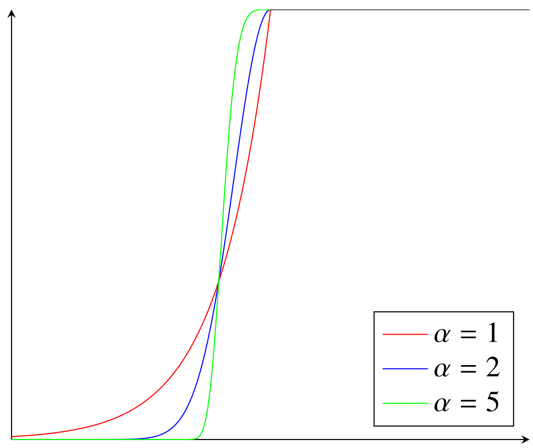



Figure 1.3: Weibull.

only on two values, 0 and 1. As (1.9) suggests, the tails of the probability distribution function $F$ should play an important role. The following theorem provides necessary and sufficient conditions for the existence of a non-degenarate limit and determines the limiting distribution, depending only on the tail of the common distribution function $F$.

**Theorem 1.3.2.** *Set $x_F := \sup\{x : F(x) < 1\}$. The following conditions are necessary and sufficient for a distribution function, $F$, to belong to the domain of attraction of one of the three extremal types:*

1. ***Fréchet:*** $x_F = \infty$,

$$\lim_{t\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \quad \forall x \in \mathbb{R}, \alpha > 0. \tag{1.12}$$

2. ***Weibull:*** $x_F \leq \infty$,

$$\lim_{t\downarrow 0} \frac{1 - F(x_F - tx)}{1 - F(x_F - t)} = x^{\alpha}, \quad \forall x \in \mathbb{R}, \alpha > 0. \tag{1.13}$$

3. ***Gumbel:*** $\exists\, g(t) > 0$,

$$\lim_{t\uparrow x_F} \frac{1 - F(t + xg(t))}{1 - F(t)} = e^{-x}, \quad \forall x \in \mathbb{R}. \tag{1.14}$$

Theorem 1.3.1 and Theorem 1.3.2 settle the questions concerning the maximum value in the case of independent identically distributed random variables. However, we are more generally interested in extremal values, i.e. all values that are in some sense close to the maximum value. To study these, it makes sense to centre and scale each random variable as being done for the maximum value. A convenient way to study the joint distribution of extremal particles turns out to be by means of the point process,

$$\mathcal{E}_{X,N} := \sum_{i=1}^{N} \delta\left(\frac{X_i - b_N}{a_N}\right), \tag{1.15}$$

which is also called *extremal process*. A point process is a random variable taking values in the set of point measures. The set of all point measures on an interval, $A \subset \mathbb{R}$, we denote by $M_p(A)$. Due to the identical centring and scaling as applied to the maximum value, one should expect that most points in (1.15) vanish to $-\infty$, as $N \to \infty$, and that we retain only points close to maximal one. The basic question is:

*Does the sequence of point processes, $\left\{\mathcal{E}_{X,N}\right\}_{N\in\mathbb{N}}$, converge (to a point process)?*

And if the answer is to the affirmative, one should further ask:

*Can we characterize the possible limit distributions?*

At this stage, it makes sense to discuss the notion of convergence of point processes. As point processes are probability distributions on the space of point measures, it is natural to think of weak convergence of probability distributions. The choice of the *vague-topology* turns the space of points measures equipped with the Borel-sigma algebra into a complete, separable metric space, which allows us to discuss questions of weak convergence. For further details on this we refer to [22, Chapter 2]. The following theorem settles both questions of convergence of the extremal process and of the characterization of its possible limit distributions in the case of iid random variables.

**Theorem 1.3.3.** *(cp. [65, Theorem 2.2]) Let $\{X_i\}_{i\in\mathbb{N}}$ be a family of independent identically distributed random variables and let $\{a_N\}_{N\in\mathbb{N}}$, $\{b_N\}_{N\in\mathbb{N}}$ satisfy* (1.11) *in Theorem 1.3.1 for some non-degenerate distribution function G. Then, $\mathcal{E}_{X,N}$ converges weakly, as $N \to \infty$, with respect to the vague topology on the space of $\sigma-$finite measures to a Poisson point process (PPP) whose intensity measure is determined by its extremal type distribution, G. In particular, if* (1.11) *holds with G*

1. *the Gumbel-distribution, then $\mathcal{E}_{X,N}$ converges weakly to a PPP$\left(e^{-x}\mathrm{d}x\right)$ in $M_p\left((-\infty,\infty]\right)$.*
2. *the Fréchet-distribution, then $\mathcal{E}_{X,N}$ converges weakly to a PPP$\left(x^{-\alpha}\mathbb{1}_{x>0}\mathrm{d}x\right)$ in $M_p\left((0,\infty]\right)$.*
3. *the Weibull-distribution, then $\mathcal{E}_{X,N}$ converges weakly to a PPP$\left((-x)^{-\alpha}\mathbb{1}_{x\leq 0}\mathrm{d}x\right)$ in $M_p\left((-\infty,0]\right)$.*

To conclude this subsection, in all three cases the extremal process is a Poisson point process with a certain intensity which is determinded by the tails of the common distribution function, $F$.

### 1.3.2 Independent identically distributed Gaussian random variables

As all models we consider in the following sections are Gaussian, we state as a reference the results in the case of independent and identically distributed Gaussian random variables, directly in the framework that is also relevant in the context of our study of the scale-inhomogeneous two-dimensional discrete Gaussian free field. Take $X_i^{(N)} \sim \mathcal{N}(0,\log N)$, for $i = 1,\dots,N^2$. In the context of the (scale-inhomogeneous) two-dimensional discrete Gaussian free field one should think of index set being the lattice box of side length $N$, $V_N = [0,N)^2 \cap \mathbb{Z}^2$. We want to find the correct centring and scaling, the limiting distribution of the maximum value and the corresponding limiting extremal process. As in (1.6),

$$\mathbb{P}\left(\max_{1\leq i\leq N^2} X_i^{(N)} \leq a_N x + b_N\right) = \left(\mathbb{P}\left(X_1^{(N)} \leq a_N x + b_N\right)\right)^{N^2} = \left(1 - \frac{N^2\mathbb{P}\left(X_1^{(N)} > a_N x + b_N\right)}{N^2}\right)^{N^2}. \quad (1.16)$$

For the right hand side to converge to a determinstic non-trivial function in $x$, $N^2\mathbb{P}\left(X_1^{(N)} > a_N x + b_N\right)$ has to converge to a non-degenerate function, $x \mapsto g(x)$, as $N \to \infty$. Thus, we need good bounds for the last probability in (1.16). Let

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{y^2}{2}\right] \mathrm{d}y \tag{1.17}$$

be the cumulative distribution function of a standard Gaussian random variable. Then, by Mills' ratio bound [62, Eq. (10)]

$$\frac{x}{(x^2+1)\sqrt{2\pi}} e^{-\frac{x^2}{2}} \leq 1 - \Phi(x) \leq \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{1.18}$$

Using this crucial estimate in (1.16) and applying Theorem 1.3.3, one can show the following theorem (cp. e.g. [22, Section 4.2.2.]).

**Theorem 1.3.4.** *Let* $\{X_i\}_{i\in\mathbb{N}}$ *be independent centred Gaussians with variance* $\log N$. *Let*

$$b_N = 2\log N - \frac{1}{4}\log\log N - \frac{1}{4}\log(2\pi) \qquad \text{and} \qquad a_N = 1. \tag{1.19}$$

*Then,*

1. *The rescaled maximum converges to a Gumbel distribution,*

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{1\leq i\leq N^2} X_i - b_N \leq x\right) = e^{-e^{-2x}}, \quad x \in \mathbb{R}. \tag{1.20}$$

2. *The limiting extremal process is a Poisson point process (PPP) with intensity* $e^{-2x}\mathrm{d}x$*,*

$$\lim_{N\to\infty} \sum_{i=1}^{N^2} \delta_{X_i - b_N} = PPP\left(e^{-2x}\mathrm{d}x\right). \tag{1.21}$$

## 1.4 Gaussian processes indexed by trees

The extremal values of log-correlated (Gaussian) fields that exhibit a hierarchical structure, such as the branching random walk or branching Brownian motion, can be considered as well understood. One major reason for this is that their correlations are encoded in a hierarchical structure which simplifies their analysis. In particular, these processes can be realized on a Galton-Watson tree. Two important properties to name here are the *splitting* and *self-similarity*. If we pick two leaves of the tree and trace back their branches to the root, their paths will meet at some point. By splitting, we mean that remaining increments after two particles' branches have split on the tree, are independent. Self-similarity simply means that all increments have identical distributions. Log-correlated fields satisfy these properties in an approximate manner. Thus, it is reasonable to first study the extreme values for log-correlated models with an explicit hierarchical order. In fact, one common idea in the analysis of the extremes of log-correlated *Gaussian* fields is to use Gaussian comparison in order to compare the actual model to a model that exhibits an explicit hierarchical structure and prove that in the limit, their extremes have identical distributions. As we will see in Section 1.6, this is also the basic underlying idea for our analysis of the scale-inhomogeneous two-dimensional discrete Gaussian free field. Suitable hierarchical models for comparison are models that can be indexed by the leaves of a tree and in which the correlations are given as functions of the tree distance between leaves. In the following, we shortly present three examples that are the most relevant to us in the sense of comparison. We start with the probably simplest Gaussian process that can be indexed by the leaves of a tree, the *random energy model* (REM). This also allows us to hint at the motivation for our results coming from spin glass theory.

### 1.4.1 The Random Energy Model

The random energy model (REM) was introduced in [40] by Derrida in 1980 as a toy model to study more complicated spin glass models such as the Sherrington-Kirckpatrick model. Spin glasses are spin systems with competing random interactions. The key objects of mathematical interest are random functions of the spin configurations, called *Hamiltonians*. In REM, different spin configurations are distributed according to the Gibbs distribution, namely, their probabilities are proportional to an exponential function of their negative energies. Of great interest in studying such models is to understand the *ground states* which, in the interesting case when the Gibbs measure feels the geometry of the random Hamiltonians, corresponds to understanding the extremes of the Hamiltonians, i.e. its minima/maxima. For easier comparison in the following, we consider REM on a 4-ary tree of depth $n$, denoted by $\mathcal{T}_n$, with leaves $v \in T_n$. It is a stochastic process, $\{X_v^n\}_{v\in T_n}$, indexed by the leaves of the 4-ary tree, $\mathcal{T}_n$, of depth $n \in \mathbb{N}$. To each leave, $v \in T_n$, we attach an independent random variable, $X_v^n \sim \mathcal{N}(0, \log N)$. Setting $N = 2^n$, this allows to apply Theorem 1.3.4 with centring and scaling for the maximum value,

$$b_n^{REM} = 2\log N - \frac{1}{4}\log\log N - \frac{1}{4}\log(2\pi) \quad \text{and} \quad a_n^{REM} \equiv 1, \tag{1.22}$$

to obtain the following:

**Corollary 1.** *In the random energy model on the 4-ary tree and rescaling as in* (1.22)*, we have, as* $n \to \infty$*,*

1. *the rescaled maximum converges to a Gumbel distribution,*

$$\mathbb{P}\left(\max_{v\in T_n} \frac{X_v^n - b_n^{REM}}{a_n^{REM}} \le x\right) \to e^{-e^{-2x}}, \quad x \in \mathbb{R}. \tag{1.23}$$

2. *the limiting extremal process is a Poisson point process (PPP) with intensity* $e^{-2x}\mathrm{d}x$*, i.e.*

$$\sum_{i=1}^{n} \delta\left(\frac{X_i^n - b_n^{REM}}{a_n^{REM}}\right) \to PPP(e^{-2x}\mathrm{d}x). \tag{1.24}$$

Even though REM seems trivial as a statistical mechanics model, its structure is sufficiently rich such that its associated Gibbs measure exhibits a phase transition [41]. Being possibly the simplest Gaussian process on a tree, its main advantage lies in the fact that it poses a workable example that can be studied in full details while its features are not entirely trivial.

### 1.4.2 The Generalized Random Energy Model

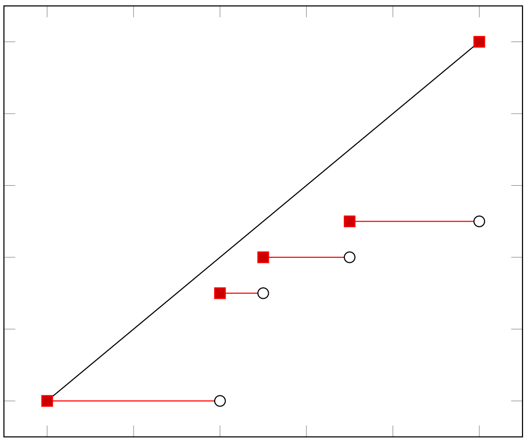
We turn to Gaussian models indexed by trees in which the random variables are hierarchically ordered, instead of being independent. The *generalized random energy model* (GREM) was introduced in [42] by Derrida in 1985 as a generalization of the REM. We restrict our considerations to GREM on a tree. In GREM, correlations between spin configurations are given as functions of the tree distance of pairs of leaves on the tree. In this sense, and when recalling that the spin configurations in REM are indexed by the leaves of the tree, GREM extends REM. As we consider GREM on a tree, correlations can be easily incorporated into the model by extending the process from the leaves of the tree to the entire tree. This requires the notion of a distance of leaves on the tree. As in REM, we consider GREM on the 4-ary tree $\mathcal{T}_n$ of depth $n \in \mathbb{N}$. Let $T_n$ be the set of leaves of $\mathcal{T}_n$ and note that there are $4^n$ leaves at generation $n$. We denote by

$$d_n(v, w) = \text{the generation/time of the most recent common ancestor of leaves } v \text{ and } w, \qquad (1.25)$$

for any two leaves $v, w \in T_n$ at generation $n$. A natural distance between two leaves, $v$ and $w$, on the tree is then be given by $n - d_n(v, w)$, i.e. the number of independent generations. As GREM is a Gaussian process it suffices to describe its mean and covariances. Let $A : [0, 1] \to [0, 1]$ be an increasing step-wise function with finitely many steps and satisfying $A(0) = 0$, $A(1) = 1$. GREM on the tree $\mathcal{T}_n$ is a Gaussian process, $\{X_v^n\}_{v \in T_n}$, with mean 0 and correlations given by

$$\mathbb{E}\left[X_v^n X_w^n\right] = \log\left(|T_n|\right) A(d_n(v, w)/n) = \log(4^n) A(d_n(v, w)/n). \qquad (1.26)$$

Note that GREM on the tree can be realized as a time-inhomogeneous branching random walk



(a) An example of a step-wise function, $A : [0, 1] \mapsto [0, 1]$, satisfying $A(x) < x$, for $(0, 1)$.

(BRW) with Gaussian increments defined on the same tree. As a Gaussian process is determined by its mean and covariance, it suffices to construct a Gaussian branching random walk on the tree $\mathcal{T}_n$ with mean zero and covariances that match those in (1.3.3). This can be realized by attaching to each edge of the tree an independent centred Gaussian random variable with variance $A(t/n)$, with the edge starting at generation $t - 1 \in \{1, \ldots, n\}$. The case when $A$, instead of a step-wise function, can be an arbitrary probability distribution function and the process being defined on a continuous-time

Galton-Watson tree, is referred to as *continuum random energy model* (CREM). The extremes of GREM and CREM were analysed by Bovier and Kurkova in [26, 27]. There are three possible regimes which are determined by the function $A$. To avoid overburdening notation we provide an informal formulation of the following two theorems which are taken from [65].

**Theorem 1.4.1.** *In GREM with $A(x) < x$, for $x \in (0,1)$, the following is true:*

1. *The level of the maximum coincides with the one in the REM.*
2. *The maximum rescaled as in* (1.22) *converges in law to a Gumbel distribution.*
3. *The extremal process converges in law to the same Poisson point process as in* (1.24).

**Theorem 1.4.2.** *In GREM, where $A(x) > x$, for some $x \in (0,1)$, the following is true:*

1. *The first order of the maximum depends on the concave hull of $A$, which we denote by $\hat{A}$. In particular,*

$$\frac{M_n}{2\log(4^n)} \int_0^1 \left(\hat{A}\right)'(x)\mathrm{d}x \to 1, \tag{1.27}$$

   *as $n \to \infty$ in probability.*

2. *The maximum can be rescaled such that it converges in law to a randomly shifted Gumbel random variable.*
3. *The properly rescaled extremal process converges in law to a cascade of Poisson point processes.*

A cascade of Poisson point processes is a concatenation of different Poisson point processes. First, one generates the first Poisson point process. At each Poisson point in the first Poisson point process one generates and attaches independent second generation Poisson point processes and so forth. A mathematically precise construction of such a process was carried out by Ruelle in [80]. Comparing Theorem 1.4.1 with Theorem 1.4.2, one should note the drastic changing behaviour of the leading order term of the maximum in GREM once $A$ crosses the straight line. In particular, the integral in (1.27) is strictly smaller than 1 and thus, the leading order term is strictly smaller compared to the case when $A(x) < x$, for $x \in (0,1)$, in which it coincides with the one in REM.

Of major importance in the analysis and understanding of the extremal process is the *genealogical structure* of extremal particles. Pick two extremal particles at generation $n$ and follow their paths backwards to the root. The key question to ask here is:

*At which generation do their paths meet?*

It turns out that the answer heavily depends on the function $A$:

1. In the setting of Theorem 1.4.1 the particles' paths will meet close to generation 0 or $n$ with high probability.
2. In the setting of Theorem 1.4.2 the particles' paths can meet at any discontinuity point $h$ of the concave hull of $A$ with $A(h) > h$. This leads to a concatenation of independent extremal processes that are initiated at each such point. In particular, an extremal particle at generation $n$ must already be extremal at those intermediate generations $nh$, for which $A(h) > h$.

Regarding this dramatically different behaviour depending on $A$, it is of natural interest to study the case of more general functions $A$, in particular the critical case in continuous time, in which $A(x) = x$ for $x \in [0, 1]$. This leads us to (variable-speed) *branching Brownian motion* (BBM) on the continuous-time Galton-Watson tree.

### 1.4.3 (Variable-speed) Branching Brownian motion

*Branching Brownian motion* (BBM) was introduced in in the late 1950's and early 1960's. It is a classical object in probability theory, which itself combines two fundamental objects, Brownian motion and the Galton-Watson tree. Notable contributions in the study of its maximum value and its connection to the Fisher-Kolmogorov-Petrovsky-Piscounov (F-KPP) equation were made by McKean [72], Bramson [28], Lalley and Selke [67] and Chauvin and Rouault [33, 34] from the 1970's and until the 1990's. The F-KPP equation is a classical equation in population dynamics and was studied earlier in 1937 by Kolmogorov, Petrovksy and Piscounov [66] and Fisher [53]. In the last decade there has been a renewed interest in branching Brownian motion, mainly initiated by the understanding of its extremal process by Arguin, Bovier and Kistler [5, 6, 7], as well as by Aidekon, Berestycki, Brunet and Shi [1]. More detailed questions concerning its extreme level sets, of all particles within $O(1)$ to the global maximum, are investigated in [36].

*Variable-speed branching Brownian motion* was introduced by Derrida and Spohn [43]. It allows for a richer class of covariances than BBM and coincides with CREM on the continuous time Galton-Waton tree. The extremes of variable-speed BBM were analysed by Fang Zeitouni [49], Maillard Zeitouni [71] and Bovier and Hartung in [23, 24]. In [23, 24], convergence of the maximum and the extremal process in the weakly correlated regime is proved.

We start with a definition of the model. Fix a time horizon $t > 0$ and let $n(t)$ be the number of particles in the Galton-Watson tree up to time $t$. To be consistent with the literature, we assume here that the number of offspring for each particle on the tree has mean 2 and is of finite variance. We collect these particles in the set $\{i_k(t) : k \leq t\}$. Analogously to (1.25), for two particles $i_k(t), i_l(t)$, we set

$$d(i_k(t, i_l(t))) = \text{ time of the most recent common ancestor of } i_k(t) \text{ and } i_l(t). \tag{1.28}$$

Let $A : [0, 1] \mapsto [0, 1]$ be a non-decreasing function that satisfies, $A(0) = 0$, $A(1) = 1$. *Variable-speed branching Brownian motion* on the Galton-Watson tree is a centred Gaussian process, $\{x_k^A(t) : k \leq n(t)\}_{t\geq 0}$, with covariance

$$\mathbb{E}\left[x_k^A(t)x_l^A(t)\right] = tA(d(i_k(t), i_l(t))/t). \tag{1.29}$$

Usual BBM is the special and critical case when $A(x) = x$, for $x \in [0, 1]$, which we call $\{x_k(t) : k \leq n(t)\}_{t\geq 0}$. In case of BBM, the following is known:

**Theorem 1.4.3.** *Let $\{x_k(t) : k \leq n(t)\}_{t\geq 0}$ be BBM and set $m_t^{BBM} := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$. Then,*

1. *The level of the maximum coincides in the leading order with the one in REM whereas its sub-leading logarithmic correction is smaller.*

2. *The maximum of BBM at time t centred by $m_t^{BBM}$ converges in law, as $t \to \infty$, to a randomly shifted Gumbel. The random shift depends on the number of particles at the very beginning that*

*can become extremal at time t, and additionally weights their positions.*

3. *The extremal process converges in law to a cluster Cox process.*

In the case of weak correlations, i.e. when $A$ stays strictly below the straight line, we have the following result. For additional necessary technical assumptions see [24].

**Theorem 1.4.4.** *Assume that* $A(x) < x$*, for* $x \in (0,1)$ *and* $A'(0) < 0$ *as well as* $A'(0) > 1$. *Then the following is true:*

1. *The level of the maximum is identical to the independent setting, i.e.* $\tilde{m}_t := \sqrt{2}t - \frac{1}{2\sqrt{2}} \log t$.

2. *The properly centred maximum converges in law to a randomly shifted Gumbel. The random shift accounts for the random number of particles at the very beginning that can become extremal at time t.*

3. *The extremal process converges in law to cluster Cox process. The limit is universal in the sense that the law of the clusters depends on A only by* $A'(1)$*, whereas the random shift depends on the function A only by* $A'(0)$.

As in the case of GREM, one should ask for the *time when two extremal particles split* with high probability. The answer in the cases of the two theorems is identical, at the very beginning or the very end. This phenomenon is also a crucial reason why the extremal process in both cases takes the form of a *cluster Cox process*. A Cox process is a Poisson process whose intensity measure itself is random. Thus, to generate $PPP(\mu)$, for a non-negative random measure $\mu$, one first samples $\mu$ and then generates the Poisson process conditional on $\mu$. In a cluster Cox process, one attaches to each Poisson point in the Cox process an independent copy of the cluster process. In the context of (variable-speed) BBM, the random Poisson points correspond to the relative heights of extreme local maxima and whose mutual genealogical distances are large, whereas the clusters are formed by those particles on the tree whose genealogical distance to a chosen extreme local maximum is small, i.e. that recently branched off the spine of the extreme local maximum.

In case when $A(x) > x$, for some $x \in (0,1)$, we have to distinguish two cases. If the concave hull of $A$ is a piecewise linear function, then the maximum and the extremal process are simply concatenations of the maxima, respectively extremal processes, on the intervals on which the concave hull is linear. The maxima and extremal processes on these sub-intervals are given by Theorem 1.4.3 and Theorem 1.4.4. Which case is present depends only on whether $A$ stays below its concave hull or coincides with it on the respective interval, see also [23]. If the concave hull is instead strictly concave much less is known. As in GREM (1.27), the first order of the maximum is determined by its concave hull, which in this case is strictly smaller than 1, and thereby, strictly smaller than in the other two cases. This is commonly referred to as *slowdown* of the maximum. Concerning its second-order correction to the maximum value, instead of being logarithmic, it is known to be a power of $1/3$ [49, 71]. Furthermore, convergence of the properly centred maximum to a solution of a time-inhomogeneous F-KPP equation is proved in [71]. The correct centring, however, is implicit and its existence part of the statement. Convergence of the extremal process remains an open question.

Note that there is an apparent discontinuity of the sub-leading order correction that occurs when the covariance function, $A$, crosses the straight line. In the case in which the concave hull of $A$ is piecewise linear, Bovier and Hartung [25] proved that it is possible to continuously interpolate between the different second order corrections of the maximum value. This works by allowing the variance function, $A$, to additionally depend on the time-horizon, $t$. The principle reason leading to different sub-leading order corrections is a localization of extremal particles' paths.

## 1.5 The two-dimensional discrete Gaussian Free Field

The study of the (two-dimensional discrete) Gaussian free field was initiated in the 1970s [74, 77, 31]. The two-dimensional discrete Gaussian free field is a special instance of a larger family of random surface models known as *Gibbs-gradient (random) fields* and probably the simplest non-trivial random height function on a two-dimensional lattice. Moreover, it is a very prominent example of a log-correlated Gaussian random field. Its relevance stems from its connection to many interesting objects in mathematics as well as in physics. One important reason for this is that its scaling limit is the two-dimensional *continuum Gaussian free field*, which itself is *scale-invariant* and a natural two-dimensional-time analog of the Brownian bridge. It connects to multiple objects of mathematical interest, e.g. Kahane's theory of Gaussian multiplicative chaos, Liouville quantum gravity, Schramm-Loewner evolutions, conformal loop ensembles or Liouville first passage percolation. For further information on these and their connection to the Gaussian free field, we refer to introductory lecture notes by Werner [89], Berestycki [13], Berestycki and Norris [14], Sheffield [82] and by Rhodes and Vargas [79]. In the physics literature, the (discrete) Gaussian free field is often referred to as the *harmonic crystal* or the *Euclidean bosonic massless free field*. As a statistical mechanics model of random interfaces, understanding its extremal values is of natural interest as the associated Gibbs measure at low temperature concentrates on the states with the lowest energy levels. The study of the extremal values of the two-dimensional discrete Gaussian free field with zero boundary conditions was initiated in 2001 when Bolthausen, Deuschel and Giacomin [20] determined the first order of the maximum value and moreover, proved that if the entire field is conditioned to be non-negative, it is pushed up by the leading order of the maximum of the unconditioned field. This phenomenon is usually referred to as *entropic repulsion* of the 2d DGFF. The extremes of the two-dimensional discrete Gaussian free field were investigated in various constellations mainly by Biskup, Bolthausen, Bramson, Deuschel, Ding, Giacomin, Louidor and Zeitouni [20, 37, 21, 30, 45, 47, 10, 29, 17, 18] and are by now well understood, i.e. one knows that both the properly centred maximum as well as its extremal process converge. There are lecture notes on the extremal values of the two-dimensional discrete Gaussian free field, very extensive ones by Biskup [16], those by Louidor [70] and by Zeitouni [90]. The latter also discusses in large parts the analysis of the maximum value of the branching random walk, which turns out to be very instructive in the study of the maximum of the two-dimensional discrete Gaussian free field.

Before entering a more detailed discussion of the model and the theory on its extremal values, we shortly present one of the key tools in the analysis of the (scale-inhomogeneous) 2d discrete Gaussian free field, *Gaussian comparison*. To keep things simple, we restrict ourselves to two inequalities that are of the greatest relevance in this context. For a more general and detailed treatment of Gaussian comparison we refer to [22, 16].

**Gaussian comparison:** For two given centred Gaussian fields, $X$ and $Y$, indexed by the same index set, $\mathcal{T}$, such that the first has more intrinsic independence, i.e. for any $s, t \in \mathcal{T}$,

$$\mathbb{E}\left[(X_t - X_s)^2\right] \leq \mathbb{E}\left[(Y_t - Y_s)^2\right], \tag{1.30}$$

we know that

$$\mathbb{E}\left[\max_{t\in\mathcal{T}} X_t\right] \leq \mathbb{E}\left[\max_{t\in\mathcal{T}} Y_t\right]. \tag{1.31}$$

In words, if we know that one of the centred fields has a pairwise smaller dependence, measured as a larger variance of the pairwise difference, we know that the expectation of its maximum value is larger. The statement in (1.31) is known as the inequality of *Sudakov-Fernique*. If, in addition, at every point we have equal variances, i.e. $\mathbb{E}\left[X_t^2\right] = \mathbb{E}\left[Y_t^2\right]$, for all $t \in \mathcal{T}$, we know that the maximum of the field with larger intrinsic independence stochastically dominates the maximum of the one with smaller pairwise independence, i.e.

$$\mathbb{P}\left(\max_{t\in\mathcal{T}} X_t \geq x\right) \leq \mathbb{P}\left(\max_{t\in\mathcal{T}} Y_t \geq x\right), \quad \forall x \in \mathbb{R}. \tag{1.32}$$

This statement is also known as *Slepian's lemma.* The idea how to use these inequalities in order to understand the extremes of the (scale-inhomogeneous) DGFF is straightforward: Construct centred Gaussian processes that have pairwise larger or smaller correlations compared to the (scale-inhomogeneous) DGFF and whose extremes we are able to analyse. Natural candidates for such processes are Gaussian processes on trees such as (variable-speed) branching Brownian motion or the (time-inhomogeneous) branching random walk.

In the following, we define the two-dimensional discrete Gaussian free field with zero boundary conditions on a box, explain how Gaussian comparison comes into play and discuss results on its extremal values.

**Definition 1.** Let $N \in \mathbb{N}$, set $V_N = [0,N)^2 \cap \mathbb{Z}^2$ and let $\{S_k\}_{k\in\mathbb{N}}$ be the simple random walk on the lattice $\mathbb{Z}^2$. Under the measure $\mathbb{P}_v$, $\{S_k\}_{k\in\mathbb{N}}$ is a simple random walk on $\mathbb{Z}^2$, started at $v \in V_N$ and $\tau_{\partial V_N} = \inf\{k \geq 0 : S_k \notin V_N\}$ denotes the first time it exits $V_N$. Let

$$G_{V_N}(v,w) = \frac{\pi}{2}\mathbb{E}_v\left[\sum_{k=0}^{\tau_{\partial V_N}-1} \mathbb{1}_{S_k=w}\right], \quad v,w \in V_N \tag{1.33}$$

be the Green kernel associated with simple random walk.

The *discrete Gaussian free field* on $V_N$ is a centred Gaussian field, $\{\phi_v^{V_N}\}_{v\in V_N}$, with correlations given by the Green kernel, i.e. $\mathbb{E}\left[\phi_v^{V_N}\phi_w^{V_N}\right] = G_{V_N}(v,w)$. We set $\phi_v^{V_N} = 0$, for $v \in \mathbb{Z}^2 \setminus V_N$ and write $\phi_v^N = \phi_v^{V_N}$, for $v \in \mathbb{Z}^2$.

Let $\delta \in (0,1/2)$ and denote by $V_N^\delta = (\delta N, (1-\delta)N)^2 \cap \mathbb{Z}^2$ the set of vertices that are at least $\delta N$ away from the boundary. It is a well-known fact, see e.g. [30, Lemma 2.2], that the covariance, for vertices $v,w \in V_N^\delta$, is of the form

$$\mathbb{E}\left[\phi_v^N\phi_w^N\right] = \log N - \log_+(\|v-w\|_2) + O(1), \tag{1.34}$$

where the constant order term $O(1)$ can be bounded by a constant, $C(\delta) > 0$, which is uniform in $N$ and $v,w \in V_N^\delta$, and with $\log_+(x) = \log(\max(x,1))$, for $x \in \mathbb{R}_+$. Based on various contributions by Bolthausen, Bramson, Deuschel, Ding and Zeitouni [21, 30, 45, 47], Bramson, Ding and Zeitouni

[29] proved convergence of the properly centred maximum with centring

$$b_N = 2\log N - \frac{3}{4}\log\log N - \frac{1}{4}\log(2\pi) \tag{1.35}$$

and scaling $a_N \equiv 1$. A common choice as suitable centring is $m_N^{DGFF} := 2\log N - \frac{3}{4}\log\log N$. Of notable interest is the factor 3/4 in front of the logarithmic correction which is completely analogue to the factor in front of the sub-leading order correction in BBM or the braching random walk (BRW), see e.g. Theorem 1.4.3. In particular, it differs from the 1/4 present in the setting of independent random variables, see (1.22). The analogy to the sub-leading order correction in the case of the BRW is of no great surprise, since in the 2d DGFF there is an approximate tree structure present which allows to use *Gaussian comparison* to relate the maximum of the 2d DGFF with the maximum of a suitable branching random walk. In the following subsection on the scale-inhomogeneous DGFF, we provide a more detailed explanation for its occurrence.

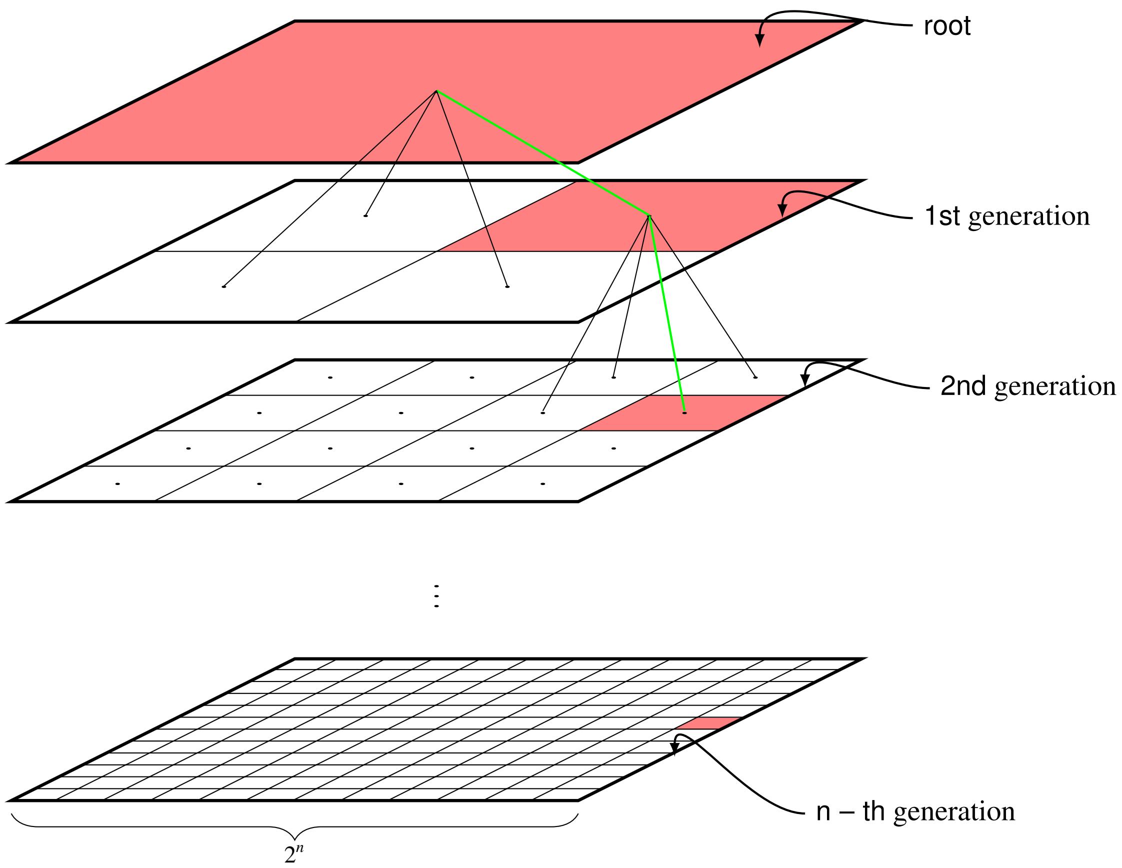



Figure 1.5: Tree decomposition of a box with side length $2^n$. The red boxes contain a chosen vertex.

A convenient way to apply Gaussian comparison to gain information on the maximum value of the 2d DGFF is to construct and compare it to a suitable Gaussian branching random walk on a 4-ary tree. A possible construction of such a branching random walk is illustrated in Figure 1.5 and which is what we describe next. One chooses the side length of the box $V_N$ to be a natural power of 2, e.g. $N = 2^n$.

This is a purely technical assumption, which at every step ensures that each box can be subdivided into four equal sized ones, and such that for each vertex at the $n-$th step there is exactly one box of side length 1 including it. We connect at every step each new box to its parent box by an edge. To each edge one attaches an independent standard Gaussian random variable. The branching random walk, $\{X_v\}_{v\in V_N}$, indexed by the vertices $v \in V_N$ is then defined by summing all random variables along the shortest path from each "leaf", a box of side length 1 containing the vertex $v$, to the root, with "root" being the box of side length $N$, see also Figure 1.5. Its covariance is given by

$$\mathbb{E}\left[X_v X_w\right] = \log N - \log_+ d_T(v,w), \quad v,w \in V_N, \tag{1.36}$$

where $d_T(v,w)$ denotes the distance on the corresponding tree between two vertices $v,w \in V_N$ and which is given by the total number of generations, $\log N$, minus the generation of their most recent common ancestor. Regarding the decomposition depicted in Figure 1.5, the generation of their most recent common ancestor corresponds to the largest integer $k \leq \log N$ such that $v,w \in V_N$ are contained in the same box of side length $2^k$. In order to use *Gaussian comparison*, one needs that the two centred Gaussian processes share their index set and have a similar correlation structure. The branching random walk comparison suffices to obtain the correct leading order of the maximum, see [20]. However, this simple approximation does not suffice to obtain the correct sub-leading order of the

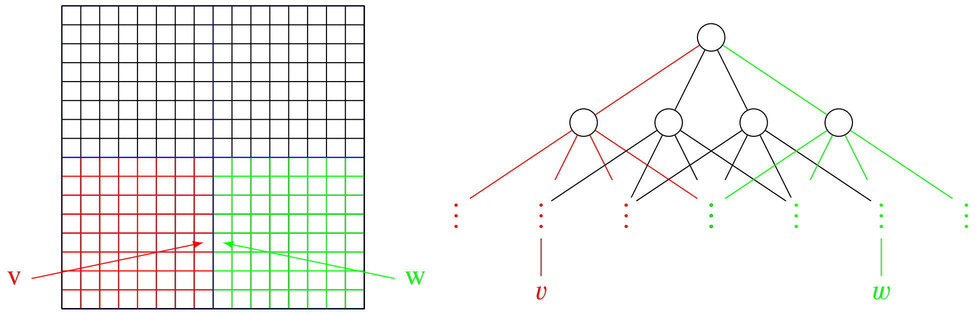



Figure 1.6: Two vertices, $v$ and $w$, with small Euclidean distance, but large distance on the associated tree.

maximum. This is due to major defects in the correlation approximation by the BRW construction which does not allow for a suitable lower bound on the maximum value by the use of Gaussian comparison. The covariance of two leaves $v,w \in V_N$ in the BRW is given by the generation of their most recent common ancestor. This leads to the effect that there are lots of vertices that are much less correlated in the BRW than in the DGFF. Indeed, if one picks two vertices that lie after the first step of the decomposition, depicted in Figure 1.6, in opposing boxes but close to their common boundary, then their distance on the tree is very large, whereas their Euclidean distance is extremely small. For a more precise approximation one can take the uniform average of all possible branching random walk decompositions when considering each box in the decomposition as a torus. See Figure 1.7 for two possible decompositions of one box. The process that is obtained by taking the uniform average of all these branching random walks is called *modified branching random walk* (MBRW). In particular, one can show that this allows for a $O(1)$ precise approximation of the covariances of vertices that are $\delta N$ away from the boundary $\partial V_N$, for any fixed $\delta \in (0,1/2)$. Using Gaussian comparison one then deduces that the maximum of the 2d DGFF can be approximated by the maximum of the corresponding modified branching random walk up to constant order, see [30]. A simple argument,

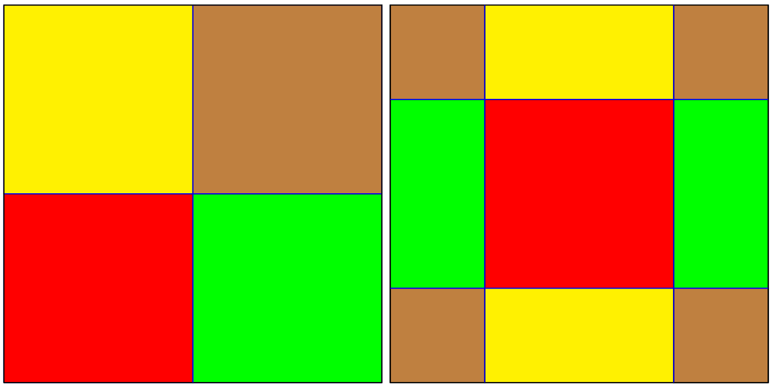


Figure 1.7: Two possible decompositions of the same box, considered as a torus, into four equal-sized sub-boxes.

which goes back to Dekking and Host [38], applied in this context then immediately yields tightness of the centred maximum [30]. Obtaining more precise results such as the convergence of the maximum, however, is much more involved. Convergence of the centred maximum value was proved by Bramson, Ding and Zeitouni [29]. In particular, the limit takes the form of a randomly shifted Gumbel random variable [17, 29], which we state as:

**Theorem 1.5.1.** *Let $\{\phi_v^N\}_{v\in V_N}$ be a 2d DGFF on $V_N$. Then, for any $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{v\in V_N} \phi_v^N - m_N^{DGFF} \leq x\right) = \mathbb{E}\left[e^{-CZe^{-2x}}\right], \tag{1.37}$$

*where $Z$ is an a.s. positive random variable, and $C > 0$ a constant.*

Apart from just considering the maximum value, one is more generally interested in the joint distribution of vertices above a certain level below the global maximum, in particular, in their properly centred height and their spatial distribution on the two-dimensional grid. A first step towards an understanding of these is the observation, due to Ding and Zeitouni [47], that there exists a finite constant $c > 0$ such that

$$\lim_{r\to\infty}\lim_{N\to\infty} \mathbb{P}\left(\exists v, w \in V_N : \|v-w\|_2 \in (r, N/r) \text{ and } \phi_v^N, \phi_w^N \geq m_N^{DGFF} - c\log\log r\right) = 0. \tag{1.38}$$

From (1.38) we see that with high probability vertices that exceed an extremal height, here $m_N^{DGFF} - c\log\log r$, are either within Euclidean distance $O(1)$ or at least $N/O(1)$ apart. In particular, extremal vertices congregate in clusters of diameter $O(1)$ and these clusters are $N/O(1)$ apart. The fact that vertices within distance $O(1)$ to an extreme local maximum are extremal themselves is very likely since the difference in height of one such vertex to the extreme local maximum is given by an independent centred Gaussian with variance $O(1)$. More interesting is the fact that the diameter of such clusters is essentially finite and furthermore, that any two such clusters are $N/O(1)$ apart, which suggests that the clusters, conditioned on the extreme local maxima, are asymptotically independent. This motivates to study the joint distribution of height and spatial location of extreme local maxima and the clusters around them. The suitable object to capture this behaviour is the following point measure, also known as *full/structured extremal process*,

$$\eta_{N,r} = \sum_{v\in V_N} \mathbb{1}_{v \text{ r-loc. max.}} \delta_{v/N} \otimes \delta_{\phi_v^N - m_N^{DGFF}} \otimes \delta_{\{\phi_v^N - \phi_w^N :\, w\in\mathbb{Z}^2\}}. \tag{1.39}$$

For each fixed $N$ and $r$, $\eta_{N,r}$ is a random point measure on $[0,1]^2\times\mathbb{R}\times\mathbb{R}^{\mathbb{Z}^2}$. In light of (1.6) one should think in (1.39) of $N\in\mathbb{N}$ being much larger than $r\in\mathbb{R}$. The indicator function in (1.39), $\mathbb{1}_{v \text{ r-loc. max.}}$, picks out the local maximum in an $r-$environment, which is the local maximum of vertices within Euclidean distance $r$ on the grid. The first coordinate in (1.39) gives their normalized position on the grid, the second their relative height and the last is the field seen from the chosen local maximum. Note that the indicator does not ensure that the vertex is within finite distance to the global maximum, and thus an extreme local maximum. However, subtracting the order of the maximum, $m_N^{DGFF}$, from its height, most point measures in the sum tend to the Dirac measure at $-\infty$ in their height coordinate, as $N\to\infty$. Endowing the space of Radon measures with the vague topology turns it into a Polish space and ensures that the limit, if it exists, is a proper point process.

In fact, Biskup and Louidor [17, 18] proved convergence of the *structured extremal process*, $\{\eta_{N,r_N}\}_{N\in\mathbb{N}}$, to a *cluster Cox process*.

**Theorem 1.5.2.** *There exists a non-trivial random Borel measure, $Z$, on $V=[0,1]^2$, with $Z(V)<\infty$ a.s. and such that, for any sequence $\{r_N\}_{N\in\mathbb{N}}$ satisfying both $r_N\to\infty$ and $r_N/N\to 0$, as $N\to\infty$,*

$$\lim_{N\to\infty} \eta_{N,r_N} = PPP\Big(Z(\mathrm{d}x)\otimes e^{-2h}\mathrm{d}h\otimes\theta(\mathrm{d}\nu)\Big), \tag{1.40}$$

*with cluster law, $\theta$, being a probability measure on $[0,\infty)^{\mathbb{Z}^2}$. Convergence in (1.40) is in law with respect to the vague topology of Radon measures on $[0,1]^2\times\mathbb{R}\times\mathbb{R}^{\mathbb{Z}^2}$.*

We remark that the cluster law, $\theta$, admits an explicit representation: Let $\phi$ be the discrete Gaussian free field on $\mathbb{Z}^2\setminus\{0\}$. Equivalently, $\phi$ is the discrete Gaussian free field on $\mathbb{Z}^2$ conditioned to be zero at 0, which is also called the *pinned discrete Gaussian free field*. The cluster law, $\theta$, is given as the weak limit

$$\theta(\cdot) = \lim_{r\to\infty} \mathbb{P}\Big(\phi+\mathfrak{a}\in\cdot\,\Big|\,\phi(x)+\mathfrak{a}\geq 0 : |x|\leq r\Big), \tag{1.41}$$

with $\mathfrak{a}$ being the potential kernel of the simple random walk on $\mathbb{Z}^2$. Note that the conditioning (1.41) ensures that the field is centred at a $r-$local maximum at 0. Moreover, the statement of Theorem 1.5.2 is true for much more general domains $V_N$, restricted only by the regularity of the boundaries of the sequence of domains, see [18]. The relevance of this fact becomes manifest when characterizing the random intensity measure $Z$ under conformal transformations of the underlying domains. In particular, its law can be identified with the critical *Liouville Quantum Gravity* measure associated with the continuum Gaussian free field [19], which is a key object in the study of random conformally-invariant geometry. One should note that in both cases, in BBM and in the 2d DGFF, correlations affect the sub-leading order correction to the maximum value, the properly centred maximum converges in law to a randomly shifted Gumbel random variable and their extremal processes to cluster Cox processes.

## 1.6 The scale-inhomogeneous two-dimensional discrete Gaussian Free Field

The *scale-inhomogeneous discrete Gaussian free field* in dimension two was first introduced in 2015 by Arguin and Zindy [10] as a tool in order to prove Poisson-Dirichlet statistics of the extremes of the two-dimensional discrete Gaussian free field at the level of the Gibbs measure at low temperature, in which case the Gibbs measure should be supported essentially on the minima of the field. Nevertheless, it is an object of its own interest. It is the analogue model, in the context of the 2d DGFF, to variable-speed branching Brownian motion on the Galton-Watson tree in the context of usual branching Brownian motion. In fact, with regards to the 2d DGFF, it allows for more general correlation functions and thus, a study of its extremal values is of natural interest. A key difference to the case of (variable-speed) BBM is that correlations of the (scale-inhomogeneous) 2d DGFF are ordered only approximately in a hierarchical fashion. The study of its maximum value was initiated in 2016 by Arguin and Ouimet [9] who determined the leading order of the maximum in the case of finitely many scales. In particular, they showed that in this case, just as in variable-speed BBM and GREM, the first order of the maximum value is determined by the concave hull of the variance function, see Theorem 1.4.1 and Theorem 1.4.2. Moreover, they determined the log-number of high points, i.e. the logarithm of the number of vertices being above a positive fraction, $\alpha \in (0, 1)$, of the leading order of the global maximum. Ouimet [75] used these results to analyse the geometry of the associated limiting Gibbs measure at low temperature in the case of finitely many scales.

We start with a definition of the model and then discuss our contributions in the study of its extremal values. Let $N \in \mathbb{N}$, and $\{\phi_v^N\}_{v \in V_N}$ be a discrete Gaussian free field on $V_N$. For $v \in V_N$ and $\lambda \in [0, 1]$, let $[v]_\lambda^N$ be the box centred at $v$ of side length $N^{1-\lambda}$ and set

$$\phi_v^N(\lambda) = \mathbb{E}\left[\phi_v^N | \sigma\left(\phi_w^N : w \notin [v]_\lambda^N\right)\right]. \tag{1.42}$$

$\phi_v^N(\lambda)$ is the expected field value at vertex $v$ conditioned on the values of the field outside the box $[v]_\lambda^N$. We denote by $\nabla\phi_v^N(\lambda)$ the partial derivative, $\partial_\lambda \phi_v^N(\lambda)$, with respect to $\lambda$. The 2d DGFF satisfies the so-called *Gibbs-Markov property*, i.e. for two sets $A \subset B$, we can decompose the DGFF on $B$ as follows:

$$\phi^B \overset{\text{law}}{=} \phi^A + \mathbb{E}\left[\phi^B | \phi_v^B : v \in B \setminus A\right], \tag{1.43}$$

where the fields on the right-hand side are independent. In particular, for any $v \in V_N$,

$$\phi_v^N \overset{\text{law}}{=} \phi_v^N(\lambda) + \phi_v^{[v]_\lambda^N}, \tag{1.44}$$

where the fields on the right-hand side are independent. This allows to decompose the DGFF at each vertex $v$ concentrically along scales, $s \in [0, 1]$, as follows

$$\phi_v^N = \int_0^1 \nabla\phi_v^N(s) \mathrm{d}s. \tag{1.45}$$

Now, let $s \mapsto \sigma(s)$, for $s \in [0, 1]$ be a non-negative function such that $\int_0^1 \sigma^2(s)\mathrm{d}s = 1$. We write $\mathcal{I}_{\sigma^2}(x) = \int_0^x \sigma^2(s)\mathrm{d}s$. The function $s \mapsto \sigma(s)$ is the so-called “variance function”.

**Definition 2.** The *scale-inhomogeneous discrete Gaussian free field on* $V_N$, $\{\psi_v^N\}_{v\in V_N}$, with variance $\sigma$ is defined as

$$\psi_v^N := \int_0^1 \sigma(s)\nabla\phi_v^N(s)\mathrm{d}s, \quad v \in V_N. \tag{1.46}$$

In particular, it is a centred Gaussian field with covariances given by

$$\mathbb{E}\left[\psi_v^N\psi_w^N\right] = \log N \mathcal{I}_{\sigma^2}\left(\frac{\log N - \log_+ \|v-w\|_2}{\log N}\right) + O\left(\sqrt{\log N}\right), \quad \forall v, w \in V_N^\delta, \tag{1.47}$$

where $\delta \in (0,1)$ is arbitrary but fixed. When neglecting the big-O error term, (1.47) should be read in the sense that up to normalization, correlations are given as functions of the spatial Euclidean distance. Comparing this to the correlations of variable-speed BBM as given in (1.29), here the map $s \mapsto \mathcal{I}_{\sigma^2}(s)$ takes the role of $s \mapsto A(s)$. In both cases, correlations are, up to normalization, given as functions of the underlying metric distance. In particular, one should think of the scale parameter, $s$, in the scale-inhomogeneous 2d DGFF being the analogue to the time parameter, normalized to $[0,1]$, in variable-speed BBM. These observations motivate the notion of the scale-inhomogeneous 2d DGFF being the analog model in the context of the 2d DGFF of variable-speed BBM in the context of BBM. In the previous subsections, we observed that the extremal values of BBM and the 2d DGFF are structurally very similar and belong to the same class of models, i.e. their properly centred maxima converge in law to randomly shifted Gumbel variables, with their centrings being completely analogue, and their extremal processes to cluster Cox processes. With regards to the extreme value theory for variable-speed BBM and considering their similar correlation structures, it is natural to ask whether such an analogy can also be observed between the extremal values of the scale-inhomogeneous 2d DGFF and those of variable-speed BBM.

In the remainder, we give a brief outline of the results obtained in chapters 2-4, which are a confirmative partial answer to this question. In particular, we discuss on a heuristic level the results and provide basic ideas of the proofs. For full details and rigorous arguments we refer to chapters 2-4, which are all available as preprints [50, 51, 52]. The contents of chapter 3 (see [51]) and chapter 4 (see [52]) is joint work with Lisa Hartung.

### 1.6.1 Subleading order and tightness of the maximum.

In Chapter 2, we pick up the study of the maximum in the case of finitely many scales initiated in [9], in which the leading order term of the maximum was established. First, we introduce additional notation that allows us to state and discuss our findings. We denote by $s \mapsto \hat{\mathcal{I}}_{\sigma^2}(s)$ the concave hull of the function $s \mapsto \mathcal{I}_{\sigma^2}(s)$. In the case when $s \mapsto \hat{\mathcal{I}}_{\sigma^2}(s)$ is piecewise linear, we can number its different slopes. The first slope we call $\bar{\sigma}_1^2$, the second $\bar{\sigma}_2^2$ and so forth. The length of the first interval with slope $\bar{\sigma}_1^2$ we denote by $\lambda^1$, the end scale of the second interval where the slope is $\bar{\sigma}_2^2$ we call $\lambda^2$ and so on. Thus, the length of the $i-$th interval is given by $\lambda^i - \lambda^{i-1}$. The parameters $\{\bar{\sigma}_i\}_{i\geq 1}$ are called *effective variance parameters* and the corresponding scales, $\{\lambda^i\}_{i\geq 1}$, are called *effective scale parameters*, see also Figure 1.8 for an example. Note that the effective variances are decreasing, i.e. $0 \leq \bar{\sigma}_{i+1} \leq \bar{\sigma}_i$, for $i \geq 1$.

In [9] the leading order of the maximum value in the case of $m$ effective parameters is established,

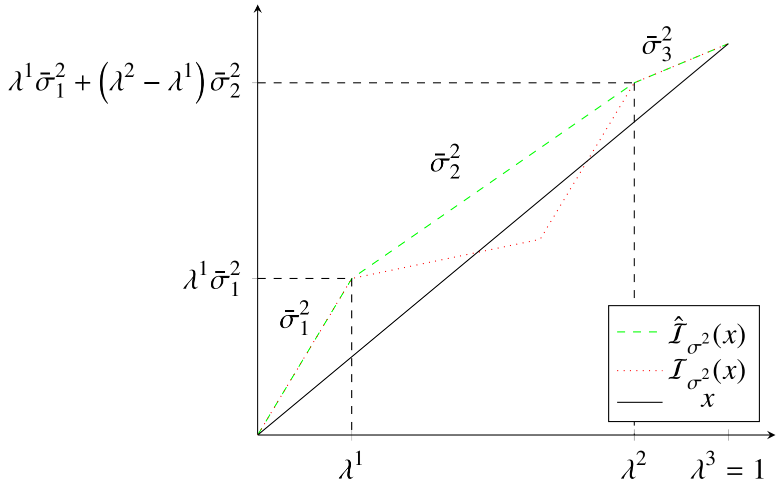



Figure 1.8: An example of effective variances.

i.e. it is shown that, in probability,

$$\lim_{N\to\infty} \frac{\max_{v\in V_N} \psi_v^N}{2\log N} = \sum_{i=1}^{m} \bar{\sigma}_i \left(\lambda^i - \lambda^{i-1}\right). \tag{1.48}$$

Note that this value is also obtained when taking the sum of the leading orders of maxima of $m$ independent 2d DGFFs on boxes of side length $\left(\lambda^i - \lambda^{i-1}\right)\log N$ and whose covariances are scaled by $\bar{\sigma}_i$. However, the exact same leading order is also attained when assuming that these $m$ scaled 2d DGFFs were completely uncorrelated centred Gaussian fields with idential variances as those of the scaled DGFFs. With regards to Gaussian comparison, these two very different kind of models, if set up correctly, should pose possible extreme candidates for comparison. While this yields the correct leading order of the maximum value, on the level of the sub-leading order correction it only gives trivial upper and lower bounds. Thus, the correct sub-leading order to the maximum value remains, at this stage, a completely open question.

To explain what happens on the level of the sub-leading order correction of the maximum value, we first consider the case when there is exactly *one effective scale*, i.e. we assume $\bar{\sigma}_1 = 1$ and $\lambda^1 = 1$. Using Gaussian comparison, we can bound the maximum value from below by that of the usual 2d DGFF and from above by an uncorrelated 2d DGFF. By an uncorrelated DGFF, we denote a centred Gaussian field having at each vertex identical variances as the usual DGFF, and else being uncorrelated. Using Gaussian comparison we deduce that the sub-leading order to the maximum lies in the interval $\left[-\frac{3}{4}\log\log N, -\frac{1}{4}\log\log N\right]$. One of the main consequences of what we prove in Chapter 2 is that, in the case of *one effective scale* and under the additional assumption of $\mathcal{I}_{\sigma^2}(x) < \hat{\mathcal{I}}_{\sigma^2}(x)$, for $x \in (0,1)$, the *sub-leading order correction* to the maximum value is as if the field was independent, namely $-\frac{1}{2}\frac{\log\log N}{2}$.

The fact that we do not see any difference on the sub-leading order correction to the case of independent random variables is quite remarkable as the field which we consider has slowly decaying correlations. The other case we consider is when $\mathcal{I}_{\sigma^2}(x) = \hat{\mathcal{I}}_{\sigma^2}(x) = x$, for $x \in [0,1]$. This is simply the usual 2d DGFF with known correction, $-\frac{3}{2}\frac{\log\log N}{2}$. Note that in variable-speed BBM and the time-inhomogeneous branching random walk the same correction factors, $1/2$ and $3/2$, in the analogue regimes can be observed, see e.g. Theorem 1.4.3 and Theorem 1.4.4. One might wonder:

*Is this analogy to the time-inhomogeneous BRW simply superficial or can it be made precise?*

In fact, our proof draws heavily on this intuition and this is what we explain next. We adapt the idea from Bramson and Zeitouni in the case of the 2d DGFF, which consists in using Gaussian comparison to argue that it suffices to study the maximum of the *modified branching random walk* (MBRW) which itself can be studied similarly to the maximum of the BRW. Recall from the discussion of the discrete Gaussian free field that the MBRW is obtained by taking uniform averages of independent BRWs. We replace the BRWs in this construction by independent *time-inhomogeneous* BRWs with variances corresponding to those of the scale-inhomogeneous DGFF. Taking the uniform average over these independent time-inhomogeneous BRWs we obtain what we call the *modified inhomogeneous branching random walk* (MIBRW). It turns out that its covariance structure away from the boundary matches that of the scale-inhomogeneous DGFF, up to constant order. This allows, in a first step, to use Gaussian comparison to reduce the necessary analysis to the MIBRW. For the study of the maximum value of the MIBRW, we use a truncated second moment method. We explain the heuristics in the special case when there are exactly two scales with variance parameters $0 < \sigma_1 < 1 < \sigma_2$. This allows us to keep things simple, while capturing the essential ideas. For the truncation in the second moment analysis we draw on a path analysis for extremal vertices of the MIBRW and which is identical to the one for extremal particles of the time-inhomogeneous BRW. See also Figure 1.9 for an illustration. It turns out that for a vertex reaching the maximum at "time" $\log N$, it has to be at height

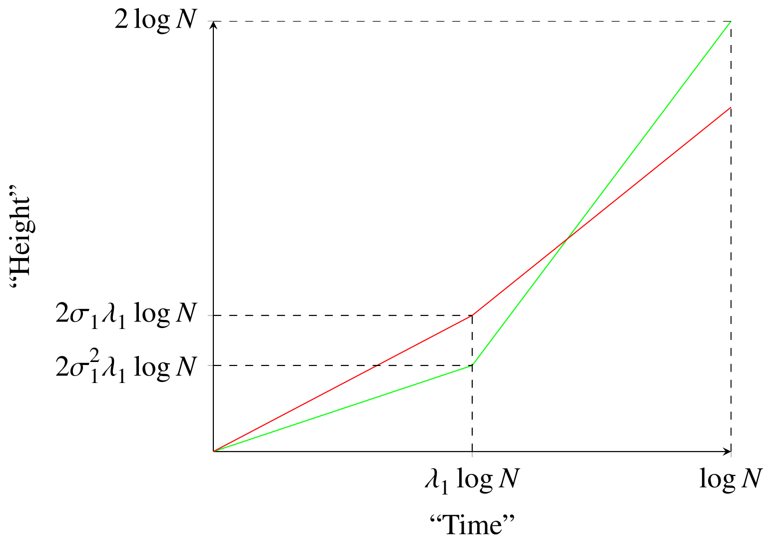



Figure 1.9: Path of an extremal particle vs path of a typical particle that is always close to the running maximum in the case when $\sigma_1 < \sigma_2$.

$2\sigma_1^2\lambda_1 \log N \pm O\left(\sqrt{\log N}\right)$ at the "time" of change in variance, which is $\lambda_1 \log N$. Note that this is significantly lower than vertices maximal at this moment which locate at $2\sigma_1\lambda_1 \log N \pm O\left(\sqrt{\log N}\right)$, a linear order above the vertices that are extremal at the end. The simple reason for this is that at the intermediate "time" $\lambda_1 \log N$, there are effectively $N^{2\lambda_1}$ particles, each with variance $\sigma_1^2\lambda_1 \log N$, for which we know what their maximum is. However, at this intermediate time point there are essentially only finitely many particles at the running maximum, and out of which at least one has to have some descendent that reaches the overall maximum level, $2\log N$, at the end. The probability of this event is much smaller than demanding one of the exponentially many particles being a linear order below the running maximum at "time" $\lambda_1 \log N$ and having a descendent that reaches the overall maximum at the end. Thus, the usual truncation applied in this context, demanding vertices staying below the running maximum at all times does not affect the extremal vertices at all in the case when $0 < \sigma_1 < 1 < \sigma_2$. Therefore, up to a linear drift, what we see is akin to a *time-inhomogeneous random walk bridge* whose probability in this case is of order $1/(\log N)^{1/2}$. This exponent leads to the $1/2$ correction factor for the sub-leading order term of order $O(\log\log N)$.

At this stage, we are also able to understand what changes in the homogeneous case, when $\mathcal{I}_{\sigma^2}(x) = x$, for $x \in [0,1]$. Recall that we have the constraint that no particle should be larger than the running maximum at any time. A similar path analysis shows that for a vertex to become maximal at the end, its path has to stay $(\log N)^{1/2}$ below the running maximum for most of the time. This effect is known as *entropic repulsion* of the BRW. What we basically see is a *random walk bridge conditioned to stay below the straight line.* The probability of such an event is of order $1/(\log N)^{3/2}$, which gives the $3/2$ factor in front of the sub-leading order correction. We now move to the case of finitely many effective scales.

As in GREM or in variable-speed BBM, this leads us to distinguish three cases:

$$a)\, \mathcal{I}_{\sigma^2}(x) < x, \text{ for } x \in (0,1). \tag{1.49}$$

$$b)\, \mathcal{I}_{\sigma^2}(x) = x, \text{ for } x \in [0,1]. \tag{1.50}$$

$$c)\, \mathcal{I}_{\sigma^2}(x) > x, \text{ for some } x \in (0,1). \tag{1.51}$$

The case in (1.49) is called *weak correlation regime*, in which correlations are such that the correct centring of the maximum value is as in the case of independent identically distributed random variables, see Theorem 1.3.4. The case in (1.50) is usually called *critical case*, which is the usual 2d DGFF. It is *critical* in the sense that correlations are such that they affect the centering on the level of the sub-leading order correction. As we have already seen in (1.48), once the concave hull of the variance function, $A$, crosses the straight line as in (1.51), even the first order of the maximum is affected by variance profile. This is usually referred to as *supercritical case*.

One of the main consequences of what we prove is: Up to a uniformly bounded constant, the maximum is a concatenation of the maxima over the effective scales in their corresponding regimes. Moreover, the centred maximum is a tight sequence of random variables. We summarize the above observations in the following, more formal statement.

**Theorem 1.6.1.** *Let $\{\psi_v^N\}_{v\in V_N}$ be a 2d scale-inhomogeneous DGFF on $V_N$ with finitely many effective scales.*
*i) In the case when $\mathcal{I}_{\sigma^2}(s) < s$, for $s \in (0,1)$,*

$$\mathbb{E}\left[\max_{v\in V_N} \psi_v^N\right] = 2\log N - \frac{1}{4}\log\log N + O(1), \tag{1.52}$$

*where the term $O(1)$ is bounded by a constant, uniformly in $N$.*

*ii) In case when $\mathcal{I}_{\sigma^2}(s) = s$, for $s \in [0,1]$, we have*

$$\mathbb{E}\left[\max_{v\in V_N} \psi_v^N\right] = \mathbb{E}\left[\max_{v\in V_N} \phi_v^N\right] = 2\log N - \frac{3}{4}\log\log N + O(1), \tag{1.53}$$

*where the term $O(1)$ is bounded by a constant, uniformly in $N$.*
*iii) Finally, in the case if, for some $s \in (0,1)$, $\mathcal{I}_{\sigma^2}(s) > s$ and if there are $m$ effective parameters, $\{\bar{\sigma}_i\}_{1\leq i\leq m}$ and $0 = \lambda^0 < \cdots < \lambda^m = 1$, it holds that*

$$\mathbb{E}\left[\max_{v\in V_N} \psi_v^N\right] = \sum_{i=1}^m 2\bar{\sigma}_i(\lambda^i - \lambda^{i-1})\log N - \frac{1+2\delta_i}{4}\bar{\sigma}_i \log\log N + O(1), \tag{1.54}$$

*where the $\delta_i$ equals 1 if, for any $s \in [\lambda^{i-1}, \lambda^i]$, $\hat{\mathcal{I}}_{\sigma^2}(s) = \mathcal{I}_{\sigma^2}(s)$, and where the term $O(1)$ is bounded by a constant, uniformly in $N$.*
*In all three cases, the centred maximum, $\max_{v\in V_N} \psi_v^N - \mathbb{E}\left[\max_{v\in V_N}\psi_v^N\right]$, is tight as a sequence of real random variables.*

This is a consequence of our main result in Chapter 2, in which we directly control the tails of the properly centred maximum, $\max_{v\in V_N}\psi_v^N - m_N$, with centring

$$m_N := \sum_{i=1}^m 2\bar{\sigma}_i(\lambda^i - \lambda^{i-1})\log N - \frac{1+2\delta_i}{4}\bar{\sigma}_i \log\log N. \tag{1.55}$$

Here, we assume that there are $M \in \mathbb{N}$ scales with $m \leq M$ effective scales and set $\delta_i$, for $i = 1, \ldots, m$, as in (1.54). The main result in Chapter 2 deals with the tails of the centred maximum. In fact, the truncated second moment computation for the MIBRW that we depicted in combination with Gaussian comparison allows to prove that the right tail, i.e. the probability to exceed $m_N$ by a positive value $x$, has exponential tails. For the left tail, i.e. the probability of the maximum to be smaller than $m_N - x$, we obtain an upper bound of exponential decay. The idea we use to prove this is to bootstrap the estimate for the right-tail and which is what we outline in the following. One first decomposes the entire box into $\exp(O(x))$ many identical sub-boxes and then rewrites the MIBRW on the entire box, called $S^N$, as a sum of iid MIBRWs, $\left\{Y^{(i)}\right\}_i$, one for each sub-box, and an independent centred Gaussian field, $X$, that encodes their correlations. A possible strategy to have a small maximum value is to require either all MIBRWs on the sub-boxes being sufficiently small or to demand that the field, $X$, which encodes their common increment, has to be small. Using independence, one deduces that

$$\mathbb{P}\left(\max_{v\in V_N} S_v^N \leq m_N - x\right) \leq \prod_i \mathbb{P}\left(\max_v Y_v^{(i)} \leq m_N - x\right) + \mathbb{P}\left(X \leq -x\right). \tag{1.56}$$

The latter probability can be bounded from above using a Gaussian tail bound. To bound the first, we write $\mathbb{P}\left(\max_v Y_v^i < m_N - x\right) = 1 - \mathbb{P}\left(\max_v Y_v^{(i)} \geq m_N - x\right)$, and use the lower bound on the right-tail of the maximum to bound this quantity from below by a constant, $\delta > 0$. This together with the fact that there are $\exp(O(x))$ many independent of such factors, implies that the probability of the event that each maximum of the MIBRWs on the sub-boxes stays below $m_N - x$ decays exponentially fast in $-x$. More formally, we prove the following.

**Theorem 1.6.2.** *Let $\{\psi_v^N\}_{v\in V_N}$ be a 2d scale-inhomogeneous DGFF on $V_N$. Assume that on each interval $[\lambda^{i-1}, \lambda^i]$ and $i = 1, \ldots, m$, we have either $\mathcal{I}_{\sigma^2} \equiv \mathcal{I}_{\bar{\sigma}^2}$ or $\mathcal{I}_{\sigma^2} < \mathcal{I}_{\bar{\sigma}^2}$. There exist constants $C, c > 0$ such that, for any $x \in [0, \sqrt{\log N}]$,*

$$C^{-1}\left(1 + x\mathbb{1}_{\sigma_1=\bar{\sigma}_1}\right)e^{-x\frac{2}{\bar{\sigma}_1}} \leq \mathbb{P}\left(\max_{v\in V_N}\psi_v^N \geq m_N + x\right) \leq C(1 + x\mathbb{1}_{\sigma_1=\bar{\sigma}_1})e^{-x\frac{2}{\bar{\sigma}_1}}. \tag{1.57}$$

*Moreover, for any $0 \leq \lambda \leq (\log\log N)^{2/3}$,*

$$\mathbb{P}\left(\max_{v\in V_N}\psi_v^N \leq m_N - \lambda\right) \leq Ce^{-c\lambda}. \tag{1.58}$$

Note that the result for the right-tail in (1.57) is precise up to a multiplicative constant. Moreover, it differs in a multiplicative factor, $x$, depending on the parameters up to the first effective scale. To control the tails for larger deviations, e.g. for $x > \sqrt{\log N}$, one can use Borell's concentration inequality, which implies that there is a constant $c_\sigma \in (0, \infty)$, depending only on the variance parameter $\sigma$, such that

$$\mathbb{P}\left(|\psi_N^* - m_N| \geq x\right) \leq 2e^{-c_\sigma x^2/\log(N)} \quad \forall x \geq 0. \tag{1.59}$$

### 1.6.2 The case of weak correlations.

In Chapter 3 and Chapter 4, we consider the scale-inhomogeneous discrete Gaussian free field in the case of *weak correlations*. More precisely, we make the following assumptions:

$$\boxed{\sigma'(0) \text{ and } \sigma'(1) \text{ exist, } \mathcal{I}'_{\sigma^2}(0) < 1,\ \mathcal{I}'_{\sigma^2}(1) > 1,\ \mathcal{I}_{\sigma^2}(x) < x \text{ for } x \in (0,1) \text{ and } \mathcal{I}_{\sigma^2}(1) = 1.} \tag{1.60}$$

In words, we want the variance function to stay beneath the straight line and require some additional regularity at the very beginning and the very end. In this setting, by the first statement of Theorem 1.6.1 the order of the maximum is as if the random variables were independent and moreover, it implies tightness of the centred maximum which implies the existence of a convergent sub-sequence. Thus, the first question one should answer concerns the convergence of the properly centred maximum and secondly, the convergence of the extremal process.

#### Convergence of the maximum and genealogy of extremal vertices.

In Chapter 3, we show that the centred maximum value converges in law to a randomly shifted Gumbel random variable and obtain information on the genealogy of extremal particles. In order to prove convergence of the maximum value a simple refinement of our previous strategy, which principally consisted in comparing the maximum value of the field to the maximum of a suitably constructed MIBRW and analysing the latter, seems unfeasible. Indeed, with regards to Gaussian comparison and in order to obtain convergence of the centred maximum by comparison to a MIBRW, one would have to be able to approximate the covariance structure asymptotically correct which simply is beyond the scope of this method. Given the answer, an instructive question one should ask instead is:

*Why should we expect a randomly shifted Gumbel as limit distribution?*

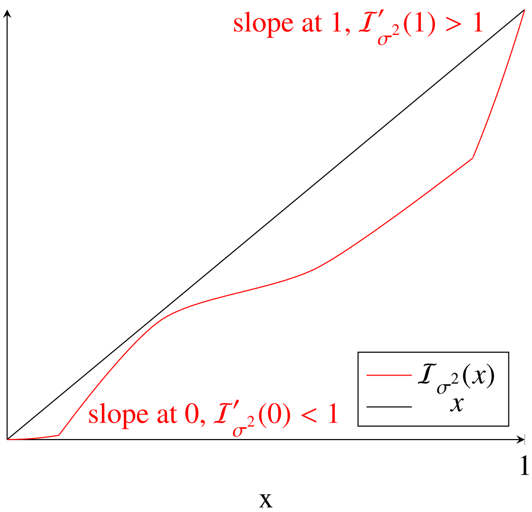



Figure 1.10: An example of $x \mapsto \mathcal{I}_{\sigma^2}(x)$ in the weak correlation regime.

This raises the question of what properties lead to this kind of limit shape? Based on the extreme value theory for independent random variables, see e.g. Theorem 1.3.2, the emergence of the Gumbel distribution should be due to taking the maximum of a growing number of independent identically distributed random variables with exponential right tails. The random shift accounts for the random number of such iid random variables with exponential tails. The additional randomness in the number of particles is usually due to a restriction that has to be verified for the random variables to be considered. One should ask:

*How to obtain suitable approximating fields that capture these two effects?*

The key observation one makes is that extreme local maxima are at mutual distance of at least $N/O(1)$, i.e. more precisely

$$\lim_{r\to\infty}\lim_{N\to\infty} \mathbb{P}\Big(\exists v, w \in V_N : \|v-w\|_2 \in (r, N/r) \wedge \psi_v^N, \psi_w^N \geq m_N - c\log\log r\Big) = 0. \tag{1.61}$$

This motivates to decompose the box $V_N$ into $K^2$ equal sized sub-boxes, $\{V_{N/K,i}\}_{1\leq i\leq K^2}$, each of side length $N/K$. We choose $K \ll N$ and take limits in the order $N \to \infty$ and then $K \to \infty$. As "fine fields" we choose the scale-inhomogeneous DGFF restricted to the interior of the boxes of side length $N/K$ minus the scale-inhomogeneous DGFF conditioned on the boundary of these boxes of side length $N/K$. By the Gibbs-Markov property (1.42), on the sub-boxes of side length $N/K$, these are $K^2$ independent copies of each other. In particular, they are multivariate Gaussian as conditioned Gaussians. Regarding the discussion incidental to the previous question, taking the maximum of the maxima of these $K^2$ independent fine fields accounts for the Gumbel limit shape, provided we can prove asymptotically exponential right-tails for their maxima. In light of Theorem 1.6.2 this seems within reach. The "global field" is then simply the harmonic extension of the values of the scale-inhomogeneous DGFF on the boundary of the $K^2$ boxes of side length $N/K$ into the entire box $V_N$. Without getting precise here, the random shift is due to a localization of the global field, i.e. only those fine fields for which the associated global field has a height within a certain interval will be counted.

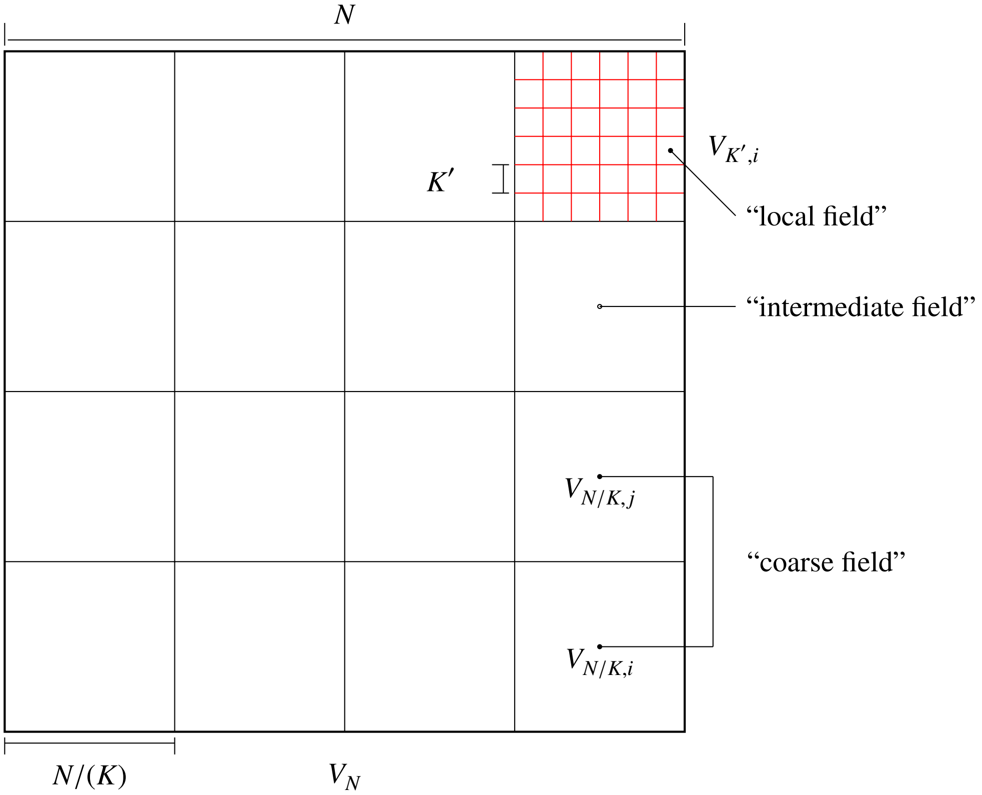



Figure 1.11: 3-field decomposition.

However, to work directly with these fields is delicate, mainly because of two issues: First, neither the global nor the fine field as defined above is constant in the $V_{N/K,i}$ boxes, and so the correct extreme local maxima depend in their positions and heights on both the global and the fine field. The second issue is that the variance parameter, $\sigma$, causes both global and fine field to be inhomogeneous, which technically complicates their analysis further.

The idea to circumvent both problems simultaneously is to use Gaussian comparison to show that one can approximate the fine and global field by auxiliary Gaussian fields that are structurally simpler and hence, easier to analyse, while both having identical limiting laws for their maximum values. Regarding (1.61) the approximating fields must have asymptotically identical correlations at both macroscopic and microscopic scale. We achieve this by approximating the global field by a scaled instance of the usual DGFF, $\{\sigma(0)\phi_v^K\}_{v\in V_K}$, which we refer to as “coarse field” and approximate the fine field further by independent copies of “local fields”, which are realized as scaled DGFFs, $\{\sigma(1)\phi_v^{K'}\}_{v\in V_{K'}}$, and a collection of modified inhomogeneous branching random walks (MIBRW) $\{S_v^{N,K,K',i}\}_{v\in V_{N/K,i}}$, capturing intermediate scales. Figure 1.11 shows a corresponding decomposition of the box $V_N$. Here we make use of the additional regularity assumptions in (1.60). An apparent advantage in this construction is that it addresses the first issue in the sense that the coarse field is constant in each sub-box $V_{N/K,i}$ and the MIBRW is constant in each small sub-box $V_{K',i}$. Moreover, it addresses the second issue in the sense that coarse and local fields are homogeneous. At this point we remark that a similar decomposition was previously employed in [46] in the context of log-correlated

Gaussian fields. As a first step in the proof, we use Gaussian comparison to show that the maximum of the auxiliary process has the same limit as the maximum of the scale-inhomogeneous DGFF, provided the limit exists.

Assuming that this is a valid approximation, we provide the heuristic picture behind the details of the limit shape of the maximum value. Having in mind the universality in the weak correlation regime in variable-speed BBM, see Theorem 1.4.4, one should should ask:

*How does the limit distribution depend on the parameters?*

Under the assumptions of weak correlations in (1.60) we show that the limiting law is *universal* in the sense that it only depends on the parameter $\sigma(0)$ through a random variable $Y$ and on a constant $C$, which solely depends on $\sigma(1)$. In the following, we explain why this is reasonable and along the way, we see more explicitely how the randomly shifted Gumbel distribution emerges from the approximation. The first key ingredient is the genealogical structure of the extremes in (1.61), which implies that if we pick two vertices whose height is extremely large then they have to be at distance of order $N/O(1)$ or $O(1)$. This implies that extreme local maxima are correlated only on scales of order $N/O(1)$ and thus, these correlations asymptotically depend solely on the coarse field with initial variance parameter $\sigma(0)$.

As a second key ingredient, we prove that the right-tail of the maximum satisfies asymptotics which depend only on the last variance parameter, $\sigma(1)$, through a constant $C_K = C_K(\sigma(1))$. In particular, we show that

$$\lim_{x\to\infty}\lim_{K',N\to\infty}\left|\mathbb{P}\left(\max_{v\in V_{N/K}} S_v^{N,K,K'}+\sigma(1)\phi_v^{K'}\geq m_N-2\sigma^2(0)\log K+x\right)-C_K(\sigma(1))e^{-2x}\right|=0. \quad (1.62)$$

The proof of this is based on a modified second moment computation in which one uses a localization of the local field at extremal vertices. This localization is the reason why $C_K$ depends only on $\sigma(1)$. To explain how the parameters enter into the limit shape we depict a heuristic computation, which is inspired by the simple calculation in the case of independent random variables, see (1.16). Mimicking (1.16), we condition on the large scales, i.e. on $\phi^K$, and obtain

$$\begin{aligned}&\mathbb{P}\left(\max_{v\in V_N}\sigma(0)\phi_v^K+S_v^{N,K,K'}+\sigma(1)\phi_v^{K'}\leq m_N+x\right)\\&=\mathbb{E}\left[\prod_{i=1}^{K^2}\left(1-\mathbb{P}\left(\max_{v\in V_{N/K}} S_v^{N,K,K'}+\sigma(1)\phi_v^{K'}\geq m_N+x-\sigma(0)\phi_v^K\middle|\phi^K\right)\right)\right].\end{aligned} \quad (1.63)$$

The third key ingredient is the simple but crucial observation that, for extremal vertices, the coarse field, $\sigma(0)\phi^K$, localizes in a window of size $O(\sqrt{\log K})$ around $2\sigma^2(0)\log K$. We collect the indices, for which this localization is satisfied, in the set $A$ and note that there are exponentially many such indices. Inserting this localization into (1.63) allows to drop the conditioning on $\phi^K$, which adds a multiplicative error of size $1+o(1)$. Furthermore, we observe that the field $S^{N,K,K'}+\sigma(1)\phi^{K'}$ is independent of the conditioning by construction. Using these observations, (1.63) can be rewritten, up to a multiplicative error of size $(1+o(1))$, as

$$\mathbb{E}\left[\prod_{i\in A}\left(1-C_K(\sigma(1))e^{-2x+2\sigma(0)\left(\phi_i^K-2\log K\right)}\right)\right]=(1+o(1))\mathbb{E}\left[e^{-C_K(\sigma(1))Y_K(\sigma(0))e^{-2x}}\right]. \quad (1.64)$$

What remains to show is that the expression and the quantities in (1.64) converge as $K \to \infty$. We summarize with a precise statement on the convergence in law of the centred maximum in the weakly correlated regime.

**Theorem 1.6.3.** *Under the assumptions of weak correlations, i.e.* (1.60)*, there exists a constant* $C = C(\sigma(1)) > 0$*, depending on the parameters only through* $\sigma(1)$ *and a random variable* $Y = Y(\sigma(0))$*, which is almost surely positive, finite and depends on the parameters only through* $\sigma(0)$*, such that*

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{v\in V_N} \psi_v^N - m_N \leq x\right) = \mathbb{E}\left[\exp\left(-C(\sigma(1))Y(\sigma(0))e^{-2x}\right)\right]. \tag{1.65}$$

**Convergence of the extremal process.**

Having established convergence in law of the maximum value, it is a natural next step to study the joint law of height and spatial distribution of all vertices that come close to the global maximum. This is the contents of Chapter 4, in which we prove convergence in law of the *full or structured extremal process*,

$$\eta_{N,r} = \sum_{v\in V_N} \mathbb{1}_{v \text{ r-loc max}} \delta_{v/N} \otimes \delta_{\psi_v^N - m_N} \otimes \delta_{\{\psi_v^N - \psi_w^N :\, w\in\mathbb{Z}^2\}}, \tag{1.66}$$

where $0 < r < N$. Let us first take a closer look at the point process defined in (1.66). It captures the following different aspects: the distribution of the location of $r-$local maxima normalized to $[0, 1]^2$ in its first coordinate, the relative height of local maxima in the second and in the last, the field centred at the corresponding extreme local maximum, i.e. the cluster around the chosen local maximum. The first question to answer here is:

*Why is* $\eta_{N,r}$ *the correct process to consider?*

Our goal is to describe the limiting joint law of all points that are in a sense close to the maximum value. With regards to the separation of extreme local maxima as in (1.61), it makes sense to rescale the box $V_N$ onto the unit square $[0, 1]^2$. In fact, as extreme local maxima are at distance $N/O(1)$, rescaling their positions to the unit square their points get mapped to distinct points in $[0, 1]^2$, which persits when taking the limit $N \to \infty$. As by (1.61) all other points whose height is in a sense close to the global maximum are spatially within distance $O(1)$ of an extreme local maximum. By rescaling all their spatial positions onto the unit square $[0, 1]^2$, their spatial positions get mapped onto the location of their closest extreme local maximum. By substracting $m_N$ from each local maximum, vertices that are not *extreme* local maxima, have relative heights tending to $-\infty$, and thus we retain only extreme local maxima. A visualization of such a process is given in Figure 1.12. As already mentioned, clusters points are spatially within distance $O(1)$ of an extreme local maximum and thus, in order to capture both their spatial distribution and relative height with respect to their corresponding extreme local maximum, one should look at the scale-inhomogeneous DGFF at its original spatial scaling and centred at an extreme local maximum, as done in the third coordinate in (1.66). Thus, we see that it is natural to consider the point process, $\{\eta_{N,r}\}_{N\geq r\geq 1}$, as it captures both spatial distribution and height of all extremal values.

The second question concerns its limit shape, as $N \to \infty$ followed by $r \to \infty$. We identify the limit process as a *cluster Cox process*, with a random intensity measure on $[0, 1]^2$, denoted by $Y$,

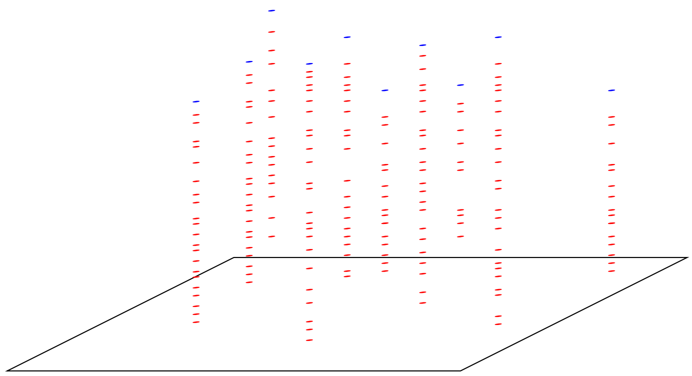



Figure 1.12: Visualization of the structured extremal process when collapsing the clusters to one spatial location: Poisson points of extreme local maxima, with corresponding cluster points beneath.

that depends on the variance parameters only through the initial value, $\sigma(0)$, and with cluster law depending on the variance parameters only through its final value, $\sigma(1)$.

In the following, we explain the heuristic picture behind this. We first ignore the last coordinate in (1.66) and explain why we obtain a Cox process as limit of the point process of extreme local maxima,

$$\tilde{\eta}_{N,r} = \sum_{v \in V_N} \mathbb{1}_{v \text{ r-loc max}} \delta_{v/N} \otimes \delta_{\psi_v^N - m_N}. \tag{1.67}$$

In a first step, we argue, based on a heuristic computation, that extreme local maxima satisfy a superposition principle. In the second step, this allows to deduce the correct limit shape, using a general result which is due to Ligget [69]. The principle argument is due to Biskup and Louidor [17] who used it in the case of the 2d DGFF.

Take $\psi, \psi'$ two independent copies of $\psi^N$ and let $t \in \mathbb{R}_+$. By Gaussian interpolation and in law,

$$\psi^N = \sqrt{1 - \frac{t}{2 \log N}} \psi + \sqrt{\frac{t}{2 \log N}} \psi'. \tag{1.68}$$

By Taylor expansion, of the first root and using that $\max \psi = O(\log N)$ with high probability,

$$\psi^N = \psi - \frac{1}{2} \frac{t}{2 \log N} \psi + \sqrt{\frac{t}{2 \log N}} \psi' + o(1). \tag{1.69}$$

Pick $v \in V_N$ such that $\psi_v^N \geq m_N - \lambda$ or $\psi_v \geq m_N - \lambda$. We consider the $r$−neighbourhood of $v$, which we denote by $\Lambda_r(v)$, and note that, for $w \in \Lambda_r(v)$, we have $\psi'_w - \psi'_v = O(1)$. Thus, with high probability,

$$\psi_w^N = \psi_w - \frac{1}{2} \frac{t}{2 \log N} \psi_w + \sqrt{\frac{t}{2 \log N}} \psi'_v + o(1), \quad \forall w \in \Lambda_r(v). \tag{1.70}$$

Similarly, and using once again $\max \psi = O(\log N)$, we have with high probability both $\psi_w^N - m_N = O(1)$ and $\psi_w - m_N = O(1)$, for $w \in \Lambda_r(v)$. Replacing the second occurrence of $\psi$ in (1.70) by $m_N + O(1)$, we deduce that in law,

$$\psi_w^N = \psi_w - \frac{1}{2}t + \sqrt{\frac{t}{2\log N}}\psi_v' + o(1), \quad \forall w \in \Lambda_r(v). \tag{1.71}$$

Next, we note that the term, $\sqrt{\frac{t}{2\log N}}\psi_w'$ is asymptotically distributed as a centred Gaussian random variable with variance $t/2$. Using this in (1.71) and the fact that local maxima are achieved at unique points, the maxima of $\psi^N$ and $\psi$ are attained at the same point with high probability. Further, as extreme local maxima are at distance of order $N/O(1)$, for two such extreme local maxima $v$ and $w$, $\mathbb{E}\left[\psi_v'\psi_w'\right] = O(1)$. Considering their normalization by $\sqrt{\frac{t}{2\log N}}$ in (1.71), we deduce that the extreme local maxima of $\psi^N$ and $\psi$ are related by mutually independent random shifts of the form $B_{t/2} - t/2$, with $B_t$ being a centred Gaussian random variable with variance $t$. In particular,

$$\sum_{v\in V_N} \mathbb{1}_{v \text{ r-loc max}}\delta_{v/N} \otimes \delta_{\psi_v^N - m_N} = \sum_{v\in V_N} \mathbb{1}_{v \text{ r-loc max}}\delta_{v/N} \otimes \delta_{\psi_v^N - t/2 + B_{t/2}^{(v)} - m_N}, \tag{1.72}$$

with $\left(B_{t/2}^{(v)}\right)_v$ being iid centred Gaussians with variance $t/2$ and equality being in law. Having established a superposition principle for extreme local maxima, we may use a general result by Ligget [69] that characterizes any possible limit of the point process $\tilde{\eta}_{N,r}$, when $N \to \infty$ followed by $r \to \infty$, as a Cox process. Note that this heuristic argument only uses that we are dealing with a Gaussian process whose extreme local maxima are well separated and that correlations decay sufficiently fast. Uniqueness of the law of the Cox process follows from uniqueness of the random intensity measure. The latter follows from proving joint convergence of extreme local maxima on a generating class of the Borel $\sigma-$algebra of $[0, 1]^2$. In the proof of the latter, we generalize arguments from the proof of convergence of the global maximum. In summary, we obtain in law,

$$\lim_{r\to\infty}\lim_{N\to\infty} \tilde{\eta}_{N,r} = PPP(Y(\sigma(0)) \otimes C(\sigma(1))e^{-2x}\mathrm{d}x), \tag{1.73}$$

with $Y$ being a random Borel measure on $[0, 1]^2$, depending only on the initial variance, $\sigma(0)$, and whose total mass, $Y([0, 1]^2)$, coincides in law with the random variable $Y$ from Theorem 1.6.3. The constant, $C(\sigma(1))$, coincides with the one in Theorem 1.6.3.

Knowing that the point process of extreme local maxima converges in law to a Cox process puts us into the position to include the law of the *cluster points*. Here again, the separation of extreme local maxima comes into play, i.e. (1.61). The idea is that due to correlations, around each extreme local maximum there are many points in a $O(1)$ neighbourhood that reach heights which are within distance $O(1)$ below the local maximum. Conditioning on the extreme local maxima and outside these $O(1)$ neighbourhoods, the fields in these $O(1)$ neighbourhoods around each local maximum are mutually independent. In particular, it turns out that they asymptotically share the same law. As we centre the field around an extreme local maximum, all points in its closest vicinity have to be smaller in height. Using the assumption that $\sigma$ is differentiable at 1 (see (1.60)), we are able to approximate the scale-inhomogeneous Gaussian free field in those $O(1)$ neighbourhoods by Gaussian fields that only depend on $\sigma(1)$. In fact, we show that the *cluster law* is given by the weak limit

$$\nu(\cdot) = \lim_{r\to\infty} \mathbb{P}\left(\phi^{\mathbb{Z}^2\setminus\{0\}} + 2\sigma(1)\mathfrak{a} \in \cdot \middle| \phi_v^{\mathbb{Z}^2\setminus\{0\}} + 2\sigma(1)\mathfrak{a}(v) \geq 0 : \|v\|_1 \leq r\right), \tag{1.74}$$

where $\phi^{\mathbb{Z}^2\setminus\{0\}}$ is the discrete Gaussian free field on $\mathbb{Z}^2 \setminus \{0\}$, which is equal to the 2d DGFF conditioned to be 0 at the origin, and where $\mathfrak{a}$ is the potential kernel of a simple random walk on $\mathbb{Z}^2$. One should compare it to the cluster law in case of the usual 2d DGFF in (1.41). In particular, the assumption, $\sigma(1) > 1$, ensures that the conditioning in (1.74) is not singular as it is in the case of the 2d DGFF, see (1.41). We summarize with our main result of Chapter 4:

**Theorem 1.6.4.** *There exists a random Borel measure, $Y$, on $[0,1]^2$, that depends only on $\sigma(0)$ and satisfies almost surely $Y([0,1]^2) < \infty$, as well as $Y(A) > 0$, for any open and non-empty $A \subset [0,1]^2$. Moreover, the weak limit in (1.74) exists and for each sequence $r_N$ with $r_N \to \infty$ and $r_N/N \to 0$, as $N \to \infty$,*

$$\eta_{N,r_N} \to PPP\left(Y(\mathrm{d}x) \otimes C(\sigma(1))e^{-2h}\mathrm{d}h \otimes \nu(\mathrm{d}\theta)\right), \tag{1.75}$$

*where the constant $C(\sigma(1)) > 0$ is the one from Theorem 1.6.3. The convergence is in law with respect to the vague convergence of Radon measures on $[0,1]^2 \times \mathbb{R} \times \bar{\mathbb{R}}^{\mathbb{Z}^2}$.*

As a simple consequence, we obtain a description of what is usually called *extremal process*, i.e. we can drop the indicator of being a local maximum in (1.67) and describe the limit law by means of a *cluster process*. Let $\{(x_i, h_i : i \in \mathbb{N})\}$ enumerate the points of a sample of $PPP(Y(\mathrm{d}x) \otimes C(\sigma(1))e^{-2h}\mathrm{d}h)$. Let $\{\Theta_w^{(i)} : w \in \mathbb{Z}^2\}$, for $i \in \mathbb{N}$, be independent samples of the measure $\nu$. Then, as $N \to \infty$,

$$\sum_{v\in V_N} \delta_{v/N} \otimes \delta_{\psi_v^N - m_N} \to \sum_{i\in\mathbb{N}} \sum_{w\in\mathbb{Z}^2} \delta_{(x_i, h_i - \Theta_w^{(i)})}. \tag{1.76}$$

## 1.7 Open problems

In this subsection, we shortly discuss possible further research directions in the context of (variable-speed) BBM and the (scale-inhomogeneous) DGFF that directly connects to the work previously presented in the introduction. The study of the extreme values in the case of a *strictly concave speed function*, $A$, in variable-speed BBM as well as in the scale-inhomogeneous DGFF is still an open problem. In the case of variable-speed BBM, it is known that the second order correction to the maximum value is no longer logarithmic but a power of $1/3$ [49, 71]. Furthermore, it is proved in [71] that the properly centred maximum converges to a solution of a time-inhomogeneous F-KPP equation. However, the centring in the statement is implicit. Understanding the maximum value with an more explicit centring up to $o(1)$ precision and moreover, the extremal process in this case is of major interest, in particular, since already the genealogical structure of extremal particles is more complicated. In the cases we have discussed so far, extremal particles are allowed to split only at the very beginning, the very end or when the concave hull changes its slope. In the case when $A$ is strictly concave, its concave hull is changing its slope at all times, which suggests that extremal particles can basically split at any time. It should be possible to approach this problem by a precise study of trajectories of particles killed at certain space-time curves, similar to works on BBM with absorption [12].

In case of the 2d DGFF the random measure, $Z$, which governs the extremal process, was characterized in multiple ways in [17, 18, 19]. In particular, it is known to coincide with the *critical Liouville Quantum Gravity measure.* In this respect, it would be of interest to further study the random measure, $Y$, that appears in the scale-inhomogeneous DGFF. We believe that it can be related to the *sub-critical Liouville Quantum Gravity measure.* Moreover, a thorough study of *extreme level sets* of the (scale-inhomogeneous) 2d DGFF, i.e. all points above a finite level $\lambda$ below the global maximum, as in the case of BBM [36] appears to be desirable. A typical question in this regard is, whether there exist certain clusters that contribute a substantial amount to the extreme level set, and if so, is this quantifiable in parameters of the model?

## 1.8 Beyond the 2d DGFF or other log-correlated (Gaussian) fields

At the end of the introduction, we want to hint at related models, for which one expects a similar behaviour concerning their extreme values. On the one hand, there is the class of *logarithmic correlated Gaussian fields*, which includes BBM and the 2d DGFF. In this case, under fairly general regularity assumptions on their correlation structure, convergence of their maximum value to a randomly shifted Gumbel was proved by Ding, Roy and Zeitouni [46]. They were also able to show that the genealogical structure of extremes in these models are all of the type we have seen in the cases of BBM and the 2d DGFF. Having these two key ingredients at hand and with regards to the heuristic computation we provided in Section 1.6.2, it seems very plausible that the extremal process for each model in this general class of models should converge to a cluster Cox process. However, this remains an open problem. The main reason for this is that there are non-trivial technical difficulties to overcome, as a simple adaptation of the fairly general proofs in case of the 2d DGFF [17, 18] is impossible since the models in this general class lack a Gibbs-Markov property which cannot be easily replaced. One should instead try to use a certain self-similarity present in these models. However, for certain important models that belong to this class and that possess a Gibbs-Markov property, e.g. the $4d-$membrane model [81], it should be feasible to adopt the proofs from [17, 18] and obtain convergence of their full extremal processes to cluster Cox processes.

On the other hand, there are log-correlated models such as the field of hitting times of Brownian motion on the torus [39, 44, 11] whose maximum is related to the cover time of Brownian motion on the torus, the randomized Riemann zeta function on the critical line [3, 76, 73, 64, 8, 4] or characteristic polynomials of random unitary matrices [2, 35]. In the last decade, the study of the extremes of these models has attracted a lot of attention. One of the major reasons for this is that their behaviour is conjectured to strongly resemble the one observed in the Gaussian case [58, 59]. Important technical tools that play major roles in the analysis of Gaussian log-correlated fields, such as *Gaussian interpolation* or *Gaussian integration by parts*, are not available in these models. Much of the analysis in the above models is based on a refined second moment method suggested by Kistler [60], for which one needs to establish a hidden branching structure, and which is usually combined with an analysis of extremal particles' trajectories. In general, a detailed understanding of the paths of extremal particles is a major key for a precise understanding of the individual models [25, 36, 4].

# Bibliography

[1] E. Aïdékon, J. Berestycki, E. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Related Fields*, 157(1-2):405–451, 2013.

[2] L.-P. Arguin, D. Belius, and P. Bourgade. Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.*, 349(2):703–751, 2017.

[3] L.-P. Arguin, D. Belius, and A. J. Harper. Maxima of a randomized Riemann zeta function, and branching random walks. *Ann. Appl. Probab.*, 27(1):178–215, 2017.

[4] L.-P. Arguin, P. Bourgade, and M. Radziwiłł. The Fyodorov-Hiary-Keating Conjecture. I. *arXiv E-print:2007.00988*, 2020.

[5] L.-P. Arguin, A. Bovier, and N. Kistler. Genealogy of extremal particles of branching Brownian motion. *Comm. Pure Appl. Math.*, 64(12):1647–1676, 2011.

[6] L.-P. Arguin, A. Bovier, and N. Kistler. Poissonian statistics in the extremal process of branching Brownian motion. *Ann. Appl. Probab.*, 22(4):1693–1711, 2012.

[7] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Related Fields*, 157(3-4):535–574, 2013.

[8] L.-P. Arguin, L. Hartung, and N. Kistler. High points of a random model of the Riemann-zeta function and Gaussian multiplicative chaos. *arXiv E-print:1906.08573*, 2019.

[9] L.-P. Arguin and F. Ouimet. Extremes of the two-dimensional Gaussian free field with scale-dependent variance. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13(2):779–808, 2016.

[10] L.-P. Arguin and O. Zindy. Poisson-Dirichlet statistics for the extremes of the two-dimensional discrete Gaussian free field. *Electron. J. Probab.*, 20:no. 59, 19, 2015.

[11] D. Belius and N. Kistler. The subleading order of two dimensional cover times. *Probab. Theory Related Fields*, 167(1-2):461–552, 2017.

[12] J. Berestycki, N. Berestycki, and J. Schweinsberg. The genealogy of branching Brownian motion with absorption. *Ann. Probab.*, 41(2):527–618, 2013.

[13] N. Berestycki. Introduction to the Gaussian Free Field and Liouville Quantum Gravity, 2016. Available at: `http://www.statslab.cam.ac.uk/~beresty/Articles/oxford4.pdf`.

[14] N. Berestycki and J. Norris. Lectures on Schramm–Loewner Evolution, 2014. Available at: `http://www.statslab.cam.ac.uk/~beresty/Articles/sle.pdf`.

[15] N. Bernoulli. *Dissertatio inauguralis mathematico-juridica: De usu artis conjectandi in jure*. a Mechel, Basel, 1709.

[16] M. Biskup. Extrema of the two-dimensional discrete Gaussian free field. In *Random graphs, phase transitions, and the Gaussian free field*, volume 304 of *Springer Proc. Math. Stat.*, pages 163–407. Springer, Cham, 2020.

[17] M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *Comm. Math. Phys.*, 345(1):271–304, 2016.

[18] M. Biskup and O. Louidor. Full extremal process, cluster law and freezing for the two-dimensional discrete Gaussian free field. *Adv. Math.*, 330:589–687, 2018.

[19] M. Biskup and O. Louidor. Conformal symmetries in the extremal process of two-dimensional discrete Gaussian free field. *Comm. Math. Phys.*, 375(1):175–235, 2020.

[20] E. Bolthausen, J.-D. Deuschel, and G. Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.*, 29(4):1670–1692, 2001.

[21] E. Bolthausen, J. D. Deuschel, and O. Zeitouni. Recursions and tightness for the maximum of the discrete, two dimensional Gaussian free field. *Electron. Commun. Probab.*, 16:no. 11, 15, 2011.

[22] A. Bovier. *Gaussian processes on trees*, volume 163 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. From spin glasses to branching Brownian motion.

[23] A. Bovier and L. Hartung. The extremal process of two-speed branching Brownian motion. *Electron. J. Probab.*, 19:no. 18, 28, 2014.

[24] A. Bovier and L. Hartung. Variable speed branching Brownian motion 1. Extremal processes in the weak correlation regime. *ALEA Lat. Am. J. Probab. Math. Stat.*, 12(1):261–291, 2015.

[25] A. Bovier and L. Hartung. From 1 to 6: A finer analysis of perturbed branching Brownian motion. *Comm. Pure Appl. Math.*, 73(7):1490–1525, 2020.

[26] A. Bovier and I. Kurkova. Derrida's generalised random energy models. I. Models with finitely many hierarchies. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(4):439–480, 2004.

[27] A. Bovier and I. Kurkova. Derrida's generalized random energy models. II. Models with continuous hierarchies. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(4):481–495, 2004.

[28] M. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.

[29] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 69(1):62–123, 2016.

[30] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 65(1):1–20, 2012.

[31] J. Bricmont and F. Debacker-Mathot. The Wegner approximation of the plane rotator model as a massless, free, lattice, Euclidean field. *J. Math. Phys.*, 18(1):37–40, 1977.

[32] D. Carpentier and P. Doussal. Glass transition of a particle in a random potential, front selection in nonlinear renormalization group, and entropic phenomena in Liouville and sinh-Gordon models. *Phys. Rev. E*, 63(026110):1–33, 2000.

[33] B. Chauvin and A. Rouault. KPP equation and supercritical branching Brownian motion in the subcritical speed area. Application to spatial trees. *Probab. Theory Related Fields*, 80(2):299–314, 1988.

[34] B. Chauvin and A. Rouault. Supercritical branching Brownian motion and K-P-P equation in the critical speed-area. *Math. Nachr.*, 149:41–59, 1990.

[35] R. Chhaibi, T. Madaule, and J. Najnudel. On the maximum of the $C\beta E$ field. *Duke Math. J.*, 167(12):2243–2345, 2018.

[36] A. Cortines, L. Hartung, and O. Louidor. The structure of extreme level sets in branching Brownian motion. *Ann. Probab.*, 47(4):2257–2302, 2019.

[37] O. Daviaud. Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.*, 34(3):962–986, 2006.

[38] F. M. Dekking and B. Host. Limit distributions for minimal displacement of branching random walks. *Probab. Theory Related Fields*, 90(3):403–426, 1991.

[39] A. Dembo, Y. Peres, J. Rosen, and O. Zeitouni. Cover times for Brownian motion and random walks in two dimensions. *Ann. of Math. (2)*, 160(2):433–464, 2004.

[40] B. Derrida. Random-energy model: Limit of a family of disordered models. *Phys. Rev. Lett.*, 45(2):79, 1980.

[41] B. Derrida. Random-energy model: An exactly solvable model of disordered systems. *Phys. Rev. B*, 24(5):2613, 1981.

[42] B. Derrida. A generalization of the random energy model which includes correlations between energies. *J. Phy. Lett.*, 46(9):401–407, 1985.

[43] B. Derrida and H. Spohn. Polymers on disordered trees, spin glasses, and traveling waves. *J. Statist. Phys.*, 51:817–840, 1988. New directions in statistical mechanics (Santa Barbara, CA, 1987).

[44] J. Ding. On cover times for 2D lattices. *Electron. J. Probab.*, 17:no. 45, 18, 2012.

[45] J. Ding. Exponential and double exponential tails for maximum of two-dimensional discrete Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):285–299, 2013.

[46] J. Ding, R. Roy, and O. Zeitouni. Convergence of the centered maximum of log-correlated Gaussian fields. *Ann. Probab.*, 45(6A):3886–3928, 2017.

[47] J. Ding and O. Zeitouni. Extreme values for two-dimensional discrete Gaussian free field. *Ann. Probab.*, 42(4):1480–1515, 2014.

[48] E. L. Dodd. The greatest and the least variate under general laws of error. *Trans. Amer. Math. Soc.*, 25(4):525–539, 1923.

[49] M. Fang and O. Zeitouni. Slowdown for time inhomogeneous branching Brownian motion. *J. Stat. Phys.*, 149(1):1–9, 2012.

[50] M. Fels. Extremes of the 2d scale-inhomogeneous discrete Gaussian free field: Sub-leading order and tightness. *arXiv E-print:1910.09915*, 2019.

[51] M. Fels and L. Hartung. Extremes of the 2d scale-inhomogeneous discrete Gaussian free field: Convergence of the maximum in the regime of weak correlations. *arXiv E-print:1912.13184*, 2019.

[52] M. Fels and L. Hartung. Extremes of the 2d scale-inhomogeneous discrete Gaussian free field: Extremal process in the weakly correlated regime. *arXiv E-print:2002.00925*, 2020.

[53] R. Fisher. The wave of advance of advantageous genes. *Ann. Eugen.*, 7:355 – 369, 1937.

[54] R. A. Fisher and L. H. C. Tippett. Limiting forms of the frequency distribution of the largest or smallest member of a sample. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 24, pages 180–190. Cambridge University Press, 1928.

[55] M. Fréchet. Sur la loi de probabilité de l'écart maximum. *Ann. Soc. Polon. Math.*, 6:93–116, 1927.

[56] Y. Fyodorov, G. Hiary, and J. Keating. Freezing transition, characteristic polynomials of random matrices, and the riemann zeta function. *Phys. Rev. Lett.*, 108:no. 170601, 4, 2012.

[57] Y. V. Fyodorov and J.-P. Bouchaud. Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential. *J. Phys. A*, 41(37):no. 372001, 12, 2008.

[58] Y. V. Fyodorov and J. P. Keating. Freezing transitions and extreme values: random matrix theory, and disordered landscapes. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 372(20120503):1–32, 2014.

[59] Y. V. Fyodorov and N. J. Simm. On the distribution of the maximum value of the characteristic polynomial of GUE random matrices. *Nonlinearity*, 29(9):2837–2855, 2016.

[60] V. Gayrard and N. Kistler, editors. *Correlated random systems: five different methods*, volume 2143 of *Lecture Notes in Mathematics*. Springer, Cham; Société Mathématique de France, Paris, 2015. Lecture notes from the 1st CIRM Jean-Morlet Chair held in Marseille, Spring 2013, CIRM Jean-Morlet Series.

[61] B. Gnedenko. Sur la distribution limite du terme maximum d'une série aléatoire. *Ann. of Math. (2)*, 44:423–453, 1943.

[62] R. D. Gordon. Values of mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. *Ann. Math. Statist.*, 12(3):364–366, 1941.

[63] E. J. Gumbel. *Statistics of extremes*. Columbia University Press, New York, 1958.

[64] A. J. Harper. On the partition function of the Riemann zeta function, and the Fyodorov–Hiary–Keating conjecture. *arXiv E-print:1906.05783*, 2019.

[65] L. B. Hartung. *Extremal processes in branching Brownian motion and friends*. Hochschulschrift, Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, 2016.

[66] A. Kolmogorov, I. Petrovskii, and N. Piscounov. Étude de l'équation de la diffusion avec croissance de la quantité de matiére et son application a un probléme biologique. *Moscou Univ. Math. Bull.*, 1, 1937.

[67] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15(3):1052–1061, 1987.

[68] M. R. Leadbetter, G. Lindgren, and H. Rootzén. *Extremes and related properties of random sequences and processes*. Springer Series in Statistics. Springer-Verlag, New York-Berlin, 1983.

[69] T. M. Liggett. Random invariant measures for Markov chains, and independent particle systems. *Z. Wahrsch. Verw. Gebiete*, 45(4):297–313, 1978.

[70] O. Louidor. Lecture Notes on Large and Extreme Values of the Discrete Gaussian Free Field KAIST Summer School in Probability (KSSP) 2018, 2018. Available at: `https://ie.technion.ac.il/~olouidor/KAIST/KAIST_Notes.pdf`.

[71] P. Maillard and O. Zeitouni. Slowdown in branching Brownian motion with inhomogeneous variance. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(3):1144–1160, 2016.

[72] H. P. McKean. A correction to: "Application of Brownian motion to the equation of Kolmogorov-Petrovskiĭ-Piskonov" (Comm. Pure Appl. Math. **28** (1975), no. 3, 323–331). *Comm. Pure Appl. Math.*, 29(5):553–554, 1976.

[73] J. Najnudel. On the extreme values of the Riemann zeta function on random intervals of the critical line. *Probab. Theory Related Fields*, 172(1-2):387–452, 2018.

[74] E. Nelson. The free Markoff field. *J. Funct. Anal.*, 12:211–227, 1973.

[75] F. Ouimet. Geometry of the Gibbs measure for the discrete 2D Gaussian free field with scale-dependent variance. *ALEA Lat. Am. J. Probab. Math. Stat.*, 14(2):851–902, 2017.

[76] F. Ouimet. Poisson-Dirichlet statistics for the extremes of a randomized Riemann zeta function. *Electron. Commun. Probab.*, 23:no. 46, 15, 2018.

[77] M. Reed and L. Rosen. Support properties of the free measure for Boson fields. *Comm. Math. Phys.*, 36:123–132, 1974.

[78] S. I. Resnick. *Extreme values, regular variation and point processes*. Springer Series in Operations Research and Financial Engineering. Springer, New York, 2008. Reprint of the 1987 original.

[79] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and Liouville quantum gravity. In *Stochastic processes and random matrices*, pages 548–577. Oxford Univ. Press, Oxford, 2017.

[80] D. Ruelle. A mathematical reformulation of Derrida's REM and GREM. *Comm. Math. Phys.*, 108(2):225–239, 1987.

[81] F. Schweiger. The maximum of the four-dimensional membrane model. *Ann. Probab.*, 48(2):714–741, 2020.

[82] S. Sheffield. Gaussian free fields for mathematicians. *arXiv E-print:0312099*, 2003.

[83] L. Tippett. On the extreme individuals and the range of samples taken from a normal population. *Biometrika*, pages 364–387, 1925.

[84] L. von Bortkiewicz. *Das Gesetz der kleinen Zahlen*. B.G. Teubner, Leipzig, 1898.

[85] L. von Bortkiewicz. *Variationsbreite und mittlerer Fehler*. Berliner Mathematische Gesellschaft, 1921.

[86] L. Von Bortkiewicz. Die Variationsbreite beim Gauss' schen Fehlergesetz. *Nordisk Statistisk Tidskrift*, 1(1138):193–220, 1922.

[87] R. Von Mises. Über die Variationsbreite einer Beobachtungsreihe. *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, 22:3–8, 1923.

[88] R. Von Mises. La distribution de la plus grande de n valuers. *Rev. math. Union interbalcanique*, 1:141–160, 1936.

[89] W. Werner. Some recent aspects of random conformally invariant systems. In *Mathematical statistical physics*, pages 57–99. Elsevier B. V., Amsterdam, 2006.

[90] O. Zeitouni. Branching random walks and Gaussian fields. In *Probability and statistical physics in St. Petersburg*, volume 91 of *Proc. Sympos. Pure Math.*, pages 437–471. Amer. Math. Soc., Providence, RI, 2016.

CHAPTER 2

# Subleading-order and tightness of the maximum of the scale-inhomogeneous two-dimensional discrete Gaussian Free Field

# EXTREMES OF THE 2D SCALE-INHOMOGENEOUS DISCRETE GAUSSIAN FREE FIELD: SUB-LEADING ORDER AND EXPONENTIAL TAILS

MAXIMILIAN FELS

Abstract. This is the first of a three paper series in which we present a comprehensive study of the extreme value theory of the scale-inhomogeneous discrete Gaussian free field. This model was introduced by Arguin and Ouimet in [7] in which they computed the first order of the maximum. In this first paper we establish tail estimates for the maximum value, which allow to deduce the log-correction to the order of the maximum and tightness of the centred maximum. Our proofs are based on the second moment method and Gaussian comparison techniques.

## 1. Introduction

In recent years, so-called log-correlated (Gaussian) processes have received considerable attention, see e.g. [4, 5, 10, 15, 25, 34, 47]. One of the reasons for this is that their correlation structure becomes relevant for the properties of the extremes of the processes. Some prominent examples that fall into this class are branching Brownian motion (BBM), the two-dimensional discrete Gaussian free field ( 2d DGFF), local maxima of the randomised Riemann zeta function on the critical line and cover times of Brownian motion on the torus. The 2d DGFF is one of the well understood non-hierarchical log-correlated fields (see [9, 10, 11, 19]). For simplicity, consider the 2d DGFF on a square lattice box of side length $N$. It turns out that the maximum can be written as a first order term which is proportional to the logarithm of the volume of the box, a second order correction which is proportional to the logarithm of the first order and stochastically bounded fluctuations. If one considers an uncorrelated Gaussian field on the same box with identical variances, a simple computation shows that the first order of the maximum coincides with the one of the DGFF, whereas the constant in front of the second order correction differs. In [7], Arguin and Ouimet introduced the scale-inhomogeneous 2d DGFF, the analogue model of variable speed BBM [47], which allows to consider different variance profiles. They determined the first order of the maximum. In this paper we continue the study of the maximum, find tail estimates on the maximum value which allow us to deduce the second order correction and tightness of the centred maximum. In the other two papers in this series, we prove, in the regime of weak correlations, convergence of the centred maximum [29] and convergence of the extremal process [30]. Both are joint work with Hartung.

1.1. **The 2d discrete Gaussian free field.** Let $V_N := ([0, N) \cap \mathbb{Z})^2$. The interior of $V_N$ is defined as $V_N^o := ([1, N-1] \cap \mathbb{Z})^2$ and the boundary of $V_N$ is denoted by $\partial V_N := V_N \setminus V_N^o$. Moreover, for points $u, v \in V_N$ we write $u \sim v$, if and only if $\|u - v\|_2 = 1$, where $\|.\|_2$ is the Euclidean norm. Let $\mathbb{P}_u$ be the law of a SRW $\{W_k\}_{k\in\mathbb{N}}$ starting at $u \in \mathbb{Z}^2$. The normalised Green kernel is given by

$$G_{V_N}(u, v) := \frac{\pi}{2}\mathbb{E}_u\left[\sum_{i=0}^{\tau_{\partial V_N}-1} \mathbb{1}_{\{W_i=v\}}\right], \text{ for } u, v \in V_N. \tag{1.1}$$

M.F. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - project-id 211504053 - SFB 1060 and Germany's Excellence Strategy – GZ 2047/1, project-id 390685813 – "Hausdorff Center for Mathematics" at Bonn University.
Keywords: extreme value theory, Gaussian free field, inhomogeneous environment, branching Brownian motion, branching random walk.

Here, $\tau_{\partial V_N}$ is the first hitting time of the boundary $\partial V_N$ by $\{W_k\}_{k\in\mathbb{N}}$. For $\delta > 0$, we set $V_N^\delta := (\delta N, (1-\delta)N)^2 \cap \mathbb{Z}^2$. By [22, Lemma 2.1], we have for $\delta \in (0,1)$ and $u, v \in V_N^\delta$,

$$G_{V_N}(u,v) = \log N - \log(\|u-v\|_2 \vee 1) + O(1). \tag{1.2}$$

**Definition 1.1.** The 2d discrete Gaussian free field (DGFF) on $V_N$, $\phi^N := \{\phi_v^N\}_{v\in V_N}$, is a centred Gaussian field with covariance matrix $G_{V_N}$ and entries $G_{V_N}(x,y) = \mathbb{E}[\phi_x^N \phi_y^N]$, for $x, y \in V_N$.

From Definition 1.1 it follows that $\phi_v^N = 0$ for $v \in \partial V_N$, i.e. we have Dirichlet boundary conditions.

1.2. **The 2d scale-inhomogeneous discrete Gaussian free field.**

**Definition 1.2.** (2d scale-inhomogeneous discrete Gaussian free field).
Let $\phi^N = \{\phi_v^N\}_{v\in V_N}$ be a 2d DGFF on $V_N$. For $v = (v_1, v_2) \in V_N$, let $[v]_\lambda^N$ be the box of side length $N^{1-\lambda}$ centred at $v$, namely

$$[v]_\lambda \equiv [v]_\lambda^N := \left(\left[v_1 - \frac{1}{2}N^{1-\lambda}, v_1 + \frac{1}{2}N^{1-\lambda}\right] \times \left[v_2 - \frac{1}{2}N^{1-\lambda}, v_2 + \frac{1}{2}N^{1-\lambda}\right]\right) \cap V_N \tag{1.3}$$

and set $[v]_0^N := V_N$ and $[v]_1^N := \{v\}$. We denote by $[v]_\lambda^o$ the interior of $[v]_\lambda$. Let $\mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c} := \sigma\left(\{\phi_v^N, v \notin [v]_\lambda^o\}\right)$ be the $\sigma-$algebra generated by the random variables outside $[v]_\lambda^o$. We define $\phi_v^N(\lambda)$ by conditioning on the DGFF outside the box $[v]_\lambda^N$, i.e.

$$\phi_v^N(\lambda) = \mathbb{E}\left[\phi_v^N | \mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c}\right], \quad \lambda \in [0,1]. \tag{1.4}$$

We denote by $\nabla\phi_v^N(\lambda)$ the derivative $\partial_\lambda \phi_v^N(\lambda)$ of the DGFF at vertex $v$ and scale $\lambda$. Further, let $s \mapsto \sigma(s)$ be a non-negative function such that $\mathcal{I}_{\sigma^2}(\lambda) := \int_0^\lambda \sigma^2(x)\mathrm{d}x$ is a non-decreasing function on $[0,1]$ with $\mathcal{I}_{\sigma^2}(0) = 1$ and $\mathcal{I}_{\sigma^2}(1) = 1$. Then the 2d scale-inhomogeneous DGFF on $V_N$ is a centred Gaussian field $\psi^N := \{\psi_v^N\}_{v\in V_N}$ defined as

$$\psi_v^N := \int_0^1 \sigma(s)\nabla\phi_v^N(s)\mathrm{d}s. \tag{1.5}$$

In this paper, we consider the case when $\sigma$ is a right-continuous step function taking $M \in \mathbb{N}$ values. Thus, there are variance parameters $(\sigma_1, \ldots, \sigma_M) \in [0,\infty)^M$ and scale parameters $(\lambda_1, \ldots, \lambda_M) \in (0,1]^M$ with $0 =: \lambda_0 < \lambda_1 \ldots < \lambda_M := 1$, such that

$$\sigma(s) = \sum_{i=1}^M \sigma_i \mathbb{1}_{[\lambda_{i-1}, \lambda_i)}(s), \ s \in [0,1]. \tag{1.6}$$

In this case, the scale-inhomogeneous 2d DGFF or 2d $(\sigma,\lambda)-$DGFF in (1.5) takes the form

$$\psi_v^N = \sum_{i=1}^M \sigma_i(\phi_v^N(\lambda_i) - \phi_v^N(\lambda_{i-1})). \tag{1.7}$$

Similarly to (1.4), we set for $v \in V_N$ and $\lambda \in [0,1]$,

$$\psi_v^N(\lambda) := \mathbb{E}\left[\psi_v^N \middle| \mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c}\right]. \tag{1.8}$$

Next, we compute the covariances of $\{\psi_v^N\}_{v\in V_N}$. We fix $\delta \in (0, 1/2)$ and $\lambda \in (4\delta/\log N, 1/\sqrt{\log N})$. For $N \in \mathbb{N}$ and $v, w \in V_N$, set $q_N(v,w) := \frac{\log N - \log\|v-w\|_2}{\log N}$. For $v, w \in V_N^\delta$, we write $\mathbb{E}\left[\psi_v^N \psi_w^N\right] = \mathbb{E}\left[\left(\psi_v^N - \psi_v^N(\lambda)\right)\psi_w^N + \psi_v^N(\lambda)\psi_w^N\right]$. By choice of $\delta$ and $\lambda$, it holds that $[v]_\lambda^N \cap \partial V_N = \emptyset$ and $[w]_\lambda^N \cap \partial V_N = \emptyset$. Therefore, we may deduce as in [50, (A.41), (A.42)],

$$\mathbb{E}\left[\left(\psi_v^N - \psi_v^N(\lambda)\right)\psi_w^N\right] = \left[\mathcal{I}_{\sigma^2}(q_N(v,w)) - \mathcal{I}_{\sigma^2}(\min\{\lambda, q_N(v,w)\})\right]\log N + O(\sqrt{\log N}), \tag{1.9}$$

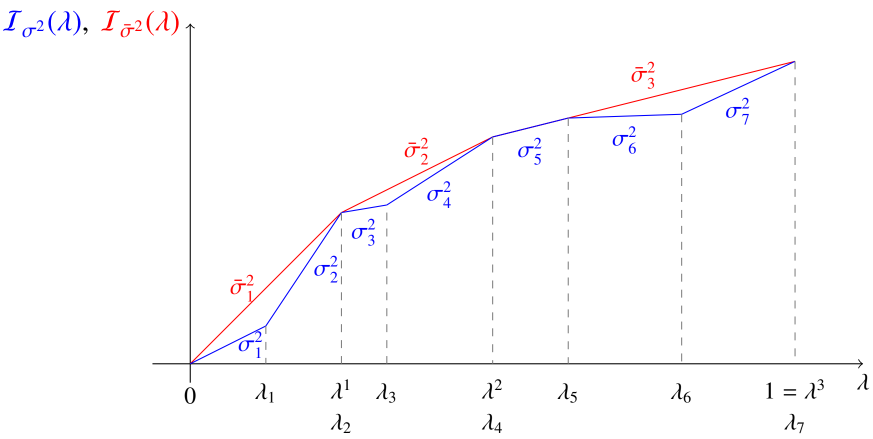



FIGURE 1. An example of variance and effective variance.

and

$$\left|\mathbb{E}\left[\psi_v^N(\lambda)\psi_w^N\right]\right| \le O(\sqrt{\log N}). \tag{1.10}$$

Using (1.9) and (1.10), we obtain for $v, w \in V_N^\delta$,

$$\mathbb{E}\left[\psi_v^N\psi_w^N\right] = \log N \mathcal{I}_{\sigma^2}\left(\frac{\log N - \log(\|v-w\|_2 \vee 1)}{\log N}\right) + O(\sqrt{\log(N)}). \tag{1.11}$$

## 2. Main result

The main result of this paper are tail estimates for the maximum of the scale-inhomogeneous 2d DGFF when there are finitely many scales. As simple consequences, we deduce the correct second order correction and tightness of the centred maximum. We start with some notation. Let $\hat{\mathcal{I}}_{\sigma^2}(s)$ be the concave hull of $\mathcal{I}_{\sigma^2}(s)$. There exists a unique non-increasing, right-continuous step function $s \to \bar\sigma(s)$, which we call 'effective variance', such that

$$\hat{\mathcal{I}}_{\sigma^2}(s) = \int_0^s \bar\sigma^2(r)\mathrm{d}r =: \mathcal{I}_{\bar\sigma^2}(s) \quad \text{for all } s \in [0,1]. \tag{2.1}$$

The points where $\bar\sigma$ jumps on $[0,1]$ we call

$$0 =: \lambda^0 < \lambda^1 < \ldots < \lambda^m := 1, \tag{2.2}$$

where $m \le M$. To be consistent with previous notation (cf.(1.6)), we write $\bar\sigma_l := \bar\sigma(\lambda^{l-1})$. We denote the maximum by $\psi_N^* := \max_{v\in V_N}\psi_v^N$. For any, possibly finite, sequence $\{x_i\}_{i\ge0}$ of real numbers we denote by $\Delta x_i = x_i - x_{i-1}$ the discrete increment. It turns out that the concave hull of $\mathcal{I}_{\sigma^2}$, denoted $\hat{\mathcal{I}}_{\sigma^2}$, gives the desired control for the first order of the maximum. Arguin and Ouimet [7, Theorem 1.2] determined the correct first order behaviour, i.e. they showed that in probability,

$$\lim_{N\to\infty}\frac{\psi_N^*}{2\log(N)} = \mathcal{I}_{\bar\sigma}(1) = \sum_{i=1}^m \bar\sigma_i\Delta\lambda^i. \tag{2.3}$$

In the following, the goal is to prove a second order correction and tightness of the maximum around its mean. Let $\pi_j$ be the unique index such that for $1 \le j \le m$ we have $\lambda^j = \lambda_{\pi_j}$. Moreover, we write $t^j = \lambda^j\frac{\log N}{\log 2}$ as well as $t_j = \lambda_j\frac{\log N}{\log 2}$. We set

$$m_N := \sum_{j=1}^m 2\log 2\bar\sigma_j\Delta t^j - \frac{(w_j\bar\sigma_j\log(\Delta t^j))}{4}, \tag{2.4}$$

where

$$w_j = \begin{cases} 3, & \mathcal{I}_{\bar{\sigma}^2}|_{(\lambda^{j-1},\lambda^j]} \equiv \mathcal{I}_{\sigma^2}|_{(\lambda^{j-1},\lambda^j]} \\ 1, & else \end{cases} \tag{2.5}$$

The following theorem establishes tail estimates of the maximum centred by $m_N$.

**Theorem 2.1.** *Let $N \in \mathbb{N}$ and $\{\psi_v^N\}_{v\in V_N}$ be a 2d $(\sigma, \lambda)$-DGFF on $V_N$ with $M \in \mathbb{N}$ scales. Assume that on each interval $[\lambda^{i-1}, \lambda^i]$ and $i = 1, \ldots, m$, we have either $\mathcal{I}_{\sigma^2} \equiv \mathcal{I}_{\bar{\sigma}^2}$ or $\mathcal{I}_{\sigma^2} < \mathcal{I}_{\bar{\sigma}^2}$ There exist constants $C, c > 0$ such that for any $x \in [0, \sqrt{\log N}]$,*

$$C^{-1}\left(1 + x\mathbb{1}_{\sigma_1=\bar{\sigma}_1}\right) e^{-x\frac{2}{\bar{\sigma}_1}} \leq \mathbb{P}\left(\max_{v\in V_N} \psi_v^N \geq m_N + x\right) \leq C(1 + x\mathbb{1}_{\sigma_1=\bar{\sigma}_1})e^{-x\frac{2}{\bar{\sigma}_1}}. \tag{2.6}$$

*and for any $0 \leq \lambda \leq (\log\log N)^{2/3}$,*

$$\mathbb{P}\left(\max_{v\in V_N} \psi_v^N \leq m_N - \lambda\right) \leq Ce^{-c\lambda}. \tag{2.7}$$

Note that the result for the right-tail in (2.6) is precise up to a multiplicative constant. For values $x > \sqrt{\log N}$, by Borell's inequality (see Theorem A.1) and [7, Lemma A.3], there is a constant $c_\sigma \in (0, \infty)$, depending only on the variance parameter $\sigma$, such that

$$\mathbb{P}\left(|\psi_N^* - m_N| \geq x\right) \leq 2e^{-c_\sigma x^2/\log(N)}. \tag{2.8}$$

As a simple consequence of Theorem 2.1, we obtain the following corollary.

**Corollary 2.2.** *Under the same assumptions of Theorem 2.1, the sequence of the centred maximum $\{\psi_N^* - m_N\}_{N\geq 0}$ is tight. In particular,*

$$\mathbb{E}\left[\psi_N^*\right] = m_N + O(1), \tag{2.9}$$

*where the term $O(1)$ is bounded by a constant which is uniform in $N$.*

An interesting fact is that the profile of the variance matters both for the leading term and the logarithmic correction. This phenomenon was first observed in the context of the GREM by Bovier and Kurkova [36, 17, 18], and in the context of the time-inhomogeneous branching Brownian motion/branching random walk by Bovier and Hartung [13, 14], Fang and Zeitouni [27], Maillard and Zeitouni [46] and Mallein [48].

*Remark* 2.3. Regarding the additional assumption on the variance profile in Theorem 2.1, we expect that in general there are essentially two properties which determine the logarithmic correction. For each interval $[\lambda^{j-1}, \lambda^j]$ one has to see whether the effective variance and the real variance coincide in a neighbourhood at the beginning or the end of the interval. If neither is the case we have the $1/2$ correction. If it coincides in a neighbourhood at exactly one end point, we expect the factor to be $2/2$ and if it coincides in neighbourhoods at the beginning and the end, the correction factor should be $3/2$. If one considers the case of strictly decreasing variance $\sigma$ in (1.5), we expect the second order correction to be proportional to $\log^{1/3}(N)$ as observed in the analogue setting for variable-speed BBM [27].

2.1. **Overview of related results.** In the case when $\sigma \equiv 1$, the 2d scale-inhomogeneous DGFF simply is the 2d DGFF. The maximum and more generally the extremal process of the DGFF has been the subject of intense investigations. Let $\phi_N^* := \max_{v\in V_N} \phi_v^N$ be the maximum of the DGFF. Through the works of Bolthausen, Deuschel and Giacomin [11] as well as Bramson and Zeitouni [20] one obtains,

$$\phi_N^* = 2\log N - \frac{3}{4}\log\log N + Y, \tag{2.10}$$

where $Y$ is random variable of order $o(\log\log N)$ in probability. Bramson and Zeitouni further deduced that the centred maximum $\phi_N^* - \mathbb{E}\left[\phi_N^*\right]$ is tight as a sequence of real random variables. Convergence

of the centred maximum was then shown by Bramson, Ding and Zeitouni in [19]. In [9, 10], Biskup and Louidor proved that the extremal process converges to a cluster Cox process.
Another closely related model is (variable-speed) branching Brownian motion (BBM). It can be considered as the analogue model to the scale-inhomogeneous DGFF in the context of BBM. It first appeared in a paper by Derrida and Spohn [23]. To define variable-speed BBM, fix a Galton Watson tree, a time horizon $t > 0$ and let $A : [0,1] \to [0,1]$, strictly increasing with $A(0) = 0$, $A(1) = 1$ and bounded second derivatives. The overlap $d(v,w)$ for leaves $v,w$ in the tree is the time of their most recent common ancestor. Variable-speed BBM in time $t$ and with time change $tA(\cdot/t)$ can then be defined as a centred Gaussian process $x$ indexed by the leaves of the tree and covariance $tA(d(v,w)/t)$, where $v$ and $w$ are leaves. BBM is the special case when $A(x) = x$ for $x \in [0,1]$, and coincides with the generalized random energy model (GREM) on the Galton-Watson tree. Compared to the 2d DGFF, its hierarchical structure makes it easier to analyse and the extremes of BBM are particularly well understood (see [3, 6, 15, 21]). The extreme values and more general the extremal process for variable-speed BBM were investigated in [13, 14, 27, 28, 46]. In particular, the first order and second order correction of the maximum in the regime of weak correlations, i.e. when $A(s) < s$ for $s \in (0,1)$, is identical to the uncorrelated regime. In this regime, convergence of the extremal process was proved by Bovier and Hartung in [13, 14]. In the case of decreasing speed with finitely many changes in speed, the global maximum is a simple concatenation of the maximum at speed change. When the speed is strictly decreasing, i.e. when $A'' < 0$, Bovier and Kurkova [17, 18] showed that the first order is as in all other cases determined by the concave hull of $A$. The second order correction is no longer logarithmic but proportional to $t^{1/3}$, which was shown by Maillard and Zeitouni in [46], building upon the work by Fang and Zeitouni in [28].
In the discrete analogue model of (variable-speed) BBM, the (time-inhomogeneous) branching random walk (BRW) on the Galton Watson tree, there are results on the first and second order correction by Fang and Zeitouni [27], Mallein [47] and Ouimet [51]. A notable difference in the context of (time-inhomogeneous) BRW is that one does not need to assume that increments are Gaussian (see [47]). For the usual BRW, Aïdékon proved convergence of the centred maximum [2] and Madaule of the extremal process [45].

2.2. **Idea of proof.** The main idea to prove Theorem 2.1 is to use Gaussian comparison to compare the distribution of the centred maximum of the scale-inhomogeneous DGFF with the distribution of two auxiliary Gaussian fields, a time-inhomogeneous BRW (IBRW) and an modified inhomogeneous branching random walk (MIBRW). The time-inhomogeneous BRW is constructed in such a way that it is slightly less correlated than the scale-inhomogeneous DGFF which allows to use an available upper bound on the right tail of the maximum of the time-inhomogeneous BRW. The MIBRW has correlations that differ from those of the scale-inhomogeneous DGFF inside the field only up to a uniformly bounded constant. This allows, in a first step, to use Gaussian comparison to reduce the remaining lower bound on the right tail of the maximum to a corresponding lower bound on the right tail of the maximum of the MIBRW. In a second step, we prove the lower bound on the right tail of the centred maximum of the MIBRW that, together with the so-called "sprinkling method", also allows to deduce the upper bound on the left tail. The remaining lower bound on the right tail is achieved by a modified second moment analysis.

*Outline of the paper:* In the next section we define two auxiliary Gaussian processes, the time-inhomogeneous branching random walk (IBRW) and the modified time-inhomogeneous branching random walk (MIBRW), and estimate their covariance structure. In Section 4 we provide the necessary tail estimates that allow us to deduce Theorem 2.1. We start with the upper bound on the right tail, then prove the lower bound on the right tail and finally, show the upper bound on the left tail. In Appendix A we provide the Gaussian comparison theorems we use in the proof and Borell's Gaussian concentration inequality. In Appendix B we prove the covariance estimates stated in Section 3.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Anton Bovier, Lisa Hartung and an anonymous referee for their very careful reading and for their valuable comments that led to improvements of this paper.

## 3. AUXILIARY PROCESSES AND COVARIANCE ESTIMATES

Consider $N = 2^n$ for some $n \in \mathbb{N}$. For $k = 0, 1, \ldots, n$ let $\mathcal{B}_k$ denote the collection of subsets of $\mathbb{Z}^2$ consisting of squares of side length $2^k - 1$ with corners in $\mathbb{Z}^2$ and let $\mathcal{BD}_k$ denote the subset of $\mathcal{B}_k$ consisting of squares of the form $([0, 2^k - 1] \cap \mathbb{Z})^2 + (i2^k, j2^k)$. We remark that the collection $\mathcal{BD}_k$ partitions $\mathbb{Z}^2$ into disjoint squares. For $v \in V_N$, let $\mathcal{B}_k(v)$ denote those elements $B \in \mathcal{B}_k(v)$ with $v \in B$. Likewise define $\mathcal{BD}_k(v)$, i.e. for $v \in V_N$, $B \in \mathcal{BD}_k(v)$ if and only if $v \in B$. One should note that $\mathcal{BD}_k(x)$ contains exactly one element, whereas $\mathcal{B}_k(x)$ contains $2^{2k}$ elements.

**Definition 3.1** (Time-inhomogeneous branching random walk (IBRW))**.** Let $\{a_{k,B}\}_{k\geq 0, B\in\mathcal{BD}_k}$ be an i.i.d. family of standard Gaussian random variables. We define the time-inhomogeneous branching random walk $\{R^N_z\}_{z\in V_N}$ by

$$R^N_z(t) := \sum_{k=n-t}^{n} \sum_{B\in\mathcal{BD}_k(z)} \sqrt{\log(2)}\tilde{\sigma}\left(\frac{n-k}{n}\right) a_{k,B}, \tag{3.1}$$

where $0 \leq t \leq n$, $t \in \mathbb{N}$ and $s \mapsto \tilde{\sigma}(s)$ is a non-negative function, for $s \in [0, 1]$. We specify the function $s \mapsto \tilde{\sigma}(s)$ later in the proof (see p. 8).

It turns out that, due to its hierarchical structure, the IBRW is less correlated than the scale-inhomogeneous DGFF, which is beneficial to obtain upper bounds using Gaussian comparison. But this also makes it unsuitable to obtain sufficient lower bounds on the maximum value. We therefore introduce another auxiliary process whose covariance structure is much closer to the scale-inhomogeneous DGFF, and is defined by taking uniform averages of IBRWs. For $v \in V_N$, let $\mathcal{B}^N_k(v)$ be the collection of subsets of $\mathbb{Z}^2$ consisting of squares of size $2^k$ with lower left corner in $V_N$. For two sets $B, B' \subset \mathbb{Z}^2$ we write $B \sim_N B'$, if there exist integers $i, j$ such that $B' = B + (iN, jN)$. Let $\{b_{k,B}\}_{k\geq 0, B\in\mathcal{B}^N_k}$ denote an i.i.d. family of centred Gaussian random variables with unit variance and set

$$b^N_{k,B} := \begin{cases} b_{k,B}, \ B \in \mathcal{B}^N_k, \\ b_{k,B'}, \ B \sim_N B' \in \mathcal{B}^N_k. \end{cases} \tag{3.2}$$

**Definition 3.2** (Modified inhomogeneous branching random walk (MIBRW))**.** The modified inhomogeneous branching random walk (MIBRW) $\{S^N_v\}_{v\in V_N}$ is defined by

$$S^N_z(t) := \sum_{k=n-t}^{n} \sum_{B\in\mathcal{B}^N_k(z)} 2^{-k}\sigma\left(\frac{n-k}{n}\right) b^N_{k,B}, \tag{3.3}$$

where $0 \leq t \leq n$, $t \in \mathbb{N}$ and $\sigma$ is defined as in (1.6).

### 3.1. Covariance estimates.

In order to be able to apply Gaussian comparison, we need to compare the correlations of the processes introduced previously. We write $\log_+(x) = \max(0, \log_2(x))$. Further, let $\|\cdot\|_2$ be the usual Euclidean distance and $\|\cdot\|_\infty$ the maximum distance. As we are working in two dimensions, they satisfy the relation $\|x - y\|_\infty \leq \|x - y\|_2 \leq \sqrt{2}\|x - y\|_\infty$. In addition, we introduce for $v, w \in V_N$ two distances on the torus induced by $V_N$,

$$d^N(v, w) := \min_{z:\, z-w\in(N\mathbb{Z})^2} \|v - z\|_2, \qquad d^N_\infty(v, w) := \min_{z:\, z-w\in(N\mathbb{Z})^2} \|v - z\|_\infty. \tag{3.4}$$

Note that the Euclidean distance on the torus is smaller than the standard Euclidean distance, i.e. for all $v, w \in V_N$, it holds $d^N(v, w) \leq \|v - w\|_2$. However, equality trivially holds if one restricts oneself on a smaller box, e.g. if $v, w \in (N/4, N/4) + V_{N/2} \subset V_N$. In the following we call $\{\tilde{S}^N_v\}_{v\in V_N}$ the homogeneous version of the process $\{S^N_v\}_{v\in V_N}$ which was introduced in [20], i.e. we assume that there is only one scale $\lambda_1 = 1$ with variance parameter $\sigma_1 = 1$.

**Lemma 3.3.** *There exists a constant $C$ independent of $N = 2^n$ such that for any $v, w \in V_N$,*

i. $\left|\mathbb{E}\left[\tilde{S}^N_v \tilde{S}^N_w\right] - (n - \log_+(d^N(x,y)))\right| \le C,$

ii. $\left|\mathbb{E}\left[S^N_v S^N_w\right] - n\mathcal{I}_{\sigma^2}\left(\frac{n-\log_+ d^N(v,w)}{n}\right)\right| \le C.$

*Further, for any $x, y \in V_N + (2N, 2N) \subset V_{4N}$,*

iii. $\left|\mathbb{E}\left[\phi^{4N}_v \phi^{4N}_w\right] - \log(2)(n - \log_+(\|v-w\|_2))\right| \le C,$

iv. $\left|\mathbb{E}\left[\psi^{4N}_v \psi^{4N}_w\right] - \log(2)\mathbb{E}\left[S^N_v S^N_w\right]\right| \le C.$

*Proof.* See Appendix B. □

*Remark* 3.4. The assumption $N = 2^n$ for $n \in \mathbb{N}$ mainly simplifies notation and also the proof, however without removing essential difficulties.

An important tool in the analysis of the scale-inhomogeneous DGFF is the Gibbs-Markov property of the DGFF. For two sets $U \subset V \subset \mathbb{Z}^2$ the DGFF on V can be decomposed into a sum of a DGFF on U and an independent Gaussian field, i.e.

$$\phi^V_u \overset{d}{=} \phi^U_u + \mathbb{E}\left[\phi^V_u | \sigma\left(\phi^V_v : v \in V \setminus U^o\right)\right], \quad u \in V. \tag{3.5}$$

Further, if $A, B \subset V$ such that $A^o \cap B^o = \emptyset$, then $\{\phi^V_u - \mathbb{E}[\phi^V_u|\mathcal{F}_{\partial A}]\}_{u\in A}$ is a DGFF on A, independent of the DGFF on B $\{\phi^V_u - \mathbb{E}[\phi^V_u|\mathcal{F}_{\partial B}]\}_{u\in B}$.

## 4. Tail estimates and tightness

The following analysis provides the necessary estimates to conclude Theorem 2.1.

**Lemma 4.1.** *There is a constant $\alpha_0 > 0$ such that for sufficiently large $N \in \mathbb{N}$ and any $v, w \in V_N$, we have*

$$\mathrm{Var}\left[\psi^N_v\right] \le \log N \mathcal{I}_{\sigma^2}(1) + \alpha_0 = \log N \sum_{i=1}^M \sigma_i^2 \Delta\lambda_i + \alpha_0, \tag{4.1}$$

*and*

$$\mathbb{E}\left[(\psi^N_v - \psi^N_w)^2\right] \le 2\log N\left[\mathcal{I}_{\sigma^2}(1) - \mathcal{I}_{\sigma^2}\left(\frac{n - \lceil \log_+ \|v-w\|_2\rceil}{n}\right)\right] - \left|\mathrm{Var}\left[\psi^N_v\right] - \mathrm{Var}\left[\psi^N_w\right]\right| + 4\alpha_0. \tag{4.2}$$

*Proof.* Recall Definition 1.2 and note that we have an underlying discrete Gaussian free field $\{\phi^N_v\}_{v\in V_N}$ such that $\psi^N_v = \sum_{i=1}^M \sigma_i\left(\phi^N_v(\lambda_i) - \phi^N_v(\lambda_{i-1})\right)$, where $\phi^N_v(\lambda_i) - \phi^N_v(\lambda_{i-1})$ for $i = 1, \dots, M$ are independent Gaussian free fields increments. A short computation shows that the variance of $\Delta\phi^N_v(\lambda_i)$ is up to constants given by the difference of Green kernels on the boxes, that is $G_{[v]_{\lambda_i}}(v,v) - G_{[v]_{\lambda_{i-1}}}(v,v)$, for which we have a sufficient bound (see [57, Lemma 3.10]), and (4.1) follows.

For (4.2), let $b_N(v,w) := \max(\lambda \in [0,1] : [v]_\lambda \cap [w]_\lambda \neq \emptyset)$ be the branching scale for particles $v, w \in V_N$. For scales $\mu_i > \mu'_i \ge b_N(v,w)$ and $i = 1, 2$, increments $\phi^N_v(\mu_1) - \phi^N_v(\mu'_1)$ are independent of $\phi^N_w(\mu_2) - \phi^N_w(\mu'_2)$. We define a set of representatives at scale $\lambda \in [0,1]$, denoted $R_\lambda$, such that it contains the centre of boxes that form a decomposition of $V_N$ into disjoint boxes with side length $N^{1-\lambda}$. Now, fix $v, w \in V_N$. There exists a set of representatives $R_\lambda$ at scale $\lambda = b_N(v,w) - \frac{4}{\log N}$, such that there is a common representative for $v$ and $w$, which we call $u_\lambda$. By [7, Lemma A.6], there is a universal constant $C > 0$ such that for $N$ large enough,

$$\max_{u\in\{v,w\}} \mathbb{E}\left[\left(\psi^N_u(\lambda) - \psi^N_{u_\lambda}(\lambda)\right)^2\right] \le C, \tag{4.3}$$

We further note that increments of $v$ and $w$ beyond $b_N(v,w)$ are independent and that, by Cauchy-Schwarz,

$$\mathbb{E}\left[\left(\psi_v^N(b_N(v,w)) - \psi_v^N(\lambda)\right)\left(\psi_v^N(b_N(v,w) - \psi_v^N(\lambda))\right)\right] \leq \tilde{C} \tag{4.4}$$

as well as

$$\max_{u\in\{v,w\}} \mathbb{E}\left[\left(\psi_u^N(b_N(v,w)) - \psi_u^N(\lambda)\right)^2\right] \leq \tilde{C}, \tag{4.5}$$

for some $\tilde{C} > 0$. Thus, writing

$$\begin{aligned}\psi_v^N - \psi_w^N =& \psi_v^N(\lambda) - \psi_{u_\lambda}^N(\lambda) + \psi_{u_\lambda}^N(\lambda) - \psi_w^N(\lambda) + \psi_v^N(b_N) - \psi_v^N(\lambda) + \psi_w^N(b_N) - \psi_w^N(\lambda) + \psi_v^N - \psi_v^N(b_N)\\ &+ \psi_w^N + \psi_w^N(b_N),\end{aligned} \tag{4.6}$$

we can bound $\mathbb{E}\left[\left(\psi_v^N - \psi_w^N\right)^2\right]$ from above using (4.3), (4.4), (4.5), Green kernel estimates as for the first statement (4.1), as well as independence of increments beyond $b_N(v,w)$, which then implies the upper bound in (4.2). □

We begin with an upper bound on the right tail.

**Proposition 4.2.** *There is a constant $C = C(\alpha_0)$, independent of $N$ such that for all $N \in \mathbb{N}$ and $x > 0$,*

$$\mathbb{P}\left(\max_{v\in V_N} \psi_v^N \geq m_N + x\right) \leq C(1 + x\mathbb{1}_{\sigma_1=\bar{\sigma}_1})e^{-x\frac{2}{\bar{\sigma}_1}}. \tag{4.7}$$

The principal idea to prove Proposition 4.2 is to use Gaussian comparison and compare the maximum of the scale-inhomogeneous DGFF to the maximum of suitable inhomogeneous branchig random walk. To obtain the correct upper bound we need to choose the variance of the IBRW appropriately. Here, we need to distinguish two cases. If there exists exactly one effective variance parameter, then we choose $s \mapsto \tilde{\sigma}(s)$, such that $s \mapsto \mathcal{I}_{\tilde{\sigma}^2}(s)$ is the lower convex envelope of the function $s \mapsto \mathcal{I}_{\sigma^2}(s)$. Else, if there are at least two effective scale parameter,s we introduce a parameter $0 < \kappa \ll n$. We set $\sigma_{min} = \min_{1\leq i\leq M} \sigma_i$ and $\sigma_{max} = \max_{1\leq i\leq M} \sigma_i$. We pick $\tilde{\lambda}^1 \equiv \tilde{\lambda}^1(\kappa) = \lambda^1 \frac{n}{n+k}$ as first effective scale and as first effective variance, $\bar{\sigma}_1$. Next, we set $\tilde{\lambda}_1 = \tilde{\lambda}_1(\kappa) = \tilde{\lambda}^1 \frac{\sigma_{max}^2 - \bar{\sigma}^1}{\sigma_{max}^2 - \sigma_{min}^2}$, $\tilde{\lambda}_2 = \tilde{\lambda}^1$ and $\tilde{\lambda}_3 = \frac{\tilde{\lambda}^1(\bar{\sigma}_1 - \sigma_{min}^2) + (\sigma_{max}^2 - 1)}{\sigma_{max}^2 - \sigma_{min}^2}$. For $s \in [0,1]$, we define the variance function as follows:

$$\tilde{\sigma}(s) = \left(\sigma_{min}\mathbb{1}_{s\in[0,\tilde{\lambda}_1)} + \sigma_{max}\mathbb{1}_{s\in[\tilde{\lambda}_1,\tilde{\lambda}_2)}\right)\mathbb{1}_{\sigma_1\neq\bar{\sigma}_1} + \bar{\sigma}_1\mathbb{1}_{\sigma_1=\bar{\sigma}_1} + \sigma_{min}\mathbb{1}_{s\in[\tilde{\lambda}_2,\tilde{\lambda}_3)} + \sigma_{max}^2\mathbb{1}_{s\in[\tilde{\lambda}_3,1]}. \tag{4.8}$$

In both cases our choice ensures that the first effective variances coincide, that $(n+\kappa)\mathcal{I}_{\tilde{\sigma}^2}\left(\frac{n-x}{n+\kappa}\right) \leq n\mathcal{I}_{\sigma^2}\left(\frac{n-x}{n}\right)$, for $x \in [0,n]$ and such that $\mathcal{I}_{\tilde{\sigma}^2}(1) = 1$. Before proving Proposition 4.2, we need one more lemma.

**Lemma 4.3.** *There is an integer $\kappa = \kappa(\alpha_0) > 0$ such that for all $N \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $A \subset V_N$,*

$$\mathbb{P}\left(\max_{v\in A} \psi_v^N \geq \lambda\right) \leq 2\mathbb{P}\left(\max_{v\in 2^\kappa A} R_v^{2^\kappa N} \geq \lambda\right). \tag{4.9}$$

*Proof.* By Lemma 4.1, we can choose a sufficiently large constant $\kappa$ that depends only on $\alpha_0$, such that $\mathrm{Var}\left[\psi_v^N\right] \leq \log(2)\mathrm{Var}\left[R_{2^\kappa v}^{2^\kappa N}\right]$ for all $v \in V_N$. Thus,

$$a_v^2 := \log(2)\mathrm{Var}\left[R_{2^\kappa v}^{2^\kappa N}\right] - \mathrm{Var}\left[\psi_v^N\right] \tag{4.10}$$

are non-negative. Let $X$ be a standard Gaussian. Since $\mathrm{Var}\left[R_v^N\right] = \mathrm{Var}\left[R_w^N\right]$, for all $v, w \in V_N$, we get

$$\begin{aligned}\mathbb{E}\left[(\psi_v^N + a_v X - \psi_w^N - a_w X)^2\right] &= \mathbb{E}\left[(\psi_v^N - \psi_w^N)^2\right] + (a_v - a_w)^2 \\ &= \mathbb{E}\left[(\psi_v^N - \psi_w^N)^2\right] + \left|\mathrm{Var}\left[\psi_v^N\right] - \mathrm{Var}\left[\psi_w^N\right]\right| \\ &\le 2\log(N)\left[1 - \mathcal{I}_{\sigma^2}\left(\frac{n - \lceil \log_+ \|v - w\|_2 \rceil}{n}\right)\right] + 4\alpha_0, \end{aligned} \tag{4.11}$$

by Lemma 3.3. On the other hand by our choice of $\tilde{\sigma}$ in (4.8), $\mathrm{Var}\left[R_{2^\kappa v}^{2^\kappa N}\right] = \log(N) + \log(2)\kappa$ grows linearly in $\kappa$, whereas $\mathbb{E}[R_{2^\kappa u}^{2^\kappa N} R_{2^\kappa v}^{2^\kappa N}] = (\log(N) + \log(2)\kappa)\, \mathcal{I}_{\tilde{\sigma}^2}\left(\frac{n - \log_+ d^N(u,v)}{n+\kappa}\right)$. By our choice in (4.8) and taking into account that for two vertices $u$ and $v$, $\log_+ d^N(u, v) \ge \log_+ \|u - v\|_2$,

$$\mathbb{E}\left[(R_{2^\kappa v}^{2^\kappa N} - R_{2^\kappa w}^{2^\kappa N})^2\right] \ge 2\left(\log(N) + \log(2)\kappa\right)\left[1 - \mathcal{I}_{\tilde{\sigma}^2}\left(\frac{n - \lceil \log_+ \|v - w\|_2 \rceil}{n + \kappa}\right)\right]. \tag{4.12}$$

Combining (4.12) with the upper bound in (4.11), it follows that we may choose $\kappa(\alpha_0)$ such that for all $v, w \in V_N$,

$$\mathbb{E}\left[(\psi_v^N - \psi_w^N)^2\right] \le \mathbb{E}\left[(\psi_v^N + a_v X - \psi_w^N - a_w X)^2\right] \le \mathbb{E}\left[(R_{2^\kappa v}^{2^\kappa N} - R_{2^\kappa w}^{2^\kappa N})^2\right]. \tag{4.13}$$

Applying Slepian's Lemma, we obtain for any $\lambda \in \mathbb{R}_+$ and $A \subset V_N$,

$$\mathbb{P}\left(\max_{v \in A} \psi_v^N + a_v X \ge \lambda\right) \le \mathbb{P}\left(\max_{v \in 2^\kappa A} R_v^{2^\kappa N} \ge \lambda\right). \tag{4.14}$$

By independence and symmetry of $X$,

$$\mathbb{P}\left(\max_{v \in A} \psi_v^N \ge \lambda\right) \le 2\mathbb{P}\left(\max_{v \in 2^\kappa A} R_v^{2^\kappa N} \ge \lambda\right). \tag{4.15}$$

□

*Proof of Proposition 4.2.* [47, Theorem 4.1] gives us

$$\mathbb{P}\left(\max_{v \in V_N} R_v^N \ge m_N + x\right) \le C(1 + x\mathbb{1}_{\sigma_1 = \bar{\sigma}_1}) e^{-x\frac{2}{\bar{\sigma}_1}}, \quad \forall x \ge 0. \tag{4.16}$$

The claim follows from a combination of Lemma 4.3 and (4.16). □

Next, we prove a corresponding lower bound on the right tail.

**Lemma 4.4.** *There is an integer $\kappa > 0$ such that for all $N \in \mathbb{N}$ and $\lambda \in \mathbb{R}$,*

$$\frac{1}{2}\mathbb{P}\left(\max_{v \in V_{2^{-\kappa}N}} \sqrt{\log(2)} S_v^{2^{-\kappa}N} \ge \lambda\right) \le \mathbb{P}\left(\max_{v \in V_N} \psi_v^N \ge \lambda\right). \tag{4.17}$$

*Proof.* Note that $(\frac{N}{4}, \frac{N}{4}) + 2^{\kappa-3} V_{2^{-\kappa}N} \subset (\frac{N}{4}, \frac{N}{4}) + V_{\frac{N}{8}} \subset V_N$. By Lemma 3.3 *ii.* and *iv.*, there is a constant $C > 0$, independent of $N$, such that

$$\left|\mathrm{Var}\left[\psi^N_{(\frac{N}{4},\frac{N}{4})+2^{\kappa-3}u}\right] - \mathrm{Var}\left[\psi^N_{(\frac{N}{4},\frac{N}{4})+2^{\kappa-3}v}\right]\right| \le C, \quad \forall u, v \in V_{2^{-\kappa}N}. \tag{4.18}$$

Moreover, by *iv.* in Lemma 3.3

$$\mathrm{Var}\left[\psi^N_{(\frac{N}{4},\frac{N}{4})+2^{\kappa-3}v}\right] \ge \log(2)\mathrm{Var}\left[S_v^{2^{-\kappa}N}\right], \quad \forall v \in V_{2^{-\kappa}N}, \tag{4.19}$$

for $\kappa > 0$ large enough, independent of $N$. Thus, we can find a family of positive real numbers $\{a_v : v \in V_{2^{-\kappa}N}\}$ that satisfy $|a_u - a_v| \le \sqrt{C}$ for a constant $C > 0$, such that for $u, v \in V_N$ and an independent standard Gaussian random variable $X$,

$$\mathrm{Var}\left[\psi^N_{(\frac{N}{4},\frac{N}{4})+2^{\kappa-3}v}\right] = \log(2)\mathrm{Var}\left[S_v^{2^{-\kappa}N} + a_v X\right], \quad \forall v \in V_{2^{-\kappa}N}. \tag{4.20}$$

Using Lemma 3.3 *iv.*, and choosing $\kappa$ large enough, we have for $u, v \in V_{2^{-\kappa}N}$,

$$\mathbb{E}\left[(\psi^N_{(\frac{N}{4},\frac{N}{4})+2^{\kappa-3}u} - \psi^N_{(\frac{N}{4},\frac{N}{4})+2^{\kappa-3}v})^2\right] \geq \log(2)\mathbb{E}\left[(S^{2^{-\kappa}N}_u - S^{2^{-\kappa}N}_v + (a_u - a_v)X)^2\right]. \tag{4.21}$$

Hence, by Slepian's Lemma we have for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}\left(\max_{v\in V_{2^{-\kappa}N}} \psi^N_{(\frac{N}{4},\frac{N}{4})+2^{\kappa-3}v} \geq \lambda\right) &\geq \mathbb{P}\left(\sqrt{\log(2)}\max_{v\in V_{2^{-\kappa}N}}(S^{2^{-\kappa}N}_v + a_v X) \geq \lambda\right)\\ &\geq \frac{1}{2}\mathbb{P}\left(\sqrt{\log(2)}\max_{v\in V_{2^{-\kappa}N}} S^{2^{-\kappa}N}_v \geq \lambda\right),\end{aligned} \tag{4.22}$$

as $X$ is an independent standard Gaussian. □

**Lemma 4.5.** *Set $M^*_N := m_N/\sqrt{\log(2)}$. There is a constant $C > 0$ such that for any $N \in \mathbb{N}$ and $y \in [0, \sqrt{\log N}]$,*

$$\mathbb{P}\left(\max_{v\in V_N} S^N_v > M^*_N + y\right) \geq C\,(1 + y\mathbb{1}_{\sigma_1=\bar{\sigma}_1})\,e^{-\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}y}. \tag{4.23}$$

Recall the notation, i.e. $\pi_j$ is the unique index such that, for $1 \leq j \leq m$, we have $\lambda^j = \lambda_{\pi_j}$ and that we write $t^j = \lambda^j \frac{\log N}{\log 2}$ as well as $t_j = \lambda_j \frac{\log N}{\log 2}$. Moreover, we set

$$M^*_N(t) := \sum_{j=1}^m \frac{t \wedge t^j - t^{j-1}}{\Delta t^j}\left[2\sqrt{\log 2}\bar{\sigma}_j \Delta t^j - \frac{(w_j \bar{\sigma}_j \log(\Delta t^j))}{4\sqrt{\log(2)}}\right], \quad t \in \mathbb{R}_+. \tag{4.24}$$

The proof of Lemma 4.5 is based on a second moment computation. We introduce suitable events that control the paths that reach the maximum. For $v \in V'_N = V_{N/2} + (N/4, N/4) \subset V_N$, $x \in \mathbb{R}$, $0 \leq k \leq n$ and $0 < i \leq m$, let

$$s_{k,n}(x) := \begin{cases} \frac{\mathcal{I}_{\sigma^2}(k/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)}(x), & \text{if } 0 \leq k \leq \lambda^1,\\ \frac{\mathcal{I}_{\sigma^2}(k/n,\lambda^i)}{\mathcal{I}_{\sigma^2}(\lambda^{i-1},\lambda^i)}(x), & \text{if } \lambda^{i-1} < k \leq \lambda^i \end{cases} \tag{4.25}$$

be the 'optimal path' followed by extremal particles and

$$f_{k,n} := \begin{cases} C_f(\mathcal{I}_{\sigma^2}(k/n)n)^{2/3}, & \text{if } 0 \leq k \leq t_1,\\ C_f(\mathcal{I}_{\sigma^2}(k/n, \lambda^1)n)^{2/3}, & \text{if } t_1 < k \leq t^1,\\ C_f(\mathcal{I}_{\sigma^2}(\lambda^i, k/n)n)^{2/3}, & \text{if } t^i < k \leq t_{\pi_i+1} : i \in \{1,\dots,m-1\}\\ C_f(\mathcal{I}_{\sigma^2}(k/n, \lambda^{i+1})n)^{2/3}, & \text{if } t_{\pi_i+1} < k \leq t^{i+1} : i \in \{1,\dots,m-1\} \end{cases} \tag{4.26}$$

be the concave barrier. The constant $C_f$ depends on the parameters and will be fixed later in the proof. For $v \in V_N$, $x \in \mathbb{R}$, $\infty > y > 0$ and $0 \leq k \leq n$, let

$$I^y_n(1) := [\Delta M^*_N(t^1) + y - 1, \Delta M^*_N(t^1) + y], \tag{4.27}$$

$$I^y_n(i) := [\Delta M^*_N(t^i) - 1, \Delta M^*_N(t^i)], \text{ for } 1 < i \leq m \tag{4.28}$$

$$I_{k,n}(x) := [s_{k,n}(x) - f_{k,n}, s_{k,n}(x) + f_{k,n}], \tag{4.29}$$

$$\begin{aligned}C^{N,y}_v(r) := \{&\Delta S^N_v(t^i) \in I^y_n(i),\ S^N_v(k + t^{i-1}) - S^N_v(t^{i-1}) \in I_{k,n}(\Delta S^N_v(t^i))\\ &\forall 0 < k < t^{i+1} - t^i,\ 0 < i \leq m :\ k + t^{i-1} \leq r\},\end{aligned} \tag{4.30}$$

$$h_N(y) := \sum_{v\in V'_N} \mathbb{1}_{C^{N,y}_v(t^m)}. \tag{4.31}$$

$f_{k,n}$ and $s_{k,n}(x)$ are defined as before (see (4.25) and (4.26)). We can restrict the proof to the case of $m = 1$ and to the assumption that $\mathcal{I}_{\sigma^2}(s) < \mathcal{I}_{\bar{\sigma}^2}(s)$ holds for all $0 < s < 1$. The statement in case of equality is given by [24, Theorem 1.1]. The lower bound then follows using the independence of increments and the fact that on the intervals, when $i > 1$, we choose $y = 0$, compare (4.27) with (4.28). This implies that there is a constant $C > 0$, such that we obtain as lower bound

$$C\,(1 + y\mathbb{1}_{\sigma_1=\bar{\sigma}_1})\,e^{-\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}y}\prod_{i=2}^{m} e^{-\frac{2\sqrt{\log(2)}}{\bar{\sigma}_i}0} \geq C\,(1 + y\mathbb{1}_{\sigma_1=\bar{\sigma}_1})\,e^{-\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}y}. \tag{4.32}$$

Thus, until the end of the proof of Lemma 4.5, we restrict ourselves to the case when $m = 1$ and $\mathcal{I}_{\sigma^2}(s) < \mathcal{I}_{\bar{\sigma}^2}(s)$ holds for all $0 < s < 1$.

**Lemma 4.6.** *There are constants $C, c > 0$ such that it holds for all $N \in \mathbb{N}$ sufficiently large and $y \in [0, \sqrt{\log N}]$,*

$$c \geq \mathbb{E}\left[h_N(y)\right] \geq Ce^{-\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}y}. \tag{4.33}$$

**Lemma 4.7.** *There is a constant $\tilde{C} > 0$ independent of $N$, such that, for $y \in [0, \sqrt{\log N}]$,*

$$\mathbb{E}\left[h_N^2(y)\right] \leq \mathbb{E}\left[h_N(y)\right]^2 + (1 + \tilde{C})\mathbb{E}\left[h_N(y)\right]. \tag{4.34}$$

*Proof of Lemma 4.6.* In the following, we write $M_N^*$ instead of $M_N^*(t^1)$. By linearity of expectations,

$$\mathbb{E}\left[h_N\right] = \frac{1}{4}2^{2t^1}\mathbb{P}(S_v^N(t^1) \in I_n(1),\, S_v^N(k) \in I_{k,n}(S_v^N(t^1)) \text{ for } 0 < k < t^1). \tag{4.35}$$

Note that $\mathbb{E}\left[s_{k,n}(S_v^N(t^1))\left(S_v^N(k) - s_{k,n}(S_v^N(t^1))\right)\right] = 0$, and so

$$\mathrm{Var}\left[S_v^N(k) - s_{k,n}(S_v^N(t^1))\right] = \mathrm{Var}\left[S_v^N(k) - s_{k,n}(S_v^N(t^1))\right] = n\mathcal{I}_{\sigma^2}\left(\frac{k}{n}\right)\left(1 - \frac{\mathcal{I}_{\sigma^2}(k/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right). \tag{4.36}$$

In particular, $\mathbb{E}\left[S_v^N(k) - s_{k,n}(S_v^N(t^1))\right] = 0$. Under our assumptions, we have . By conditioning the last event in (4.35) on $S_v^N(t^1)$, using that this is independent of $\{S_v^N(k) - s_{k,n}(S_v^N(t^1))\}_{k=0}^{t^1}$, we have

$$\mathbb{E}\left[h_N(y)\right] = \frac{1}{4}2^{2n}\mathbb{P}\left(S_v^N(t^1) \in [M_N^* + y - 1, M_N^* + y]\right)\mathbb{P}\left(S_v^N(k) \in I_{k,n}(S_v^N(t^1),\, 0 < k < t^1\right). \tag{4.37}$$

To estimate the first probability in (4.37), note that $S_v^N(t^1) \sim \mathcal{N}\left(0, \bar{\sigma}_1^2 t^1\right)$ and that the assumptions imply the identity $M_N^* = 2\sqrt{\log(2)}\bar{\sigma}_1 n - \frac{1}{4\sqrt{\log(2)}}\log(n)\bar{\sigma}_1$. Thus, by a standard Gaussian estimate,

$$\mathbb{P}\left(S_v^N(t^1) \in I_n^y(1)\right) = \int_{M_N^*+y-1}^{M_N^*+y} \frac{\exp\left[-x^2/(2\bar{\sigma}_1^2 t^1)\right]}{\sqrt{2\pi\bar{\sigma}_1^2 t^1}}\mathrm{d}x \geq \frac{\exp\left[-(M_N^* + y)^2/(2\bar{\sigma}_1^2 t^1)\right]}{\sqrt{2\pi\bar{\sigma}_1^2 t^1}}. \tag{4.38}$$

By expanding the square in (4.38) and bounding all terms in the exponential that tend to 0 as $n \to \infty$ by a constant, we can find a constant $C > 0$ such that

$$\mathbb{P}\left(S_v^N(t^1) \in I_n^y(1)\right) \geq CN^{-2}e^{-y\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}}. \tag{4.39}$$

We turn to the second probability in (4.37). By subadditivity of measures and using (4.36),

$$\begin{aligned}\mathbb{P}(S_v^N(k) \in I_{k,n}(S_v^N(t^1)),\, 0 < k < t^1) &\geq 1 - 2\sum_{k=1}^{t^1-1}\mathbb{P}(S_v^N(k) - s_{k,n}(S_v^N(t^1)) > f_{k,n})\\ &\geq 1 - 2\sum_{k=1}^{t^1-1} C\exp\left[-\frac{1}{2}\frac{f_{k,n}^2}{\mathcal{I}_{\sigma^2}(k/n)n(1 - \frac{\mathcal{I}_{\sigma^2}(k/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)})}\right].\end{aligned} \tag{4.40}$$

By definition of the concave barrier in (4.26), we may split and bound the sum in (4.40) from above by

$$\sum_{k=1}^{t_1} C\exp\left[-\frac{1}{2}C_f^2\sigma_1^{2/3}k^{1/3}\right]\mathbb{1}_{\sigma_1\neq 0}+\sum_{k=t_1+1}^{t^1-1} C\exp\left[-\frac{1}{2}C_f^2\min_{i\in\{2,\dots,\pi_1\}:\sigma_i>0}(\sigma_i)^{2/3}(t^1-k)^{1/3}\right]<\frac{c}{2}, \tag{4.41}$$

where $0<c<1$ is a constant independent of $n$, if $C_f$ is large enough. Inserting (4.41) into (4.40) gives

$$\mathbb{P}(S_v^N(k)\in I_{k,n}(S_v^N(t^1)),\ \forall 0<k<t^1)>1-c=c_2>0. \tag{4.42}$$

Inserting (4.42) and (4.39) into (4.37) finishes the proof of the lower bound in (4.33). To get an upper bound in (4.37) we bound the second probability by 1. For the first probability, as for the lower bound, we get

$$\mathbb{P}\left(S_v^N(t^1)\in I_n^y(1)\right)\le CN^{-2}\exp\left[-(y-1)\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}\right]. \tag{4.43}$$

Inserting this into (4.37), we obtain the upper bound in (4.33). □

*Proof of Lemma 4.7.* As in the proof of Lemma 4.6, we write $M_N^*$ instead of $M_N^*(t^1)$. Recall that, for $v,w\in V_N$, $r(v,w)=n-\lceil\log_2(d_N^\infty(v,w)+1)\rceil$ denotes the number of scales of independent increments of the processes $S_v^N(k)$ and $S_w^N(k')$. By decomposing the second moment along $r(\cdot,\cdot)$ and using independence of the increments,

$$\begin{aligned}\mathbb{E}\left[h_N^2(y)\right]&=\sum_{v,w\in V_N'}\mathbb{P}\left(C_v^{N,y}(t^1)\cap C_w^{N,y}(t^1)\right)=\sum_{k=0}^{n}\sum_{\substack{v,w\in V_N'\\ r(v,w)=k}}\mathbb{P}\left(C_v^{N,y}(t^1)\cap C_w^{N,y}(t^1)\right)\\ &\le\mathbb{E}\left[h_N(y)\right]^2+\mathbb{E}\left[h_N(y)\right]+\sum_{k=1}^{n-1}\sum_{\substack{v,w\in V_N'\\ r(v,w)=k}}\mathbb{P}\left(C_v^{N,y}(t^1)\cap C_w^{N,y}(t^1)\right).\end{aligned} \tag{4.44}$$

To bound the double sum from above, we bound each summand from above. Fix $v,w\in V_N'$ with $r(v,w)=r=k\in\{1,\dots,n-1\}$. We set $B_{k,n}(x):=[x-s_{k,n}(x)-f_{r,n},x-s_{k,n}(x)+f_{r,n}]$. Dropping the constraint for $w$ up to time $r$, we have

$$\begin{aligned}\mathbb{P}\left(C_v^{N,y}(t^1)\cap C_w^{N,y}(t^1)\right)&\le\mathbb{P}\left(C_v^{N,y}(t^1)\cap C_w^{N,y}(r)\right)\max_{x\in I_n(1)}\mathbb{P}\left(S_w^N(t^1)-S_w^N(r)\in B_{r,n}(x)\right)\\ &\le\mathbb{P}\left(C_v^{N,y}(t^m)\right)\max_{x\in I_n(1)}\mathbb{P}\left(S_w^N(t^1)-S_w^N(r)\in B_{r,n}(x)\right).\end{aligned} \tag{4.45}$$

For fixed $v\in V_N'$, the number of points $w\in V_N'$ that satisfy $d_N^\infty(v,w)\in[2^k,2^{k+1}]$, is bounded by $c_1 2^{2k}=2^{2(t^1-r)}$ for some $c_1>0$. Therefore, we can bound the last summand in (4.44) from above by

$$c_1\mathbb{E}\left[h_N(y)\right]\sum_{r=1}^{n-1}2^{2(t^1-r)}\max_{\substack{x\in I_n^y(1)\\ v\in V_N}}\mathbb{P}\left(S_v^N(t^1)-S_v^N(r)\in x+I_{r,n}(x)\right). \tag{4.46}$$

To bound the probability in (4.46), we use that for any $x\in I_n^y(1)$,

$$\begin{aligned}A_{r,n,x}^y&:=\mathbb{P}\left(S_v^N(t^1)-S_v^N(r)\in x+I_{r,n}(x)\right)=\int_{x-s_{r,n}(x)-f_{r,n}}^{x-s_{r,n}(x)+f_{r,n}}\frac{\exp\left[-\frac{1}{2}\frac{z^2}{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n}\right]}{\sqrt{2\pi\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n}}\mathrm{d}z\\ &\le\frac{2f_{r,n}}{\sqrt{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n}}\exp\left[-\frac{1}{2}\frac{(M_N^*+y-s_{r,n}(M_N^*+y)-f_{r,n})^2}{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n}\right].\end{aligned} \tag{4.47}$$

Noting that $n = t^1$ and using (4.25), we bound from below the square in the exponential in (4.47) by

$$(M_N^* + y)^2 \left(1 - \frac{\mathcal{I}_{\sigma^2}(r/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right)^2 - 2f_{r,n}\frac{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)}(M_N^* + y) = (M_N^*)^2 \left(1 - \frac{\mathcal{I}_{\sigma^2}(r/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right)^2 - 2f_{r,n}M_N^*\frac{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)}$$
$$+(2M_N^* y + y^2)\left(1 - \frac{\mathcal{I}_{\sigma^2}(r/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right)^2 - 2yf_{r,n}\frac{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)}. \tag{4.48}$$

Inserting (4.48) into (4.47), dropping the term involving $y^2$, and noting that we can bound the term $\exp\left[\frac{2yf_{r,n}\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)}}{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n}\right]$ by a constant, we obtain that (4.47) is bounded from above by

$$\frac{Cf_{r,n}}{\sqrt{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)n}} \exp\left[-\frac{1}{2}\frac{(M_N^*)^2\left(1 - \frac{\mathcal{I}_{\sigma^2}(r/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right)^2 - 2f_{r,n}M_N^*\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)} + 2yM_N^*\left(1 - \frac{\mathcal{I}_{\sigma^2}(r/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right)^2}{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)n}\right]$$
$$\leq \frac{Cf_{r,n}}{\sqrt{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)n}} \exp\left[-2\log(2)t^1\frac{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)} + \frac{\left(1 + \frac{y}{4\sqrt{\log(2)t^1}}\right)}{2}\log(t^1)\frac{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right.$$
$$\left. -2y\sqrt{\log(2)}\frac{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)} - \frac{\log(t^1)^2}{32\log(2)t^1}\frac{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)} + \frac{C_f\sigma_1^{4/3}r^{2/3}}{(4\log(2))^{-1/2}\bar{\sigma}_1}\right]. \tag{4.49}$$

Let $i$ be minimal such that $\sigma_i > 0$. We distinguish the cases $0 < r \leq t_i$ and $t_i < k < t^1$. We may assume that $\sigma_1 > 0$.

Case 1: In this case, we have $\mathrm{Var}[S_v^N(t^1) - S_v^N(r)] = \mathcal{I}_{\sigma^2}(r/n, \lambda^1)n$ and $f_{r,n} = C_f(\sigma_1^2 r)^{2/3}$. Since $r \leq t_1$, $\frac{\frac{1}{\lambda_1}\mathcal{I}_{\sigma^2}(\lambda_1)}{\frac{1}{\lambda^1}\mathcal{I}_{\sigma^2}(\lambda^1)} = \frac{\sigma_1^2}{\bar{\sigma}_1^2} \in (0, 1)$, and so there is an $\eta_1 < 1$, independent of $r$ and $n$, such that

$$\frac{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)}t^1 = t^1 - t^1\frac{\mathcal{I}_{\sigma^2}(r/n)}{\mathcal{I}_{\sigma^2}(\lambda^1)} = t^1 - r\frac{\frac{1}{r/n}\mathcal{I}_{\sigma^2}(r/n)}{\frac{1}{\lambda^1}\mathcal{I}_{\sigma^2}(\lambda^1)} = t^1 - r\frac{\frac{1}{\lambda_1}\mathcal{I}_{\sigma^2}(\lambda_1)}{\frac{1}{\lambda^1}\mathcal{I}_{\sigma^2}(\lambda^1)} = t^1 - \eta_1 r. \tag{4.50}$$

Similarly, we have $\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)} \geq 1 - \frac{\sigma_1^2}{\bar{\sigma}_1^2}\frac{\lambda_1}{\lambda^1}$. Using these facts in (4.49), we get

$$A_{r,n,x}^y \leq Cr^{2/3}\exp\left(\tilde{C}r^{2/3}\right)2^{-2(t^1-\eta_1 r)}\frac{\exp\left[\log(t^1)\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{2\mathcal{I}_{\sigma^2}(\lambda^1)} - 2y\sqrt{\log(2)}\left(1 - \frac{\sigma_1^2}{\bar{\sigma}_1^2}\frac{\lambda_1}{\lambda^1}\right)\right]}{\sqrt{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)t^1}}$$
$$\times\exp\left[-\log(t^1)\frac{\mathcal{I}_{\sigma^2}(r/n, \lambda^1)}{2t^1\mathcal{I}_{\sigma^2}(\lambda^1)}\left(\frac{\log(t^1) - 4\sqrt{\log(2)}y}{16\log(2)}\right)\right]. \tag{4.51}$$

Note that we have $\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)} < 1$ and $\left(\frac{\log(t^1)-4\sqrt{\log(2)}y}{16\log(2)}\right) \geq 0$. Thus, in the case $0 < r \leq t_1$, we have

$$A_{r,n,x}^y \leq Cr^{2/3}2^{-2(t^1-\eta_1 r)}\exp\left[\tilde{C}r^{2/3} - \frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}\left(1 - \frac{\sigma_1^2}{\bar{\sigma}_1^2}\frac{\lambda_1}{\lambda^1}\right)y\right]$$
$$\leq C2^{-2(t^1-\eta_1 r)+o(r)}\exp\left[-\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}\left(1 - \frac{\sigma_1^2}{\bar{\sigma}_1^2}\frac{\lambda_1}{\lambda^1}\right)y\right]. \tag{4.52}$$

Note that for the last factor in the exponent we know $0 < 1 - \frac{\sigma_1^2}{\bar{\sigma}_1^2}\frac{\lambda_1}{\lambda^1} < 1$, which guarantees that we have the correct sign to have sufficient decay in $y$.

Case 2: The same computation as in (4.47), now in the case of $t_1 < r < t^1$, $f_{r,n} = C_f(\mathcal{I}_{\sigma^2}(r/n, \lambda^1))^{2/3}$ and $x \in I_n^y(1)$, yields

$$\begin{aligned} A^y_{r,n,x} &\leq \frac{2f_{r,n}}{\sqrt{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n}} \exp\left[-\frac{1}{2}\frac{(M_N^*(t^1)+y-s_{r,n}(M_N^*(t^1)+y)-f_{r,n})^2}{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)t^1}\right] \\ &\leq C2^{-2t^1\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)}}(\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n)^{1/6}\exp\left[\log(t^1)\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{2\mathcal{I}_{\sigma^2}(\lambda^1)} + \frac{C_f(\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n)^{2/3}}{(4\log(2))^{-1/2}\bar{\sigma}_1}\right] \\ &\quad\times\exp\left[-y\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)} - \log(t^1)\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{2t^1\mathcal{I}_{\sigma^2}(\lambda^1)}\left(\frac{\log(t^1)-4\sqrt{\log(2)}y}{16\log(2)}\right)\right]. \end{aligned} \tag{4.53}$$

As $y \in [0, \sqrt{\log N}]$, $\left(\frac{\log(t^1)-4\sqrt{\log(2)}y}{16\log(2)}\right) \geq 0$. Moreover, for $t_1 < r < t^1$,

$$t^1\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)} = \frac{\frac{1}{\lambda^1 - r/n}\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\frac{1}{\lambda^1}\mathcal{I}_{\sigma^2}(\lambda^1)}(t^1-r) \geq \eta_2(t^1-r), \tag{4.54}$$

for a constant $\eta_2 > 1$ that is independent of $r$ and $n$. Using these facts in (4.53), we obtain

$$\begin{aligned} A^y_{r,n,x} &\leq C2^{-\eta_2(t^1-r)}(\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n)^{2/3}\exp\left[C_f(\mathcal{I}_{\sigma^2}(r/n,\lambda^1)n)^{2/3}2\sqrt{\log(2)} - y\left(\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right)\right] \\ &\leq C2^{-2\eta_2(t^1-r)+o(t^1-r)}\exp\left[-y\left(\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}\frac{\mathcal{I}_{\sigma^2}(r/n,\lambda^1)}{\mathcal{I}_{\sigma^2}(\lambda^1)}\right)\right]. \end{aligned} \tag{4.55}$$

Combining the bounds in (4.52) and (4.55) and observing that both $(1-\eta_1) > 0$ and $(1-\eta_2) < 0$ hold and using $y \geq 0$, allows us to bound the sum in (4.46) by an absolute constant $C_2 > 0$, i.e.

$$\sum_{r=1}^{n-1} 2^{2(t^1-r)} \max_{x\in I_n^y(1)} A^y_{r,n,x} \leq C\left[\sum_{r=1}^{t_1} 2^{-2r(1-\eta_1)+o(r)} + \sum_{r=t_1+1}^{t^1-1} 2^{2(1-\eta_2)(t^1-r)+o(t^1-r)}\right] \leq C_2. \tag{4.56}$$

Inserting (4.56) into (4.44) concludes the proof. □

*Proof of Lemma 4.5.* Combining Lemma 4.6 with Lemma 4.7 shows that there are constants, $\tilde{C}, C, c > 0$, such that

$$\begin{aligned} \mathbb{P}\left(\max_{v\in V_N} S_v^N > M_N^* + y\right) &\geq \mathbb{P}(h_N(y) \geq 1) \geq \frac{(\mathbb{E}[h_N(y)])^2}{\mathbb{E}\left[h_N^2(y)\right]} \geq \frac{\mathbb{E}[h_N(y)]^2}{\mathbb{E}[h_N(y)]^2 + (1+\tilde{C})\mathbb{E}[h_N(y)])} \\ &\geq \frac{\mathbb{E}[h_N(y)]}{1+c} \geq Ce^{-y\frac{2\sqrt{\log(2)}}{\bar{\sigma}_1}}. \end{aligned} \tag{4.57}$$

□

The goal in the following is to provide an upper bound on the left tail of the centred maximum of the $(\sigma,\lambda)$−DGFF. We start with a bound on the left tail of $S_N^* - M_N^*$.

**Lemma 4.8.** *There exist constants $C, c > 0$, such that, for all $N \in \mathbb{N}$, and $0 \leq \lambda \leq (\log\log N)^{2/3}$,*

$$\mathbb{P}\left(\max_{v\in V_N} S_v^N \leq M_N^* - \lambda\right) \leq Ce^{-c\lambda}. \tag{4.58}$$

*Proof.* By Lemma 4.5, there are $\beta > 0$ and $\delta_0 \in (0,1)$ such that, for all $N \in \mathbb{N}$,

$$\mathbb{P}\left(\max_{v\in V_N} S_v^N \geq m_N/\sqrt{\log(2)} - \beta\right) \geq \delta_0. \tag{4.59}$$

In particular, there is a $\kappa > 0$ such that, for all $N \geq N' \geq 4$,

$$\begin{aligned} 2\sqrt{\log(2)}\mathcal{I}_{\bar\sigma}(1)\log\left(\frac{N}{N'}\right) - \frac{3}{4\sqrt{\log(2)}}\sum_{j=1}^m \bar\sigma_j \log\left(\log\left(\frac{N}{N'}\right)\right) - \kappa \leq M_N^* - M_{N'}^* \\ \leq 2\sqrt{\log(2)}\mathcal{I}_{\bar\sigma}(1)\log\left(\frac{N}{N'}\right) + \kappa. \end{aligned} \tag{4.60}$$

We now pick $\lambda' = \frac{\lambda}{2}$, $N' = N\exp\left[-\frac{1}{2\sqrt{\log(2)}\mathcal{I}_{\bar\sigma}(1)}(\lambda' - \beta - \kappa - 4)\right]$ and set $n' = \log_2 N'$. With this choice, we deduce from (4.60) that $M_N - M_{N'} \leq \lambda' - \beta$. We divide $V_N$ into disjoint boxes by placing at each position $(3iN', 3jN')$ a box of size $N'$, for $1 \leq i, j \leq \frac{N}{N'}$. We call this collection of boxes $\mathcal{B}$ and note that the pairwise distances between two boxes are at least $2N'$. This implies independence of the processes $\{S_v^{N'}\}_{v\in B}$ on pairwise disjoint boxes. This allows us to bound the number of boxes $B \in \mathcal{B}$ from below by

$$\frac{N}{3N'} \geq \frac{1}{3}\exp\left[\frac{1}{2\sqrt{\log(2)}\mathcal{I}_{\bar\sigma}(1)}(\lambda' - \beta - \kappa - 4)\right]. \tag{4.61}$$

Let $\tilde S_v^N = S_v^{N'} + X$, for $v \in B$ and $B \in \mathcal{B}$, where $X \sim \mathcal{N}(0, s^2)$ is an independent random variable and with $s^2$ such that $\mathrm{Var}(S_v^N) = \mathrm{Var}(\tilde S_v^N)$. For $u, v \in \bigcup_{B\in\mathcal{B}} B$, we then have

$$\mathbb{E}\left[(\tilde S_u^N - \tilde S_v^N)^2\right] = \mathbb{E}\left[(S_u^{N'} - S_v^{N'})^2\right] \leq \mathbb{E}\left[(S_u^N - S_v^N)^2\right]. \tag{4.62}$$

An application of Slepian's Lemma gives that, for any $t \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{v\in V_N} S_v^N \leq t\right) \leq \mathbb{P}\left(\max_{v\in\cup_{B\in\mathcal{B}}B} S_v^N \leq t\right) \leq \mathbb{P}\left(\max_{v\in\cup_{B\in\mathcal{B}}B} \tilde S_v^N \leq t\right). \tag{4.63}$$

Using $M_N^* - \lambda' \leq M_{N'}^* - \beta$ and (4.59), one obtains, for each $B \in \mathcal{B}$,

$$\mathbb{P}\left(\max_{v\in B} S_v^{N'} \geq M_N^* - \lambda'\right) \geq \mathbb{P}\left(\max_{v\in B} S_v^{N'} \geq M_{N'}^* - \beta\right) \geq \delta_0. \tag{4.64}$$

By (4.64) and the independence of $\{S_v^{N'}\}_{v\in B}$ and $\{S_v^{N'}\}_{v\in B'}$, for different $B, B' \in \mathcal{B}$,

$$\mathbb{P}\left(\max_{v\in\cup_{B\in\mathcal{B}}B} S_v^{N'} < M_N^* - \lambda'\right) \leq (1-\delta_0)^{|\mathcal{B}|}. \tag{4.65}$$

As $\delta_0 \in (0,1)$, by (4.61), there are constants, $C, c > 0$, such that

$$(1-\delta_0)^{|\mathcal{B}|} \leq \exp\left[\frac{\log(1-\delta_0)}{3}\exp\left(\frac{1}{2\sqrt{\log(2)}\mathcal{I}_{\bar\sigma}(1)}(\lambda' - \beta - \kappa - 4)\right)\right] \leq Ce^{-c\lambda'}. \tag{4.66}$$

Using (4.63), we can bound $\mathbb{P}\left(\max_{v\in V_N} S_v^N \leq M_N^* - \lambda\right)$ from above by

$$\mathbb{P}\left(\max_{v\in\cup_{B\in\mathcal{B}}B} S_v^{N'} < M_N^* - \lambda'\right) + \mathbb{P}(\theta \leq -\lambda') \leq Ce^{-c\lambda'}, \tag{4.67}$$

where the last bound follows from (4.66) and a Gaussian tail bound. □

Lemma 4.8 allows us to deduce the upper bound on the left tail of the centred maximum.

**Lemma 4.9.** *There exist constants, $C, c > 0$, so that, for all $N \in \mathbb{N}$, and $0 \le \lambda \le (\log\log N)^{2/3}$,*

$$\mathbb{P}\left(\max_{v\in V_N} \psi_v^N \le m_N - \lambda\right) \le Ce^{-c\lambda}. \tag{4.68}$$

*Proof.* Following the proof of Lemma 4.3, we see that, instead of $\mathbb{P}\left(\max_{v\in V_N} \psi_v^N \le m_N - \lambda\right)$, it suffices to bound $\mathbb{P}\left(\max_{v\in 2^\kappa V_N} \psi^{2^{\kappa+2}N}_{v+(2^{\kappa+1}N,2^{\kappa+1}N)} \le m_N - \lambda\right)$. By Lemma 3.3 *iv*., there is a constant $\kappa_0 > 0$, such that, for all $\kappa \ge \kappa_0$,

$$\mathrm{Var}\left[\psi^{2^{\kappa+2}N}_{2^\kappa v+(2^{\kappa+1}N,2^{\kappa+1}N)}\right] \le \log(2)\mathrm{Var}\left[S_v^{2^{2\kappa}N}\right], \quad \forall v \in V_N. \tag{4.69}$$

Therefore, we can choose a collection of positive numbers, $\{a_v : v \in V_N\}$, and an independent standard Gaussian random variable, $X$, so that, for any $N$ and $u, v \in V_N$,

$$\mathrm{Var}\left[\psi^{2^{\kappa+2}N}_{2^\kappa v+(2^{\kappa+1}N,2^{\kappa+1}N)} + a_v X\right] = \log(2)\mathrm{Var}\left[S_v^{2^{2\kappa}N}\right], \quad \forall v \in V_N. \tag{4.70}$$

As $\mathrm{Var}\left[S_v^{2^{2\kappa}N}\right] = \mathrm{Var}\left[S_v^{2^{2\kappa}N}\right]$, for all $v, w \in V_{2^{2\kappa}N}$, and by the uniform bound in Lemma 3.3 *ii*., there is a constant $C_1 > 0$, such that

$$|a_u - a_v| \le C_1. \tag{4.71}$$

Writing $\tilde{u} = 2^\kappa u + (2^{\kappa+1}N, 2^{\kappa+1}N)$ and using Lemma 3.3 *ii* and *iv*., we get

$$\begin{aligned}\mathbb{E}\left[\psi_{\tilde{u}}^{2^{\kappa+2}N}\psi_{\tilde{v}}^{2^{\kappa+2}N}\right] &\ge \log(2)(n+\kappa)\mathcal{I}_{\sigma^2}\left(\frac{n+\kappa-\log_+\|2^\kappa u - 2^\kappa v\|_2}{n+\kappa}\right) - c \\ &= \log(2)(n+\kappa)\mathcal{I}_{\sigma^2}\left(\frac{n-\log_+\|u-v\|_2}{n+\kappa}\right) - c,\end{aligned} \tag{4.72}$$

where $c > 0$ is a constant. Further, taking into account that the Euclidean distance on the torus is bounded by the usual Euclidean distance, we have by Lemma 3.3 *ii*.,

$$\mathbb{E}\left[S_u^{2^\kappa N} S_v^{2^\kappa N}\right] \le (n+2\kappa)\mathcal{I}_{\sigma^2}\left(\frac{n+2\kappa-\log_+\|u-v\|_2}{n+2\kappa}\right) + C, \tag{4.73}$$

where $C > 0$ is another constant. Comparing (4.72) and (4.73), one deduces, using (4.70) that there is a $\kappa_0$, such that, for $\kappa \ge \kappa_0$,

$$\mathbb{E}\left[\left(\psi^{2^{\kappa+2}N}_{2^\kappa u+(2^{\kappa+1}N,2^{\kappa+1}N)} + a_u X\right)\left(\psi^{2^{\kappa+2}N}_{2^\kappa v+(2^{\kappa+1}N,2^{\kappa+1}N)} + a_v X\right)\right] \le \log(2)\mathbb{E}\left[S_u^{2^\kappa N} S_v^{2^\kappa N}\right]. \tag{4.74}$$

Using (4.74) and (4.70), we can apply Slepian's lemma to obtain

$$\begin{aligned}&\mathbb{P}\left(\max_{v\in V_N} \psi^{2^{\kappa+2}N}_{2^\kappa v+(2^{\kappa+1}N,2^{\kappa+1}N)} \le m_N - \lambda\right) \\ \le &\mathbb{P}\left(\max_{v\in V_N} \psi^{2^{\kappa+2}N}_{2^\kappa v+(2^{\kappa+1}N,2^{\kappa+1}N)} + a_v X \le m_N - \frac{\lambda}{2}\right) + \mathbb{P}\left(X \le -\frac{\lambda}{C_\kappa}\right) \\ \le &\mathbb{P}\left(\max_{v\in V_N} S_v^{2^{2\kappa}N} \le M_N^* - \frac{\lambda}{2\sqrt{\log(2)}}\right) + \mathbb{P}\left(X \le -\frac{\lambda}{C_\kappa}\right),\end{aligned} \tag{4.75}$$

where $C_\kappa > 0$ is a constant that solely depends on $\kappa$. Note that there is a collection of boxes $\mathcal{V}$, consisting of at most $2^{8\kappa}$ translated copies of $V_N$, such that $V_{2^{2\kappa}N} \subset \cup_{V\in\mathcal{V}} V$. Since

$$\left\{\max_{v\in V_{2^{2\kappa}N}} S_v^{2^\kappa N} \le M_N^* - x\right\} = \cap_{V\in\mathcal{V}}\left\{\max_{v\in V_N} S_v^{2^\kappa N} \le M_N^* - x\right\}, \tag{4.76}$$

we have, by the FKG inequality [33, Proposition 1], that

$$\mathbb{P}\left(\max_{v\in V_{2^{2\kappa}N}} S_v^{2^{2\kappa}N} \le M_N^* - \frac{\lambda}{2\sqrt{\log(2)}}\right) \ge \left(\mathbb{P}\left(\max_{v\in V_N} S_v^{2^{2\kappa}N} \le M_N^* - \frac{\lambda}{2\sqrt{\log(2)}}\right)\right)^{8\kappa}. \tag{4.77}$$

Using (4.77) and then Lemma 4.8, we bound (4.75) from above by

$$\begin{aligned}\mathbb{P}\left(\max_{v\in V_{2^\kappa N}} \psi^{2^{\kappa+2}N}_{2^\kappa v+(2^{\kappa+1}N,2^{\kappa+1}N)} \leq m_N-\lambda\right) &\leq \mathbb{P}\left(\max_{v\in V_N} \psi^{2^{\kappa+2}N}_{2^\kappa v+(2^{\kappa+1}N,2^{\kappa+1}N)} \leq m_N-\lambda\right)\\ &\leq \mathbb{P}\left(\max_{v\in V_N} S_v^{2^{2\kappa}N} \leq M_N^* - \frac{\lambda}{2\sqrt{\log(2)}}\right) + \mathbb{P}\left(X\leq -\frac{\lambda}{C_\kappa}\right)\\ &\leq \left(\mathbb{P}\left(\max_{v\in V_{2^{2\kappa}N}} S_v^{2^{2\kappa}N} \leq M_N^* - \frac{\lambda}{2\sqrt{\log(2)}}\right)\right)^{1/(8\kappa)}\\ &\quad + \mathbb{P}\left(X\leq -\frac{\lambda}{C_\kappa}\right) \leq \tilde{C}e^{-\tilde{c}\lambda},\end{aligned} \tag{4.78}$$

where $\tilde{C},\tilde{c}>0$ are constants that are independent of $N$. This concludes the proof of Lemma 4.9. □

We now have all the ingredients to finish the proof of Theorem 2.1.

*Proof of Theorem 2.1.* The upper bound on the right-tail in (2.6) follows using Proposition 4.2. A combination of Lemma 4.4 with Lemma 4.5 implies the lower bound on the right-tail in (2.6). The second statement, the upper bound for the left tail (2.7), is given by Lemma 4.9, which finishes the proof. □

## Appendix A. Gaussian comparison

**Theorem A.1** (Borell's inequality, [44, Lemma 3.1]). *Let $T$ be compact and $\{X_t\}_{t\in T}$ a centred Gaussian process on $T$ with continuous covariance. Further assume that almost surely, $X^* := \sup_{t\in T} X_t < \infty$. Then,*

$$\mathbb{E}[X^*] < \infty, \tag{A.1}$$

*and*

$$\mathbb{P}\left(\left|X^* - \mathbb{E}[X^*]\right| > x\right) \leq 2e^{-x^2/2\sigma_T^2}, \tag{A.2}$$

*where $\sigma_T^2 := \max_{t\in T}\mathbb{E}[X_t^2]$.*

**Theorem A.2** (Slepian's Lemma, [44, Theorem 3.11]). *Let $T=\{1,\dots,n\}$ and $X,Y$ be two centred Gaussian vectors. Assume that we have two subsets $A,B\subset T\times T$ satisfying*

$$\mathbb{E}[X_iX_j] \leq \mathbb{E}[Y_iY_j], \quad (i,j)\in A \tag{A.3}$$
$$\mathbb{E}[X_iX_j] \geq \mathbb{E}[Y_iY_j], \quad (i,j)\in B \tag{A.4}$$
$$\mathbb{E}[X_iX_j] = \mathbb{E}[Y_iY_j], \quad (i,j)\notin A\cup B. \tag{A.5}$$

*Further, suppose that $f:\mathbb{R}^n\to\mathbb{R}$ is a smooth function with at most exponential growth at infinity of $f$ itself, as well as its first and second derivatives, and that*

$$\partial_{ij}f \geq 0, \quad (i,j)\in A \tag{A.6}$$
$$\partial_{ij}f \leq 0, \quad (i,j)\in B. \tag{A.7}$$

*Then,*

$$\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]. \tag{A.8}$$

We use Slepian's Lemma in a particular setting, i.e. we assume that $\mathbb{E}\left[X_i^2\right]=\mathbb{E}\left[Y_i^2\right]$ and $\mathbb{E}\left[X_iX_j\right]\geq\mathbb{E}\left[Y_iY_j\right]$ for all $i,j\in T$. We then have for any $x\in\mathbb{R}$,

$$\mathbb{P}\left(\max_{i\in T} X_i > x\right) \leq \mathbb{P}\left(\max_{i\in T} Y_i > x\right). \tag{A.9}$$

In particular, $\mathbb{E}[\max_{i\in T} X_i] \leq \mathbb{E}[\max_{i\in T} Y_i]$. If we only want to compare the expectation of maxima we do not need the equality of variances. This is a result due to Sudakov and Fernique.

**Theorem A.3** (Sudakov-Fernique, [31]). *Let $I$ be an arbitrary set of finite size $n$, $\{X_i\}_{i\in I}, \{Y_i\}_{i\in I}$ be two centred Gaussian vectors. Define $\gamma^X_{ij} := \mathbb{E}[(X_i - X_j)^2]$, $\gamma^Y_{ij} := \mathbb{E}[(Y_i - Y_j)^2]$. Let $\gamma := \max_{i,j} |\gamma^X_{ij} - \gamma^Y_{ij}|$. Then,*

$$\left|\mathbb{E}[X^*] - \mathbb{E}[Y^*]\right| \leq \sqrt{\gamma \log(n)}. \tag{A.10}$$

*If $\gamma^X_{ij} \leq \gamma^Y_{ij}$ for any $i, j \in I$, then*

$$\mathbb{E}[X^*] \leq \mathbb{E}[Y^*]. \tag{A.11}$$

In particular, if $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ are independent centred Gaussian fields without any additional assumptions on their correlations, one deduces

$$\mathbb{E}\left[\max_{i\in I}(X_i + Y_i)\right] \geq \mathbb{E}\left[\max_{i\in I} X_i\right]. \tag{A.12}$$

## Appendix B. Covariance estimates

For particles $v, w \in V_N$, let

$$b_N(v,w) := \max\{\lambda \in [0,1] : [v]^N_\lambda \cap [w]^N_\lambda \neq \emptyset\} \tag{B.1}$$

denote the branching scale. The key point is that beyond $b_N(v,w)$, increments are independent, that is for $1 \geq \lambda' > \lambda > b_N(v,w)$, $\phi^N_v(\lambda') - \phi^N_v(\lambda)$ is independent of $\phi^N_w(\lambda') - \phi^N_w(\lambda)$, whereas increments before the branching scale are correlated. Further, for some $B \subset V_N$, we set

$$\phi^N_v(B) := \mathbb{E}\left[\phi^N_v | \sigma\left(\phi^N_w : w \in B^c\right)\right]. \tag{B.2}$$

Recall that for $\lambda \in [0,1]$, we also write $\phi^N_v(\lambda) = \phi^N_v([v]^N_\lambda)$.

**Lemma B.1.** *Let $\delta \in (0, 1/2)$ and $N \in \mathbb{N}$ such that $\min_{1\leq i\leq M} 2^{\frac{2}{\Delta\lambda_i}} \leq N$, as well as $N^{\lambda_1} > \delta^{-1}$. Let $v, w \in V^\delta_N$ and assume that the branching scale $b_N(v,w)$ coincides with a scale parameter, i.e. $b_N(v,w) = \lambda_i$ for some $i \in \mathbb{N}$. Then for any $0 \leq i, j \leq M$ with $\lambda_i, \lambda_j \leq b_N(v,w)$, we have*

$$\mathbb{E}\left[\Delta\phi^N_v(\lambda_i)\Delta\phi^N_w(\lambda_j)\right] = \Delta\lambda_i \log(N)\mathbb{1}_{i=j} + O(1). \tag{B.3}$$

*Proof.* For $v = w$ the statement is contained in [7, Lemma A.2]. Let us assume $v \neq w$ throughout the proof . We start with the case $i = j$. More, we assume $[v]_{\lambda_i} \cap [w]_{\lambda_i} \neq \emptyset$, i.e. the boxes should intersect at least at the boundary. If this is not the case, we can subdivide the scales further and use that beyond $b_N(v,w)$ the respective increments are independent. This implies that $\|v - w\|_2 \leq \sqrt{2}N^{1-\lambda_i}$. We now pick a box $B$ of side length $2N^{1-\lambda_i}$, centred at the middle of the line connecting the vertices $v$ and $w$. This ensures the inclusion

$$\sigma\left(\phi^N_u : u \in B^c\right) \subset \sigma\left(\phi^N_u : u \in [v]^c_{\lambda_i}\right), \sigma\left(\phi^N_u : u \in [w]^c_{\lambda_i}\right). \tag{B.4}$$

Next we pick a box $\tilde{B}$ of side length $\frac{1}{2}N^{1-\lambda_{i-1}}$ with the same centre as $B$. For $N$ as in the assumption, this implies in particular that $\sigma\left(\phi^N_u : u \in \tilde{B}^c\right) \subset \sigma\left(\phi^N_u : u \in B^c\right)$, as well as

$$\sigma\left(\phi^N_u : u \in [v]^c_{\lambda_{i-1}}\right), \sigma\left(\phi^N_u : u \in [w]^c_{\lambda_{i-1}}\right) \subset \sigma\left(\phi^N_u : u \in \tilde{B}^c\right). \tag{B.5}$$

We write $\Delta\phi_v^N(B) = \phi_v^N(B) - \phi_v^N(\tilde{B})$ and compute,

$$
\begin{aligned}
\mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\Delta\phi_w^N(\lambda_i)\right] &= \mathbb{E}\Big[\left(\phi_v^N(\lambda_i) - \phi_v^N(B) + \nabla\phi_v^N(B) + \phi_v^N(\tilde{B}) - \phi_v^N(\lambda_{i-1})\right) \\
&\quad \times\left(\phi_w^N(\lambda_i) - \phi_w^N(B) + \nabla\phi_w^N(B) + \phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)\Big] \\
&= \mathbb{E}\left[\Delta\phi_v^N(B)\Delta\phi_w^N(B)\right] && \text{(B.6)} \\
&\quad + \mathbb{E}\left[\Delta\phi_v^N(B)\left(\phi_w^N(\lambda_i) - \phi_w^N(B) + \phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)\right] && \text{(B.7)} \\
&\quad + \mathbb{E}\left[\left(\phi_v^N(\lambda_i) - \phi_v^N(B)\right)\left(\phi_w^N(\lambda_i) - \phi_w^N(B) + \phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)\right] && \text{(B.8)} \\
&\quad - \mathbb{E}\left[\left(\phi_v^N(\lambda_{i-1}) - \phi_v^N(\tilde{B})\right)\left(\phi_w^N(\lambda_i) - \phi_w^N(B) + \phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)\right]. && \text{(B.9)}
\end{aligned}
$$

Using the conditional covariance identity

$$
\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{A}\right]\mathbb{E}\left[Y|\mathcal{A}\right]\right] = \mathbb{E}\left[XY\right] - \mathbb{E}\left[\left(X - \mathbb{E}\left[X|\mathcal{A}\right]\right)\left(Y - \mathbb{E}\left[Y|\mathcal{A}\right]\right)\right], \tag{B.10}
$$

with $X = \phi_v^N(1) - \phi_v^N(\tilde{B})$, $Y = \phi_w^N(1) - \phi_w^N(\tilde{B})$ and $\mathcal{A} = \sigma\left(\phi_u^N : u \notin B^o\right)$, along with noting that by the Gibbs-Markov property of the DGFF $\phi_v^N(1) - \phi_v^N(\tilde{B}) \overset{d}{=} \phi_v^{\tilde{B}}$, we can write the first term (B.6) as

$$
\begin{aligned}
\mathbb{E}\left[\phi_v^B\phi_w^B\right] - \mathbb{E}\left[\phi_v^{\tilde{B}}\phi_w^{\tilde{B}}\right] &= \log\left(N^{1-\lambda_i+\log(2)/\log(N)}\right) - \log(\|v-w\| \vee 1) - \log\left(N^{1-\lambda_i-\log(2)/\log(N)}\right) \\
&\quad + \log(\|v-w\| \vee 1) + O(1) = \Delta\lambda_i\log(N) + O(1). && \text{(B.11)}
\end{aligned}
$$

For the remaining terms we need to show that they are at most of constant order. As the last two terms (B.8) and (B.9) can be estimated the same way, we only deal with (B.8). Using Cauchy-Schwarz,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\phi_v^N(\lambda_i) - \phi_v^N(B)\right)\left(\phi_w^N(\lambda_i) - \phi_w^N(B) - \phi_w^N(\lambda_{i-1}) + \phi_w^N(\tilde{B})\right)\right] \\
\leq&\mathbb{E}\left[\left(\phi_v^N(\lambda_i) - \phi_v^N(B)\right)^2\right]^{1/2}\left(\mathbb{E}\left[\left(\phi_w^N(\lambda_i) - \phi_w^N(B)\right)^2\right]^{1/2} + \mathbb{E}\left[\left(\phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)^2\right]^{1/2}\right) \\
=&(\log(2) + c_1)(\log(2) + c_2 + \log(2) + c_3) = O(1). && \text{(B.12)}
\end{aligned}
$$

To estimate (B.7) we make exhaustive use of our choice of boxes and use the relations (B.4) and (B.5) along with the tower property for conditional expectations and the law of total expectation, i.e. we first observe that both $\mathbb{E}\left[\phi_v^N(B)\phi_w^N(\lambda_i)\right] = \mathbb{E}\left[\phi_v^N(B)\phi_w^N(B)\right]$ and $\mathbb{E}\left[\phi_v^N(\tilde{B})\phi_w^N(\lambda_i)\right] = \mathbb{E}\left[\phi_v^N(\tilde{B})\phi_w^N(\tilde{B})\right]$ hold. Using this, we reformulate (B.7), i.e.

$$
\begin{aligned}
&\mathbb{E}\left[\Delta\phi_v^N(B)\left(\phi_w^N(\lambda_i) - \phi_w^N(B) + \phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)\right] \\
=&\mathbb{E}\left[\phi_v^N(B)\left(\phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)\right] - \mathbb{E}\left[\phi_v^N(\tilde{B})\left(\phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)\right] \\
=&\mathbb{E}\left[\phi_v^N\left(\phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)\right] - \mathbb{E}\left[\phi_v^N\left(\phi_w^N(\tilde{B}) - \phi_w^N(\lambda_{i-1})\right)\right] = 0. && \text{(B.13)}
\end{aligned}
$$

For the remaining case $i \neq j$, we note that for $|i-j| \geq 2$ increments are independent as the difference of the boxes do not intersect for any $v, w \in V_N$, as we assume $N$ to be sufficiently large. The only remaining case is $j = i-1$. Note that in this case, the increment $\Delta\phi_v^N(\lambda_i)$ is independent of the increment $\phi_w^N(\lambda_{i-1} - \frac{\log(4)}{\log(N)}) - \phi_w^N(\lambda_{i-2})$, as the annuli of the corresponding boxes do not intersect. This gives,

$$
\begin{aligned}
\mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\Delta\phi_w^N(\lambda_{i-1})\right] =&\mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\left(\phi_w^N(\lambda_{i-1}) - \phi_w^N\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right) + \phi_w^N\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right) - \phi_w^N(\lambda_{i-2})\right)\right] \\
=&\mathbb{E}\left[\Delta\phi_v^N(\lambda_i)\left(\phi_w^N(\lambda_{i-1}) - \phi_w^N\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right)\right)\right] \\
=&\mathbb{E}\Bigg[\left(\phi_v^N(\lambda_i) - \phi_v^N([w]_{\lambda_i}) + \phi_v^N([w]_{\lambda_i}) - \phi_v^N([w]_{\lambda_{i-1}}) + \phi_v^N([w]_{\lambda_{i-1}}) - \phi_v^N(\lambda_{i-1})\right) \\
&\times\left(\phi_w^N(\lambda_{i-1}) - \phi_w^N\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right)\right)\Bigg]. && \text{(B.14)}
\end{aligned}
$$

Provided $N$ is large, we have $[v]^c_{\lambda_i} \cup [w]^c_{\lambda_i} \supset [w]^c_{\lambda_{i-1}} \supset [w]^c_{\lambda_{i-1}-\frac{\log(4)}{\log(N)}}$ and so by the tower property and the law of total expectation, we deduce

$$\begin{aligned}
&\mathbb{E}\left[\left(\phi^N_v(\lambda_i) - \phi^N_v([w]_{\lambda_i})\right)\left(\phi^N_w(\lambda_{i-1}) - \phi^N_w\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right)\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\phi^N_v\left(\phi^N_w(\lambda_{i-1}) - \phi^N_w\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right)\right)\Big|\sigma\left(\phi^N_u : u \in [v]^c_{\lambda_i}\right)\right]\right] \\
&\quad - \mathbb{E}\left[\mathbb{E}\left[\phi^N_v\left(\phi^N_w(\lambda_{i-1}) - \phi^N_w\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right)\right)\Big|\sigma\left(\phi^N_u : u \in [w]^c_{\lambda_i}\right)\right]\right] = 0. \qquad \text{(B.15)}
\end{aligned}$$

As the annuli $[w]_{\lambda_{i-1}} \backslash [w]_{\lambda_i}$ and $[w]_{\lambda_{i-1}-\frac{\log(4)}{\log(N)}} \backslash [w]_{\lambda_{i-1}}$ do not intersect, we have independence of the corresponding increments, i.e.

$$\mathbb{E}\left[\left(\phi^N_v([w]_{\lambda_i}) - \phi^N_v([w]_{\lambda_{i-1}})\right)\left(\phi^N_w(\lambda_{i-1}) - \phi^N_w\left(\lambda_{i-1} - \frac{\log(4)}{\log(N)}\right)\right)\right] = 0. \qquad \text{(B.16)}$$

The remaining term in (B.14) can be bounded in a first step by the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}\left[\left(\phi^N_v([w]_{\lambda_{i-1}}) - \phi^N_v(\lambda_{i-1})\right)\left(\phi^N_w(\lambda_{i-1}) - \phi^N_w\left(\lambda_{i-1} - \log(4)/\log(N)\right)\right)\right] \\
&\leq c\sqrt{\log(4)}\mathbb{E}\left[\left(\phi^N_v([w]_{\lambda_{i-1}}) - \phi^N_v(\lambda_{i-1})\right)^2\right]^{1/2}. \qquad \text{(B.17)}
\end{aligned}$$

In order to bound the expectation on the right hand side, we consider a box $B$ centred at the middle of the line connecting $v$ and $w$ of side length $N^{1-\lambda_{i-1}} - \sqrt{2}N^{1-\lambda_i}$. The assumption $\|v - w\|_\infty \leq \sqrt{2}N^{1-\lambda_i}$ ensures the inclusion $B \subset [v]_{\lambda_{i-1}} \cap [w]_{\lambda_{i-1}}$. This allows us to compute in a similar fashion as in the first case (B.6), i.e.

$$\begin{aligned}
\mathbb{E}\left[\left(\phi^N_v([w]_{\lambda_{i-1}}) - \phi^N_v(\lambda_{i-1})\right)^2\right] &= \mathbb{E}\left[\left(\phi^N_v([w]_{\lambda_{i-1}}) - \phi^N_v(B) + \phi^N_v(B) - \phi^N_v(\lambda_{i-1})\right)^2\right] \\
&\leq 4\max\left(\mathbb{E}\left[\left(\phi^N_v([w]_{\lambda_{i-1}}) - \phi^N_v(B)\right)^2\right], \mathbb{E}\left[\left(\phi^N_v(B) - \phi^N_v(\lambda_{i-1})\right)^2\right]\right) \\
&\leq 4(c + \log(N^{1-\lambda_{i-1}}) - \log(N^{1-\lambda_{i-1}}(1 - \sqrt{2}N^{-\Delta\lambda_i}))) \leq C. \qquad \text{(B.18)}
\end{aligned}$$

The constants $c, C > 0$ can be chosen uniformly in $N$, however depending on the scale parameters. Altogether, we obtain

$$\mathbb{E}\left[\Delta\phi^N_v(\lambda_i)\Delta\phi^N_w(\lambda_j)\right] \leq C, \qquad \text{(B.19)}$$

for some constant $C > 0$ that is uniform in $N$, which finishes the proof. □

*Proof of Lemma 3.3.* For a proof of the statements *i.* and *iii.*, we refer to [20, Lemma 2.2]. We have that $\log_+(d^N_\infty(v,w)) \leq \log_+(d^N(v,w)) \leq \log_+(d^N_\infty(v,w)) + 1$. We begin with the proof of the second statement. Note that if $1 \leq k < \log_+(d^N_\infty(v,w) + 1)$, there are no boxes of size $2^k$ that cover both $v$ and $w$. Thus, if $B, \tilde{B}$ are boxes such that one covers $v$ but not $w$ and the other $w$ but not $v$, the associated random variables $b_{k,B}, b_{k,\tilde{B}}$ are independent. And so, only random variables $b_{k,B}$ associated to boxes of size $2^k$ with $k > \left\lceil \log_2(d^N_\infty(v,w) + 1)\right\rceil$ contribute to the covariance. For $v = (v_1, v_2)$, $w = (w_1, w_2)$ and $i = 1, 2$, we write $r_i(v,w) = \min(|v_i - w_i|, |v_i - w_i - N|, |v_i - w_i + N|)$. Using the fact that the number of common boxes for $v, w \in V_N$ is given by $[2^k - r_1(v,w)][2^k - r_2(v,w)]$,

$$\begin{aligned}
\mathbb{E}\left[S^N_v S^N_w\right] &= \sum_{k=\lceil \log_+(d^N_\infty(v,w))\rceil}^{n} 2^{-2k}\sigma^2\left(\frac{n-k}{n}\right)[2^k - r_1(v,w)][2^k - r_2(v,w)] \qquad \text{(B.20)} \\
&= \sum_{k=\lceil \log_+(d^N_\infty(v,w))\rceil}^{n} \left[\left(1 - \frac{r_1(v,w)}{2^k} - \frac{r_2(v,w)}{2^k} + \frac{r_1(v,w)r_2(v,w)}{2^{2k}}\right)\left(\sum_{i=1}^{M} \mathbb{1}_{n-k\in(\lambda_{i-1}n,\lambda_i n]}\sigma_i^2\right)\right].
\end{aligned}$$

We note that since $a+b-ab\geq 0$ for $0\leq a,b\leq 1$, we get

$$\begin{aligned}
\mathbb{E}\left[S_v^N S_w^N\right] &\leq n\sum_{i=1}^M \sigma_i^2\Delta\lambda_i - \sum_{i=1}^M \sigma_i^2[n\Delta\lambda_i \mathbb{1}_{n-\lceil \log_+(d_\infty^N(v,w))\rceil\leq \lambda_i n}\\
&\quad + \left[\lambda_i n - \left(n-\left\lceil \log_+(d_\infty^N(v,w))\right\rceil\right)\right]\mathbb{1}_{\lambda_{i-1}n<n-\lceil \log_+(d_\infty^N(v,w))\rceil<\lambda_i n}]\\
&= 2\sum_{i=1}^M \sigma_i^2 + \sum_{i=1}^M \sigma_i^2[n\Delta\lambda_i \mathbb{1}_{n-\lceil \log_+(d^N(v,w))\rceil\geq \lambda_i n}\\
&\quad + \left((1-\lambda_{i-1})n\left\lceil \log_+(d^N(v,w))\right\rceil\right)\mathbb{1}_{\lambda_{i-1}n<n-\lceil \log_+(d^N(v,w))\rceil<\lambda_i n}]\\
&= 2\mathcal{I}_{\sigma^2}(1) + n\mathcal{I}_{\sigma^2}\left(\frac{n-\left\lceil \log_+\left(d^N(v,w)\right)\right\rceil}{n}\right).
\end{aligned}\tag{B.21}$$

On the other hand, since $a+b-ab\leq a+b$ for $a,b\geq 0$, we get

$$\begin{aligned}
\mathbb{E}\left[S_v^N S_w^N\right] &\geq \sum_{k=\lceil \log_+(d_\infty^N(v,w))\rceil}^{n} \sigma^2\left(\frac{n-k}{n}\right) - \max_{1\leq i\leq M}\sigma^2\left(\frac{i}{n}\right)2^{-k+1}d_\infty^N(v,w)\\
&\geq \sum_{i=1}^M \sigma_i^2\left[n\Delta\lambda_i \mathbb{1}_{n-\lceil \log_+(d^N(v,w))\rceil\geq\lambda_i n} + ((1-\lambda_{i-1})n\right.\\
&\quad \left.-\left\lceil \log_+(d^N(v,w))\right\rceil\right)\mathbb{1}_{\lambda_{i-1}n<n-\lceil \log_+(d^N(v,w))\rceil<\lambda_i n}\Big] - C\\
&= n\mathcal{I}_{\sigma^2}\left(\frac{n-\left\lceil \log_+\left(d^N(v,w)\right)\right\rceil}{n}\right) - C,
\end{aligned}\tag{B.22}$$

where in the second step we did a rescaling from $[0,n]$ onto the unit interval $[0,1]$ and where $C>0$ is a constant independent of $N$ with $C>2\max_{1\leq i\leq M}\sigma^2(i/M)$ that deals with the second part of the sum. To prove the last statement *iv.*, we note that beyond the branching scale, $N$ being sufficiently large (see assumptions of Lemma B.1) and by the Gibbs-Markov property, increments are independent as the annuli of the corresponding boxes do not intersect (see for instance [7, Section 2]). Moreover, by a refinement of the scale parameters and possibly allowing for an additional uniformly bounded constant, we can assume that the branching scale coincides with a scale parameter. With this we can apply Lemma B.1 and obtain the result, i.e.

$$\begin{aligned}
\mathbb{E}\left[\psi_x^{4N}\psi_y^{4N}\right] &= \mathbb{E}\left[\sum_{i=1}^M\sum_{j=1}^M \sigma_i\sigma_j\Delta\phi_x^{4N}(\lambda_i)\Delta\phi_y^{4N}(\lambda_j)\right] = \sum_{i=1}^M \sigma_i^2\mathbb{E}\left[(\Delta\phi_x^{4N}(\lambda_i))^2\mathbb{1}_{n-\lceil \log_+(\|v-w\|_2)\rceil\geq\lambda_i}\right.\\
&\quad \left.+\left(\phi_x^{4N}\left(\frac{n-\lceil \log_+(\|x-y\|_2)\rceil}{n}\right)-\phi_x^{4N}(\lambda_{i-1})\right)\mathbb{1}_{\lambda_{i-1}n<n-\lceil \log_+(\|x-y\|_2)\rceil<\lambda_i n}\right] + O(1)\\
&= \log(2)\sum_{i=1}^M \sigma_i^2\left[n\Delta\lambda_i\mathbb{1}_{n-\lceil \log_+\|x-y\|_2\rceil\geq\lambda_i n} + ((1-\lambda_{i-1})n\right.\\
&\quad \left.-\lceil \log_+\|x-y\|_2\rceil\mathbb{1}_{\lambda_{i-1}n<n-\lceil \log_+\|x-y\|_2\rceil<\lambda_i n})\right] + O(1)\\
&= \log(2)n\mathcal{I}_{\sigma^2}\left(\frac{n-\lceil \log_+\|x-y\|_2\rceil}{n}\right) + O(1),
\end{aligned}\tag{B.23}$$

where $O(1)$ is uniform in N. □

## References


[1] R. J. Adler. *An introduction to continuity, extrema, and related topics for general Gaussian processes*. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 12. Institute of Mathematical Statistics, Hayward, CA, 1990.

[2] E. Aïdékon. Convergence in law of the minimum of a branching random walk. *Ann. Probab.*, 41(3A):1362–1426, 2013.

[3] E. Aïdékon, J. Berestycki, E. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Related Fields*, 157(1-2):405–451, 2013.

[4] L.-P. Arguin, D. Belius, and P. Bourgade. Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.*, 349(2):703–751, 2017.

[5] L.-P. Arguin, D. Belius, P. Bourgade, M. Radziwił ł, and K. Soundararajan. Maximum of the Riemann zeta function on a short interval of the critical line. *Comm. Pure Appl. Math.*, 72(3):500–535, 2019.

[6] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Related Fields*, 157(3-4):535–574, 2013.

[7] L.-P. Arguin and F. Ouimet. Extremes of the two-dimensional Gaussian free field with scale-dependent variance. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13(2):779–808, 2016.

[8] M. Biskup. Extrema of the two-dimensional discrete Gaussian free field. In *Random graphs, phase transitions, and the Gaussian free field*, volume 304 of *Springer Proc. Math. Stat.*, pages 163–407. Springer, Cham, 2020.

[9] M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *Comm. Math. Phys.*, 345(1):271–304, 2016.

[10] M. Biskup and O. Louidor. Full extremal process, cluster law and freezing for the two-dimensional discrete Gaussian free field. *Adv. Math.*, 330:589–687, 2018.

[11] E. Bolthausen, J.-D. Deuschel, and G. Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.*, 29(4):1670–1692, 2001.

[12] E. Bolthausen, J.-D. Deuschel, and O. Zeitouni. Recursions and tightness for the maximum of the discrete, two dimensional Gaussian free field. *Electron. Commun. Probab.*, 16:114–119, 2011.

[13] A. Bovier and L. Hartung. The extremal process of two-speed branching Brownian motion. *Electron. J. Probab.*, 19:no. 18, 28, 2014.

[14] A. Bovier and L. Hartung. Variable speed branching Brownian motion 1. Extremal processes in the weak correlation regime. *ALEA Lat. Am. J. Probab. Math. Stat.*, 12(1):261–291, 2015.

[15] A. Bovier and L. Hartung. Extended convergence of the extremal process of branching Brownian motion. *Ann. Appl. Probab.*, 27(3):1756–1777, 2017.

[16] A. Bovier and L. Hartung. From 1 to 6: A finer analysis of perturbed branching brownian motion. *Communications on Pure and Applied Mathematics*, 73(7):1490–1525, 2020.

[17] A. Bovier and I. Kurkova. Derrida's generalised random energy models. I. Models with finitely many hierarchies. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(4):439–480, 2004.

[18] A. Bovier and I. Kurkova. Derrida's generalized random energy models. II. Models with continuous hierarchies. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(4):481–495, 2004.

[19] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 69(1):62–123, 2016.

[20] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 65(1):1–20, 2012.

[21] M. D. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.

[22] O. Daviaud. Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.*, 34(3):962–986, 2006.

[23] B. Derrida and H. Spohn. Polymers on disordered trees, spin glasses, and traveling waves. volume 51, pages 817–840. 1988. New directions in statistical mechanics (Santa Barbara, CA, 1987).

[24] J. Ding. Exponential and double exponential tails for maximum of two-dimensional discrete Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):285–299, 2013.

[25] J. Ding, R. Roy, and O. Zeitouni. Convergence of the centered maximum of log-correlated Gaussian fields. *Ann. Probab.*, 45(6A):3886–3928, 2017.

[26] J. Ding and O. Zeitouni. Extreme values for two-dimensional discrete Gaussian free field. *Ann. Probab.*, 42(4):1480–1515, 2014.

[27] M. Fang and O. Zeitouni. Branching random walks in time inhomogeneous environments. *Electron. J. Probab.*, 17:no. 67, 18, 2012.

[28] M. Fang and O. Zeitouni. Slowdown for time inhomogeneous branching Brownian motion. *J. Stat. Phys.*, 149(1):1–9, 2012.

[29] M. Fels and L. Hartung. Extremes of the 2d scale-inhomogeneous discrete gaussian free field: Convergence of the maximum in the regime of weak correlations. *arXiv:1912.13184*, 2019.

[30] M. Fels and L. Hartung. Extremes of the 2d scale-inhomogeneous discrete gaussian free field: Extremal process in the weakly correlated regime. *arXiv:2002.00925*, 2020.
[31] X. Fernique. Regularité des trajectoires des fonctions aléatoires gaussiennes. In *École d'Été de Probabilités de Saint-Flour, IV-1974*, pages 1–96. Lecture Notes in Math., Vol. 480. Springer, Berlin, 1975.
[32] P. L. Ferrari and H. Spohn. Constrained Brownian motion: fluctuations away from circular and parabolic barriers. *Ann. Probab.*, 33(4):1302–1325, 2005.
[33] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, 22(2):89–103, 1971.
[34] Y. Fyodorov, G. Hiary, and J. Keating. Freezing transition, characteristic polynomials of random matrices, and the riemann zeta function. *Physical review letters*, 108:170601, 04 2012.
[35] E. Gardner and B. Derrida. Magnetic properties and function $q(x)$ of the generalized random energy model. *J. Phys. C*, 19:5783–5798, 1986.
[36] E. Gardner and B. Derrida. Solution of the generalized random energy model. *J. Phys. C*, 19:2253–2274, 1986.
[37] V. Gayrard and N. Kistler, editors. *Correlated random systems: five different methods*, volume 2143 of *Lecture Notes in Mathematics*. Springer, Cham; Société Mathématique de France, Paris, 2015. Lecture notes from the 1st CIRM Jean-Morlet Chair held in Marseille, Spring 2013, CIRM Jean-Morlet Series.
[38] R. D. Gordon. Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. *Ann. Math. Statist.*, 12(3):364–366, 09 1941.
[39] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.
[40] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15(3):1052–1061, 1987.
[41] G. F. Lawler. *Random walk and the heat equation*, volume 55 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2010.
[42] G. F. Lawler. *Intersections of random walks*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2013. Reprint of the 1996 edition.
[43] G. F. Lawler and V. Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.
[44] M. Ledoux and M. Talagrand. *Probability in Banach spaces*. Classics in Mathematics. Springer-Verlag, Berlin, 2011. Isoperimetry and processes, Reprint of the 1991 edition.
[45] T. Madaule. Convergence in law for the branching random walk seen from its tip. *J. Theoret. Probab.*, 30(1):27–63, 2017.
[46] P. Maillard and O. Zeitouni. Slowdown in branching Brownian motion with inhomogeneous variance. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(3):1144–1160, 2016.
[47] B. Mallein. Maximal displacement in a branching random walk through interfaces. *Electron. J. Probab.*, 20:no. 68, 40, 2015.
[48] B. Mallein. Maximal displacement of a branching random walk in time-inhomogeneous environment. *Stochastic Process. Appl.*, 125(10):3958–4019, 2015.
[49] B. Mallein. Genealogy of the extremal process of the branching random walk. *ALEA Lat. Am. J. Probab. Math. Stat.*, 15(2):1065–1087, 2018.
[50] F. Ouimet. Geometry of the Gibbs measure for the discrete 2D Gaussian free field with scale-dependent variance. *ALEA Lat. Am. J. Probab. Math. Stat.*, 14(2):851–902, 2017.
[51] F. Ouimet. Maxima of branching random walks with piecewise constant variance. *Braz. J. Probab. Stat.*, 32(4):679–706, 2018.
[52] L. D. Pitt. Positively correlated normal variables are associated. *Ann. Probab.*, 10(2):496–499, 1982.
[53] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and Liouville quantum gravity. In *Stochastic processes and random matrices*, pages 548–577. Oxford Univ. Press, Oxford, 2017.
[54] S. Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007.
[55] R. K. Sundaram. *A first course in optimization theory*. Cambridge University Press, Cambridge, 1996.
[56] D. Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, extended edition, 2000.
[57] O. Zeitouni. Branching random walks and Gaussian fields. In *Probability and statistical physics in St. Petersburg*, volume 91 of *Proc. Sympos. Pure Math.*, pages 437–471. Amer. Math. Soc., Providence, RI, 2016.

M. Fels, Institut für Angewandte Mathematik, Rheinische Friedrich-Wilhelms-Universität, Endenicher Allee 60, 53115 Bonn, Germany
*Email address*: `fels@iam.uni-bonn.de, wt.iam.uni-bonn.de/maximilian-fels`

CHAPTER 3

# Convergence of the maximum of the scale-inhomogeneous two-dimensional discrete Gaussian Free Field in the weakly correlated regime

# EXTREMES OF THE 2D SCALE-INHOMOGENEOUS DISCRETE GAUSSIAN FREE FIELD: CONVERGENCE OF THE MAXIMUM IN THE REGIME OF WEAK CORRELATIONS

MAXIMILIAN FELS, LISA HARTUNG

Abstract. We continue the study of the maximum of the scale-inhomogeneous discrete Gaussian free field in dimension two that was initiated in [36] and continued in [37]. In this paper, we consider the regime of weak correlations and prove the convergence in law of the centred maximum to a randomly shifted Gumbel distribution. In particular, we obtain limiting expressions for the random shift. As in the case of variable speed branching Brownian motion, the shift is of the form $CY$, where $C$ is a constant that depends only on the variance at the shortest scales, and $Y$ is a random variable that depends only on the variance at the largest scales. Moreover, we investigate the geometry of highest local maxima. We show that they occur in clusters of finite size that are separated by macroscopic distances. The poofs are based on Gaussian comparison with branching random walks and second moment estimates.

## 1. Introduction

In recent years, log-correlated (Gaussian) processes have received considerable attention, see e.g. [3, 4, 15, 19, 32, 41, 55]. Some prominent examples that fall into this class are branching Brownian motion (BBM), the branching random walk (BRW), the 2d discrete Gaussian free field (DGFF), local maxima of the randomised Riemann zeta function on the critical line and cover times of Brownian motion on the torus. One of the reasons why these processes are interesting is that their correlation structure is such that it becomes relevant for the properties of the extremes of the processes. The 2d scale-inhomogeneous discrete Gaussian free field first appeared in [10], where it served as a tool in order to prove Poisson-Dirichlet statistics of the extreme values of the 2d DGFF. Moreover, it is the natural analogue model of variable-speed BBM or the time-inhomogeneous BRW in the context of the two-dimensional DGFF. To be more precise, we start with a formal definition of the model studied in this paper and then, present our new results on the maximum value.

1.1. **The discrete Gaussian free field.** Let $V_N := ([0, N) \cap \mathbb{Z})^2$. The interior of $V_N$ is defined as $V_N^o := ([1, N-1] \cap \mathbb{Z})^2$ and the boundary of $V_N$ is denoted by $\partial V_N := V_N \setminus V_N^o$. Moreover, for points $u, v \in V_N$ we write $u \sim v$, if and only if $\|u - v\|_2 = 1$, where $\|.\|_2$ is the Euclidean norm. Let $\mathbb{P}_u$ be the law of a SRW $\{W_k\}_{k\in\mathbb{N}}$ starting at $u \in \mathbb{Z}^2$. The normalised Green kernel is given by

$$G_{V_N}(u, v) := \frac{\pi}{2}\mathbb{E}_u\left[\sum_{i=0}^{\tau_{\partial V_N}-1} \mathbb{1}_{\{W_i=v\}}\right], \text{ for } u, v \in V_N. \tag{1.1}$$

Here, $\tau_{\partial V_N}$ is the first hitting time of the boundary $\partial V_N$ by $\{W_k\}_{k\in\mathbb{N}}$. For $\delta > 0$, we set $V_N^\delta := (\delta N, (1-\delta)N)^2 \cap \mathbb{Z}^2$. By [29, Lemma 2.1], we have, for $\delta \in (0, 1)$ and $u, v \in V_N^\delta$,

$$G_{V_N}(u, v) = \log N - \log_+ \|u - v\|_2 + O(1), \tag{1.2}$$

where $\log_+(x) = \max\{0, \log(x)\}$.

M.F. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - project-id 211504053 - SFB 1060 and Germany's Excellence Strategy – GZ 2047/1, project-id 390685813 – "Hausdorff Center for Mathematics" at Bonn University.
Keywords: extreme value theory, Gaussian free field, inhomogeneous environment, branching Brownian motion, branching random walk.

**Definition 1.1.** The 2d discrete Gaussian free field (DGFF) on $V_N$, $\phi^N := \{\phi_v^N\}_{v\in V_N}$, is a centred Gaussian field with covariance matrix $G_{V_N}$ and entries $G_{V_N}(x,y) = \mathbb{E}[\phi_x^N \phi_y^N]$, for $x, y \in V_N$.

From Definition 1.1 it follows that $\phi_v^N = 0$ for $v \in \partial V_N$, i.e. we have Dirichlet boundary conditions.

### 1.2. The two-dimensional scale-inhomogeneous discrete Gaussian free field.

**Definition 1.2.** (The 2d scale-inhomogeneous discrete Gaussian free field).
Let $\phi^N = \{\phi_v^N\}_{v\in V_N}$ be a 2d DGFF on $V_N$. For $v = (v_1, v_2) \in V_N$ and $\lambda \in (0,1)$, let

$$[v]_\lambda \equiv [v]_\lambda^N := \left(\left[v_1 - \frac{1}{2}N^{1-\lambda}, v_1 + \frac{1}{2}N^{1-\lambda}\right] \times \left[v_2 - \frac{1}{2}N^{1-\lambda}, v_2 + \frac{1}{2}N^{1-\lambda}\right]\right) \cap V_N, \tag{1.3}$$

and set $[v]_0^N := V_N$ and $[v]_1^N := \{v\}$. We denote by $[v]_\lambda^o$ the interior of $[v]_\lambda$. Let $\mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c} := \sigma\left(\{\phi_v^N, v \notin [v]_\lambda^o\}\right)$ be the $\sigma-$algebra generated by the random variables outside $[v]_\lambda^o$. For $v \in V_N$, let

$$\phi_v^N(\lambda) = \mathbb{E}\left[\phi_v^N | \mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c}\right], \quad \lambda \in [0,1]. \tag{1.4}$$

We denote by $\nabla\phi_v^N(\lambda)$ the derivative $\partial_\lambda \phi_v^N(\lambda)$ of the DGFF at vertex $v$ and scale $\lambda$. Moreover, let $s \mapsto \sigma(s)$ be a non-negative function such that $\mathcal{I}_{\sigma^2}(\lambda) := \int_0^\lambda \sigma^2(x)\mathrm{d}x$ is a function on $[0,1]$ with $\mathcal{I}_{\sigma^2}(0) = 1$ and $\mathcal{I}_{\sigma^2}(1) = 1$. Then the 2d scale-inhomogeneous DGFF on $V_N$ is a centred Gaussian field $\psi^N := \{\psi_v^N\}_{v\in V_N}$ defined as

$$\psi_v^N := \int_0^1 \sigma(s)\nabla\phi_v^N(s)\mathrm{d}s. \tag{1.5}$$

In the case when $\sigma$ is a right-continuous step function taking finitely many values, [36, (1.11)] shows that it is a centred Gaussian field with covariance given by

$$\mathbb{E}\left[\psi_v^N \psi_w^N\right] = \log N \mathcal{I}_{\sigma^2}\left(\frac{\log N - \log_+ \|v-w\|_2}{\log N}\right) + O(\sqrt{\log(N)}), \quad \text{for } v, w \in V_N^\delta. \tag{1.6}$$

## 2. Main results

In the case of finitely many scales, Arguin and Ouimet [9] showed the first order of the maximum and the size of the level sets.

**Assumption 1.** *In the rest of the paper, $\{\psi_v^N\}_{v\in V_N}$ is always a 2d scale-inhomogeneous DGFF on $V_N$. Moreover, we assume that $\mathcal{I}_{\sigma^2}(x) < x$, for $x \in (0,1)$, and that $\mathcal{I}_{\sigma^2}(1) = 1$, with $s \mapsto \sigma(s)$ being differentiable at 0 and 1, such that $\sigma(0) < 1$ and $\sigma(1) > 1$.*

In [36], we proved, in the case when $s \mapsto \mathcal{I}_{\sigma^2}$ is piecewise linear, that the maximum centred by $m_N$ has exponential tails. In particular, in the case of the right-tail, our results are precise up to a multiplicative constant. As a simple consequence we obtained the sub-leading logarithmic correction to the maximum value . Provided Assumption 1, there are right-continuous, non-negative step functions, $s \mapsto \sigma_1(s)$, $s \mapsto \sigma_2(s)$, taking finitely many values, such that, for $x \in (0,1)$,

$$\mathcal{I}_{\sigma_1^2}(x) \leq \mathcal{I}_{\sigma^2}(x) \leq \mathcal{I}_{\sigma_2^2}(x) < x, \tag{2.1}$$

and such that $\mathcal{I}_{\sigma_1^2}(1) = \mathcal{I}_{\sigma_2^2}(1) = 1$. [36] shows that for scale-inhomogeneous DGFFs with parameters $\sigma_1$ or $\sigma_2$, the maximum value is given by $2\log N - \frac{1}{4}\log\log N + O_P(1)$, where $O_P(1)$ means that remainder is stochastically bounded and that the centred maxima are tight. (2.1), Sudakov-Fernique and [36] imply that the maximum value under Assumption 1 is given by

$$\psi_N^* := \max_{v\in V_N} \psi_v^N = 2\log N - \frac{1}{4}\log\log N + O_P(1). \tag{2.2}$$

In particular, the maximum, $\psi_N^*$, centred by $m_N := 2\log N - \frac{\log\log N}{4}$ is tight. Our main result in this paper is convergence in distribution of the centred maximum.

**Theorem 2.1.** *Let $\{\psi_v^N\}_{v\in V_N}$ satisfy Assumption 1. Then, the sequence $\left\{\psi_N^* - m_N\right\}_{N\geq 0}$ converges in distribution. In particular, there is a constant $\beta(\sigma(1)) > 0$ depending only on the final variance, and a random variable $Y(\sigma(0))$ which is almost surely non-negative, finite and depends only on the initial variance, such that, for any $z \in \mathbb{R}$,*

$$\mathbb{P}\left(\psi_N^* - m_N \leq +z\right) \overset{N\to\infty}{\to} \mathbb{E}\left[\exp\left[-\beta(\sigma(1))Y(\sigma(0))e^{-2z}\right]\right], \quad \text{as } N \to \infty. \tag{2.3}$$

Note that the limiting law is universal in the sense that only $\sigma(0)$ and $\sigma(1)$ affect the limiting law. In particular, the choice of $\sigma(s)$, for $s \in (0,1)$, does not affect the law, as long as $\mathcal{I}_{\sigma^2}(x) < x$, for $x \in (0,1)$. In the proof of Theorem 2.1 one needs to understand the genealogy of particles close to the maximum. Since this is of independent interest, we state it as a separate theorem.

**Theorem 2.2.** *Let $\{\psi_v^N\}_{v\in V_N}$ satisfy Assumption 1. Then, there exists a constant $c > 0$, such that*

$$\lim_{r\to\infty}\lim_{N\to\infty} \mathbb{P}\left(\exists u, v \in V_N \text{ with } r \leq \|u-v\|_2 \leq \frac{N}{r} \text{ and } \psi_u^N, \psi_v^N \geq m_N - c\log\log r\right) = 0. \tag{2.4}$$

As the field is strongly correlated, Theorem 2.2 implies that local maxima of the scale-inhomogeneous DGFF are surrounded by very heigh points in $O(1)$ neighbourhoods. Moreover, the local maxima are at distance $O(N)$ to each other and therefore, almost independent.

2.1. **Related work.** The special case $\sigma(x) \equiv 1$, for $x \in [0,1]$, is the usual 2d DGFF. In this case, building upon work by Bolthausen, Bramson, Deuschel, Ding, Giacomin and Zeitouni [18, 26, 31, 33], Bramson, Ding and Zeitouni [25] proved convergence in law of the centred maximum. Generalizing this approach, Ding, Roy and Zeitouni [32] proved convergence of the centred maximum for more general log-correlated Gaussian fields. In the 2d DGFF, Biskup and Louidor [14, 15] proved convergence of the full extremal process to a cluster Cox process. Moreover, they derived several properties of the random intensity measure appearing in the Cox process, which they identified as the so-called critical Liouville quantum gravity measure.

Another closely related model is (variable-speed) branching Brownian motion (BBM). Variable-speed BBM, introduced by Derrida and Spohn [30], is the natural analogue model of the 2d scale-inhomogeneous DGFF in the context of BBM. In order to define the model, fix a time horizon $t > 0$, a super-critical (continuous time) Galton-Watson tree and a strictly increasing function $A : [0,1] \to [0,1]$, with $A(0) = 0$, $A(1) = 1$. For two leaves $v$ and $w$, we denote by $d(v,w)$ their overlap, which is the time of their most recent common ancestor. Variable-speed BBM in time $t$, is a centred Gaussian process, indexed by the leaves of a super-critical (continuous time) Galton-Watson tree, and covariance $tA(d(v,w)/t)$. BBM is the special case when $A(x) = x$, for $x \in [0,1]$. It coincides with the continuous random energy model (CREM) on the Galton-Watson tree [42, 43, 24]. The extremal process of BBM was investigated in [27, 48, 2, 5, 6, 8, 7, 21], and those of variable-speed BBM in [19, 20, 35, 54]. In the weakly correlated regime, i.e. when $A(x) < x$, for $x \in (0,1)$, $A'(0) < 1$ and $A'(1) > 1$, Bovier and Hartung [19, 20] proved convergence of the extremal process to a cluster Cox process. They identified the random intensity measure as the so-called "McKean-martingale" which differs from the random intensity measure, the "derivate-martingale", which appears in BBM. Works by Bovier and Kurkova [24] for general variance profiles show that in the context of GREM the first order of the maximum is determined by the concave hull of $A$. Building upon results obtained by Fang and Zeitouni [35], Maillard and Zeitouni [54] proved in the case variable-speed BBM with strictly decreasing speed, that the 2nd order correction is proportional to $t^{1/3}$. As also in the case of the 2d scale-inhomogeneous DGFF all variances profiles can be achieved, studying its extremes in the analogue setting of strictly decreasing speed would be of great interest.

2.2. **Outline of proof.** We start to explain the proof of Theorem 2.2 as these ideas are also used in the proof of Theorem 2.1. In order to prove Theorem 2.2, we have to show with high probability, that there cannot be two vertices in $V_N$ at "intermediate distance" to each other, i.e. in between $O(1)$ and $O(N)$, and both reaching an extremal height. We therefore study the sum of two vertices, under the additional restriction that their distance is "intermediate", i.e. such that $r \leq \|u - v\| \leq N/r$ with $r \ll N$. The idea here is, if both vertices reach extreme heights, their sum must exceed twice an extremal threshold. This reasoning works, since tightness of the centred maximum implies that there cannot be a vertex being considerably larger than the expected maximum. To analyse the maximum of the sum of particles of the scale-inhomogeneous DGFF, we prove a variant of Slepian's lemma which allows to compare this quantity with the maximum of the sum of particles of corresponding inhomogeneous branching random walks. We show that using a truncated second moment method.

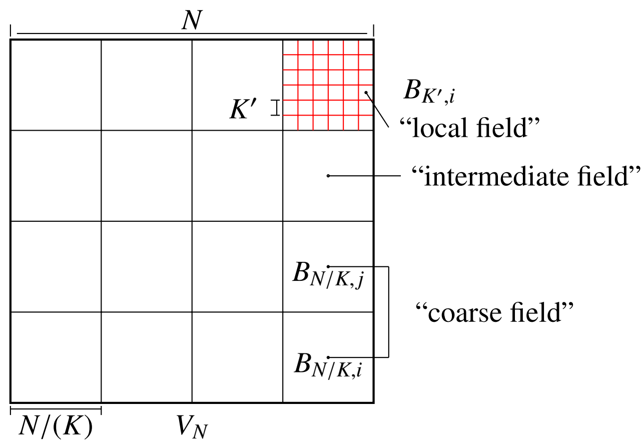



Figure 1: 3-field decomposition

Theorem 2.2 suggests that to understand the law of the centred maximum, it suffices to consider local maxima in "small" $O(1)$ neighbourhoods, while the "small" neighbourhoods are far, i.e. $O(N)$, apart. The fact that these neighbourhoods are very far apart, makes them correlated only on the level of the first increments, $\phi_v^N(\lambda_1) - \phi_v^N(0)$, for some $\lambda_1 > 1$, as boxes of side length $N^{1-\lambda_1}$ and centred at local maxima do not overlap. In particular, the remaining increments, $\phi_v^N(\lambda) - \phi_v^N(\mu)$, for $\lambda > \mu \geq \lambda_1$, for distinct such neighbourhoods are independent. We split these two different contributions by studying the sum of two independent Gaussian fields. To do so, decompose the box $V_N$ into $K^2$ boxes $B_{N/K,i}$ and $(N/K')^2$ boxes $B_{K',j}$ with side lengths $N/K$ and $K'$, where $K, K' \ll N$. One of the Gaussian fields is the "coarse field", which is defined such that it is constant in each box $B_{N/K,i}$. It encodes initial increments and correlations of the field between different boxes $B_{N/K,i}$. The other Gaussian field is the "fine field". It is independent between different boxes $B_{N/K,i}$, and encodes the remaining increments, including the local neighbourhoods. The "fine field" is then decomposed further into a field capturing the "intermediate" increments and an independent "local field", which captures the increments in the small neighbourhoods, $B_{K',j}$, that carry the local maxima. Instead of working directly with the scale-inhomogeneous 2d DGFF, we define a Gaussian field, $\{S_v^N\}_{v \in V_N}$, as a sum of four independent Gaussian fields, with covariance structure of the "coarse field", "local field", "intermediate field" and an additional independent Gaussian field. The additional field is defined such that variances of the scale-inhomogeneous DGFF and the approximating field match asymptotically, which is crucial in order to use Gaussian comparison to reduce the proof of Theorem 2.1 to show convergence of the centred maximum of the approximating process, $\{S_v^N\}_{v \in V_N}$. The "coarse and local field" are instances of appropriately scaled 2d DGFFs, the "intermediate field" is a collection of modified branching random walks (MIBRW). The advantage of working with the approximating process is that the "coarse field" is constant in large boxes, which substantially simplifies the analysis. To justify this approximation, it

is essential to control its covariance structure, and how it differs from that of the scale-inhomogeneous DGFF. In particular, one needs to understand the influence of this difference on the law of the centred maximum. This is done similarly as in [32], adapting an idea from [14], to show a certain invariance principle: Partition $V_N$ into sub-boxes $V_L$, where $L$ can be either of order $K$ or $N/K$, with $K \ll N$. If one adds i.i.d. Gaussians of bounded variance to each sub-box $V_L$, i.e. the same random variable to each vertex in a sub-box, then the law of the centred maximum is given by a deterministic shift of the original law. Moreover, the shift can be stated explicitly. This is the contents of Lemma 5.5 and its proof uses Theorem 2.2 and Gaussian comparison.
Another key step in the proof of convergence in law of the centred maximum of the approximating process, $\{S_v^N\}_{v\in V_N}$, is to understand the correct right-tail asymptotics of the (auxiliary) process. This is provided in Proposition 5.8, which is proved using a truncated second moment method. The truncation uses a localizing property of vertices reaching extreme heights, similar to the one observed in variable speed BBM. The idea is that intermediate increments of extremal vertices have to stay far below the maximum possible increment. For vertices to become very heigh at the end, this is then compensated by extraordinarily huge final increments. Based on a localization of increments of the auxiliary process for vertices that are local extremes (cp. Proposition 4.2), one is able to define random variables with the correct tails and distributions, whose parameters are determined through those of the "coarse and local field", and therefore independent of $N$. This is done in (5.44), (5.45) and (5.46). These are then coupled to the auxiliary process and allow to obtain convergence in law of the centred maximum, and further, for an explicit description of the limit distribution.

*Outline of the paper:* In Section 3 we recall the definition of the corresponding inhomogeneous branching random walk (IBRW) and the modified inhomogeneous branching random walk (MIBRW), introduced in [36], and state covariance estimates. The proof of Theorem 2.2 is provided in Section 4 and the proof of Theorem 2.1 in Section 5. In Appendix A we state Gaussian comparison tools such as Slepian's lemma, the inequality of Sudakov-Fernique and provide proofs of the additional covariance estimates. Lemma 5.5 and Lemma 5.6 are proved in Appendix B, and the proof of the right-tail asymptotics, i.e. Proposition 5.8, is provided in Appendix C.

Acknowledgements

The authors want to thank Anton Bovier for his careful reading and for his valuable comments that led to improvements in the presentation of this paper. We would also like to thank both authors' home institutions for their hospitality.

## 3. Frequently occurring auxiliary processes

3.1. **Inhomogeneous branching random walk.** Let $n \in \mathbb{N}$ and set $N = 2^n$. For $k = 0, 1, \dots, n$, let $\mathcal{B}_k$ denote the collection of subsets of $\mathbb{Z}^2$ consisting of squares of side length $2^k$ with corners in $\mathbb{Z}^2$, and let $\mathcal{BD}_k$ denote the subset of $\mathcal{B}_k$ consisting of squares of the form $([0, 2^k - 1] \cap \mathbb{Z})^2 + (i2^k, j2^k)$. Note that the collection $\mathcal{BD}_k$ partitions $\mathbb{Z}^2$ into disjoint squares. For $v \in V_N$, let $\mathcal{B}_k(v)$ denote the set of elements $B \in \mathcal{B}_k$ with $v \in B$. Let $B_k(v)$ be the unique box $B_k(v) \in \mathcal{BD}_k$ that contains $v$.

**Definition 3.1** (Inhomogeneous branching random walk (IBRW))**.** Let $\{a_{k,B}\}_{k\geq 0, B\in\mathcal{BD}_k}$ be an i.i.d. family of standard Gaussian random variables. Define the inhomogeneous branching random walk $\{R_v^N\}_{v\in V_N}$, by

$$R_v^N(t) := \sum_{k=n-t}^{n} \sqrt{\log(2)} a_{k,B_k(v)} \int_{n-k-1}^{n-k} \sigma\left(\frac{s}{n}\right) \mathrm{d}s, \tag{3.1}$$

where $0 \leq t \leq n$, $t \in \mathbb{N}$.

3.2. **Modified inhomogeneous branching random walk.** For $N = 2^n$, $v \in V_N$, let $\mathcal{B}_k^N(v)$ be the collection of subsets of $\mathbb{Z}^2$ consisting of squares of size $2^k$ with lower left corner in $V_N$ and containing $v$. Note that the cardinality of $\mathcal{B}_k^N(v)$ is $2^k$. For two sets $B, B'$, write $B \sim_N B'$ if there are integers, $i, j$, such that $B' = B + (iN, jN)$. Let $\{b_{k,B}\}_{k\geq 0, B\in\mathcal{B}_k^N}$ denote an i.i.d. family of centred Gaussian random variables with unit variance, and define

$$b_{k,B}^N := \begin{cases} b_{k,B},\ B \in \mathcal{B}_k^N, \\ b_{k,B'},\ B \sim_N B' \in \mathcal{B}_k^N. \end{cases} \tag{3.2}$$

**Definition 3.2** (Modified inhomogeneous branching random walk (MIBRW))**.** The modified inhomogeneous branching random walk (MIBRW) $\{\tilde{S}_v^N\}_{v\in V_N}$ is defined by

$$\tilde{S}_v^N(t) := \sum_{k=n-t}^{n} \sum_{B\in\mathcal{B}_k^N(v)} 2^{-k}\sqrt{\log(2)}\, b_{k,B}^N \int_{n-k-1}^{n-k} \sigma\left(\frac{s}{n}\right) \mathrm{d}s, \tag{3.3}$$

where $0 \leq t \leq n$, $t \in \mathbb{N}$.

3.3. **Covariance estimates.** In order to compare the auxiliary processes with the scale-inhomogeneous DGFF, one needs estimates on their covariances, which are provided in this section. Let $\|\cdot\|_2$ be the usual Euclidean distance and $\|\cdot\|_\infty$ the maximum distance. In addition, introduce the following two distances on the torus induced by $V_N$, i.e. for $v, w \in V_N$,

$$d^N(v,w) := \min_{z:\, z-w\in(N\mathbb{Z})^2} \|v-z\|_2, \qquad d_\infty^N(v,w) := \min_{z:\, z-w\in(N\mathbb{Z})^2} \|v-z\|_\infty. \tag{3.4}$$

Note that the Euclidean distance on the torus is smaller than the standard Euclidean distance, i.e. for all $v, w \in V_N$, it holds $d^N(v,w) \leq \|v-w\|_2$. Equality holds if $v, w \in (N/4, N/4) + V_{N/2} \subset V_N$.

**Lemma 3.3.** [36, Lemma 3.3] *For any $\delta > 0$, there exists a constant $\alpha > 0$ independent of $N = 2^n$, such that the following estimates hold: For any $v, w \in V_N$,*

i. $\left|\mathbb{E}\left[\tilde{S}_v^N \tilde{S}_w^N\right] - \log N \mathcal{I}_{\sigma^2}\left(1 - \frac{\log_+ d^N(v,w)}{\log N}\right)\right| \leq \alpha.$

*Further, for any $u, v \in V_N^\delta$, and any $x, y \in V_N + (2N, 2N) \subset V_{4N}$ :*

ii. $\left|\mathbb{E}\left[\psi_u^N \psi_v^N\right] - \log N \mathcal{I}_{\sigma^2}\left(1 - \frac{\log_+ \|u-v\|_2}{\log N}\right)\right| \leq \alpha$

iii. $\left|\mathbb{E}\left[\psi_x^{4N} \psi_y^{4N}\right] - \mathbb{E}\left[\tilde{S}_x^{4N} \tilde{S}_y^{4N}\right]\right| \leq \alpha.$

*Proof.* The proof is given in Subsection A.1. □

In the following lemma, we identify the asymptotic behaviour of covariances of the scale-inhomogeneous 2d DGFF close to the diagonal and for two vertices at macroscopic distance, i.e. at distance of order of the side length of the underlying box.

**Lemma 3.4.** *There are continuous functions, $f : (0,1)^2 \mapsto \mathbb{R}$ and $h : [0,1]^2 \backslash \{(x,x) : x \in [0,1]\} \mapsto \mathbb{R}$, and a function, $g : \mathbb{Z}^2 \times \mathbb{Z}^2 \mapsto \mathbb{R}$, such that the following two statements hold:*

i. *For all $L, \epsilon, \delta > 0$, there exists an integer $N_0 = N_0(\epsilon, \delta, L) > 0$ such that, for all $x \in [0,1]^2$ with $xN \in V_N^\delta$, $u, v \in [0,L]^2$ and $N \geq N_0$, we have*

$$\left|\mathbb{E}\left[\psi_{xN+u}^N \psi_{xN+v}^N\right] - \log(N) - \sigma^2(0) f(x) - \sigma^2(1) g(u,v)\right| < \epsilon. \tag{3.5}$$

ii. *For all $L, \epsilon, \delta > 0$, there exists an integer $N_1 = N_1(\epsilon, \delta, L) > 0$ such that, for all $x, y \in [0,1]^2$ with $xN, yN \in V_N^\delta$ as well as $|x-y| \geq 1/L$ and $N \geq N_1$, we have*

$$\left|\mathbb{E}\left[\psi_{xN}^N \psi_{yN}^N\right] - \sigma^2(0) h(x,y)\right| < \epsilon. \tag{3.6}$$

*Proof.* The proof is given in Subsection A.1. □

## 4. Proof of Theorem 2.2

In order to prove Theorem 2.2, we have to show with high probability that there cannot be two vertices at "intermediate distance" to each other and both reaching an extremal height. We therefore study the sum of two vertices, under the additional restriction that their distance is "intermediate". For such sums, we first prove a version of Slepian's lemma, which relates these functionals of the scale-inhomogeneous DGFF to the same functionals of a suitable IBRW.

**Lemma 4.1.** *Let $\{\chi_v^N\}_{v\in V_N}$ and $\{\eta_v^N\}_{v\in V_N}$ be two centred Gaussian processes, such that*

$$\mathbb{E}\left[\eta_u^N\eta_v^N\right] \leq \mathbb{E}\left[\chi_u^N\chi_v^N\right] \quad \forall u,v\in V_N, \tag{4.1}$$

$$\mathrm{Var}\left[\eta_u^N\right] = \mathrm{Var}\left[\chi_u^N\right] \qquad \forall u\in V_N. \tag{4.2}$$

*Let $\Omega_{m,r} := \{A\subset V_N : |A| = m, u,v\in A \Rightarrow r\leq \|u-v\|_2 \leq N/r\}$. For any $r\geq 0$, $N>r$ and any $\lambda\in\mathbb{R}$, it holds that*

$$\mathbb{P}\left(\max_{A\in\Omega_{m,r}}\sum_{v\in A}\eta_v^N\leq\lambda\right)\leq\mathbb{P}\left(\max_{A\in\Omega_{m,r}}\sum_{v\in A}\chi_v^N\leq\lambda\right). \tag{4.3}$$

*Proof.* The idea is to use Gaussian interpolation. We first introduce the necessary set-up. For $h\in[0,1]$ and $u\in V_N$, let

$$X_u^h = \sqrt{h}\eta_u^N + \sqrt{1-h}\chi_u^N \tag{4.4}$$

be a Gaussian random variable, interpolating between the scale-inhomogeneous DGFF and the time-inhomogeneous BRW. Moreover, let $s>0$, set $\Phi_s(x) = \frac{1}{\sqrt{2\pi s^2}}\int_{-\infty}^x \exp\left[-\frac{z^2}{2s^2}\right]\mathrm{d}z$ and write $x_A = \sum_{v\in A} x_v$, for $A\subset V_N$. We define

$$F_s(x_1,\dots,x_{4^n}) = \prod_{A\in\Omega_{m,r}}\Phi_s(\lambda-x_A). \tag{4.5}$$

Clearly, $F_s$ is bounded uniformly in $s$, smooth for all $s>0$, and converges pointwise to $F(x_1,\dots,x_{4^n}) = \mathbb{1}_{x_A\leq u,\in A\in\Omega_{m,r}}$ at all continuity points of $F$. We have that, for $i\neq j$,

$$\begin{aligned}\frac{\partial^2 F_s}{\partial x_i\partial x_j}(x_1,\dots,x_{4^n}) &= \sum_{\substack{A\in\Omega_{m,r}\\ x_i,x_j\in A}}\Phi_s''(\lambda-x_A)\prod_{B\in\Omega_{m,r},B\neq A}\Phi_s(\lambda-x_B)\\ &\quad+\sum_{\substack{A\in\Omega_{m,r}\\ x_i\in A}}\sum_{\substack{B\in\Omega_{m,r}\\ x_j\in B,B\neq A}}\Phi_s'(\lambda-x_A)\Phi_s'(\lambda-x_B)\prod_{C\in\Omega_{m,r},C\neq A,B}\Phi_s(\lambda-x_C).\end{aligned} \tag{4.6}$$

Observe that, by dominated convergence, for $A\in\Omega_{m,r}$,

$$\mathbb{E}\left[\Phi_s''(\lambda-X_A^h)\right] = \int\phi_{h,A}(x)\frac{\lambda-x}{\sqrt{2\pi s^2}s^2}\exp\left[-\frac{(\lambda-x)^2}{2s^2}\right]\mathrm{d}x\to 0, \tag{4.7}$$

as $s\to 0$, and where $\phi_{h,A}$ is the density of the centred Gaussian $\sum_{v\in A}X_v^h$. By (4.7) and as $\prod_{B\in\Omega_{m,r},B\neq A}\Phi_s(\lambda-x_B)\leq 1$,

$$\sum_{\substack{A\in\Omega_{m,r}\\ x_i,x_j\in A}}\mathbb{E}\left[\Phi_s''(\lambda-X_A^h)\prod_{B\in\Omega_{m,r},B\neq A}\Phi_s(\lambda-X_B^h)\right]\to 0, \tag{4.8}$$

as $s \to 0$. Next, we turn to the second sum in (4.6). For $A, B \in \Omega_{m,r}$, we have

$$\mathbb{E}\left[\Phi_s'(\lambda - X_A^h)\Phi_s'(\lambda - X_B^h) \prod_{C \in \Omega_{m,r}, C \neq A,B} \Phi_s(\lambda - X_C^h)\right] \leq \mathbb{E}\left[\Phi_s'(\lambda - X_A^h)\Phi_s'(\lambda - X_B^h)\right]$$
$$= \int \phi_{h,A,B}(x,y)\frac{1}{2\pi s^2}\exp\left[-\frac{(\lambda-x)^2+(\lambda-y)^2}{2s^2}\right]\mathrm{d}x\mathrm{d}y \to \phi_{h,A,B}(\lambda,\lambda), \tag{4.9}$$

as $s \to 0$ and where

$$\phi_{h,A,B}(x,y) = \frac{1}{2\pi\sigma_A^h\sigma_B^h\sqrt{1-\varrho_{h,A,B}^2}}\exp\left[-\frac{1}{2(1-\varrho_{h,A,B}^2)}\left(\frac{x^2}{(\sigma_A^h)^2}+\frac{y^2}{(\sigma_B^h)^2}-2\varrho_{h,A,B}\frac{xy}{\sigma_A^h\sigma_B^h}\right)\right] \tag{4.10}$$

with $(\sigma_A^h)^2 = \mathrm{Var}\left[\sum_{v\in A} X_v^h\right]$, $(\sigma_B^h)^2 = \mathrm{Var}\left[\sum_{v\in B} X_v^h\right]$ and $\varrho_{h,A,B} = \frac{\mathbb{E}[(\sum_{v\in A} X_A^h)(\sum_{v\in B} X_B^h)]}{\sqrt{\mathrm{Var}[\sum_{v\in A} X_A^h]\mathrm{Var}[\sum_{v\in B} X_v^h]}}$. $\phi_{h,A,B}(x,y)$ is the density of the bivariate distributed random vector $\left(\sum_{v\in A} X_v^h, \sum_{v\in B} X_v^h\right)$. Observe that,

$$\phi_{h,A,B}(x,y) \leq \frac{1}{2\pi\sqrt{1-\varrho_{A,B}^2}}\exp\left[-\frac{x^2+y^2}{2(1+\varrho_{A,B})}\right], \tag{4.11}$$

where $\varrho_{A,B} = \max\left(\mathbb{E}\left[\left(\sum_{v\in A}\eta_v\right)\left(\sum_{v\in B}\eta_v\right)\right], \mathbb{E}\left[\left(\sum_{v\in A}\chi_v^N\right)\left(\sum_{v\in B}\chi_v^N\right)\right]\right)$. Inserting (4.11) into (4.9) and using this with (4.8) in (4.6) and letting $s \to 0$, allows to use Kahane's theorem [47], to obtain

$$\mathbb{P}\left(\forall A \in \Omega_{m,r} : \sum_{v\in A}\eta_v \leq \lambda\right) - \mathbb{P}\left(\forall A \in \Omega_{m,r} : \sum_{v\in A}\chi_v^N \leq \lambda\right)$$
$$\leq \frac{1}{2\pi}\sum_{1\leq i<j\leq 4^n}\sum_{\substack{A\in\Omega_{m,r}\\ x_i\in A}}\sum_{\substack{B\in\Omega_{m,r}\\ x_j\in B, B\neq A}}\frac{(\Lambda_{A,B}^1-\Lambda_{A,B}^0)_+}{\sqrt{1-\varrho_{A,B}^2}}\exp\left[-\frac{2\lambda^2}{2(1+\varrho_{A,B})}\right], \tag{4.12}$$

with $\Lambda_{A,B}^0 = \mathbb{E}\left[\left(\sum_{v\in A}\chi_v^N\right)\left(\sum_{v\in B}\chi_v^N\right)\right]$ and $\Lambda_{A,B}^1 = \mathbb{E}\left[\left(\sum_{v\in A}\eta_v\right)\left(\sum_{v\in B}\eta_v\right)\right]$. By (4.1), $(\Lambda_{A,B}^1 - \Lambda_{A,B}^0)_+ = 0$, and thus, (4.12) implies (4.3). □

In the following proposition, we determine the position of extremal particles of an inhomogeneous BRW at the times when its variance changes. This is a direct consequence of [19, Proposition 2.1] in the weakly-correlated regime of variable speed BBM. Set $i(t,n) := t \wedge (n-t)$.

**Proposition 4.2.** *Let $\{R_v^N\}_{v\in V_N}$ be a inhomogeneous BRW with $\mathcal{I}_{\sigma^2}(x) < x$, for $x \in (0,1)$ and $\sigma(0) < 1 < \sigma(1)$. Let $s \in \mathbb{R}$. Then, there is a constant $r_0 > 0$ such that for any $r > r_0$, $N = 2^n$, $N$ sufficiently large, and any $\gamma \in (1/2, 1)$,*

$$\mathbb{P}\left(\exists v \in V_N,\ t \in [\log r, n - \log r] : R_v^N \geq m_N - s, R_v^N(t) - 2\log 2\mathcal{I}_{\sigma^2}\left(\frac{t}{n}\right)n \notin [-i^\gamma(t,n), i^\gamma(t,n)]\right)$$
$$\leq Ce^{2s}\sum_{k=\lfloor\log r\rfloor}^{\infty} k^{\frac{1}{2}-\gamma}\exp\left[-k^{\frac{2\gamma-1}{2}}\right], \tag{4.13}$$

*where $C \leq \frac{8}{\sqrt{\log 2 - \frac{\log n + 4s}{4n}}}$.*

By Gaussian comparison and since we have $\mathcal{I}_{\sigma^2}(x) < x$, for $x \in (0,1)$, it turns out that for our purposes, it suffices to consider a two-speed branching random walk, $(X_v^N(j))_{v\in V_N, 0\leq j\leq n}$. We choose the first speed to be 0 and the second to be $\sigma_{max}^2$, where $\sigma_{max} = \mathrm{ess\,sup}\{\sigma(s) : 0 \leq s \leq 1\}$. Note that $\sigma_{max} < \infty$, as $\mathcal{I}_{\sigma^2}(x) < x$, for $x \in (0,1)$. To match variances, the change of speed occurs at scale $1 - 1/\sigma_{max}^2$. Write $u \underset{j}{\sim} v$, for $u, v \in V_N$, if $j$ is the largest integer such that $\mathcal{BD}_{n-j}(u) \cap \mathcal{BD}_{n-j}(v) \neq \emptyset$, i.e. in the language of BRW the "splitting-time" of $u$ and $v$ is $j$. The following Proposition is the analogue statement to Theorem 2.2 for the two-speed BRW and key in the proof of Theorem 2.2.

**Proposition 4.3.** *There is a constant $C > 0$, such that for any constant $c > 0$ and any $z \geq 0$,*

$$\mathbb{P}\left(\exists j \in (\log r, n - \log r),\ \exists u \underset{j}{\sim} v :\ X_u^N, X_v^N \geq m_N - c \log\log r + z\right) \tag{4.14}$$
$$\leq C\left(4^{-\log r} \exp\left[-4z + 4c \log\log r\right] + \log(r)^{-1/2} \exp\left[2\log 2(1-\sigma_{max}^2)\log r + 2c\log\log r - 2z\right]\right).$$

*In particular, there are $c, r_0 > 0$, such that for all $r > r_0$ and $n$ sufficiently large,*

$$\mathbb{E}\left[\max_{u \underset{s}{\sim} v, s\in\{\log r, \dots, n-\log r\}} X_u^N + X_v^N\right] \leq 2m_N - c\log\log r. \tag{4.15}$$

*Proof.* We first consider the case when $u \underset{j}{\sim} v$ and $j < n/\sigma_{max}^2$. In this case, the particles split before the change in speed occurs. The speed change occurs at scale $1 - \lambda = 1 - 1/\sigma_{max}^2$. Note that there are $4^{2n-j}$ such pairs, and as the initial speed is zero, $X_u^N, X_v^N$ are independent. Hence,

$$\mathbb{P}\left(\exists j \in (\log r, \lfloor n(1-1/\sigma_{max}^2)\rfloor),\ \exists u \underset{j}{\sim} v :\ X_u^N, X_v^N \geq m_N - c\log\log r + z\right)$$
$$\leq \sum_{j=\log r}^{\lfloor n(1-1/\sigma_{max}^2)\rfloor} 4^{2n-j}\mathbb{P}\left(X_u^N \geq m_N - c\log\log r + z\right)^2 \leq \tilde{C} \sum_{j=\log r}^{\lfloor n(1-1/\sigma_{max}^2)\rfloor} 4^{2n-j} \frac{\log(2)n}{(m_N - c\log r + z)^2}$$
$$\times \exp\left[-4\log(2)n + \log n + 4(z - c\log\log r)\right] \leq C4^{-\log r}\exp\left[-4z + 4c\log\log r\right]. \tag{4.16}$$

where $\tilde{C}, C > 0$ are finite constants and the last inequality follows from a Gaussian tail bound. Next, we treat the case when particles split after the change of speed. Let $\gamma \in (1/2, 1)$ and set $i(j,n) := (n - \sigma_{max}^2(n-j)) \wedge (\sigma_{max}^2(n-j))$ and $A_1(j) := \{x \in \mathbb{R} :\ |x - \frac{n-\sigma_{max}^2(n-j)}{n} m_N| \leq i^\gamma(j,n)\}$. As the extremal particles of the BRW stay with high probability in $A_1(j)$, for $j \in \{\log r, \dots, n - \log r\}$ (see Proposition 4.2 for a precise statement), we can compute as follows:

$$\mathbb{P}\left(\exists s \in (\lfloor n(1-1/\sigma_{max}^2)\rfloor + 1, n - \log r),\ \exists u \underset{s}{\sim} v : X_u^N, X_v^N \geq m_N - c\log\log r + z\right)$$
$$\leq C \sum_{j=\lfloor n(1-1/\sigma_{max}^2)\rfloor+1}^{n-\log r} \int_{A_1(j)} 4^{2n-j}\mathbb{P}\left(X_v^N(n) - X_v^N(j) \geq m_N - c\log\log r + z - x\right)^2$$
$$\times \frac{1}{\sqrt{2\pi\log 2(n - \sigma_{max}^2(n-j))}} \exp\left[-\frac{x^2}{2\log 2(n-\sigma_{max}^2(n-j))}\right] \mathrm{d}x + \epsilon. \tag{4.17}$$

By a Gaussian tail bound and using that by the integral restriction, $(m_N - x)^2 \geq (\frac{\sigma_{max}^2(n-j)}{n} m_N - i(j,n)^\gamma)^2$, the summand in (4.17) is bounded from above by

$$C\frac{\sigma_{max}^2(n-j)\exp\left[-\frac{(m_N - c\log\log r + z)^2}{\log 2(2n - \sigma_{max}^2(n-j))}\right]}{\sqrt{2\pi\log 2(n-\sigma_{max}^2(n-j))}(\frac{\sigma_{max}^2(n-j)}{n} m_N - c\log\log r + z - i^\gamma(j,n))^2}$$
$$\times 4^{2n-j}\int_{A_1(j)} \exp\left[-\frac{\left(x - (m_N - c\log\log r + z)\frac{2(n-\sigma_{max}^2(n-j))}{2n-\sigma_{max}^2(n-j)}\right)^2}{2\log 2\frac{(n-\sigma_{max}^2(n-j))\sigma_{max}^2(n-j)}{2n-\sigma_{max}^2(n-j)}}\right]\mathrm{d}x. \tag{4.18}$$

Changing variables, i.e. $x = \sqrt{\log 2\sigma^2_{max}(n-j)\frac{n-\sigma^2_{max}(n-j)}{2n-\sigma^2_{max}(n-j)}}y + \frac{2(m_N - c\log\log r + z)(n-\sigma^2_{max}(n-j))}{2n-\sigma^2_{max}(n-j)}$, and neglecting the upper restriction in $A_1(j)$, (4.18) is bounded from above by

$$C\frac{(\sigma^2_{max}(n-j))^{3/2}}{\left(\frac{\sigma^2_{max}(n-j)}{n}m_N - c\log\log r + z - i^\gamma(j,n)\right)^2\sqrt{2\pi\log 2(2n-\sigma^2_{max}(n-j))}}$$
$$\times\exp\left[-\frac{(m_N - c\log\log r + z)^2}{\log 2(2n-\sigma^2_{max}(n-j))}\right]4^{2n-j}\int_{A_1'(j)}\exp\left[-y^2/2\right]\mathrm{d}y, \tag{4.19}$$

with $A_1'(j) = \left[-\frac{m_N}{n}\tilde{\sigma}(n,j) - (z - c\log\log r)\sqrt{\frac{n-\sigma^2_{max}(n-j)}{\log 2\sigma^2_{max}(n-j)(2n-\sigma^2_{max}(n-j))}}) - \frac{i^\gamma(j,n)}{\tilde{\sigma}(n,j)}, +\infty\right]$, and where $\tilde{\sigma}(n,j) = \sqrt{\frac{\sigma^2_{max}(n-j)(n-\sigma^2_{max}(n-j))}{\log 2(2n-\sigma^2_{max}(n-j))}}$. By a Gaussian tail bound applied to the integral, (4.19) is bounded from above by

$$O\left(\frac{1}{(n-j)\sqrt{n-\sigma^2_{max}(n-j)}}\right)4^{2n-j}\exp\left[-\frac{(m_N - c\log\log r + z)^2}{\log 2(2n-\sigma^2_{max}(n-j))} - \frac{i^{2\gamma}(j,n)\log 2(2n-\sigma^2_{max}(n-j))}{2\sigma^2_{max}(n-j)(n-\sigma^2_{max}(n-j))}\right]$$
$$\times\exp\left[-\frac{m_N i^\gamma(j,n)}{n} - \frac{m_N^2\sigma^2_{max}(n-j)(n-\sigma^2_{max}(n-j))}{2n^2\log 2(2n-\sigma^2_{max}(n-j))} - \frac{m_N(z-c\log\log r)}{n}\frac{n-\sigma^2_{max}(n-j)}{2n-\sigma^2_{max}(n-j)}\right]. \tag{4.20}$$

Keeping only the dominant terms, one sees that the exponential is bounded from above by

$$\exp\left[2\log 2(n-j)(1-\sigma^2_{max}) + 2c\log\log r - 2z + \frac{\sigma^2_{max}\frac{n-j}{n}+1}{2}\log n - c_1 i^\gamma(j,n) - c_2 i^{2\gamma-1}(j,n)\right], \tag{4.21}$$

where $c_1, c_2 > 0$ are some finite constants. Inserting (4.21) into (4.20), allows to bound (4.17) from above by

$$\sum_{j=\lfloor n(1-1/\sigma^2_{max})\rfloor+1}^{n-\log r} O\left(\frac{1}{(n-j)\sqrt{n-\sigma^2_{max}(n-j)}}\right)\exp\left[2(n-j)(1-\sigma^2_{max})\log 2 - 2z + \frac{\sigma^2_{max}\frac{n-j}{n}+1}{2}\log n\right.$$
$$\left.+2c\log\log r - c_1 i^\gamma(j,n) - c_2 i^{2\gamma-1}(j,n)\right] \leq O\left(\frac{1}{\sqrt{\log r}}\right)\exp\left[2\log 2(1-\sigma^2_{max})\log r + 2c\log\log r - 2z\right]. \tag{4.22}$$

Since $\sigma_{max} > 1$, (4.22) tends to zero, as $n \to \infty$. (4.15) is an immediate consequence of (4.16) and (4.22). This concludes the proof of Proposition 4.3. □

Similarly, as for the IBRW, we have a localization for extremal particles of the MIBRW, which is the statement of the following lemma.

**Lemma 4.4.** *Let $\{\tilde{S}^N_v\}_{v\in V_N}$ be the MIBRW, defined in* (3.3)*. Let $s \in \mathbb{R}$. Then, for any $\epsilon > 0$, there is a constant $r_0 > 0$ such that for any $r > r_0$, $N = 2^n$, $N$ sufficiently large, and any $\gamma \in (1/2, 1)$,*

$$\mathbb{P}\left(\exists v \in V_N,\ t \in [\log r, n-\log r] : \tilde{S}^N_v \geq m_N - s,\ \tilde{S}^N_v(t) \notin [-i^\gamma(t,n), i^\gamma(t,n)]\right)$$
$$\leq Ce^{2s}\sum_{k=\lfloor\log r\rfloor}^{\infty} k^{\frac{1}{2}-\gamma}\exp\left[-k^{\frac{2\gamma-1}{2}}\right], \tag{4.23}$$

*where $C \leq \frac{8}{\sqrt{\log 2 - \frac{\log n + 4s}{4n}}}$.*

We do not give a proof here, as it is basically identical to the one of Proposition 4.2.

*Proof of Theorem 2.2.* Note that the tree distance of two vertices $u, v \in V_N$ on the underlying tree of the IBRW, $\{X_v^N\}_{v\in V_N}$, is up to an additional constant smaller than the Euclidean distance. Hence, by Lemma 3.3 *ii*. there is a $\kappa \in \mathbb{N}$ and non-negative constants $\{a_v\}_{v\in V_N}$ such that, for all $N \in \mathbb{N}$ and all $u, v \in V_N$,

$$\mathbb{E}\left[X_{2^k u}^{2^k N} X_{2^k v}^{2^k N}\right] \leq \mathbb{E}\left[\psi_u^N \psi_v^N\right] + a_u a_v, \tag{4.24}$$

and

$$\mathrm{Var}\left[X_{2^k u}^{2^k N}\right] = \mathrm{Var}\left[\psi_u^N\right] + a_u^2. \tag{4.25}$$

Thus, we may apply Lemma 4.1 with $m = 2$ and obtain, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{P}\Big((\exists u, v \in V_N,\ r \leq \|u-v\|_2 \leq N/r:\ \psi_u^N + \psi_v^N \geq \lambda\Big) \\
&\quad \leq \mathbb{P}\Big((\exists u, v \in V_N,\ r \leq \|u-v\|_2 \leq N/r:\ \psi_u^N a_u G + \psi_v^N + a_v G \geq \lambda\Big) \\
&\quad \leq \mathbb{P}\Big((\exists u, v \in V_N,\ r \leq \|u-v\|_2 \leq N/r:\ X_{2^\kappa u}^{2^\kappa N} + X_{2^\kappa v}^{2^\kappa N} \geq \lambda\Big) \\
&\quad \leq \mathbb{P}\Big((\exists u, v \in V_{2^\kappa N},\ r \leq \|u-v\|_2 \leq 2^\kappa N/r:\ X_u^{2^\kappa N} + X_v^{2^\kappa N} \geq \lambda\Big),
\end{aligned} \tag{4.26}$$

where $G$ is an independent standard Gaussian. Choosing $\lambda = m_N - c \log\log r$ and applying Proposition 4.3 to last probability in (4.26) yields (2.4), which concludes the proof of Theorem 2.2. □

## 5. Proof of Theorem 2.1

The following proposition allows to control the right tail of the maximum over subsets.

**Proposition 5.1.** *Let $\epsilon > 0$ and $\{\bar{\psi}_v^N\}_{v\in V_N}$ be a centred Gaussian field such that, for all $v, w \in V_N$, $|\mathbb{E}\left[\bar{\psi}_v^N \bar{\psi}_w^N\right] - \mathbb{E}\left[\psi_v^N \psi_w^N\right]| \leq \epsilon$. If $N$ is sufficiently large, then, for any $A \subset V_N$ and for all $z \geq 1, y \geq 0$, we have*

$$\mathbb{P}\left(\max_{v\in A} \bar{\psi}_v^N \geq m_N + z - y\right) \leq C \frac{|A|}{|V_N|} e^{-2(z-y)}. \tag{5.1}$$

*Proof of Proposition 5.1.* By the covariance assumptions and Lemma 3.3 *i*., *iii*. one can apply Slepian's lemma, to deduce that there exists $k \in \mathbb{N}$, such that for all sufficiently large $N \in \mathbb{N}$ and any $\lambda \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{v\in A} \bar{\psi}_v^N \geq \lambda\right) \leq \mathbb{P}\left(\max_{v\in 2^k A} R_v^{2^k} \geq \lambda\right). \tag{5.2}$$

Thus, it suffices to show (5.1) with $R^N$ instead of $\bar{\psi}^N$. Note that for any $v \in V_N$, $R_v^N \sim \mathcal{N}(0, n\log 2)$. Thus, by a first moment bound and a standard Gaussian tail estimate,

$$\begin{aligned}
\mathbb{P}\left(\max_{v\in A} R_v^N \geq m_N + y - z\right) &\leq C|A| \frac{n\log 2}{(m_N + z - y)\sqrt{2\pi n \log 2}} \exp\left[-\frac{(m_N + z - y)^2}{2n\log 2}\right] \\
&\leq C|A| \frac{n\log 2}{(m_N + z - y)\sqrt{2\pi n \log 2}} \exp\left[-2n\log 2 + 1/2\log n - 2(z-y)\right] \\
&\leq C\frac{|A|}{|V_N|} \exp\left[-2(z-y)\right],
\end{aligned} \tag{5.3}$$

where the constant $C > 0$ may change from line to line and where we used that $|V_N| = 2^{2n}$. □

5.1. **Approximation via an auxiliary field.** Let $N = 2^n$ be an integer, much larger as any other integers forthcoming. For two integers $L = 2^l$ and $K = 2^k$, partition $V_N$ into a disjoint union of $(KL)^2$ boxes, with each of side length $N/KL$, and denote the partition by $\mathcal{B}_{N/KL} = \{B_{N/KL,i} : i = 1, \dots, (KL)^2\}$. Let $v_{N/KL,i} \in V_N$ be the left bottom corner of box $B_{N/KL,i}$ and write $w_i = \frac{v_{N/KL,i}}{N/KL}$. This allows to consider the grid points $\{w_i\}_{i=1,\dots,(KL)^2}$ as elements of $V_{KL}$. Analogously, let $K' = 2^{k'}$ and $L' = 2^{l'}$ be another two integers and let $\mathcal{B}_{K'L'} = \{B_{K'L',i} : i = 1, \dots, [N/(K'L')]^2\}$ be a partitioning of $V_N$ with boxes $B_{K'L',i}$, each of side length $K'L'$. The left bottom corner of a box $B_{K'L',i}$ is denoted by $v_{K'L',i}$. One should think of $N/KL$ being much larger than $K'L'$. Considering Lemma 3.4, it turns out that this allows to define the corresponding approximating fields in such a way that they have only a fixed variance parameter, which makes them easier to analyse. The macroscopic or "coarse field", $\{S_v^{N,c} : v \in V_N\}$, is defined as a centred Gaussian field on $V_N$ with covariance matrix $\Sigma^c$ and entries given by

$$\Sigma^c_{u,v} := \sigma^2(0)\mathbb{E}\left[\phi^{KL}_{w_i}\phi^{KL}_{w_j}\right], \quad \text{for } u \in B_{N/KL,i},\ v \in B_{N/KL,j}, \tag{5.4}$$

where $\{\phi^{KL}_v\}_{v\in V_{KL}}$ is a standard 2d DGFF on $V_{KL}$. This field captures the macroscopic dependence. The microscopic or "bottom field", $\{S_v^{N,b} : v \in V_N\}$, is a centred Gaussian field with covariance matrix $\Sigma^b$ defined entry-wise as

$$\Sigma^b_{u,v} := \begin{cases} \sigma^2(1)\mathbb{E}\left[\phi^{K'L'}_{u-v_{K'L',i}}\phi^{K'L'}_{v-v_{K'L',i}}\right], & \text{if } u, v \in B_{K'L',i} \\ 0, & \text{else,} \end{cases} \tag{5.5}$$

where $\{\phi^{K'L'}_v\}_{v\in V_{K'L'}}$ is a 2d DGFF on $V_{K'L'}$. This field is supposed to capture the "local" correlations. The third Gaussian field, $\{S_v^{N,m} : v \in V_N\}$, is a collection of MIBRWs on $B_{N/KL,i}$, $i = 1, \dots, (KL)^2$, i.e.

$$S_v^{N,m} := \sum_{j=l'+k'}^{n-l-k} \sum_{B\in\mathcal{B}_j(v_{K'L',i'})} 2^{-j}\sqrt{\log(2)}b^N_{i,j,B}\int_{n-j-1}^{n-j}\sigma\left(\frac{s}{n}\right)\mathrm{d}s, \quad \text{for } v \in B_{N/KL,i} \cap B_{K'L',i'}, \tag{5.6}$$

with $\{b^N_{i,j,B} : i = 1, \dots, (KL)^2,\ j \geq 0,\ B \in \mathcal{B}^N_j\}$ being a family of independent standard Gaussian random variables. Recall that $\mathcal{B}_j(v_{K'L',i'})$ is the collection of boxes $B \subset V_N$, of side length $2^j$, that contain the element $v_{K'L',i'}$. This field is supposed to capture the "intermediate" correlations. To obtain sufficiently precise covariance estimates, one needs to avoid boundary effects, which can achieved working on a suitable subset of $V_N$. Consider therefore the partitioning into $N/L$- and $L-$boxes, i.e. $\mathcal{B}_{N/L} = \{B_{N/L,i} : 1 \leq i \leq L^2\}$ and $\mathcal{B}_L = \{B_{L,i} : 1 \leq i \leq (N/L)^2\}$. Analogously, let $v_{N/L,i}$ and $v_{L,i}$ be the left bottom corners of boxes $B_{N/L,i}$, $B_{L,i}$ containing $v$. For a box $B$, let $B^\delta \subset B$ the set $B^\delta = \{v \in B : \min_{z\in\partial B}\|v - z\| \geq \delta l_B\}$, where $l_B$ denotes the side length of the box $B$. Finally, set

$$V^*_{N,\delta} := \{\bigcup_{1\leq i\leq L^2} B^\delta_{N/L,i}\} \cap \{\bigcup_{1\leq i\leq (KL)^2} B^\delta_{N/KL,i}\} \cap \{\bigcup_{1\leq i\leq (N/L)^2} B^\delta_{L,i}\} \cap \{\bigcup_{1\leq i\leq (N/KL)^2} B^\delta_{KL,i}\}. \tag{5.7}$$

As $|V^*_{N,\delta}| \geq (1 - 16\delta)|V_N|$, and using Proposition 5.1 with $A = (V^*_{N,\delta})^c$, we have

$$\mathbb{P}\left(\max_{v\in\left(V^*_{N,\delta}\right)^c} S^N_v \geq m_N + z\right) \leq 16\delta\,\mathbb{P}\left(\max_{v\in V_N} S^N_v \geq m_N + z\right), \tag{5.8}$$

which tends to 0, as $\delta \to 0$. Thus, it suffices to consider the maximum of the field on the set $V^*_{N,\delta}$.

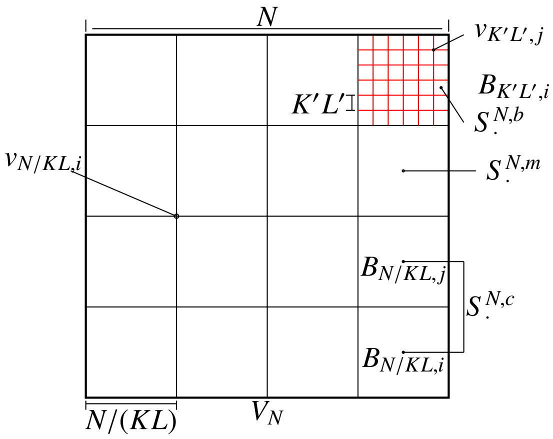



Figure 2: 3-field decomposition

Using Gaussian comparison, we reduce the proof of Theorem 2.1 to showing convergence in law of the centred maximum of an auxiliary field. Therefore, we need to have precise estimates on the variances and covariances, which is what we provide in the following. In order to use Slepian's lemma, we actually need, for each $v \in V_N$, equality of variances. This is usually achieved by adding suitable independent Gaussian random variables, which is done in the following lemma. In particular, the lemma states that one can choose the constants in such a way, that, asymptotically, they only depend on the "fine scales", i.e. they live on boxes $B_{K'L',i}$,. In the rest of the paper, limits are taken in the order $N, K', L', K$ and then $L$, for which we write $(L, K, L', K', N) \Rightarrow \infty$.

**Lemma 5.2.** *Let $\{\Phi_j\}_{1\leq j\leq (N/K'L')^2}$ be a family of i.i.d. standard Gaussian random variables. For $v \in B_{K'L',j}$, $j = 1, \ldots, (N/K'L')^2$ and $v \equiv \bar{v} \mod K'L'$, i.e. $\bar{v} = v - v_{K'L',j}$, there exists a collection of non-negative constants $\{a_{K'L',\bar{v}}\}_{K'L',\bar{v}}$, such that if we set*

$$S_v^N := S_v^{N,c} + S_v^{N,b} + S_v^{N,m} + a_{K'L',\bar{v}}\Phi_j, \tag{5.9}$$

*then*

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \left|\mathrm{Var}\left(S_v^N\right) - \mathrm{Var}\left(\psi_v^N\right) - 4\alpha\right| = 0. \tag{5.10}$$

*Proof.* Considering Lemma 3.3 ii., (5.4), (5.5) and (5.6), a simple computation shows that, for any $v \in V^*_{N,\delta}$,

$$\mathrm{Var}\left(S_v^{N,c}\right) + \mathrm{Var}\left(S_v^{N,b}\right) + \mathrm{Var}\left(S_v^{N,m}\right) = \log N + O_N(1), \tag{5.11}$$

where the term $O_N(1)$ means that the constants are uniformly bounded in $N$. In particular, by Lemma 3.3 *iii.* one has

$$\left|\mathrm{Var}\left(S_v^{N,c}\right) + \mathrm{Var}\left(S_v^{N,b}\right) + \mathrm{Var}\left(S_v^{N,m}\right) - \mathrm{Var}\left(\psi_v^N\right)\right| \leq 4\alpha. \tag{5.12}$$

By (5.12), there exist non-negative constants $\{a_{N,v}\}_{v\in B_{N/KL,i}}$, $1 \leq i \leq (KL)^2$, such that

$$\mathrm{Var}\left(S_v^{N,c} + S_v^{N,b} + S_v^{N,m}\right) + a_{N,v}^2 = \mathrm{Var}\left(\psi_v^N\right) + 4\alpha. \tag{5.13}$$

Note that $\{a_{N,v}\}_{v\in B_{N/KL,i}}$ implicitly depend on $KL$ and by (5.12), one gets

$$\max_{v\in V^*_{N,\delta}} a_{N,v} \leq \sqrt{8\alpha}. \tag{5.14}$$

For $v \in B^{\delta}_{N/KL,i} \cap V^{\delta}_N$, writing $v \equiv \bar{v} \mod K'L'$, where $\bar{v} = v - v_{N/KL,i}$, for $v \in B_{N/KL,i}$, and using Lemma 3.4 *i.* and [13, (1.29)],

$$\begin{aligned} a^2_{N,v} &= 4\alpha + \mathrm{Var}\left(\psi^N_v\right) - \sigma^2(0)\mathrm{Var}\left(\phi^{KL}_{w_i}\right) - \sigma^2(1)\mathrm{Var}\left(\phi^{K'L'}_{\bar{v}}\right) - \mathcal{I}_{\sigma^2}\left(\frac{l+k}{n}, \frac{n-l'-k'}{n}\right)\log(N) \\ &= 4\alpha + \sigma^2(0)f(v/N) - \sigma^2(0)f(w_i/(KL)) - \sigma^2(1)f(\bar{v}/(K'L')) + \epsilon_{N,KL,K'L'}(v), \end{aligned} \tag{5.15}$$

which is non-negative. By Lemma 3.4 *i.*, $f$ is continuous and using $\|\frac{v}{N} - \frac{w_i}{KL}\| = \|\frac{v - v_{KL,i}}{N}\| \to 0$, as $(L, K, N) \Rightarrow \infty$, we have in the same limit, $|f(v/N) - f(w_i/(KL))| \to 0$. Moreover, by using [13, (1.29)], Lemma 3.4 *i.* and (5.13) in the first line of (5.15), it follows that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \sup_{v\in V^*_{N,\delta}} \epsilon_{N,KL,K'L'}(v) = 0. \tag{5.16}$$

Regarding (5.15), (5.16), and that $\mathrm{Var}\left[\phi^{K'L'}_v\right] \le \log(K'L') + \alpha$, for all $v \in V_N$, there exist non-negative $a_{K'L',\bar{v}}$, such that

$$a^2_{N,v} = a^2_{K'L',\bar{v}} + \epsilon_{N,KL,K'L'}(v). \tag{5.17}$$

Using [15, Lemma B.3,Lemma B.4,Lemma B.5], one obtains

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \sup_{u,v\in V^*_{N,\delta}:\|u-v\|_\infty\le L'} \left|\mathrm{Var}\left(\phi^{K'L'}_u\right) - \mathrm{Var}\left(\phi^{K'L'}_v\right)\right| = 0, \tag{5.18}$$

which, together with (5.15) and (5.16), implies

$$\left|a^2_{K'L',\bar{u}} - a^2_{K'L',\bar{v}}\right| \le \sup_{v\in V^*_{N,\delta}} \epsilon_{N,KL,K'L'}(v), \quad \forall u, v \in V^*_{N,\delta} : \|u - v\|_\infty \le L'. \tag{5.19}$$

For $v \in B_{K'L',j}$, $j = 1, \dots, (N/K'L')^2$ and $v \equiv \bar{v} \mod K'L'$, set

$$S^N_v := S^{N,c}_v + S^{N,b}_v + S^{N,m}_v + a_{K'L',\bar{v}}\Phi_j. \tag{5.20}$$

By (5.13) and (5.17), it holds that, for $v \in V^*_{N,\delta}$,

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \left|\mathrm{Var}\left(S^N_v\right) - \mathrm{Var}\left(\psi^N_v\right) - 4\alpha\right| = 0, \tag{5.21}$$

which concludes the proof of Lemma 5.2. □

The next goal is to show that it suffices to prove convergence of the centred maximum of the approximating process, $\{S^N_v\}_{v\in V_N}$, defined in (5.9). This can be done by using Gaussian comparison. The previous lemma, Lemma 5.2, provides asymptotically equal variances, and the following lemma provides covariance estimates for $\{S^N_v\}_{v\in V_N}$. Crucially, for vertices close-by or at macroscopic distance, the covariances coincide asymptotically.

**Lemma 5.3.** *There exists a non-negative sequence* $\{\epsilon'_{N,KL,K'L'}\}_{N,K,L,K',L'\ge 0}$*, and bounded constants* $C_\delta, C > 0$*, such that* $\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \epsilon'_{N,KL,K'L'} = 0$*, and for all* $u, v \in V^*_{N,\delta}$ :

i. *If* $u, v \in B_{L',i}$*, then* $\left|\mathbb{E}\left[\left(S^N_u - S^N_v\right)^2\right] - \mathbb{E}\left[\left(\psi^N_u - \psi^N_v\right)^2\right]\right| \le \epsilon'_{N,KL,K'L'}$.
ii. *If* $u \in B_{N/L,i}$, $v \in B_{N/L,j}$ *and* $i \ne j$*, then* $\left|\mathbb{E}\left[S^N_u S^N_v\right] - \mathbb{E}\left[\psi^N_u \psi^N_v\right]\right| \le \epsilon'_{N,KL,K'L'}$.
iii. *In all other cases, i.e. if* $u, v \in B_{N/L,i}$ *but* $u \in B_{L',i'}$ *and* $v \in B_{L',j'}$*, for some* $i' \ne j'$*, it holds that* $\left|\mathbb{E}\left[S^N_u S^N_v\right] - \mathbb{E}\left[\psi^N_u \psi^N_v\right]\right| \le C_\delta + 40\alpha$.

*Proof.* See Subsection A.1. □

We use the Lévy-Prokhorov metric, $d(\cdot,\cdot)$, which is, for two probability measures on $\mathbb{R}$, $\mu$ and $\nu$, given by

$$d(\mu,\nu) := \inf\{\delta > 0 : \mu(B) \le \nu(B^\delta) + \delta, \text{ for all open sets B}\}, \tag{5.22}$$

where $B^\delta = \{y \in \mathbb{R} : |x-y| < \delta, \text{ for some } x \in B\}$. Moreover, let

$$\tilde{d}(\mu,\nu) = \inf\{\delta > 0 : \mu((x,\infty)) \le \nu((x-\delta,\infty)) + \delta, \text{ for all } x \in \mathbb{R}\}. \tag{5.23}$$

and observe that if $\tilde{d}(\mu,\nu) = 0$, then $\nu$ stochastically dominates $\mu$. For random variables $X, Y$ with laws $\mu_X, \mu_Y$, write $d(X,Y)$ instead of $d(\mu_X,\mu_Y)$, and likewise for $\tilde{d}(\cdot,\cdot)$. The following lemma reduces the proof of Theorem 2.1 to show convergence in law of $S_N^* := \max_{v\in V_N} S_v^N$.

**Lemma 5.4.** *Let $\{S_v^N\}_{v\in V_N}$ be the field defined in* (5.9)*. Then,*

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} d(\psi_N^* - m_N, S_N^* - m_N - 4\alpha) = 0. \tag{5.24}$$

The proof of Lemma 5.4 is based on the two following lemmas, whose proofs are postponed and given in Appendix B. The overall idea is the following: Having asymptotically precise covariance estimates for vertices close-by or at macroscopic distance, and in order to use Slepian's lemma, we would like to add independent Gaussian fields living on those scales and control how the laws of the corresponding centred maxima change under such perturbations. It turns out, that this leads to a deterministic shift (see Lemma 5.5). Having this control, we can then prove Lemma 5.4. First, introduce additional notation.

Fix a positive integer $r \in \mathbb{N}$ and let $\mathcal{B}_r$ a partition of $V_{\lfloor N/r\rfloor r}$ into sub-boxes of side length $r$. Let $\mathcal{B} = \cup_{r\in\mathbb{N}, r\le N}\mathcal{B}_r$ and $\{g_B\}_{B\in\mathcal{B}}$ be a collection of i.i.d. standard Gaussian random variables. For $v \in V_N$, denote by $B_r(v) \in \mathcal{B}_r$ the box containing $v$. For $s = (s_1, s_2) \in \mathbb{R}_+^2$, and two positive integers, $r_1, r_2$, define

$$\tilde{\psi}^N_{v,s,r_1,r_2} = \psi_v^N + s_1 g_{B_{r_1}(v)} + s_2 g_{B_{N/r_2}(v)}. \tag{5.25}$$

Set $\tilde{\psi}^*_{N,s,r_1,r_2} = \max_{v\in V_N} \tilde{\psi}^N_{v,s,r_1,r_2}$ and similarly, $\tilde{S}^N_{v,s,r_1,r_2} = S_v^N + s_1 g_{B_{r_1}(v)} + s_2 g_{B_{N/r_2}(v)}$, and $\tilde{S}^*_{N,s,r_1,r_2} = \max_{v\in V_N} \tilde{S}^N_{v,s,r_1,r_2}$.

**Lemma 5.5.** *Let $\{S_v^N\}_{v\in V_N}$ be the field defined in* (5.9)*. Then,*

$$\limsup_{r_1,r_2\to\infty} \limsup_{N\to\infty} d\left(\psi_N^* - m_N, \tilde{\psi}^*_{N,s,r_1,r_2} - m_N - \|s\|_2^2\right) = 0, \tag{5.26}$$

*and*

$$\limsup_{r_1,r_2\to\infty} \limsup_{N\to\infty} d\left(S_N^* - m_N, \tilde{S}^*_{N,s,r_1,r_2} - m_N - \|s\|_2^2\right) = 0. \tag{5.27}$$

**Lemma 5.6.** *Let $\{\bar{\psi}_v^N\}_{v\in V_N}$ be a centred Gaussian field such that, for all $u, v \in V_N$, $N \in \mathbb{N}$ and some arbitrary $\epsilon > 0$, $|\mathrm{Var}\left(\psi_v^N\right) - \mathrm{Var}\left(\bar{\psi}_v^N\right)| \le \epsilon$. Set $\bar{\psi}_N^* := \max_{v\in V_N} \bar{\psi}_v^N$. Then there is a function, $l = l(\epsilon)$, with $l(\epsilon) \to 0$, as $\epsilon \to 0$, such that, if $\mathbb{E}\left[\bar{\psi}_u^N \bar{\psi}_v^N\right] \le \mathbb{E}\left[\psi_u^N \psi_v^N\right] + \epsilon$,*

$$\limsup_{N\to\infty} \tilde{d}\left(\psi_N^* - m_N, \bar{\psi}_N^* - m_N\right) \le l(\epsilon). \tag{5.28}$$

*Else if $\mathbb{E}\left[\bar{\psi}_u^N \bar{\psi}_v^N\right] + \epsilon \ge \mathbb{E}\left[\psi_u^N \psi_v^N\right]$, then*

$$\limsup_{N\to\infty} \tilde{d}\left(\bar{\psi}_N^* - m_N, \psi_N^* - m_N\right) \le l(\epsilon). \tag{5.29}$$

Lemma 5.5 and Lemma 5.6 allow to prove Lemma 5.4.

*Proof of Lemma 5.4:* As in (5.25), we write

$$\tilde{\psi}^N_{v,s,r_1,r_2} = \psi^N_v + s_1 g_{B_{r_1}(v)} + s_2 g_{B_{N/r_2}(v)}, \tag{5.30}$$

and analogously,

$$\tilde{S}^N_{v,s,r_1,r_2} = S^N_v + s_1 g_{B_{r_1}(v)} + s_2 g_{B_{N/r_2}(v)}, \tag{5.31}$$

where $s = (s_1, s_2) \in (0,\infty)^2$, $r_1, r_2 \in \mathbb{N}_+$ and $\{g_B\}_B$ being a collection of i.i.d. Gaussian random variables. Recall that $\mathcal{B}_r$ is a collection of sub-boxes of side length $r$ and that this forms a partition of $V_{\lfloor N/r \rfloor r}$. By (5.8), we only need to show that, for any $\delta > 0$,

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} d\left(\max_{v\in V^*_{N,\delta}} \psi^N_v - m_N, \max_{v\in V^*_{N,\delta}} S^N_v - m_N - 4\alpha\right) = 0. \tag{5.32}$$

Thus, fix $\delta > 0$ and let $\sigma_*^2 = C_\delta + 40\alpha$ with the constant $C_\delta$ as in Lemma 5.3, $\sigma_{lw} = (0, \sqrt{\sigma_*^2 + 4\alpha})$ and $\sigma_{up} = (\sigma_*, 0)$. We consider the two Gaussian fields $\left\{\tilde{\psi}^N_{v,L',0,L,\sqrt{\sigma_*^2+4\alpha}}\right\}_{v\in V^*_{N,\delta}}$ and $\left\{\tilde{S}^N_{v,L',\sigma_*,L,0}\right\}_{v\in V^*_{N,\delta}}$. By Lemma 5.3 *i.*, *ii.*, *iii.* and (5.10), one gets for $u, v \in V^*_{N,\delta}$,

$$\left|\operatorname{Var}\left(\tilde{\psi}^N_{v,L',0,L,\sqrt{\sigma_*^2+4\alpha}}\right) - \operatorname{Var}\left(\tilde{S}^N_{v,L',\sigma_*,L,0}\right)\right| \le \bar{\epsilon}_{N,KL,K'L'}, \tag{5.33}$$

and

$$\mathbb{E}\left[\tilde{S}^N_{u,L',\sigma_*^2,L,0}\tilde{S}^N_{v,L',\sigma_*,L,0}\right] \le \mathbb{E}\left[\tilde{\psi}^N_{u,L',0,L,\sqrt{\sigma_*^2+4\alpha}}\tilde{\psi}^N_{v,L',0,L,\sqrt{\sigma_*^2+4\alpha}}\right] + \bar{\epsilon}_{N,KL,K'L'}, \tag{5.34}$$

where $\limsup_{(L,K,L',K',N)\Rightarrow\infty} \bar{\epsilon}_{N,KL,K'L'} = 0$. Lemma 5.5 implies both

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} d\left(\max_{v\in V^*_{N,\delta}} \tilde{\psi}^N_{v,L',0,L,\sqrt{\sigma_*^2+4\alpha}} - m_N - (\sigma_*^2 + 4\alpha), \max_{v\in V^*_{N,\delta}} \psi^N_v - m_N\right) = 0, \tag{5.35}$$

and

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} d\left(\max_{v\in V^*_{N,\delta}} \tilde{S}^N_{v,L',\sigma_*,L,0} - m_N - \sigma_*^2, \max_{v\in V^*_{N,\delta}} S^N_v - m_N\right) = 0. \tag{5.36}$$

Having (5.33) and (5.34), Lemma 5.6 implies that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \tilde{d}\left(\max_{v\in V^*_{N,\delta}} \tilde{\psi}^N_{v,L',0,L,\sqrt{\sigma_*^2+4\alpha}} - m_N, \max_{v\in V^*_{N,\delta}} \tilde{S}^N_{v,L',\sigma_*,L,0} - m_N\right) = 0. \tag{5.37}$$

A combination of (5.35), (5.36) and (5.37), and using the triangle-inequality, gives stochastic domination in one direction, i.e.

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \tilde{d}\left(\max_{v\in V^*_{N,\delta}} \psi^N_v - m_N, \max_{v\in V^*_{N,\delta}} S^N_v - m_N - 4\alpha\right) = 0. \tag{5.38}$$

For the proof of the other direction of stochastic domination, consider instead the Gaussian fields $\left\{\tilde{\psi}^N_{v,L',\sqrt{\sigma_*^2+4\alpha},L,0}\right\}_{v\in V_N}$ and $\left\{\tilde{S}^N_{v,L',0,L,\sigma_*}\right\}$. This switches the roles in (5.34) and the rest of the proof carries out analogously, which concludes the proof of Lemma 5.4. □

5.2. **Convergence in law of the auxiliary field.** A key step in the proof of Theorem 2.1 is to establish a precise right-tail estimate for the maximum of the auxiliary process, which is provided in the following proposition. Before we state it, we introduce additional notation and make a preliminary observation. For $a, b \in [0,1]$, we write $\mathcal{I}_{\sigma^2}(a,b) = \int_a^b \sigma^2(x)\mathrm{d}x$. Let $\{S_v^N\}_{v\in V_N}$ be the field defined in (5.9), and set $S_v^{N,f} := S_v^N - S_v^{N,c}$. Recall the tail-bounds from [36, (2.6) in Theorem 2.1]. By Lemma 5.3 and applying Slepian's lemma, these carry over to $\{S_v^N\}_{v\in V_N}$. In particular, [36, (2.6) in Theorem 2.1] implies that there are constants $c_\alpha, C_\alpha > 0$ such that for $z \geq 0$,

$$c_\alpha e^{-2z} \leq \mathbb{P}\left(\max_{v\in V_N} S_v^N \geq m_N + z\right) \leq C_\alpha e^{-2z}. \tag{5.39}$$

**Lemma 5.7.** *Let $\gamma \in (1/2, 1)$ and fix $A > 0$. Then, for $z \in \mathbb{R}$,*

$$\mathbb{P}\left(\exists v \in V_N : S_v^N \geq m_N + z, S_v^{N,c} - 2\log(2)\sigma^2(0)(k+l) \notin [-A(k+l)^\gamma, A(k+l)^\gamma]\right) \leq Ce^{-\frac{A^2(k+l)^{2\gamma-1}}{2\log(2)\sigma^2(0)}}. \tag{5.40}$$

*Proof.* Denote by $\nu_{v,N}^c(\cdot)$ the density such that for any interval $I \subset \mathbb{R}$,

$$\int_I \nu_{v,N}^c(y)\mathrm{d}y = \mathbb{P}\left(S_v^{N,c} - 2\log(2)\sigma^2(0)(k+l) \in I\right). \tag{5.41}$$

For any $v \in V_N^\delta$, using a union bound the probability in (5.40) is bounded from above by

$$\begin{aligned}
&2^{2n}\int_{[-A(k+l)^\gamma, A(k+l)^\gamma]^c} \nu_{v,N}^c(x)\mathbb{P}\left(S_v^{N,f} \geq 2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{k+l}{n},1\right)n - \log(n)/4 + z - x\right)\mathrm{d}x \\
&= 2^{2n}\int_{[-A(k+l)^\gamma, A(k+l)^\gamma]^c} \frac{\exp\left[-2\log(2)\sigma_1^2(k+l) - 2x - \frac{x^2}{2\log(2)\sigma^2(0)(k+l)}\right]}{\sqrt{2\pi\log(2)\sigma^2(0)(k+l)}} \\
&\quad\times \exp\left[-2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{k+l}{n},1\right)n - 2\left(z - x - \frac{\log(n)}{4}\right) - \frac{\left(z-x-\frac{\log(n)}{4}\right)^2}{2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{k+l}{n},1\right)n}\right] \\
&\quad\times \frac{\sqrt{2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{k+l}{n},1\right)n}}{2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{k+l}{n},1\right)n - \frac{\log(n)}{4} + z - x}\mathrm{d}x.
\end{aligned} \tag{5.42}$$

The latter integral decays with $e^{-A^2(k+l)^{2\gamma-1}/(2\log(2)\sigma^2(0))}$, which allows to conclude the proof. □

Write $\bar{k} = k + l$ and $M_n(k,t) = 2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{k}{n},\frac{t}{n}\right)n - \frac{((t)\wedge(n-\bar{l}))\log(n)}{4(n-\bar{l})}$, for $t \in [k,n]$. Note that $m_N = M_n(0,n)$, for $n = \log_2 N$.

**Proposition 5.8.** *Let $\{S_v^N\}_{v\in V_N}$ be the field defined in* (5.9), *and set $S_v^{N,f} := S_v^N - S_v^{N,c}$. Then, there are constants $C_\alpha, c_\alpha > 0$, depending only on $\alpha$, and constants $c_\alpha \leq \beta^*_{K',L'} \leq C_\alpha$, such that*

$$\lim_{z\to\infty}\limsup_{(L',K',N)\Rightarrow\infty} |e^{2\log(2)(\bar{k})(1-\sigma^2(0))}e^{-2\bar{k}^\gamma}e^{2z}\,\mathbb{P}\left(\max_{v\in B_{N/KL,i}} S_v^{N,f} \geq M_n(\bar{k},n) - \bar{k}^\gamma + z\right) - \beta^*_{K',L'}| = 0. \tag{5.43}$$

*In particular, $\{\beta^*_{K',L'}\}_{K',L'\geq 0}$ depends on the variance parameters only through $\sigma(1)$.*

Note that, unlike previous tail estimates obtained in [36, Theorem 2.1], the estimates in Proposition 5.8 are precise estimates for the maximum far in front of the expected maximum. Nevertheless, the proofs are technically similar, i.e. both rely on a truncated second moment computation. The proof of Proposition 5.8 is postponed to Appendix C, as we first want to use it to finish the proof of Theorem 2.1. Proposition 5.8 allows to construct the limiting law of $(\max_{v\in V_N} S_v^N - m_N)_{N\geq 0}$, which is the contents of the following: Partition $[0,1]^2$ into $R = (KL)^2$ disjoint, equal-sized boxes. Let

$\{\beta^*_{K',L'}\}_{K',L'\geq 0}$ be given by Proposition 5.8. Then, there is a function, $\rho : \mathbb{R} \to \mathbb{R}$, that grows to infinity arbitrarily slowly, and such that

$$\lim_{z'\to\infty} \limsup_{(L',K',N)\Rightarrow\infty} \sup_{z'\leq z\leq\rho(K'L')} \left| e^{2z} e^{-2\bar{k}^\gamma} e^{2\log(2)\bar{k}(1-\sigma^2(0))} \mathbb{P}\left( \max_{v\in B_{N/KL,i}} S_v^{N,f} \geq M_n(\bar{k},n) + z - \bar{k}^\gamma \right) - \beta^*_{K',L'} \right| = 0. \tag{5.44}$$

Let $\{\varrho_{R,i}\}_{1\leq i\leq R}$ be independent Bernoulli random variables with

$$\mathbb{P}\left(\varrho_{R,i} = 1\right) = \beta^*_{K',L'} e^{2\bar{k}^\gamma} 2^{2\log(2)\bar{k}(\sigma^2(0)-1)}. \tag{5.45}$$

In addition, consider independent random variables $\{Y_{R,i}\}_{1\leq i\leq R}$ satisfying

$$\mathbb{P}\left(Y_{R,i} \geq x\right) = e^{-2x} e^{-2\bar{k}^\gamma}, \quad x \geq -\bar{k}^\gamma, \tag{5.46}$$

and let $\{Z_{R,i}\}_{1\leq i\leq R}$ be an independent Gaussian field with the same distribution as $\{S_v^{N,c}\}_{v\in V_N}$. Set

$$G_{R,i} \coloneqq \varrho_{R,i}(Y_{R,i} + 2\log(KL)(1-\sigma^2(0))) + (Z_{R,i} - 2\log(KL)), \tag{5.47}$$

and

$$G^*_{K,L,K',L'} \coloneqq \max_{\substack{1\leq i\leq R\\ \varrho_{R,i}=1}} G_{R,i}. \tag{5.48}$$

Let $\bar{\mu}_{K,L,K',L'}$ be the distribution of $G^*_{K,L,K',L'}$. Note that it is independent of $N$, which is essential for the proof of convergence in law. The following theorem reduces the proof of convergence in law of $\max_{v\in V_N} S_v^N - m_N$, to proving convergence of the sequence $\{\bar{\mu}_{K,L,K',L'}\}_{K,L,K',L'}$.

**Theorem 5.9.** *Let $\mu_N$ = law of $\left(\max_{v\in V_N} S_v^N - m_N\right)$. Then,*

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} d\left(\mu_N, \bar{\mu}_{K,L,K',L'}\right) = 0. \tag{5.49}$$

*In particular, there exists $\mu_\infty$ such that $\lim_{N\to\infty} d(\mu_N, \mu_\infty) = 0$.*

*Proof.* Denote by $\tau = \arg\max_{v\in V_N} S_v^N$ the (unique) particle achieving the maximal value. The correlation estimates in Lemma 5.3, together with Slepian's lemma and (2.2), imply that $\max_{v\in V_N} S_v^N - m_N$, as a sequence in $n$, is tight. Using this fact and the localization of $\{S_v^{N,c}\}_{v\in V_N}$ in Lemma 5.7, one obtains

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \mathbb{P}\left(S_\tau^{N,f} \geq M_n(\bar{k},n) - \bar{k}^\gamma\right) = 1. \tag{5.50}$$

Thus, assume that $S_\tau^{N,f} \geq M_n(\bar{k},n) - \bar{k}^\gamma$ holds. To exclude that $\max_{v\in V_N} S_v^{N,f}$ is too large, consider the event $\mathcal{E} = \cup_{i=1}^R \{\max_{v\in B_{N/KL,i}} S_v^{N,f} \geq M_n(\bar{k},n) + KL + \bar{k}^\gamma\}$. By a union and a Gaussian tail bound,

$$\begin{aligned}\mathbb{P}(\mathcal{E}) &\leq 2^{2\bar{k}} \mathbb{P}\left(\max_{v\in B_{N/KL,i}} S_v^{N,f} \geq M_n(\bar{k},n) + KL + \bar{k}^\gamma\right) \leq 2^{2n} \mathbb{P}\left(S_v^{N,f} \geq M_n(\bar{k},n) + KL + \bar{k}^\gamma\right) \\ &\leq C\exp\left[2\log(2)\sigma^2(0)(k+l) - 2KL - 2\bar{k}^\gamma\right].\end{aligned} \tag{5.51}$$

Thus, one obtains

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \mathbb{P}(\mathcal{E}) = 0. \tag{5.52}$$

Analogously, a union bound on the event $\mathcal{E}' = \cup_{i=1}^R \{Y_{R,i} \geq KL + \bar{k}^\gamma\}$ yields

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \limsup_{N\to\infty} \mathbb{P}(\mathcal{E}') = 0. \tag{5.53}$$

As a next step, we couple the centred fine field, $M^f_{n,i} := \max_{v\in B_{N/kl,i}} S^{N,f}_v - M_n(\bar{k}, n)$, to the approximating process $G_{R,i}$ defined in (5.47). By Proposition 5.8, there are $\epsilon^*_{N,KL,K'L'} > 0$, satisfying $\limsup_{(L,K,L',K',N)\Rightarrow\infty} \epsilon^*_{N,KL,K'L'} = 0$, and such that, for some $|\epsilon^\diamond| \leq \epsilon^*_{N,KL,K'L'}/4$,

$$\mathbb{P}\left(-A\bar{k}^\gamma + \epsilon^\diamond \leq M^f_{n,i} \leq KL + \bar{k}^\gamma\right) = \mathbb{P}\left(\varrho_{R,i} = 1,\ Y_{R,i} \leq KL + \bar{k}^\gamma\right) \tag{5.54}$$

and such that, for all $t$ with $-\bar{k}^\gamma - 1 \leq t \leq KL + \bar{k}^\gamma$,

$$\begin{aligned}\mathbb{P}\left(\varrho_{R,i} = 1, Y_{R,i} \leq t - \epsilon^*_{N,KL,K'L'}/2\right) &\leq \mathbb{P}\left(-\bar{k}^\gamma + \epsilon^\diamond \leq M^f_{n,i} \leq t\right)\\ &\leq \mathbb{P}\left(\varrho_{R,i} = 1, Y_{R,i} \leq t + \epsilon^*_{N,KL,K'L'}/2\right).\end{aligned} \tag{5.55}$$

Thus, there is a coupling of $\{M^f_{n,i} : 1 \leq i \leq R\}$ and $\{(\varrho_{R,i}, Y_{R,i}) : 1 \leq i \leq R\}$, such that, on the event $(\mathcal{E}\cup\mathcal{E}')^c$,

$$\varrho_{R,i} = 1,\ |Y_{R,i} - M^f_{n,i}| \leq \epsilon^*_{N,KL,K'L'}, \qquad \text{if } M^f_{n,i} \geq \epsilon^*_{N,KL,K'L'} \tag{5.56}$$

$$|Y_{R,i} - M^f_{n,i}| \leq \epsilon^*_{N,KL,K'L'}, \qquad \text{if } \varrho_{R,i} = 1. \tag{5.57}$$

Note that, for each $N$, one possibly needs a different coupling, since $M^f_{n,i}$ depends on $N$, whereas $(\varrho_{R,i}, Y_{R,i})$ does not. A short argument for the existence of such couplings is as follows: In the event $\mathcal{E}^c \cap \mathcal{E}'^{,c}$, (5.54) becomes

$$\mathbb{P}\left(-\bar{k}^\gamma + \epsilon^\diamond \leq M^f_{n,i}\right) = \mathbb{P}\left(\varrho_{R,i} = 1\right). \tag{5.58}$$

By (5.55) and since the random variables have distributions that are absolutely continuous with respect to the Lebesgue measure, there is an increasing function, $g : \mathbb{R} \to \mathbb{R}$, with $g(t) \in [t - \epsilon^*/2, t + \epsilon^*/2]$, for $-\bar{k}^\gamma - 1 \leq t \leq KL + \bar{k}^\gamma$, and such that

$$\mathbb{P}\left(\varrho_{R,i} = 1, Y_{R,i} \leq g(t)\right) = \mathbb{P}\left(-\bar{k}^\gamma + \epsilon^\diamond \leq M^f_{n,i} \leq t\right). \tag{5.59}$$

Let $-\bar{k}^\gamma - 1 = t_0 < \ldots < t_D = KL + \bar{k}^\gamma$ be an arbitrary partition. Define sets

$$A_j := \{\omega : \varrho_{R,i}(\omega) = 1, Y_{R,i}(\omega) \in [g(t_j), g(t_{j+1}))\}, \tag{5.60}$$

$$B_j := \{\omega : \epsilon^\diamond \leq M^f_{n,i}(\omega) \in [t_j, t_{j+1})\}. \tag{5.61}$$

In particular, for any $0 \leq j < D$, $\mathbb{P}\left(A_j\right) = \mathbb{P}\left(B_j\right)$. Define random variables $(\varrho'_{R,i}, Y'_{R,i})$, i.e for $\omega \in B_j \cap (\mathcal{E}\cup\mathcal{E}')^c$, set $Y'_{R,i}(\omega) = g(M^f_{n,i}(\omega))$ and such that, for all $\omega \in (\mathcal{E}\cup\mathcal{E}')^c \cap \left(\cup_j B_j\right)$, $\varrho'_{R,i}(\omega) = 1$. For $\omega \in \mathcal{E}\cup\mathcal{E}'$, set $\varrho'_{R,i}(\omega) = \varrho_{R,i}(\omega)$ and $Y'_{R,i}(\omega) = Y_{R,i}(\omega)$. Then $(\varrho'_{R,i}, Y'_{R,i}) \overset{d}{=} (\varrho_{R,i}, Y_{R,i})$, and $(\varrho'_{R,i}, Y'_{R,i})$ additionally satisfies both (5.56) and (5.57). Concerning the coarse field, one can couple such that $S^{N,c}_v = Z_{R,i}$, for $v \in B_{N/KL,i}$, $1 \leq i \leq R$, simply as they have the same law. Thus, there are couplings, such that, outside an event of vanishing probability as $(L, K, L', K', N) \Rightarrow \infty$,

$$\max_{v\in V_N}\left(S^N_v - m_N\right) - G^*_{K,L,K',L'} \leq 2\epsilon^*_{N,KL,K'L'}. \tag{5.62}$$

Let $\tau' = \arg\max_{1\leq i\leq R} G_{R,i}$. In the following, we exclude the case that the maximum of $G_{R,i}$ is achieved at $i = \tau'$ and when at the same time, $\varrho_{R,\tau'} = 0$. The first order of the maximum of $\{S^{N,c}_v\}_{v\in V_N}$ is given by $2\log(KL)\sigma(0)$ (see [17]), which is of order $O(\log(KL))$ less than subtracted in (5.47), and so, $Z_{R,i} - 2\log(KL) \to -\infty$, as $(L, K) \Rightarrow \infty$. Having this in mind, considering (5.62) and since $(\max_{v\in V_N} S^N_v - N)_{N\geq 0}$ is tight, it follows that

$$\limsup_{(L,K,L',K',N)\Rightarrow\infty} \mathbb{P}\left(\varrho_{R,\tau'} = 1\right) = 1. \tag{5.63}$$

By (5.56), (5.57) and (5.63), there are couplings, such that outside a set with probability tending to 0, as $(L, K, L', K', N) \Rightarrow \infty$, it holds that

$$\left|\max_{v\in V_N} S_v^N - m_N - G^*_{K,L,K',L'}\right| \leq 2\epsilon^*_{N,KL,K'L'}, \tag{5.64}$$

which proves (5.49). Moreover, (5.64) implies that $\mu_N$ is a Cauchy sequence and that there is $\mu_\infty$, such that $\lim_{N\to\infty} d(\mu_N, \mu_\infty) = 0$, which concludes the proof of Theorem 5.9. □

*Proof of Theorem 2.1:* Recall that $G^*_{K,L,K',L'}$ is a random variable with law $\bar{\mu}_{K,L,K',L'}$. The goal is to construct a sequence of random variables, $\{D_{K,L}\}_{K,L\geq 0}$, which are measurable with respect to $\mathcal{F}^c := \sigma\left(\{Z_{R,i}\}\right)_{i=1}^R$, with $R := (KL)^2$, and so that, for any $x \in \mathbb{R}$,

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty, x])}{\mathbb{E}\left[\exp(-\beta^*_{K',L'} D_{K,L} e^{-2x})\right]} = \liminf_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty, x])}{\mathbb{E}\left[\exp(-\beta^*_{K',L'} D_{K,L} e^{-2x})\right]} = 1. \tag{5.65}$$

Regarding (5.63), assume $\varrho_{R,\tau'} = 1$. Moreover, let

$$S_{R,i} := 2\log(KL)(1 + \sigma^2(0)) - Z_{R,i}, \quad \text{for } i = 1, \ldots, R. \tag{5.66}$$

For $x \in \mathbb{R}$, it holds

$$\begin{aligned} \bar{\mu}_{K,L,K',L'}((-\infty, x]) &= \mathbb{P}\left(G^*_{K,L,K',L'} \leq x\right) \\ &= \mathbb{E}\left[\prod_{i=1}^R \left(1 - \mathbb{P}\left(\varrho_{R,i}\left(Y_{R,i} + 2\log(KL)(1 - \sigma^2(0))\right) > 2\log(KL) - Z_{R,i} + x\right)\right) | \mathcal{F}^c\right]. \end{aligned} \tag{5.67}$$

A union bound on $\mathcal{D}^c = \{\min_{1\leq i\leq R} 2\log(KL) - Z_{R,i} \geq 0\}^c$, shows that $\limsup_{KL\to\infty} \mathbb{P}(\mathcal{D}) = 1$. Thus, on the event $\mathcal{D}$ and using (5.45), (5.46), (5.66), one deduces

$$\mathbb{P}\left(\varrho_{R,i} Y_{R,i} > 2\log(KL)\sigma^2(0) - Z_{R,i} + x | \mathcal{F}^c\right) = \beta^*_{K',L'} e^{-2(S_{R,i}+x)}. \tag{5.68}$$

Note that (5.68) tends to 0, as $KL \to \infty$. Using the fact that $e^{-\frac{x}{1-x}} \leq 1 - x \leq e^{-x}$, for $x < 1$, and inserting for $x$ the probability in (5.68), it follows that there is a non-negative sequence $\{\epsilon_{K,L}\}_{K,L\geq 0}$, satisfying $\limsup_{KL\to\infty} \epsilon_{K,L} = 0$, and such that

$$\begin{aligned} \exp\left(-(1 + \epsilon_{K,L})\beta^*_{K',L'} e^{-2(S_{R,i}+x)}\right) &\leq \mathbb{P}\left(\varrho_{R,i} Y_{R,i} \leq 2\log(KL)\sigma^2(0) - Z_{R,i} + x | \mathcal{F}^c\right) \\ &\leq \exp\left(-(1 - \epsilon_{K,L})\beta^*_{K',L'} e^{-2(S_{R,i}+x)}\right). \end{aligned} \tag{5.69}$$

Plugging (5.69) into (5.67) yields (5.65). Combining (5.65) with Theorem 5.9 implies that there is a constant $\beta^*$, such that

$$\limsup_{(K',L')\Rightarrow\infty} |\beta^*_{K',L'} - \beta^*| = 0. \tag{5.70}$$

Set

$$D_{K,L} = \sum_{i=1}^R e^{-2S_{R,i}}. \tag{5.71}$$

Combining (5.70) with (5.65), it follows that

$$\limsup_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty, x])}{\mathbb{E}\left[\exp(-\beta^* D_{K,L} e^{-2x})\right]} = \liminf_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{K,L,K',L'}((-\infty, x])}{\mathbb{E}\left[\exp(-\beta^* D_{K,L} e^{-2x})\right]} = 1. \tag{5.72}$$

Theorem 5.9 and (5.72) imply that $D_{K,L}$ converges weakly to a random variable $D$, as $(L, K) \Rightarrow \infty$. (5.71) shows that $D_{K,L}$ depends solely on $(KL)^2 = R$. Moreover, as $\bar{\mu}_{K,L,K',L'}$ is a tight sequence of laws, it follows that almost surely, $D > 0$. This concludes the proof of Theorem 2.1. □

Note that the random variables $\{D_{K,L}\}_{K,L\geq 0}$, defined in (5.71), are the analogue of the "McKean martingale" in variable-speed BBM (see [20, (1.14)]).

## Appendix A. Gaussian comparison and covariance estimates

**Theorem A.1** (Slepian's Lemma, [52, Theorem 3.11])**.** *Let $T = \{1,\dots,n\}$ and $X, Y$ be two centred Gaussian vectors. Assume further that it exist two subsets $A, B \subset T\times T$, so that*

$$\mathbb{E}[X_iX_j] \leq \mathbb{E}[Y_iY_j], \quad (i,j)\in A \tag{A.1}$$

$$\mathbb{E}[X_iX_j] \geq \mathbb{E}[Y_iY_j], \quad (i,j)\in B \tag{A.2}$$

$$\mathbb{E}[X_iX_j] = \mathbb{E}[Y_iY_j], \quad (i,j)\notin A\cup B. \tag{A.3}$$

*Suppose $f:\mathbb{R}^n\to\mathbb{R}$ is smooth, with at most exponential growth at infinity of $f$ and its first and second derivatives , and*

$$\partial_{ij} f \geq 0, \quad (i,j)\in A \tag{A.4}$$

$$\partial_{ij} f \leq 0, \quad (i,j)\in B. \tag{A.5}$$

*Then,*

$$\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]. \tag{A.6}$$

*Remark* A.2. We use Slepian's Lemma in a very particular setting: Assume that $\mathbb{E}\left[X_i^2\right] = \mathbb{E}\left[Y_i^2\right]$ and $\mathbb{E}\left[X_iX_j\right] \geq \mathbb{E}\left[Y_iY_j\right]$, for all $i,j\in T$. Then, for any $x\in\mathbb{R}$,

$$\mathbb{P}\left(\max_{i\in T} X_i > x\right) \leq \mathbb{P}\left(\max_{i\in T} Y_i > x\right), \tag{A.7}$$

and

$$\mathbb{E}\left[\max_{i\in T} X_i\right] \leq \mathbb{E}\left[\max_{i\in T} Y_i\right]. \tag{A.8}$$

**Theorem A.3** (Sudakov-Fernique, [38, Sudakov-Fernique])**.** *Let $I$ be an arbitrary set with cardinality $|I| = n$, $\{X_i\}_{i\in I}, \{Y_i\}_{i\in I}$ be two centred Gaussian vectors. Define $\gamma_{ij}^X := \mathbb{E}[(X_i-X_j)^2]$, $\gamma_{ij}^Y := \mathbb{E}[(Y_i-Y_j)^2]$. Let $\gamma := \max_{i,j}|\gamma_{ij}^X - \gamma_{ij}^Y|$. Then,*

$$\left|\mathbb{E}[X^*] - \mathbb{E}[Y^*]\right| \leq \sqrt{\gamma\log(n)}. \tag{A.9}$$

$$\text{If } \gamma_{ij}^X \leq \gamma_{ij}^Y \text{ for all } i,j \text{ then } \mathbb{E}[X^*] \leq \mathbb{E}[Y^*]. \tag{A.10}$$

In particular, if $\{X_i\}_{i\in I}, \{Y_i\}_{i\in I}$ are independent centred Gaussian fields, then

$$\mathbb{E}\left[\max_{i\in I}(X_i+Y_i)\right] \geq \mathbb{E}\left[\max_{i\in I} X_i\right]. \tag{A.11}$$

### A.1. **Covariance estimates.**

*Proof of Lemma 3.3.* The proof of statement *i.* is a simple adaptation of the proof of the analogue statement for finitely many scales [36, Lemma 3.3]. The third statement follows by a combination of *i.* with *ii.*. In the following, we prove statement *ii.*. Let $u,v\in V_N^\delta$ and denote by $b_N(u,v) = 1 - \frac{\log_+\|u-v\|_2}{\log N}$ the "branching scale". By the Gibbs-Markov property of the DGFF, increments $\nabla\phi_u^N(s)$, $\nabla\phi_v^N(s)$ beyond $b_N(u,v)$ are independent. By (1.5), one has

$$\mathbb{E}\left[\psi_u^N\psi_v^N\right] = \int_0^1\int_0^1 \sigma(s)\sigma(t)\mathbb{E}\left[\nabla\phi_u^N(s)\nabla\phi_v^N(t)\right]\mathrm{d}s\mathrm{d}t. \tag{A.12}$$

To compute the discrete gradients, it suffices to consider $\mathbb{E}\left[\phi_u^N(s)\phi_v^N(t)\right]$, for $s,t\in[0,1]$. Let $S$ be a simple random walk with hitting times $\tau_{\partial A} = \inf\{r\geq 0 : S_r\in\partial A\}$, for $A\subset\mathbb{Z}^2$. Let $c:\partial[-\frac{1}{2},\frac{1}{2}]^2\to\mathbb{R}^2$ be the continuous function, encoding the relative position on the boundary, such that, for $x\in(0,1)$,

$u \in \mathbb{Z}^2$ and $z \in \partial[xN+u]_{\lambda_i}$, $z = xN + u + c(z)N^{1-\lambda_i}$. In particular, the function $c$ is in both components absolutely bounded away from zero by $1/2$ and from above by $\sqrt{1/2}$. For $0 \le s < t \le 1$, we have

$$\begin{aligned}\mathbb{E}\left[\phi_u^N(s)\phi_v^N(t)\right] &= \sum_{\substack{x\in\partial[u]_s\\ y\in\partial[v]_t}} \mathbb{P}_u\left(S_{\tau_{\partial[u]_s}} = x\right)\mathbb{P}_v\left(S_{\tau_{\partial[v]_t}} = y\right)\mathbb{E}\left[\phi^N_{u+c(x)N^{1-s}}\phi^N_{v+c(y)N^{1-t}}\right]\\ &= \sum_{\substack{x\in\partial[u]_s\\ y\in\partial[v]_t}} \mathbb{P}_u\left(S_{\tau_{\partial[u]_s}} = x\right)\mathbb{P}_v\left(S_{\tau_{\partial[v]_t}} = y\right)\Big[-\mathfrak{a}\left(u - v + N^{1-s}(c(x) - c(y)N^{s-t})\right)\\ &\quad + \sum_{z\in\partial V_N} \mathbb{P}_{u+c(x)N^{1-s}}\left(S_{\tau_{\partial V_N}} = z\right)\mathfrak{a}(z - v - c(y)N^{1-t})\Big],\end{aligned} \tag{A.13}$$

where $\mathfrak{a}$ denotes the Potential kernel, which satisfies the asymptotics

$$\mathfrak{a}(x) = \log\|x\|_2 + c_0 + O(\|x\|_2^{-2}), \tag{A.14}$$

as $\|x\|_2 \to \infty$. Using this asymptotics and the approximate uniformity of the harmonic measure away from the boundary [15, Lemma B.5], the second sum in is about $\log(N) + O(1)$, and the first is about $\log(N^{1-s}) + O(1)$ if $s < t$ and if $\|u - v\|_2 \ll N^{1-s}$, i.e. $b_N(u, v) \le s - \epsilon_N$ with $\epsilon_N = 4/\log N$. In particular,

$$\int_{s+\epsilon_N}^1 \mathbb{E}\left[\phi_u^N(s)\nabla\phi_v^N(t)\right]\mathrm{d}t = 0, \tag{A.15}$$

and, if $\|u - v\|_2 < N^{1-t}$,

$$\int_0^{t-\epsilon_N} \mathbb{E}\left[\nabla\phi_u^N(s)\phi_v^N(t)\right]\mathrm{d}s = (t - \epsilon_N)\log(N) + O(1), \tag{A.16}$$

where the constant order term is uniform in $N$. (A.15) and (A.16) imply that the integral in (A.12) concentrates on the diagonal. Then, by independence of increments beyond the branching scale,

$$\begin{aligned}\mathbb{E}\left[\psi_u^N\psi_v^N\right] = \int_0^1 \sigma^2(s)\mathbb{E}\left[\nabla\phi_u^N(s)\nabla\phi_v^N(s)\right]\mathrm{d}s &= \int_0^{b_N(u,v)-\epsilon_N} \sigma^2(s)\mathbb{E}\left[\nabla\phi_u^N(s)\nabla\phi_v^N(s)\right]\mathrm{d}s\\ &\quad + \int_{b_N(u,v)-\epsilon_N}^{b_N(u,v)} \sigma^2(s)\mathbb{E}\left[\nabla\phi_u^N(s)\nabla\phi_v^N(s)\right]\mathrm{d}s.\end{aligned} \tag{A.17}$$

By Cauchy-Schwarz, the second integral in (A.17) is absolutely bounded by a constant $C$ which depends on $\sigma$ but is independent of $N$. To bound the first integral in (A.17) with $s = t$, note that in (A.13) there are $2\pi\|u - v\|_2$ many pairs, $x \in \partial[u]_s$, $y \in [v]_s$ that have distance less than $\|u - v\|_2$ at scale $b_N(u, v) - \epsilon_N$. By [15, Lemma B.5] the harmonic measures evaluate to approximately $1/4\|u-v\|_2$. Thus, the sum over these particles is at most of order $O\left(\frac{\log_+\|u-v\|}{\|u-v\|_2}\right) = O(1)$. For summands $x \in \partial[u]_s$, $y \in [v]_s$ and $\|x - y\|_2 \ge \|u - v\|_2$, we use (A.14) and [15, Lemma B.5], to deduce that the first integral in (A.17) equals

$$\log N \int_0^{b_N(u,v)-\epsilon_N} \sigma^2(s)\mathrm{d}s + O(1) = \log N \mathcal{I}_{\sigma^2}\left(1 - \frac{\log_+(\|u - v\|_2)}{\log N}\right) + O(1). \tag{A.18}$$

This concludes the proof of the extension. □

*Proof of Lemma 3.4.* We start with the proof of the first statement. First note that by Lemma 3.3 *ii.*, for $xN + u, xN + v \in V_N^\delta$, it holds that

$$\mathbb{E}\left[\psi^N_{xN+u}\psi^N_{xN+v}\right] = \log(N) + O(1). \tag{A.19}$$

Thus, one has to show that, as $N \to \infty$, the constant order contribution may depend on $u, v$, but not on $x$ and apart from this, has fluctuations which vanish as $N \to \infty$. By (1.5), one has

$$\begin{aligned}\mathbb{E}\left[\psi^N_{xN+u}\psi^N_{xN+v}\right] = \int_0^1 \sigma^2(s)\mathbb{E}\left[\nabla\phi^N_{xN+u}(s)\nabla\phi^N_{xN+v}(s)\right]\mathrm{d}s = \int_0^{\lambda_0} \sigma^2(s)\mathbb{E}\left[\nabla\phi^N_{xN+u}(s)\nabla\phi^N_{xN+v}(s)\right]\mathrm{d}s \\ + \int_{\lambda_0}^{1-\lambda_1} \sigma^2(s)\mathbb{E}\left[\nabla\phi^N_{xN+u}(s)\nabla\phi^N_{xN+v}(s)\right]\mathrm{d}s + \int_{1-\lambda_1}^1 \sigma^2(s)\mathbb{E}\left[\nabla\phi^N_{xN+u}(s)\nabla\phi^N_{xN+v}(s)\right]\mathrm{d}s.\end{aligned} \tag{A.20}$$

We choose $\lambda_0, \lambda_1 = O\left(\frac{\log\log N}{\log N}\right)$, such that

$$\sigma^2(0)\lambda_0 + \sigma^2(1)\lambda_1 + \int_{\lambda_0}^{1-\lambda_1} \sigma^2(s)\mathrm{d}s = 1. \tag{A.21}$$

Note that we have by assumptions $\|u - v\|_2 \leq L$, for $L \ll N$ and thus, we can assume $b_N(xN + u, xN + v) > 1 - \lambda_1$. For the first integral in (A.20), we use a Taylor expansion of $\sigma$ at 0, i.e. $\sigma(s) = \sigma(0) + \sigma'(0)s + o(\sigma'(0)s)$, for $s \geq 0$ small. Thus, the first integral becomes

$$\begin{aligned}\int_0^{\lambda_0} \sigma^2(0)\mathbb{E}\left[\nabla\phi^N_{xN+u}(s)\nabla\phi^N_{xN+v}(s)\right]\mathrm{d}s + O(\lambda_0^2\log N\sigma(0)\sigma'(0)) \\ = \sigma^2(0)\mathbb{E}\left[\phi^N_{xN+u}(\lambda_0)\phi^N_{xN+v}(\lambda_0)\right] + O(\lambda_0^2\log N\sigma(0)\sigma'(0)),\end{aligned} \tag{A.22}$$

where the error term vanishes as $N \to \infty$, since $\lambda_0^2 \log N = O\left(\frac{\log\log N}{\log N}\right)$. Similarly, by a Taylor expansion of $\sigma$ at 1, i.e. $\sigma(s) = \sigma(1) - \sigma'(1)(1 - s) + o(\sigma'(1)(1 - s))$, for $s < 1$ close to one, the last integral in (A.20) can be computed as

$$\begin{aligned}&\int_{1-\lambda_1}^1 \sigma^2(1)\mathbb{E}\left[\nabla\phi^N_{xN+u}(s)\nabla\phi^N_{xN+v}(s)\right]\mathrm{d}s + O(\lambda_1^2\log N\sigma(1)\sigma'(1)) \\ &= \sigma^2(1)\mathbb{E}\left[\left(\phi^N_{xN+u}(1) - \phi^N_{xN+u}(1-\lambda_1)\right)\left(\phi^N_{xN+v}(1) - \phi^N_{xN+v}(1-\lambda_1)\right)\right] + O(\lambda_1^2\log N\sigma(1)\sigma'(1)).\end{aligned} \tag{A.23}$$

Similarly as in (A.22), the error term vanishes as $N \to \infty$. In all three cases in (A.20), using (A.22) and (A.23), it suffices to compute quantities of the form $\mathbb{E}\left[\phi^N_{xN+u}(s)\phi^N_{xN+v}(s)\right]$. The case when $s = 0$ is trivial since, for any $v \in V_N$, $\phi^N_v(0) = 0$, as the harmonic average of the value zero is zero. Note that by [15, (B.5),(B.6),(B.7)] one has, for $v, w \in V_N$,

$$\mathbb{E}\left[\phi^N_v\phi^N_w\right] = -\mathfrak{a}(v - w) + \sum_{z\in\partial V_N} \mathbb{P}_v\left(S_{\tau_{\partial V_N}} = w\right)\mathfrak{a}(z - w), \tag{A.24}$$

where $\mathfrak{a}$ denotes the potential kernel, with representation as in (A.14). First, consider the case when $0 < s < 1$. Note that the discrete harmonic measure converges weakly to the harmonic measure associated to Brownian motion [13, Lemma 1.23], i.e. to the measure $\Pi(x, A) := \mathbb{P}_x\left(B_{\tau_{\partial[0,1]^2}} \in A\right)$, where $(B_t)_{t\geq 0}$ is Brownian motion in $\mathbb{R}^2$ killed upon exiting $[0, 1]^2$. Moreover, since the logarithm is continuous and bounded in a neighbourhood of $\partial[0, 1]^2$, using (A.24) and the weak convergence of the discrete harmonic measure, one obtains

$$\mathbb{E}\left[\phi^N_{xN+u}(s)\phi^N_{xN+v}(s)\right] = \sum_{\substack{z\in\partial[xN+u]_s\\ y\in\partial[xN+v]_s}} \mathbb{P}_{xN+u}\left(S_{\tau_{\partial[xN+u]_s}} = z\right)\mathbb{P}_{xN+v}\left(S_{\tau_{\partial[xN+v]_{\lambda_i}}} = y\right)\mathbb{E}\left[\phi^N_z\phi^N_y\right]$$
$$= \sum_{\substack{z\in\partial[xN+u]_s\\ y\in\partial[xN+v]_s}} \mathbb{P}_{xN+u}\left(S_{\tau_{\partial[xN+u]_s}} = z\right)\mathbb{P}_{xN+v}\left(S_{\tau_{\partial[xN+v]_s}} = y\right)\Big(-\mathfrak{a}(u-v+N^{1-s}(c(z)-(y)))$$
$$+\sum_{w\in\partial V_N}\mathbb{P}_{xN+u+N^{1-s}}\left(S_{\tau_{\partial V_N}} = w\right)\mathfrak{a}(w-xN-v-N^{1-s}c(y))\Big)$$
$$= -\log N^{1-s} + \log N + f(x) + o(1) = s\log N + f(x) + o(1), \tag{A.25}$$

where $f(x) = \int_{z\in\partial[0,1]^2}\Pi(x,\mathrm{d}z)\log\|z-x\|_2$. In particular, $f$ is continuous. Using (A.25) and (A.22), the first integral in (A.20) can be rewritten as

$$\sigma^2(0)\left(\lambda_0\log N + f(x)\right) + o(1). \tag{A.26}$$

For the remaining case, $s=1$, call $e_i$ the $i-$th unit vector. By (A.24) and using weak convergence of the discrete harmonic measure [13, Lemma 1.23],

$$\mathbb{E}\left[\phi^N_{xN+u}(1)\phi^N_{xN+v}(1)\right] = \mathbb{E}\left[\phi^N_{xN+u}\phi^N_{xN+v}\right] = \log N + f(x) - \mathfrak{a}(u,v) + o(1). \tag{A.27}$$

Using (A.23) and (A.27) allows to rewrite the third integral in (A.20) as

$$\sigma^2(1)\left(\lambda_1\log N - \mathfrak{a}(u,v)\right) + o(1). \tag{A.28}$$

Inserting (A.26), (A.28) into (A.20), using (A.25), (A.21) and $\mathcal{I}_{\sigma^2}(1)=1$, one obtains,

$$\mathbb{E}\left[\psi^N_{xN+u}\psi^N_{xN+v}\right] = \log N + \sigma(0)^2 f(x) + \sigma(1)^2 g(u,v) + o(1), \tag{A.29}$$

with $g(u,v) = -\mathfrak{a}(u,v)$ and where $o(1)\to 0$, as $N\to\infty$. This concludes the proof of statement $i.$ in Lemma 3.4.

The covariances in the off-diagonal case, i.e. when $x\neq y\in(0,1)^2$, $\|x-y\|_2\geq 1/L$, can be computed similarly, now by Taylor expansion of the variance $\sigma(s)$ at 0. First note that, for $\lambda = \frac{\log\log N}{\log N}$ and $N$ large enough, $\lambda > b_N(xN,yN)$. Thus,

$$\mathbb{E}\left[\psi^N_{xN}\psi^N_{yN}\right] = \int_0^\lambda \sigma^2(s)\mathbb{E}\left[\nabla\phi^N_{xN}(s)\nabla\phi^N_{yN}(s)\right]\mathrm{d}s = \sigma^2(0)\mathbb{E}\left[\phi^N_{xN}(\lambda)\phi^N_{yN}(\lambda)\right] + O(\sigma(0)\sigma'(0)\lambda^2\log N). \tag{A.30}$$

By choice of $\lambda$, $O\left(\sigma(0)\sigma'(0)\lambda\log N\right) = O\left(\sigma(0)\sigma'(0)\frac{\log\log N}{\log N}\right) = o(1)$.

$$\sigma^2(0)\mathbb{E}\left[\phi^N_{xN}(\lambda)\phi^N_{yN}(\lambda)\right] = \sigma^2(0)\sum_{\substack{u\in\partial[xN]_\lambda\\ v\in\partial[xN]_\lambda}}\mathbb{P}_{xN}\left(S_{\tau_{\partial[xN]_\lambda}} = u\right)\mathbb{P}_{yN}\left(S_{\tau_{\partial[yN]_{\lambda_1}}} = v\right)\mathbb{E}\left[\phi^N_u\phi^N_v\right]. \tag{A.31}$$

Using (A.24) and previous notation allows to reformulate (A.31) as

$$\sigma^2(0)\sum_{\substack{u\in\partial[xN]_\lambda\\ v\in\partial[xN]_\lambda}}\mathbb{P}_{xN}\left(S_{\tau_{\partial[xN]_\lambda}} = u\right)\mathbb{P}_{yN}\left(S_{\tau_{\partial[yN]_\lambda}} = v\right)\Big(-\mathfrak{a}(N(x-y+N^{-\lambda}(c(u)-c(v))))$$
$$+\sum_{w\in\partial V_N}\mathbb{P}_{xN}\left(S_{\tau_{\partial V_N}} = w\right)\mathfrak{a}(w-yN)\Big). \tag{A.32}$$

Using (A.14), we rewrite (A.32)

$$\sigma^2(0) \sum_{\substack{u\in\partial[xN]_\lambda\\ v\in\partial[xN]_\lambda}} \mathbb{P}_{xN}\left(S_{\tau_{\partial[xN]_\lambda}} = u\right)\mathbb{P}_{yN}\left(S_{\tau_{\partial[yN]_\lambda}} = v\right)(-\log N - \log\|x-y\|_2 - c_0 + o(1)$$

$$+ \sum_{w\in\partial V_N} \mathbb{P}_{xN}\left(S_{\tau_{\partial V_N}} = w\right)(\log N + \log\|c(w)-y\|_2 + c_0 + o(1))\Bigg)$$

$$= \sigma^2(0)h(x,y) + o(1), \tag{A.33}$$

where $h(x,y) = -\log\|x-y\|_2 + \int_{\partial[0,1]^2}\Pi(x,\mathrm{d}z)\log\|z-y\|_2$, by the weak convergence of the harmonic measure to $\Pi$. In particular, $h$ is continuous on $[0,1]^2\setminus\{(x,x):\ x\in[0,1]\}$. This concludes the proof of the second statement and thus, of Lemma 3.4. □

*Proof of Lemma 5.3.* : We start with the proof of $(i)$. Let $i'$ be such that $u,v\in B_{L',i}\subset B_{K'L',i'}$. By (5.9), one has

$$\begin{aligned} S_u^N - S_v^N &= \left(S_u^{N,c} - S_v^{N,c}\right) + \left(S_u^{N,m} - S_v^{N,m}\right) + \left(S_u^{N,b} - S_v^{N,b}\right) + \Phi_{i'}\left(a_{K'L',\bar u} - a_{K'L',\bar v}\right)\\ &= S_u^{N,b} - S_v^{N,b} + \Phi_{i'}\left(a_{K'L',\bar u} - a_{K'L',\bar v}\right). \end{aligned} \tag{A.34}$$

In particular, by (5.19), $|a_{K'L',\bar u} - a_{K'L',\bar v}| \le \epsilon_{N,KL,K'L'}$, and so

$$\begin{aligned} &\left|\mathbb{E}\left[\left(S_u^N - S_v^N\right)^2\right] - \mathbb{E}\left[\left(\psi_u^N - \psi_v^N\right)^2\right]\right|\\ &\le 4\epsilon_{N,KL,K'L'} + \left|\sigma^2(1)\mathbb{E}\left[\left(\phi^{K'L'}_{u-v_{K'L',i'}} - \phi^{K'L'}_{v-v_{K'L',i'}}\right)^2\right] - \mathbb{E}\left[\left(\psi_u^N - \psi_v^N\right)^2\right]\right|. \end{aligned} \tag{A.35}$$

Using the tower property of conditional expectation, conditioning $\{\psi_v^N\}_{v\in V_N}$ on $\sigma\left(\phi_w^N : w\in[v_{K'L',i'}]^c_{K'L'}\right)$ and using (A.27) and Lemma 3.4 *ii.*, it follows that

$$\limsup_{(L,K,L',K',N)\Rightarrow\infty}\ \sup_{\substack{u,v\in B_{L',i}\cap V^*_{N,\delta}\\ 1\le i\le (N/L')^2}} \left|\sigma^2(1)\mathbb{E}\left[\left(\phi^{K'L'}_{u-v_{K'L',i'}} - \phi^{K'L'}_{v-v_{K'L',i'}}\right)^2\right] - \mathbb{E}\left[\left(\psi_u^N - \psi_v^N\right)^2\right]\right| = 0. \tag{A.36}$$

Statement *i.* follows from (A.36) together with (A.35). Next, we prove *ii.*. Let $i'\neq j'$ be such that $u\in B_{N/KL,i'}$, $v\in B_{N/KL,j'}$ and assume without loss of generality that $N\gg K'\gg L'\gg K\gg L\gg 1/\delta$. Since vertices $u$ and $v$ belong to distinct boxes of side length $N/KL$ and thus, also to distinct $K'L'$−boxes, both $\mathbb{E}\left[S_u^{N,m}S_v^{N,m}\right] = 0$ and $\mathbb{E}\left[S_u^{N,b}S_v^{N,b}\right] = 0$. Using these observations, scaling the DGFF from $V_{KL}$ to $V_N$ and by (A.24),

$$\mathbb{E}\left[S_u^NS_v^N\right] = \mathbb{E}\left[S_u^{N,c}S_v^{N,c}\right] = \sigma^2(0)\mathbb{E}\left[\phi^{KL}_{w_{i'}}\phi^{KL}_{w_{j'}}\right] = \sigma^2(0)\mathbb{E}\left[\phi^N_{v_{N/KL,i'}}\phi^N_{v_{N/KL,j'}}\right] + o(1). \tag{A.37}$$

Since $\|\frac{v_{N/KL,i'}-u}{N}\|_2, \|\frac{v_{N/KL,j'}-v}{N}\|_2 = O\left(\frac{1}{KL}\right)$, [15, Lemma B.14] implies

$$\limsup_{(L,K,L',K',N)\Rightarrow\infty}\ \sup_{\substack{u\in B_{N/KL,i'}\cap V^*_{N,\delta}\\ v\in B_{N/KL,j'}\cap V^*_{N,\delta},\, i'\neq j'}} \left|\mathbb{E}\left[S_u^NS_v^N\right] - \sigma^2(0)\mathbb{E}\left[\phi_u^N\phi_v^N\right]\right| = 0. \tag{A.38}$$

On the other hand, the vertices $u,v$ are at distance of order $N/KL$ away from each other. Since considering limits of the form $(L,K,L',K',N)\Rightarrow\infty$, one can assume that $N/KL\gg N^{1-\lambda_1}$, and thus $\mathbb{E}\left[\phi_u^N\phi_v^N\right] = \mathbb{E}\left[\phi_u^N(\lambda_1)\phi_v^N(\lambda_1)\right]$. Therefore, by a Taylor expansion of $\sigma$ at 0 as in (A.30),

$$\left|\mathbb{E}\left[\psi_u^N\psi_v^N\right] - \sigma^2(0)\mathbb{E}\left[\phi_u^N\phi_v^N\right]\right| = \left|\sigma^2(0)\mathbb{E}\left[\phi_u^N(\lambda_1)\phi_v^N(\lambda_1)\right] - \sigma^2(0)\mathbb{E}\left[\phi_u^N\phi_v^N\right]\right| + o(1)\to 0, \tag{A.39}$$

as $N\to\infty$. (A.38) together with (A.39) implies statement *ii.*. Note that for statement *iii.*, one has $\|u-v\|_2 = O(N/L)$. This allows to approximate as in (A.39). Note that in this case, there is a constant

$L \geq c(u,v) > 0$, such that the leading order of the first covariance is given by $\log(\|u-v\|_2 + N^{1-\lambda_1}) - \log(\|u-v\|_2) = \log\left(1+\frac{cL}{N^{\lambda_1}}\right)$. In the following, we distinguish three cases:

(1) $u,v \in B_{K'L',i}$ but $u \in B_{L',i'}$ and $v \in B_{L',j'}$
(2) $u,v \in B_{N/KL,i}$, but $u \in B_{K'L',\tilde{i}}$ and $v \in B_{K'L',\tilde{j}}$
(3) $u \in B_{N/KL,i} \cap B_{L',i'}$ and $v \in B_{N/KL,j} \cap B_{L',j'}$.

In case (1), $S_u^{N,c} = S_v^{N,c}$ and $S_u^{N,m} = S_v^{N,m}$ and so, using notation from the proof of Lemma 3.4, by (A.24), (5.13), (5.17) and as in (A.27),

$$\begin{aligned}\mathbb{E}\left[S_u^N S_v^N\right] &= \mathrm{Var}\left[S_u^{N,c}S_v^{N,c}\right] + \mathrm{Var}\left[S_u^{N,m}\right] + \mathbb{E}\left[S_u^{N,b}S_v^{N,b}\right] + a_{K'L',\bar{u}}a_{K'L',\bar{v}} + o(1)\\ &= \log N + \sigma^2(0) f\left(\frac{u}{N}\right) + \sigma^2(1)\left(-\mathfrak{a}(u-v) + \int_{\partial[0,1]^2} \Pi\left(\frac{u}{N},\mathrm{d}z\right)\mathfrak{a}\left(z - \frac{v}{K'L'}\right)\right)\\ &\quad + a_{K'L',\bar{u}}a_{K'L',\bar{v}} + o(1). \end{aligned} \tag{A.40}$$

Since $u,v \in V^*_{N,\delta}$ are away from the boundary, the integral in (A.40) is bounded by a constant $C_\delta$, depending on $\delta$. Thus, (A.40) can be written as $\log N - \sigma^2(1)\log_+\|u-v\|_2 + O(1)$, where the constant order term is bounded by $8\alpha + C_\delta$. By Lemma 3.3 *ii.*, $\mathbb{E}\left[\psi_u^N\psi_v^N\right] = \log N - \sigma^2(1)\log_+\|u-v\|_2 + O(1)$, where the constant order term is bounded by $\alpha$. Thus, statement *ii.* follows in case (1). In case (2), $\mathbb{E}\left[S_u^{N,b}S_v^{N,b}\right] = 0$. Thus, there is a constant $c_1 > 0$, such that

$$\mathbb{E}\left[S_u^N S_v^N\right] = \mathbb{E}\left[S_u^{N,c}S_v^{N,c}\right] + \mathbb{E}\left[S_u^{N,m}S_v^{N,m}\right] + c_1. \tag{A.41}$$

To estimate the first covariance in (A.41), apply (A.24) and for the second, note that $\{S_v^{N,m}\}_{v\in V_N}$ is a MIBRW, and thus, using Lemma 3.3 *i.* and *ii.*, statement *ii.* follows, in case (2). In case (3), $\mathbb{E}\left[S_u^{N,m}S_v^{N,m}\right] = 0$ and $\mathbb{E}\left[S_u^{N,b}S_v^{N,b}\right] = 0$. By scaling the DGFF as in (A.37) and using (A.24),

$$\mathbb{E}\left[S_u^N S_v^N\right] = \mathbb{E}\left[S_u^{N,c}S_v^{N,c}\right] = \sigma^2(0)\left(\log(N) - \log_+(\|u-v\|_2)\right) + c + o(1), \tag{A.42}$$

where $c$ is a bounded constant depending on $\delta$ and where the error $o(1)$ vanishes as $N \to \infty$. The same reasoning applied to $\mathbb{E}\left[\psi_u^N\psi_v^N\right]$ as in (A.39), implies the claim in this remaining case and thereby concludes the proof Lemma 5.3. □

## Appendix B. Proof of Lemma 5.5 and Lemma 5.6

We prove Lemma 5.5 in the case of the scale-inhomogeneous DGFF. The proof for the approximating field, $\{S_v^N\}_{V\in V_N}$, is essentially identical. This is due to Lemma 5.3, which allows to use Gaussian comparison to reduce the proof to the one we provide.

**Lemma B.1.** *Let $\{g_v^N : u \in V_N\}$ be a collection of random variables, independent of the centred Gaussian field, $\{\bar{\psi}_u^N : u \in V_N\}$, and the 2d scale-inhomogeneous DGFF, $\{\psi_u^N : u \in V_N\}$, such that*

$$\mathbb{P}\left(g_u^N \geq 1 + y\right) \leq e^{-y^2} \quad \forall u \in V_N. \tag{B.1}$$

*Assume further that there is some $\delta > 0$, such that, for all $v,w \in V_N$, $\mathbb{E}\left[\bar{\psi}_v^N\bar{\psi}_w^N\right] - \mathbb{E}\left[\psi_v^N\psi_w^N\right]| \leq \delta$. Then, there is a constant $C = C(\alpha)$, such that, for any $\epsilon > 0$, $N \in \mathbb{N}$ and $x \geq -\sqrt{\epsilon}$,*

$$\mathbb{P}\left(\max_{v\in V_N}\left(\bar{\psi}_v^N + \epsilon g_v^N\right) \geq m_N + x\right) \leq \mathbb{P}\left(\max_{v\in V_N}\bar{\psi}_v^N \geq m_N + x - \sqrt{\epsilon}\right)\left(Ce^{-C^{-1}\epsilon^{-1}}\right). \tag{B.2}$$

*Proof.* Let $\Gamma_y := \{v \in V_N : y/2 \leq \epsilon g_v^N \leq y\}$. Then,

$$\begin{aligned}\mathbb{P}\left(\max_{v\in V_N}\left(\bar{\psi}_v^N + \epsilon g_v^N\right) \geq m_N + x\right) \leq& \mathbb{P}\left(\max_{v\in V_N}\bar{\psi}_v^N \geq m_N + x - \sqrt{\epsilon}\right)\\ &+ \sum_{i=0}^\infty \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\max_{v\in\Gamma_{2^i\sqrt{\epsilon}}}\bar{\psi}_v^N \geq m_N + x - 2^i\sqrt{\epsilon}}\middle|\Gamma_{2^i\sqrt{\epsilon}}\right]\right]. \end{aligned} \tag{B.3}$$

By Proposition 5.1, the last sum in (B.3) can be bounded from above by

$$\tilde{c}e^{-2x}\sum_{i=0}^{\infty}\mathbb{E}\left[|\Gamma_{2^i\sqrt{\epsilon}}|/|V_N|\right]e^{2^{i+1}\sqrt{\epsilon}}, \tag{B.4}$$

with $\tilde{c}>0$ being a finite constant. By assumption (B.1), one has

$$\mathbb{E}\left[|\Gamma_{2^i\sqrt{\epsilon}}|/|V_N|\right]\leq e^{-4^i(C\epsilon)^{-1}}. \tag{B.5}$$

Thus, (B.4) is bounded from above by $\tilde{c}e^{-2x}e^{-(C\epsilon)^{-1}}$. This concludes the proof of Lemma B.1. □

**Proposition B.2.** *Let $\{\varphi_v^N\}_{v\in V_N}, \{\tilde{\varphi}_v^N\}_{v\in V_N}$ be two independent centred Gaussian fields satisfying the covariance estimates in Lemma 5.3, and let $\{g_B : B\subset V_N\}$ be a family of independent standard Gaussians. Moreover, let $\tilde{\sigma}=(\tilde{\sigma}_1,\tilde{\sigma}_2)\in\mathbb{R}_+^2$ and $\{\varphi_v^{N,r,\tilde{\sigma}} : v\in V_N\}$ and $\{\varphi_v^{N,\tilde{\sigma},*} : v\in V_N\}$ be two centred Gaussian fields, given by*

$$\varphi_v^{N,r_1,r_2,\tilde{\sigma}}=\varphi_v^N+\tilde{\sigma}_1 g_{B_{v,r_1}}+\tilde{\sigma}_2 g_{B_{v,N/r_2}}, \tag{B.6}$$

*and*

$$\varphi_v^{N,\tilde{\sigma},*}=\varphi_v^N+\sqrt{\frac{\|\tilde{\sigma}\|_2^2}{\log N}}\tilde{\varphi}_v^N, \tag{B.7}$$

*for $v\in V_N$. Set $M_{N,r_1,r_2,\tilde{\sigma}}=\max_{v\in V_N}\varphi_v^{N,r_1,r_2,\tilde{\sigma}}$, and likewise, $M_{N,\tilde{\sigma},*}=\max_{v\in V_N}\varphi_v^{N,\tilde{\sigma},*}$. Then, for any fixed $\tilde{\sigma}\in(0,\infty)^2$,*

$$\lim_{r_1,r_2\to\infty}\limsup_{N\to\infty} d\left(M_{N,r_1,r_2,\tilde{\sigma}}-m_N, M_{N,\tilde{\sigma},*}-m_N\right)=0. \tag{B.8}$$

*Proof.* Partition $V_N$ into boxes of side length $N/r_2$ and denote by $\mathcal{B}$ the collection of these boxes. Fix arbitrary $\delta>0$, for $B\in\mathcal{B}$ denote by $B_\delta$ the box with the same centre as $B$, but with side length $(1-\delta)N/r_2$. The union of such restricted boxes, we call $V_{N,\delta}=\bigcup_{B\in\mathcal{B}}B_\delta$. The maxima over these sets, we denote by $M_{N,r_1,r_2,\tilde{\sigma},\delta}=\max_{v\in V_{N,\delta}}\varphi_v^{N,r_1,r_2,\tilde{\sigma}}$ and $M_{N,\tilde{\sigma},*,\delta}=\max_{v\in V_{N,\delta}}\varphi_v^{N,\tilde{\sigma},*}$. By Proposition 5.1,

$$\lim_{\delta\to0}\lim_{N\to\infty}\mathbb{P}\left(M_{N,r_1,r_2,\tilde{\sigma},\delta}\neq M_{N,r_1,r_2,\tilde{\sigma}}\right)=\lim_{\delta\to0}\lim_{N\to\infty}\mathbb{P}\left(M_{N,\tilde{\sigma},*,\delta}\neq M_{N,\tilde{\sigma},*}\right)=0. \tag{B.9}$$

Thus, it suffices to show equation (B.8) with $M_{N,r_1,r_2,\tilde{\sigma},\delta}-m_N$ and $M_{N,\tilde{\sigma},*,\delta}-m_N$. Next, we show that the main contribution to the maximum is given by $\{\varphi_v^N\}_{v\in V_N}$, while the perturbation fields only have a negligible influence. For $B\in\mathcal{B}$, let $z_b\in B$ the maximizing element, i.e. $\max_{v\in B_\delta}\varphi_v^N=\varphi_{z_B}^N$. The claim is that

$$\begin{aligned}\lim_{r_1,r_2\to\infty}\limsup_{N\to\infty}&\mathbb{P}\left(|M_{N,r_1,r_2,\tilde{\sigma},\delta}-\max_{B\in\mathcal{B}}\varphi_{z_B}^{N,r_1,r_2,\tilde{\sigma}}|\geq\frac{1}{\log n}\right)\\&=\limsup_{N\to\infty}\mathbb{P}\left(|M_{N,\tilde{\sigma},*,\delta}-\max_{B\in\mathcal{B}}\varphi_{z_B}^{N,\tilde{\sigma},*}|\geq\frac{1}{\log n}\right)=0.\end{aligned} \tag{B.10}$$

We first show how Proposition B.2 follows from (B.10). Assuming (B.10), conditioning on the positions of the maximum, $\{z_B\}_{B\in\mathcal{B}}$, one deduces that the centred Gaussian field $\left\{\sqrt{\|\tilde{\sigma}\|_2^2/\log N}\tilde{\varphi}_{z_B}^N\right\}_{B\in\mathcal{B}}$ has pairwise correlations of order at most $O(1/\log N)$. Thus, the conditional covariance matrices of $\left\{\sqrt{\frac{\|\tilde{\sigma}\|_2^2}{\log(N)}}\tilde{\varphi}_{z_B}^N\right\}_{B\in\mathcal{B}}$ and $\{\tilde{\sigma}_1 g_{B_{z_B,r_1}}+\tilde{\sigma}_2 g_{B_{z_B,N/r_2}}\}_{B\in\mathcal{B}}$ are within $O(1/\log N)$ of each other entry-wise. In combination with (B.10) this proves Proposition B.2. It remains to prove (B.10). Suppose that on the contrary, either of the events considered in the probabilities in (B.10) occurs. By (2.2) and Gaussian comparison, we know that $E_1=E_1(C)=\{\omega : M_{N,r_1,r_2,\tilde{\sigma},\delta}\notin(m_N-C,m_N+C)\}\cup\{M_{N,\tilde{\sigma},*,\delta}\notin$

$(m_N - C, m_N + C)\}$ has a probability tending to 0, i.e. $\lim_{C\to\infty} \limsup_{N\to\infty} \mathbb{P}(E_1) = 0$. Moreover, Theorem 2.2 implies that also the event $E_2 = \{\omega : \exists u, v \in V_N : \|u - v\|_2 \in (r, N/r)$ and $\min(\varphi_u^N, \varphi_v^N) > m_N - c \log\log r\}$ cannot occur, i.e. $\lim_{r\to\infty} \limsup_{N\to\infty} \mathbb{P}(E_2) = 0$. Note that Theorem 2.2 is stated only for the scale-inhomogeneous DGFF. However, using the covariance assumptions and Gaussian comparison, it is possible to replace $\{\psi_v^N\}_{v\in V_N}$ with $\{\varphi_v^N\}_{v\in V_N}$ throughout the proof of Theorem 2.2. This allows to assume the event $E_1^c \cap E_2^c$. To show (B.10), we consider the following events:

- $E_3 = \tilde{E}_3 \cup E_3^*$, where $\tilde{E}_3 = \{\omega : \exists v \in V_N : \varphi_=^{N,r_1,r_2,\tilde{\sigma}} M_{N,r_1,r_2,\tilde{\sigma},\delta},\ \varphi_v^N \leq m_N - c \log\log r\}$ and $E_3^* = \{\omega : \exists v \in V_N : \varphi_v^{N,\tilde{\sigma},*} = M_{N,\tilde{\sigma},*,\delta},\ \varphi_v^N \leq m_N - c\log\log r\}$.
- $E_4 = \{\omega : \exists v \in B, B \in \mathcal{B} : \varphi_v^N \geq m_N - c \log\log r$ and $\sqrt{\frac{\|\tilde{\sigma}\|_2^2}{\log N}} \left(\tilde{\varphi}_v^N - \tilde{\varphi}_{z_B}^N\right) \geq 1/\log n\}$.

E3: Let $\Gamma_x = \{v \in V_N : \varphi_v^{N,r_1,r_2,\tilde{\sigma}} - \varphi_v^N \in (x, x+1)\}$. The idea is that, by localizing and conditioning on the difference of the two Gaussian fields through the set $\Gamma_x$, one can use Proposition 5.1 to bound $\max_{v\in\Gamma_x} \varphi_v^N$ from above, i.e.

$$\begin{aligned}\mathbb{P}\left(E_1^c \cap \tilde{E}_3\right) &\leq \mathbb{P}\Big(\max_{x\geq c\log(n)-C} \max_{v\in\Gamma_x} \varphi_v^{N,r_1,r_2,\tilde{\sigma}} \geq m_N - C\Big) \leq \sum_{x\geq c\log(n)-C} \mathbb{P}\Big(\max_{v\in\Gamma_x} \varphi_v^{N,r_1,r_2,\tilde{\sigma}} \geq m_N - C\Big) \\ &\leq \sum_{x\geq c\log(n)-C} \mathbb{E}\left[\mathbb{P}\left(\max_{v\in\Gamma_x} \varphi_v^N \geq m_N - x - C|\Gamma_x\right)\right] \leq \tilde{c} \sum_{x\geq c\log(n)-C} \mathbb{E}\left[|\Gamma_x|/|V_N|\right] e^{2x}. \end{aligned} \tag{B.11}$$

By a first moment bound for Gaussian random variables, one has

$$\begin{aligned}\mathbb{E}\left[|\Gamma_x|^{1/2}/|V_N|^{1/2}\right] &\leq \mathbb{E}\left[|\{v \in V_N : \tilde{\sigma}_1 g_{B_{v,r_1}} + \tilde{\sigma}_2 g_{B_{v,N/r_2}} \in (x, x+1)\}|^{1/2}\right]/|V_N|^{1/2} \\ &\leq \mathbb{P}\left(\tilde{\sigma}_1 g_{B_{v,r_1}} + \tilde{\sigma}_2 g_{B_{v,N/r_2}} \in (x, x+1)\right)^{1/2} \leq e^{-c' x^2}/c',\end{aligned} \tag{B.12}$$

for some constant $c' = c'(\sigma, \tilde{\sigma}) > 0$. Thus,

$$\limsup_{C\to\infty} \limsup_{r\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(E_1^c(C) \cap \tilde{E}_3\right) = 0. \tag{B.13}$$

In the same way, one can prove an analogue estimate for $E_3^*$ in place of $\tilde{E}_3$, which gives

$$\limsup_{C\to\infty} \limsup_{r\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(E_1^c(C) \cap E_3\right) = 0. \tag{B.14}$$

E4: Let $\Gamma_r' = \{v \in V_N : \varphi_v^N \geq m_N - c\log\log r\}$. In $V_N$, there can be at most $r^2$ particles at minimum distance $N/r$, and around each of these, one can find approximately $r^2$ particles in $V_N$ which are within distance $r$. Thus, on $E_2^c$, one has $|\Gamma_r'| \leq 2r^4$. Further, for each $v \in B \cap \Gamma_r'$ and in the event of $E_2^c$, one has $\|v - z_B\|_2 \leq r$. Thus, by independence between the Gaussian fields $\{\varphi_v^N\}_{v\in V_N}$ and $\{\varphi_v^{N,'}\}_{v\in V_N}$, and using 2nd order Chebychev's inequality,

$$\mathbb{P}\left(\sqrt{\frac{\|\tilde{\sigma}\|_2^2}{\log N}} \left(\varphi_v^{N,'} - \varphi_{z_B}^{N,'}\right) \geq \frac{1}{\log\log N}\right) \leq \frac{(\tilde{c}(\sigma,\tilde{\sigma})\log r + c_1)(\log\log N)^2}{\log N}, \tag{B.15}$$

where $\tilde{c}, c_1 > 0$ are finite constants. Therefore, and by a union bound,

$$\limsup_{r\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(E_4 \cap E_2^c\right) \leq \limsup_{r\to\infty} \limsup_{N\to\infty} 2r^4[\tilde{c}(\sigma,\tilde{\sigma})\log r + c_1] \frac{(\log\log N)^2}{\log N} = 0. \tag{B.16}$$

This concludes the proof of equation (B.10) and thereby, the proof of Proposition B.2. □

*Proof of Lemma 5.5:* We prove Lemma 5.5 in the case of the scale-inhomogeneous DGFF. Lemma 5.5 for the approximating field follows from Gaussian Define $\bar{\psi}^{N,\tilde{\sigma}} = \left(1 + \frac{\|\tilde{\sigma}\|_2^2}{2\log N}\right)\psi_v^N$, for $v \in V_N$, and set $M_N = \max_{v\in V_N} \psi_v^N$ and $\bar{M}_{N,\tilde{\sigma}} = \max_{v\in V_N} \bar{\psi}^{N,\tilde{\sigma}}$. One has $\bar{M}_{N,\tilde{\sigma}} = \left(1 + \frac{\|\tilde{\sigma}\|_2^2}{\log N}\right)M_N$. Using (2.2), this gives us both

$$\mathbb{E}\left[\bar{M}_{N,\tilde{\sigma}}\right] = \mathbb{E}\left[M_N\right] + 2\|\tilde{\sigma}\|_2^2 + o(1), \tag{B.17}$$

and

$$\lim_{N\to\infty} d\left(M_N - \mathbb{E}\left[M_N\right], \bar{M}_{N,\tilde{\sigma}} - \mathbb{E}\left[\bar{M}_{N,\tilde{\sigma}}\right]\right) = 0. \tag{B.18}$$

Further, let $\{\psi_v^{N,\tilde{\sigma},*} : v \in V_N\}$ be defined as in (B.7) and set $M_{N,\tilde{\sigma},*} = \max_{v\in V_N} \psi_v^{N,\tilde{\sigma},*}$. In the distributional sense, $\left\{\bar{\psi}_v^{N,\tilde{\sigma}}\right\}_{v\in V_N}$ can be considered as a sum of $\left\{\psi_v^{N,\tilde{\sigma},*}\right\}_{v\in V_N}$ and an independent centred Gaussian field with variances of order $O((1/\log N)^3)$. Thus, by Gaussian comparison, it follows that

$$\mathbb{E}\left[\bar{M}_{N,\tilde{\sigma}}\right] = \mathbb{E}\left[M_{N,\tilde{\sigma},*}\right] + o(1), \tag{B.19}$$

as well as

$$\lim_{N\to\infty} d\left(\bar{M}_{N,\tilde{\sigma}} - \mathbb{E}\left[\bar{M}_{N,\tilde{\sigma}}\right], M_{N,\tilde{\sigma},*} - \mathbb{E}\left[M_{N,\tilde{\sigma},*}\right]\right) = 0. \tag{B.20}$$

By (B.20), Proposition B.2, and using the triangle inequality, one concludes the proof of Lemma 5.5. □

*Proof of Lemma 5.6.* Recall that we want to prove asymptotic stochastic domination. The basic idea is to use Slepian's Lemma. Let $\Phi$, $\{\Phi_v^N\}_{v\in V_N}$ be independent standard Gaussian random variables and for some $\epsilon^* > 0$, set

$$\psi_v^{N,lw,\epsilon^*} = \left(1 - \frac{\epsilon^*}{\log N}\right)\psi_v^N + \epsilon_v^{N,\prime}\Phi \tag{B.21}$$

$$\bar{\psi}_v^{N,up,\epsilon^*} = \left(1 - \frac{\epsilon^*}{\log N}\right)\bar{\psi}_v^N + \epsilon_v^{N,\prime\prime}\Phi_v^N, \tag{B.22}$$

where $\epsilon_v^{N,\prime} = \epsilon_v^{N,\prime}(\epsilon, \epsilon^*)$ and $\epsilon_v^{N,\prime\prime} = \epsilon_v^{N,\prime\prime}(\epsilon, \epsilon^*)$ are chosen such that

$$\mathrm{Var}\left[\psi_v^{N,lw,\epsilon^*}\right] = \left(1 - \frac{\epsilon^*}{\log N}\right)^2 \mathrm{Var}\left[\psi_v^N\right] + (\epsilon_v^{N,\prime})^2 = \mathrm{Var}\left[\psi_v^N\right] + \epsilon \tag{B.23}$$

and

$$\mathrm{Var}\left[\bar{\psi}_v^{N,up,\epsilon^*}\right] = \left(1 - \frac{\epsilon^*}{\log N}\right)^2 \mathrm{Var}\left[\bar{\psi}_v^N\right] + (\epsilon_v^{N,\prime\prime})^2 = \mathrm{Var}\left[\psi_v^N\right] + \epsilon. \tag{B.24}$$

Solving for $\epsilon_v^{N,\prime}$ in (B.23), gives

$$(\epsilon_v^{N,\prime})^2 = \frac{\epsilon^*}{\log N}\mathrm{Var}\left[\psi_v^N\right] + \epsilon. \tag{B.25}$$

Moreover, for $u \neq v \in V_N$,

$$\mathbb{E}\left[\psi_u^{N,lw,\epsilon^*}\psi_v^{N,lw,\epsilon^*}\right] = \left(1 - \frac{\epsilon^*}{\log N}\right)^2 \mathbb{E}\left[\psi_u^N\psi_v^N\right] + \epsilon_u^{N,\prime}\epsilon_v^{N,\prime} \tag{B.26}$$

and by (B.24),

$$\mathbb{E}\left[\bar{\psi}_u^{N,up,\epsilon^*}\bar{\psi}_v^{N,up,\epsilon^*}\right] = \left(1 - \frac{\epsilon^*}{\log N}\right)^2 \mathbb{E}\left[\bar{\psi}_u^N\bar{\psi}_v^N\right] \leq \left(1 - \frac{\epsilon^*}{\log N}\right)^2 \mathbb{E}\left[\psi_u^N\psi_v^N\right] + \epsilon\left(1 - \frac{\epsilon^*}{\log N}\right)^2. \tag{B.27}$$

We want that, for all $u, v \in V_N$, $\mathbb{E}\left[\psi_u^{N,lw,\epsilon^*}\psi_v^{N,lw,\epsilon^*}\right] \geq \mathbb{E}\left[\bar{\psi}_u^{N,up,\epsilon^*}\bar{\psi}_v^{N,up,\epsilon^*}\right]$. Considering (B.26) and (B.27), this holds, provided

$$\epsilon_u^{N,\prime}\epsilon_v^{N,\prime} \geq \epsilon\left(1 - \frac{\epsilon^*}{\log N}\right)^2. \tag{B.28}$$

Combining (B.28) with (B.25) and as $\epsilon \to 0$, one sees that it is possible to choose first $\epsilon^*(\epsilon)$ and then both $\{\epsilon_v^{N,\prime}(\epsilon, \epsilon^*)\}_{v\in V_N}$ and $\{\epsilon_v^{N,\prime\prime}(\epsilon, \epsilon^*)\}_{v\in V_N}$, such that $\epsilon^* \to 0$, and that at the same time, all requirements (B.23), (B.24) and (B.28) hold. Observe further, that in this case, by (B.23) and (B.24), $\max_{v\in V_N}\epsilon_v^{N,\prime} \to 0$, as well as $\max_{v\in V_N}\epsilon_v^{N,\prime\prime} \to 0$. With this choice, one can apply Slepian's lemma to obtain

$$\tilde{d}\left(\max_{v\in V_N}\psi_v^{N,lw,\epsilon^*} - m_N, \max_{v\in V_N}\bar{\psi}_v^{N,up,\epsilon^*} - m_N\right) = 0. \tag{B.29}$$

As $\epsilon \to 0$, the distribution of the Gaussian field $\{\psi_v^{N,lw,\epsilon^*}\}_{v\in V_N}$ tends to that of $\{\psi_v^N\}_{v\in V_N}$. Applying Lemma B.1 to $\{\bar{\psi}_v^{N,up,\epsilon^*}\}_{v\in V_N}$, one deduces

$$\mathbb{P}\left(\max_{v\in V_N}\bar{\psi}_v^{N,up,\epsilon^*} - m_N \geq x\right) \leq \mathbb{P}\left(\max_{v\in V_N}\bar{\psi}_v^N - m_N \geq x - \sqrt{\max_{w\in V_N}\epsilon_w^{N,\prime\prime}}\right)\left(Ce^{-(C\max_{w\in V_N}\epsilon_w^{N,\prime\prime})^{-1}}\right). \tag{B.30}$$

Since $\max\limits_{w\in V_N}\epsilon_w^{N,\prime\prime} \to 0$, as $\epsilon \to 0$, this allows to conclude the proof of (5.28). (5.29) can be proved in the same way, by switching the roles of $\{\psi_v^N\}_{v\in V_N}$ and $\{\bar{\psi}_v^N\}_{v\in V_N}$ in the proof above. Further details are omitted. □

## Appendix C. Proof of Proposition 5.8

We outline the strategy of the proof: First, we localize the position of $S_v^{N,m}$, for particles $v \in V_N$ that satisfy $S_v^N \geq m_N + z$. This reduces the computation of the asymptotic right-tail distribution to the computation of an expectation of a sum of indicators, which is significantly simpler, as it essentially boils down to computing a single probability. In the second step, we prove that the asymptotic behaviour of the right-tail of the maximum of the auxiliary field does not depend on the parameter $N$, so that any possible constant also depends only on the remaining parameters, $K', L'$ and $z$. In the third step, we investigate how the limit scales in $z$, which allows us to factorize the dependence on the variable $z$ in the above obtained constants, reducing the dependence of the constants to the parameters, $K', L'$. We further show that the constants can be bounded uniformly from below and from above, which then concludes the proof. Recall that $S_v^{N,f} = S_v^N - S_v^{N,c}$, for $v \in V_N$. For the entire proof, fix the index $i$ along with a box $B_{N/KL,i}$. The field $\{S_v^{N,f}\}_{v\in B_{N/KL,i}}$ is constructed in such a way (see (5.9)), that it is independent of the integers $K, L$ and $i$. In particular, the sequence $\{\beta^*_{K',L'}\}_{K'L'}$ does not depend on these. For a fixed $v \in B_{N/KL,i}$, and for $S_v^{N,m}$, consider $X_v^N$ as the associated variable speed Brownian motion. To be more precise, recall the definition of $S_v^{N,m}$ in (5.6). To each Gaussian random variable $b_{i,j,B}^N$ in (5.6), associate an independent Brownian motion $b_{i,j,B}^N(t)$ that runs for $2^{-2j}$ time with rate $\sigma\left(\frac{n-j}{n}\right)$ and ends at the value of $\sigma\left(\frac{n-j}{n}\right)b_{i,j,B}^N$. Each variable speed Brownian motion, $\{X_v^N(t)\}_{0\leq t\leq n-k-l-k'-l'}$, is defined by concatenating the Brownian motions associated to earlier times, which correspond to larger scales. Until the end of the proof, in order to shorten notation, simply write $\bar{N} = N/KL$, $n^* = n-k-l-k'-l'$ and analogously, $\bar{n} = n-k-l$ as well as $\bar{l} = l' + k'$, $\bar{k} = k + l$. As in (5.5), we consider the partitioning of $B_{N/KL,i}$ into a collection of $K'L'$-boxes $\mathcal{B}_{K'L'}$ and refer to $B_{K'L'}(v) \in \mathcal{B}_{K'L'}$ as the unique $K'L'$−box that contains $v$. The set of all left bottom corners of these $K'L'$−boxes is called $\Xi_{\bar{N}}$. We further write

$M_n(k,t) = 2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{k}{n},\frac{t}{n}\right)n - \frac{((t)\wedge(n-\bar{l}))\log(n)}{4(n-\bar{l})}$, for $t \in [k,n]$. Let

$$E_{v,N}(z) = \Big\{X_v^N(t) - M_n(\bar{k},t) \in [-i^\gamma(t,n^*), \max(i^\gamma(t,n^*),z)],\ \forall 0 \le t \le n^*,$$
$$\max_{u\in B_{K'L'}(v)} Y_u^N \ge 2\log(2)\mathcal{I}_{\sigma_2}\left(\frac{\bar{k}}{n},1\right)n - \log(n)/4 - \bar{k}^\gamma + z - X_v^N(n^*)\Big\}, \tag{C.1}$$

where $Y_u^N \overset{\text{law}}{\sim} S_u^N - S_u^{N,c} - S_u^{N,m} = S_u^{N,f} - S_u^{N,m}$ is an independent Gaussian field. The first restriction is that all particles have to stay within a tube around $2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n},\frac{\bar{k}+t}{n}\right)n$, which is due to Proposition 4.2. Moreover, it ensures that at the beginning, particles cannot be too large. The second event ensures that there are particles reaching the relevant level. We consider the number of particles satisfying the event $E_{v,N}(z)$, namely

$$\Lambda_N(z) := \sum_{v\in\Xi_{\bar{N}}} \mathbb{1}_{E_{v,N}(z)} \tag{C.2}$$

and claim that

$$\limsup_{z\to\infty}\ \limsup_{(L,K,L',K',N)\Rightarrow\infty} \left| \frac{\mathbb{P}\left(\max_{v\in B_{N/KL,i}} S_v^{N,f} \ge M_n(\bar{k},n) + z - k^\gamma\right)}{\mathbb{E}\left[\Lambda_N(z)\right]} \right| = 1. \tag{C.3}$$

This reduces the analysis to compute the asymptotics of the expectation, which is much simpler, as this only needs precise right-tail asymptotic of a single vertex. We start proving the claim (C.3). By a first moment bound and using Lemma 4.4, one obtains

$$\limsup_{z\to\infty}\ \limsup_{(L,K,L',K',N)\Rightarrow\infty} \mathbb{P}\left(\max_{v\in B_{N/KL,i}} S_v^{N,f} \ge M_n(\bar{k},n) + z - \bar{k}^\gamma\right) \le \mathbb{E}\left[\Lambda_N(z)\right], \tag{C.4}$$

which implies that the quotient is bounded from above by 1. In order to obtain equality, one shows

$$\limsup_{z\to\infty}\ \limsup_{(L,K,L',K',N)\Rightarrow\infty} \mathbb{E}\left[\Lambda_N(z)^2\right]/\mathbb{E}\left[\Lambda_N(z)\right] = 1. \tag{C.5}$$

Assuming (C.5) and using the Cauchy-Schwarz inequality, one has

$$\mathbb{P}\left(\max_{v\in B_{N/KL,i}} S_v^{N,f} \ge M_n(\bar{k},n) + z\right) \ge \mathbb{E}\left[\Lambda_N(z)\right], \tag{C.6}$$

which, together with (C.4), then implies (C.3). Thus, we turn to the proof of equation (C.5). First, decompose the second moment along the branching scale, $b_N(v,w) = \max\{\lambda \ge 0 : [v]_\lambda \cap [w]_\lambda \neq \emptyset\}$, beyond which increments are independent, i.e.

$$\begin{aligned}\mathbb{E}\left[\Lambda_N(z)^2\right] &= \mathbb{E}\left[\Lambda_N(z)\right] + \sum_{v,w\in\Xi_{\bar{N}}} \mathbb{P}\left(E_{v,N}(z) \cap E_{w,N}(z)\right) \\ &= \mathbb{E}\left[\Lambda_N(z)\right] + \sum_{t_s=0}^{n^*-1} \sum_{v,w:d(v,w)=t_s} \mathbb{P}\left(E_{v,N}(z) \cap E_{w,N}(z)\right)\end{aligned} \tag{C.7}$$

Note that, for $v \in \Xi_{\bar{N}}$ fixed, there are $2^{2(n^*-(\bar{k}+t_s))}$ many $w \in \Xi_{\bar{N}}$ with $d(v,w) = t_s$. The probabilities in (C.7) can be bounded from above by

$$\begin{aligned}\mathbb{P}\left(E_{v,N}(z) \cap E_{w,N}(z)\right) \leq \sum_{\substack{x_s \in [-i^\gamma(\bar{k}+t_s,n^*),\max(i^\gamma(\bar{k}+t_s,n^*),z)] \\ x_1,x_2 \in [-\bar{l}^\gamma,\bar{l}^\gamma]}} & \mathbb{P}\left(X_v^N(t_s) - M_n(\bar{k},t_s) \in [x_s-1,x_s]\right) \\ & \times \mathbb{P}\left(X_v^N(n^*) - X_v^N(t_s) - M_n(t_s, n-\bar{l}) + \bar{k}^\gamma + x_s \in [x_1-1,x_1]\right) \\ & \times \mathbb{P}\left(\max_{u \in B_{K'L'}(v)} Y_u^N \geq 2\log(2)\sigma^2(1)\bar{l} + z - x_1\right) \\ & \times \mathbb{P}\left(X_w^N(n^*) - X_w^N(t_s) - M_n(t_s, n-\bar{l}) + \bar{k}^\gamma + x_s \in [x_2-1,x_2]\right) \\ & \times \mathbb{P}\left(\max_{u \in B_{K'L'}(w)} Y_u^N \geq 2\log(2)\sigma^2(1)\bar{l} + z - x_2\right) \end{aligned} \tag{C.8}$$

Similarly, one can expand $\mathbb{E}\left[\Lambda_N(z)\right]$, i.e.

$$\begin{aligned}\mathbb{E}\left[\Lambda_N(z)\right] = 2^{2n^*} \sum_{\substack{x_s \in [-i^\gamma(\bar{k}+t_s,n^*),\max(i^\gamma(\bar{k}+t_s,n^*),z)] \\ x_1,x_2 \in [-\bar{l}^\gamma,\bar{l}^\gamma]}} & \mathbb{P}\left(X_v^N(t_s) - M_n(\bar{k},t_s) \in [x_s-1,x_s]\right) \\ & \times \mathbb{P}\left(X_v^N(n^*) - X_v^N(t_s) - M_n(\bar{k}+t_s, n-\bar{l}) + \bar{k}^\gamma + x_s \in [x_1-1,x_1]\right) \\ & \times \mathbb{P}\left(\max_{u \in B_{K'L'}(v)} Y_u^N \geq 2\log(2)\sigma^2(1)\bar{l} + z - x_1\right). \end{aligned} \tag{C.9}$$

For each summand, there is an additional factor appearing in (C.8) compared to (C.9). If one can show that all these vanish uniformly over $x_s$, when summing over $t_s$ and then taking the limits, $(z, \bar{L}, N) \Rightarrow \infty$, one obtains (C.5), and thereby (C.3). Thus, one needs to estimate the additional factors,

$$\begin{aligned} & \sum_{x_2 \in [-\bar{l}^\gamma,\bar{l}^\gamma]} \mathbb{P}\left(X_w^N(n^*) - X_w^N(t_s) - M_n(\bar{k}+t_s, n-\bar{l}) + \bar{k}^\gamma + x_s \in [x_2-1,x_2]\right) \\ & \quad \times \mathbb{P}\left(\max_{u \in B_{K'L'}(w)} Y_u^N \geq 2\log(2)\sigma^2(1)\bar{l} + z - x_2\right) \\ & \leq 2^{-2(n^*-(\bar{k}+t_s))} \sum_{x_2 \in [-\bar{l}^\gamma,\bar{l}^\gamma]} \frac{2\log(2)\bar{l}\sigma(1) + \frac{z-x_2}{\sigma(1)}}{\sqrt{2\pi\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}+t_s}{n}, \frac{n-\bar{l}}{n}\right)n}\sqrt{\bar{l}\log 2}} \exp\left[-2\log(2)(\bar{k}+t_s - \mathcal{I}_{\sigma^2}\left(\frac{\bar{k}+t_s}{n}\right)n)\right] \\ & \quad \times \exp\left[-2\log(2)\bar{l} - 2\left(z - x_s - \frac{n-\bar{k}-\bar{l}-t_s}{4(n-\bar{k}-\bar{l})}\log(n) - \bar{k}^\gamma\right) - \frac{\left(\frac{z-x_2}{\sigma(1)}\right)^2}{2\log(2)\bar{l}}\right] \\ & \quad \times \exp\left[-\frac{\left(x_2 - x_s - \frac{n-\bar{k}-\bar{l}-t_s}{4(n-\bar{k}-\bar{l})}\log(n) - \bar{k}^\gamma\right)^2}{2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}+t_s}{n}, \frac{n-\bar{l}}{n}\right)n}\right]. \end{aligned} \tag{C.10}$$

Note that there are $2^{2(n^*-(\bar{k}+t_s))}$ vertices $w \in \Xi_{\bar{N}}$ with $d(v,w) = t_s$, for fixed $v \in \Xi_{\bar{N}}$, which cancels with the prefactor in (C.10) when taking the sum in (C.7). To show that the sum in $t_s$ is finite, first note that the relevant term in (C.10) is given by $\exp\left[-2\log(2)(\bar{k}+t_s - \mathcal{I}_{\sigma^2}\left(\frac{\bar{k}+t_s}{n}\right)n)\right]$. Recall the assumption $\mathcal{I}_{\sigma^2}(x) < x$, for $x \in (0,1)$. In particular, for any $\delta > 0$, there exists $\epsilon > 0$ such that $\mathcal{I}_{\sigma^2}(x) < x - \epsilon$, for $x \in (\delta, 1-\delta)$. Since one is interested in the limit, as $(z, K', L', N) \Rightarrow \infty$, it is possible to assume $\frac{\bar{k}(1-\sigma^2(0))}{n} < \epsilon/2$ and $\frac{\bar{l}(\sigma^2(1)-1)}{n} < \epsilon/2$. In this case it holds, for $t_s \in (0, n-\bar{k}-\bar{l})$,

$$\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}+t_s}{n}\right) < \frac{\bar{k}+t_s}{n} - \epsilon/2. \tag{C.11}$$

Using (C.11) in (C.10), implies that (C.10) is summable in $t_s \in (0, n - \bar{k} - \bar{l})$, when considering limits $(z, K', L', N) \Rightarrow \infty$. The sum in $x_2$ in (C.10) is bounded by its number of summands, i.e. one gets a prefactor of leading order $4\log(2)\bar{l}^{\gamma+1/2}\sigma(1)$, where one can choose $\gamma \in (\frac{1}{2}, 1)$. Note that there is still the term $\exp\left[-2\log(2)\bar{l}\right]$ which ensures that (C.10) tends to zero, as $(z, K', L', N) \Rightarrow \infty$. Altogether, this proves (C.5). In the second step, we show that it is possible to choose the sequence of constants independently of $N$. More explicitly, in the following, we show that there are constants $\beta_{K',L',z} > 0$, such that

$$\lim_{z\to\infty} \limsup_{(L',K',N)\Rightarrow\infty} \frac{\mathbb{E}\left[\Lambda_N(z)\right]}{\beta_{K',L',z}} = \lim_{z\to\infty} \liminf_{(L',K',N)\Rightarrow\infty} \frac{\mathbb{E}\left[\Lambda_N(z)\right]}{\beta_{K',L',z}} = e^{2\log(2)\bar{k}(\sigma^2(0)-1)}e^{2\bar{k}^\gamma}. \tag{C.12}$$

Since $X_v^N(n^*) \sim \mathcal{N}\left(0, \log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, \frac{n^*}{n}\right)n\right)$, and using Lemma 4.4, which allows to ignore the restriction to stay below the maximum at all times, $\mathbb{E}\left[\Lambda_N(z)\right]$ reads

$$\begin{aligned}
&2^{2(n-\bar{k}-\bar{l})}\mathbb{P}\left(X_v^N(\bar{n}) - M_n(\bar{k}, n-\bar{l}) \in [-\bar{l}^\gamma, \bar{l}^\gamma], \max_{u\in B_{K'L'}(v)} Y_u^N \geq M_n(\bar{k}, n) - X_v^N(n^*) - \bar{k}^\gamma + z\right)\\
&= \int_{-\bar{l}^\gamma}^{\bar{l}^\gamma} \frac{2^{2(n-\bar{k}-\bar{l})}}{\sqrt{2\pi\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, \frac{n-\bar{l}}{n}\right)n}} \exp\left[-\frac{\left(M_n(\bar{k}, n-\bar{l}) + x\right)^2}{2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, \frac{n-\bar{l}}{n}\right)n}\right]\\
&\quad\times \mathbb{P}\left(\max_{u\in B_{K'L'}(v)} Y_u^N \geq 2\log(2)\bar{l}\sigma^2(1) + z - \bar{k}^\gamma - x\right)\mathrm{d}x\\
&= \int_{-\bar{l}^\gamma}^{\bar{l}^\gamma} \frac{2^{2\bar{k}(\sigma^2(0)-1)}2^{2\bar{l}(\sigma^2(1)-1)}\sqrt{n}}{\sqrt{2\pi\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}\frac{n-\bar{l}}{n}\right)n}} \exp\left[-2x - \frac{\left(x - \frac{\log(n)}{4}\right)^2}{2\log(2)(n - \sigma^2(0)\bar{k} - \sigma^2(1)\bar{l})}\right]\\
&\quad\times \mathbb{P}\left(\max_{u\in B_{K'L'}(v)} Y_u^N \geq 2\log(2)\bar{l}\sigma^2(1) + z - \bar{k}^\gamma - x\right)\mathrm{d}x.
\end{aligned} \tag{C.13}$$

By definition of $S_u^N$ (see (5.9)), $\max_{u\in B_{K'L'}(v)} Y_u^N$ has the same law as $\max_{u\in V_{K'L'}} S_u^{N,b} + a_{K'L',\bar{u}}\Phi_j$ and is therefore independent of $N$ (cp. (5.5) and (5.9)). Note further that $\frac{\sqrt{n}}{\sqrt{\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, \frac{n-\bar{l}}{n}\right)n}} \overset{n\to\infty}{\to} 1$, and by Borell's inequality for Gaussian processes (see [52, Lemma 3.1]),

$$\mathbb{P}\left(\left|\max_{u\in B_{K'L'}(v)} Y_u^N \geq 2\log(2)\bar{l}\sigma^2(1) + z - x - \bar{k}^\gamma\right|\right) \leq C2^{-2\bar{l}(\sigma(1)-1)^2}\bar{l}^{-\frac{3}{2}(\sigma(1)-1)}\exp\left[-2\frac{\sigma(1)-1}{\sigma(1)}(z - \bar{k}^\gamma)\right]. \tag{C.14}$$

As $\sigma(1) > 1$, (C.14), together with (C.13), implies (C.12) and thus, the third claim. In particular, one can read off (C.13) that the sequence $\{\beta_{K',L',z}\}$ depends only on the very last variance parameter and on $\bar{k}^\gamma$. In the last step, we analyse how the right tail probability scales in $z$, namely we want to show

$$\lim_{z_1,z_2\to\infty} \limsup_{(\bar{L},N)\Rightarrow\infty} \frac{e^{-2z_2}\mathbb{E}\left[\Lambda_N(z_1)\right]}{e^{-2z_1}\mathbb{E}\left[\Lambda_N(z_2)\right]} = \lim_{z_1,z_2\to\infty} \liminf_{(\bar{L},N)\Rightarrow\infty} \frac{e^{-2z_2}\mathbb{E}\left[\Lambda_N(z_1)\right]}{e^{-2z_1}\mathbb{E}\left[\Lambda_N(z_2)\right]} = 1. \tag{C.15}$$

For $v \in V_N$, set $\nu_{v,N}(\cdot)$ be the density, such that for any interval $I \subset \mathbb{R}$,

$$\int_I \nu_{v,N}(y)\mathrm{d}y = \mathbb{P}\left(X_v^N(n^*) \in I + M_n(\bar{k}, n-\bar{l})\right). \tag{C.16}$$

Using this notation, we can rewrite

$$\mathbb{P}\left(E_{v,N}(z)\right) = \int_{-\bar{l}^\gamma}^{\bar{l}^\gamma} \nu_{v,N}(z+x)\mathbb{P}\left(\max_{u\in B_{K'L'}(v)} Y_u^N \geq 2\log(2)\bar{l}\sigma^2(1) - \bar{k}^\gamma - x\right)\mathrm{d}x. \tag{C.17}$$

Note that in (C.17) only $\nu_{v,N}(z+x)$ depends on $z$. For $z_1, z_2 > 0$, one has to compute the quotient $\mathbb{E}\left[\Lambda_N(z_1)\right]/\mathbb{E}\left[\Lambda_N(z_2)\right]$, for which we use the reformulation in (C.17). The strategy is to compute the asymptotic limit of the integral involving $z_1$ in terms of the integral involving $z_2$ and an additional correction factor. As $\bar{l} \to \infty$, prior to $z_1, z_2 \to \infty$, there is no need to shift the limits of the integrals. For the remaining factors in both integrals, one obtains the relative density with respect to $z_1, z_2$, i.e.

$$\frac{\nu_{v,N}(z_1+x)}{\nu_{v,N}(z_2+x)} = \exp\left[-2(z_1-z_2) - \frac{z_1^2 - z_2^2 - (z_1-z_2)\frac{\log(n)}{2}}{2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, \frac{n-\bar{l}}{n}\right)n} - x\frac{(z_1-z_2)}{\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, \frac{n-\bar{l}}{n}\right)n}\right]. \tag{C.18}$$

Thus, we can rewrite $\mathbb{P}\left(E_{v,N}(z_1)\right)$ as

$$\int_{-\bar{l}^\gamma}^{\bar{l}^\gamma} \nu_{v,N}(z_2+x)e^{2(z_1-z_2)}\mathbb{P}\left(\max_{u\in B_{K'L'}(v)} Y_u^N \geq 2\log(2)\bar{l}\sigma^2(1) - \bar{k}^\gamma - x\right)$$
$$\times \exp\left[\frac{z_1^2 - z_2^2 - (z_1+z_2)\frac{\log(n)}{2}}{2\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, \frac{n-\bar{l}}{n}\right)n} + x\frac{(z_1-z_2)}{\log(2)\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, \frac{n-\bar{l}}{n}\right)n}\right]\mathrm{d}x, \tag{C.19}$$

where the last factor tends to 1, as $(\bar{L}, N) \Rightarrow \infty$. Computing the quotient $\mathbb{E}\left[\Lambda_N(z_1)\right]/\mathbb{E}\left[\Lambda_N(z_2)\right]$ using (C.19) and summing over all vertices, one obtains, when turning to limits, that (C.15) holds. Combining the above steps, in particular (C.15) with (C.12), completes the proof of (5.43), with some non-negative sequence $\{\beta_{K',L'}\}_{K',L'\geq 0}$. In the final step, we show that this sequence is bounded. Using Lemma 5.7, one has for some $\epsilon > 0$, being at most of order $O\left(e^{-2\bar{k}^{2\gamma-1}/(2\sigma^2(0)\log 2)}\right)$,

$$c_\alpha e^{-2z} \leq \int_{-\bar{k}^\gamma}^{\bar{k}^\gamma} \nu_{v,N}^c(x)2^{2\bar{k}}\mathbb{P}\left(\max_{v\in B_{N/KL,i}} S_v^{N,f} \geq 2\log 2\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, 1\right)n - \frac{\log n}{4} + z - x\right) + \epsilon. \tag{C.20}$$

Using the asymptotics (C.13) for the probability in the integral in (C.20), one can instead compute the integral

$$\int_{-\bar{k}^\gamma}^{\bar{k}^\gamma} \frac{\exp\left[-\frac{\left(2\log 2\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}\right)n + x\right)^2}{2\log 2\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}\right)n}\right]}{\sqrt{2\pi\log 2\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}\right)n}} 2^{2\bar{k}}\beta_{K',L'}e^{-2z+2x+2\log 2\bar{k}(\sigma^2(0)-1)}\mathrm{d}x = \beta_{K',L'}e^{-2z}\int_{-\bar{k}^\gamma}^{\bar{k}^\gamma} \frac{\exp\left[-\frac{x^2}{2\log 2\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}\right)n}\right]}{\sqrt{2\pi\log 2\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}\right)n}}\mathrm{d}x. \tag{C.21}$$

The integral in (C.21) is bounded by 1 and thus, when considering the lower bound in (C.20), one can deduce that $c_\alpha \leq \beta_{K',L'}$, for $K', L' \geq 0$. The upper bound, i.e. $\beta_{K',L'} \leq C_\alpha$, for $K', L' \geq 0$ and for some constant $C_\alpha > 0$, follows from a union and a Gaussian tail bound, i.e.

$$\mathbb{P}\left(\max_{v\in B_{N/KL,i}} S_v^{N,f} \geq 2\log 2\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, 1\right)n - \frac{\log n}{4} + z - \bar{k}^\gamma\right)$$
$$\leq C_\alpha\frac{2^{2(n-\bar{k})}}{\sqrt{n}}\exp\left[-2\log 2\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, 1\right)n - 2\left(z - \bar{k}^\gamma + \frac{\log n}{4}\right) - \frac{\left(z - \bar{k}^\gamma - \frac{\log}{4}\right)^2}{2\log 2\mathcal{I}_{\sigma^2}\left(\frac{\bar{k}}{n}, 1\right)n}\right]$$
$$\leq C_\alpha\exp\left[2\log(2)\bar{k}(\sigma^2(0)-1) + 2\bar{k}^\gamma - 2z\right], \tag{C.22}$$

This concludes the proof of Proposition 5.8. □

## References


[1] R. J. Adler. *An introduction to continuity, extrema, and related topics for general Gaussian processes*. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 12. Institute of Mathematical Statistics, Hayward, CA, 1990.

[2] E. Aïdékon, J. Berestycki, E. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Related Fields*, 157(1-2):405–451, 2013.

[3] L.-P. Arguin, D. Belius, and P. Bourgade. Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.*, 349(2):703–751, 2017.

[4] L.-P. Arguin, D. Belius, P. Bourgade, M. Radziwiłł, and K. Soundararajan. Maximum of the Riemann zeta function on a short interval of the critical line. *Comm. Pure Appl. Math.*, 72(3):500–535, 2019.

[5] L.-P. Arguin, A. Bovier, and N. Kistler. Genealogy of extremal particles of branching Brownian motion. *Comm. Pure Appl. Math.*, 64(12):1647–1676, 2011.

[6] L.-P. Arguin, A. Bovier, and N. Kistler. Poissonian statistics in the extremal process of branching Brownian motion. *Ann. Appl. Probab.*, 22(4):1693–1711, 2012.

[7] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Related Fields*, 157(3-4):535–574, 2013.

[8] L.-P. Arguin, A. Bovier, and N. Kistler. An ergodic theorem for the extremal process of branching Brownian motion. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(2):557–569, 2015.

[9] L.-P. Arguin and F. Ouimet. Extremes of the two-dimensional Gaussian free field with scale-dependent variance. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13(2):779–808, 2016.

[10] L.-P. Arguin and O. Zindy. Poisson-Dirichlet statistics for the extremes of the two-dimensional discrete Gaussian free field. *Electron. J. Probab.*, 20:no. 59, 19, 2015.

[11] N. Berestycki. Introduction to the Gaussian free field and Liouville quantum gravity. *Lecture notes – http://www.statslab.cam.ac.uk/ beresty/Articles/oxford4.pdf*, 2016.

[12] N. Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.*, 22:Paper No. 27, 12, 2017.

[13] M. Biskup. Extrema of the two-dimensional discrete Gaussian free field. In *Random graphs, phase transitions, and the Gaussian free field*, volume 304 of *Springer Proc. Math. Stat.*, pages 163–407. Springer, Cham, 2020.

[14] M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *Comm. Math. Phys.*, 345(1):271–304, 2016.

[15] M. Biskup and O. Louidor. Full extremal process, cluster law and freezing for the two-dimensional discrete Gaussian free field. *Adv. Math.*, 330:589–687, 2018.

[16] M. Biskup and O. Louidor. Conformal Symmetries in the Extremal Process of Two-Dimensional Discrete Gaussian Free Field. *Comm. Math. Phys.*, 375(1):175–235, 2020.

[17] E. Bolthausen, J.-D. Deuschel, and G. Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.*, 29(4):1670–1692, 2001.

[18] E. Bolthausen, J. D. Deuschel, and O. Zeitouni. Recursions and tightness for the maximum of the discrete, two dimensional Gaussian free field. *Electron. Commun. Probab.*, 16:114–119, 2011.

[19] A. Bovier and L. Hartung. The extremal process of two-speed branching Brownian motion. *Electron. J. Probab.*, 19:no. 18, 28, 2014.

[20] A. Bovier and L. Hartung. Variable speed branching Brownian motion 1. Extremal processes in the weak correlation regime. *ALEA Lat. Am. J. Probab. Math. Stat.*, 12(1):261–291, 2015.

[21] A. Bovier and L. Hartung. Extended convergence of the extremal process of branching Brownian motion. *Ann. Appl. Probab.*, 27(3):1756–1777, 2017.

[22] A. Bovier and L. Hartung. From 1 to 6: A finer analysis of perturbed branching brownian motion. *Communications on Pure and Applied Mathematics*, 73(7):1490–1525, 2020.

[23] A. Bovier and I. Kurkova. Derrida's generalised random energy models. I. Models with finitely many hierarchies. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(4):439–480, 2004.

[24] A. Bovier and I. Kurkova. Derrida's generalized random energy models. II. Models with continuous hierarchies. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(4):481–495, 2004.

[25] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 69(1):62–123, 2016.

[26] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 65(1):1–20, 2012.

[27] M. D. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.

[28] D. Capocaccia, M. Cassandro, and P. Picco. On the existence of thermodynamics for the generalized random energy model. *J. Statist. Phys.*, 46(3-4):493–505, 1987.

[29] O. Daviaud. Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.*, 34(3):962–986, 2006.

[30] B. Derrida and H. Spohn. Polymers on disordered trees, spin glasses, and traveling waves. volume 51, pages 817–840. 1988. New directions in statistical mechanics (Santa Barbara, CA, 1987).

[31] J. Ding. Exponential and double exponential tails for maximum of two-dimensional discrete Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):285–299, 2013.

[32] J. Ding, R. Roy, and O. Zeitouni. Convergence of the centered maximum of log-correlated Gaussian fields. *Ann. Probab.*, 45(6A):3886–3928, 2017.

[33] J. Ding and O. Zeitouni. Extreme values for two-dimensional discrete Gaussian free field. *Ann. Probab.*, 42(4):1480–1515, 2014.
[34] M. Fang and O. Zeitouni. Branching random walks in time inhomogeneous environments. *Electron. J. Probab.*, 17:no. 67, 18, 2012.
[35] M. Fang and O. Zeitouni. Slowdown for time inhomogeneous branching Brownian motion. *J. Stat. Phys.*, 149(1):1–9, 2012.
[36] M. Fels. Extremes of the 2d scale-inhomogeneous discrete Gaussian free field: Sub-leading order and exponential tails. *arXiv:1910.09915*, 2019.
[37] M. Fels and L. Hartung. Extremes of the 2d scale-inhomogeneous discrete gaussian free field: Extremal process in the weakly correlated regime. *arXiv:2002.00925*, 2020.
[38] X. Fernique. Regularité des trajectoires des fonctions aléatoires gaussiennes. In *École d'Été de Probabilités de Saint-Flour, IV-1974*, pages 1–96. Lecture Notes in Math., Vol. 480. Springer, Berlin, 1975.
[39] P. L. Ferrari and H. Spohn. Constrained Brownian motion: fluctuations away from circular and parabolic barriers. *Ann. Probab.*, 33(4):1302–1325, 2005.
[40] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre. Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.*, 22(2):89–103, 1971.
[41] Y. Fyodorov, G. Hiary, and J. Keating. Freezing transition, characteristic polynomials of random matrices, and the riemann zeta function. *Physical review letters*, 108:170601, 04 2012.
[42] E. Gardner and B. Derrida. Magnetic properties and function $q(x)$ of the generalized random energy model. *J. Phys. C*, 19:5783–5798, 1986.
[43] E. Gardner and B. Derrida. Solution of the generalized random energy model. *J. Phys. C*, 19:2253–2274, 1986.
[44] V. Gayrard and N. Kistler, editors. *Correlated random systems: five different methods*, volume 2143 of *Lecture Notes in Mathematics*. Springer, Cham; Société Mathématique de France, Paris, 2015. Lecture notes from the 1st CIRM Jean-Morlet Chair held in Marseille, Spring 2013, CIRM Jean-Morlet Series.
[45] R. D. Gordon. Values of mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. *Ann. Math. Statist.*, 12(3):364–366, 09 1941.
[46] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.
[47] J.-P. Kahane. Une inégalité du type de Slepian et Gordon sur les processus gaussiens. *Israel J. Math.*, 55(1):109–110, 1986.
[48] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15(3):1052–1061, 1987.
[49] G. F. Lawler. *Random walk and the heat equation*, volume 55 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2010.
[50] G. F. Lawler. *Intersections of random walks*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2013. Reprint of the 1996 edition.
[51] G. F. Lawler and V. Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.
[52] M. Ledoux and M. Talagrand. *Probability in Banach spaces*. Classics in Mathematics. Springer-Verlag, Berlin, 2011. Isoperimetry and processes, Reprint of the 1991 edition.
[53] T. M. Liggett. Random invariant measures for markov chains, and independent particle systems. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 45(4):297–313, Dec 1978.
[54] P. Maillard and O. Zeitouni. Slowdown in branching Brownian motion with inhomogeneous variance. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(3):1144–1160, 2016.
[55] B. Mallein. Maximal displacement in a branching random walk through interfaces. *Electron. J. Probab.*, 20:no. 68, 40, 2015.
[56] B. Mallein. Maximal displacement of a branching random walk in time-inhomogeneous environment. *Stochastic Process. Appl.*, 125(10):3958–4019, 2015.
[57] F. Ouimet. Geometry of the Gibbs measure for the discrete 2D Gaussian free field with scale-dependent variance. *ALEA Lat. Am. J. Probab. Math. Stat.*, 14(2):851–902, 2017.
[58] F. Ouimet. Maxima of branching random walks with piecewise constant variance. *Braz. J. Probab. Stat.*, 32(4):679–706, 2018.
[59] L. D. Pitt. Positively correlated normal variables are associated. *Ann. Probab.*, 10(2):496–499, 1982.
[60] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and Liouville quantum gravity. In *Stochastic processes and random matrices*, pages 548–577. Oxford Univ. Press, Oxford, 2017.
[61] S. Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007.
[62] R. K. Sundaram. *A first course in optimization theory*. Cambridge University Press, Cambridge, 1996.
[63] D. Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, extended edition, 2000.

[64] W. Werner. Topics on the two-dimensional Gaussian Free Field. *Lecture Notes – https://people.math.ethz.ch/ wewerner/GFFln.pdf*, 2014.
[65] O. Zeitouni. Branching random walks and Gaussian fields. In *Probability and statistical physics in St. Petersburg*, volume 91 of *Proc. Sympos. Pure Math.*, pages 437–471. Amer. Math. Soc., Providence, RI, 2016.

M. Fels, Institut für Angewandte Mathematik, Rheinische Friedrich-Wilhelms-Universität, Endenicher Allee 60, 53115 Bonn, Germany
*Email address*: fels@iam.uni-bonn.de, wt.iam.uni-bonn.de/maximilian-fels

L. Hartung, Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, 55099 Mainz, Germany
*Email address*: lhartung@uni-mainz.de, https://sites.google.com/view/lisahartung/home

CHAPTER 4

# Full extremal process of the scale-inhomogeneous two-dimensional discrete Gaussian Free Field in the weakly correlated regime

# EXTREMES OF THE 2D SCALE-INHOMOGENEOUS DISCRETE GAUSSIAN FREE FIELD: EXTREMAL PROCESS IN THE WEAKLY CORRELATED REGIME

MAXIMILIAN FELS, LISA HARTUNG

Abstract. We prove convergence of the full extremal process of the two-dimensional scale-inhomogeneous discrete Gaussian free field in the weak correlation regime. The scale-inhomogeneous discrete Gaussian free field is obtained from the 2d discrete Gaussian free field by modifying the variance through a function $\mathcal{I} : [0,1] \to [0,1]$. The limiting process is a cluster Cox process. The random intensity of the Cox process depends on the $\mathcal{I}'(0)$ through a random measure $Y$ and on the $\mathcal{I}'(1)$ through a constant $\beta$. We describe the cluster process, which only depends on $\mathcal{I}'(1)$, as points of a standard 2d discrete Gaussian free field conditioned to be unusually high.

## 1. Introduction

Log-correlated processes have received a lot of attention in recent years, see e.g. [1, 6, 27, 10, 15, 36, 32, 35, 2, 3]. Prominent examples are branching Brownian motion (BBM), the two-dimensional discrete Gaussian free field (DGFF), cover times of Brownian motion on the torus, characteristic polynomials of random unitary matrices or local maxima of the randomized Riemann zeta function on the critical line. One of the key features in these models is that their correlations are such that they start to become relevant for the extreme values of the processes. In particular, one is interested in the structure of the extremal processes that arises when the size of the index set tends to infinity. In the case of the 2d DGFF, one considers the field indexed by the vertices of a lattice box of side length $N$, where $N$ is taken to infinity. In this paper, we study the extremal process of the scale-inhomogeneous 2d DGFF in the weakly correlated regime. The model first appeared as a tool to prove Poisson-Dirichlet statistics of the extreme values of the 2d DGFF [8]. In the context of the 2d DGFF, it is the natural analogue model of the variable-speed BBM or time-inhomogeneous branching random walk (BRW). We start with a precise definition of the model we consider in the following.

**Definition 1.1** (2d discrete Gaussian free field (DGFF))**.** Let $N \in \mathbb{N}$ and $V_N = [0,N)^2 \cap \mathbb{Z}^2$. Then, the centred Gaussian field $\{\phi_v^N\}_{v\in V_N}$ with correlations given by the Green kernel

$$\mathbb{E}\left[\phi_v^N \phi_w^N\right] = G_{V_N}(v,w) := \frac{\pi}{2}\mathbb{E}_v\left[\sum_{k=0}^{\tau_{\partial V_N}-1} \mathbb{1}_{S_k=w}\right], \text{ for } v,w \in V_N \tag{1.1}$$

is called DGFF on $V_N$. Here, $\mathbb{E}_v$ is the expectation with respect to the SRW $\{S_k\}_{k\geq 0}$ on $\mathbb{Z}^2$ started in $v$ and $\tau_{\partial V_N}$ denotes the stopping time of the SRW hitting the boundary $\partial V_N$.

**Definition 1.2** (2d scale-inhomogeneous DGFF)**.** Let $\{\phi_v^N\}_{v\in V_N}$ be a DGFF on $V_N$. For $v = (v_1, v_2) \in V_N$ and $\lambda \in (0,1)$, set

$$[v]_\lambda \equiv [v]_\lambda^N := \left(\left[v_1 - \frac{1}{2}N^{1-\lambda}, v_1 + \frac{1}{2}N^{1-\lambda}\right] \times \left[v_2 - \frac{1}{2}N^{1-\lambda}, v_2 + \frac{1}{2}N^{1-\lambda}\right]\right) \cap V_N. \tag{1.2}$$

M.F. is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - project-id 211504053 - SFB 1060 and Germany's Excellence Strategy – GZ 2047/1, project-id 390685813 – "Hausdorff Center for Mathematics" at Bonn University.

Keywords: Gaussian free field, inhomogeneous environment, extreme values, extremal processes, branching Brownian motion, branching random walk.

We set $[v]_0^N := V_N$ and $[v]_1^N := \{v\}$. We denote by $[v]_\lambda^o$ the interior of $[v]_\lambda$. Let $\mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c} := \sigma\left(\{\phi_v^N, v \notin [v]_\lambda^o\}\right)$ be the $\sigma-$algebra generated by the random variables outside $[v]_\lambda^o$. For $v \in V_N$, let

$$\phi_v^N(\lambda) = \mathbb{E}\left[\phi_v^N | \mathcal{F}_{\partial[v]_\lambda \cup [v]_\lambda^c}\right], \quad \lambda \in [0,1]. \tag{1.3}$$

We denote by $\nabla\phi_v^N(\lambda)$ the derivative $\partial_\lambda \phi_v^N(\lambda)$ of the DGFF at vertex $v$ and scale $\lambda$. Moreover, let $s \mapsto \sigma(s)$ be a non-negative function such that $\mathcal{I}_{\sigma^2}(\lambda) := \int_0^\lambda \sigma^2(x)\mathrm{d}x$ is a function on $[0,1]$ with $\mathcal{I}_{\sigma^2}(0) = 1$ and $\mathcal{I}_{\sigma^2}(1) = 1$. The 2d scale-inhomogeneous DGFF on $V_N$ is a centred Gaussian field, $\psi^N := \{\psi_v^N\}_{v \in V_N}$, defined as

$$\psi_v^N := \int_0^1 \sigma(s)\nabla\phi_v^N(s)\mathrm{d}s. \tag{1.4}$$

For $\delta > 0$, let $V_N^\delta = [\delta N, (1-\delta)N)^2 \cap \mathbb{Z}^2$. [31, Lemma 3.3 (ii)] shows that it is a centred Gaussian field with covariance given by

$$\mathbb{E}\left[\psi_v^N \psi_w^N\right] = \log N \mathcal{I}_{\sigma^2}\left(\frac{\log N - \log_+ \|v-w\|_2}{\log N}\right) + O(1), \quad \text{for } v, w \in V_N^\delta, \tag{1.5}$$

with $\log_+ = \max\{0, \log(x)\}$.

**Assumption 1.** *In the rest of the paper, $\{\psi_v^N\}_{v\in V_N}$ is always a 2d scale-inhomogeneous DGFF on $V_N$. Moreover, we assume that $\mathcal{I}_{\sigma^2}(x) < x$, for $x \in (0,1)$, and that $\mathcal{I}_{\sigma^2}(1) = 1$, with $s \mapsto \sigma(s)$ being differentiable at* 0 *and* 1*, such that $\sigma(0) < 1$ and $\sigma(1) > 1$.*

Under Assumption 1 we proved in [30, 31], building on work by Arguin and Ouimet [7], the subleading order correction, tightness and convergence of the appropriately centred maximum. More explicitely, there exists a constant, $\beta = \beta(\sigma(1))$, which depends only on the final variance $\sigma(1)$, and a random variable, $Y = Y(\sigma(0))$, depending only on the initial variance $\sigma(0)$, such that, for any $z \in \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{v\in V_N} \psi_v^N \le m_N - z\right) = \mathbb{E}\left[\exp\left[-\beta Y e^{-2z}\right]\right], \tag{1.6}$$

where $m_N := 2\log N - \frac{\log\log N}{4}$. In particular, the limiting law solely depends on $\sigma(0)$ and $\sigma(1)$ and is therefore universal in the considered regime. Note that $m_N$ is also the maximum of $N^2$ i.i.d. $\mathcal{N}(0, \log N)$. Moreover, we proved in [31, Theorem 2.2] that under Assumption 1, points whose height is close to the maximum are either $O(N)$ apart or within distance $O(1)$. In particular, there is a constant $c > 0$, such that

$$\lim_{r\to\infty}\lim_{N\to\infty} \mathbb{P}\left(\exists u, v \in V_N \text{ with } r \le \|u-v\|_2 \le \frac{N}{r} \text{ and } \psi_u^N, \psi_v^N \ge m_N - c\log\log r\right) = 0. \tag{1.7}$$

To state our results, we introduce some additional notation. Let $A \subset [0,1]^2$ and $B \subset \mathbb{R}$ be two Borel sets. For $v \in \mathbb{Z}^2$ and $r > 0$, let its $r-$neighbourhood be $\Lambda_r(v) = \{w \in \mathbb{Z}^2 : \|v-w\|_1 \le r\}$. Then, define

$$\eta_{N,r}(A\times B) := \sum_{v\in V_N} \mathbb{1}_{\psi_v^N = \max_{u\in\Lambda_r(v)}\psi_u^N} \mathbb{1}_{x/N\in A}\mathbb{1}_{\psi_v^N - m_N\in B}. \tag{1.8}$$

$\eta_{N,r}$ is a point measure encoding both position and relative height of extreme local maxima in $r-$neighbourhoods. To study distributional limits of these point measures, we equip the space of point measures on $[0,1]^2 \times \mathbb{R}$ with the vague topology.

**Theorem 1.3.** *Let $\{\psi_v^N\}_{v\in V_N}$ be a scale-inhomogeneous DGFF satisfying Assumption 1. Then, there is a random measure $Y(\mathrm{d}x)$ on $[0,1]^2$ which depends only on the initial variance $\sigma(0)$ and satisfies almost surely $Y([0,1]^2) < \infty$ and $Y(A) > 0$, for any open and non-empty $A \subset [0,1]^2$. Moreover, there* *is a constant $\beta = \beta(\sigma(1)) > 0$, depending only on the final variance $\sigma(1)$, such that, for any sequence $r_N$ with $r_N \to \infty$ and $r_N/N \to 0$, as $N \to \infty$,*

$$\eta_{N,r_N} \overset{N\to\infty}{\to} PPP\Big(Y(\mathrm{d}x) \otimes \beta e^{-2h}\mathrm{d}h\Big), \tag{1.9}$$

*where convergence is in law with respect to the vague convergence of Radon measures on* $[0,1]^2 \times \mathbb{R}$.

As the field at nearby vertices is strongly correlated, around each local maximum there will naturally be plenty of particles being close to it. Together with location and height of $r-$local maxima, we encode them in the point process

$$\mu_{N,r} := \sum_{v\in V_N} \mathbb{1}_{\psi_v^N = \max_{u\in\Lambda_r(v)} \psi_u^N}\, \delta_{x/N} \otimes \delta_{\psi_v^N - m_N} \otimes \delta_{\{\psi_v^N - \psi_{v+w}^N :\, w\in\mathbb{Z}^2\}}. \tag{1.10}$$

These are Radon measures on $[0,1]^2 \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2}$. We consider this space equipped with the topology of vague convergence. The following theorem shows convergence of $\mu_{N,r}$, the full extremal process.

**Theorem 1.4.** *There is a probability measure $\nu$ on $[0,\infty)^{\mathbb{Z}^2}$ such that for each $r_N$ with $r_N \to \infty$ and $r_N/N \to 0$, as $N \to \infty$,*

$$\mu_{N,r_N} \to PPP\Big(Y(\mathrm{d}x) \otimes \beta e^{-2h}\mathrm{d}h \otimes \nu(\mathrm{d}\theta)\Big). \tag{1.11}$$

*The convergence is in law with respect to the vague convergence of Radon measures on* $[0,1]^2\times\mathbb{R}\times\bar{\mathbb{R}}^{\mathbb{Z}^2}$. *Moreover, $\nu$ is given by the weak limit,*

$$\nu(\cdot) = \lim_{r\to\infty} \mathbb{P}\Big(\phi^{\mathbb{Z}^2\setminus\{0\}} + 2\sigma(1)\mathfrak{a} \in \cdot | \phi_w^{\mathbb{Z}^2\setminus\{0\}} + 2\sigma(1)\mathfrak{a}(w) \geq 0,\ \forall \|w\|_1 \leq r\Big), \tag{1.12}$$

*with $\mathfrak{a}(w) = \lim_{N\to\infty} G_{V_{2N}}[(N,N),(N,N)] - G_{V_{2N}}[(N,N),(N,N)+w]$ being the potential kernel. In addition, $\theta_0 = 0$ and $|\{w \in \mathbb{Z}^2 : \theta_w \leq c\}| < \infty$, $\nu-$a.s. for each $c > 0$.*

As a consequence of Theorem 1.4, we obtain convergence of the extremal process

$$\eta_N := \sum_{v\in V_N} \delta_{v/N} \otimes \delta_{\psi_v^N - m_N}. \tag{1.13}$$

**Corollary 1.5.** *Let $\{(x_i,h_i) : i \in \mathbb{N}\}$ enumerate the points in a sample of $PPP\Big(Y(\mathrm{d}x) \otimes \beta e^{-2h}\mathrm{d}h\Big)$. Let $\{\theta_w^{(i)} : w \in \mathbb{Z}^2\}$, $i \in \mathbb{N}$, be independent samples from the measure $\nu$, independent of $\{(x_i,h_i) : i \in \mathbb{N}\}$. Then, as $N \to \infty$,*

$$\eta_N \to \sum_{i\in\mathbb{N}} \sum_{w\in\mathbb{Z}^2} \delta_{(x_i, h_i - \theta_w^{(i)})}. \tag{1.14}$$

*The convergence is in law with respect to the vague convergence of Radon measures on* $[0,1]^2 \times \mathbb{R}$. *Moreover, the measure on the right-hand side of* (1.14) *is locally finite on* $[0,1]^2 \times \mathbb{R}$ *a.s.*

1.1. **Related work.** Choosing $\sigma(x) \equiv 1$, for $x \in [0,1]$, in (1.4) gives the 2d DGFF. Its maximum value was investigated by Bolthausen, Bramson, Daviaud, Deuschel, Ding, Giacomin and Zeitouni [12, 24, 13, 21, 26, 28, 20], which culminated in the proof of convergence of the maximum [20]. Biskup and Louidor proved convergence of the extremal point process encoding local maxima and the field centred at those, to a cluster Cox process [9, 10]. The random intensity measure is identified with the so-called Liouville quantum gravity measure [11]. The cluster law of the 2d DGFF admits a closely related formulation to the one we obtain in Theorem 1.4, namely

$$\nu_{DGFF} = \lim_{r\to\infty} \mathbb{P}\Big(\phi^{\mathbb{Z}^2\setminus\{0\}} + 2\mathfrak{a} \in \cdot | \phi_w^{\mathbb{Z}^2\setminus\{0\}} + 2\mathfrak{a}(w) \geq 0,\ \forall \|w\|_1 \leq r\Big). \tag{1.15}$$

The slight, however important difference, is that the factor $\sigma(1)$ in (1.12) is equal to one. This causes the conditioning in (1.15) to be asymptotically singular. There is another possible regime in the scale-inhomogeneous DGFF, i.e. when $\mathcal{I}_{\sigma^2}(x) > x$, for some $x \in (0,1)$. When $x \mapsto \mathcal{I}_{\sigma^2}(x)$ is piecewise linear, the leading and sub-leading order of the maximum, as well as exponential tails of the centred maximum, in particular tightness, are known [7, 30].

Variable-speed branching Brownian motion (BBM), which first appeared in a paper by Derrida and Spohn [25], is the natural analogue in the context of BBM of the scale-inhomogeneous DGFF. It is a centred Gaussian process indexed by the leaves of the super-critical Galton-Watson tree, and covariance given by $tA(d(v,w)/t)$, where $d(v,w)$ is the time of the most recent common ancestor of two leaves $v$ and $w$. $A(x) \equiv 1$ corresponds to standard BBM. Its extremal process was investigated in [1, 6, 17, 22, 34, 4, 5, 23]. In [1, 6], the cluster process was shown to be BBM conditioned on the maximum being larger than $\sqrt{2}t$, or alternatively given as the limiting distribution of the neighbours of a local maximum. The extremal process of variable-speed BBM was investigated in [15, 16, 35, 29, 18]. In the regime of weak correlations, i.e. when $A(x) < x$, for $x \in (0,1)$, $A'(0) < 1$ and $A'(1) > 1$, Bovier and Hartung [15, 16] proved convergence of the extremal process to a cluster Cox process. The cluster law can be described by the law of BBM in time $t$, conditioned on the maximum being larger than $\sqrt{2}A'(1)t$, which is a perfect match to the one in the weakly correlated regime of the scale-inhomogeneous DGFF in (1.12). In the regime when $A$ is strictly concave, Bovier and Kurkova [19] showed that the first order of the maximum depends only on the concave hull of $A$. Moreover, Maillard and Zeitouni [35] proved that the 2nd order correction is proportional to $t^{1/3}$.

Note that there are other models such as the BRW [37] or first passage percolation [33] where it was proven that the extremal process converges to a (cluster) Cox process.

1.2. **Outline of Proof.** We start to explain the proof of Theorem 1.3. First, we deduce tightness of $\eta_{N,r}$ from (1.6), (1.7) and a uniform exponential upper bound on extreme level sets, which is proven in Proposition 2.1. Then, we characterize possible limit laws as a Cox processes using a superposition principle as in [9]. Finally, we need to show uniqueness of the random intensity measure. This follows from the convergence in distribution of multiple local maxima over disjoint subsets (see Theorem 2.5).

Next, we explain the proof of Theorem 1.4. By (1.7), we know that extreme local maxima have to be separated at distance $O(N)$ and, due to correlations, are surrounded by $O(1)$ neighbourhoods of high points. We need to show that the $O(1)$ neighbourhoods of extreme local maxima converge to independent samples of a cluster law. Using (1.7) we know that also the $O(1)$ neigbourhoods must be at macroscopic distance, i.e. at distance of $O(N)$. To obtain independence of the clusters, we decompose the field into a sum of independent "local fields" that are zero outside the $O(1)$ neighbourhoods and a "binding field", which captures the contributions from outside the neighbourhoods. The requirement of being a cluster around a local maximum then translates into the local field being smaller than the value at its centre. We then show convergence of the laws of the local fields conditioned on a local maximum at their centre. In particular, we deduce that the clusters are i.i.d. samples of a common cluster law. Together with convergence of the extremal process of local maxima, Theorem 1.3, this yields Theorem 1.4.

*Structure of the paper:* In Section 2, we prove Theorem 1.3. The necessary ingredient, convergence of multiple local maxima over disjoint subsets, i.e. Theorem 2.5, is proved in Section 4. The proof of Theorem 1.4 is provided in Section 3. The appendix recalls Gaussian comparison tools.

## 2. Proof of Theorem 1.3

It turns out that we are able to follow and use large parts of the proof for the DGFF by Biskup and Louidor [9]. As depicted in [9, 14], the fact that the limiting point process takes the particular form of a generalized Poisson point process, is a consequence of a superposition property, which is due to its Gaussian nature along with certain properties of the field such as the separation of local maxima [31] and tightness of extreme level sets. The main ingredient we need, in order to apply the machinery from [9] to obtain the distributional invariance and thus Poisson limit laws, is tightness of the point processes, which is a consequence of the following proposition and previous results in [31]. For $y \in \mathbb{R}$, we denote by

$$\Gamma_N(y) = \left\{ v \in V_N : \psi_v^N \geq m_N - y \right\}, \tag{2.1}$$

the level set above $m_N - y$.

**Proposition 2.1.** *There exists a constant $C > 0$, such that, for all $z > 1$ and all $\kappa$,*

$$\sup_{N \geq 1} \mathbb{P}\left(|\Gamma_N(y)| > e^{\kappa z}\right) \leq C e^{2y - \kappa z}. \tag{2.2}$$

*Proof.* By a first order Chebychev inequality and a standard Gaussian tail bound,

$$\mathbb{P}\left(|\Gamma_N(y)| > e^{\kappa z}\right) \leq \tilde{C} \frac{\sqrt{\log N}}{m_N - \lambda} N^2 \exp\left[-\frac{(m_N - y)^2}{2 \log N}\right] \leq C \exp\left[2y - \kappa z\right], \tag{2.3}$$

which shows (2.2). □

Proposition 2.1 together with [31, Theorem 2.1] implies tightness of $\{\eta_{N,r_N}\}_{N \in \mathbb{N}}$, as the right-hand side of (2.2) tends to zero as $N \to \infty$

2.1. **Distributional Invariance.** Let $(W_t)_{t \geq 0}$ be an independent standard Brownian motion started in 0. Given a measurable function $f : [0,1] \times \mathbb{R} \to [0, \infty)$, let

$$f_t(x, h) = -\log \mathbb{E}^0 \left[ e^{-f(x, h + W_t - \frac{1}{2}t)} \right], \quad t \geq 0, \tag{2.4}$$

where $\mathbb{E}^0$ is the expectation with respect to the Brownian motion $(W_t)_{t \geq 0}$.

**Theorem 2.2.** *(cp.* [9, Theorem 3.1]*) Let $\eta$ be any sub-sequential distributional limit of the processes $\{\eta_{N,r_N}\}_{N \geq 1}$, for some $r_N \to \infty$ with $r_N / N \to 0$. Then, for any continuous function $f : [0,1]^2 \times \mathbb{R} \to [0, \infty)$ with compact support and all $t \geq 0$,*

$$\mathbb{E}\left[e^{-<\eta, f>}\right] = \mathbb{E}\left[e^{-<\eta, f_t>}\right]. \tag{2.5}$$

*Proof.* The proof of Theorem 2.2 is a rerun of the one in the case of the 2d DGFF [9, Theorem 3.1]. We therefore omit details here. It essentially uses convergence of the maximum obtained in [31] together with expontential bounds on level sets, see Proposition 2.1. □

*Remark* 2.3. As we think that the interpretation of the statement by Biskup and Louidor in [9] is enlightening, we reproduce it here. Picking a sample, $\eta$, of the limit process, we know by tightness that $\eta(C) < \infty$ almost surely for any compact $C$. This allows us to write

$$\eta = \sum_{i \in \mathbb{N}} \delta_{(x_i, h_i)}, \tag{2.6}$$

where $\{(x_i, h_i) \in [0,1] \times \mathbb{R} \cup \{-\infty\} : i \in \mathbb{N}\}$ enumerate the points. Let $\{W_t^{(i)} : i \in \mathbb{N}\}$ be a collection of independent standard Brownian motions, independent of $\eta$, and set

$$\eta_t := \sum_{i \in \mathbb{N}} \delta_{(x_i, h_i + W_t^{(i)} - \frac{1}{2}t)}, \quad t \geq 0. \tag{2.7}$$

Using Fubini and dominated convergence, we have for all non-negative functions $f$,

$$\mathbb{E}\left[e^{-<\eta, f_t>}\right] = \mathbb{E}\left[e^{-<\eta_t, f>}\right]. \tag{2.8}$$

Theorem 2.2 then implies,

$$\eta_t \overset{d}{=} \eta, \quad t \geq 0. \tag{2.9}$$

We borrow from [9] a short heuristic argument why Theorem 2.2 should hold. Let $\psi$ be a scale-inhomogeneous DGFF on $V_N$ satisfying Assumption 1 and let $\psi', \psi''$ be two independent copies of it. Fix some $t > 0$. Then,

$$\psi \overset{d}{=} \sqrt{1 - \frac{t}{\log N}}\psi' + \sqrt{\frac{t}{\log N}}\psi'' = \psi' - \frac{t}{2\log N}\psi' + \sqrt{\frac{t}{\log N}}\psi'' + o(1), \tag{2.10}$$

where we have used a Taylor expansion of the first square root, which has an error term $O(t^2/\log^2 N)$. Using the fact, that the first order of the maximum of the scale-inhomogeneous DGFF is $\log N$, we obtain an error $o(1)$. If we take $v \in V_N$ away from the boundary, where $\psi_v \geq m_N - y$ or $\psi'_v \geq m_N - y$ and consider the $r-$neighbourhood $\Lambda_r(v)$, we first note that, for $w \in \Lambda_r(v)$, $\psi''_w - \psi''_v = O(1)$, and so by the prefactor, we may write,

$$\psi_w \overset{d}{=} \psi'_w - \frac{t}{2\log N}\psi'_w + \sqrt{\frac{t}{\log N}}\psi''_v + o(1), \quad w \in \Lambda_r(v). \tag{2.11}$$

Similarly, we know that $\psi_w - m_N = O(1)$ and $\psi'_w - m_N = O(1)$, for $w \in \Lambda_r(v)$, and thus, we may replace $\frac{t}{2\log N}\psi'_w$ by $\frac{t}{2\log N}(m_N + O(1)) = t + o(1)$, to obtain

$$\psi_w \overset{d}{=} \psi'_w - t + \sqrt{\frac{t}{\log N}}\psi''_v + o(1), \quad w \in \Lambda_r(v). \tag{2.12}$$

Finally, we see that $\sqrt{\frac{t}{\log N}}\psi''$ is asymptotically distributed as $W_t$, where $(W_t)_{t\geq 0}$ is a Brownian motion. Further, we know from [31, Theorem 2.2], that local extremes are at distance of order $N$ and so the field $\psi''$ in two such neighbourhoods has correlation of order $O(1)$. The normalizing factor $\sqrt{\frac{t}{\log N}}$ then implies that two such neighbourhoods are asymptotically independent. Thus, for $N$ large, we have a one-to-one correspondence between local maxima of $\psi$ and local maxima of $\psi'$ by a shift in their height through independent Brownian motions with drift $-1$.

2.2. **Poisson limit law.** Just as in [9], distributional invariance, Theorem 2.2, allows to extract a Poisson limit law for every such subsequence, i.e. for any sub-sequential limit of the extremal process. In our setting, we can directly apply [9, Theorem 3.2].

**Theorem 2.4.** [9, Theorem 3.2] *Suppose that $\eta$ is a sub-sequential limit of the process $\eta_{N,r_N}$, that is a point process on $[0,1]^2 \times \mathbb{R}$ such that, for some $t > 0$, and all continuous functions $f : [0,1]^2 \times \mathbb{R} \to [0,\infty)$ with compact support, it holds, as in Theorem 2.2,*

$$\mathbb{E}\left[e^{-<\eta,f>}\right] = \mathbb{E}\left[e^{-<\eta,f_t>}\right]. \tag{2.13}$$

*Moreover, assume that almost surely $\eta([0,1]^2 \times [0,\infty)) < \infty$ and $\eta([0,1]^2 \times \mathbb{R}) > 0$. Then, there is a random Borel measure $Y$ on $[0,1]^2$, satisfying $Y([0,1]^2) \in (0,\infty)$ almost surely, such that*

$$\eta \overset{d}{=} PPP\left(Y(\mathrm{d}x) \otimes \beta e^{-2h}\mathrm{d}h\right). \tag{2.14}$$

2.3. **Uniqueness.** In this section, we show uniqueness of the extremal process of local extremes, i.e. of the limit $\lim_{N\to\infty} \eta_{N,r_N}$. In light of Theorem 2.4, we do this by showing uniqueness of the random measure $Y(\mathrm{d}x)$. The proof is a generalization of the proof of uniqueness of the random variable $Y$ in [31, Theorem 2.1]. We show that the joint law of local maxima converges in law and that this law can be written as a Laplace transform of the random measure $Y(\mathrm{d}x)$, which then implies uniqueness of $Y(\mathrm{d}x)$. For a set $A \subset [0,1]$, we write $\psi^*_{N,A} = \max\left\{\psi^N_v : v \in V_N, v/N \in A\right\}$.

**Theorem 2.5.** *Let $(A_1,\ldots,A_p)$ be a collection of disjoint non-empty open subsets of $[0,1]^2$. Then the law of $\left(\max\{\psi^N_v : v \in V_N, v/N \in A_l\} - m_N\right)_{l=1}^p$ converges weakly as $N \to \infty$. More precisely, there are random variables $Y_{A_1},\ldots,Y_{A_p}$ depending only on the initial variance $\sigma(0)$, satisfying $Y_{A_i} > 0$ almost* *surely, for* $1 \leq i \leq p$*, and there is a constant* $\beta > 0$*, depending only on the final variance* $\sigma(1)$*, such that*

$$\lim_{N\to\infty} \mathbb{P}\left(\psi^*_{N,A_l} - m_N \leq x_l : l = 1, \ldots, p\right) = \mathbb{E}\left[\exp\left(-\beta \sum_{l=1}^{p} e^{-2x_l} Y_{A_l}\right)\right]. \tag{2.15}$$

The constant $\beta$ in Theorem 2.5 is identical to the one appearing in (1.6). Next, we prove Theorem 1.3. The proof of Theorem 2.5 is given in Section 4.

*Proof of Theorem 1.3 using Theorem 2.5.* Let $r_N \to \infty$ with $r_N/N \to 0$ be now a fixed sequence. Denote by $\eta$ a corresponding sub-sequential limit of the extremal process $\{\eta_{N,r_N}\}_{N\geq 1}$. By Theorem 2.4, there is a corresponding random measure $\tilde{Y}(\mathrm{d}x)$ such that $\eta \overset{d}{=} PPP\left(\tilde{Y}(\mathrm{d}x) \otimes \beta e^{-2h}\mathrm{d}h\right)$. Note that, as a trivial consequence of Theorem 2.5, for any open and non-empty $A \subset [0,1]^2$, $\psi^*_{N,A} - m_N$ is a tight sequence. Fix an arbitrary collection, $(A_1, \ldots, A_p)$, of disjoint, open and non-empty subsets of $[0,1]^2$, with $\tilde{Y}(\partial A_l) = 0$, for any $l \in \{1, \ldots, p\}$. By Theorem 2.5, there is a dense subset $R \subset \mathbb{R}$ such that, for any $x_1, \ldots, x_p \in R$,

$$\mathbb{E}\left[\exp\left(-\beta \sum_{l=1}^{p} e^{-2x_l} \tilde{Y}(A_l)\right)\right] = \lim_{N\to\infty} \mathbb{P}\left(\psi^*_{N,A_l} - m_N \leq x_l : l = 1, \ldots, p\right). \tag{2.16}$$

Again by Theorem 2.5, the right-hand side of (2.16) is the same for all subsequences. Using continuity in $x$ of the left hand side, we can deduce from convergence on the dense subset $R$, convergence on $\mathbb{R}$. Along with a standard approximation argument of continuous functions on $[0,1]^2$ via non-negative simple functions, this implies uniqueness of the Laplace transform of the random measure $\tilde{Y}(\mathrm{d}x)$ on the disjoint collection $(A_1, \ldots, A_p)$, regardless of the subsequence considered. As $p \in \mathbb{N}$ and $A_1, \ldots, A_p$ are arbitrary, it follows that $\tilde{Y}(\mathrm{d}x)$ is the same for all sub-sequences. Therefore, we obtain a random Borel measure $Y(\mathrm{d}x)$ whose masses of any countable collection of open sets $A_1, \ldots, A_p$ are given by $Y_{A_1}, \ldots, Y_{A_p}$ from Theorem 2.5, depending only on $\sigma(0)$. We conclude, that the law of the measure $Y(\mathrm{d}x)$ also depends only on initial variance, $\sigma(0)$. Further, note that by Proposition 2.1,

$$\mathbb{P}\left(\eta([0,1]^2 \times [-y, \infty]) > e^{ky}\right) \leq C e^{-y(\kappa-2)}. \tag{2.17}$$

In combination with Theorem 2.4, (2.17) implies that the total mass of $Y$ is almost surely finite. Moreover, Theorem 2.5 implies that, for any non-empty and open $A \subset [0,1]^2$, we have almost surely $Y(A) > 0$. □

## 3. Proof of Theorem 1.4

In the following, we assume that $V_N$ is centred at the origin. Let $\mu$ be a Radon measure on $[0,1]^2 \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2}$ and $f : [0,1]^2 \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2} \to [0,\infty)$ be a measurable function with compact support. We write

$$\langle \mu, f \rangle := \int \mu(\mathrm{d}x\mathrm{d}h\mathrm{d}\theta) f(x, h, \theta). \tag{3.1}$$

Further, let

$$\Theta_{N,r} := \{v \in V_N : \psi^N_v = \max_{u \in \Lambda_r(v)} \psi^N_u\} \tag{3.2}$$

be the set of $r-$local maxima.

**Lemma 3.1.** *For any* $r_N \to \infty$ *with* $r_N/N \to 0$ *and any continuous function* $f : [0,1]^2 \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2}$ *with compact support,*

$$\lim_{r\to\infty} \limsup_{N\to\infty} \max_{M: r \leq M \leq N/r} \left| \mathbb{E}\left[e^{-\langle \mu_{N,r_N} \rangle}\right] - \mathbb{E}\left[e^{-\langle \mu_{N,M}, f \rangle}\right] \right| = 0. \tag{3.3}$$

*Proof.* Let $\lambda > 0$ be such that $f(x,h,\theta) = 0$, for $h \geq \lambda$. If $\langle \mu_{N,r_N}, f\rangle \neq \langle \mu_{N,M}, f\rangle$, for some $M$ with $r \leq M \leq N/r$, then $\Theta_{N,r_N} \triangle \Theta_{N,M} \cap \Gamma_N(\lambda) \neq \emptyset$. Thus, there are $u, v \in \Gamma_N(\lambda)$ such that $\min(M, r_N) \leq \|u-v\|_2 \leq \max(M, r_N)$. For $N$ being so large that $r_N > r$ and $r_N \leq N/r$, this implies

$$\max_{M: r\leq M\leq N/r} \left| \mathbb{E}\left[e^{-\langle \mu_{N,r_N}, f\rangle}\right] - \mathbb{E}\left[e^{-\langle \mu_{N,M}, f\rangle}\right]\right| \leq \mathbb{P}\left(\exists u, v \in \Gamma_N(\lambda) : \, r \leq \|u-v\|_2 \leq N/r\right), \tag{3.4}$$

which by [31, Theorem 2.2] tends to zero. This shows (3.3). □

We set $M := \min\{k : 2^k > r\}$. In light of Lemma 3.1, we work with $\mu_{N,M}$ instead of $\mu_{N,r_N}$. Suppose that the local maximum is taken at $v \in V_N$. We decompose into two fields. The idea is, for fixed $v \in V_N$, to use the Gibbs-Markov property of the underlying DGFF to write the field into independent components. One that captures the field inside $\Lambda_M(v)$ and another that captures the field outside, i.e. in $\Lambda^c_M(v)$. $v \in V_N$ later plays the role of a local maximum. Thus, we write

$$\psi^N_w = \Phi^{M,v}_w + \tilde{\psi}^{\Lambda_M(v)}_w, \qquad \text{for } w \in \Lambda_M(v), \tag{3.5}$$

where

$$\Phi^{M,v}_w := \int_0^{1-\frac{\log M + log_+ \|v-w\|_2}{\log N}} \sigma(s)\nabla\phi^N_w(s)\mathrm{d}s + \int_{1-\frac{\log M + log_+ \|v-w\|_2}{\log N}}^1 \sigma(s)\nabla\mathbb{E}\left[\phi^N_w | \sigma\left(\phi^N_y : y \in \partial[w]_s \cap \Lambda^c_M(v)\right)\right]. \tag{3.6}$$

and where

$$\tilde{\psi}^{\Lambda_M(v)}_w = \int_{1-\frac{\log M + \log_+ \|v-w\|_2}{\log N}}^1 \sigma(s)\phi^{\Lambda_M(v)}_w(s)\mathrm{d}s. \tag{3.7}$$

The field in (3.6) encodes the increments when conditioning outside the local maximum $v \in V_N$ and its $M-$neighbourhood, $\Lambda_M(v)$. The field in (3.7) encodes the remaining increments within $\Lambda_M(v)$. The following lemma points out the key idea behind the definitions in (3.6) and (3.7).

**Lemma 3.2.** *Suppose $v \in V_N$ such that $\Lambda_M(v) \subset V_N$ and let $M = 2^k$. Consider the sigma-algebra*

$$\mathcal{F}_{M,v} := \sigma\left(\phi^N_w : \, w \in \{v\} \cup \Lambda_M(v)^c\right). \tag{3.8}$$

*Then, for Lebesgue almost every $t \in \mathbb{R}$,*

$$\mathbb{P}\left(\psi^N_{v+\cdot} - \Phi^{M,v}_{v+\cdot} \in \cdot | \mathcal{F}_{M,v}\right) = \mathbb{P}\left(\tilde{\psi}^{\Lambda_M(v)}_{v+\cdot} \in \cdot | \tilde{\psi}^{\Lambda_M(v)}_v = t - \Phi^{M,v}_v\right), \qquad \text{on } \{\psi^N_v = t\}. \tag{3.9}$$

*Proof.* It is an immediate consequence using (3.5). □

The following proposition is used to localize the initial increments, $\Phi^{M,v}_v$, of a local maximum at $v \in V_N$.

**Proposition 3.3.** *Let $t \in \mathbb{R}$. There is $r_0 \in \mathbb{N}$ such that, for any $\delta \in (0,1)$, $r \geq r_0$, $N \in \mathbb{N}$, sufficiently large, $M \in (r, N/r)$ and $\gamma \in (0, 1/2)$, there is a constant $C_\delta > 0$, depending only on $\delta$,*

$$\mathbb{P}\left(\exists v \in V_N : \, \psi^N_v \geq m_N - t, \Phi^{M,v}_v - 2\log N \mathcal{I}_{\sigma^2}\left(1 - \frac{\log M}{\log N}\right) \notin [-\log^\gamma(M), \log^\gamma(M)]\right)$$
$$\leq C_\delta e^{2s} \sum_{k=\lfloor \log M\rfloor}^{\infty} k^{\frac{1}{2}-\gamma} \exp\left[-k^{\frac{2\gamma-1}{2}}\right]. \tag{3.10}$$

*Proof.* As in (3.5),

$$\psi^N_v = \Phi^{M,v}_v + \tilde{\psi}^{\Lambda_M(v)}_v, \tag{3.11}$$

where the fields on the right hand side are independent. Using [31, Lemma 3.1 (i)] for the first and the last field in (3.11), as well as by Green function asymptotics, see e.g. [10, (3.47), (B.5)], we deduce that, for any $\delta > 0$, there is a constant $c_\delta > 0$, such that

$$\sup_{v\in V_N^\delta} \mathrm{Var}\left[\Phi_v^{M,v}\right] \leq 2\log N \mathcal{I}_{\sigma^2}\left(1-\frac{\log M}{\log N}\right) + c_\delta. \tag{3.12}$$

Moreover, $\{\Phi_v^{M,v}\}_{v\in V_N}$ is a centred Gaussian field. Thus, we can rerun the proof of [31, Proposition 4.2], where the constant on the right of [31, (4.13)] may now depend on $\delta$. This concludes the proof of Proposition 3.3. □

The following lemma allows us to reduce the local field defined in (3.7) to a usual DGFF with a constant parameter.

**Lemma 3.4.** *Let $v \in V_N^\delta$ and let $\{\tilde{\psi}_w^{\Lambda_M(v)} : w \in \Lambda_M(v)\}$ be the centred Gaussian field defined in* (3.7). *Then,*

$$\lim_{M\to\infty} \tilde{\psi}^{\Lambda_M(v)} - \sigma(1)\phi^{\Lambda_M(v)} = 0 \quad a.s. \tag{3.13}$$

*Proof.* Note that for some $\epsilon > 0$, by an Taylor expansion at $s = 1$, we have $\sigma(s) = \sigma(1) - \sigma'(1)(1 - s) + o(\sigma'(1)(1 - s))$, for $s \in (1 - \epsilon, 1]$. In particular, for any $v \in V_N$ and $w \in \Lambda_M(v)$,

$$\tilde{\psi}_w^{\Lambda_M(v)} - \sigma(1)\phi_w^{\Lambda_M(v)} = \int_{1-\frac{\log M+\log_+\|v-w\|_2}{\log N}}^1 \sigma'(1)(1-s)\nabla\phi_w^{\Lambda_M(v)}(s)\mathrm{d}s + o(1), \tag{3.14}$$

which is a centred Gaussian and where the error term vanishes, as $N \to \infty$. By Cauchy-Schwarz and asymptotics of the potential kernel, e.g. [10, (2.7), (B.6)], the covariances of the field on the right-hand side of (3.14) is bounded by a uniform constant times $\log^2 M/\log^{3/2} N$, which tends to zero uniformly, as $N \to \infty$. This shows (3.13) □

*Remark* 3.5. With regard to Proposition 3.3, the cluster law around around a local maximum $v \in V_N^\delta$ can be written in the form $\mathbb{P}\left(\tilde{\psi}^{\Lambda_M(v)} \in \cdot | \tilde{\psi}_v^{\Lambda_M(v)} = 2\log N\mathcal{I}_{\sigma^2}\left(1-\frac{\log M}{\log N},1\right)+t, \tilde{\psi}_w^{\Lambda_M(v)} \leq \tilde{\psi}_v^{\Lambda_M(v)}\right)$. Lemma 3.4 shows that this has the same weak limit, as $M \to \infty$ after $N \to \infty$, as

$$\nu^{(M,t)}(\cdot) \coloneqq \mathbb{P}\left(\sigma(1)\left(\phi_0^{\Lambda_M(0)} - \phi^{\Lambda_M(0)}\right) \in \cdot | \sigma(1)\phi_0^{\Lambda_M(0)} = 2\sigma^2(1)\log M + t, \sigma(1)\phi^{\Lambda_M(0)} \leq \sigma(1)\phi_0^{\Lambda_M(0)}\right). \tag{3.15}$$

In the following lemma we show that the the cluster limit of the law $\nu^{(M,t)}$ exists in a suitable sense.

**Lemma 3.6.** *Fix $r, j \geq 1$ and let $c_1 \in (0,\infty)$. For $M = \min\{k : 2^k > r\}$, uniformly in $f \in C_b\left(\mathbb{R}^{\Lambda_j}\right)$ and $t = o(\log M)$,*

$$\lim_{M\to\infty} \mathbb{E}_{\nu^{(M,t)}}[f] = \mathbb{E}_\nu[f], \tag{3.16}$$

*where $\nu(\cdot) \coloneqq \lim_{r\to\infty} \nu_r(\cdot)$,*

$$\nu_r(\cdot) \coloneqq \mathbb{P}\left(\phi^{\mathbb{Z}^2\setminus\{0\}} + 2\sigma(1)\mathfrak{a} \in \cdot | \phi_v^{\mathbb{Z}^2\setminus\{0\}} + 2\sigma(1)\mathfrak{a}(v) \geq 0 : \|v\|_1 \leq r\right) \tag{3.17}$$

*and $\mathfrak{a}$ being the potential kernel.*

*Proof.* Convergence of the finite dimensional distributions of the measures $\nu_r(\cdot)$ is a simple consequence of the DGFF satisfying the strong FKG-inequality, which implies that $r \mapsto \nu_r$ is stochastically increasing. Thus, $\lim_{r\to\infty} \nu_r(A)$ exists for any event $A$, depending on only a finite number of coordinates. Next, we prove that $\{\nu_r\}_r$ is tight, which then implies that $\nu$ is a distribution on $\mathbb{R}^{\mathbb{Z}^2}$. By a union and a Gaussian tail bound, for any $r \geq k_0 > 0$, there are constants $C, \tilde{C} > 0$ such that

$$\mathbb{P}\left(\exists v, k_0 \leq \|v\|_1 \leq r : \phi_v^{\mathbb{Z}^2\setminus\{0\}} > 2\sigma(1)\log\|v\|\right) \leq \sum_{k=k_0}^{r} 4k\mathbb{P}\left(\sup_{\|v\|_1=k} \phi_v^{\mathbb{Z}^2\setminus\{0\}} > 2\sigma(1)\log k + \frac{1}{2}\log(2)\right)$$
$$\leq C\sum_{k=k_0}^{r}\frac{4k}{\sqrt{\log k}}\exp\left[-\sigma^2(1)\log k + c_0\right] \leq \tilde{C}\sum_{k=k_0}^{\infty}\frac{1}{\sqrt{\log k}}\exp\left[-[\sigma^2(1)-1]\log k\right]. \tag{3.18}$$

As the sum converges and vanishes, as $k_0 \to \infty$, we deduce tightness of $(\nu_r)_{r\in\mathbb{N}}$ and so $\nu(\mathbb{R}^{\mathbb{Z}^2}) = 1$. In the last step, we show that it takes the particular form as in (3.17). We have that $\phi^{\Lambda_M(0)}$ conditioned on $\phi_0^{\Lambda_M(0)} = 2\sigma(1)\log M$ shifts the mean of $\phi_0^{\Lambda_M(0)} - \phi^{\Lambda_M(0)}$ by a quantity with asymptotic

$$(2\sigma(1)\log M + t)(1 - g_M(v)) \to 2\sigma(1)\mathfrak{a}(v), \tag{3.19}$$

as $M \to \infty$, and where $g_M(x)$ is discrete harmonic with $g_M(0) = 1$ and $g_M(x) = 0$, for $x \notin \Lambda_M(0)$. In particular, the law of $v \mapsto \phi_0^{\Lambda_M(0)} - \phi_v^{\Lambda_M(0)}$ conditioned on $\phi_0^{\Lambda_M(0)} = 2\sigma(1)\log M$ converges in the sense of finite dimensional distributions to

$$\phi_v^{\mathbb{Z}^2\setminus\{0\}} + 2\sigma(1)\mathfrak{a}(v), \tag{3.20}$$

where $\{\phi_v^{\mathbb{Z}^2\setminus\{0\}}\}_{v\in\mathbb{Z}^2\setminus\{0\}}$ is the pinned DGFF, which is a centred Gaussian field with covariances as in [10, (2.7)]. This concludes the proof of Lemma 3.6. □

Having weak convergence of the auxiliary cluster law, $\nu_r$, we are now in a position to prove convergence of the full extremal process.

*Proof of Theorem 1.4.* First note that by Lemma 3.1 we can work with $M$ instead of $r_N$. Let $f : [0,1]^2 \times \mathbb{R} \times \mathbb{R}^{\mathbb{Z}^2} \mapsto [0,\infty)$ be a continuous function with compact support. In addition, assume that, for any $x \in [0,1]^2$ and $t \in \mathbb{R}$, $f(x,t,\phi)$ depends only on $\{\phi_y : y \in \Lambda_M(x)\}$. Let $V_N = \cup_{i=1}^{(N/M)^2} V_{M,i}$ be a decomposition of $V_N$ into disjoint shifts of $V_M$. Moreover, let $\delta \in (0,1)$ and set

$$\mu_{N,M,\delta} := \sum_{v\in\cup_{i=1}^{(N/M)^2} V_{M,i}^{\delta}} \mathbb{1}_{v\in\Theta_{N,M}}\delta_{v/N}\otimes\delta_{\psi_v^N - m_N}\otimes\delta_{\{\psi_v^N-\psi_{v+w}^N :\, w\in\mathbb{Z}^2\}}. \tag{3.21}$$

By Proposition 3.3, [31, Proposition 5.1] and [31, Theorem 2.2], it suffices to compute

$$\lim_{\delta\to 0}\lim_{M\to\infty}\lim_{N\to\infty}\mathbb{E}\left[e^{-\langle\mu_{N,M,\delta},f\rangle}\mathbb{1}_{N\|v-w\|_2>4M : v,w\in\Theta_{N,M}}\mathbb{1}_{\{\Phi_v^{N,v}-2\log N\mathcal{I}_{\sigma^2}\left(1-\frac{\log M}{\log N}\right)\in[-\log^{\gamma}(M),\log^{\gamma}(M)] :\, v\in\Theta_{N,M}\}}\right]. \tag{3.22}$$

Set

$$f_{N,M}(v/N,t) :=$$
$$-\log\mathbb{E}\left[\exp\left[-f\left(x,t,\left(\psi_v^N - \Phi_v^{M,v} - \psi_{v+w}^N + \Phi_{v+w}^{M,v} : w\in\mathbb{Z}^2\right)\right)\right]|\psi_v^N = m_N + t, v\in\Theta_{N,M}\right]. \tag{3.23}$$

Conditioning on position, $x_i N$, and height, $m_N + t_i$, of local maxima in $\cup_{i=1}^{(N/M)^2} V_{M,i}^{\delta}$ and on the sigma-algebra $\sigma\left(\phi_w^N : w \in \cup\partial\Lambda_M(x_i N)\right)$, using Lemma 3.2 and the Taylor approximation for the cluster process as in Remark 3.5, we can rewrite (3.22) as

$$\mathbb{E}\left[\prod_{i=1}^{(N/M)^2} e^{-f_{N,M}(x_i,t_i)}\mathbb{1}_{N\|x_j-x_k\|_2>4M : x_jN,x_kN\in\Theta_{N,M}}\mathbb{1}_{\{\Phi_v^{N,v}-2\log N\mathcal{I}_{\sigma^2}\left(1-\frac{\log M}{\log N}\in[-\log^{\gamma}(M),\log^{\gamma}(M)]\right) :\, v\in\Theta_{N,M}\}}\right]. \tag{3.24}$$

On $\{\Phi_v^{N,v} - 2\log N \mathcal{I}_{\sigma^2}\left(1-\frac{\log M}{\log N}\right) \in [-\log^\gamma(M), \log^\gamma(M)] : v \in \Theta_{N,M}\}$, Lemma 3.2, Lemma 3.4, Remark 3.5 and Lemma 3.6 imply

$$\lim_{M\to\infty}\lim_{N\to\infty} f_{N,M}(x,t) = f_v(x,t) := -\log \mathbb{E}_v\left[e^{-f(x,t,\phi)}\right]. \tag{3.25}$$

In particular, the convergence in (3.25) is uniform in $x \in \cup_{i=1}^{(N/M)^2} V_{M,i}^\delta$ and $t \in \mathbb{R}$. Using (3.24) and Proposition 3.3, we can rewrite (3.22) as

$$\mathbb{E}\left[e^{-\langle \eta_{N,M}, f_v\rangle}\right] + o(1). \tag{3.26}$$

Applying Theorem 1.3 to (3.26), we obtain

$$\begin{aligned}\lim_{M\to\infty}\lim_{N\to\infty} \mathbb{E}\left[e^{-\langle \mu_{N,M}, f_v\rangle}\right] &= \mathbb{E}\left[\exp\left[-\int_{[0,1]^2\times\mathbb{R}} Y(\mathrm{d}x)\otimes\beta e^{-2h}\mathrm{d}h\left(1-e^{-f_v(x,h)}\right)\right]\right] \\ &= \mathbb{E}\left[\exp\left[-\int_{[0,1]^2\times\mathbb{R}\times\mathbb{R}^{\mathbb{Z}^2}} Y(\mathrm{d}x)\otimes\beta e^{-2h}\mathrm{d}h\otimes\nu(\mathrm{d}\phi)\left(1-e^{-f(x,h,\phi)}\right)\right]\right]. \end{aligned}\tag{3.27}$$

Noting that the last line in (3.27) is the Laplace transform of a Poisson point process with intensity $\beta Y(\mathrm{d}x)\otimes e^{-2h}\mathrm{d}h\otimes\nu(\mathrm{d}\phi)$, concludes the proof. □

## 4. Proof of Theorem 2.5

First, we recall the 3−field approximation used in [31] to prove convergence in law of the centred maximum.

4.1. 3−**field approximation.** We first decompose the underlying grid $V_N$. Assume $N = 2^n$ to be much larger than any other forthcoming integers. Next, pick two large integers $L = 2^l$ and $K = 2^k$. Partition $V_N$ in a disjoint union of $(KL)^2$ boxes, $\mathcal{B}_{N/KL} = \{B_{N/KL,i} : i = 1,\dots,(KL)^2\}$, each of side length $N/KL$. Let $v_{N/KL,i} \in V_N$ be the left bottom corner of box $B_{N/KL,i}$ and write $w_i = \frac{v_{N/KL,i}}{N/KL}$. We consider $\{w_i\}_{i=1,\dots,(KL)^2}$ as the vertices of a box $V_{KL}$. Analogously, let $K' = 2^{k'}$ and $L' = 2^{l'}$ be two integers, such that $K'L'$ divides $N$. Let $\mathcal{B}_{K'L'} = \{B_{K'L',i} : i = 1,\dots,[N/(K'L')]^2\}$ be a disjoint partitioning of $V_N$ with boxes $B_{K'L',j}$, each of side length $K'L'$. The left bottom corner of a box $B_{K'L',i}$ we call $v_{K'L',i}$. We take limits in the order $N, L, K, L'$ and then $K'$, for which we write $(N, L, K, L', K') \Rightarrow \infty$. The macroscopic field, $\{S_v^{N,c}\}_{v\in V_N}$, is a centred Gaussian field with covariance matrix $\Sigma^c$, with entries given by

$$\Sigma_{u,v}^c := \sigma^2(0)\mathbb{E}\left[\phi_{w_i}^{KL}\phi_{w_j}^{KL}\right], \quad \text{for } u \in B_{N/KL,i},\ v \in B_{N/KL,j}, \tag{4.1}$$

where $\{\phi_v^{KL}\}_{v\in V_{KL}}$ is a DGFF on $V_{KL}$. It captures the macroscopic dependence. The microscopic or "bottom field", $\{S_v^{N,b}\}_{v\in V_N}$, is a centred Gaussian field with covariance matrix $\Sigma^b$ defined entry-wise as

$$\Sigma_{u,v}^b := \begin{cases}\sigma^2(1)\mathbb{E}\left[\phi_{u-v_{K'L',i}}^{K'L'}\phi_{v-v_{K'L',i}}^{K'L'}\right], & \text{if } u,v \in B_{K'L',i}\\ 0, & \text{else,}\end{cases} \tag{4.2}$$

where $\{\phi_v^{K'L'}\}_{v\in V_{K'L'}}$ is a DGFF on $V_{K'L'}$. It captures "local" correlations. The third centred Gaussian field, $\{S_v^{N,m}\}_{v\in V_N}$, approximates the "intermediate" scales. It is a modified inhomogeneous branching random walk, defined pointwise as

$$S_v^{N,m} := \sum_{j=k'+l'}^{n-l-k}\ \sum_{B\in\mathcal{B}_j(v_{K'L',i'})} 2^{-j}\sqrt{\log 2}\, b_{i,j,B}^N \int_{n-j-1}^{n-j}\sigma\left(\frac{s}{n}\right)\mathrm{d}s, \quad \text{for } v \in B_{N/KL,i}\cap B_{K'L',i'}, \tag{4.3}$$

with $\{b_{i,j,B}^N : B \in \cup_{i'}\mathcal{B}_j(v_{K'L',i'}), i = 1,\dots,(KL)^2,\ j = 1,\dots,(N/K'L')^2,\ \}$ being a family of independent standard Gaussian random variables and where $\mathcal{B}_j(v_{K'L',i'})$ is the collection of boxes, $B \subset V_N$, of side length $2^j$ and lower left corner in $V_N$, that contain the element $v_{K'L',i'}$. In order to avoid boundary effects, we restrict our considerations onto a slightly smaller set, which is defined next.

Consider the disjoint union of $N/L-$ and $L-$boxes, that is $\mathcal{B}_{N/L} = \{B_{N/L,i} : i = 1\dots, L^2\}$ and $\mathcal{B}_L = \{B_{L,i} : i = 1,\dots,(N/L)^2\}$. Analogously, let $v_{N/L,i}$ and $v_{L,i}$ be the bottom left corners of the boxes $B_{N/L,i}$, $B_{L,i}$ containing $v$. For a box $B$, let $B^\delta \subset B$ be the set $B^\delta = \{v \in B : \min_{z\in\partial B} \|v - z\| \geq \delta l_B\}$, where $l_B$ denotes the side length of the box $B$. Finally, let

$$V^*_{N,\delta} := \{\bigcup_{1\leq i\leq L^2} B^\delta_{N/L,i}\} \cap \{\bigcup_{1\leq i\leq (KL)^2} B^\delta_{N/KL,i}\} \cap \{\bigcup_{1\leq i\leq (N/L)^2} B^\delta_{L,i}\} \cap \{\bigcup_{1\leq i\leq (N/KL)^2} B^\delta_{KL,i}\}. \tag{4.4}$$

The next lemma ensures that the sum of the three fields, $\{S^{N,c}_v\}_{v\in V_N}$, $\{S^{N,m}_v\}_{v\in V_N}$, $\{S^{N,b}_v\}_{v\in V_N}$, approximates well the scale-inhomogeneous DGFF, $\{\psi^N_v\}_{v\in V_N}$.

**Lemma 4.1.** [31, Lemma 5.2, Lemma 5.3] *There are non-negative uniformly bounded sequences of constants $a_{K'L',\bar{v}}$ and a family of i.i.d. Gaussians $\{\Theta_j\}_{j=1,\dots,(N/K'L')^2}$, such that, for $v \in B_{K'L',j}$, $v \equiv \bar{v}$* mod $K'L'$, *i.e. $\bar{v} = v - v_{K'L',j}$, and when setting*

$$S^N_v := S^{N,c}_v + S^{N,m}_v + S^{N,b}_v + a_{K',L',j}\Theta_j, \tag{4.5}$$

*we have*

$$\limsup_{(N,L,K,L',K')\Rightarrow\infty} \left|\mathrm{Var}\left(S^N_v\right) - \mathrm{Var}\left(\psi^N_v\right) - 4\alpha\right| = 0, \tag{4.6}$$

*for some $\alpha > 0$. Further, there exists a sequence $\{\epsilon'_{N,KL,K'L'} \geq 0\}$ with $\limsup_{(N,L,K,L',K')\Rightarrow\infty} \epsilon'_{N,KL,K'L'} = 0$ and bounded constants $C_\delta, C > 0$, such that for all $u, v \in V^*_{N,\delta}$ :*

1. *If $u, v \in B_{L',i}$, then $\left|\mathbb{E}\left[\left(S^N_u - S^N_v\right)^2\right] - \mathbb{E}\left[\left(\psi^N_u - \psi^N_v\right)^2\right]\right| \leq \epsilon'_{N,KL,K'L'}$.*
2. *If $u \in B_{N/L,i}$, $v \in B_{N/L,j}$ with $i \neq j$, then $\left|\mathbb{E}\left[S^N_u S^N_v\right] - \mathbb{E}\left[\psi^N_u \psi^N_v\right]\right| \leq \epsilon'_{N,KL,K'L'}$.*
3. *In all other cases, that is if $u, v \in B_{N/L,i}$ but $u \in B_{L',i'}$ and $v \in B_{L',j'}$ for some $i' \neq j'$, it holds that $\left|\mathbb{E}\left[S^N_u S^N_v\right] - \mathbb{E}\left[\psi^N_u \psi^N_v\right]\right| \leq C_\delta + 40\alpha$.*

The field, $\{S^N_v\}_{v\in V_N}$, defined in (4.5) is the approximating 3−field we work with.

4.2. **Reduction to approximating field.** In the following, we generalize the approximation results from [31] to the case of countably many local maxima. We show that the local maxima of $\{\psi^N_v\}_{v\in V_N}$ are well approximated by those of $\{S^N_v\}_{v\in V_N}$. As we need to compare probability measures on $\mathbb{R}^p$, we use the Lévy-Prokhorov metric $d(\cdot,\cdot)$, to measure distances between probability measures on $\mathbb{R}^p$. For two probability measures, $\mu$ and $\nu$, it is given by

$$d(\mu, \nu) = \inf\{\delta > 0 : \mu(B) \leq \nu(B^\delta) + \delta \text{ for all open sets } B\}, \tag{4.7}$$

where $B^\delta = \{y \in \mathbb{R}^p : |x - y| < \delta, \text{ for some } x \in B\}$. Further, let

$$\tilde{d}(\mu, \nu) = \inf\{\delta > 0 : \mu((x_1, \infty), \dots, (x_p, \infty)) \leq \nu((x_1 - \delta, \infty), \dots, (x_p - \delta, \infty)) + \delta, \forall (x_1, \dots, x_p) \in \mathbb{R}^p\}, \tag{4.8}$$

which is a measure for stochastic domination. In particular, if $\tilde{d}(\mu, \nu) = 0$, then $\nu$ stochastically dominates $\mu$. Note, unlike $d(\cdot,\cdot)$, $\tilde{d}(\cdot,\cdot)$ is not symmetric. Abusing notation, we write for random vector $X, Y$ with laws $\mu_X, \mu_Y$, $d(X, Y)$ instead of $d(\mu_X, \mu_Y)$ and likewise for $\tilde{d}$. Fix $r \in \mathbb{N}$ and let $\mathcal{B}_r$ of $V_{\lfloor N/r\rfloor r}$ into sub-boxes of side length $r$. Let $\mathcal{B} = \cup_{r\in\mathbb{N}, r\leq N}\mathcal{B}_r$ and $\{g_b\}_{B\in\mathcal{B}}$ be a collection of i.i.d. standard Gaussian random variables. For $v \in V_N$, denote by $B_r(v) \in \mathcal{B}_r$ the box containing $v$. For $r_1, r_2 \in \mathbb{N}$, $r_1, r_2 \leq N$, $A \subset [0, 1]^2$, $s_1.s_2 \in \mathbb{R}_+$, we write

$$\bar{\psi}^*_{N,A} := \max_{v\in V_N : v/N\in A} \psi^N_v + s_1 g_{B_{v,r_1}} + s_2 g_{B_{v,N/r_2}}, \tag{4.9}$$

and for a general field $\{g^N_v\}_{v\in V_N}$,

$$g^*_{N,A} := \max_{v\in V_N : v/N\in A} g^N_v. \tag{4.10}$$

Fix $p \in \mathbb{N}$ and disjoint, open, non-empty, simply connected sets $A_1, \ldots, A_p \subset [0,1]^2$.

**Lemma 4.2.** *For $s = (s_1, s_2) \in \mathbb{R}_+^2$, it holds*

$$\limsup_{r_1, r_2 \to \infty} \limsup_{N \to \infty} d((\psi^*_{N,A_i} - m_N)_{1 \le i \le p}, (\bar{\psi}^*_{N,A_i} - m_N - \|s\|_2^2)_{1 \le i \le p}) = 0. \tag{4.11}$$

For the proof of Lemma 4.2 we need some additional estimates.

**Lemma 4.3.** *Let $\{\bar{\psi}^N_v\}_{v \in V_N}$ be a centred Gaussian field and $c > 0$ a constant, such that, for any $v, w \in V_N$, $\left|\mathbb{E}\left[\bar{\psi}^N_v \bar{\psi}^N_w\right] - \mathbb{E}\left[\psi^N_v \psi^N_w\right]\right| \le c$. Moreover, let $A \subset [0,1]^2$ be an open, non-empty subset and $\{g^N_v\}_{v \in V_N}$ be a collection of independent random variables, such that*

$$\mathbb{P}\left(g^N_v \ge 1 + y\right) \le e^{-y^2} \quad \text{for } v \in V_N. \tag{4.12}$$

*Then, there is a constant $C = C(\alpha) > 0$ such that, for any $\epsilon > 0$, $N \in \mathbb{N}$ and $x \ge -\epsilon^{1/2}$,*

$$\mathbb{P}\left(\max_{v \in V_N : v/N \in A} (\bar{\psi}^N_v + \epsilon g^N_v) \ge m_N + x\right) \le \mathbb{P}\left(\max_{v \in V_N : v/N \in A} \bar{\psi}^N_v \ge m_N + x - \sqrt{\epsilon}\right)(Ce^{-C^{-1}\epsilon^{-1}}). \tag{4.13}$$

*Proof.* Set $\Gamma_y := \{v \in V_N : v/N \in A, y/2 \le \epsilon g^N_v \le y\}$. Then,

$$\begin{aligned}\mathbb{P}\left(\max_{v \in V_N : v/N \in A} (\bar{\psi}^N_v + \epsilon g^N_v) \ge m_N + x\right) &\le \mathbb{P}\left(\max_{v \in V_N : v/N \in A} \bar{\psi}^N_v \ge m_N + x - \sqrt{\epsilon}\right) \\ &+ \sum_{i=0}^{\infty} \mathbb{E}\left[\mathbb{P}\left(\max_{v \in \Gamma_{2^i \sqrt{\epsilon}}} \bar{\psi}^N_v \ge m_N + x - 2^i \sqrt{\epsilon} |\Gamma_{2^i \sqrt{\epsilon}}\right)\right].\end{aligned} \tag{4.14}$$

By [31, Proposition 5.1], the second term on the right hand side in (4.14) is bounded from above by

$$\sum_{i=0}^{\infty} \mathbb{E}\left[\mathbb{P}\left(\max_{v \in V_N : v/N \in A} \bar{\psi}^N_v \ge m_N + x - 2^i \sqrt{\epsilon} |\Gamma_{2^i \sqrt{\epsilon}}\right)\right] \le \tilde{c} e^{-cx} \sum_{i=0}^{\infty} \mathbb{E}\left[|\Gamma_{2^i \sqrt{\epsilon}}| / |\{v \in V_N : v/N \in A\}|\right] e^{c 2^i \sqrt{\epsilon}}, \tag{4.15}$$

where $\tilde{c} > 0$ is a finite constant. By assumption (4.12), one has

$$\mathbb{E}\left[|\Gamma_{2^i \sqrt{\epsilon}}| / |\{v \in V_N : v/N \in A\}|\right] \le e^{-4^i (C\epsilon)^{-1}}. \tag{4.16}$$

Thus, using (4.16), (4.15) is bounded from above by

$$\tilde{c} e^{-cx} e^{-(C\epsilon)^{-1}}. \tag{4.17}$$

This concludes the proof of Lemma 4.3. □

**Lemma 4.4.** *Let $\{\tilde{\psi}^N_v\}_{v \in V_N}$ be a centred Gaussian field satisfying*

$$|\mathrm{Var}\, \psi^N_v - \mathrm{Var}\, \tilde{\psi}^N_v| \le \epsilon. \tag{4.18}$$

*Further, fix some $p \in \mathbb{N}$, and disjoint open, non-empty sets $A_1, \ldots, A_p \subset [0,1]^2$. If*

$$\mathbb{E}\left[\tilde{\psi}^N_v \tilde{\psi}^N_w\right] \le \mathbb{E}\left[\psi^N_v \psi^N_w\right] + \epsilon, \tag{4.19}$$

*then*

$$\limsup_{N \to \infty} \tilde{d}\left((\psi^*_{N,A_1} - m_N, \ldots, \psi^*_{N,A_p} - m_N), (\tilde{\psi}^*_{N,A_1} - m_N, \ldots, \tilde{\psi}^*_{N,A_p} - m_N)\right) \le l(\epsilon), \tag{4.20}$$

*and else if,*

$$\mathbb{E}\left[\tilde{\psi}^N_v \tilde{\psi}^N_w\right] + \epsilon \ge \mathbb{E}\left[\psi^N_v \psi^N_w\right], \tag{4.21}$$

*then*

$$\limsup_{N \to \infty} \tilde{d}\left((\tilde{\psi}^*_{N,A_1} - m_N, \ldots, \tilde{\psi}^*_{N,A_p} - m_N), (\psi^*_{N,A_1} - m_N, \ldots, \psi^*_{N,A_p} - m_N)\right) \le l(\epsilon), \tag{4.22}$$

*where $l(\epsilon) \to 0$ as $\epsilon \to 0$.*

*Proof.* Let $\{\psi_v^N\}_{v\in V_N}$, $\{\tilde{\psi}_v^N\}_{v\in V_N}$ satisfy relations (4.18) and (4.19). Let $\Phi, \{\Phi_v^N\}_{v\in V_N}$ two independent standard Gaussian random variables, and $\epsilon^*(\epsilon) > 0$. For $v \in V_N$, set

$$\psi_v^{N,lw,\epsilon^*} = \left(1 - \frac{\epsilon^*}{\log N}\right)\psi_v^N + \epsilon^{N,\prime}\Phi, \tag{4.23}$$

$$\tilde{\psi}_v^{N,up,\epsilon^*} = \left(1 - \frac{\epsilon^*}{\log N}\right)\tilde{\psi}_v^N + \epsilon_v^{N,\prime\prime}\Phi_v^N, \tag{4.24}$$

where we can choose, as in the proof of [31, Lemma 5.6], $\epsilon^*$, $\epsilon_v^{N,\prime} = \epsilon_v^{N,\prime}(\epsilon,\epsilon^*)$ and $\epsilon_v^{N,\prime\prime} = \epsilon_v^{N,\prime\prime}(\epsilon,\epsilon^*)$ all non-negative and tending to 0 as $\epsilon \to 0$, such that

$$\mathrm{Var}\left[\psi_v^{N,lw,\epsilon^*}\right] = \mathrm{Var}\left[\tilde{\psi}_v^{N,up,\epsilon^*}\right] = \mathrm{Var}\left[\psi_v^N\right] + \epsilon, \quad \forall v \in V_N \tag{4.25}$$

and

$$\mathbb{E}\left[\psi_v^{N,lw,\epsilon^*}\psi_w^{N,lw,\epsilon^*}\right] \geq \mathbb{E}\left[\tilde{\psi}_v^{N,up,\epsilon^*}\tilde{\psi}_w^{N,up,\epsilon^*}\right], \quad \forall v, w \in V_N. \tag{4.26}$$

An application of Slepian's lemma for vectors (Theorem 5.2), gives

$$\tilde{d}\left((\psi^*_{N,lw,\epsilon^*,A_1} - m_N, \ldots, \psi^*_{N,lw,\epsilon^*,A_p} - m_N), (\tilde{\psi}^*_{N,up,\epsilon^*,A_1} - m_N, \ldots, \tilde{\psi}^*_{N,up,\epsilon^*,A_1} - m_N)\right) = 0. \tag{4.27}$$

By Lemma 4.3, we obtain, for $x_1, \ldots, x_p \in \mathbb{R}$,

$$\mathbb{P}\left(\tilde{\psi}^*_{N,up,\epsilon^*,A_i} - m_N \geq x_i,\ 1 \leq i \leq p\right) \leq \mathbb{P}\left(\psi^*_{N,A_i} - m_N \geq x_i - \sqrt{\max_{w\in V_N} \epsilon_w^{N,\prime\prime}},\ 1 \leq i \leq p\right) \times Ce^{-(C\max_{w\in V_N}\epsilon_w^{N,\prime\prime})^{-1}}. \tag{4.28}$$

Since $\lim_{\epsilon\to 0}\max_{w\in V_N}\epsilon_w^{N,\prime\prime} = 0$ this implies (4.20). (4.22) can be proved the same way by switching the roles of $\{\psi_v^N\}_{v\in V_N}$ and $\{\tilde{\psi}_v^N\}_{v\in V_N}$. We omit further details. □

**Proposition 4.5.** *Let $\tilde{\sigma} \in (0,\infty)^2$, $r = (r_1, r_2) \in (0,\infty)^2$, and $\{\psi_v^{N,r,\tilde{\sigma}} : v \in V_N\}$ as well as $\{\psi_v^{N,\tilde{\sigma},*} : v \in V_N\}$ be two Gaussian fields given by*

$$\psi_v^{N,r,\tilde{\sigma}} = \psi_v^N + \tilde{\sigma}_1 g_{B_{v,r_1}} + \tilde{\sigma}_2 g_{B_{v,N/r_2}}, \quad \text{for } v \in V_n \tag{4.29}$$

*and*

$$\psi_v^{N,\tilde{\sigma},*} = \psi_v^N + \sqrt{\frac{\|\tilde{\sigma}\|_2^2}{\log(N)}}\tilde{\psi}_v^N, \quad \text{for } v \in V_N \tag{4.30}$$

*where $\{\psi_v^N\}_{v\in V_N}$, $\{\tilde{\psi}_v^N\}_{v\in V_N}$ are two independent scale-inhomogeneous DGFFs, satisfying Assumption 1, and where $\{g_B\}_{B\in\mathcal{B}}$ is a collection of independent standard Gaussians. For a set $A \subset [0,1]^2$, we write $M_{N,A,r_1,r_2,\tilde{\sigma}} = \max_{v\in V_N: v/N\in A}\psi_v^{N,r,\tilde{\sigma}}$ and likewise, $M_{N,A,\tilde{\sigma},*} = \max_{v\in V_N: v/N\in A}\psi_v^{N,\tilde{\sigma},*}$. Then, for any $p \in \mathbb{N}$, and any collection of disjoint, open and non-empty $A_1, \ldots, A_p \subset [0,1]^2$,*

$$\limsup_{N\to\infty} d\left((M_{N,A_1,r_1,r_2,\tilde{\sigma}} - m_N, \ldots, M_{N,A_p,r_1,r_2,\tilde{\sigma}} - m_N), (M_{N,A_1,\tilde{\sigma},*} - m_N, \ldots, M_{N,A_p,\tilde{\sigma}*} - m_N)\right) = 0, \tag{4.31}$$

*as $r_1, r_2 \to \infty$.*

*Proof.* The proof is a straightforward adaptation of the proof of [31, Proposition B.2]. Decompose $V_N$ into boxes $B$ of side length $N/r_2$ and call their collection $\mathcal{B}$. Further, for $\delta \in (0,1)$ and $B \in \mathcal{B}$, let $B_\delta$ be the box with the identical centre as $B$, and reduced side length $(1-\delta)N/r_2$. Then, we set $V_{N,\delta} = \cup_{B\in\mathcal{B}}B_\delta$. The corresponding maxima over are called $M_{N,A,r_1,r_2,\tilde{\sigma},\delta} = \max_{v\in V_{N,\delta}: v/N\in A}\psi_v^{N,r,\tilde{\sigma}}$ and

$M_{N,A,\tilde{\sigma},*} = \max_{v\in V_{N,\delta}:v/N\in A} \psi_v^{N,\tilde{\sigma},*}$. [31, Proposition 5.1] shows that it suffices to consider the maxima on the slightly smaller sets, i.e. one has

$$\lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{P}\left(M_{N,A_1,r_1,r_2,\tilde{\sigma},\delta} \neq M_{N,A_1,r_1,r_2,\tilde{\sigma}},\dots,M_{N,A_p,r_1,r_2,\tilde{\sigma},\delta} \neq M_{N,A_p,r_1,r_2,\tilde{\sigma}}\right)$$
$$= \lim_{\delta\to 0}\lim_{N\to\infty} \mathbb{P}\left(M_{N,A_1,\tilde{\sigma},*,\delta} \neq M_{N,A_1,\tilde{\sigma},*},\dots,M_{N,A_p,\tilde{\sigma},*,\delta} \neq M_{N,A_p,\tilde{\sigma},*}\right) = 0. \quad (4.32)$$

Next, we claim that the maximum is essentially determined by the maximum of the unperturbed scale-inhomogeneous DGFF, $\{\psi_v^N\}_{v\in V_N}$. For $B \in \mathcal{B}$, let $z_B$ be the unique element, such that

$$\psi_{z_B}^N = \max_{v\in B_\delta} \psi_v^N. \quad (4.33)$$

The claim is that

$$\lim_{r_1,r_2\to\infty}\lim_{N\to\infty} \mathbb{P}\left(|M_{N,A_i,r_1,r_2,\tilde{\sigma},\delta} - \max_{B\in\mathcal{B},B\subset NA_i} \psi_{z_B}^{N,r,\tilde{\sigma}}| \geq \frac{1}{\log n} \,:\, 1\leq i\leq p\right)$$
$$= \limsup_{N\to\infty} \mathbb{P}\left(|M_{N,A_i,\tilde{\sigma},*,\delta} - \max_{B\in\mathcal{B},B\subset NA_i} \psi_{z_B}^{N,\tilde{\sigma},*}| \geq \frac{1}{\log n} \,:\, 1\leq i\leq p\right) = 0. \quad (4.34)$$

In the following, we show that none of the events in the probabilities in (4.34) can occur. It suffices to show that none of the following events can happen. For $i \in \{1,\dots,p\}$, let

$$E_1^{(i)} = \{M_{N,A_i,r_1,r_2,\tilde{\sigma},\delta} \notin (m_N - C, m_N + C)\} \cup \{M_{N,A_i,\tilde{\sigma},*,\delta} \notin (m_N - C, m_N + C)\} \quad (4.35)$$
$$E_2^{(i)} = \{\exists u,v \in V_N : u,v/N \in A_i, \|u-v\| \in (r, N/r) \text{ and } \min(\psi_u^N,\psi_v^N) > m_N - c\log n\} \quad (4.36)$$
$$E_3^{(i)} = \tilde{E}_3^{(i)} \cup \bar{E}_3^{(i)}, \text{ where } \tilde{E}_3^{(i)} = \{\omega : \exists v\in V_N,\ v/N\in A_i : \psi_v^{N,r,\tilde{\sigma}} = M_{N,A_i,r_1,r_2,\tilde{\sigma},\delta},\ \psi_v^N \leq m_N - c\log n\},$$
$$\bar{E}_3^{(i)} = \{\omega : \exists v\in V_N,\ v/N\in A_i : \psi_v^{N,\tilde{\sigma},*} = M_{N,A_i,\tilde{\sigma},*,\delta},\ \psi_v^N \leq m_N - c\log n\} \quad (4.37)$$
$$E_4^{(i)} = \left\{\exists v\in B\in\mathcal{B}\subset NA_i : \psi_v^N \geq m_N - c\log n \text{ and } \sqrt{\frac{\|\tilde{\sigma}\|_2^2}{\log N}}\tilde{\psi}_v^N - \sqrt{\frac{\|\tilde{\sigma}\|_2^2}{\log N}}\tilde{\psi}_{z_B}^N \geq 1/\log n\right\}. \quad (4.38)$$

The events $E_2, E_3$ and $E_4$ in the proof of [31, Proposition B.2] include the corresponding events, $E_2^{(i)}, E_3^{(i)}, E_4^{(i)}$, we are considering here, and so we know that the probability of their occurrence tends to zero. So, we are left with bounding the events $E_1^{(i)}$. First note that it suffices to consider the scale-inhomogeneous DGFF, as the other terms are centred Gaussians with uniformly bounded variance. Since maximizing over a subset, we have, for any $i \in \{1,\dots,p\}$,

$$\mathbb{P}\left(\max_{v\in V_N:\,v/N\in A_i} \psi_v^N > m_N + C\right) \leq \mathbb{P}\left(\max_{v\in V_N} \psi_v^N > m_N + C\right). \quad (4.39)$$

By tightness of the centred maximum [31, (2.2)], (4.39) tends to 0 as $C\to\infty$, uniformly in $N$. Hence to show (4.34), it suffices to prove, for any $i \in \{1,\dots,p\}$,

$$\lim_{C\to\infty}\lim_{N\to\infty} \mathbb{P}\left(\max_{v\in V_N:v/N\in A_i} \psi_v^N \leq m_N - C\right) = 0. \quad (4.40)$$

Assume otherwise, then there is a subsequence $\{N_k\}_{k\in\mathbb{N}}$, a sequence $C_N\to\infty$ as $N\to\infty$ and a constant $\epsilon > 0$, such that, for any $k\in\mathbb{N}$,

$$\mathbb{P}\left(\max_{v\in V_{N_k}:\,v/N_k\in A_i} \psi_v^{N_k} \leq m_{N_k} - C_{N_k}\right) \geq \epsilon. \quad (4.41)$$

We can further assume that $A_i \subset [0,1]^2$ is a box, otherwise pick the largest box that fits into $A_i$. We can decompose $[0,1]^2$ into disjoint translations of $A_i^{(j)}$, that we possible need to cut with $[0,1]^2$. For each $A_i^{(j)}N$ we consider an independent copy of $\{\psi_v^N\}_{v\in V_N}$, called $\{\psi_v^{N,j}\}_{v\in V_N}$. By translation invariance, for each of these (4.41) holds. By Gaussian comparison, independence and (4.41), we have

$$\mathbb{P}\left(\max_{v\in V_{N_k}} \psi_v^{N_k} \leq m_{N_k} - C_{N_k}\right) \geq \mathbb{P}\left(\max_j \max_{v\in A_i^{(j)}N_k} \psi_v^{N_k,j} \leq m_{N_k} - C_{N_k}\right) > 0. \tag{4.42}$$

By tightness of $\{\max_{v\in V_N} \psi_v^N - m_N\}_{N\in\mathbb{N}}$, the left-hand side of (4.42) tends to zero, which is a contradiction. Thus, this yields (4.40), which concludes the proof of Proposition 4.5. □

Lemma 4.4 and Proposition 4.5 allow us to prove Lemma 4.2.

*Proof of Lemma 4.2:* Define for $v \in V_N$, $\bar{\psi}_v^{N,\tilde{\sigma}} = \left(1 + \frac{\|\tilde{\sigma}\|^2}{\log(N)}\right)\psi_v^N$, and for $A \subset [0,1]^2$ open and non-empty, $\bar{M}_{N,A,\tilde{\sigma}} = \max_{v\in V_N: v/N\in A} \bar{\psi}_v^{N,\tilde{\sigma}}$ and set $M_{N,A} = \max_{v\in V_N: v/N\in A} \psi_v^N$. (4.39) together with tightness of the centred maximum [31, (2.2)] and (4.40) implies,

$$\mathbb{E}\left[\bar{M}_{N,A_i,\tilde{\sigma}}\right] = \mathbb{E}\left[M_{N,A_i}\right] + 2\|\tilde{\sigma}\|_2^2 + o(1), \tag{4.43}$$

and

$$\lim_{N\to\infty} d(M_{N,A_i} - \mathbb{E}\left[M_{N,A_i}\right], \bar{M}_{N,A_i,\tilde{\sigma}} - \mathbb{E}\left[\bar{M}_{N,A_i,\tilde{\sigma}}\right]) = 0. \tag{4.44}$$

Next, we consider the field, $\{\psi_v^{N,\tilde{\sigma},*}\}_{v\in V_N}$, defined in (4.30). For $i \in \{1,\ldots,p\}$, set $M_{N,A_i,\tilde{\sigma},*} = \max_{v\in V_N:\, v/N\in A_i} \psi_v^{N,\tilde{\sigma},*}$. In distribution, $\{\psi_v^{N,\tilde{\sigma},*}\}_{v\in V_N}$ can be written as a sum of $\{\bar{\psi}_v^{N,\tilde{\sigma}}\}_{v\in V_N}$ and an independent centred Gaussian field with variances of order $O((1/\log N)^3)$. Thus, by Gaussian comparison,

$$\mathbb{E}\left[\bar{M}_{N,A_i,\tilde{\sigma}}\right] = \mathbb{E}\left[M_{N,A_i,\tilde{\sigma},*}\right] + o(1) \tag{4.45}$$

and

$$\lim_{N\to\infty} d\left(\left(\bar{M}_{N,A_i,\tilde{\sigma}} - \mathbb{E}\left[\bar{M}_{N,A_i,\tilde{\sigma}}\right]\right)_{1\leq i\leq p}, \left(\bar{M}_{N,A_i,\tilde{\sigma},*} - \mathbb{E}\left[\bar{M}_{N,A_i,\tilde{\sigma},*}\right]\right)_{1\leq i\leq p}\right) = 0. \tag{4.46}$$

Combining (4.46) with Proposition 4.5 and applying the triangle inequality, one concludes the proof of Lemma 4.2. □

Finally, we are able to deduce the key result in this subsection.

**Lemma 4.6.** *Let $p \in \mathbb{N}$, and $A_1,\ldots,A_p \subset [0,1]^2$ be disjoint, open and non-empty. Then,*

$$\limsup_{(N,L,K,L',K')\Rightarrow\infty} d\left((\psi^*_{N,A_i} - m_N)_{1\leq i\leq p}, (S^*_{N,A_i} - m_N - 4\alpha)_{1\leq i\leq p}\right) = 0. \tag{4.47}$$

*Proof.* We refrain from giving the proof, as it follows in complete analogy to [31, Lemma 5.4]. Instead of using [31, Lemma 5.6] in the proof, one replaces it by its multi-dimensional analogue, Lemma 4.4. □

This reduces the proof of convergence in law of multiple local maxima of the scale-inhomogeneous DGFF to the structurally simpler field, $\{S_v^N\}_{v\in V_N}$, as it decouples microscopic and macroscopic dependence.

4.3. **Coupling to independent random variables.** Recall $\underline{A} = (A_1,\ldots,A_p)$ is a collection of disjoint open, non-empty, simply-connected subsets of $[0,1]^2$, for some fixed $p \in \mathbb{N}$. Further, we have tiled $V_N$ with boxes $B_{N/KL,i}$ of side length $N/KL$. Instead of considering the maximum over the sets $\{v \in V_N : v/N \in A_i\}$, we want to work with the $B_{N/KL}$-boxes. Thus, for any $i \in \{1,\ldots,p\}$, let $T_i^{(KL)} \subset \{1,\ldots,(KL)^2\}$ denote the maximal index set, such that $j \in T_i^{(KL)}$ implies $B_{N/KL,j}/N \subset A_i$, i.e.

$$\cup_{j\in T_i^{(KL)}} B_{N/KL,j}/N \subset A_i. \tag{4.48}$$

Further, it is immediate to see that for all $1 \leq i \leq p$

$$\frac{|NA_i \setminus \cup_{j\in T_i^{(KL)}} B_{N/KL,j}|}{|NA_i|} \to 0, \tag{4.49}$$

as we let $N, K, L$ tend to infinity in this order. In particular,

$$\mathbb{P}\left(\max_{v\in\cup_i i=1^p\left(A_i\setminus\cup_{j\in T_i^{(KL)}} B_{N/KL,j}\right)} \psi_v^N \geq m_N + z\right) \leq \sum_{i=1}^{p} |NA_i \setminus \cup_{j\in T_i^{(KL)}} B_{N/KL,j}| \sup_{v\in V_N} \mathbb{P}\left(\psi_v^N \geq m_N + z\right)$$
$$\leq C \sum_{i=1}^{p} \frac{|NA_i \setminus \cup_{j\in T_i^{(KL)}} B_{N/KL,j}|}{N^2} e^{-2z}, \tag{4.50}$$

which, by (4.49), converges to zero as $N \to \infty$. Next, we construct random variables that do not depend on $N$ and that we couple to the local maxima of $\{S_v^N\}_{v\in V_N}$ on $\cup_{j\in T_1^{(KL)}} B_{N/KL,j}, \ldots, \cup_{j\in T_p^{(KL)}} B_{N/KL,j}$. We set $A_i' := \cup_{j\in T_i^{(KL)}} B_{N/KL,j}$, and $S_v^{N,f} := S_v^N - S^{N,c}v$, for $v \in V_N$. Let $\{\varrho_{R,i} : 1 \leq i \leq R\}$ be a collection of independent Bernoulli random variables with

$$\mathbb{P}(\varrho_{R,i} = 1) = \beta^*_{K',L'} e^{2\bar{k}^\gamma} e^{2\bar{k}(\sigma^2(0)-1)}, \tag{4.51}$$

where, by using [31, Proposition 5.8], the constants $\beta^*_{K',L'}$ are such that they satisfy,

$$\lim_{z\to\infty} \limsup_{(L',K',N)\Rightarrow\infty} \left| e^{2\log(2)\bar{k}(1-\sigma^2(0))} e^{-2\bar{k}^\gamma} e^{2z} \mathbb{P}\left(\max_{v\in B_{N/KL,i}} S_v^{N,f} \geq m_N(\bar{k}, n) - \bar{k}^\gamma + z\right) - \beta^*_{K',L'}\right| = 0. \tag{4.52}$$

Moreover, there are constans $c_\alpha, C_\alpha > 0$ such that $c_\alpha \leq \beta^*_{K',L'} \leq C_\alpha$, where $\alpha$ is as in Lemma 4.1, and the collection $\{\beta^*_{K',L'}\}_{K',L'\geq 0}$ depends on the variance only through $\sigma(1)$. In addition, we specify an independent family of exponential random variables, $\{Y_{R,i} : 1 \leq i \leq R\}$,

$$\mathbb{P}(Y_{R,i} \geq x) = e^{-2x} e^{2\bar{k}^\gamma}, \quad \text{for } x \geq -\bar{k}^\gamma. \tag{4.53}$$

Also, let $\{Z_{R,i}\}_{1\leq i\leq R}$ be a centered Gaussian field with correlation kernel $\Sigma^c$. For each $i \in \{1, \ldots, p\}$, set

$$G^{(i)}_{L,K,L',K'} := \max_{\substack{j\in T_i^{(KL)}\\ \varrho_{R,j}=1}} (Y_{R,j} + 2\log(KL)(1 - \sigma^2(0))) + (Z_{R,j} - 2\log(KL)). \tag{4.54}$$

We collect these in the vector

$$G^*_{\underline{A},L,K,L',K'} := \left(G^{(1)}_{L,K,L',K'}, \ldots, G^{(p)}_{L,K,L',K'}\right). \tag{4.55}$$

We denote the law of the random vector defined in (4.55) by $\bar{\mu}_{L,K,L',K',\underline{A}}$, which does not depend on $N$. Next, we show that $\bar{\mu}_{L,K,L',K',\underline{A}}$ converges to the same limit as $\mu_{N,\underline{A}}$, the law of

$$\left(\max_{v\in A_1'} S_v^N - m_N, \ldots, \max_{v\in A_p'} S_v^N - m_N\right). \tag{4.56}$$

Set $m_N(k,t) := 2\log N\mathcal{I}_{\sigma^2}\left(\frac{k}{n}, \frac{t}{n}\right) - \frac{(t\wedge(n-\bar{l}))\log n}{4(n-\bar{l})}$, for $k \leq n$ and $t \in [k, n]$.

**Theorem 4.7.** *It holds that*

$$\limsup_{(N,L,K,L',K')\Rightarrow\infty} d(\mu_{N,\underline{A}}, \bar{\mu}_{L,K,L',K',\underline{A}}) = 0. \tag{4.57}$$

*In particular, there exists $\mu_{\infty,\underline{A}}$ such that* $\lim_{N\to\infty} d(\mu_{N,\underline{A}}, \mu_{\infty,\underline{A}}) = 0$.

*Proof.* We follow the proof of [31, Theorem 5.9] that deals with the global maximum. Denote by $\tau_i' = \arg\max_{v\in B_{N/KL,i}} S_v^N$, the a.s. unique point where the local maximum is achieved. By [31, (5.50)], we have, for $1 \le i \le p$,

$$\limsup_{(N,L,K,L',K')\Rightarrow\infty} \mathbb{P}\left(S^{N,f}_{\tau_i'} \ge m_N(\bar{k},n) - \bar{k}^\gamma\right) = 1. \tag{4.58}$$

Moreover, we know that the fine field values cannot be too large, i.e. let

$$\mathcal{E} = \cup_{1\le i\le R}\{\max_{v\in B_{N/KL,i}} S_v^{N,f} \ge m_N(\bar{k},n) + KL + \bar{k}^\gamma\},\ \text{and}\ \mathcal{E}' = \cup_{1\le i\le R}\{Y_{R,i} \ge KL + \bar{k}^\gamma\}. \tag{4.59}$$

By [31, (5.51)] respectively [31, (5.53)], we deduce

$$\limsup_{(N,L,K,L',K')\Rightarrow\infty} \mathbb{P}(\mathcal{E}) = 0 \text{ and } \limsup_{(N,L,K,L',K')\Rightarrow\infty} \mathbb{P}(\mathcal{E}') = 0. \tag{4.60}$$

This allows to couple the centred fine field, $\tilde{M}^f_{N,i} = \max_{v\in B_{N/KL,i}} S_i^{N,f} - m_N(\bar{k},n)$, to the approximating process $G^{(i)}_{L,K,L',K'}$, defined in (4.54). By [31, Proposition 5.8], there are $\epsilon^*_{N,KL,K'L'} > 0$ with

$$\limsup_{(N,L,K,L',K')\Rightarrow\infty} \epsilon^*_{N,KL,K'L'} = 0, \tag{4.61}$$

such that, for some $|\overset{\diamond}{\epsilon}| \le \epsilon^*_{N,KL,K'L'}/4$,

$$\mathbb{P}\left(-\bar{k}^\gamma + \overset{\diamond}{\epsilon} \le \tilde{M}^f_{N,i} \le KL + \bar{k}^\gamma\right) = \mathbb{P}\left(\varrho_{R,i} = 1,\ Y_{R,i} \le KL + \bar{k}^\gamma\right), \tag{4.62}$$

and such that for all $t$ with $-\bar{k}^\gamma - 1 \le t \le KL + \bar{k}^\gamma$,

$$\mathbb{P}\left(\varrho_{R,i} = 1,\ Y_{R,i} \le t - \epsilon^*_{N,KL,K'L'}\right) \le \mathbb{P}\left(-\bar{k}^\gamma + \overset{\diamond}{\epsilon} \le \tilde{M}^f_{N,i} \le t\right) \le \mathbb{P}\left(\varrho_{R,i} = 1,\ Y_{R,i} \le t + \epsilon^*_{N,KL,K'L'}/2\right). \tag{4.63}$$

Thus, by the same argument given in the proof of [31, Theorem 5.9], there is a coupling between $\{\tilde{M}^f_{N,i} :\ 1 \le i \le R\}$ and $\{(\varrho_{R,i}, Y_{R,i}) :\ 1 \le i \le R\}$ such that on the event $(\mathcal{E} \cup \mathcal{E}')^c$:

$$\varrho_{R,i} = 1, |Y_{R,i} - \tilde{M}^f_{N,i}| \le \epsilon^*_{N,KL,K'L'}, \quad \text{if } \tilde{M}^f_{N,i} \ge \epsilon^*_{N,KL,K'L'} \tag{4.64}$$

$$|Y_{R,i} - \tilde{M}^f_{N,i}| \le \epsilon^*_{N,KL,K'L'}, \quad \text{if } \varrho_{R,i} = 1. \tag{4.65}$$

As $\{Z_{R,i}\}_{1\le i\le R}$ and $\{S_v^{N,c}\}_{v\in V_N}$ have the same law, one can couple such that $S_v^{N,c} = Z_{R,i}$, for $v \in B_{N/KL,i}$ and $1 \le i \le R$. Using [31, (5.63)], we deduce

$$\limsup_{(N,L,K,L',K')\Rightarrow\infty} \mathbb{P}(\varrho_{R,\bar{\tau}_i} = 1) = 1, \tag{4.66}$$

and thereby exclude that the local maximum is achieved in a box $T_j^{(KL)}$ when at the same time $\varrho_{R,j} = 0$. Thus, there are couplings, such that outside an event of vanishing probability as $(N, L, K, L', K') \Rightarrow \infty$, we have

$$\left\|\left((\max_{v\in A_1'} S_v^N - m_N) - G^{(1)}_{L,K,L',K'}, \ldots, (\max_{v\in A_p'} S_v^N - m_N) - G^{(p)}_{L,K,L',K'}\right)\right\|_\infty \le 2\epsilon^*_{N,KL,K'L'}, \tag{4.67}$$

which proves Theorem 4.7. □

Next, we prove Theorem 2.5.

*Proof of Theorem 2.5:* By Lemma 4.6, (4.50) and Theorem 4.7, we can reduce the proof to proving convergence of the laws $\bar{\mu}_{L,K,L',K',\underline{A}}$. Recall that we write $R = KL$. In the following, we construct random variables $\{D_{KL}(A_i) : 1 \leq i \leq p\}_{K,L\geq 0}$ that are measurable with respect to $\mathcal{F}^C := \sigma(Z_{R,i})_{i=1}^R$, so that for any $x_1, \ldots, x_p \in \mathbb{R}$, the following limit exists

$$\lim_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{L,K,L',K',\underline{A}}((-\infty, x_1], \ldots, (-\infty, x_p])}{\mathbb{E}\left[\exp(-\beta^*_{K',L'} \sum_{i=1}^p D_{KL}(A_i)e^{-2x_i})\right]}, \tag{4.68}$$

and is equal to one. Regarding (4.66), assume $\varrho_{R,\tilde{\tau}_i}$, for $1 \leq i \leq p$. Conditioning on $\mathcal{F}^c$, we have, for any $x_1, \ldots, x_p \in \mathbb{R}$ ,

$$\begin{aligned}\bar{\mu}_{L,K,L',K'}((-\infty, x_1], \ldots, (-\infty, x_p]) &= \mathbb{P}\left(G^{(i)}_{L,K,L',K'} \leq x_i : i = 1, \ldots, p\right) \\ &= \mathbb{E}\left[\prod_{i=1}^p \left(1 - \mathbb{P}\left(\varrho_{R,j}(Y_{R,j} + 2\log(KL)(\sigma_1^2 - 1)) > x_i + 2\log(KL) - Z_{R,j}|\mathcal{F}^c\right)\right)^{|T^{(KL)_i}|}\right].\end{aligned} \tag{4.69}$$

A union bound on $\mathcal{D}^c = \{\min_{1\leq i\leq R} 2\log(KL) - Z_{R,i} \geq 0\}^c$, shows that

$$\limsup_{(L,K)\implies\infty} \mathbb{P}(\mathcal{D}) = 1. \tag{4.70}$$

Thus, on the event $\mathcal{D}$, and by (4.51), (4.53) and (4.73), one deduces

$$\mathbb{P}\left(\varrho_{R,j}Y_{R,j} \geq 2\log(KL)\sigma^2(0) - Z_{R,j} + x_i|\mathcal{F}^c\right) = \beta^*_{K',L'}e^{-2(2(1+\sigma^2(0))\log(KL)-Z_{R,j}+x_i)} \tag{4.71}$$

In particular, note that (4.71) tends to zero as $KL \to \infty$. Using $e^{-\frac{x}{1-x}} \leq 1 - x \leq e^{-x}$, for $x < 1$, and inserting for $x$ the probability in (4.71) with $K, L$ large, implies that there is non-negative sequence $\{\epsilon_{K,L}\}_{K,L\geq 0}$, with $\limsup_{(K,L)\Rightarrow\infty} \epsilon_{K,L} = 0$, such that

$$\begin{aligned}\exp\left(-(1 + \epsilon_{K,L})\beta^*_{K',L'}e^{-2((1+\sigma^2(0))\log(KL)-Z_{R,j}+x_i)}\right) &\leq \mathbb{P}\left(\varrho_{R,j}Y_{R,j} \leq 2\log(KL)\sigma^2(0) - Z_{R,j} + x_i|\mathcal{F}^c\right) \\ &\leq \exp\left(-(1 - \epsilon_{K,L})\beta^*_{K',L'}e^{-2((1+\sigma^2(0))\log(KL)-Z_{R,j}+x_i)}\right).\end{aligned} \tag{4.72}$$

Plugging (4.72) into (4.69) gives (4.68), with

$$D_{K,L}(A_i) = \sum_{j\in T_i^{(KL)}} e^{-2(2(1+\sigma^2(0))\log(KL)-Z_{R,j})}. \tag{4.73}$$

(4.68) combined with Theorem 4.7, implies that there is a constant $\beta^* > 0$, such that

$$\limsup_{(K',L')\Rightarrow\infty} |\beta^*_{K',L'} - \beta^*| = 0. \tag{4.74}$$

Inserting (4.74) into (4.68), we obtain

$$\lim_{(L,K,L',K')\Rightarrow\infty} \frac{\bar{\mu}_{L,K,L',K',\underline{A}}((-\infty, x_1], \ldots, (-\infty, x_p])}{\mathbb{E}\left[\exp(-\beta^* \sum_{i=1}^p D_{KL}(A_i)e^{-2x_i})\right]} = 1. \tag{4.75}$$

Theorem 4.7 in combination with (4.75), implies that $\{D_{KL}(A_i) : 1 \leq i \leq p\}$ converge weakly to random variables $\{D(A_i) : 1 \leq i \leq p\}$, as $K, L \to \infty$. Moreover, as the sequence of laws, $\{\bar{\mu}_{L,K,L',K',\underline{A}}\}_{L,K,L',K'\geq 0}$, is tight, it follows that almost surely, $D(A_i) > 0$, for $i \in \{1, \ldots, p\}$. This concludes the proof. □

## 5. Appendix

5.1. **Gaussian comparison.** We need a vector version of Kahane's theorem.

**Theorem 5.1.** *Let $f \in C^2(\mathbb{R}^n;\mathbb{R}^k)$ with sub-Gaussian growth in every component of the second derivatives. Further let $\{X_i\}_{1\le i\le n}$, $\{Y_i\}_{1\le i\le n}$ be two centred Gaussian fields satisfying*

$$\mathbb{E}\left[Y_iY_j\right] > \mathbb{E}\left[X_iX_j\right] \implies \frac{\partial f}{\partial x_i \partial x_j}(x) \ge 0, \quad x \in \mathbb{R}, \tag{5.1}$$

*where the inequality is to be understood component-wise. Then,*

$$\mathbb{E}\left[f(Y)\right] \le \mathbb{E}\left[f(X)\right], \tag{5.2}$$

*again to be understood as an inequality valid in each component.*

*Proof.* The proof is an immediate adaptation of the original proof, as each component of $f$ is a function $f_i \in C^2(\mathbb{R}^n)$ with sub-Gaussian growth in its second derivatives, for which Kahane's theorem holds. In particular, each component of the map $f$ can be treated separately. □

This allows us to deduce a vector version of Slepian's inequality.

**Theorem 5.2.** *Let $T$ be a countable index set, $\{X_i\}_{i\in T}$, $\{Y_i\}_{i\in T}$ be two centred Gaussian fields satisfying*

$$\mathrm{Var}\left[X_i\right] = \mathrm{Var}\left[Y_i\right] \ \forall i \in T \quad \text{and} \quad \mathbb{E}\left[X_iX_j\right] \le \mathbb{E}\left[Y_iY_j\right],\ \forall i,j \in T. \tag{5.3}$$

*Then, for any disjoint collection of subsets $T_1,\dots,T_k \subset T$ and real numbers $x_1,\dots,x_k \in \mathbb{R}$,*

$$\mathbb{P}\left(\max_{i\in T_1} Y_i \le x_1,\dots,\max_{i\in T_k} Y_i \le x_k\right) \le \mathbb{P}\left(\max_{i\in T_1} X_i \le x_1,\dots,\max_{i\in T_m} X_i \le x_k\right). \tag{5.4}$$

*Proof.* The proof is basically only a vector version of the original, which is why we just give a sketch. Assume for simplicity $|T| = n$. One takes a sequence of maps $f_l : \mathbb{R}^n \to \mathbb{R}^k$ of the form

$$f_l = \begin{pmatrix} \prod_{i\in A_1} g_i^l(x_i) \\ \prod_{i\in A_2} g_i^l(x_i) \\ \vdots \\ \prod_{i\in A_k} g_i^l(x_i) \end{pmatrix} \tag{5.5}$$

where $g_i^l(x_j)$ are smooth, non-increasing and converge from above to $\mathbb{1}_{(-\infty,x_j]}$. One notices that the requirements of Theorem 5.1 are met, and an application of it finishes the proof. □

## References


[1] E. Aïdékon, J. Berestycki, E. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Related Fields*, 157(1-2):405–451, 2013.

[2] L.-P. Arguin, D. Belius, and P. Bourgade. Maximum of the characteristic polynomial of random unitary matrices. *Comm. Math. Phys.*, 349(2):703–751, 2017.

[3] L.-P. Arguin, D. Belius, P. Bourgade, M. Radziwiłł, and K. Soundararajan. Maximum of the Riemann zeta function on a short interval of the critical line. *Comm. Pure Appl. Math.*, 72(3):500–535, 2019.

[4] L.-P. Arguin, A. Bovier, and N. Kistler. Genealogy of extremal particles of branching Brownian motion. *Comm. Pure Appl. Math.*, 64(12):1647–1676, 2011.

[5] L.-P. Arguin, A. Bovier, and N. Kistler. Poissonian statistics in the extremal process of branching Brownian motion. *Ann. Appl. Probab.*, 22(4):1693–1711, 2012.

[6] L.-P. Arguin, A. Bovier, and N. Kistler. The extremal process of branching Brownian motion. *Probab. Theory Related Fields*, 157(3-4):535–574, 2013.

[7] L.-P. Arguin and F. Ouimet. Extremes of the two-dimensional Gaussian free field with scale-dependent variance. *ALEA Lat. Am. J. Probab. Math. Stat.*, 13(2):779–808, 2016.

[8] L.-P. Arguin and O. Zindy. Poisson-Dirichlet statistics for the extremes of the two-dimensional discrete Gaussian free field. *Electron. J. Probab.*, 20:no. 59, 19, 2015.

[9] M. Biskup and O. Louidor. Extreme local extrema of two-dimensional discrete Gaussian free field. *Comm. Math. Phys.*, 345(1):271–304, 2016.

[10] M. Biskup and O. Louidor. Full extremal process, cluster law and freezing for the two-dimensional discrete Gaussian free field. *Adv. Math.*, 330:589–687, 2018.

[11] M. Biskup and O. Louidor. On intermediate level sets of two-dimensional discrete Gaussian free field. *Ann. Inst. Henri Poincaré Probab. Stat.*, 55(4):1948–1987, 2019.

[12] E. Bolthausen, J.-D. Deuschel, and G. Giacomin. Entropic repulsion and the maximum of the two-dimensional harmonic crystal. *Ann. Probab.*, 29(4):1670–1692, 2001.

[13] Bolthausen, Erwin and Deuschel, Jean Dominique and Zeitouni, Ofer. Recursions and tightness for the maximum of the discrete, two dimensional Gaussian free field. *Electron. Commun. Probab.*, 16:114–119, 2011.

[14] A. Bovier. *Gaussian processes on trees*, volume 163 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. From spin glasses to branching Brownian motion.

[15] A. Bovier and L. Hartung. The extremal process of two-speed branching Brownian motion. *Electron. J. Probab.*, 19:no. 18, 28, 2014.

[16] A. Bovier and L. Hartung. Variable speed branching Brownian motion 1. Extremal processes in the weak correlation regime. *ALEA Lat. Am. J. Probab. Math. Stat.*, 12(1):261–291, 2015.

[17] A. Bovier and L. Hartung. Extended convergence of the extremal process of branching Brownian motion. *Ann. Appl. Probab.*, 27(3):1756–1777, 2017.

[18] A. Bovier and L. Hartung. From 1 to 6: A finer analysis of perturbed branching brownian motion. *Communications on Pure and Applied Mathematics*, 73(7):1490–1525, 2020.

[19] A. Bovier and I. Kurkova. Derrida's generalized random energy models. II. Models with continuous hierarchies. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(4):481–495, 2004.

[20] M. Bramson, J. Ding, and O. Zeitouni. Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 69(1):62–123, 2016.

[21] M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.*, 65(1):1–20, 2012.

[22] M. D. Bramson. Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.*, 31(5):531–581, 1978.

[23] A. Cortines, L. Hartung, and O. Louidor. The structure of extreme level sets in branching Brownian motion. *Ann. Probab.*, 47(4):2257–2302, 2019.

[24] O. Daviaud. Extremes of the discrete two-dimensional Gaussian free field. *Ann. Probab.*, 34(3):962–986, 2006.

[25] B. Derrida and H. Spohn. Polymers on disordered trees, spin glasses, and traveling waves. volume 51, pages 817–840. 1988. New directions in statistical mechanics (Santa Barbara, CA, 1987).

[26] J. Ding. Exponential and double exponential tails for maximum of two-dimensional discrete Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):285–299, 2013.

[27] J. Ding, R. Roy, and O. Zeitouni. Convergence of the centered maximum of log-correlated Gaussian fields. *Ann. Probab.*, 45(6A):3886–3928, 2017.

[28] J. Ding and O. Zeitouni. Extreme values for two-dimensional discrete Gaussian free field. *Ann. Probab.*, 42(4):1480–1515, 2014.

[29] M. Fang and O. Zeitouni. Slowdown for time inhomogeneous branching Brownian motion. *J. Stat. Phys.*, 149(1):1–9, 2012.

[30] M. Fels. Extremes of the 2d scale-inhomogeneous discrete gaussian free field: Sub-leading order and exponential tails. *arXiv:1910.09915*, 2019.

[31] M. Fels and L. Hartung. Extremes of the 2d scale-inhomogeneous discrete gaussian free field: Convergence of the maximum in the regime of weak correlations. *arXiv:1912.13184*, 2019.

[32] Y. Fyodorov, G. Hiary, and J. Keating. Freezing transition, characteristic polynomials of random matrices, and the riemann zeta function. *Physical review letters*, 108:170601, 04 2012.

[33] N. Kistler, A. Schertzer, and M. A. Schmidt. Oriented first passage percolation in the mean field limit. 2. The extremal process. *Ann. Appl. Probab.*, 30(2):788–811, 2020.

[34] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15(3):1052–1061, 1987.

[35] P. Maillard and O. Zeitouni. Slowdown in branching Brownian motion with inhomogeneous variance. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(3):1144–1160, 2016.

[36] B. Mallein. Maximal displacement in a branching random walk through interfaces. *Electron. J. Probab.*, 20:no. 68, 40, 2015.

[37] B. Mallein. Genealogy of the extremal process of the branching random walk. *ALEA Lat. Am. J. Probab. Math. Stat.*, 15(2):1065–1087, 2018.

M. Fels, Institut für Angewandte Mathematik, Rheinische Friedrich-Wilhelms-Universität, Endenicher Allee 60, 53115 Bonn, Germany

*Email address*: fels@iam.uni-bonn.de, wt.iam.uni-bonn.de/maximilian-fels

L. Hartung, Institut für Mathematik, Johannes Gutenberg-Universität Mainz, Staudingerweg 9, 55099 Mainz, Germany

*Email address*: lhartung@uni-mainz.de, https://sites.google.com/view/lisahartung/home