On the membrane model and the discrete Bilaplacian

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Dedicated to the memory of Barbara and Josef Schweiger, and of Albert Zandtner

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Contents

1 Introduction

1.1 Overview

This thesis is concerned with two objects from seemingly different fields: on the one hand, the membrane model from probability theory, and on the other hand, the Green's function of the discrete Bilaplacian from the theory of partial differential equations (PDEs) or numerical analysis. As it turns out, though, these two objects are closely related, and an improved understanding of one of them can also help in the study of the other.

The membrane model is an example of a random interface model. Such models arise when studying the thermal fluctuations in interfaces in physics, chemistry and biology. The membrane model, specifically, is relevant when this interface is very flexible so that the bending modes dominate. It is used, for instance, to describe the behaviour of biomembranes such as those in the walls of cells.

There are also purely mathematical reasons to be interested in the membrane model. Namely, it is a natural variant of the most fundamental example of a random interface model, the so-called discrete Gaussian free field. This model has some particularly nice features (such as random walk representations and correlation inequalities), and in the last three decades this model has been very well understood. On the other hand, there are general classes of random interface models where very little is known. The membrane model might now serve as a stepping stone to investigate these more complicated models. Namely, it shares some of the features of the discrete Gaussian free field, but not all of them. Thus, even though the membrane model is expected to behave similarly to the discrete Gaussian free field, one is forced to develop new techniques to make this rigorous.

In this thesis we study various features of the membrane model, filling several gaps in the literature. In each case the answer had previously been known for the discrete Gaussian free field, and we extend these results to the case of the membrane model.

In Chapter 3 we study the effect of a hard wall that forces the field to be positive on the interface, focussing on dimensions 2 and 3. This chapter is based on the publication [BDKS19] which is joint work with Simon Buchholz, Jean-Dominique Deuschel and Noemi Kurt and has appeared in the **Electronic Communications in Probability**.

In Chapter 4 we investigate the behaviour of the maximum of the field in dimension 4. This chapter is based on the publication [Sch20a] which has appeared in **The Annals of Probability**.

Chapter 6 is dedicated to the effects that a small attractive potential has on the interface, in dimensions 4 and above. This potential localizes the field, and we study how this localization manifests itself. This chapter is based on the preprint [Sch20b] that will be submitted for publication.

The starting point for all these results is the fact that the covariance function of the membrane model is the Green's function of the discrete Bilaplacian. Heuristically, this object should behave similarly to the Green's function of the continuous Bilaplacian. A major part of this thesis will be to make this heuristic rigorous. To that end we combine various

methods from PDE theory with methods from numerical analysis.

In fact, in Chapter 2 we prove estimates for the Green's function in dimensions 2 and 3 using a compactness argument and results for continuous elliptic equations in domains with singularities. This chapter is the basis for the results in Chapter 3. It is based on the publication [MS19], which is joint work with Stefan Müller and has appeared in the **Vietnam Journal of Mathematics**.

Furthermore, as a part of Chapter 4 we prove estimates for the Green's function in dimension 4, using estimates for finite difference schemes and preexisting results on the continuous Green's function.

For the application in Chapter 4 we do not need the full strength of the estimates for the finite difference scheme we use. Optimizing such estimates is, however, very interesting for numerical analysis itself. In Chapter 5 we improve the best known estimates on the approximation quality of the scheme we use in Chapter 4 and some other schemes. This chapter is based on the publication [MSS20] which is joint work with Stefan Müller and Endre Süli and has appeared in the **SIAM Journal on Numerical Analysis**.

In this introductory chapter we will lay the foundation for these results and discuss the necessary background, and we will give a more detailed description of the results in the following chapters.

In Section 1.2 we discuss random interface models and their basic properties. We begin by describing the motivation from physics and biochemistry for the study of these models. Each random interface model is given as a probability measure, the so-called Gibbs measure, for a certain Hamiltonian, and so we explain the physical background as well as the mathematical theory underlying these measures. We then introduce some important examples of random interface models (including the membrane model), and describe some of their basic properties. In particular, we discuss the existence of subcritical, critical and supercritical dimensions. We also survey the most important mathematical tools used to study random interface models. Finally, we describe how to simulate random interface models on a computer, and how the images throughout this introduction were generated.

In Section 1.3 we then give more details on the membrane model. We compare it with a few other random interface models, namely the discrete Gaussian free field (or gradient model), the $\nabla\varphi$-model with strictly convex $\mathcal{V}$ and the $\nabla\varphi$-model with slightly non-convex $\mathcal{V}$. We discuss various aspects of these models, reviewing the existing results in the literature and describing the new contributions of this thesis. We begin with infinite volume limits of the interfaces, and then discuss the maximum of the fields. Afterwards, we discuss the phenomena of entropic repulsion, pinning, and wetting. Finally, we mention a few further interesting questions. As part of this section, we outline the results of Chapters 3, 4 and 6 on entropic repulsion for the subcritical membrane model, the maximum of the critical membrane model, and pinning for the critical and supercritical membrane model, respectively.

In Section 1.4 we describe the connection between the membrane model and the Green's function of the discrete Bilaplacian, and discuss discrete Green's functions more generally. We begin with a summary of some facts from elliptic PDE theory and numerical analysis. We then focus on discrete Green's functions and describe the tools available to study them. As part of this we summarize the results of Chapters 2 and 4 on the subcritical and critical Green's function of the discrete Bilaplacian. The estimates from Chapter 4 are based on estimates for a certain finite difference scheme. We explain this connection, and describe further results on such schemes that are contained in Chapter 5.

Most of the content of this introduction is an exposition of well-known results in the literature. Other than the summaries of the results of the later chapters, the only slightly original parts are the description of some algorithms to generate samples from the membrane model in Section 1.2.7, and the discussion of Hessian Gibbs measures in Section 1.3.1.

1.2 Random interface models

In this section we will give some background on random interface models, describe some examples and discuss basic properties.

1.2.1 Motivation

Macroscopic interfaces

In physics there are many systems that can form sharp interfaces. Let us discuss two main examples.

As a first example, consider a substance that can be in the solid, liquid or gaseous phase. Under certain circumstances two or more of these phases can coexist, and there will be interfaces between them. For example, for water at 0° Celsius and standard pressure both ice and liquid water can appear. More generally, a variety of materials can form stable crystals within a surrounding liquid. We assume that the system is in equilibrium. This assumption is not always reasonable (e.g. ice crystals look very different than the conjectured equilibrium shape), but for some materials such as small crystals of certain metals it aligns well with experiments [RW84]. Under this equilibrium assumption, the macroscopic theory of interfaces for crystals was pioneered by Wulff [Wul01]. He proposed that the atoms in the crystal arrange themselves in a shape $U \subset \mathbb{R}^3$ so that U minimizes the "Wulff functional"

$$\mathcal{W}(U) = \int_{\partial U} \sigma(n(x))\, \mathrm{d}\mathcal{H}^2(x)$$

under the constraint that the volume of U is fixed. Here $n \in \mathbb{S}^2$ is a normal vector to ∂U, and σ is the so-called surface tension. This variational problem is an anisotropic variant of the isoperimetric problem. Its minimizer (the Wulff shape) can be constructed using the so-called Wulff construction, and in practice this variational problem is well understood. Note that if we write ∂U locally as the graph of a function $u\colon A \subset R^2 \to \mathbb{R}$, then the integrand in the Wulff functional becomes a certain functional of ∇u.

Our second example of interfaces is from biology, and for details on the following see [Lip95]. Many biological membranes, such as the wall of a cell, are formed by bilayers of lipids. Lipids are molecules with a hydrophilic head and a hydrophobic tail, and in a solution they can arrange themselves in quite stable double layers with the hydrophobic tails pointing inwards. Such a structure is called a bilayer. The lipid molecules in the bilayer are typically in the liquid phase, meaning that the single lipids can move almost freely around in the membrane while maintaining the bilayer structure. The lipid bilayers are quite resistant to stretching (instead they rupture before stretching significantly) but have low resistance to bending. This suggests that the energy of a bilayer should depend mainly on the curvature of the bilayer. Indeed, Helfrich [Hel73] (cf. also [BWW17]) proposed a variational problem for the surface Σ occupied by a closed bilayer, namely that it minimizes the "Helfrich functional"

$$\mathrm{H}(\Sigma) = \int_\Sigma \frac{k_c}{2}(H - c_0)^2 + \bar{k} K \mathrm{d}\, \mathcal{H}^2(x)\,.$$

Here H and K are the mean and Gaussian curvature, respectively, k_c and $\bar{k}$ are bending moduli and c_0 is the spontaneous curvature. By the Gauss-Bonnet theorem the integral over $\bar{k}K$ evaluates to $2\pi\bar{k}\chi(\Sigma)$, where $\chi(\Sigma)$ is the Euler characteristic of Σ. Thus, for fixed topology of the bilayer we can neglect the second summand and study

$$\tilde{\mathrm{H}}(\Sigma) = \int_\Sigma \frac{k_c}{2}(H - c_0)^2 \mathrm{d}\mathcal{H}^2(x)$$

instead. We can again write Σ locally as the graph of a function $u\colon A \subset R^2 \to \mathbb{R}$. If the spontaneous curvature is close to 0 and the bilayer is locally almost planar, we can neglect ∇u, and the dominant term looks like $\frac{k_c}{8}|\nabla^2 u|^2$. In particular, the functional now involves only second derivatives.

Microscopic interfaces

The next question then is to analyse the interfaces that we just described on a microscopic level, to study thermic fluctuations and to derive the respective functionals from atomistic theories. The tools for this issue are provided by statistical mechanics (cf. the next section).

For our first example of a solid-to-liquid transition this line of research was initiated by Dobrushin, Kotecký and Shlosman [DKS92], who analysed macroscopic interfaces arising from the Ising model, and since then there have been many works in that direction, cf. [BIV00]. On a macroscopic level, the Wulff shape is a deterministic subset of $\mathbb{R}^3$, and we can represent its boundary locally as the graph of a function $\mathbb{R}^2 \to \mathbb{R}$. On the microscopic scale, however, the interface will not be stationary, and there will be fluctuations. We make the rather strong simplifying assumption that these fluctuations can locally be represented as the graph of a random function $\Lambda \subset \mathbb{R}^2 \to \mathbb{R}$ (and thus we in particular assume that there are no holes or overhangs in our interface). As we are interested in atomistic models, it is reasonable to assume that Λ is discrete, and we take it to be a subset of the lattice $\mathbb{Z}^2$ or $(h\mathbb{Z})^2$. We are thus left to study random height functions on Λ. The probability to observe a certain height function (i.e. a certain microscopic configuration) will depend on the temperature of the system and on the energy of that configuration. It turns out that this energy is (at least approximately) a certain functional of the discrete gradient. This is unsurprising when one compares it with the Wulff functional itself.

For our second example of bilayer membranes one can, in principle, proceed similarly. We are not aware of any rigourous mathematical derivation of the properties of membrane bilayers from atomistic models, but there many results in the physics literature on thermal fluctuations in bilayers and how they influence the macroscopic properties of the membrane (see e.g. [DGT06, NP87, Lip95, HL97, RCMS05]). On a microscopic level these fluctuations are once again governed by an energy consisting of discrete curvature terms, and so we arrive once again at the problem of studying the arising probability distribution of height functions.

More generally, a random interface model will be a probability distribution on functions $\Lambda \to \mathbb{R}$, where $\Lambda \subset \mathbb{R}^{\mathrm{d}}$. Before we discuss the actual random interface models that we are interested in, let us discuss the physical and mathematical background necessary to choose and define these probability distributions.

1.2.2 Statistical mechanics

In our examples of interfaces we encountered the situation that the interface takes a certain deterministic macroscopic equilibrium shape. Its microscopic state, meanwhile, is hard or rather impossible to predict, and moreover it will change incredibly fast, so that it does not really make sense to speak of "a" microscopic state, but rather of an ensemble of such. This idea can best be formalized by using concepts of statistical mechanics.

More generally, statistical mechanics is concerned with the study of systems with many degrees of freedom, and with deriving their behaviour from their microscopic structure. The subject was introduced in the late 19th century by Boltzmann, Gibbs, Maxwell and others, and has since developed into an important part of modern physics. A landmark reference is [Gib02], and a comprehensive treatment can be found in [LL58]. We will briefly discuss the notions that are most important to us, mostly following [Tho72, FV18].

We would like to describe the macroscopic behaviour of a physical system consisting of a large number of constituents (e.g. the atoms in a crystal or the molecules that form a gas). In classical physics, such a system can be parametrized by the positions and momenta of all the constituents, and knowing these, the Hamilton equations describe the state of the systems for all future times. In practice, however, this is completely infeasible: there is no practical way to know the initial state of a system. Even if one did, the evolution of the system would be incredibly complicated due to the huge number of constituents. Furthermore, we are not actually interested in the detailed evolution of the constituents, but rather in the evolution of some macroscopic quantities.

The starting point of statistical mechanics is thus to replace the given microscopic initial state ω that we have no way to know with a probability distribution $\mathbb{P}$ over the set of all microstates Ω. Of course, this probability distribution should be supported only on those microstates that are compatible with our knowledge about the macrostate. For our analysis we need a Hamiltonian $H\colon \Omega \to \mathbb{R}$ that gives the energy of each microstate, and an a priori measure λ on Ω (typically the Hausdorff measure). We restrict ourselves to the case that our systems are static on a macroscopic scale.

Suppose that all we know about our system is that it consists of N particles that are located in some $\Lambda \subset \mathbb{R}^{\mathrm{d}}$ with volume $|\Lambda| = V$ and that the energy of our system is some constant E. Denoting by $\Omega_{\Lambda,N}$ the set of all microstates compatible with the assumptions on the number of particles and their occupied volume, it has then been postulated by Gibbs that if we have no further information the equilibrium measure on $\Omega_{\Lambda,N}$ should be given by

$$\mathbb{P}^{\mathrm{mic}}_{\Lambda,E,N}(\mathrm{d}\omega) = \frac{1}{Z_{\Lambda,E,N}} \mathbb{1}_{H(\omega)=E} \mathbb{1}_{\omega\in\Omega_{\Lambda,N}} \lambda(\mathrm{d}\omega)\,.$$

Here $\mathbb{1}_s$ is equal to 1 if s is true and otherwise 0. This is the so-called microcanonical ensemble, and the normalization factor $Z_{\Lambda,E,N}$ is the so-called microcanonical partition function.

Of greater interest to us, however, is a different ensemble where we do not fix the energy, but the temperature. Here we need to proceed differently as it is not clear how to define the temperatue $T(\omega)$ of a microstate ω. Physically, one way to prescribe the temperature is to assume that our system is in contact with a heat reservoir, i.e. with another system with which it can exchange energy. If that other system is very large, this will lead to both systems being at approximately the temperature of the heat reservoir. Assuming that both systems together are described by the microcanical ensemble, one can, in principle, calculate the marginal distribution of the system we are interested in. When one pursues this calculation

at least on a heuristic level, one obtains the equilibrium measure

$$\mathbb{P}^{\mathrm{can}}_{\Lambda,\beta,N}(\mathrm{d}\omega) = \frac{1}{Z_{\Lambda,\beta,N}} \exp(-\beta H(\omega)) \mathbb{1}_{\omega\in\Omega_{\Lambda,N}} \lambda(\mathrm{d}\omega)\,.$$

Here β is proportional to $\frac{1}{T}$, and we choose units in such a way that the proportionality constant is equal to 1. This is the so-called canonical ensemble, and $Z_{\Lambda,\beta,N}$ is the canonical partition function. The factor $\exp(-\beta H(\omega))$ is called Boltzmann weight.

Another viewpoint on the canonical ensemble is that it is chosen in such a way as to maximize the relative entropy of $\mathbb{P}^{\mathrm{can}}_{\Lambda,\beta,N}$ with respect to λ under the constraint that the expected value of the energy takes some fixed value.

The canonical ensemble is also called the (canonical) Gibbs measure. The Boltzmann weight is the larger, the smaller $H(\omega)$ is. If β is large (i.e. the temperature is low) then the Gibbs measure will mostly be supported on those states with small energy. On the other hand, if β is small (i.e. the temperature is high) then the Gibbs measure does not discriminate as much between states with lower and higher energy.

Often one does not know (or care about) the precise value of N beyond the fact that it is very large. This suggests that one should directly study the system in the limit $N \to \infty$. Then the volume V needs to grow simultaneously in such a way that the particle density $\frac{N}{V}$ has a finite limit. This procedure is called taking the thermodynamic limit.

1.2.3 Spin systems, Gibbs measures and random interface models

We now apply the theory of the previous section to the case of random interface models, aiming for mathematical rigor. We follow [Geo88, Bov06, FV18]. In particular, in [Geo88] the theory is described in much greater generality.

We are looking for a discretized mathematical model of an interface. This interface is formed by a set of particles in $\mathbb{R}^{\mathsf{d}+1}$ such that the first d coordinates of each particle are fixed, while the $(\mathsf{d}+1)$-th coordinate is free.

We thus consider an at most countable $S \subset \mathbb{R}^{\mathsf{d}}$ as the parameter set (we will take $S = \mathbb{Z}^{\mathsf{d}}$ or $S = (h\mathbb{Z})^{\mathsf{d}}$ for some $h > 0$), and a set $E \subset \mathbb{R}$ (we will take $E = \mathbb{R}$ or $E = \mathbb{Z}$) with the Borel σ-algebra $\mathcal{E}$ and a reference measure λ as the single spin space. We consider the set $\Omega = E^S$ of all possible configurations, equipped with the product σ-algebra $\mathcal{F} = \mathcal{E}^S$. Let $\mathcal{P}(\Omega, \mathcal{F})$ be the set of all probability measures on Ω. For a $\psi \in \Omega$ we write ψ_x for the value of ψ at $x \in S$, and for $\Lambda \subset S$ we let $\mathcal{F}_\Lambda = \sigma(\psi_x \colon x \in \Lambda)$.

Given a $\mathcal{F}$-measurable Hamiltonian $H \colon \Omega \to \mathbb{R} \cup \{+\infty\}$ we would now like to define the canonical ensemble as in the previous section, i.e. the measure

$$\mathbb{P}(\mathrm{d}\psi) = \frac{1}{Z_\beta} \exp(-\beta H(\psi)) \prod_{x\in S} \lambda(\mathrm{d}\psi_x)\,.$$

If S is finite this definition works well. However, reasonable Hamiltonians will be infinite when S is infinite, and so we need a different construction in that case. The crucial idea here, due to Dobrushin, Lanford and Ruelle [Dob68, Dob70, LR69], is to define a measure μ as a Gibbs measure for H, if for every finite $\Lambda \subset S$ the law of the field under μ conditioned on the values of the field outside of Λ is the correct one.

We restrict ourselves to finite-range interactions. That is, we consider an interaction of the form $\Phi = \{\Phi_A\}_{A\Subset S}$, where $\Phi_A \colon S \to \mathbb{R}$ is $\mathcal{F}_A$-measurable (and by $A \Subset S$ we denote

that A is a compact, i.e. finite, subset of S). We also assume $\sup_{A \Subset \mathbb{Z}^d} \|\Phi_A\|_{L^\infty} < \infty$, and that $\Phi_A = 0$ when $|A| > R$ for some R. Then for $\Lambda \Subset S$ we can define the Hamiltonian

$$H_\Lambda(\psi) = \sum_{\substack{A \Subset S \\ A \cap \Lambda \neq \varnothing}} \Phi_A(\psi)$$

and for $\varphi \in \Omega$ the Gibbs specification

$$\mathbb{P}^{(\varphi)}_{\Phi,\Lambda,\beta}(\mathrm{d}\psi) = \frac{1}{Z^{(\varphi)}_{\Phi,\Lambda,\beta}} \exp(-\beta H_\Lambda(\psi)) \prod_{x \in \Lambda} \lambda(\mathrm{d}\psi_x) \prod_{x \in S \setminus \Lambda} \delta_{\varphi_x}(\mathrm{d}\psi_x)\,.$$

One can check that this definition is self-consistent in the sense that if $\Lambda \subset \Lambda'$ then

$$\mathbb{P}^{(\varphi)}_{\Phi,\Lambda',\beta}(\mathrm{d}\varphi')\mathbb{P}^{(\varphi')}_{\Phi,\Lambda,\beta} = \mathbb{P}^{(\varphi)}_{\Phi,\Lambda',\beta} \quad \text{for each } \varphi \in \Omega\,. \tag{1.2.1}$$

We then define that a probability measure $\mathbb{P}_{\Phi,\beta}$ on $(\Omega, \mathcal{F})$ is a Gibbs measure for Φ if

$$\mathbb{P}_{\Phi,\beta}(E \mid \mathcal{F}_{\Lambda^c}) = \mathbb{P}^{(\cdot)}_{\Phi,\Lambda,\beta}(E) \quad \mathbb{P}_{\Phi,\beta}\text{-a.s. for each } E \in \mathcal{F} \text{ and } \Lambda \Subset \mathbb{Z}^d\,.$$

One can show that $\mathbb{P}_{\Phi,\beta}$ is a Gibbs measure if and only if it satisfies the analogue of (1.2.1) in infinite volume, i.e. if and only if for any $\Lambda \subset S$

$$\mathbb{P}_{\Phi,\beta}(\mathrm{d}\varphi')\mathbb{P}^{(\varphi')}_{\Phi,\Lambda,\beta} = \mathbb{P}_{\Phi,\beta}\,. \tag{1.2.2}$$

This equation is called the DLR equation after Dobrushin, Lanford and Ruelle. The relation (1.2.2) formalizes the intuition that a system is in equilibrium if any of its subsystems is in equilibrium and thus distributed according to the canonical Gibbs measure.

Of course, this raises the question of existence and uniqueness of Gibbs measures. In our case the spin space is non-compact, and so neither question is trivial. In fact, it may happen that there is no infinite volume Gibbs measure or that there are infinitely many. One may hope to construct a Gibbs measure by choosing a specific sequence of domains (e.g. $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$) together with a choice of boundary data (e.g. $\varphi = 0$) and considering a weak limit of the corresponding sequence of finite volume Gibbs measures (the so-called thermodynamic limit). If such a weak limit exists, it is easy to see that in our setting it will be a Gibbs measure.

We will discuss the question of existence and uniqueness of Gibbs measures in more detail once we have introduced some examples of random interface models.

1.2.4 Examples of random interface models

Now that we have laid the theoretical foundations, we can introduce and describe some important examples of random interface models. As explained in the previous section, to describe an interface model we need an interaction Φ and an inverse temperature β. From a physical point of view it would be important to treat these two objects separately. However, all the models we consider have Hamiltonians that allow an arbitrary positive prefactor (and so we can include β in that prefactor) or have a Hamiltonian that is positively homogenous of some degree (and so a change of β only scales the field by a deterministic factor). Thus, there is no loss when we set $\beta = 1$ in all of the following and omit it in our notation.

We choose $S = \mathbb{Z}^d$ as our parameter space. In most of the following, we describe continuous models in the sense that we take the single spin space $E = \mathbb{R}$. At the end we will briefly mention discrete models where $E = \mathbb{Z}$. For now, we always take the reference measure $\lambda(\mathrm{d}\psi)$ to be the Lebesgue measure (simply denoted $\mathrm{d}\psi$). We also write $d(x, \Lambda)$ for the (Euclidean) distance from x to Λ.

Discrete Gaussian free field

The first and most important example of a random interface model is the discrete Gaussian free field (also called gradient model or harmonic crystal). This model is given by the interaction $\Phi_A(\psi) = \frac{1}{2}\sum_{x\in A}|\nabla_1\psi_x|^2$, where $\nabla_1\psi_x := (D_i^1\psi_x)_{i=1}^{\mathsf{d}} := (\psi_{x+e_i} - \psi_x)_{i=1}^{\mathsf{d}}$. This yields the measure

$$\mathbb{P}^{(\varphi)}_{\nabla,\Lambda}(\mathrm{d}\psi) = \frac{1}{Z^{(\varphi)}_{\nabla,\Lambda}} \exp\left(-\frac{1}{2}\sum_{\substack{x\in\mathbb{Z}^{\mathsf{d}}\\ d(x,\Lambda)\le 1}} |\nabla_1\psi_x|^2\right) \prod_{x\in\Lambda}\mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus\Lambda}\delta_{\varphi_x}(\mathrm{d}\psi_x)\,. \tag{1.2.3}$$

If $\varphi = 0$ we can remove the restriction on x in the sum, so that the measure with zero boundary values takes the form

$$\mathbb{P}_{\nabla,\Lambda}(\mathrm{d}\psi) = \frac{1}{Z_{\nabla,\Lambda}} \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^{\mathsf{d}}} |\nabla_1\psi_x|^2\right) \prod_{x\in\Lambda}\mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus\Lambda}\delta_0(\mathrm{d}\psi_x) \tag{1.2.4}$$

where we drop the superscript (0) for brevity.

The Hamiltonian here is the discrete L^2-norm of the gradient of ψ. It thus penalizes large slopes in ψ, in line with what we expected for the solid-liquid-interface models in Section 1.2.1. This model is particularly nice from a mathematical point of view. Namely, the Hamiltonian is a quadratic function of ψ, and so the measure is Gaussian.

The discrete Gaussian free field has a continuous analogue, the (continuous) Gaussian free field. Informally, this is the measure on functions $\psi\colon \mathbb{R}^{\mathsf{d}} \to \mathbb{R}$ with density $\frac{1}{Z}\exp\left(-\frac{1}{2}\|\nabla\psi\|_{L^2}^2\right)\mathrm{d}\psi$, but the actual definition is as a Gaussian measure on a negative Sobolev space, see e.g. [She07]. This measure appears as a scaling limit of a variety of models in probability, e.g. the dimer model in integrable probability [Ken01] or fields in random matrix theory [RV07]. It is also related to quantum field theory, where one tries to construct operator-valued Gaussian and non-Gaussian fields. This explains the name "free", since the Gaussian field corresponds to systems without interaction in that setting (cf. [GJ87]).

The discrete Gaussian free field will be one important example for us. We will mostly call it the gradient model, as this emphasizes the contrast to the membrane model (to be defined shortly).

$\nabla\varphi$-interface models

In the context of the application to solid-to-liquid phase transitions in 1.2.1 there is no reason to assume that slopes are penalized precisely by $\frac{1}{2}|\cdot|^2$. If we instead use an arbitrary even function $\mathcal{V}\colon \mathbb{R}\to\mathbb{R}$, we obtain the (Ginzburg-Landau) $\nabla\varphi$-model

$$\mathbb{P}^{(\varphi)}_{\mathcal{V}(\nabla),\Lambda}(\mathrm{d}\psi) = \frac{1}{Z^{(\varphi)}_{\mathcal{V}(\nabla),\Lambda}} \exp\left(-\sum_{\substack{x\in\mathbb{Z}^{\mathsf{d}}\\ d(x,\Lambda)\le 1}}\sum_{i=1}^{\mathsf{d}}\mathcal{V}(D_i^1\psi_x)\right) \prod_{x\in\Lambda}\mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus\Lambda}\delta_{\varphi_x}(\mathrm{d}\psi_x) \tag{1.2.5}$$

and its variant with zero boundary data

$$\mathbb{P}_{\mathcal{V}(\nabla),\Lambda}(\mathrm{d}\psi) = \frac{1}{Z_{\mathcal{V}(\nabla),\Lambda}} \exp\left(-\sum_{x\in\mathbb{Z}^{\mathsf{d}}}\sum_{i=1}^{\mathsf{d}}\mathcal{V}(D_i^1\psi_x)\right) \prod_{x\in\Lambda}\mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus\Lambda}\delta_0(\mathrm{d}\psi_x)\,. \tag{1.2.6}$$

As we will explain in the following sections, this model behaves quite similarly to the discrete Gaussian free field when $\mathcal{V}$ is strictly convex and satisfies some other mild regularity assumptions. Physically, this should not be surprising, as the convexity of $\mathcal{V}$ ensures that mixtures of different slopes are energetically unfavourable in comparison to pure slopes. The regularity assumptions one needs for $\mathcal{V}$ change from application to application. Typically one requires $\mathcal{V} \in C^2(\mathbb{R})$ and $c \leq \mathcal{V}''(x) \leq C$, but occasionally results in the literature require, e.g., that $\mathcal{V} \in C^\infty(\mathbb{R})$. In the following we will not be precise in this regard, and just speak of the $\nabla\varphi$-model with strictly convex $\mathcal{V}$.

When $\mathcal{V}$ is not convex, but in a suitable sense close to being convex, the model still behaves similarly to the Gaussian free field, although much less is known. Rigorous results on this are perturbative, and they require that $\mathcal{V}$ is close to a strictly convex function in a sufficiently strong norm. Again we will be rather vague, and speak of the $\nabla\varphi$-model with slightly non-convex $\mathcal{V}$ in the following.

For the case that $\mathcal{V}$ is far from being convex the model behaves very differently, see the discussion below.

The membrane model

Our discussion of lipid bilayers suggests that one should also study interface models involving second instead of first derivatives. The easiest such model is the membrane model where one considers the interaction $\Phi_A(\psi) = \frac{1}{2}\sum_{x\in A} |\Delta_1\psi_x|^2$, where $\Delta_1\psi_x := \sum_{i=1}^{\mathsf{d}} \psi_{x+e_i} - 2\psi_x + \psi_{x-e_i}$. In principle $\frac{1}{2}\sum_{x\in A} |\nabla_1^2\psi_x|^2$, where $\nabla_1^2\psi_x := (D_i^1 D_{-j}^1\psi_x)_{i,j=1}^{\mathsf{d}}$, would be an equally natural choice, but a discrete integration by parts shows that this leads to exactly the same model. In any case, one obtains the probability measure

$$\mathbb{P}_{\Delta,\Lambda}^{(\varphi)}(\mathrm{d}\psi) = \frac{1}{Z_{\nabla,\Lambda}^{(\varphi)}} \exp\left(-\frac{1}{2}\sum_{\substack{x\in\mathbb{Z}^{\mathsf{d}}\\ d(x,\Lambda)\leq 2}} |\Delta_1\psi_x|^2\right) \prod_{x\in\Lambda} \mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus\Lambda} \delta_{\varphi_x}(\mathrm{d}\psi_x) \tag{1.2.7}$$

or the zero boundary variant

$$\mathbb{P}_{\Delta,\Lambda}(\mathrm{d}\psi) = \frac{1}{Z_{\nabla,\Lambda}} \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^{\mathsf{d}}} |\Delta_1\psi_x|^2\right) \prod_{x\in\Lambda} \mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus\Lambda} \delta_0(\mathrm{d}\psi_x)\,. \tag{1.2.8}$$

Once again this is a Gaussian measure. The Hamiltonian here penalizes high curvature.

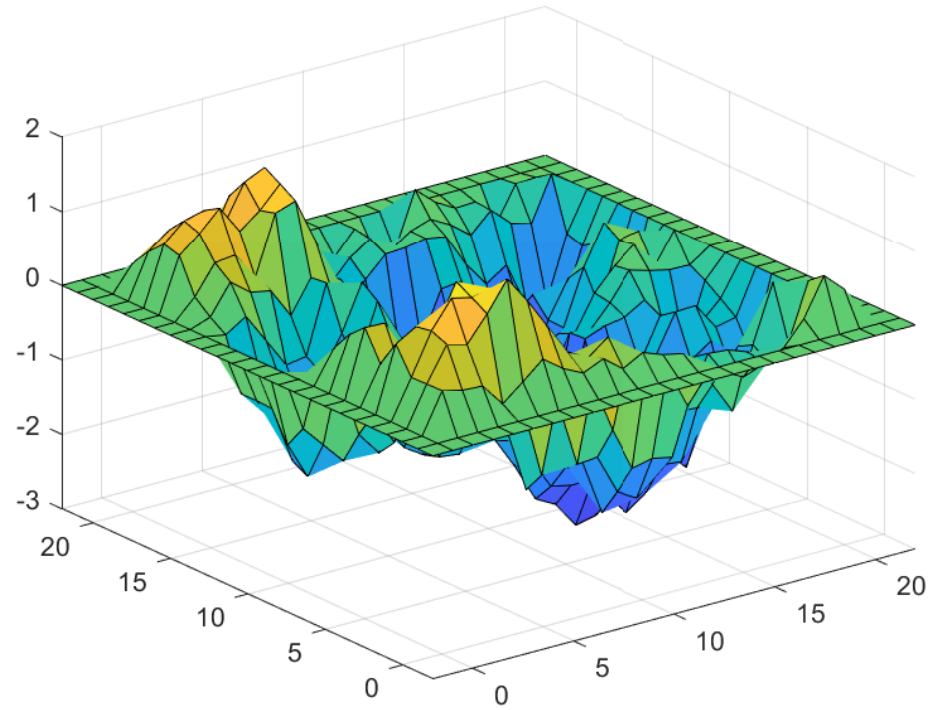
This model also has a continuous relative, the (continuous) membrane model (see e.g. [CDH19] for a precise definition). In recent years this model has also been identified as the scaling limit of several fields in probability theory, e.g. the odometer in divisible sandpiles [CHR18] or certain spanning trees related to the loop-erased random walk [LSW19].

See Figure 1.1 for pictures of samples of the membrane model in various dimensions.

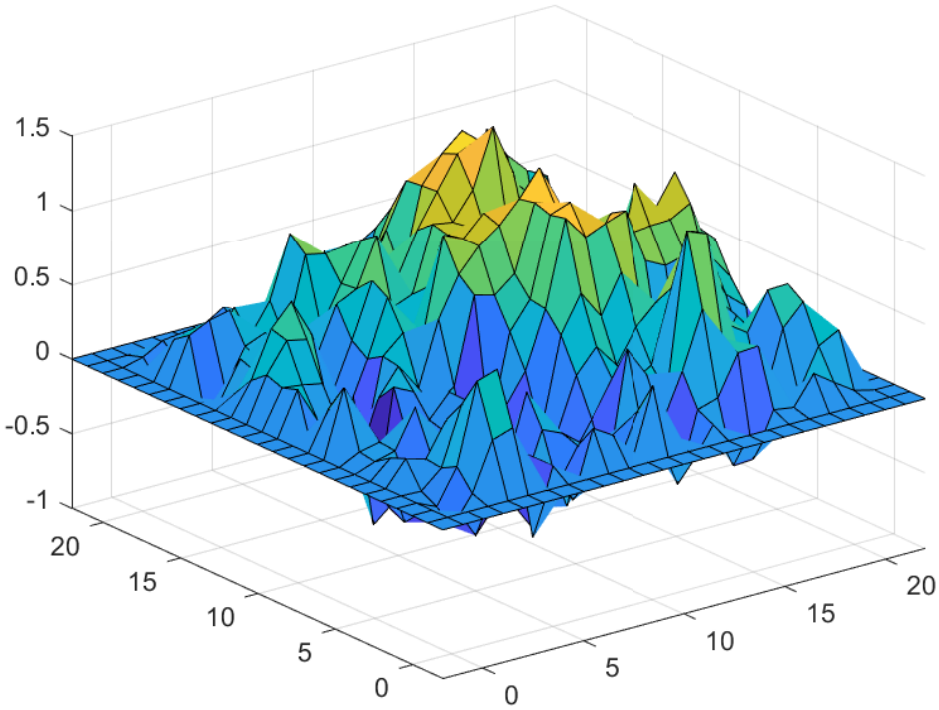
Other examples

Let us mention some other examples of random interface models. For simplicity we only give each model with zero boundary condition.

First of all, it remains to discuss the $\nabla\varphi$-interface with a potential $\mathcal{V}$ that is far from being convex. In that case the random interface can behave very differently from the models that we have mentioned so far. In particular, there can be phase transitions in the sense that there

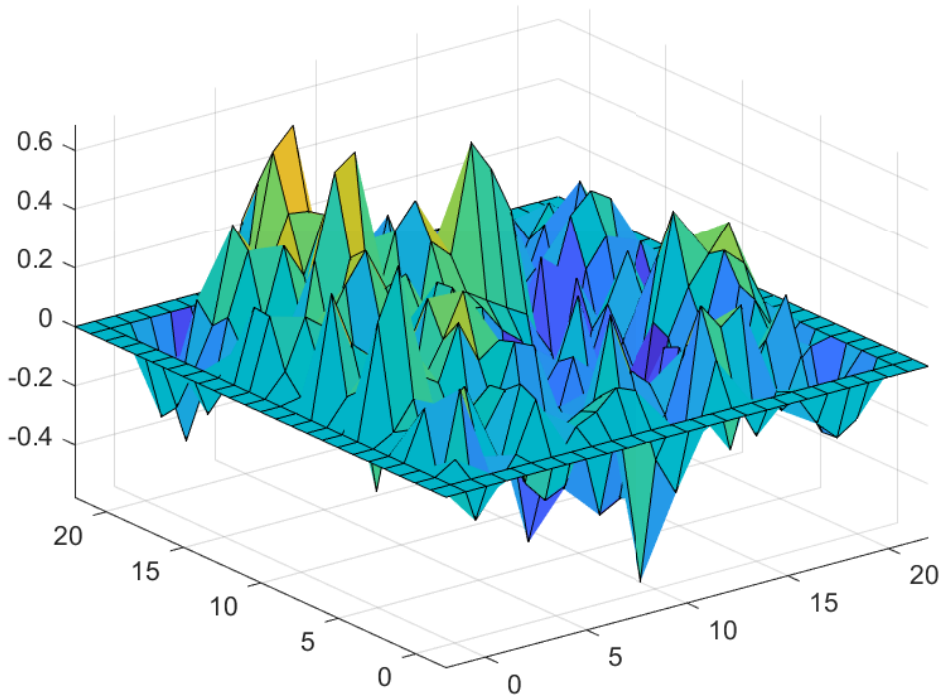


(a) $d = 2$

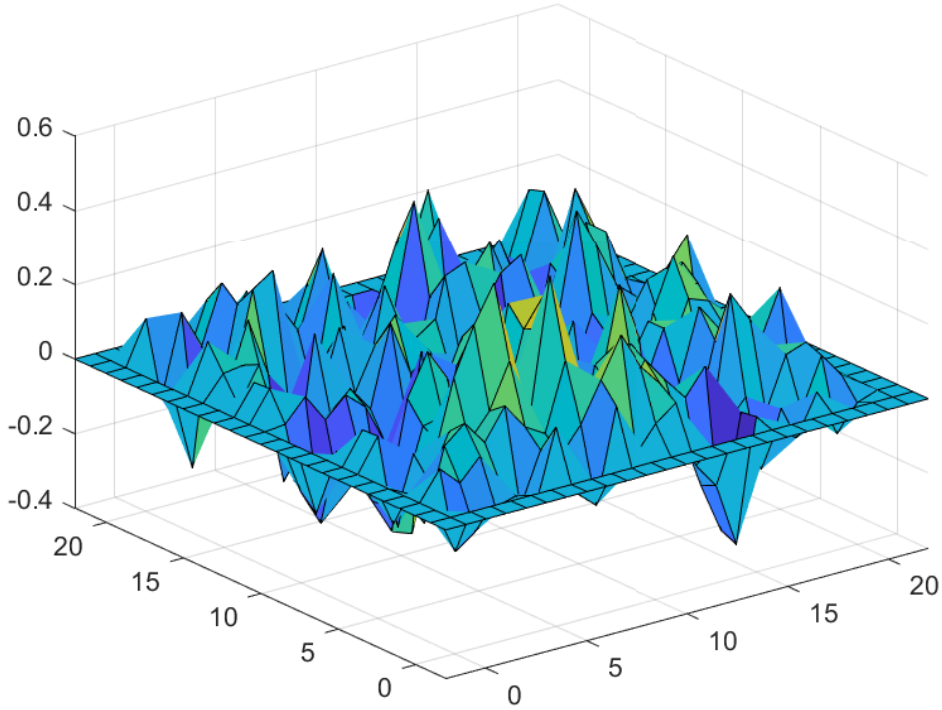


(b) $d = 3$

Figure 1.1: Samples of the membrane model in dimension $d \in \{2, 3, 4, 5\}$ on the domain $\{0, \ldots, 20\}^d$. The pictures show the values of the sample on the slice $\{0, \ldots, 20\}^2 \times \{10\}^{d-2}$. See Section 1.2.7 for a description how the samples were generated.



(c) d = 4



(d) d = 5

Figure 1.1: Samples of the membrane model (continued)

may be more than one Gibbs measure with the same tilt. We will not discuss this further, but see [BK07, Buc19].

Next, just like the $\nabla\varphi$-interface models are a generalization of the discrete Gaussian free field, one can consider the generalization of the membrane model where one uses an arbitrary even function $\mathcal{V}\colon \mathbb{R} \to \mathbb{R}$ instead of $\frac{1}{2}|\cdot|^2$. This yields the model

$$\mathbb{P}_{\Delta,\Lambda}(\mathrm{d}\psi) = \frac{1}{Z_{\nabla,\Lambda}} \exp\left(-\sum_{x\in\mathbb{Z}^d}\sum_{i=1}^{d} \mathcal{V}(\Delta_1\psi_x)\right) \prod_{x\in\Lambda} \mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^d\setminus\Lambda} \delta_{\varphi_x}(\mathrm{d}\psi_x)\,. \tag{1.2.9}$$

This model should behave similarly to the membrane model when $\mathcal{V}$ is strictly convex or a small perturbation of a strictly convex function, while the case where $\mathcal{V}$ is far from being convex is presumably very complicated. The only work where this model is studied is [Kur12].

Of course, it is also reasonable to consider models where first and second derivatives are mixed. One such model has been studied very recently in [CDH20]. It is given by the probability measure

$$\mathbb{P}_{\Delta,\Lambda}(\mathrm{d}\psi) = \frac{1}{Z_{\nabla,\Lambda}} \exp\left(-\sum_{x\in\mathbb{Z}^d} |\nabla_1\psi_x|^2 + a_\Lambda|\Delta_1\psi_x|^2\right) \prod_{x\in\Lambda} \mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^d\setminus\Lambda} \delta_{\varphi_x}(\mathrm{d}\psi_x) \tag{1.2.10}$$

Here a_Λ is a scalar that may depend on Λ. In the limit $\Lambda \nearrow \infty$ it depends on the choice of a_Λ whether the behaviour of this model resembles that of the discrete Gaussian free field, that of the membrane model, or shows some genuinely new mixed behaviour.

It is also reasonable to not only use first and second order difference operators, but higher polyharmonic difference operators or various linear combinations. Such models are not studied much in the literature (an exception is [Sak03]). The reason for this probably is that these models do not add much from a mathematical point of view, while the notation becomes increasingly complicated. In particular, all the results for the membrane models to be discussed in the following should have analogues for the case of higher-order polyharmonic operators in place of Δ_1^2.

All the models discussed so far have in common that the specification Ψ is translation-invariant. There are also interesting examples where this is not the case. In particular, one can sample the $(\Psi_{A+x})_{x\in\mathbb{Z}^d}$ from some random distribution. This leads to disordered random interface models. One can investigate their properties either for almost every realization of the specification, or averaged over the randomness of the specifications. The disorder can drastically change the behaviour of the field (see [Vel06], or e.g. [GL18] for recent work).

It is also possible to consider all the models considered so far with the continuous spin space $E = \mathbb{R}$ (with the Lebesgue measure) replaced by the discrete spin space $E = \mathbb{Z}$ (with the counting measure). The phenomenology is generally rather similar for those discrete models (at least when β is small enough), but there are also new phenomena when β is large and there is a so-called roughening transition (see e.g. [She05, Vel06]).

1.2.5 Critical dimensions and $\log$-correlated fields

In Figure 1.1 one can see that the membrane model becomes progressively rougher as the dimension increases. This phenomenon is not unique to the membrane model. In fact, for most of the models discussed above, there is a critical dimension so that the behaviour of

the field is very different depending on whether d is less than, equal or larger than that dimension. These regimes are called subcritical, critical and supercritical, respectively.

Typically, in the subcritical dimensions the variance of ψ_x for some $x \in \Lambda$ is unbounded and grows like a power of $d(x, \partial\Lambda)$, the distance of x to the boundary of Λ. In the critical dimension the variance of ψ_x is still unbounded but only grows like the logarithm of $d(x, \partial\Lambda)$, while in the supercritical dimensions the variance of ψ_x is bounded uniformly in x and Λ. In the following sections we describe a number of implications of this.

For now let us mention that the critical dimension of both the discrete Gaussian free field and the $\nabla\varphi$-interface models is $\mathsf{d} = 2$, while the critical dimension of the membrane model is $\mathsf{d} = 4$. This already gives a hint of the heuristic that the d-dimensional membrane model behaves in many aspects like the $(\mathsf{d} - 2)$-dimensional discrete Gaussian free field.

Of particular interest is the case of the critical dimension. This case is in many aspects borderline between the very different sub- and supercritical dimensions. It turns out that in the critical dimension not only the variances grow like the logarithm of the distance to the boundary, but also the covariances decay like the logarithm of the distance between the respective sites. Discrete random fields with these properties are called log-correlated fields. Of course one can also define (continuous) log-correlated fields. Both discrete and continuous log-correlated fields have been a topic of intensive study in the past years. In particular, there are various predictions of universality in this class, meaning that various properties of the field do not depend on the precise structure of the correlations, but only the fact that these decay logarithmically. See [DRSV14, DRSV17] for an overview.

The interest in log-correlated fields is fueled by the fact that these arise in a variety of contexts. Beyond the random interface models that we have already described, let us mention branching random walks and branching Brownian motion (see [Bov17] for an overview), the characteristic polynomial of certain random matrices and (closely related) the values of the Riemann ζ function on the critical line (see [FHK12, FK14] for important conjectures and [CMN18, ABB$^+$19] and the references therein for recent rigorous results).

1.2.6 Mathematical tools to study random interface models

Before describing in detail what is known about the membrane model and the other random interface models, we will outline the main mathematical techniques that have been used to study these models. The message that we want to transmit here is the following: For the discrete Gaussian free field there exist many powerful techniques one can use to study it. The $\nabla\varphi$-interface model with strictly convex $\mathcal{V}$ and the membrane model both are more difficult than the discrete Gaussian free field. However, this difficulty manifests itself in different ways: for each of the two models only some of the tools survive while others can no longer be applied. Finally, the $\nabla\varphi$-interface model with slightly non-convex $\mathcal{V}$ is the most difficult model, and there are few existing techniques that can be applied. See Figure 1.2 for a schematic drawing of the relations between the models.

Markov property

The (domain) Markov property is a rather obvious, but nonetheless useful consequence of our definition of random interface models. It applies to all random interface models in the sense of our definition. For the following see [Fun05, Bis20]. Recall that we have assumed that the terms Φ_A in our interaction are 0 when $\operatorname{diam} \Phi > R$. Then the field values in some $A \subset \Lambda$ depend only on the boundary data on the sites that are at most R away from A.

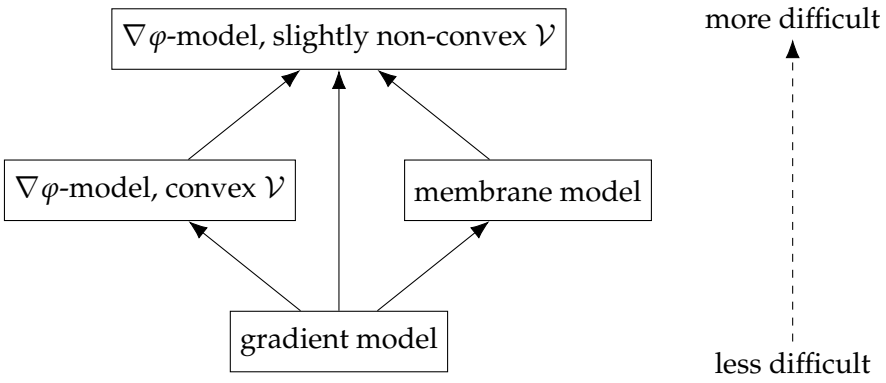


Figure 1.2: Relations between the random interface models

More formally, let $B = \{x \in \Lambda \setminus A\colon \operatorname{dist}(x, A) \leq R\}$. Then

$$\mathbb{P}^{(\varphi)}_{\Phi,\Lambda}(E \mid \mathcal{F}_{A^c}) = \mathbb{P}^{(\varphi)}_{\Phi,\Lambda}(E \mid \mathcal{F}_{B\setminus A}) \quad \text{for all } E \in \mathcal{F}_A\,.$$

For Gaussian measures such as the discrete Gaussian free field and the membrane model we can even give a somewhat explicit description of $\mathbb{P}^{(\varphi)}_{\Phi,\Lambda}(\cdot \mid \mathcal{F}_{B\setminus A})$. Suppose that ψ is distributed according to $\mathbb{P}^{(\varphi)}_{\Phi,\Lambda}$ and let $h_x = \mathbb{E}^{(\varphi)}_{\Phi,\Lambda}(\psi_x \mid \mathcal{F}_{B\setminus A})$ for $x \in A$. Then $\{\psi_x - h_x\colon x \in A\}$ is independent of $\mathcal{F}_{A^c}$ and its law is given by $\mathbb{P}_{\Phi,A}$.

Random walk representations

Random walk representations exist for the discrete Gaussian free field and the $\nabla\varphi$-interface models with strictly convex $\mathcal{V}$. We begin with the former case, where the situation is less complicated.

Recall that the discrete Gaussian free field is a centred Gaussian measure, and as such it is determined by its covariance matrix. It is easy to see (cf. e.g. [Fun05]) that this covariance matrix is equal to the Green's function of the symmetric random walk on $\mathbb{Z}^{\mathsf{d}}$ killed when leaving Λ. That is, when $(X_t)_{t\geq 0}$ is the path of a (continuous time) symmetric random walk on $\mathbb{Z}^{\mathsf{d}}$ that jumps at rate $2\mathsf{d}$ to a uniformly chosen neighbour and $\mathbb{E}^x$ denotes the expectation with respect to the law of $(X_t)_{t\geq 0}$ when $X_0 = x$, then

$$\mathbb{P}_{\nabla,\Lambda}(\psi_x\psi_y) = \mathbb{E}^x\left(\int_0^{T_{\Lambda^c}} \mathbb{1}_{X_t=y}\mathrm{d}t\right) \tag{1.2.11}$$

where $T_{\Lambda^c} = \inf\{n \geq 0\colon X_n \notin \Lambda\}$. This representation allows to use estimates for random walks to conclude estimates on the behaviour of the covariance.

For $\nabla\varphi$-interface models we do not have a representation as simple as (1.2.11). However, there is a more complicated version due to Helffer and Sjöstrand [HS94] that allows to write

$$\mathbb{P}_{\mathcal{V}(\nabla),\Lambda}(\psi_x\psi_y) = \mathbb{E}^{\delta_x\otimes\mathbb{P}_{\mathcal{V}(\nabla),\Lambda}}\left(\int_0^{T_{\Lambda^c}} \mathbb{1}_{X_t=y}\mathrm{d}t\right)\,. \tag{1.2.12}$$

Here $(X_t)_{t\geq 0}$ describes a continuous time random walk in a time-dependent random environment given by the Langevin dynamics of the field. This random walk exists when $\mathcal{V}$ is strictly convex. See [Fun05] for the details. Actually (1.2.12) is only a special case of the full Helffer-Sjöstrand representation, which applies to the covariances of arbitrary observables of the field.

Of course, this representation is only useful if one is also able to control the right-hand side. This amounts to understanding random walks in a random environment, a subject where the methods of quantitative stochastic homogenization apply. The Helffer-Sjöstrand representation was first used in the study of $\nabla\varphi$-interface models by Naddaf and Spencer in [NS97] with later refinements in [DGI00, GOS01]. Recently Armstrong and Wu [AW19] have made further progress by systemically using the emerging theory of quantitative stochastic homogenization as described in [AKM19].

Let us note an important fact: clearly the existence of a random walk representation for the covariance as in (1.2.11) or (1.2.12) implies that the covariance is pointwise nonnegative, and indeed this is the case for the discrete Gaussian free field as well as the $\nabla\varphi$-interface model with strictly convex $\mathcal{V}$. For the membrane model, however, the covariance is, in general, not pointwise nonnegative, and so there is no random walk representation. One way to see this is to explicitly compute the covariance on some small domains. For example, for $\mathsf{d} = 2$ and $\Lambda = \{0, \dots, 20\}^2$ the Green's function is negative in the corners of the domain. Closely related to the question of nonnegativity of the covariances of the membrane model is the question of nonnegativity of the Green's function of the continuous Bilaplacian. We will discuss the latter in Section 1.4.1. See [Gia01, Appendix A.2] for a more detailed discussion of random walk representations.

PDE estimates for the Green's function

For some of the Gaussian models, in particular the discrete Gaussian free field and the membrane model, one can alternatively apply estimates from PDE theory and numerical analysis. As they are Gaussian measures, they are determined by their covariance matrix. The point is that an easy calculation (cf. again [Fun05]) reveals that the covariance matrix of the discrete Gaussian free field is also given by the Green's function of $-\Delta_1$ on Λ with zero boundary data, i.e. that $\mathbb{P}_{\nabla,\Lambda}(\psi_x\psi_y) = G_{\nabla,\Lambda}(x,y)$, where for each $y \in \Lambda$ the function $G_{\nabla,\Lambda}(\cdot, y)$ is the solution of the partial differential equation

$$\begin{aligned} -\Delta_1 G_{\nabla,\Lambda}(\cdot, y) &= \delta_y && \text{in } \Lambda \\ G_{\nabla,\Lambda}(\cdot, y) &= 0 && \text{on } \mathbb{Z}^{\mathsf{d}} \setminus \Lambda\,. \end{aligned} \tag{1.2.13}$$

Similarly, one can show that $\mathbb{P}_{\Delta,\Lambda}(\psi_x\psi_y) = G_{\Delta,\Lambda}(x,y)$, where $G_{\Delta,\Lambda}(\cdot, y)$ is the Green's function of Δ_1^2, i.e. the solution of the partial differential equation

$$\begin{aligned} \Delta_1^2 G_{\Delta,\Lambda}(\cdot, y) &= \delta_y && \text{in } \Lambda \\ G_{\Delta,\Lambda}(\cdot, y) &= 0 && \text{on } \mathbb{Z}^{\mathsf{d}} \setminus \Lambda\,. \end{aligned} \tag{1.2.14}$$

Thus, one can hope to use methods from the theory of partial differential equations to derive results for $G_{\nabla,\Lambda}(\cdot, y)$ and $G_{\Delta,\Lambda}(\cdot, y)$. Furthermore, one can consider (1.2.13) and (1.2.14) as finite difference schemes for the Laplacian and Bilaplacian, respectively, and so apply tools from numerical analysis. In fact, these are the main approaches used in this thesis. We postpone a more detailed exposition to Section 1.4.

Correlation inequalities

Another very important tool in the study of random interface models are correlation inequalities. We describe two of them: The FKG inequality and the Gaussian correlation inequality. The former can be applied to the discrete Gaussian free field and the $\nabla\varphi$-interface models,

the latter to all Gaussian measures (and so in particular to the discrete Gaussian free field and the membrane model). These are not the only correlation inequalities important for the study of random interface models, though. Let us mention the Brascamp-Lieb inequality (cf. [Fun05, Section 4.2] or [Gia01, Appendix A.2]) and the GKS inequality (cf. [DMRR92, Appendix A]).

However, we focus on the FKG-inequality and the Gaussian correlation inequality. We begin with the former, introduced by (and named after) Fortuin, Kasteleyn and Ginibre [FKG71]. In the context of random interface models this is the statement that whenever A, B are events that are increasing (i.e. if $\psi \in A$ and $\psi' \geq \psi$ pointwise then $\psi' \in A$, and similarly for B) then

$$\mathbb{P}_\Lambda(A \cap B) \geq \mathbb{P}_\Lambda(A)\,\mathbb{P}_\Lambda(B) \ .$$

This inequality is an extremely powerful tool. One typical application is to use that conditioning on some increasing event (say, the field being large on some subset of Λ) increases the probability of some other increasing event (say, the field being large on another subset of Λ). For $\nabla\varphi$-interface models with stricty convex $\mathcal{V}$ the FKG inequality follows easily from the Helffer-Sjöstrand representation, cf. [Fun05]. For Gaussian interface models it is even easier to decide whether the FKG inequality holds: According to a criterion of Pitt [Pit82] this is the case if and only if the correlation matrix is elementwise nonnegative. As already mentioned that is the case for the discrete Gaussian free field but not for the membrane model. So we see that the membrane model does not satisfy the FKG inequality.

The other correlation inequality that we want to discuss, the Gaussian correlation inequality, looks quite similar. This is the statement that whenever A, B are closed events that are symmetric around 0 then

$$\mathbb{P}_\Lambda(A \cap B) \geq \mathbb{P}_\Lambda(A)\,\mathbb{P}_\Lambda(B) \ .$$

It was a longstanding open conjecture that this inequality holds for all Gaussian measures $\mathbb{P}_\Lambda$. This conjecture was settled in 2014 by Royen [Roy14] (see also [LM17] for an exposition of the proof). There is no reason to believe that this correlation inequality holds for general (log-concave) probability measures such as the $\nabla\varphi$-interface models with stricty convex $\mathcal{V}$. For instance, a counterexample can be constructed out of a suitable Gaussian measure conditioned to be small, similar to Remark 6.2.1.

Other techniques

Let us mention a few other techniques that have been employed in the study of the membrane model. We have already briefly mentioned the Langevin dynamics associated to a $\nabla\varphi$-interface model with strictly convex $\mathcal{V}$. Beyond their occurrence in the Helffer-Sjöstrand representation they can also be used directly to understand the underlying Gibbs measure. This was pioneered by Funaki and Spohn [FS97]. For that model also some other techniques such as Sheffield's cluster swapping [She05] exist.

Finally, we need to mention some techniques available for the $\nabla\varphi$-interface model with slightly non-convex $\mathcal{V}$. The main technical tools used for that model are various implementations of the renormalization group from theoretical physics, ranging from a one-step renormalization scheme in [CDM09] to a very subtle multi-step renormalization scheme in [ABKM19].

1.2.7 Sampling from the models

Before discussing more detailed properties of the models, let us describe how to efficiently generate samples of a random interface model. In particular, we want to explain how the samples in Figure 1.1 and in Figures 1.3 and 1.4 were generated. The problem is that while we have an explicit expression for the Boltzmann weight $\exp(-H)$, the partition function is given by a high-dimensional integral that is very hard to calculate explicitly or numerically, so a direct computation of the probability density seems infeasible. Let us describe some alternative approaches.

Gibbs sampler

A general method for sampling from a Gibbs measure $\mathbb{P}$ is to use a Markov chain Monte Carlo method, or, more precisely, a Gibbs sampler (see e.g [LPW09, Section 3.3]. The idea here is that while the joint density of all field heights is too complicated to understand, the conditional density of the height of a single site while all other heights are fixed is easy to compute. This suggests an algorithm where we iteratively resample the field at a (randomly chosen) site. This resampling defines a Markov chain $(\psi^{(k)})_{k=0}^{\infty}$ whose unique stationary measure is the Gibbs measure we want to sample from. Under some weak assumptions on the Markov chain, for any initial configuration $\psi^{(0)}$ the distribution of $\psi^{(k)}$ converges (as $k \to \infty$) in total variation distance to $\mathbb{P}$. In practice, one needs to decide how to choose k. For this one would need to know how far the distribution of $\psi^{(k)}$ still is from $\mathbb{P}$, i.e. how fast the Markov chain mixes. Unfortunately, this is in general a very hard problem (see again [LPW09] for a review of what is known), and so in practice one often has to make a guess.

We have implemented this algorithm in Matlab to generate samples from the membrane model on the domains $\{0, \dots, N\}^{\mathsf{d}}$. It is clear that we need to resample each site at least once to have a chance to see the actual behaviour of the model, and so we need at least $(N+1)^{\mathsf{d}}$ iterations. In practice, for $N = 20$ and $\mathsf{d} = 4$ it seems that $10^2 N^{\mathsf{d}}$ iterations are not quite enough (as the maximum of the field is not as high as the theory predicts), while $10^3 N^{\mathsf{d}}$ iterations seem reasonable. This means that to sample the membrane model for $N = 20$ and $\mathsf{d} = 4$ we should use at least $10^3 N^{\mathsf{d}} \approx 2 \cdot 10^8$ iterations, while for $\mathsf{d} = 5$ one would need already approximately $4 \cdot 10^9$ iterations. For comparison, the author's laptop computer was able to process about 10^6 iterations per second.

We have generated Figures 1.3 and 1.4 using this method (with $2 \cdot 10^8$ iterations), as its flexibility allowed to easily include the single site potentials that will appear there. For Figure 1.1 we used another approach to be described below.

Naive direct sampling for Gaussian measures

When sampling from a Gaussian measure such as the discrete Gaussian free field or the membrane model, it is also possible to use the Gaussian structure of the measure to directly sample from the measure. This has the advantage that we can be sure that our sample has the correct distribution, and we do not need to guess how many iterations to use.

We focus on the membrane model, but the situation is similar for other Gaussian Gibbs measures. The probability distribution (1.2.8) of the membrane model can be rewritten as

$$\mathbb{P}_{\Delta,\Lambda}(\mathrm{d}\psi) = \frac{1}{Z_{\nabla,\Lambda}} \exp\left(-\frac{1}{2}(\psi_x, \Delta_1^2|_{\Lambda\times\Lambda}\psi_x)\right)_{L^2(\Lambda)} \prod_{x\in\Lambda} \mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus\Lambda} \delta_{\varphi_x}(\mathrm{d}\psi_x)$$

where $(\cdot,\cdot)_{L^2(\Lambda)}$ denotes the standard scalar product on $\mathbb{R}^\Lambda$. This means we need to sample a centred normal random variable with covariance $(\Delta_1^2|_{\Lambda\times\Lambda})^{-1}$. Such a sample is given by MX, where $X \in \mathbb{R}^\Lambda$ is a vector of i.i.d. standard Gaussians and $M \in \mathbb{R}^{\Lambda\times\Lambda}$ is a matrix such that $MM^t = (\Delta_1^2|_{\Lambda\times\Lambda})^{-1}$. An efficient way to compute a possible M is to compute the Cholesky decomposition $\Delta_1^2|_{\Lambda\times\Lambda} = LL^t$ with a lower triangular matrix L and then choosing $M = L^{-1}$.

We have implemented this algorithm in Matlab. It works well for small domains or for small dimensions, but for $\mathsf{d} = 5$ and $N = 20$, we would have $|\Lambda| = (N+1)^{\mathsf{d}} \approx 4 * 10^6$, and computing the Cholesky decomposition of a matrix this large is infeasible (in fact, Matlab runs out of memory quickly when attempting this on a laptop computer).

Improved direct sampling for Gaussian measures

For Gaussian models there is also a better direct sampling algorithm. We again focus on the membrane model with zero boundary data. For this we use ideas of Sheffield [She07, Section 4.4] for the discrete Gaussian free field that can easily be adapted to the membrane model. The first observation is that the membrane model is much easier to sample when we first sample a complex version and our domain is a torus $(\mathbb{Z}/N'\mathbb{Z})^{\mathsf{d}}$, as we can then use discrete Fourier analysis. For that end consider the scalar product $\langle\psi,\psi'\rangle := (\Delta_1\psi, \Delta_1\psi')_{L^2((\mathbb{Z}/N'\mathbb{Z})^{\mathsf{d}})}$ on the space H of lattice functions on $(\mathbb{Z}/N'\mathbb{Z})^{\mathsf{d}}$ with average 0. Our goal is to sample from the Gaussian measure with density $\frac{1}{Z}\exp\left(-\frac{1}{2}\langle\psi,\psi\rangle\right)\mathrm{d}\psi$, and one easily checks (cf. [She07, Proposition 2.1] that a sample ψ from that measure has the same law as $\sum_{j=1}^{N'^{\mathsf{d}}-1}\varphi_j X_j$, where $(\varphi_j)_{j=1}^{N'^{\mathsf{d}}-1}$ is a fixed orthonormal basis of H, and the X_j are i.i.d. standard Gaussians. We can generate such an orthonormal basis using the eigenfunctions of the discrete Bilaplacian. We take the functions

$$\psi_{k_1,\ldots,k_{\mathsf{d}}}(x) = \alpha_{k_1,\ldots,k_{\mathsf{d}}}\exp\left(i\frac{k_1x_1+\ldots+k_{\mathsf{d}}x_{\mathsf{d}}}{N'}\right)$$

for $k_l \in \{0,\ldots,\mathsf{d}-1\}$, not all 0, where

$$\alpha_{k_1,\ldots,k_{\mathsf{d}}} = \frac{1}{4N'^{\mathsf{d}/2}\left(\sin^2\left(\frac{k_1\pi}{N'}\right)+\ldots+\sin^2\left(\frac{k_1\pi}{N'}\right)\right)}.$$

Thus, a fast way to generate a sample of the membrane model on a torus is to compute the array $(\alpha_{k_1,\ldots,k_{\mathsf{d}}}X_{k_1,\ldots,k_{\mathsf{d}}})_{k_1,\ldots,k_{\mathsf{d}}=1}^{\mathsf{d}}$, where $X_{0,\ldots,0} = 0$ and all other X are i.i.d. standard Gaussians, and then take the multidimensional discrete Fourier transform of that array. Both steps can be done extremely fast (in a few seconds even if $\mathsf{d} = 5$ and $N = 20$).

Of course, this is not yet quite what we wanted, as we were looking for zero and not periodic boundary conditions. For this we use another observation from [She07, Section 4.4], namely, that the domain Markov property still applies. Thus, we embed our domain $\{0,\ldots,N\}^{\mathsf{d}}$ into $(\mathbb{Z}/N'\mathbb{Z})^{\mathsf{d}}$ for some $N' \geq N+5$. Now, given a sample of the membrane model on $(\mathbb{Z}/N'\mathbb{Z})^{\mathsf{d}}$, we only need to subtract the conditional expectation of the field given its values outside of $\{0,\ldots,N\}^{\mathsf{d}}$ to obtain a sample of the membrane model with zero boundary data on $\{0,\ldots,N\}^{\mathsf{d}}$. It remains to compute that conditional expectation, i.e. the biharmonic extension of the boundary values. For that purpose we need to solve the system $\Delta_1^2|_{\Lambda\times\Lambda}u = f$ for one right-hand side. This is a large system (about $4\cdot 10^6$ unknowns if $\mathsf{d} = 5$ and $N = 20$), but the matrix Δ_1^2 is very sparse, and we can use an iterative solver such as

Matlab's preconditioned conjugate gradient method to compute a very good approximative solution within several minutes of computation.

We have used this algorithm to generate the samples in Figure 1.1.

1.3 Properties of the membrane model and other interface models

We will now consider specific aspects of random interface models and describe what is known about them. We focus on the models described in Section 1.2.4, that is, the discrete Gaussian free field (or gradient model), the $\nabla\varphi$-model with strictly convex $\mathcal{V}$, the membrane model, and the $\nabla\varphi$-model with slightly non-convex $\mathcal{V}$. The reader should keep in mind the relation between these models illustrated in Figure 1.2. For most of the properties that we will mention, the answers for the gradient model are known. For quite a few of them the answer is also known for the $\nabla\varphi$-model with strictly convex $\mathcal{V}$ and the membrane model (and this thesis makes some further progress in the case of the membrane model). For the $\nabla\varphi$-model with slightly non-convex $\mathcal{V}$, only little is known. In fact, in some of the following sections, we will not even mention it, as there are no results. It is a long-term hope that progress on models such as the membrane model or the $\nabla\varphi$-model with strictly convex $\mathcal{V}$ also furthers the understanding of this model.

For the following sections most of the material on the gradient model and the $\nabla\varphi$-model with strictly convex $\mathcal{V}$ is taken from [Fun05, Vel06].

1.3.1 Gibbs measures and scaling limits

Gibbs measures

After the discussion in Section 1.2.3 an obvious question is that of the existence of Gibbs measures for our random interface models. Because all the specifications we have considered are invariant under shifts of the field by a constant, it is clear that if there is a Gibbs measure, then there are infinitely many. Thus, a Gibbs measure is never unique, and the interesting question is whether one exists at all.

It turns out that this is the case if and only if the dimension is supercritical. Indeed, in supercritical dimensions the variance at a single site is uniformly bounded, as can be seen for the gradient model and the membrane model from straightforward estimates on the Green's function, for the $\nabla\varphi$-model with strictly convex $\mathcal{V}$ from the Brascamp-Lieb inequality and for $\nabla\varphi$-model with slightly non-convex $\mathcal{V}$ from the techniques in [ABKM19, Hil19]. Now the boundedness of the variances is easily seen to imply tightness of the finite volume Gibbs measures, and each subsequential weak limit will be a Gibbs measure. In the critical and subcritical dimensions the variances diverge as Λ grows, and so there is no hope for the existence of a Gibbs measure (cf. e.g. [Geo88] for a rigorous proof in the case of the gradient model).

Gradient Gibbs measures and Hessian Gibbs measures

It is of course quite unsatisfactory that there is no infinite volume Gibbs measure in the critical and subcritical dimensions. In the case of the gradient model and the $\nabla\varphi$-model one alternative way to make sense of an infinite volume limit of the field is to use so-called gradient Gibbs measures. The idea here is to consider not the field heights ψ_x but their gradients $\nabla_1\psi_x$, and to pass to a limit of the field of gradients. The limit field should then

also satisfy some variant of the DLR equations (1.2.2). Making this intuition rigorous is somewhat technical and we refer to [FS97] and in particular [She05, Section 3.1] for the details.

The point for the gradient model is that while $\mathrm{Var}_{\nabla,\Lambda}(\psi_x) = G_{\nabla,\Lambda}(x,x)$ diverges in dimensions $\mathsf{d} \leq 2$, we have

$$\begin{aligned}\mathrm{Var}_{\nabla,\Lambda}(D_i^1\psi_x) &= \mathbb{E}_{\nabla,\Lambda}(\psi_{x+e_i} - \psi_x)^2 \\ &= G_{\nabla,\Lambda}(x+e_i, x+e_i) - 2G_{\nabla,\Lambda}(x, x+e_i) + G_{\nabla,\Lambda}(x,x) \\ &= D_{i,x}^1 D_{i,y}^1 G_{\nabla,\Lambda}(x,x)\end{aligned}$$

where $D_{i,x}^1$ and $D_{i,y}^1$ denote the discrete derivative with respect to the first and second variable, respectively. That is, by passing to the gradients of ψ_x we have gained two derivatives. Now, $\nabla_{1,x}\nabla_{1,y}G_{\nabla,\Lambda}(x,x)$ is uniformly bounded for all $\mathsf{d} \geq 1$, and so one can see that there is a gradient Gibbs measure for the gradient model in all dimensions. Choosing boundary data $\varphi_x = a \cdot x$ for some $a \in \mathbb{R}^{\mathsf{d}}$, we can construct a gradient Gibbs measure with given tilt (i.e. expected value of the gradient at each single site) a. One can show that every shift invariant ergodic gradient Gibbs measure $\mathbb{P}$ which is tempered (i.e. satisfies $\mathbb{E}(D_i^1\psi_x)^2 < \infty$ for all $x \in \mathbb{Z}^{\mathsf{d}}$, $i \in \{1,\dots,\mathsf{d}\}$) is equal to one of these gradient Gibbs measures with the corresponding tilt. Indeed, this follows easily from the uniform boundedness of $\nabla_{1,x}\nabla_{1,y}G_{\nabla,\Lambda}(x,x)$ together with [Geo88, Theorem 13.24 and Theorem 13.26] and the well-known fact that every bounded discrete harmonic function is constant.

This characterization of gradient Gibbs measure for the gradient model is actually a special case of a result by Funaki and Spohn [FS97] who have extended the above considerations to the $\nabla\varphi$-model with strictly convex $\mathcal{V}$. Again gradient Gibbs measures exist in every dimension, and the tempered ones are parametrised by their tilt.

For the membrane model one can proceed similarly. But in view of the fact that the Hamiltonian now involves second derivatives instead of first ones, it is more natural to consider not the gradients of the field but their Hessians, i.e. consider "Hessian Gibbs measures". While it seems that this has not been worked out in detail anywhere in the literature, the construction can proceed analogously to [She05, Section 3.1], so we only sketch the outcome. A short calculation as above shows that

$$\mathrm{Var}_{\Delta,\Lambda}(D_i^1 D_{-j}^1\psi_x) = \mathbb{E}_{\nabla,\Lambda}(\psi_{x+e_i} - \psi_x - \psi_{x+e_i-e_j} + \psi_{x-e_j})^2 = D_{i,x}^1 D_{-j,x}^1 D_{i,y}^1 D_{-j,y}^1 G_{\Delta,\Lambda}(x,x)$$

That is, by passing to Hessian Gibbs measures we have gained four derivatives, and so Hessian Gibbs measures can be constructed in all dimensions. For every $A \in \mathbb{R}^{\mathsf{d}\times\mathsf{d}}$ by choosing the boundary values $\varphi_x = \frac{1}{2}x^t Ax$ we can find a Hessian Gibbs measure with expected value of the Hessian at every site equal to A. Just as for the gradient model, one sees that all shift-invariant ergodic Hessian Gibbs measures which are tempered (i.e. satisfy $\mathbb{E}(D_i^1 D_{-j}^1\psi_x)^2 < \infty$ for all $x \in \mathbb{Z}^{\mathsf{d}}$, $i,j \in \{1,\dots,\mathsf{d}\}$) are given this way. For the proof one needs the fact that every bounded discrete biharmonic function on $\mathbb{Z}^{\mathsf{d}}$ is constant. This fact might not be completely obvious, but it can be shown, for example, by combining Lemma 2.3.1 and Lemma 2.5.1 from Chapter 2 to see that such a function needs to be affine.

Scaling limits

A different, but closely related question is whether it is possible to extract a scaling limit of the field. To do so, we first need to define an appropriate rescaling of the field. We choose

some bounded domain $D \subset \mathbb{R}^{\mathsf{d}}$ and consider the domains $\Lambda_N = ND \cap \mathbb{Z}^{\mathsf{d}}$. Then, if ψ_N is distributed according to $\mathbb{P}_{\Psi,\Lambda_N}$ and $\alpha \in \mathbb{R}$, we can consider the field $\tilde{\psi}_N(x) = N^\alpha \psi_N(Nx)$ as a random function on $D \cap \left(\frac{1}{N}\mathbb{Z}\right)^{\mathsf{d}}$. Now there are various ways to interpolate $\tilde{\psi}_N$ to a function defined on all of D, and one can wonder whether for the right choice of α these random functions on D have a scaling limit, and if so, in which topologies the convergence to the scaling limit holds.

For the gradient model the answer is straightforward. The correct choice for α here is $\alpha = \frac{\mathsf{d}-2}{2}$, and the scaling limit is the continuum Gaussian free field that was already mentioned in Section 1.2.4. In the subcritical dimension $\mathsf{d} = 1$ the continuum Gaussian free field is nothing else than Brownian motion. As D is bounded and connected, it is an interval. The gradient model with zero boundary data on Λ is just a random walk bridge, and it is well-known that under rescaling with factor $N^{-\frac{1}{2}}$ and piecewise linear interpolation this random walk bridge converges to a Brownian bridge on D, where the convergence takes place in the Hölder spaces $C^{0,\gamma}$ for any $\gamma < \frac{1}{2}$. In the critical and supercritical dimensions the convergence takes place in negative Sobolev spaces. We interpolate $\tilde{\psi}_N$ in some reasonable way (e.g. piecewise affinely on a triangulation subordinate to $\mathbb{Z}^{\mathsf{d}}$). Then the interpolated fields converge in $H^{-s}(D)$ for any $s > \frac{\mathsf{d}-2}{2}$, where $H^{-s}(D)$ is the dual space of the Hilbert space $H_0^s(D)$ [She07]. It is also possible to prove analogous results for non-zero boundary conditions.

The $\nabla\varphi$-model with strictly convex $\mathcal{V}$ behaves similarly to the gradient model. In particular, we still take $\alpha = \frac{\mathsf{d}-2}{2}$, and the scaling limit is still (a scalar multiple of) the continuum Gaussian free field. The main focus in the literature has been to show the convergence in $\mathcal{D}'$, the space of distributions. This was first shown in [NS97]. Whether one can upgrade this to convergence in law in some negative Sobolev space then depends on the precise assumptions made on $\mathcal{V}$. In [GOS01] it is shown that under fairly general assumptions the convergence holds in $H^{-s}(D)$ for any $s > \mathsf{d}+1$. The situation with nonzero boundary conditions was investigated in [Mil11].

For the $\nabla\varphi$-model with slightly non-convex $\mathcal{V}$ one still has convergence to the continuum Gaussian free field [Hil16, ABKM19]. Somewhat surprisingly, this even holds for some very non-convex $\mathcal{V}$, at least in the zero-boundary case [BS11].

For the membrane model one observes a different scaling limit, namely the continuum membrane model that was also already mentioned in Section 1.2.4, and the correct choice for α turns out to be $\alpha = \frac{\mathsf{d}-4}{2}$. Other than that, the situation is very similar as for the gradient model: provided that one chooses a sufficiently smooth interpolation, the convergence holds in the subcritical dimensions in the Hölder space $C^{\lfloor\gamma\rfloor,\{\gamma\}}(D)$ for any $\gamma < \frac{4-\mathsf{d}}{2}$. This result can be found in [CDH19], with a crucial ingredient being the estimates in Chapter 2 of this thesis. In the critical and supercritical case the convergence holds in some negative Sobolev space $H^{-s}(D)$. The published result [CDH19] allows $s > s_*$ for some $s_* \approx \frac{7}{8}\mathsf{d}$, but it should be possible to improve their result to the optimal $s > \frac{\mathsf{d}-4}{2}$.

1.3.2 Extrema of the field

This section includes parts of the introduction of the author's paper [Sch20a].

Existing results

We can now turn to discuss finer properties of the random interface models. The first such property is the behaviour of the extrema of the field. All of our models are invariant under

reflection around 0, and so we only need to consider the maximum. We always consider the field on the domain $\Lambda_N = [0, N]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}}$. For $\Phi \in \{\nabla, \mathcal{V}(\nabla), \Delta\}$ we denote a sample from $\mathbb{P}_{\Phi,\Lambda_N}$ by ψ_N^Φ, and let $M_N^\Phi = \max_{x\in\Lambda_N} \psi_x$. We are interested in the asymptotics of the random variables M_N^Φ as $N \to \infty$. The answer here depends very much on whether the dimension is subcritical, critical or supercritical, and so we will discuss those cases separately.

In the subcritical dimensions ($\mathsf{d} = 1$ for the gradient model and the $\nabla\varphi$-models, $\mathsf{d} \leq 3$ for the membrane model) we have that $\frac{M_N^\nabla}{N^{\frac{2-\mathsf{d}}{2}}}$, $\frac{M_N^{\mathcal{V}(\nabla)}}{N^{\frac{2-\mathsf{d}}{2}}}$ and $\frac{M_N^\Delta}{N^{\frac{4-\mathsf{d}}{2}}}$ converge in distribution, which follows from the fact that the whole rescaled field converges weakly in C^0, as discussed in the previous section.

In the supercritical dimensions ($\mathsf{d} \geq 3$ for the gradient model and the $\nabla\varphi$-models, $\mathsf{d} \geq 5$ for the membrane model) the correlations decay rapidly, so one can expect that the maximum of the field behaves as if the ψ_N^Φ were independent. In the cases of the gradient model and the membrane model this can be made rigorous using Stein's method. Thereby it was shown in [CCH16b, CCH16a] that M_N^∇ behaves as if the $(\psi_{N,x}^\nabla)_{x\in V_N}$ were independent, i.e. that

$$\frac{\sqrt{2\mathsf{d}\log N}}{\sqrt{g_{\mathsf{d}}^\nabla}} \left(M_N^\nabla - \sqrt{2\mathsf{d} g_{\mathsf{d}}^\nabla \log N} + \frac{\sqrt{g_{\mathsf{d}}^\nabla}\,(\log(\mathsf{d}\log N) + \log 4\pi)}{\sqrt{8\mathsf{d}\log N}} \right)$$

converges in distribution to a Gumbel random variable, where x_N is a lattice point closest to the centre of $[0, N]^{\mathsf{d}}$ and $g_{\mathsf{d}}^\nabla = \lim_{N\to\infty} \mathrm{Var}(\psi_{N,x_N}^\nabla)$. The analogous statement holds true for M_N^Δ. It is likely that a similar result also holds for the $\nabla\varphi$-model with strictly convex φ, but this has not been rigorously shown yet (cf. [CCH16a, Remark 3]).

The most interesting and most subtle case is the critical one ($\mathsf{d} = 2$ for the gradient model and the $\nabla\varphi$-models, $\mathsf{d} = 4$ for the membrane model). For the gradient model, in a series of papers making successive improvements [BDG01, BDZ11, BZ12, BDZ16] it was shown that $M_N^\nabla - m_N^\nabla$ converges in distribution to a randomly shifted Gumbel variable, where $m_N^\nabla = \sqrt{\frac{2}{\pi}} \log N - \frac{3}{\sqrt{32\pi}} \log\log N$. For the $\nabla\varphi$-model with strictly convex φ convergence in law of the maximum is a challenging open problem, but it is known that $\frac{M_N^{\mathcal{V}(\nabla)}}{\log N}$ converges in probability [BW20] and that there is a deterministic subsequence $(N_k)_{k=0}^\infty$ along which $M_{N_k}^{\mathcal{V}(\nabla)} - \mathbb{E}_{\mathcal{V}(\nabla),\Lambda_N} M_{N_k}^{\mathcal{V}(\nabla)}$ is tight [WZ19]. For the membrane model previously there were only partial results. The best result [Kur09] is that $\frac{M_N^\Delta}{\log N}$ converges to $\frac{1}{\pi}$ in probability. The question whether a centred version of M_N^Δ converges in distribution was posed for example in [Roy16, CDH19]. In Chapter 4 (that is based on the publication [Sch20a]) we prove that this is the case, i.e. that $M_N^\Delta - M_N^\Delta$ converges to a randomly shifted Gumbel variable, where $m_N^\Delta = \frac{1}{\pi} \log N - \frac{3}{16\pi} \log\log N$.

The maximum of the critical membrane model

Let us give a few more details on the result of Chapter 4 on the maximum of the membrane model in dimension $\mathsf{d} = 4$. The precise result will be the following:

Theorem 1.3.1. *Let* $\mathsf{d} = 4$. *The random variable*

$$M_N^\Delta - m_N^\Delta := M_N^\Delta - \frac{1}{\pi} \log N + \frac{3}{16\pi} \log\log N$$

converges in distribution. The limit law is a randomly shifted Gumbel distribution μ_∞, given by

$$\mu_\infty((-\infty, t]) = \mathbb{E}e^{-\gamma^* \mathcal{Z} e^{-8\pi t}} \ \forall t$$

where γ^* is a constant and $\mathcal{Z}$ is a positive random variable that is the limit in law of

$$\mathcal{Z}_N = \sqrt{8} \sum_{x \in V_N} (\log N - \pi \psi_{N,x}) e^{-8(\log N - \pi \psi_{N,x})} .$$

For this result it is very important that the critical membrane model is a log-correlated Gaussian field (cf. Section 1.2.5). As discussed there, it is conjectured that these form a universality class. One example of a feature that is conjectured to be universal is the behaviour of the maximum of the field, and one expects that convergence in law of the recentred maximum holds true for general log-correlated fields. However, it is a challenging problem to verify this fact for specific examples of log-correlated fields. In recent years convergence in law of the recentred maximum has been proven for the critical gradient model, as already discussed, and also for various other models. Let us mention branching Brownian motion [Bra83], branching random walks [Aïd13], and also problems from random matrix theory (see [CMN18] for partial results).

Furthermore, there have been efforts to give sufficient criteria for convergence in law of the maximum that cover a wide range of log-correlated fields. In [Mad15] this was done for so-called $*$-scale invariant models. Most importantly for us, in [DRZ17] Ding, Roy and Zeitouni gave some sufficient conditions on the covariances that ensure that the maximum of the field converges in distribution. Their approach is based on a very subtle comparison of the interface with a modified branched random walk. The result from [DRZ17] reduces the proof of Theorem 1.3.1 to the verification of certain estimates on the Green's function of the discrete Bilaplacian. We discuss these in Section 1.4.3.

1.3.3 Entropic repulsion

This section includes parts of the introduction of the paper [BDKS19], written jointly by Simon Buchholz, Jean-Dominique Deuschel, Noemi Kurt and the author.

Existing results

In this and the following sections we will consider the effect of various single-spin potentials to the boundary. We begin with the phenomenon of entropic repulsion. That is, we restrict the interface to be non-negative on some subset of the domain Λ. In physics, this corresponds to the presence of a hard wall that the interface cannot cross. This hard wall leads to a competition between energetic and entropic effects: on the one hand it is energetically favourable for the interface to stay flat and thus close to the hard wall, on the other hand the hard wall severely limits the possible fluctuations, so that it is entropically advantageous for the interface to keep a distance from the wall. Therefore, there will be some repulsive effect of the wall, that is, its local averages will increase. We speak of entropic repulsion if the order by which the field increases is strictly larger than the order of the square root of the variances of the original field, [LM87, Gia01]. See Figure 1.3 for a sample of the membrane model under entropic repulsion, in particular in comparison to Figure 1.1a. This and related problems have also been studied in the physics literature, e.g. in [HL97].

This leads to the question how big this repulsive effect is for our models of interest, and whether there is entropic repulsion in the sense just mentioned. As in the previous

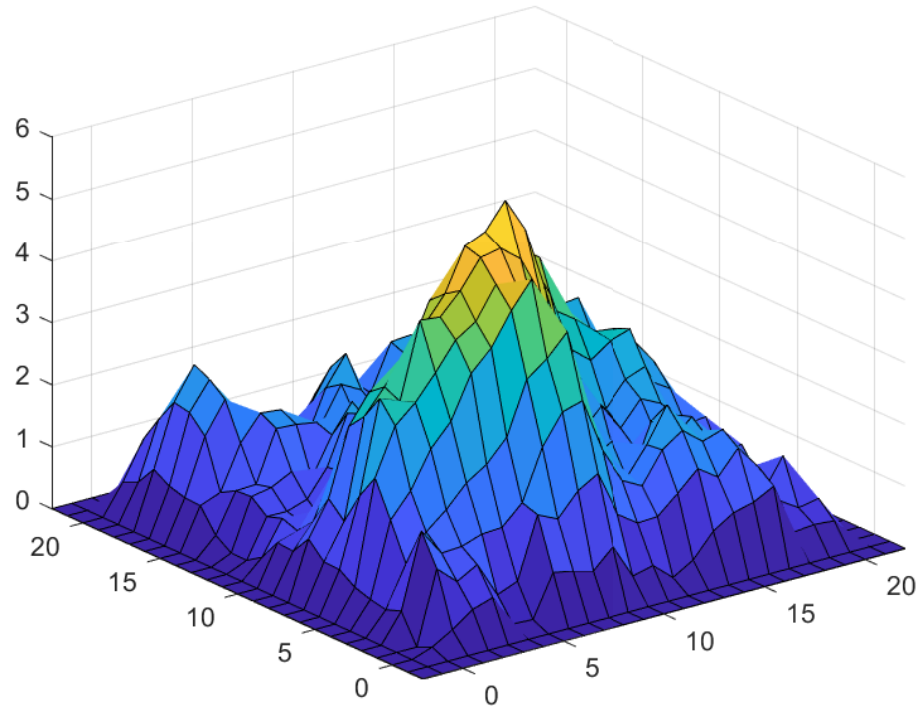


Figure 1.3: A sample of the membrane model in dimension $\mathsf{d} = 2$ on the domain $\{0, \dots, 20\}^{\mathsf{d}}$ under entropic repulsion on all of the domain. See Section 1.2.7 for a description how the sample was generated.

section, the answer strongly depends on whether the dimension is subcritical, critical or supercritical. In fact, it is known (or conjectured) that entropic repulsion happens if and only if the dimension is critical or supercritical.

We consider the field on the domain $\Lambda_N = [0, N]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}}$, and for some $D \subset [0,1]^{\mathsf{d}}$ we let $D_N = ND \cap \mathbb{Z}^{\mathsf{d}}$. We consider the event $\Omega_{D_N,+} = \{\psi \colon \psi_x \geq 0 \ \forall x \in D_N\}$, and we are interested in the behaviour of the fields when conditioned on being nonnegative on D_N. A first step to understand that behaviour will be to estimate the probability of $\Omega_{D_N,+}$. We focus on the two cases $D \Subset [0,1]^{\mathsf{d}}$ (i.e. D is compactly contained in $[0,1]^{\mathsf{d}}$), or $D = [0,1]^{\mathsf{d}}$.

In the critical and supercritical dimensions there are very precise results known for the Gaussian free field. Namely, if $D \Subset [0,1]^{\mathsf{d}}$, then the probability $\mathbb{P}_{\nabla,\Lambda_N}(\Omega_{D_N,+})$ scales like $\exp(-C_{\mathsf{d},D} N^{\mathsf{d}-2} \log N)$ if $\mathsf{d} \geq 3$, and like $\exp(-C_{\mathsf{d},D}(\log N)^2)$ if $\mathsf{d} = 2$, while the field is repelled to a height $C_{\mathsf{d},D}\sqrt{\log N}$ if $\mathsf{d} \geq 3$, and to a height $C_{\mathsf{d},D} \log N$ if $\mathsf{d} = 2$. Here the constants are explicitly known and they depend on d and on the capacity of D with respect to $[0,1]^{\mathsf{d}}$. If $D = [0,1]^{\mathsf{d}}$, then $\mathbb{P}_{\nabla,\Lambda_N}(\Omega_{D_N,+})$ scales like $\exp(-C_{\mathsf{d},D} N^{\mathsf{d}-1})$ for any $\mathsf{d} \geq 2$ (this is a boundary effect), while the field is repelled to a height of the same order as before. These results are due to [BDZ95, Deu96, BDG01]. Similar, but somewhat weaker results (namely with upper and lower bounds on the rates that differ by a constant factor) are known for the $\nabla\varphi$-interface model with strictly convex $\mathcal{V}$ [DG00]. For the membrane model only the case that $D \Subset [0,1]^{\mathsf{d}}$ has been studied. There one finds that $\mathbb{P}_{\Delta,\Lambda_N}(\Omega_{D_N,+})$ scales like $\exp(-C_{\mathsf{d},D} N^{\mathsf{d}-4} \log N)$ or $\exp(-C_{\mathsf{d},D}(\log N)^2)$ for $\mathsf{d} \geq 5$ and $\mathsf{d} = 4$, respectively, while the field is repelled to heights $C_{\mathsf{d},D}\sqrt{\log N}$ or $C_{\mathsf{d},D} \log N$, respectively [Sak03, Kur07, Kur09]. In all cases, the order of the height to which the field is repelled is larger than the square

root of the variance of the unperturbed field, and so there actually is entropic repulsion in the sense defined above.

The subcritical case is rather different. For the gradient model in dimension $\mathsf{d} = 1$ we actually observe a simple random walk bridge conditioned to be positive on some part of its domain. If $D \Subset [0,1]^{\mathsf{d}}$ this random walk has as its scaling limit a Brownian bridge conditioned to be positive on D, while if $D = [0,1]$ the scaling limit is the Brownian excursion (cf. e.g. [CC13]). In particular, $\mathbb{P}_{\nabla,\Lambda_N}(\Omega_{D_N,+}) \geq c_D$ if $D \subset [0,1]$, while $\mathbb{P}_{\nabla,\Lambda_N}(\Omega_{D_N,+})$ scales like $\frac{c_D}{N}$ (as can easily be shown using the reflection principle). In both cases the field is repelled to the height $c_D\sqrt{N}$. These statements are also still valid for the $\nabla\varphi$-interface model with strictly convex $\mathcal{V}$, as can be seen using renewal methods, cf. e.g. [FO01, Gia07]. For the membrane model in dimension $\mathsf{d} = 1$ the scaling limit is an integrated Brownian bridge, and we expect that $\mathbb{P}_{\nabla,\Lambda_N}(\Omega_{D_N,+}) \geq c_D$ if $D \subset [0,1]$, while $\mathbb{P}_{\nabla,\Lambda_N}(\Omega_{D_N,+})$ scales like $\frac{c_D}{\sqrt{N}}$, and the field is repelled to the height $c_D N^{\frac{3}{2}}$. This has rigorously been shown only for the one-sided problem [DW15] (see also the earlier results [Sin92, DDG13]), but the method should carry over.

Of course, for the membrane model $\mathsf{d} = 1$ is not the only subcritical dimension, and so we are left to discuss what happens when $\mathsf{d} \in \{2,3\}$. It is likely that just as in the other subcritical cases there is no entropic repulsion here, i.e. the field is repelled only to the height $N^{\frac{4-\mathsf{d}}{2}}$. Moreover, $\mathbb{P}_{\nabla,\Lambda_N}(\Omega_{D_N,+})$ should be bounded below for any $D \Subset [0,1]^{\mathsf{d}}$, while it should decay at a surface rate if $D = [0,1]^{\mathsf{d}}$. The only previous rigorous result on this topic, however, is [Sak16], where it is shown that for sufficiently small D we have $\mathbb{P}_{\nabla,\Lambda_N}(\Omega_{D_N,+}) \geq c_D$. In Chapter 3 (that is based on the publication [BDKS19]) we give a significant improvement and prove that $\mathbb{P}_{\nabla,\Lambda_N}(\Omega_{D_N,+})$ behaves as expected. Unfortunately, we only have partial results on the behaviour of the field when conditioned on $\Omega_{D_N,+}$.

Probability to be positive for the subcritical membrane model

We will now give a few more details on the results of Chapter 3. There we prove the following result.

Theorem 1.3.2. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$. *For* $\delta \in (0,1)$ *there is a constant* $c_\delta > 0$ *such that*

$$c_\delta \leq \mathbb{P}_{\Delta,N}(\Omega_{\Lambda_{\delta N},+}) \leq \frac{1}{2}.$$

Moreover,

$$\exp(-CN^{\mathsf{d}-1}) \leq \mathbb{P}_{\Delta,N}\left(\Omega_{\Lambda_N,+}\right) \leq \exp(-cN^{\mathsf{d}-1})$$

We can even prove a result interpolating between the two estimates above, i.e. we can take $D_N = \Lambda_{N,L_N}$ for some L_N depending on N (see Theorem 3.1.1). Theorem 1.3.2 easily implies that the field still has a scaling limit in some Hölder space when conditioned on being positive on $\Omega_{\Lambda_{\delta N},+}$ for $\delta < 1$. However, this is a soft argument that relies on the lower bound on $\mathbb{P}_N(\Omega_{\Lambda_{\delta N},+})$ being uniform in N. In the case $D = [0,1]^{\mathsf{d}}$ the probability $\mathbb{P}_N(\Omega_{\Lambda_{\delta N},+})$ is exponentially small in N, and so it is difficult to analyse what happens when one conditions on that event. For this we do not know yet how to proceed.

A crucial ingredient for the proof are estimates for the Green's function of the discrete Bilaplacian and its derivatives that are sharp up to the boundary. These estimates are shown in Chapter 2 (that is based on [MS19]), and we will give an outline in Section 1.4. Here we focus on the probabilistic aspects of the proof of Theorem 1.3.2.

The upper bound in Theorem 1.3.2 follows easily from the estimates on the Green's function. Namely, it turns out that the correlations at sites close to the boundary decay rapidly, so we can take a sparse subset of cardinality $\geq c_{\mathsf{d}} N^{\mathsf{d}-1}$ where the correlations are very small. In fact, the correlations will be so small that we can use a Gaussian comparison lemma from [LS04] to compare to the situation where the heights at our subsets are independent, and so the upper bound immediately follows.

For the lower bound we use the fact that the field is Hölder continuous up to the boundary, with a random Hölder constant for which we have tail bounds. This means that if the field is positive at a certain site x, it is positive in a neighbourhood of x with a decent probability as well. Unfortunately, in the absence of the FKG inequality there is no direct way to patch these local results together. To solve this problem, note that we can also conclude from the Hölder continuity that the field is locally small with a decent probability, and these results we can patch together using the Gaussian correlation inequality. This is not yet the result we were looking for, but one can use a change of measure argument to bound the probability that the field is close to any given macroscopic profile from below. For a sufficiently positive profile this then implies the result.

1.3.4 Pinning

This section includes parts of the introduction of the author's preprint [Sch20b].

Existing results

In the previous section we discussed the effect of a hard wall that repels the field from 0. In this section we will do the opposite, namely consider the effect of a small attractive potential. So we add a small attractive potential that rewards the field for being equal to (or close to) 0. It is clear that this should pull the interface closer to 0, and one can explore to what extent this effect happens. The physical motivation for this is mainly that it serves as a stepping stone for understanding the phenomenon of wetting, where one considers the competition between pinning and entropic repulsion. We shall discuss that problem in the next section, and keep our focus on pinning here.

Various pinning potentials have been considered in the literature. We restrict ourselves to the mathematically easiest one, namely $\varepsilon\delta_0$ with δ_0 a point-mass at 0. That is, we consider the probability measures

$$\mathbb{P}^{\varepsilon}_{\Phi,\Lambda}(\mathrm{d}\psi) = \frac{1}{Z^{\varepsilon}_{\Phi,\Lambda}} \exp(-H_\Lambda(\psi)) \prod_{x\in\Lambda} (\lambda(\mathrm{d}\psi_x) + \varepsilon\delta_0(\mathrm{d}\psi_x)) \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus\Lambda} \delta_0(\mathrm{d}\psi_x) \tag{1.3.1}$$

for some $\varepsilon > 0$.

If λ is the Lebesgue measure on $\mathbb{R}$, we cannot understand this as a Gibbs measure with a priori measure λ (as δ_0 is singular with respect to the Lebesgue measure), but the formalism from Section 1.2.3 works well if we directly take $\lambda + \varepsilon\delta_0$ as the a priori measure.

The first question about pinning is whether the additional effect of the pinning measure is strong enough to actually localize the field. In the supercritical dimensions the variances are bounded already without any pinning, so one can expect that the field stays bounded when pinned. In the critical and subcritical dimensions the answer is less easy to guess. See Figure 1.4 for a sample of the pinned membrane model in comparison to Figure 1.1c.

To give a rigorous answer we first have to define what we mean by localization, or being pinned. For each sample of $\mathbb{P}^{\varepsilon}_{\Phi,\Lambda}$ there will be some $x \in \Lambda$ with $\psi_x = 0$ which we call the

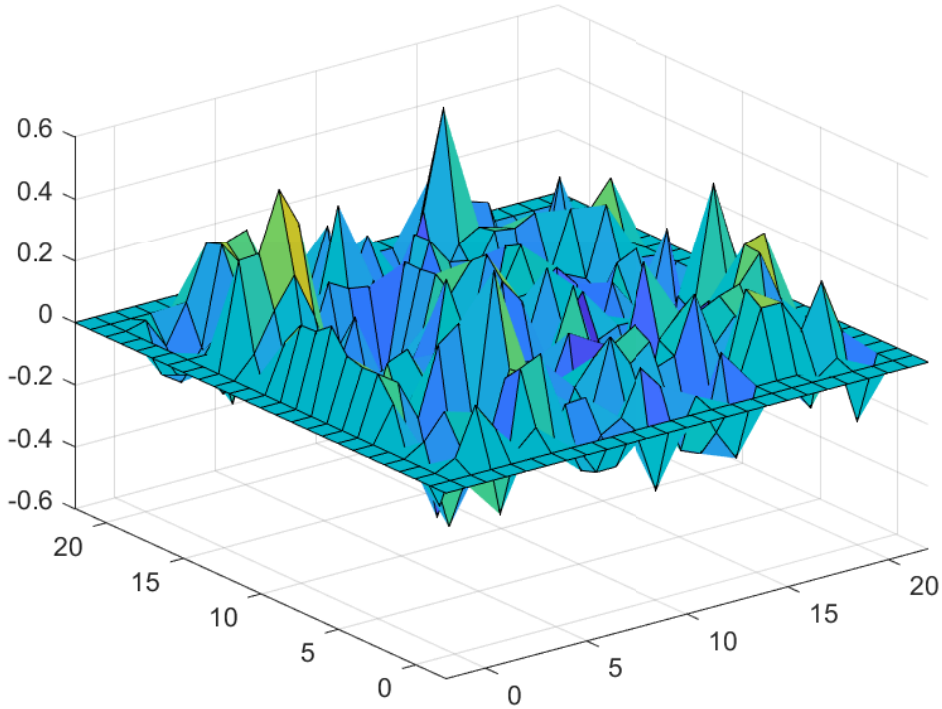


Figure 1.4: A sample of the membrane model in dimension $\mathsf{d} = 4$ on the domain $\{0, \dots, 20\}^{\mathsf{d}}$, pinned with pinning strength $\varepsilon = 1$. The picture shows the values of the sample on the slice $\{0, \dots, 20\}^2 \times \{10\}^2$. See Section 1.2.7 for a description how the sample was generated.

pinned points. We call $\mathbb{P}^{\varepsilon}_{\Phi,\Lambda}$ pinned if the expected fraction of points in Λ that are pinned is bounded below uniformly in Λ. One easily checks that being pinned is a monotonic property in ε, so there will some critical value $\varepsilon_{\text{pin},*} \in [0, \infty]$ such that the field is pinned if $\varepsilon > \varepsilon_{\text{pin},*}$ but not if $\varepsilon < \varepsilon_{\text{pin},*}$. Now for the gradient model and the $\nabla\varphi$-interface models with strictly convex $\mathcal{V}$ it turns out that the field is always pinned, i.e. $\varepsilon_{\text{pin},*} = 0$ in any dimension. For $\mathsf{d} = 1$ this follows again from renewal theory methods as in [Gia07], for $\mathsf{d} = 2$ this follows from [DMRR92, DV00], and the case $\mathsf{d} \geq 3$ is almost trivial. The situation is more exciting for the membrane model. We still have $\varepsilon_{\text{pin},*} = 0$ for $\mathsf{d} \geq 2$ [Sak12, Sak18], but somewhat surprisingly $0 < \varepsilon_{\text{pin},*} < \infty$ if $\mathsf{d} = 1$ [CD08].

Thus, in most cases the pinning effect manages to localize the field in the sense that it touches the 0-plane on a positive fraction of Λ. It is natural to ask whether this localization also manifests itself in some other ways. In particular, it is expected that the variance of the pinned field is bounded, and the covariance decays exponentially in the distance (i.e. a mass is generated). Physically speaking, this corresponds to a finite transverse and longitudinal correlation length, respectively. For the case of the gradient model, this was studied in [BB01, IV00, BDG01] in $\mathsf{d} \geq 2$. There not only finiteness of the variance and existence of a mass is known, but even the ε-asymptotic of these quantities for small ε. The former result is also known for $\nabla\varphi$-interface models with strictly convex $\mathcal{V}$ [DV00]. For $\mathsf{d} = 1$ and both models there are even better results [Gia07, Gia08].

For the membrane model very refined results are known if $\mathsf{d} = 1$ [CD08, CD09]. These include boundedness of the variance, and should easily imply exponential decay of the

covariances. It is also known that in the supercritical dimensions $\mathsf{d} \geq 5$ (where variances are trivially bounded) one has a positive mass [BCK17]. Other than that the only previous result is that in the critical dimension $\mathsf{d} = 4$ the correlations decay stretched-exponentially [BCK16]. In Chapter 6 that is based on the preprint [Sch20b] we improve these results by showing that the variances are bounded and the covariances decay exponentially (i.e. the mass is positive) also if $\mathsf{d} = 4$. We also give asymptotics for these quantities if $\mathsf{d} \geq 4$. This leaves open the cases $\mathsf{d} \in \{2, 3\}$. Already in [BCK17] the authors wondered whether exponential decay of correlations also holds in that dimensions. This seems likely, but we do not know how to prove this.

Pinning for the critical and supercritical membrane model

There are quite a few new results in Chapter 6. The following theorem summarizes the most important ones.

Theorem 1.3.3. *Let $\mathsf{d} \geq 4$, and $x \in \Lambda \Subset \mathbb{Z}^{\mathsf{d}}$. If ε is sufficiently small, the variances of the pinned field satisfy*

$$c_{\mathsf{d}} \leq \mathbb{E}^{\varepsilon}_{\Delta,\Lambda}(\psi_x^2) \leq C_{\mathsf{d}}$$

if $\mathsf{d} \geq 5$, and

$$\frac{|\log \varepsilon|}{32\pi^2} - C_4 \log|\log \varepsilon| \leq \mathbb{E}^{\varepsilon}_{\Delta,\Lambda}(\psi_x^2) \leq \frac{|\log \varepsilon|}{16\pi^2} + C_4 \log|\log \varepsilon|$$

if $\mathsf{d} \geq 4$ and x is sufficiently far from the boundary.

We also have the lower bounds for the mass m_ε (i.e. exponential rate of decay of the covariances)

$$c_{\mathsf{d}} \varepsilon^{1/4} \leq m_\varepsilon$$

if $\mathsf{d} \geq 5$, and

$$c_4 \frac{\varepsilon^{1/4}}{|\log \varepsilon|^{3/8}} \leq m_\varepsilon$$

if $\mathsf{d} = 4$.

Moreover for every $\varepsilon \geq 0$ the fields have a unique thermodynamic limit as $\Lambda \nearrow \mathbb{Z}^{\mathsf{d}}$.

The first step to establish the results of Theorem 1.3.3 is to understand the set of pinned points. The first important observation of Chapter 6 is that this set is positively correlated (as follows from the Gaussian correlation inequality). The heuristic is that this set behaves like a Bernoulli point process with a certain density. This is true in a rather strong sense if $\mathsf{d} \geq 5$, and still true in a weakened sense if $\mathsf{d} = 4$ (as we show following a two-scale argument from [BDG01]). If $\mathsf{d} \leq 3$ this breaks down completely, however, and so the arguments have no chance to work in that case.

Having established these estimates on the set of pinned points, the estimates on the variance follow easily. The estimates on the covariance are much more complicated. This problem resembles the classical problem of the homogenization of elliptic PDEs in perforated domains (see [CM97]), and the arguments are inspired from this connection. We proceed by using from [BCK17] the idea to use a Widman hole filler argument [Wid71] on random annuli. The details, however, are rather different. We use a multipolar Hardy-Rellich inequality for second derivatives (inspired by similar inequalities for first derivatives as e.g. in [CZ13]) to estimate the local effect of the pinned points. We also use a rather subtle multiscale construction to construct the required cut-off functions, and to prove that this construction can be done with sufficiently high probability. This is the most technical part of the chapter, and it is novel to the best of our knowledge.

1.3.5 Wetting

As already briefly mentioned in the previous section, one can consider a competition between the effects of a hard wall and an attractive potential. This problem is known as wetting. It should be plausible that, depending on the strength of the attractive potential, one or the other effect might win. That is, the interface could be repelled far away from the boundary or the interface could stay close to the wall and touch it at many points (the two phases are called dry/wet [Gia01] or partially wetted/wetted [Vel06] in the literature; we stick to the former).

The physical interpretation of this problem explains the terminology. Namely, this model arises when analyzing the coexistence of a liquid and a gas in a domain, where the liquid prefers to stick to the boundary of the domain due to certain molecular forces. Our height function then describes the interface between the liquid and the gas. It could happen that there are only a few drops of liquid at the wall with the rest of the wall being dry, or that the whole wall is covered by a liquid film. Which of the two phenomena occurs depends on the amount of liquid present (which itself depends on the attractive forces of the wall). See e.g. [Lip01] and the references therein for biophysical work on this problem.

Another physical interpretation arises when considering a biomembrane contained in a domain that experiences some attractive forces close to the wall of the domain, while entropic effects tend to keep it away from the wall, cf. [Lip95].

Mathematically, wetting consists in the analysis of the measure (1.3.1) conditioned on the event $\Omega_{\Lambda,+} = \{\psi\colon \psi_x \geq 0\ \forall x \in \Lambda\}$. We call those sites where $\psi_x = 0$ under the conditioned measure the dry sets, and define the field to be dry when the expected value of the fraction of points in Λ that are dry is positive (and otherwise wet). As for pinning, one can argue that there is a critical value $\varepsilon_{\mathrm{wet},*} \in [0,\infty]$ such that the field is wet if $\varepsilon > \varepsilon_{\mathrm{wet},*}$ but not if $\varepsilon < \varepsilon_{\mathrm{wet},*}$. Now one can investigate whether $\varepsilon_{\mathrm{wet},*}$ is nontrivial and how the field behaves in the dry and wet phases.

There are only a few rigorous results on this problem: For the gradient model it is known that $\varepsilon_{\mathrm{wet},*} = 0$ if $\mathsf{d} \geq 3$ [BDZ00], while $\varepsilon_{\mathrm{wet},*} > 0$ if $\mathsf{d} \leq 2$ [CV00]. It is unknown whether the same holds for the $\nabla\varphi$-interface model other than in the case $\mathsf{d} = 1$ [HV04]. Similarly for the membrane model it is only known that $\varepsilon_{\mathrm{pin},*} < \varepsilon_{\mathrm{wet},*} < \infty$ in $\mathsf{d} = 1$ [CD08]. The only pathwise results in the literature are in [Vel04] where the main result is an estimate of the typical height of the wet interface.

As mentioned, for the membrane model it is not even known whether $\varepsilon_{\mathrm{wet},*} > 0$ for some $\mathsf{d} \geq 2$, although in analogy with the gradient model one can conjecture that $\varepsilon_{\mathrm{wet},*} > 0$ if and only if $\mathsf{d} \leq 4$. Maybe a combination of the results in [Kur09] on pure entropic repulsion combined with the results of Chapter 6 on pure pinning can help shed light on this question.

1.3.6 Further aspects and open questions

There are many more interesting questions about random interface models that one can study. Let us describe three of them where there is active research right now.

Near-extrema and thick points

In view of the results on the maximum of the fields in Section 1.3.2, it is natural to wonder whether one can say more about the behaviour of the field at or near its maximum. This question is interesting mainly in the critical case. For the case of the gradient model, this has

been thoroughly addressed in [BL16, BL18, BL20]. There, convergence of the full extremal process (encoding height, location and local neighbourhoods of near-maxima) to some limiting process is shown, and the law of the locations of those near-maxima is identified as critical Liouville quantum gravity (cf. [Ber15] for an introduction).

One can also study the so-called λ-thick points of the field, i.e. those points whose height is approximately λ times the height of the maximum. This was done in [BL19], where the authors show that the locations of the λ-thick points converge to subcritical Liouville quantum gravity . This refines an earlier result, [Dav06] finding the Hausdorff dimension of that set.

Analogous results for the $\nabla\varphi$-model with strictly convex $\mathcal{V}$ seem out of reach, as one does not even know the precise height of the maximum yet. For the membrane model there is more hope in view of the results of Chapter 4. However, some of the results for the gradient model that we have just mentioned rely on its conformal invariance and thus on some special properties of the two-dimensional space, and so these probably have no replacement for the membrane model. Let us mention, though, that the results on the Hausdorff dimension of the set of thick points have already been adapted to the case of the membrane model in [Cip13].

Level surfaces

Another natural question for the random interface models is how their level surfaces look like. We focus on the zero contour, i.e. the set where the (suitably interpolated) interface intersects the 0-plane. This is mainly interesting in the critical case. For the gradient model and the $\nabla\varphi$-model with strictly convex $\mathcal{V}$ in $\mathsf{d} = 2$ it turns out that the scaling limit of the contour surfaces (which then are contour lines) is the conformal loop ensemble $CLE(4)$, a variant of the Schramm-Loewner evolution. This was shown in [SS09, Mil10].

For the membrane model in $\mathsf{d} = 4$ the analogous question is very interesting, but probably also extremely difficult. In the absence of conformal invariance it is not even heuristically clear what the scaling limit might be.

Level set percolation

A random interface model also gives rises to a strongly correlated percolation model. This is most natural in the supercritical dimensions, as then there is an infinite volume limit of the field without rescaling. One can then consider the interface on all of $\mathbb{Z}^{\mathsf{d}}$, and for some $t \in \mathbb{R}$ study the set $E_t = \{x \colon \psi_x \geq t\}$, and in particular its percolative properties. Then one can define various critical values for t. The most natural of them, t_*, is such that E_t contains almost surely an infinite connected component if $t < t_*$, but not if $t > t_*$. For the gradient model it is known that $0 < t_* < \infty$ for all $\mathsf{d} \geq 3$ [BLM87, RS13, DPR18]. This means in particular that both the set where the gradient model is positive and its complement contain an infinite connected component. This needs to be contrasted with Bernoulli percolation, where this coexistence is almost surely impossible.

In a recent breakthrough [DCGRS20] it was shown that t_* agrees with a variety of other critical values, and so there are clearly defined subcritical and supercritical phases for percolation.

As a link to Section 1.3.3, one can consider the interface conditioned on the event that for some $D \Subset [0,1]^{\mathsf{d}}$ the sets $D_N = ND \cap \mathbb{Z}^{\mathsf{d}}$ and $Z^{\mathsf{d}} \setminus [0,N]^{\mathsf{d}}$ are not connected in E_t. For $t < t_*$ this is an unlikely event, and conditioning on it will lead to a repulsion of the interface. For

the gradient model, this problem was studied in [CN20], using the result from [DCGRS20] to derive sharp estimates for the resulting entropic repulsion.

It would be very interesting to see whether one can transfer these results to the $\nabla\varphi$-model for strictly convex $\mathcal{V}$ or to the membrane model. For the former, a first result can be found in [Rod16].

1.4 Discrete Green's functions and finite difference schemes

We will now turn to the second important topic of this thesis, namely the discrete Bilaplace equation and its Green's function. We first provide some context and discuss the continuous counterparts of these objects, and then we turn to the new results of this thesis.

1.4.1 Continuous elliptic partial differential equations and Green's functions

Equations in smooth domains

Elliptic partial differential equations are a class of partial differential equations that generalize Poisson's equation. There exists a vast amount of theory on this subject, and we will just briefly mention a few notions important to us. The following results are classical (see e.g. [LM72a, Gia83]). We focus on constant coefficient operators $L = \sum_{|\alpha|\leq 2m} a_\alpha \partial^\alpha$. Such an operator is called elliptic if its principal symbol is invertible, i.e. if $\sum_{|\alpha|=2m} a_\alpha \xi^\alpha \neq 0$ for any $\xi \in \mathbb{R}^d \setminus \{0\}$, where we use the usual multi-index notation. Given a domain $\Omega \subset \mathbb{R}^d$ with C^{m-1}-boundary, one can then consider the boundary value problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega, \\ \partial_\nu^k u &= 0 \quad \text{on } \partial\Omega \quad \forall 0 \leq k \leq m-1 \end{aligned} \tag{1.4.1}$$

Elliptic regularity theory implies that, informally speaking, u is better than f by $2m$ derivatives. That is, there are interior estimates of the form $\|\nabla^{2m} u\|_X \leq C\|f\|_X$ for a variety of function spaces X. Such estimates hold locally in the interior of Ω in any case, and if the domain Ω has a sufficiently smooth boundary, they extend to global estimates. In particular, the equation is uniquely solvable for $f \in X$.

Under the stated assumptions there also exists a Green's function G for L. Formally $G(\cdot, y)$ is the solution of (1.4.1) with $f = \delta_y$, the δ-distribution at $y \in \Omega$. One can show that this is a well-defined function on $\Omega \times \Omega \setminus \{(x,x) \colon x \in \Omega\}$, and that for $f \in L^2(\Omega)$ one can represent the solution of (1.4.1) as

$$u = \int_\Omega G(\cdot, y) f(y)\, \mathrm{d}y$$

This should make it obvious that the Green's function G is closely linked to the elliptic operator L, and that it is important to understand the behaviour of G to analyse the elliptic boundary value problem associated to L. In particular, understanding the behaviour of G near its singularity at the diagonal can lead via the theory of singular integral operators to regularity estimates for the boundary value problem (1.4.1).

In the full space (i.e. $\Omega = \mathbb{R}^d$), one can analyse the Green's function using Fourier analysis and obtain an explicit expression for it. For bounded Ω one can then use some regularity theory for (1.4.1) to show that the Green's function behaves similarly to the full space Green's function. In fact, using this approach precise estimates on the Green's function and its derivatives are known in smooth domains (see e.g. [Kra67, DS04, GGS10]).

Equations in domains with singularities

If the domain Ω no longer has a smooth boundary, the situation becomes much more difficult. An important class of non-smooth domains are polyhedra (i.e. convex hulls of a finite number of points in $\mathbb{R}^d$), or, more generally, domains with piecewise smooth boundary. In that case there is still an existence and regularity theory for elliptic equations, although typically one has to introduce weighted function spaces, where the weight measures the distance to the singular part(s) of the boundary (cf. e.g. the series [KMR97, KMR01, MR10]). There are eigenvalue problems associated with the singularities, and these determine for what range of parameters the existence and regularity theory applies, and when one encounters a nontrivial kernel of L. One can still define a Green's function of (1.4.1), and derive asymptotics for its behaviour. In the interior of the domain this Green's function behaves again like the full-space one, while near the boundary it might be sensitive to the geometry of the domain.

When the boundary of Ω is no longer piecewise smooth, general results become rare, but they still exist. In particular, for general domains there is an existence and regularity theory in various spaces that are sufficiently close to the energy space $H^m(\Omega)$ (see e.g. [MM13]). There is also still a Green's function, and one can establish various estimates on it that are independent of the geometry of the domain [MM14].

The continuous Bilaplace equation

The differential operator that is most important to this thesis is the Bilaplacian operator, given as Δ^2 (where Δ is the standard Laplacian). This is a fourth-order elliptic operator, and probably the most important such operator. It arises for example in linear elasticity, fluid dynamics and the theory of phase separation.

All the general considerations from the previous paragraphs apply to the Bilaplace operator. While this operator is quite similar to the Laplacian, one important difference is the absence of a maximum principle. One can ask whether the operator is still positivity-preserving in the sense that $f \geq 0$ in (1.4.1) implies that $u \geq 0$ as well. This is equivalent to the Green's function of the Bilaplacian being nonnegative. From a physical point of view this seems quite plausible.In fact, in 1908 Hadamard [Had08] was convinced that this was the case when Ω is convex, even though he had no proof. Later it turned out that the conjecture is false. A first counterexample was found in 1949 by Duffin [Duf49], who showed that for a long thin rectangle the Green's function can become negative. Later on, many other counterexamples (including ones with smooth boundary such as certain ellipses) were found. For more on the history of this conjecture see [GGS10, Section 1.1.2].

Of course, in Section 1.2.6 we had already remarked that the Green's function of the discrete Bilaplacian can be negative, so with this in mind the failure of the conjecture should be not surprising.

1.4.2 Numerical analysis of partial differential equations

Finite differences and finite elements

While partial differential equations are a fairly universal tool to describe physical phenomena, they are ill-suited to computations. In order to actually compute an (approximate) solution, one typically wants to discretize the problem in some way, so that one obtains a

finite-dimensional problem which one can then solve exactly or approximately. This is one of the main topics of numerical analysis, see e.g. [Col60, JS14, Hac17].

One way to obtain such a discretization is to use finite differences. This means that one replaces the domain Ω by some lattice, typically $\Omega_h = \Omega \cap (h\mathbb{Z})^{\mathsf{d}}$, and the derivatives in the differential operators by some finite differences at the lattice points. In particular, one can replace the partial derivative $\partial_i u(x)$ by the forward difference $D_i^h u := \frac{1}{h}((u(x+he_i) - u(x))$, the backward difference $D_{-i}^h u := \frac{1}{h}((u(x) - u(x-he_i))$ or the central difference $D_{0,i}^h u := \frac{1}{2h}((u(x+he_i) - u(x-he_i))$. Thus, a simple finite difference scheme for the boundary value problem (1.4.1) would be

$$\begin{aligned} L_h u_h &= f_h \quad \text{in } \Omega_h, \\ u_h &= 0 \quad \text{on } (h\mathbb{Z})^{\mathsf{d}} \setminus \Omega_h \end{aligned} \tag{1.4.2}$$

where $u_h \colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, $f_h = f|_{(h\mathbb{Z})^{\mathsf{d}}}$, $L_h = \sum_{|\alpha| \leq 2m} a_\alpha D_\alpha^h$, and $D_\alpha^h = (D_1^h)^{\alpha_1} \dots (D_{\mathsf{d}}^h)^{\alpha_{\mathsf{d}}}$. This is a linear system with $|\Omega_h|$ unknowns, and one can solve it by using various efficient methods. One can devise other finite difference schemes using Taylor expansion, and in general one has to weigh simplicity of the scheme against its convergence properties. For our boundary problem (1.4.1) we could easily discretize the boundary condition by requiring u_h to be zero outside of Ω_h, but for other boundary conditions this might be difficult, in particular, when the shape of the boundary is complicated as well. Sometimes it might also be advantageous or necessary to choose f_h in some other way than just the restriction of f to the lattice.

While not important for the present thesis, we should also mention that another major method to discretize PDEs is to use a finite element method. Here one reformulates the PDE as a variational problem in some Hilbert space, and then picks a finite-dimensional subspace of that Hilbert space to solve the variational problem in. Typically one picks that finite-dimensional subspace as the span of a set of basis functions that have particularly nice properties (in particular, that most pairs of basis functions are orthogonal).

Consistency, stability and convergence

Whether by finite differences or finite elements, there are many ways to discretize a PDE. Of course, such schemes are only useful, if one can relate the solution of the discretized problem and the solution of the original PDE, i.e. if one understands the discretization error. This is again a classical topic of numerical analysis, well covered by the above references. Our presentation follows [Arn15].

It is a basic principle in this regard, dating back to [CFL28, Ger30, vNG47] that consistency and stability imply convergence of a scheme. Let us explain what these terms mean. We do so for the finite difference scheme (1.4.2). We define two Banach spaces V_h and W_h with norms $\|\cdot\|_{V_h}$ and $\|\cdot\|_{W_h}$ denoting the spaces in which u_h and w_h live (in our example the underlying vectorspace for both V_h and W_h is $\mathbb{R}^{\Omega_h}$).

First of all, to compare the solutions u and u_h of (1.4.1) and (1.4.2), we need them to live in the same space. This we can achieve by using a suitable map $u \mapsto U_h \in V_h$, e.g $U_h = u|_{\Omega_h}$. Then our goal is to control the error $\|U_h - u_h\|_{V_h}$. Now the consistency error measures how far U_h is from being a solution of (1.4.2). We define it as $\|L_h U_h - f_h\|_{W_h}$. We call the system consistent, if the consistency error tends to 0 as $h \to 0$. We also define the stability constant as the operator norm of L_h^{-1} as a map $W_h \to V_h$, and we call the scheme stable if that constant is bounded uniformly in h.

Now it is easy to check that consistency and stability imply convergence, and that we actually have a quantitative version of this result. Namely, one calculates

$$\|U_h - u_h\|_{V_h} = \left\| L_h^{-1}(L_h U_h - f_h) \right\|_{V_h} \leq \|L_h^{-1}\|_{\mathcal{L}(W_h \to V_h)} \|(L_h U_h - f_h)\|_{W_h}$$

and so the error is bounded by the product of the stability constant and the consistency error.

In practice, it is of course a minimum requirement on a finite difference scheme that it converges. One looks for finite difference schemes whose error converges as fast as possible, while the schemes stays simple enough that the required calculations are feasible. Thus, an important topic in the field is to analyse a given finite difference schemes and to estimate its approximation error.

1.4.3 Estimates for discrete Green's functions

This section includes parts of the introduction of the paper [MS19], written jointly by Stefan Müller and the author, as well as the introduction of the author's paper [Sch20a].

Overview

We consider partial difference equations, that is systems of the form (1.4.2) with L_h some discretization of an elliptic differential operator. If that discretization is chosen in such a way that L_h is elliptic (i.e. positive definite as a linear operator on $\mathbb{R}^{\Omega_h}$), then there is a Green's function G_h for L_h, i.e. $G_h(\cdot, y)$ is a solution of (1.4.2) with $f_h(x) = \delta_{h,y}(x) = \begin{cases} \frac{1}{h^d} & x = y \\ 0 & \text{else} \end{cases}$. It is cleary interesting in its own right to study the behaviour of its Green's function.

In addition to that, these Green's functions (with $h = 1$ and $L_h = -\Delta_h$ or $L_h = \Delta_h^2$) are the same Green' function as the ones discussed in Section 1.2.6, and in that Section we have already discussed the importance of these Green's functions for the study of the gradient and the membrane model.

Note that the Green's functions for different h are all related to each other via scaling, and so we can equally well study the Green's functions on $\Lambda_h := [0,1]^d \cap (h\mathbb{Z})^d$ to conclude results for the Green's functions on $[0,N]^d \cap \mathbb{Z}^d$ that arise in the study of Gaussian interface models. In fact, the former interpretation is oftentimes better suited to the application of tools from PDE theory or numerical analysis. In view of our applications to Gaussian interface models, our main focus here is on the discrete polyharmonic operators $L_h = \Delta_h^k$ for $k \geq 1$.

In order to prove results on the behaviour of the Green's function, there are a variety of approaches one can pursue. Just like in the continuous case, for simple domains such as the full-space $(h\mathbb{Z})^d$ one can apply discrete Fourier analysis to compute somewhat explicit expressions for the Green's function, and these in turn can be used to derive precise asymptotics for the Green's function. This method has led to asymptotics for the Green's function of the discrete Laplacian (e.g. [MW40, Duf53]), and more generally, for the Green's function of discrete polyharmonic operators [DS58, Sim67, Man67]. In particular, in [Man67] Mangad gives an algorithm that allows to compute the asymptotic expansion of the Green's function of Δ_h^k up to arbitrarily high order.

Apart from that, one can try to transfer some techniques available to study continuous PDEs and continuous Green's functions to the discrete setting. For some of them this seems

hopeless, e.g. flattening the boundary, or polar coordinates for corner singularities. Some other techniques carry over well, e.g. approaches based on well-chosen test functions. For example, in the author's MSc thesis [Sch16] Campanato's approach to elliptic regularity [Cam80] was adapted to the discrete setting, leading to a regularity theory for Δ_1^2 in L^p and $L^{p,\infty}$. This was based on earlier work of Dolzmann [Dol93, Dol99], who used the same approach for error estimates for finite element schemes. This approach also works for other discrete polyharmonic operators. Also in [Kur09, Cip13] a similar approach was used to derive estimates for the Green's function of the discrete Bilaplacian in $\mathsf{d} = 4$.

Alternatively, one can also try to transfer existing results for continuous Green's functions to discrete ones. For that purpose one needs a quantitative estimate that those Green's functions (or truncated versions of themselves) are close. One way to derive such estimates are estimates for the appropriate finite difference scheme, as discussed in Section 1.4.2. One can also try to use results for discrete PDEs (as in the previous paragraph) to establish this convergence.

In the following paragraphs we will describe how we have put these methods into practice.

Estimates for the discrete Bilaplacian for $\mathsf{d} \in \{2,3\}$ via continuous elliptic theory

As we have discussed, the subcritical membrane model is quite regular in the bulk. As a consequence, its behaviour at and near boundary is important for its global behaviour. In particular, as described in Section 1.3.3, the question of entropic repulsion is dominated by its boundary behaviour. Thus, one is interested in estimates for the Green's funcion of that model, i.e. the Green's function of the discrete Bilaplacian, that are sharp up to the boundary. Furthermore, to establish the Hölder continuity of the field via Kolmogorov's continuity criterion, one requires estimates for the mixed second derivatives of G, that again are valid up to the boundary.

In Chapter 2, that is based on the publication [MS19], we derive such estimates. In fact, the main result of that chapter is the following.

Theorem 1.4.1. *Let $\mathsf{d} = 2$ or $\mathsf{d} = 3$, and let $d(z)$ denote the distance of $z \in \Lambda_h = [0,1] \cap (h\mathbb{Z})^{\mathsf{d}}$ to $(h\mathbb{Z}) \setminus \Lambda_h$. Then there exist $c, C > 0$ independent of h such that for any $x, y \in (h\mathbb{Z})^{\mathsf{d}}$*

$$
\begin{aligned}
|G_h(x,y)| &\le C \min\left(d(x)^{2-\frac{\mathsf{d}}{2}} d(y)^{2-\frac{\mathsf{d}}{2}}, \frac{d(x)^2 d(y)^2}{(|x-y|+h)^{\mathsf{d}}} \right), \\
|\nabla_{h,x} G_h(x,y)| &\le C \min\left(d(y)^{3-\mathsf{d}}, \frac{(d(x)+h) d(y)^2}{(|x-y|+h)^{\mathsf{d}}} \right), \\
|\nabla_{h,x}^2 G_h(x,y)| &\le \begin{cases} C \log\left(1 + \frac{d(y)^2}{(|x-y|+h)^2}\right) & \mathsf{d} = 2 \\ C \min\left(\frac{1}{|x-y|+h}, \frac{d(y)^2}{(|x-y|+h)^3} \right) & \mathsf{d} = 3 \end{cases}, \\
|\nabla_{h,x} \nabla_{h,y} G_h(x,y)| &\le \begin{cases} C \log\left(1 + \frac{(d(x)+h)(d(y)+h)}{(|x-y|+h)^2}\right) & \mathsf{d} = 2 \\ C \min\left(\frac{1}{|x-y|+h}, \frac{(d(x)+h)(d(y)+h)}{(|x-y|+h)^3} \right) & \mathsf{d} = 3 \end{cases}.
\end{aligned}
$$

Besides its application to the problem of entropic repulsion in Chapter 3, this result has also been used in [CDH19] to rigorously establish the scaling limit of the subcritical membrane model.

Let us briefly mention how we prove Theorem 1.4.1. Our main tool are Caccioppoli (or reverse Poincaré) inequalities. That is, for some u that is discretely biharmonic on some

large ball, we want to control the L^∞-norm of its Hessian on a small ball by the L^2 norm of its Hessian on a larger ball. This is an interior estimate, and combined with the corresponding exterior estimate, and the results on the full-space discrete Green's function from [Man67], one can derive the estimates in 1.4.1 using some careful reasoning.

The main challenge thus is to derive such Caccioppoli inequalities. In the interior or near flat parts of the boundary, we could, in principle, use for that purpose the classical approach based on test functions (as is done in [Dol93, Dol99]). Near the singularities of $[0,1]^{\mathsf{d}}$, this is no longer possible. There we use that the continuous theory for biharmonic functions in domains with singularities [KMR97, MR10] predicts that these functions decay rather rapidly near the singularities. Using a compactness argument based on a discrete version of the Kolmogorov-Riesz-Fréchet criterion and the Caccioppoli inequality, we can transfer this to a discrete Caccioppolli inequality near the corresponding singularity. Since we anyhow need to introduce this compactness framework, we directly use it for the estimates in the interior or near flat parts of the boundary, as well.

Estimates for the discrete Bilaplacian for $\mathsf{d} = 4$ via finite difference schemes

For the critical membrane model, the boundary behaviour is less important. Instead, most relevant for the analysis of the field is the logarithmic correlation structure in the bulk. In particular, as discussed in Section 1.3.2, one can obtain the convergence of the maximum of the field provided one has very sharp estimates on the Green's function in the bulk. These estimates are derived in Chapter 4 which is based on the publication [Sch20a]. The main result on the Green's function there is too technical to state in full here, but we give some of the estimates.

Theorem 1.4.2. *Let $\mathsf{d} = 4$, and $\Lambda_h = [0,1]^4 \cap (h\mathbb{Z})^4$. Also let $d(x) = d(x, \partial[0,1]^{\mathsf{d}})$. Then for all $x, y \in V_h$*

$$\left| 8\pi^2 G_{\Delta,h}(x,y) - \log\left(2 + \frac{\max(d(x), d(y))}{h + |x-y|}\right)\right| \leq C\,.$$

Furthermore, there are a constant $\theta_0 > 0$, a continuous function $f_1\colon (0,1)^4 \to \mathbb{R}$ and a function $f_2\colon \mathbb{Z}^4 \times \mathbb{Z}^4 \to \mathbb{R}$ such that the following holds. For all $L, \varepsilon > 0$, $\theta > \theta_0$ there exists $N_0' = N_0'(L, \varepsilon, \theta)$ such that for all $h \leq \frac{1}{N_0'}$ with $\frac{1}{h} \in \mathbb{N}$, all $x \in \Lambda_h$ such that $d(x) \geq h|\log h|^\theta$ and for all $u, v \in [0, L]^4 \cap \mathbb{Z}^4$ we have

$$\left|8\pi^2 G_{\Delta,h}(x+hu, x+hv) + \log h - f_1(x) - f_2(u,v)\right| < \varepsilon\,.$$

Similarly, there are a constant $\theta_1 > 0$ and a continuous function $f_3\colon \mathcal{D}^4 \to \mathbb{R}$, where $\mathcal{D}^4 = \{(x,y) : x, y \in (0,1)^4, x \neq y\}$ such that the following holds. For all $L, \varepsilon > 0$, $\theta > \theta_1$ there exists $N_1' = N_1'(L, \varepsilon, \theta)$ such that for all $h \leq \frac{1}{N_1'}$ with $\frac{1}{h} \in \mathbb{N}$ and for $x, y \in \Lambda_h$ such that $\min(d(x), d(y)) \geq h|\log h|^\theta$ and $|x-y| \geq \frac{1}{L}$ we have

$$\left|8\pi^2 G_{\Delta,h}(x,y) - f_3(x,y)\right| < \varepsilon\,.$$

The compactness methods from Chapter 2 are not well-suited to be applied here for two reasons. First of all, the relevant continuous estimates have not yet been worked out for $\mathsf{d} = 4$ in the literature. In addition, the estimates obtained using compactness methods and Caccioppoli inequalities are all up to a possibly large constant, while in Theorem 1.4.2 we are interested in estimates with an error that tends to 0 as $h \to 0$.

Instead we follow another of the strategies mentioned above. Namely, we use an estimate for the approximation quality of finite difference schemes to compare truncated versions of the discrete and continuous Green's function. This estimate is similar to the one in Chapter 5 to be discussed below, although it is easier than this result and can be established using textbook methods as in [JS14]. Besides this error estimates we again use the results from [Man67] on the discrete Green's function in the full-space, as well as results similar to ones in [MM13, MM14] for the continuous Green's function in $[0,1]^4$.

1.4.4 Estimates for finite difference schemes for the Bilaplacian

This section includes parts of the introduction of the author's paper [MSS20], written jointly by Stefan Müller, Endre Süli and the author.

Overview

We have already described the importance of the Bilaplace equation, as well the relevance of finite difference schemes together with error bounds. The convergence analysis of numerical methods for the approximate solution of the biharmonic equation has therefore been of considerable interest. Some references are the early papers by Tee [Tee64], Bramble [Bra66], Smith [Smi68, Smi70], and Ehrlich [Ehr71]; see also [Col60, Ch. V, §1.5 III and Table VI in the Appendix]. For the numerical analysis of finite difference approximations of the biharmonic equation in rectangles a fast algorithm was given by Bjørstad [Bjø83]. For a modern review in the context of the approximate solution of the Navier–Stokes equations in planar domains, see [BACF13].

In these works the data and the solution to the boundary-value problems under consideration were assumed to be sufficiently smooth. This assumption, however, is quite restrictive, as in practice one often encounters right-hand sides that are rather rough. One of many examples is that of turbulence in a fluid. Another example is given by the analysis of discrete Green's function, as just discussed in Section 1.4.3.

Finite difference schemes for rough right-hand sides were considered by Lazarov [Laz81], Gavrilyuk et al. [GLMP83], and Ivanović et al. [IĬS86], for example. For a detailed survey of the relevant literature see the monograph of Jovanović and Süli [JS14], devoted to the finite difference approximation of linear partial differential equations with generalized solutions.

Consider for instance the boundary value problem

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \Gamma\,, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \tag{1.4.3}$$

where $\Omega = (0,1)^{\mathsf{d}}$. A finite difference scheme associated with this boundary value problem is given by

$$\begin{aligned} \Delta_h^2 U &= T^{h,2,\dots,2} f \quad \text{in } \Lambda_h\,, \\ U &= 0 \quad \text{on } \Gamma_h\,, \\ D_{0,\nu}^h U &= 0 \quad \text{on } \Gamma_h\,. \end{aligned} \tag{1.4.4}$$

where $T^{h,2,\dots,2} f$ is a certain smoothing operator (to be defined precisely in Chapter 5), $\Lambda_h = [0,1]^{\mathsf{d}} \cap (h\mathbb{Z})^{\mathsf{d}}$, and $\Gamma_h = (\partial[0,1]^{\mathsf{d}}) \cap (h\mathbb{Z})^{\mathsf{d}}$. The operator $T^{h,2,\dots,2} f$ regularizes the right-hand

side, and so (1.4.4) makes sense whenever $f \in H^t(\Omega)$ for $t > -\frac{3}{2}$. This scheme has been studied in [GMP83, JS14] for $\mathsf{d} = 2$ and u in the fractional Sobolev space $H^s((0,1)^{\mathsf{d}})$ (for $s = t + 4 > \frac{5}{2}$).

In [JS14, Theorem 2.69] the error bound

$$\|u - U\|_{H_h^2(\Lambda_h)} \leq Ch^{\min\{s-2,3/2\}} |\log h|^{1-|\mathrm{sgn}(s-7/2)|}) \|u\|_{H^s((0,1)^{\mathsf{d}})}$$

for $\frac{5}{2} < s \leq 4$ is established, and in [GMP83] the error bound

$$\|u - U\|_{H_h^2(\Lambda_h)} \leq Ch^{s-2} \|u\|_{H^s((0,1)^{\mathsf{d}})}$$

is shown for $\frac{5}{2} < s \leq 4$, albeit under the additional assumption that the third normal derivative of u vanishes at the boundary.

These results seem suboptimal, because the operator $D_{0,h}$ has truncation error h^2, and so one can hope that one actually has an error estimate of order h^2 unconditionally. In Chapter 5 we prove that we actually have such an estimate, and this not only if $\mathsf{d} = 2$.

Improved estimates for a finite difference scheme

In fact, in Chapter 5, that is based on the publication [MSS20], we prove the following result.

Theorem 1.4.3. *Let $\mathsf{d} \geq 2$. Suppose that $\frac{1}{2}\max(5, \mathsf{d}) < s \leq 4$, and let $u \in H^s(\Omega) \cap H_0^2(\Omega)$; then, there exists a positive constant $C = C(\mathsf{d}, s)$, independent of h, such that*

$$\|u - U\|_{H_h^2(\Lambda_h)} \leq Ch^{s-2} \|u\|_{H^s(\Omega)} .$$

The restrictions on the range of s in [JS14, Theorem 2.69] and on the third normal derivative of u in [GMP83] arise for the following reason: in order to compare the finite difference approximation with the original problem one needs an extension of the (generalized) solution u from Ω to $\mathbb{R}^{\mathsf{d}} \setminus \Omega$ that preserves the Sobolev regularity of u and has, ideally, zero discrete boundary values. The assumptions in [JS14, Theorem 2.69] and in [GMP83] permit the use of the symmetric extension of u across $\partial\Omega$ for that purpose.

In our setting, with $\frac{1}{2}\max(5, \mathsf{d}) < s \leq 4$, this is no longer possible. The main novelty of the proof of Theorem 1.4.3 is to use a different, carefully chosen, extension of u. This extension will no longer have zero boundary values, but we will show that they can be made small (in an appropriate norm, in terms of positive powers of the discretization parameter h), so that we can still close the argument. More precisely, we prove that the boundary values of u are small in a discrete version of the $H^{1/2}$-norm on the boundary. We also show that this implies that there is an extension of the boundary values back into Λ_h with small L^2-norm of the Hessian. We can then subtract this extension from $u - U$ and apply classical energy space estimates as in [JS14] to bound the remaining error terms.

1.5 Notation

We have made an attempt to keep the notation consistent throughout the whole thesis. However, this has not always been possible, and we indicate near the beginning of each chapter when the notation there deviates. If some notation is only relevant for a particular chapter, we also only introduce it there.

Let us summarize here the notation that is relevant for all of the thesis.

- We use the convention that c and C denote generic constants whose precise value can change from occurrence to occurrence. Constants that are denoted by any other Latin or Greek letter have some fixed value and keep it. By adding subscripts to a constant we emphasize that the precise value of that constant may depend on the variables in the subscript (and typically on no others).

  In a few places we use the standard Landau notation. That is, we write $a = b + O(d)$ to denote $|a - b| \leq Cd$, and $a = b + o(d)$ to denote that $\frac{a-b}{d}$ tends to zero (in a limit that will be clear from the context). Again we add subscripts to emphasize what the implied constant or the implied convergence rate depend on.

- We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^+ = \{1, 2, \ldots\}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the natural numbers, positive integers, rational numbers, real numbers, and complex numbers, respectively.

- We denote the cardinality of a set A by $|A|$.

- We use the standard notation for multiindexes $\underline{\alpha} \in \mathbb{N}^{\mathsf{d}}$. We will sometimes omit the underline, when there is no risk of confusion.

- We denote by $\mathsf{d} \in \mathbb{N}^+$ the dimension of our space. The standard basis of R^{d} is denoted $e_1, \ldots, e_{\mathsf{d}}$. We consider the lattice $Z^{\mathsf{d}} \subset \mathbb{R}^{\mathsf{d}}$, and for $h > 0$ such that $\frac{1}{h} \in \mathbb{N}^+$ also the lattices $(h\mathbb{Z})^{\mathsf{d}}$.

- We write $B_r(x)$ for the open ball of radius $r > 0$ around $x \in \mathbb{R}^{\mathsf{d}}$, and $Q_r(x) = x + (-r, r)^{\mathsf{d}}$ for the open cube of sidelength $2r$. We also use discrete cubes. That is, we define
$$Q_r^h(x) = \{y \in (h\mathbb{Z})^{\mathsf{d}} \colon |y - x|_\infty \leq r\} = \overline{Q_r(x)} \cap (h\mathbb{Z})^{\mathsf{d}}\,.$$
  When $x = 0$ we sometimes omit the x.

  As an exception to this, in Chapter 6 we will only use discrete cubes, and so we can drop the superscript in $Q_r^h(x)$ there. We will recall this in the introduction of that chapter.

- On $\mathbb{R}^{\mathsf{d}}$ we use the l^p-norms $|\cdot|_p$ for $p \in [1, \infty]$. When $p = 2$ we often drop the subscript p so that $|\cdot|$ denotes the Euclidean norm. For $x \in \mathbb{R}^{\mathsf{d}}$ and $A, A' \subset \mathbb{R}^{\mathsf{d}}$ we let $d(x, A) = \inf_{y \in A} |x - y|$ be the Euclidean distance of x to A, and $d(A, A') = \inf_{y \in A, y' \in A'} |x - y|$ be the Euclidean distance of A and A'. By adding a subscript to d we indicate that either we take the distance with respect to some other norm or that A is some fixed set (say $A = \Lambda_N$). This will be defined in detail in the corresponding chapters.

- Given $N \in \mathbb{N}^+$, Λ_N denotes a lattice square of sidelength comparable to N, with lattice width 1. Similarly for $h > 0$ such that $\frac{1}{h} \in \mathbb{N}^+$ we denote by Λ_h a lattice square of sidelength comparable to 1, with lattice width comparable $\frac{1}{h}$. The reader should think of $\Lambda_N = [0, N]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}}$ and $\Lambda_h = [0, 1]^{\mathsf{d}} \cap (h\mathbb{Z})^{\mathsf{d}}$, but see the introductions of the individual chapters for the precise definitions.

- We define the forward difference quotient $D_i^h u(x) = u(x + he_i) - u(x)$, the backward difference quotient $D_{-i}^h u(x) = u(x) - u(x - he_i)$ and the centred difference quotient $D_{0,i}^h u(x) := \frac{1}{2h}(u(x + he_i) - u(x - he_i))$. The discrete gradient is the vector $\nabla_h u(x) := (D_i^h u(x))_{i=1}^{\mathsf{d}}$, the discrete Hessian is the tuple $\nabla_h^2 u(x) := (D_i^h D_{-j}^h u(x))_{i,j=1}^{\mathsf{d}}$, the discrete Laplacian is $\Delta_h u(x) := \sum_{i=1}^{\mathsf{d}} D_i^h D_{-i}^h u(x)$, and the discrete Bilaplacian is $\Delta_h^2 := \Delta_h \circ \Delta_h$.

For a multi-index $\underline{\alpha} \in \mathbb{N}^{\mathsf{d}}$ we write $D^h_{\underline{\alpha}} u(x) = (D^h_1)^{\alpha_1} \dots (D^h_{\mathsf{d}})^{\alpha_{\mathsf{d}}} u(x)$, and for $k \in \mathbb{N}$ with $k \geq 3$ we let $\nabla^k u(x)$ be the the collection of all $D^h_{\underline{\alpha}} u(x)$ with $|\underline{\alpha}| = k$.

More generally, for a vector $a \in \mathbb{Z}^{\mathsf{d}}$ of unit length we define the forward difference quotient $D^h_a u(x) := \frac{1}{h}(u(x+ha) - u(x))$, the backward difference quotient $D^h_{-a} u(x) := \frac{1}{h}(u(x) - u(x-ha))$ and the centred difference quotient $D^h_{0,a} v(x) := \frac{1}{2h}(v(x+ha) - v(x-ha))$.

- We use the translation operators $\tau^h_{\pm i}$ defined by $\tau^h_{\pm i} u(x) = u(x \pm he_i)$. More generally, for $a \in \mathbb{Z}^{\mathsf{d}}$ we set $\tau^h_a u(x) = u(x+ha)$.

- For a domain $\Omega \subset \mathbb{R}^{\mathsf{d}}$ and $p \in [1, \infty]$ we use the standard L^p-norms $\|\cdot\|_{L^p(\Omega)}$, and for $k \in \mathbb{N}$ the Sobolev-norms $\|\cdot\|_{W^{k,p}(\Omega)}$. When $p = 2$, we also write $\|\cdot\|_{H^k(\Omega)}$ instead of $\|\cdot\|_{W^{k,2}(\Omega)}$. Furthermore, for $\alpha \in [0,1]$ we use the Hölder seminorms $[\cdot]_{C^{0,\alpha}(\Omega)}$ and the Hölder norms $\|\cdot\|_{C^{0,\alpha}(\Omega)}$. We extend these norms to vector-valued functions by taking the Euclidean norm of the norms of the components. Each of these norms comes with an associated function space.

- We also use various discrete function spaces. For $A \subset (h\mathbb{Z})^{\mathsf{d}}$ and $p < \infty$ we define a discrete L^p_h-norm
$$\|u\|^p_{L^p_h(\Omega)} = \sum_{x \in A} h^{\mathsf{d}} |u(x)|^p$$
if $p < \infty$, and
$$\|u\|_{L^\infty_h(\Omega)} = \sup_{x \in A} h^{\mathsf{d}} |u(x)|$$
and the associated function spaces. For $p = 2$ this norm is induced by the scalar product
$$(u, v)_{L^2_h(\Omega)} = \sum_{x \in A} h^{\mathsf{d}} u(x) v(x)$$
When there is no risk of confusion, we drop the subscript h. We also use various other discrete function spaces that are defined in the individual chapters.

- For $\Lambda \Subset (h\mathbb{Z})^{\mathsf{d}}$ we denote the gradient and membrane by $\mathbb{P}_{\nabla,\Lambda}$ and $\mathbb{P}_{\Delta,\Lambda}$ (as introduced in Section 1.2.4). We denote samples from these measures by $\psi_{\nabla,\Lambda}$ and $\psi_{\Delta,\Lambda}$, respectively, and write $G_{\nabla,\Lambda}$ and $G_{\Delta,\Lambda}$ for the associated Green's functions. When $\Lambda = \Lambda_N$ or $\Lambda = \Lambda_h$, we just write N or h instead of Λ_N or Λ_h. We drop the subscripts ∇ and Δ when there is no risk of confusion.

2 Estimates for the Green's function of the discrete Bilaplacian in dimensions two and three

This chapter is based on the paper [MS19], written jointly by Stefan Müller and the author, with only minor changes. A small part of the content of this chapter has already appeared in the author's M.Sc. thesis [Sch16], where a result similar to Theorem 2.1.1 was shown, but only for $\mathsf{d} = 2$ and using a different and more complicated approach.

2.1 Introduction

In this chapter we will establish estimates for the Green's function of the subcritical discrete Bilaplacian, as described in Section 1.4.3. In particular we prove Theorem 1.4.1. Actually, we prove a slightly different statement: while we still set $\Lambda_h = [0,1]^{\mathsf{d}} \cap (h\mathbb{Z})^{\mathsf{d}}$, we take the Green's function on $\operatorname{int}\Lambda_h = [h, 1-h]^{\mathsf{d}} \cap (h\mathbb{Z})^{\mathsf{d}}$ instead of Λ_h. The precise statement is Theorem 2.1.1 below. It is easy to see that this Theorem is equivalent to Theorem 1.4.1 as stated in the introduction.

We use the notation from Section 1.5. As this chapter is only concerned with the Bilaplacian, we drop all subscripts Δ right away. We define $\Lambda_N = [-N, N]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}}$.

We first state an unrescaled version of our main result (i.e. with unit lattice width).

Theorem 2.1.1. *Let $\mathsf{d} = 2$ or $\mathsf{d} = 3$, let G_N be the Green's function of the discrete Bilaplacian with zero boundary data outside Λ_N, and let $d(z) = \operatorname{dist}(z, \mathbb{Z}^{\mathsf{d}} \setminus \Lambda_N)$. Then there exist $c, C > 0$ independent of N such that G_N and its discrete derivatives satisfy the following estimates.*

i) For any $x, y \in \mathbb{Z}^{\mathsf{d}}$

$$|G_N(x,y)| \leq C \min\left(d(x)^{2-\frac{\mathsf{d}}{2}} d(y)^{2-\frac{\mathsf{d}}{2}}, \frac{d(x)^2 d(y)^2}{(|x-y|+1)^{\mathsf{d}}} \right), \tag{2.1.1}$$

$$|\nabla_x G_N(x,y)| \leq C \min\left(d(y)^{3-\mathsf{d}}, \frac{(d(x)+1)d(y)^2}{(|x-y|+1)^{\mathsf{d}}} \right), \tag{2.1.2}$$

$$|\nabla_x^2 G_N(x,y)| \leq \begin{cases} C \log\left(1 + \frac{d(y)^2}{(|x-y|+1)^2}\right) & \mathsf{d} = 2 \\ C \min\left(\frac{1}{|x-y|+1}, \frac{d(y)^2}{(|x-y|+1)^3}\right) & \mathsf{d} = 3 \end{cases}, \tag{2.1.3}$$

$$|\nabla_x \nabla_y G_N(x,y)| \leq \begin{cases} C \log\left(1 + \frac{(d(x)+1)(d(y)+1)}{(|x-y|+1)^2}\right) & \mathsf{d} = 2 \\ C \min\left(\frac{1}{|x-y|+1}, \frac{(d(x)+1)(d(y)+1)}{(|x-y|+1)^3}\right) & \mathsf{d} = 3 \end{cases}. \tag{2.1.4}$$

ii) For any $x \in \mathbb{Z}^{\mathsf{d}}$

$$G_N(x,x) \geq c d(x)^{4-\mathsf{d}}. \tag{2.1.5}$$

G_N is symmetric in x and y, so we also have the analogous estimates for $|\nabla_y G_N(x,y)|$ and $|\nabla_y^2 G_N(x,y)|$. For the optimality of these estimates, see the discussion after Theorem 2.1.3.

The estimates (2.1.1) and (2.1.5) immediately provide estimates for the variance and covariance of ψ under P_N. From the estimates (2.1.1) and (2.1.5) and a thinning procedure one can also deduce estimates on the probability of the membrane model to be positive. We give details on these arguments in Chapter 3.

In addition Theorem 2.1.1 implies the following continuity estimates.

Corollary 2.1.2. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$. *Under* P_N, *the random field* ψ *satisfies*

$$E_N(|\psi_x - \psi_y|^2) \leq \begin{cases} C|x-y|^2 \log\left(2 + \frac{N}{|x-y|}\right) & \mathsf{d} = 2\,, \\ C|x-y| & \mathsf{d} = 3 \end{cases}. \tag{2.1.6}$$

To show (2.1.6) for $\mathsf{d} = 2$ one uses the identity

$$E_N(|\psi_x - \psi_y|^2) = G_N(x,x) - G_N(x,y) - G_N(y,x) + G_N(y,y)\,, \tag{2.1.7}$$

as well as a discrete counterpart of the identity

$$H(x,x) - H(x,y) - H(y,x) + H(y,y) = \int_0^1 \int_0^1 \partial_s \partial_t H(x + s(y-x), x + t(y-x))\,\mathrm{d}s\,\mathrm{d}t\,,$$

valid for every smooth function H, and (2.1.4). For $\mathsf{d} = 3$ one uses (2.1.7) and the estimates for $G(x,x) - G(x,y)$ and $G(y,y) - G(y,x)$ provided by (2.1.2) and its analogue for the y-derivative. Since ψ is a Gaussian field the estimate (2.1.6) and the Kolmogorov continuity criterion imply that the rescaled fields $\psi'_{x'} = N^{-2+\mathsf{d}/2}\psi_{Nx'}$ are uniformly Hölder continuous with exponents $\alpha < \alpha_{\mathsf{d}}$ where $\alpha_2 = 1$ and $\alpha_3 = \frac{1}{2}$. More precisely

$$P\Big(\big\{\psi' : \sup_{x' \neq y'} \frac{|\psi'_{x'} - \psi'_{y'}|}{|x'-y'|^\alpha} \leq K\big\}\Big) \geq 1 - \varepsilon_\alpha(K)$$

with $\lim_{K\to\infty} \varepsilon_\alpha(K) = 0$. After the publication of these results, Cipriani-Dan-Hazra [CDH19] completed the argument sketched above and proved that the membrane model has a Hölder-continuous scaling limit in dimensions $\mathsf{d} \leq 3$.

In order to prove Theorem 2.1.1, we need regularity improving estimates for discrete biharmonic functions and optimal decay estimates for various norms in annuli around the singularity. The corresponding estimates for continuous biharmonic functions can be proved using well-established techniques. One insight of this chapter is that these estimates can be transferred to the discrete realm using two ingredients: a new compactness argument and the discrete version of the Caccioppoli (or reverse Poincaré) inequality. It should also be possible to transfer continuous estimate to discrete estimates by using error estimates in numerical analysis, see the discussion below Corollary 2.1.4.

In order to derive the estimates in detail and to highlight the similarities between the continuous and discrete setting, it is convenient to change notation. In particular, we rescale our lattice to have width h, while the domain is fixed. We also shift the boundary by h inwards.

Consider the lattice $(h\mathbb{Z})^{\mathsf{d}}$, where we assume $\frac{1}{h} \in \mathbb{N}$. Let $\Lambda_h = [0,1]^{\mathsf{d}} \cap (h\mathbb{Z})^{\mathsf{d}}$, $\operatorname{int}\Lambda_h = \left[\frac{1}{h}, 1 - \frac{1}{h}\right]^{\mathsf{d}} \cap (h\mathbb{Z})^{\mathsf{d}}$ and let Δ_h be the discrete Laplacian on $(h\mathbb{Z})^{\mathsf{d}}$. Let $G_h(x,y)$ be the Green's function for $\Delta_h^2 = (\Delta_h)^2$ on $\operatorname{int}\Lambda_h$ with zero boundary values on $(h\mathbb{Z})^{\mathsf{d}} \setminus \operatorname{int}\Lambda_h$. In this setting, Theorem 2.1.1 becomes

Theorem 2.1.3. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$*, and let* $d(z)$ *denote the distance of* $z \in \operatorname{int} \Lambda_h$ *to* $(h\mathbb{Z})^{\mathsf{d}} \setminus \operatorname{int} \Lambda_h$*. Then there exist* $c, C > 0$ *independent of* h *such that*

i) for any $x, y \in (h\mathbb{Z})^{\mathsf{d}}$

$$|G_h(x,y)| \leq C \min\left(d(x)^{2-\frac{\mathsf{d}}{2}} d(y)^{2-\frac{\mathsf{d}}{2}}, \frac{d(x)^2 d(y)^2}{(|x-y|+h)^{\mathsf{d}}} \right), \tag{2.1.8}$$

$$|\nabla_{h,x} G_h(x,y)| \leq C \min\left(d(y)^{3-\mathsf{d}}, \frac{(d(x)+h)d(y)^2}{(|x-y|+h)^{\mathsf{d}}} \right), \tag{2.1.9}$$

$$|\nabla^2_{h,x} G_h(x,y)| \leq \begin{cases} C \log\left(1 + \frac{d(y)^2}{(|x-y|+h)^2}\right) & \mathsf{d} = 2 \\ C \min\left(\frac{1}{|x-y|+h}, \frac{d(y)^2}{(|x-y|+h)^3}\right) & \mathsf{d} = 3 \end{cases}, \tag{2.1.10}$$

$$|\nabla_{h,x} \nabla_{h,y} G_h(x,y)| \leq \begin{cases} C \log\left(1 + \frac{(d(x)+h)(d(y)+h)}{(|x-y|+h)^2}\right) & \mathsf{d} = 2 \\ C \min\left(\frac{1}{|x-y|+h}, \frac{(d(x)+h)(d(y)+h)}{(|x-y|+h)^3}\right) & \mathsf{d} = 3 \end{cases}. \tag{2.1.11}$$

ii) for any $x \in (h\mathbb{Z})^{\mathsf{d}}$

$$G_h(x,x) \geq c d(x)^{4-\mathsf{d}}. \tag{2.1.12}$$

Theorem 2.1.1 can be easily derived from Theorem 2.1.3 if one chooses $h = \frac{1}{2N+2}$, rescales by a factor of $2N+2$ and observes that the estimates are scale-invariant. One can also obtain estimates for higher discrete derivatives, see Remark 2.8.4 below.

Comparison with the Green's function of the continuous Bilaplacian in the ball (see [Bog05, eqn. (48)] or [GGS10, eqn. (2.65) and Thm. 4.7]), a general bounded smooth set [DS04, Thm. 3 and Thm. 12] or a half-space [GGS10, eqn. (2.66)] shows that the estimates in Theorem 2.1.3 are optimal in the interior and near the regular boundary points (edges for $\mathsf{d} = 2$ and faces for $\mathsf{d} = 3$).

Near the singular boundary points (corners for $\mathsf{d} = 2$ and edges and corners for $\mathsf{d} = 3$) the continuous regularity theory gives a more rapid decay of biharmonic functions (and their derivatives) and hence a more rapid decay for the Green's function with a decay exponent γ. Our compactness argument can be used to establish a similar decay estimate for all exponents $\gamma' < \gamma$. Since the general continuum theory provides an open interval of admissible exponents γ (due to possible logarithmic terms) there is no loss in passing to the discrete estimates.

The general statement is rather tedious, so let us look instead at an illustrative example, the corner point 0 of the square $(0,1)^2$. In this case the distance of a point x from the corner point is given by $|x|$. If $|x| < \frac{1}{4}|y|$ then $|x-y| \geq \frac{1}{2}|y| \geq \frac{1}{2}d(y)$ and the continuous theory implies that

$$|G(x,y)| \leq C \left(\frac{|x|}{|y|}\right)^{2+\theta/2} d^2(y). \tag{2.1.13}$$

where $0 < \theta < \theta_0$, and $\theta_0 \approx 3.47918$. To see this use Lemma 2.5.13 and note that

$$\|\nabla^2 G(\cdot, y)\|_{L^2(Q_{|y|/2} \cap (0,1)^2)} \leq C|y|^{-1} d^2(y)$$

(this follows from the continuous counterparts of (2.8.2) and Lemma 2.6.2). Moreover we have

$$\sup_{Q_s \cap (0,1)^2} G(\cdot, y) \leq s \|\nabla^2 G(\cdot, y)\|_{L^2(Q_s \cap (0,1)^2)}$$

by the Sobolev-Poincaré inequality and scaling.

The estimate (2.1.13) is better than the estimate

$$G(x,y) \leq \frac{d^2(x)d^2(y)}{|x-y|^2} \sim C\frac{d(x)^2}{|y|^2}d^2(y)$$

if

$$\frac{d(x)}{|y|} \gg \left(\frac{|x|}{|y|}\right)^{1+\theta/4}.$$

Note that this condition holds in particular if $|x|$ and $d(x)$ are comparable and $|x| \ll |y|$. The compactness argument shows that the discrete Green's function G_h satisfies a counterpart of (2.1.13) if we replace θ by any smaller exponent θ' and C by $C_{\theta'}$.

It is also easy to show that the discrete Green's function converges to the the continuous Green's function.

Corollary 2.1.4. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$. *Let* $G(\cdot, y) \in W_0^{2,2}((0,1)^{\mathsf{d}})$ *denote the continuous Green's function, i.e., the unique weak solution of* $\Delta^2 G(\cdot, y) = \delta_y$. *Extend* $G_h(x,y)$ *to* $y \in (0,1)^{\mathsf{d}}$ *by piecewise constant interpolation in the second variable. Then for each* $y \in (0,1)^{\mathsf{d}}$ *the following assertions hold.*

i) We have

$$I_h^{pc} G_h(\cdot, y) \to G(\cdot, y) \quad \textit{uniformly},$$

where I_h^{pc} *denotes the piecewise constant interpolation in the first variable.*

ii) If $\mathsf{d} = 2$ *then* $I_h^{pc}\nabla_h G_h(\cdot, y)$ *converges uniformly to* $\nabla G(\cdot, y)$ *and* $I_h^{pc}\nabla^2 G_h(\cdot, y)$ *converges to* $\nabla^2 G(\cdot, y)$ *in* $L^p((0,1)^2)$ *for all* $p < \infty$.

iii) If $\mathsf{d} = 3$ *then* $I_h^{pc}\nabla_h G_h(\cdot, y)$ *is uniformly bounded and converges to* $\nabla G(\cdot, y)$ *in* $L^p((0,1)^3)$ *for all* $p < \infty$ *and locally uniformly in* $[0,1]^3 \setminus \{y\}$. *Moreover* $I_h^{pc}\nabla_h^2 G_h(\cdot, y)$ *converges to* $\nabla^2 G(\cdot, y)$ *in* L^p *for all* $p < 3$.

A slight variant of the argument given below shows that the convergence in i) is also uniform in y, i.e., that we have uniform convergence of the piecewise constant interpolation of G_h to G in $(0,1)^{\mathsf{d}} \times (0,1)^{\mathsf{d}}$. The proof of asssertion i) in Corollary 2.1.4 uses essentially only the elementary discrete $W^{2,2}$ estimate in Lemma 2.8.1 and the compact embedding from $W^{2,2}$ to C^0. The other two assertion follow from Theorem 2.1.3 and the local compactness argument in Section 2.5. See Section 2.8 for the details.

For $\mathsf{d} = 2$ quantitative estimates for the discrete $W^{2,2}$ norm of difference between the solutions of the discretised and the continuous biharmonic equation under weak assumptions on the regularity of the continuous solution have been obtained by Lazarov [Laz81], Gavrilyuk, Makarov and Pirnazarov [GMP83], Gavrilyuk et al. [GLMP83] and Ivanović, Jovanović and Süli [IĬS86], see also Chapter 2.7 in [JS14] which includes estimates for more general fourth order equations in divergence form with variable coefficients. More precisely, let $u \in (W_0^{2,2} \cap W^{s,2})((0,1)^2)$ and let $\hat{u}_h$ be the solution of

$$\Delta_h^2 \hat{u}_h = K_h * \Delta^2 u \quad \text{in int}\, \Lambda_h^2$$

subject to the discrete boundary conditions

$$\hat{u}_h(x) = 0 \quad \text{and} \quad \hat{u}_h(x + he_i) - \hat{u}_h(x - he_i) = 0 \quad \forall x \in \Lambda_h^2 \setminus \text{int}\, \Lambda_h^2 \quad \forall i \in \{1,2\}. \quad (2.1.14)$$

Here $K_h(x) = h^{-2}K(\frac{x}{h})$ and $K(z) = (1-|z_1|)_+(1-|z_2|)_+$. The boundary condition (2.1.14) has the advantage that it leads to a higher order of consistency compared to our boundary condition $u_h = 0$ on $(h\mathbb{Z})^2 \setminus \operatorname{int}\Lambda_h$ (this latter condition is arguably more natural from the point of view of probability and statistical mechanics). For the discrete $W^{2,2}$ norm the optimal error estimates

$$\|u - \hat{u}_h\|_{W^{2,2}(\Lambda_h)} \leq Ch^{s-2}\|u\|_{W^{s,2}((0,1)^2)} \tag{2.1.15}$$

were established in [GMP83] for $s = 3$ and in [JS14, Thm. 2.69] for $\frac{5}{2} < s < \frac{7}{2}$. In [GMP83] the estimate (2.1.15) is also proved for $s = 4$, but under the additional condition that that the symmetric extension $\tilde{u}$ of u outside $(0,1)^2$ still belongs to $W^{4,2}$. This holds only if the third normal derivatives of u (which exist in the sense of trace) vanish.

Because $K_h * \delta = \delta_h$ these estimates can be used to compare the continuous Green's function $G_y \in W_0^{2,2}$ and the discrete Green's function $\hat{G}_{h,y}$ (defined using the boundary conditions (2.1.14) rather than $G_{h,y} = 0$ on $(h\mathbb{Z})^2 \setminus \operatorname{int}\Lambda_h$) and one obtains $\|G_y - \hat{G}_{h,y}\|_{W^{2,2}(\Lambda_h)} \leq C_s h^{s-2} d^{3-s}(y)$ for $s \in (\frac{5}{2}, 3)$. More precise estimates can be obtained if one applies the error estimates to $u = G_y - \eta\tilde{G}_y$ where $\tilde{G}_y$ is a suitable Green's function in $\mathbb{R}^2$ and η is a suitable cut-off function (see below).

One can also use Theorem 2.1.3 to obtain quantitative error estimates for $G_h - G$ and its discrete derivatives.

Let us briefly discuss some other approaches to prove Theorem 2.1.3. For $\mathsf{d} = 2$ the estimates (2.1.8) and (2.1.12) as well as a discrete BMO estimate for the mixed derivative were proved in the author's M.Sc. thesis [Sch16]. There a different approach was used to obtain the estimates near the corners. One starts from a discrete biharmonic function, defines a careful interpolation to get a continuous functions which is biharmonic up to a small error and uses the continuous theory to get good estimates for that interpolation which can then be transferred back to the original discrete function. This approach can in principle be extended to $\mathsf{d} = 3$, but we found the compactness argument more flexible and more convenient to use.

Hackbusch [Hac83, Thm. 2.1] has developed a very general approach to derive discrete stability estimates on a scale of Banach spaces from the corresponding continuous estimates. One advantage of the compactness method is that it avoids the construction of suitable discrete norms and restriction and prolongation operators which is a bit delicate near the singular boundary points.

Alternatively, for $\mathsf{d} = 2$ and the symmetric boundary condition (2.1.14) one can use the optimal error estimates (2.1.15) in connection with the asymptotic expansion of the discrete Green's function $\tilde{G}_{h,y}$ on $(h\mathbb{Z})^2$ in [Man67] (see also Section 2.7). One applies the estimate (2.1.15) with $s = 3$ to $u = G_y - \eta\tilde{G}_y$ where $\tilde{G}_y$ is a suitable Green's function in $\mathbb{R}^2$. It is not difficult to estimate the additional error term $w_h = G_h - \eta\tilde{G}_h - \hat{u}_h$ in the discrete $W^{2,2}$ norm by computing $\Delta_h^2 w_h$ and testing with w_h. This yields the estimate $\|\hat{G}_{h,y} - G_y\|_{W^{2,2}(\Lambda_h^2)} \leq Ch$ and the discrete inverse estimate implies that $\|\hat{G}_{h,y} - G_y\|_{W^{2,\infty}(\Lambda_h^2)} \leq C$. Together with the known estimates for $\nabla^2 G_y$ one concludes in particular that

$$|\nabla_h^2 \hat{G}_{h,y}| \leq Cd^2(y)/(|x-y|+h)^2 \quad \text{for } |x-y| \leq Cd(y)\,. \tag{2.1.16}$$

To get the optimal estimate for $|x-y| \gg d(y)$ one may proceed as follows. From the estimate for $|x-y| \leq Cd(y)$ one can obtain the crucial discrete $L^\infty - L^2$ estimate (2.6.1) for

the second discrete derivatives for cubes of length $2r$ that touch the boundary by using the identity $u(x) = \sum_{y \in \operatorname{int} \Lambda_h} \hat{G}_h(x,y) \Delta_h^2(\eta u)(y) h^2$ for an arbitrary lattice function u and a suitable cut-off function η with $|\nabla_h^k \eta| \leq C_k r^{-k}$. For cubes which do not touch the boundary one can apply the identity $v(x) = \sum_{y \in \operatorname{int} \Lambda_h} \hat{G}_h(x,y) \Delta_h^2(\eta v)(y) h^2$ to $v(x) = u(x) - a - b \cdot x$ where a is the average of u over the cube and b is the average of $\nabla_h u$. Together with the duality argument in Lemma 2.6.2 and Theorem 2.6.3 and similar estimates for the discrete y-derivatives of $G_y - \hat{G}_{h,y}$ this yields the estimates in Theorem 2.1.3 for $\mathsf{d} = 2$ for the Green's function $\hat{G}_{h,y}$ which satisfies the modified boundary conditions (2.1.14). The same argument applies to G_h.

These estimates initially hold for $\hat{G}_{h,y}$ and not for the function $G_{h,y}$ in Theorem 2.1.3. Note, however, that $\Delta_h^2(G_{h,y} - \hat{G}_{h,y}) = 0$ in $\operatorname{int} \Lambda_h$. Using this fact as well as careful comparison of the different boundary conditions for $\hat{G}_h$ and G_h one can show that $\|\hat{G}_{h,y} - G_{h,y}\|_{W^{2,2}(\Lambda_h)} \leq Ch$. This shows that the estimate (2.1.16) also holds for G_h. For the estimates for $|x-y| \gg d(y)$ one can then argue as for $\hat{G}_h$.

The remainder of this chapter is organised as follows. In Section 2.2 we introduce some notation in the discrete setting and recall discrete counterparts of the product rule as well as Sobolev and Poincaré estimates. In Section 2.3 we give the weak and strong formulation of the discrete Bilaplace equation and prove the Caccioppoli inequality (or reverse Poincaré inequality). The proof is very similar to the argument in the continuous case based on testing the equation with a cut-off function times the solution, but due to the discrete product rule some additional terms appear. In Section 2.4 we associate to each discrete function a continuous function by discrete convolution with a B-spline and prove basic estimates of the interpolation.

Sections 2.5 and 2.6 contain the key estimates. The first key ingredient is an $L^\infty - L^2$ estimate for the discrete second derivative of discrete biharmonic functions in cubes which may intersect the boundary (see Theorem 2.6.1). This estimate is deduced from decay estimates for the second derivative of continuous biharmonic functions using a discrete version of the Kolmogorov-Riesz-Fréchet compactness criterion and the Caccioppoli inequality. The transition from continuous to discrete decay estimates is carried out in Section 2.5 separately for interior cubes, cubes near regular boundary points and cubes near singular boundary points.

The second key estimate is an L^∞ decay estimate for discretely biharmonic functions in the complement of a cube (see Lemma 2.6.2 and Theorem 2.6.3). This follows by duality from the $L^\infty - L^2$ estimate in Theorem 2.6.1. The estimates in the interior and near regular boundary points can alternatively be derived by using discrete scaled L^2 estimates, i.e., by translating the continuous Campanato regularity theory to the discrete setting (see Dolzmann [Dol93, Dol99]). For the behaviour near the singular boundary points there seems to be no argument, however, which is only based on scaled L^2-norms and testing. For ease of exposition we use the compactness approach in all three regimes: interior points, regular boundary points and singular boundary points.

In Section 2.7 we recall Mangad's [Man67] asymptotic expansion of a Green's function $\tilde{G}_h$ of the discrete biharmonic operator in $(h\mathbb{Z})^{\mathsf{d}}$. Finally in Section 2.8 we prove Theorem 2.1.3 and Corollary 2.1.4. An L^2 estimate for the second discrete derivatives of G_h is easily obtained by testing with G_h and Poincaré's inequality. We then choose a suitable cut-off function η_h and use the fact that $G_h(\cdot, y) - \eta_h(x)\tilde{G}_h(x-y)$ is biharmonic near $x = y$ to prove estimates for the mixed third discrete derivative $\nabla_{h,x}^2 \nabla_{h,y} G_h$. The estimates for the lower derivatives now follow essentially by discrete integration over suitable paths (the relevant

path are the discrete counterparts of the paths used in [DS04]). For the estimate for the first discrete derivatives for $\mathsf{d} = 3$ we directly use the discrete Sobolev embedding since integration of the second derivative would generate an unnecessary additional logarithmic term.

2.2 Preliminaries

2.2.1 Notation

In the following C denotes a constant that may change from line to line but is independent of h, unless stated otherwise.

Given $a \in \mathbb{R}^{\mathsf{d}}$, we define $\tau_a f = f(\cdot + a)$ for any f. This corresponds to shifting f by $-a$.

For a function f we denote by $[f]_\Omega = \frac{1}{|\Omega|}\int_\Omega f \,\mathrm{d}x$ its average over the bounded open set Ω.

For discrete quantities we choose notation in such a way that it resembles the continuous notation. Let $h > 0$ be the (typically small) lattice width. We consider the lattice $(h\mathbb{Z})^{\mathsf{d}} \subset \mathbb{R}^{\mathsf{d}}$.

For $r \in \mathbb{R}$ we define $\lfloor r \rfloor_h := h \left\lfloor \frac{r}{h} \right\rfloor$, the largest element of $h\mathbb{Z}$ less than or equal to r.

F

Given $A_h \subset (h\mathbb{Z})^{\mathsf{d}}$, we define a corresponding subset $(A_h)_{pc} \subset \mathbb{R}^{\mathsf{d}}$ as

$$(A_h)_{pc} = \operatorname{int}\left(A + \left[-\frac{h}{2}, \frac{h}{2}\right]^{\mathsf{d}}\right).$$

For example, for $x \in (h\mathbb{Z})^{\mathsf{d}}$, $r \in h\mathbb{N}$, $\left(Q_r^h(x)\right)_{pc} = Q_{r+\frac{h}{2}}(x)$. For a function $u_h \colon A_h \to \mathbb{R}$, we define its piecewise constant interpolation $I_h^{pc} u_h \colon A_{pc} \to \mathbb{R}$ by $I_h^{pc} u_h(y) = u_h(x)$ on each square $x + \left[-\frac{h}{2}, \frac{h}{2}\right)^{\mathsf{d}}$, where $x \in A$.

For a multi-index $\alpha \in \mathbb{N}^{\mathsf{d}}$ we define $D_{\pm h}^{\alpha} u_h(x) = (D_{\pm 1}^h)^{\alpha_1} \dots (D_{\pm \mathsf{d}}^h)^{\alpha_{\mathsf{d}}} u_h(x)$, and for $a \in \mathbb{N}$, $a > 2$ we set $\nabla_h^a u_h(x) = (D_{-i_1}^h D_{i_2}^h \dots D_{i_{\mathsf{d}}}^h u_h(x))_{i_1, i_2, \dots, i_{\mathsf{d}}}$.

The discrete product rule then takes the form

$$D_i^h(f_h g_h) = (D_i^h f_h) g_h + \tau_i^h f_h D_i^h g_h \,.$$

When dealing with functions of several variables we use a sub- or superscript to indicate the variable with respect to which a derivative is taken. So for example in $\nabla_{h,x} \nabla_{h,y} G_h(x, y)$ we take one gradient in each variable.

As mentioned in the introduction, we set $\Lambda_h = [0,1]^{\mathsf{d}} \cap (h\mathbb{Z})^{\mathsf{d}}$ and $\operatorname{int} \Lambda_h = \left[\frac{1}{h}, 1 - \frac{1}{h}\right]^{\mathsf{d}} \cap (h\mathbb{Z})^{\mathsf{d}}$. We also set $\partial \Lambda_h = \Lambda_h \setminus \operatorname{int} \Lambda_h$.

2.2.2 Function spaces and inequalities

Let $u_h, v_h \colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$. For $\Omega \subset \mathbb{R}^{\mathsf{d}}$ measurable, $p \in [1, \infty]$, $k \in \mathbb{N}$, $\alpha \in [0,1]$ we define (slightly abusing notation)

$$\begin{aligned}
\|u_h\|_{L^p(\Omega)} &:= \|I_h^{pc} u_h\|_{L^p(\Omega)}\,, \\
(u_h, v_h)_{L^2(\Omega)} &:= \left(I_h^{pc} u_h, I_h^{pc} v_h\right)_{L^2(\Omega)}\,, \\
\|u_h\|_{W^{k,p}(\Omega)} &:= \left(\sum_{|\alpha| \le k} \|I_h^{pc} D_h^{\alpha} u_h\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}\,,
\end{aligned}$$

$$[u_h]_{C_h^{0,\alpha}(\Omega)} = \sup_{\substack{x,y\in\Omega \\ |x-y|\geq h}} \frac{|I_h^{pc}u_h(x) - I_h^{pc}u_h(y)|}{|x-y|^\alpha}.$$

For $[\cdot]_{C_h^{0,\alpha}}$ we add the index h to emphasize the fact that we only take the supremum over x, y with $|x-y| \geq h$.

For $A_h \subset (h\mathbb{Z})^{\mathsf{d}}$ these definitions take a familiar form. For example, if $p < \infty$

$$\|u_h\|_{L^p((A_h)_{pc})} = \left(\sum_{x\in A_h} h^{\mathsf{d}}|u_h(x)|^p\right)^{\frac{1}{p}},$$

$$[u_h]_{C_h^{0,\alpha}((A_h)_{pc})} = \sup_{\substack{x,y\in A_h \\ x\neq y}} \frac{|u_h(x) - u_h(y)|}{|x-y|^\alpha}.$$

We extend these definitions to vector-valued functions by taking the Euclidean norm of the norms of the components.

We also set $[u_h]_\Omega = [I_h^{pc}u_h]_\Omega = \frac{1}{|\Omega|}\int_\Omega I_h^{pc}u_h$.

We then have the discrete analogues of Poincaré and Sobolev inequalities. All of them can be proved easily by applying their continuous counterpart to the piecewise multilinear interpolation of the function. We state the results that we will need.

Lemma 2.2.1 (Poincaré inequality on cubes with 0 boundary values)**.** *Let $p \in [1,\infty]$, let $u_h\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, $x \in (h\mathbb{Z})^{\mathsf{d}}$, $r \in h\mathbb{N} + \frac{h}{2}$, and suppose that $u_h = 0$ on at least one of the faces of $Q_r^h(x)$. Then*

$$\|u_h\|_{L^p(Q_r(x))} \leq Cr\|\nabla_h u_h\|_{L^p(Q_r(x))}$$

where C is independent of h and r.

Lemma 2.2.2 (Poincaré inequality on annuli with 0 boundary values)**.** *Let $p \in [1,\infty]$, $u_h\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, let $x \in (h\mathbb{Z})^{\mathsf{d}}$, $r, s \in h\mathbb{N} + \frac{h}{2}$, $s < r$ and suppose that $u_h = 0$ on at least one of the faces of $Q_r^h(x)$. Then*

$$\|u_h\|_{L^p(Q_r(x)\setminus Q_s(x))} \leq Cr\|\nabla_h u_h\|_{L^p(Q_r(x)\setminus Q_s(x))}$$

where C only depends on $\frac{s}{r}$, p and d.

Lemma 2.2.3 (Sobolev-Poincaré inequality on cubes with 0 boundary values)**.** *Let $p \in [1,\infty]$, $u_h\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, let $x \in (h\mathbb{Z})^{\mathsf{d}}$, $r \in h\mathbb{N} + \frac{h}{2}$, and suppose that $u_h = 0$ on at least one of the faces of $Q_r^h(x)$.*

If $q \in [1,\infty]$ is such that $\frac{\mathsf{d}}{q} + 1 \geq \frac{\mathsf{d}}{p}$ and $(p,q) \neq (\mathsf{d},\infty)$, then

$$\|u_h\|_{L^q(Q_r(x))} \leq Cr^{1+\frac{\mathsf{d}}{q}-\frac{\mathsf{d}}{p}}\|\nabla_h u_h\|_{L^p(Q_r(x))}$$

and if $\alpha \in (0,1]$ is such that $\alpha + \frac{\mathsf{d}}{p} \leq 1$, then

$$[u_h]_{C_h^{0,\alpha}(Q_r(x))} \leq Cr^{1-\frac{\mathsf{d}}{p}-\alpha}\|\nabla_h u_h\|_{L^p(Q_r(x))}.$$

2.3 The discrete Bilaplacian equation

2.3.1 Definitions and basic properties

We consider the space of functions

$$\Phi_h = \{u_h \colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R} \colon u_h(x) = 0 \; \forall x \in (h\mathbb{Z})^{\mathsf{d}} \setminus \operatorname{int} \Lambda_h\} .$$

The discrete Bilaplacian equation on Λ_h with 0 boundary data is the equation

$$\Delta_h^2 u_h = f_h \;\text{ in }\; \operatorname{int} \Lambda_h \tag{2.3.1}$$

where $f_h \colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$ is given and we are looking for a solution $u_h \in \Phi_h$.

This equation is the discrete analogue of the Bilaplace equation with clamped boundary conditions,

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } [0,1]^{\mathsf{d}} , \\ u &= 0 \quad \text{on } \partial[0,1]^{\mathsf{d}} , \\ D_\nu u &= 0 \quad \text{on } \partial[0,1]^{\mathsf{d}} . \end{aligned}$$

If we multiply (2.3.1) with a test function $\varphi_h \in \Phi_h$ and use summation by parts, we obtain the weak form of the Bilaplace equation

$$(\nabla_h^2 u_h, \nabla_h^2 \varphi_h)_{L^2(\mathbb{R}^{\mathsf{d}})} = (f_h, \varphi_h)_{L^2(\mathbb{R}^{\mathsf{d}})} \quad \forall \varphi_h \in \Phi_h . \tag{2.3.2}$$

It is easy to check that (2.3.1) and (2.3.2) are equivalent.

Written as a sum over lattice points, (2.3.2) becomes

$$h^{\mathsf{d}} \sum_{x \in \Lambda_h} \nabla_h^2 u_h(x) : \nabla_h^2 \varphi_h(x) = h^{\mathsf{d}} \sum_{x \in \operatorname{int} \Lambda_h} f_h(x) \varphi_h(x) .$$

Observe that the sum on the left-hand side has nonzero terms for $x \in \Lambda_h$, whereas the right-hand side has nonzero terms only for $x \in \operatorname{int} \Lambda_h$.

If we choose $\varphi_h = u_h$ in (2.3.2), we obtain

$$(\Delta_h^2 u_h, u_h)_{L^2(\mathbb{R}^{\mathsf{d}})} = (\nabla_h^2 u_h, \nabla_h^2 u_h)_{L^2(\mathbb{R}^{\mathsf{d}})} = \|\nabla_h^2 u_h\|^2_{L^2(\mathbb{R}^{\mathsf{d}})} .$$

Hence Δ_h^2, seen as a linear operator on Φ_h, is positive definite and hence invertible, and so (2.3.1) has a unique solution for any right-hand side f_h.

The discrete Green's function G_h is now defined as the inverse of Δ_h^2 (considered as a matrix operating on $\mathbb{R}^{\operatorname{int} \Lambda_h}$ with the scalar product $\langle u_h, v_h \rangle = (u_h, v_h)_{L^2(\mathbb{R}^{\mathsf{d}})}$).

Let us also give an alternative description of G_h: The discrete delta function is given as

$$\delta_{h,x}(y) = \begin{cases} \frac{1}{h^{\mathsf{d}}} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} .$$

The discrete Green's function G_h of Λ_h^{d} is then the function $(h\mathbb{Z})^{\mathsf{d}} \times (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$ such that $G_h(x,y) = 0$ when $y \notin \operatorname{int} \Lambda_h$ and such that $G_h(\cdot, y)$ is the unique solution in Φ_h of

$$\Delta_h^2 u_h = \delta_{h,y} \text{ in } \operatorname{int} \Lambda_h$$

when $y \in \operatorname{int} \Lambda_h$.

As in the continuous case one can easily show that G_h is symmetric in x and y. We will frequently denote $G_h(x, \cdot)$ and $G_h(\cdot, y)$ by $G_{h,x}$ and $G_{h,y}$ respectively.

Let us return our attention to (2.3.2) for a moment. If f_h is given in divergence form as $\operatorname{div}_h \operatorname{div}_{-h} g_h$, this equation takes the form

$$(\nabla_h^2 u_h, \nabla_h^2 \varphi_h)_{L^2(\mathbb{R}^d)} = (g_h, \nabla_h^2 \varphi_h)_{L^2(\mathbb{R}^d)}$$

and if we choose $\varphi_h = u_h$, we obtain the energy estimate

$$\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d)} \leq \|g_h\|_{L^2(\mathbb{R}^d)} .$$

2.3.2 Caccioppoli inequalities

We will need a discrete counterpart of the Caccioppoli (or reverse Poincaré) estimate for biharmonic functions (see e.g. [Cam80, Cap. II, Lemma 1.II]). It can be derived by testing $\Delta_h^2 u_h = 0$ with $\eta_h u_h$ for a suitable cut-off function η_h and some manipulations of the error terms.

Lemma 2.3.1. *Let* $\mathsf{d} \in \mathbb{N}$, $u_h \in \Phi_h$, $x \in (h\mathbb{Z})^{\mathsf{d}}$, $r > 0$ *and assume that* $\Delta_h^2 u_h(y) = 0$ *for all* $y \in Q_{r-h}^h(x) \cap \operatorname{int} \Lambda_h$. *Then for any* $0 < s \leq r - 4h$ *we have*

$$\|\nabla_h^2 u_h\|_{L^2(Q_s(x))}^2 \leq \frac{C}{(r-s)^4} \|u_h\|_{L^2(Q_r(x))}^2 + \frac{C}{(r-s)^2} \|\nabla_h u_h\|_{L^2(Q_r(x))}^2 .$$

The proof is similar to the continuous case. However, the fact that the discrete chain rule only holds up to translations generates additional error terms. Therefore we will give the somewhat lenghty proof in full detail. The proof is adapted from that of Lemma 2.9 in [Dol93].

Proof. By replacing r by $\lfloor r - \frac{h}{2} \rfloor_h + \frac{h}{2}$ and s by $\lfloor s - \frac{h}{2} \rfloor_h + \frac{3h}{2}$, we can assume that $r, s \in h\mathbb{Z} + \frac{h}{2}$ and $s \leq r - 3h$.

Choose a discrete cut-off function η_h with support in $Q_{r-2h}(x)$ that is 1 on $Q_{s+h}(x)$ und such that $|\nabla_h^\kappa \eta| \leq \frac{C}{(r-s)^\kappa}$ for $\kappa \leq 2$. Note that $\eta_h^4 u_h \in \Phi_h$, and $\eta_h^4 u_h = 0$ whenever $\Delta_h^2 u_h \neq 0$. Thus the weak form of (2.3.2) with $\varphi_h = \eta_h^4 u_h$ is

$$0 = \left(\Delta_h^2 u_h, \eta_h^4 u_h\right)_{L^2(\mathbb{R}^d)} = \left(\nabla_h^2 u_h, \nabla_h^2 (\eta_h^4 u_h)\right)_{L^2(\mathbb{R}^d)} .$$

We can expand the right-hand side and obtain

$$\begin{aligned}
0 &= \left(\nabla_h^2 u_h, \nabla_h^2 (\eta_h^4 u_h)\right)_{L^2(\mathbb{R}^d)} \\
&= \sum_{i,j}^{\mathsf{d}} \left(D_{-i}^h D_j^h u_h, \eta_h^4 D_{-i}^h D_j^h u_h\right)_{L^2(\mathbb{R}^d)} \\
&\quad + \sum_{i,j}^{\mathsf{d}} \left(D_{-i}^h D_j^h u_h, D_j^h(\eta_h^4) \tau_j^h D_{-i}^h u_h + D_{-i}^h(\eta_h^4) \tau_{-i}^h D_j^h u_h\right)_{L^2(\mathbb{R}^d)} \\
&\quad + \sum_{i,j}^{\mathsf{d}} \left(D_{-i}^h D_j^h u_h, D_{-i}^h D_j^h(\eta_h^4) \tau_{-i}^h \tau_j^h u_h\right)_{L^2(\mathbb{R}^d)} .
\end{aligned}$$

We can rewrite this as

$$\begin{aligned}
\|\eta_h^2 \nabla_h^2 u_h\|_{L^2(\mathbb{R}^d)}^2 &= \sum_{i,j}^{d} \left\| \eta_h^2 D_{-i}^h D_j^h u_h \right\|_{L^2(\mathbb{R}^d)}^2 \\
&\leq \left| \sum_{i,j}^{d} \left(D_{-i}^h D_j^h u_h, D_j^h(\eta_h^4) \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} \right| + \left| \sum_{i,j}^{d} \left(D_{-i}^h D_j^h u_h, D_{-i}^h(\eta_h^4) \tau_{-i}^h D_j^h u_h \right)_{L^2(\mathbb{R}^d)} \right| \\
&\quad + \left| \sum_{i,j}^{d} \left(D_{-i}^h D_j^h u_h, D_{-i}^h D_j^h(\eta_h^4) \tau_{-i}^h \tau_j^h u_h \right)_{L^2(\mathbb{R}^d)} \right| .
\end{aligned} \tag{2.3.3}$$

We will estimate the terms on the right-hand side separately.

Using $\frac{a^4-b^4}{a-b} = a^3 + a^2 b + ab^2 + b^3$ for $a = \eta_h^4 \circ \tau_j^h$ and $b = \eta_h^4$ we can rewrite the summands of the first term as

$$\begin{aligned}
&\left(D_{-i}^h D_j^h u_h, D_j^h(\eta_h^4) \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} \\
&= \left(D_{-i}^h D_j^h u_h, \left(\eta_h^3 + \eta_h^2 \tau_j^h \eta_h + \eta_h \tau_j^h \eta_h^2 + \tau_j^h \eta_h^3 \right) D_j^h \eta_h \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} \\
&= \left(D_{-i}^h D_j^h u_h, 4\eta_h^3 D_j^h \eta_h \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} \\
&\quad + \left(D_{-i}^h D_j^h u_h, \left(\eta_h^2(\tau_j^h \eta_h - \eta_h) + \eta_h(\tau_j^h \eta_h^2 - \eta_h^2) + (\tau_j^h \eta_h^3 - \eta_h^3) \right) D_j^h \eta_h \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} .
\end{aligned}$$

The second term here is problematic[1], because it does not contain a factor $\eta_h^2 D_{-i}^h D_j^h u_h$. We will control it by moving a factor $\frac{1}{h}$ from the left-hand side to the right-hand side, so that we are no longer taking second derivatives of u_h. We obtain

$$\begin{aligned}
&\left(D_{-i}^h D_j^h u_h, D_j^h(\eta_h^4) \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} \\
&\quad = \left(\eta_h^2 D_{-i}^h D_j^h u_h, 4\eta_h D_j^h \eta_h \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} \\
&\qquad + \left(\tau_j^h D_{-i}^h u_h - D_{-i}^h u_h, \left(\eta_h^2 D_j^h \eta_h + \eta_h D_j^h(\eta_h^2) + D_j^h(\eta_h^3) \right) D_j^h \eta_h \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} .
\end{aligned}$$

Therefore, using the Cauchy-Schwarz inequality, $ab \leq \delta a^2 + \frac{1}{4\delta} b^2$ and the pointwise bounds on η_h and its derivatives we get

$$\begin{aligned}
&\left| \sum_{i,j}^{d} \left(D_{-i}^h D_j^h u_h, D_j^h(\eta_h^4) \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} \right| \\
&\quad = \left| \sum_{i,j}^{d} \left(\eta_h^2 D_{-i}^h D_j^h u_h, 4\eta_h D_j^h \eta_h \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} \right| \\
&\qquad + \left| \sum_{i,j}^{d} \left(\tau_j^h D_{-i}^h u_h - D_{-i}^h u_h, \left(\eta_h^2 D_j^h \eta_h + \eta_h D_j^h(\eta_h^2) + D_j^h(\eta_h^3) \right) D_j^h \eta_h \tau_j^h D_{-i}^h u_h \right)_{L^2(\mathbb{R}^d)} \right| \\
&\quad \leq \frac{1}{4} \|\eta_h^2 \nabla_h^2 u_h\|_{L^2(\mathbb{R}^d)}^2 + \sum_{i,j}^{d} \left\| 4\eta_h D_j^h \eta_h \tau_j^h D_{-i}^h u_h \right\|_{L^2(Q_{r-h}(x))}^2 \\
&\qquad + \frac{1}{2(r-s)^2} \sum_{i,j}^{d} \left\| \tau_j^h D_{-i}^h u_h - D_{-i}^h u_h \right\|_{L^2(Q_{r-h}(x))}^2
\end{aligned}$$

[1] Note that in a continuous setting this term would not occur at all.

$$+ \frac{(r-s)^2}{2} \sum_{i,j}^{\mathsf{d}} \left\| \left(\eta_h^2 D_j^h \eta_h + \eta_h D_j^h(\eta_h^2) + D_j^h(\eta_h^3) \right) D_j^h \eta_h \tau_j^h D_{-i}^h u_h \right\|_{L^2(Q_{r-h}(x))}^2$$
$$\leq \frac{1}{4} \|\eta_h^2 \nabla_h^2 u_h\|_{L^2(\mathbb{R}^{\mathsf{d}})}^2 + \frac{C}{(r-s)^4} \|u_h\|_{L^2(Q_r(x))}^2 + \frac{C}{(r-s)^2} \|\nabla_h u_h\|_{L^2(Q_r(x))}^2 .$$

Analogously we can find the same upper bound for the other two terms on the right-hand side of (2.3.3). Then we obtain

$$\|\eta_h^2 \nabla_h^2 u_h\|_{L^2(\mathbb{R}^{\mathsf{d}})}^2 \leq \frac{3}{4} \|\eta_h^2 \nabla_h^2 u_h\|_{L^2(\mathbb{R}^{\mathsf{d}})}^2 + \frac{C}{(r-s)^4} \|u_h\|_{L^2(Q_r(x))}^2 + \frac{C}{(r-s)^2} \|\nabla_h u_h\|_{L^2(Q_r(x))}^2$$

and hence

$$\|\eta_h^2 \nabla_h^2 u_h\|_{L^2(\mathbb{R}^{\mathsf{d}})}^2 \leq \frac{C}{(r-s)^4} \|u_h\|_{L^2(Q_r(x))}^2 + \frac{C}{(r-s)^2} \|\nabla_h u_h\|_{L^2(Q_r(x))}^2 .$$

This implies the claim, once one notes that $\|\nabla_h^2 u_h\|_{L^2(Q_s(x))} \leq \|\eta_h^2 \nabla_h^2 u_h\|_{L^2(\mathbb{R}^{\mathsf{d}})}$. □ □

2.4 Interpolation

We want to deduce discrete estimates from their continuous counterparts using compactness arguments. To do so, we need an interpolation operator that turns discrete functions into continuous functions having similar features. The most important property of this interpolation operator that we require is that the continuous derivatives of the output are comparable to the discrete derivatives of the input.

To construct such an operator we use B-splines (cf., e.g., [Sch81, §4.4]): For $m \geq 1$, $x \in \mathbb{R}$ the m-th normalized B-spline is given by

$$N^m(x) = m \sum_{i=0}^{m} \frac{(-1)^i \binom{m}{i} \max(x-i,0)^{m-1}}{m!} .$$

The function N^m is piecewise a polynomial of degree $m-1$, has support in $[0,m]$ and satisfies $\sum_{z \in \mathbb{Z}} N^m(x-z) = 1$ for all $x \in \mathbb{R}$. Furthermore its discrete and continuous derivatives are closely related. Indeed we have

$$\partial_x N^m(x) = N^{m-1}(x) - N^{m-1}(x-1) = D_{-1}^1 N^{m-1}(x) \tag{2.4.1}$$

for all $x \in \mathbb{R}$ (see [Sch81] for proofs).

We need a multidimensional version of these splines which is also adapted to the lattice $(h\mathbb{Z})^{\mathsf{d}}$. So for $h > 0$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^{\mathsf{d}}$ with $\mu_i \geq 1$ let

$$N_h^\mu(x_1, \ldots, x_n) = N^{\mu_1}\left(\frac{x_1}{h}\right) \cdots N^{\mu_n}\left(\frac{x_n}{h}\right) .$$

It follows easily from (2.4.1) that for any $\alpha \in \mathbb{N}^{\mathsf{d}}$ with $\alpha_i < \mu_i$ for all i we have

$$D^\alpha N_h^\mu = D_{-h}^\alpha N_h^{\mu-\alpha} . \tag{2.4.2}$$

Using this, we can define our interpolation operator:

Definition 2.4.1. Let $h > 0$, $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^{\mathsf{d}}$ with $\mu_i \geq 1$ for all i. Define $J_h^\mu \colon \mathbb{R}^{(h\mathbb{Z})^{\mathsf{d}}} \to L^1_{loc}(\mathbb{R}^{\mathsf{d}})$ by

$$(J_h^\mu u_h)(x) = \sum_{z \in (h\mathbb{Z})^{\mathsf{d}}} u_h(z) N_h^\mu(x - z)$$

and extend J_h^μ to vector-valued functions component-wise.

Note that N_h^μ has compact support so that the above sum has only finitely many nonzero terms.

J_h^μ does not interpolate the values of u_h (i.e. in general we will not have $J_h^\mu u_h(x) = u_h(x)$ for all $x \in (h\mathbb{Z})^{\mathsf{d}}$). The maps $J_h^\mu u_h$ and u_h, however, share so many properties that we still call J_h^μ an interpolation operator.

Let us collect some properties of J_h^μ.

Proposition 2.4.2. *Let J_h^μ be the family of interpolation operators that we have just defined, and let $u_h \colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$.*

i) J_h^μ is linear.

ii) $J_h^\mu u_h$ is piecewise a polynomial and is in the Sobolev space $W_{loc}^{(\min_i \mu_i)-1,2}$

iii) J_h^μ is local in the sense that $(J_h^\mu u_h)(x)$ only depends on the values of u_h in $Q_{(\max_i \mu_i)h}(x)$.

iv) J_h^μ preserves constant functions, i.e. $(J_h^\mu c)(x) = c$ for any $c \in \mathbb{R}$ and any $x \in \mathbb{R}^{\mathsf{d}}$.

v) For every α with $\alpha_i < \mu_i$ we have $(D^\alpha J_h^\mu u_h)(x) = (J_h^{\mu-\alpha}(D_h^\alpha u_h))(x)$.

vi) For every α with $\alpha_i < \mu_i$ and any $p \in [1, \infty]$ there is a constant $C = C(\mu, \alpha, \mathsf{d}, p)$ such that for any $x \in \mathbb{R}^{\mathsf{d}}$ and any $r \geq s + (1 + \max_i \mu_i)h$ we have

$$\|D^\alpha J_h^\mu u_h\|_{L^p(Q_s(x))} \leq C \|D_h^\alpha u_h\|_{L^p(Q_r(x))} \tag{2.4.3}$$

and

$$\|D_h^\alpha u_h\|_{L^p(Q_s(x))} \leq C \|D^\alpha J_h^\mu u_h\|_{L^p(Q_r(x))} \,. \tag{2.4.4}$$

Proof. Properties i), ii) and iii) are obvious. Property iv) easily follows from $\sum_{z \in \mathbb{Z}} N^m(x - z) = 1$ for all $x \in \mathbb{R}$, so it remains to prove v) and vi).

For v), note that we can assume that u_h is zero far away from x by iii). This means that all sums in the following calculations have only finitely many nonzero terms. Now, using (2.4.2), we can calculate that

$$\begin{aligned}
(D^\alpha J_h^\mu u_h)(x) &= D^\alpha \left(\sum_{z \in (h\mathbb{Z})^{\mathsf{d}}} u_h(z) N_h^\mu(x - z) \right) \\
&= \sum_{z \in (h\mathbb{Z})^{\mathsf{d}}} u_h(z) D^\alpha N_h^\mu(x - z) \\
&= \sum_{z \in (h\mathbb{Z})^{\mathsf{d}}} u_h(z) D_{-h}^\alpha N_h^{\mu-\alpha}(x - z) \\
&= \sum_{z \in (h\mathbb{Z})^{\mathsf{d}}} D_h^\alpha u_h(z) N_h^{\mu-\alpha}(x - z) = (J_h^{\mu-\alpha}(D_h^\alpha u_h))(x) \,.
\end{aligned}$$

Finally we prove vi). In view of v) it is sufficient to consider the case $\alpha = 0$ here. We can also assume that $x \in (h\mathbb{Z})^{\mathsf{d}}$ and $r, s \in h\mathbb{N} + \frac{h}{2}$, $r \geq s + (\max_i \mu_i)h$ (otherwise move x to the nearest lattice point, and replace r and s by $\lfloor r - \frac{h}{2} \rfloor_h + \frac{h}{2}$ and $\lfloor s - \frac{h}{2} \rfloor_h + \frac{3h}{2}$ respectively).

Let $y \in Q_s^h(x)$. The definition of J_h^μ immediately implies

$$\|J_h^\mu u_h\|_{L^\infty(Q_{h/2}(y))} \leq C \sup_{\substack{z \in (h\mathbb{Z})^{\mathsf{d}} \\ |z-y| \leq (\max_i \mu_i)h}} |u_h(z)|$$

and thus

$$\|J_h^\mu u_h\|_{L^p(Q_{h/2}(y))}^p \leq C \sum_{\substack{z \in (h\mathbb{Z})^{\mathsf{d}} \\ |z-y| \leq (\max_i \mu_i)h}} |u_h(z)|^p \leq C \|u_h\|_{L^p(Q_{(\max_i \mu_i + 1/2)h}(y))}^p .$$

If we sum this over all $y \in Q_s^h(x)$, we easily obtain (2.4.3).

For (2.4.4), by a similar argument it suffices to show

$$|u_h(y)| \leq C \|J_h^\mu u_h\|_{L^p(Q_{h/2}(y))} \tag{2.4.5}$$

for all $y \in Q_s^h(x)$.

One can see this as follows: N_h^μ has support $[0, \mu_1] \times \cdots \times [0, \mu_n]$. This means that the values of $J_h^\mu u_h$ in $Q_{h/2}(y)$ depend on the finitely many values $\{u_h(z)\}_{z \in I_y}$, where $I_y := [y_1 - \mu_1] \times \cdots \times [y_{\mathsf{d}} - \mu_{\mathsf{d}}] \cap (h\mathbb{Z})^{\mathsf{d}}$ and no others. Furthermore by linear independence of the B-splines (see [Sch81, Theorem 4.18] for the one-dimensional case; the d-dimensional case is analogous) $J_h^\mu u_h$ is identically 0 in $Q_{h/2}(y)$ only if all $\{u_h(z)\}_{z \in I_y}$ are 0. This means that $\|J_h^\mu u_h\|_{L^p(Q_{h/2}(y))}$ is not only a seminorm on $\mathbb{R}^{I_y}$ but actually a norm. Now all norms on a finite-dimensional vector space are equivalent, so in particular

$$\|u_h\|_{l^2(I_y)} = \left(\sum_{z \in I_y} |u_h(z)|^2 \right)^{\frac{1}{2}} \leq C \|J_h^\mu u_h\|_{L^p(Q_{h/2}(y))}$$

for a constant C that is independent of y. This immediately implies (2.4.5).

□

Using these interpolation operators J_h^μ we define the two operators that we will actually use most often: One is $J_h := J_h^{(3,3,\ldots,3)}$ and the other is the matrix interpolation operator $\tilde{J}_h$ given by $(\tilde{J}_h)_{ij} = J_h^{(3,3,\ldots,3)-e_i-e_j} \circ \tau_i^h$ (for example $(\tilde{J}_h)_{11} = J_h^{(1,3,\ldots,3)} \circ \tau_1^h$).

One easily checks using parts ii) and v) of Proposition 2.4.2 that for any $f_h \colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$ we have $J_h f_h \in W_{loc}^{2,2}(\mathbb{R}^{\mathsf{d}})$ and

$$\nabla^2 J_h f_h = \tilde{J}_h \nabla_h^2 f_h . \tag{2.4.6}$$

2.5 Inner decay estimates for discrete biharmonic functions: special cases

Our goal is to prove an L^∞-L^2 estimate for discrete biharmonic functions (see Theorem 2.6.1): If $u_h \in \Phi_h$, $x \in \Lambda_h$, $r > 0$ and $\Delta_h^2 u_h(y) = 0$ for all $y \in Q_{r-h}(x) \cap \operatorname{int} \Lambda_h^{\mathsf{d}}$, then, for all $z \in Q_{\frac{r}{2}}(x) \cap \Lambda_h$,

$$|\nabla_h^2 u_h(z)| \leq \frac{C}{r^{\frac{\mathsf{d}}{2}}} \|\nabla_h^2 u_h\|_{L^2(Q_r(x))} .$$

To prove this estimate it will be necessary to distinguish where x lies in relation to $\partial\Lambda_h$: x can be far inside Λ_h, near a face, near an edge or near a vertex. In the following subsections we will study these cases separately and prove some decay estimates that we will then assemble to prove the aforementioned estimate.

2.5.1 Full space

Lemma 2.5.1. *Let* $\mathsf{d} \in \mathbb{N}$, $u_h \colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, *let* $x \in (h\mathbb{Z})^{\mathsf{d}}$, $r > 0$. *Suppose* $\Delta_h^2 u_h(y) = 0$ *for all* $y \in Q_{r-h}^h(x)$. *Then*

$$|\nabla_h^2 u_h(x)| \leq \frac{C}{r^{\frac{\mathsf{d}}{2}}} \|\nabla_h^2 u_h\|_{L^2(Q_r(x))} .$$

The main tool to prove this statement will be the following estimate:

Lemma 2.5.2. *Let* $\mathsf{d} \in \mathbb{N}$. *There exist constants* $M \in \mathbb{N}$, $0 < \rho < \frac{1}{2}$ *with the following property: Let* $u_h \colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, $r > 0$, *such that* $\Delta_h^2 u_h(y) = 0$ *for all* $y \in Q_{r-h}^h$. *Assume that* $\rho r \geq Mh$. *Then we have that*

$$\left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_{\rho r}} \right\|_{L^2(Q_{\rho r})}^2 \leq \rho^{n+1} \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_r} \right\|_{L^2(Q_r)}^2 .$$

We will prove this lemma by contradiction using a compactness argument and the following decay estimate for continuous biharmonic functions:

Lemma 2.5.3. *Let* $\mathsf{d} \in \mathbb{N}$, $0 < s \leq \frac{r}{2}$, $u \in W^{2,2}(Q_r)$ *such that* $\Delta^2 u = 0$ *weakly in* Q_r. *Then we have*

$$\left\| \nabla^2 u - \left[\nabla^2 u\right]_{Q_s} \right\|_{L^2(Q_s)}^2 \leq C \left(\frac{s}{r}\right)^{\mathsf{d}+\frac{3}{2}} \left\| \nabla^2 u - \left[\nabla^2 u\right]_{Q_r} \right\|_{L^2(Q_r)}^2 . \tag{2.5.1}$$

Proof. The estimate (2.5.1) expresses the fact that the second derivatives of biharmonic functions are in the Campanato space $\mathcal{L}^{2,\mathsf{d}+\frac{3}{2}} \simeq C^{0,3/4}$. The easiest way to show it is to use Schauder estimates for higher order elliptic equations as follows.

By scaling we can assume $r = 1$. By replacing u with $u - \frac{1}{2}\left[\nabla^2 u\right]_{Q_1} : x \otimes x$ we can assume that $\left[\nabla^2 u\right]_{Q_1} = 0$. Now by Schauder estimates (see e.g. [Mor66, Theorem 6.4.8] or [Cam80, Cap. II, Teorema 6.I]) we have that any $C^{0,\alpha}$-Hölder seminorm of $\nabla^2 u$ in $Q_{1/2}$ is bounded by the L^2-norm of $\nabla^2 u$ in Q_1. In particular, we have

$$\left[\nabla^2 u\right]_{C^{0,\frac{3}{4}}(Q_{1/2})} \leq C \left\|\nabla^2 u\right\|_{L^2(Q_1)} .$$

On the other hand, Jensen's inequality easily yields that

$$\begin{aligned} \left\| \nabla^2 u - \left[\nabla^2 u\right]_{Q_s} \right\|_{L^2(Q_s)}^2 &\leq \frac{1}{|Q_s|} \int_{Q_s} \int_{Q_s} |\nabla^2 u(y) - \nabla^2 u(y')|^2 \,\mathrm{d}y \,\mathrm{d}y' \\ &\leq C s^{\mathsf{d}+\frac{3}{2}} \left[\nabla^2 u\right]_{C^{0,\frac{3}{4}}(Q_{1/2})}^2 . \end{aligned}$$

Together with the previous estimate this yields the result. □

We will also need a local version of the well-known Kolmogorov-Riesz-Fréchet compactness theorem.

Lemma 2.5.4. *Let* $\mathsf{d} \in \mathbb{N}$, $p \in [1, \infty)$, *let* $U, V, W \subset \mathbb{R}^{\mathsf{d}}$ *be open with* U *compactly contained in* V, *and* V *compactly contained in* W. *Let* A *be a subset of* $L^p(W)$.

i) If A is bounded in $L^p(W)$ and

$$\lim_{\delta\to 0}\sup_{f\in A}\|\tau_\delta f - f\|_{L^p(V)} = 0$$

then A (or rather the restriction of the elements of A to U) is precompact in $L^p(U)$.

ii) If A is precompact in $L^p(W)$ then

$$\lim_{\delta\to 0}\sup_{f\in A}\|\tau_\delta f - f\|_{L^p(V)} = 0\,.$$

Proof. Part i) follows by applying the usual Kolmogorov-Riesz-Fréchet compactness theorem (see e.g. [Bre11, Corollary 4.27 and Exercise 4.34]) to the family $\{\eta f\colon f\in A\}$, where η is a smooth cut-off function that is 1 on U and 0 outside of V.

For part ii) let $\tilde V$ be open such that V is compactly contained in $\tilde V$ and $\tilde V$ is compactly contained in W, and let ζ be a cut-off function that is 1 on $\tilde V$ and 0 outside of W. Then the family $\{\zeta f\colon f\in A\}$ is precompact in $L^p(\mathbb{R}^{\mathsf{d}})$ and the statement is obtained by applying the converse of the Kolmogorov-Riesz-Fréchet compactness theorem to that family. □

After these preparations we can return to the proofs of Lemma 2.5.1 and Lemma 2.5.2.

Proof of Lemma 2.5.2.
Step 1: Set-up of the compactness argument
Let the constant $\rho\le\frac12$ be fixed later, and suppose that the statement for that fixed ρ is wrong. Then for any $k\in\mathbb{N}$ there exist $M_k\ge k$, $h_k>0$, $u_{h_k}\colon (h_k\mathbb{Z})^{\mathsf{d}}\to\mathbb{R}$, $r_k>0$ such that

$$\left\|\nabla^2_{h_k}u_{h_k}-\left[\nabla^2_{h_k}u_{h_k}\right]_{Q_{\rho r_k}}\right\|^2_{L^2(Q_{\rho r_k})} > \rho^{\mathsf{d}+1}\left\|\nabla^2_{h_k}u_{h_k}-\left[\nabla^2_{h_k}u_{h_k}\right]_{Q_{r_k}}\right\|^2_{L^2(Q_{r_k})}\,. \tag{2.5.2}$$

By rescaling the lattice by a factor of r_k, we can assume that all the r_k are equal to 1. Because $h_k\le\frac{\rho}{M_k}\le\frac{\rho}{k}$, we have that $h_k\to 0$. Omitting finitely many k, we can assume that all h_k are small (less than $\frac{1}{1000}$, say).

By replacing u_{h_k} with $u_{h_k}-\frac12\left[\nabla^2_{h_k}u_{h_k}\right]_{Q_1}:x\otimes x$ we can assume that $\left[\nabla^2_{h_k}u_{h_k}\right]_{Q_1}=0$, and by scaling we can assume that $\left\|\nabla^2_{h_k}u_{h_k}\right\|_{L^2(Q_1)}=1$ (note that $\nabla^2_{h_k}u_{h_k}$ cannot be identically 0, as then u_{h_k} would be affine, and so both sides of (2.5.2) would be 0). Then (2.5.2) implies that

$$\left\|\nabla^2_{h_k}u_{h_k}-\left[\nabla^2_{h_k}u_{h_k}\right]_{Q_\rho}\right\|^2_{L^2(Q_\rho)} > \rho^{\mathsf{d}+1}\,. \tag{2.5.3}$$

Finally, we replace u_{h_k} by $u_{h_k}-a_k-b_k\cdot x$, where $a_k\in\mathbb{R}$, $b_k\in\mathbb{R}^{\mathsf{d}}$ are constants that will be chosen below (such that equation (2.5.4) is satisfied). This leaves $\nabla^2_{h_k}u_{h_k}$ unaffected, so all the above statements about $\nabla^2_{h_k}u_{h_k}$ remain true.

We let $v_k=J_{h_k}u_{h_k}$, where $J_{h_k}=J^{(3,\dots,3)}_{h_k}$ is the interpolation operator introduced in Section 2.4. From $\left\|\nabla^2_{h_k}u_{h_k}\right\|_{L^2(Q_1)}=1$ and Proposition 2.4.2 vi) we immediately conclude that $\|\nabla^2 v_k\|_{L^2(Q_{13/14})}\le C$.

Now we choose a_k and b_k in such a way that

$$[v_k]_{Q_{13/14}}=0\,,\qquad [\nabla v_k]_{Q_{13/14}}=0\,. \tag{2.5.4}$$

The Poincaré inequality on $Q_{13/14}$ implies that $\|v_k\|_{W^{2,2}(Q_{13/14})} \leq C\|\nabla^2 v_k\|_{L^2(Q_{13/14})} \leq C$. Therefore the v_k are bounded in $W^{2,2}(Q_{13/14})$ and hence have a subsequence (not relabeled) that converges weakly to some $v \in W^{2,2}(Q_{13/14})$.

Step 2: $\Delta^2 v = 0$

We claim that $\Delta^2 v = 0$ weakly in $Q_{13/14}$. To prove this, let $\varphi \in C_c^\infty(Q_{13/14})$ be arbitrary and let φ_{h_k} be its restriction to $(h_k\mathbb{Z})^{\mathsf{d}}$. We need to prove that $\int_{Q_{13/14}} \nabla^2 v : \nabla^2 \varphi \,\mathrm{d}x = 0$.

We have by (2.4.6) that

$$\begin{aligned}
\int_{Q_{13/14}} \nabla^2 v_k : \nabla^2 \varphi \,\mathrm{d}x &= \int_{Q_{13/14}} \nabla^2 J_{h_k} u_{h_k} : \nabla^2 \varphi \,\mathrm{d}x \\
&= \int_{Q_{13/14}} \tilde{J}_{h_k} \nabla^2_{h_k} v_k : \nabla^2 \varphi \,\mathrm{d}x \\
&= \sum_{i,j=1}^{\mathsf{d}} \int_{Q_{13/14}} J_{h_k}^{(3,3,\dots,3)-e_i-e_j} \circ \tau_i^{h_k} D_{-i}^{h_k} D_j^{h_k} v_k D_i D_j \varphi \,\mathrm{d}x \\
&= \sum_{i,j=1}^{\mathsf{d}} \int_{Q_{13/14}} \sum_{z\in(h_k\mathbb{Z})^{\mathsf{d}}} N_{h_k}^{(3,3,\dots,3)-e_i-e_j}(x-z) D_i^{h_k} D_j^{h_k} u_{h_k}(z) D_i D_j \varphi(x) \,\mathrm{d}x \\
&= \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in(h_k\mathbb{Z})^{\mathsf{d}}} D_i^{h_k} D_j^{h_k} u_{h_k}(z) \int_{Q_{13/14}} N_{h_k}^{(3,3,\dots,3)-e_i-e_j}(x-z) D_i D_j \varphi(x) \,\mathrm{d}x\,.
\end{aligned}$$

Now Taylor expansion and the fact that $\int_{Q_{13/14}} N_{h_k}^{(3,3,\dots,3)-\delta_i-\delta_j} = 1$ imply that

$$\int_{Q_{13/14}} N_{h_k}^{(3,3,\dots,3)-e_i-e_j}(x-z) D_i D_j \varphi(x) \,\mathrm{d}x = D_i D_j \varphi(z) + O(h_k) = D_i^{h_k} D_j^{h_k} \varphi_{h_k}(z) + O(h_k)$$

In addition, from $\Delta^2_{h_k} u_{h_k} = 0$ in $Q_{13/14}$ we conclude that

$$\begin{aligned}
\sum_{i,j=1}^{\mathsf{d}} \sum_{z\in(h_k\mathbb{Z})^{\mathsf{d}}} D_i^{h_k} D_j^{h_k} u_{h_k}(z) D_i^{h_k} D_j^{h_k} \varphi_{h_k}(z) &= \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in(h_k\mathbb{Z})^{\mathsf{d}}} D_{-i}^{h_k} D_j^{h_k} u_{h_k}(z) D_{-i}^{h_k} D_j^{h_k} \varphi_{h_k}(z) \\
&= (\nabla^2_{h_k} u_{h_k}, \nabla^2_{h_k} \varphi_{h_k})_{L^2(\mathbb{R}^{\mathsf{d}})} = 0
\end{aligned}$$

and so we obtain

$$\left| \int_{Q_{13/14}} \nabla^2 v_k : \nabla^2 \varphi \,\mathrm{d}x \right| \leq C \left\| \nabla^2_{h_k} u_{h_k} \right\|_{L^2(Q_1)} h_k = C h_k\,.$$

Using weak convergence of $\nabla^2 v_k$ we can pass to the limit here and get

$$\int_{Q_{13/14}} \nabla^2 v : \nabla^2 \varphi \,\mathrm{d}x = 0\,.$$

Step 3: Strong convergence of v_k

Let $w_k = I^{pc}_{h_k} \nabla^2_{h_k} u_{h_k}$. We claim that both $\nabla^2 v_k$ and w_k converge strongly in $L^2(Q_{1/2})$ to $\nabla^2 v$.

Step 3.1: Precompactness of w_k

We first prove that $(w_k)_{k\in\mathbb{N}}$ is precompact in $L^2(Q_{4/7})$.

Because $(\nabla^2_{h_k} u_{h_k})$ is bounded in $L^2(Q_1)$, w_k is bounded in $L^2(Q_1)$. So, according to Lemma 2.5.4 i), it suffices to verify that

$$\lim_{\substack{a\in\mathbb{R}^{\mathsf{d}} \\ |a|\to 0}} \sup_{k\in\mathbb{N}} \|\tau_a w_k - w_k\|_{L^2(Q_{5/7})} = 0\,. \tag{2.5.5}$$

Let $a \in (h\mathbb{Z})^{\mathsf{d}}$ such that $|a| \leq \frac{1}{7}$. Then $\Delta^2_{h_k}(\tau_a u_{h_k} - u_{h_k}) = 0$ in $Q_{11/14}$, so by the Caccioppoli inequality we obtain

$$\begin{aligned}\|\nabla^2_{h_k}(\tau_a u_{h_k} - u_{h_k})\|^2_{L^2(Q_{5/7}(x))} &\leq C\|\tau_a u_{h_k} - u_{h_k}\|^2_{L^2(Q_{11/14}(x))} \\ &\quad + C\|\nabla_{h_k}(\tau_a u_{h_k} - u_{h_k})\|^2_{L^2(Q_{11/14}(x))}\,.\end{aligned}$$

Here the left-hand side is equal to $\|\tau_a w_k - w_k\|^2_{L^2(Q_{5/7})}$, while we can use Proposition 2.4.2 vi) to bound the right-hand side. We obtain

$$\|\tau_a w_k - w_k\|^2_{L^2(Q_{5/7})} \leq C\|\tau_a v_k - v_k\|^2_{L^2(Q_{6/7}(x))} + C\|\tau_a \nabla v_k - \nabla v_k\|^2_{L^2(Q_{6/7}(x))}\,.$$

Recall that (v_k) is bounded in $W^{2,2}(Q_{13/14})$. Hence by the compact Sobolev embedding, (v_k) and (∇v_k) are precompact in $L^2(Q_{13/14})$. Thus by Lemma 2.5.4 ii),

$$\limsup_{a\to 0}\sup_{k\in\mathbb{N}}\Big(\|\tau_a v_k - v_k\|^2_{L^2(Q_{6/7}(x))} + \|\tau_a \nabla v_k - \nabla v_k\|^2_{L^2(Q_{6/7}(x))}\Big) = 0$$

(note that this expression is defined for all $a > 0$, not just those in $(h\mathbb{Z})^{\mathsf{d}}$).

In particular,

$$\limsup_{\delta\to 0}\sup_{k\in\mathbb{N}}\sup_{\substack{a\in(h_k\mathbb{Z})^{\mathsf{d}}\\ |a|\leq\delta}}\Big(\|\tau_a v_k - v_k\|^2_{L^2(Q_{6/7}(x))} + \|\tau_a \nabla v_k - \nabla v_k\|^2_{L^2(Q_{6/7}(x))}\Big) = 0$$

and therefore

$$\limsup_{\delta\to 0}\sup_{k\in\mathbb{N}}\sup_{\substack{a\in(h_k\mathbb{Z})^{\mathsf{d}}\\ |a|\leq\delta}}\|\tau_a w_k - w_k\|_{L^2(Q_{5/7}(x))} = 0\,.$$

It remains to consider shifts τ_a where $a \notin (h_k\mathbb{Z})^{\mathsf{d}}$. This is possible because w_k is piecewise constant on cubes of sidelength h_k. This easily implies that for any $a \in \mathbb{R}^{\mathsf{d}}$ we have

$$\|\tau_a w_k - w_k\|_{L^2(Q_{9/14}(x))} \leq C\sup_{\substack{b\in(h_k\mathbb{Z})^{\mathsf{d}}\\ |b-a|\leq h_k}}\|\tau_b w_k - w_k\|_{L^2(Q_{5/7}(x))}\,.$$

Combining this with the previous estimate we find that

$$\limsup_{\delta\to 0}\sup_{k\in\mathbb{N}}\sup_{\substack{a\in\mathbb{R}^{\mathsf{d}}\\ |a|\leq\delta+h_k}}\|\tau_a w_k - w_k\|_{L^2(Q_{9/14}(x))} = 0\,.$$

Because $h_k \to 0$, this implies

$$\lim_{\substack{a\in\mathbb{R}^{\mathsf{d}}\\ |a|\to 0}}\limsup_{k\to\infty}\|\tau_a w_k - w_k\|_{L^2(Q_{9/14}(x))} = 0\,. \tag{2.5.6}$$

We finally show that (2.5.6) already implies (2.5.5). It follows from (2.5.6) that for every fixed $\varepsilon > 0$ there are $\delta > 0$, $K \in \mathbb{N}$ such that $\sup_{k\geq K}\|\tau_a w_k - w_k\|_{L^2(Q_{9/14}(x))} \leq \varepsilon$ for all a with $|a| \leq \delta$. For the finitely many $k < K$, we use that $\lim_{\substack{a\in\mathbb{R}^{\mathsf{d}}\\ |a|\to 0}}\|\tau_a w_k - w_k\|_{L^2(Q_{9/14}(x))} = 0$ to see that for a potentially smaller δ' we have $\sup_{k\in\mathbb{N}}\|\tau_a w_k - w_k\|_{L^2(Q_{9/14}(x))} \leq \varepsilon$ for all a with $|a| \leq \delta'$.

Therefore the sequence (w_k) is precompact in $L^2(Q_{4/7}(x))$. Choose a subsequence (not relabeled) converging strongly to some $w \in L^2(Q_{4/7}(x))$.

Step 3.2: Strong convergence of $(\nabla^2 v_k)$ and $w = \nabla^2 v$
We split w into a smooth part and a part with small L^2-norm. Let $\varepsilon > 0$ be arbitrary, and choose a $w^{(\varepsilon)}$ in $C_c^\infty(Q_{4/7})$ such that $\|w - w^{(\varepsilon)}\|_{L^2(Q_{4/7})} \leq \varepsilon$. We denote the restriction of $w^{(\varepsilon)}$ to $(h_k\mathbb{Z})^{\mathsf{d}}$ by $w_{h_k}^{(\varepsilon)}$. Using Taylor expansion, one immediately verifies that then $I_{h_k}^{pc} w_{h_k}^{(\varepsilon)}$ and $\tilde{J}_{h_k} w_{h_k}^{(\varepsilon)}$ converge to $w^{(\varepsilon)}$ in $L^2(Q_{4/7})$ and $L^2(Q_{1/2})$, respectively.

This means in particular that

$$\lim_{k\to\infty} \|w_{h_k}^{(\varepsilon)} - \nabla_{h_k}^2 u_{h_k}\|_{L^2(Q_{4/7})} = \|w^{(\epsilon)} - w\|_{L^2(Q_{4/7})} \leq \varepsilon\,.$$

Using Proposition 2.4.2 vi), we conclude that

$$\limsup_{k\to\infty} \left\| \tilde{J}_{h_k} \left(w_{h_k}^{(\varepsilon)} - \nabla_{h_k}^2 u_{h_k} \right) \right\|_{L^2(Q_{1/2})} \leq C\varepsilon\,.$$

The left-hand side here equals $\limsup_{k\to\infty} \|w - \nabla^2 v_k\|_{L^2(Q_{1/2})}$, and so we obtain

$$\limsup_{k\to\infty} \|w - \nabla^2 v_k\|_{L^2(Q_{1/2})} \leq C\epsilon\,.$$

Since ε was arbitrary, we conclude that $(\nabla^2 v_k)$ converges strongly in $L^2(Q_{1/2})$ to w. But we already know that $(\nabla^2 v_k)$ converges weakly in $L^2(Q_{13/14})$ to $\nabla^2 v$, so we obtain that $\nabla^2 v = w$ in $Q_{1/2}$.

Step 4: Conclusion of the argument
We proved that $w_k = I_{h_k}^{pc} \nabla_{h_k}^2 u_{h_k}$ converges strongly in $L^2(Q_{1/2})$ to $\nabla^2 v$. Because $\rho \leq \frac{1}{2}$ then also $\nabla_{h_k}^2 u_{h_k} - \left[\nabla_{h_k}^2 u_{h_k}\right]_{Q_\rho}$ converges strongly in $L^2(Q_{1/2})$ to $\nabla^2 v - \left[\nabla^2 v\right]_{Q_\rho}$, and so from (2.5.3) we conclude that

$$\left\| \nabla^2 v - \left[\nabla^2 v\right]_{Q_\rho} \right\|_{L^2(Q_\rho)}^2 \geq \rho^{\mathsf{d}+1}\,.$$

In addition, we know that $\left\|\nabla^2 v_k\right\|_{L^2(Q_{13/14})} \leq C$, and also that $\nabla^2 v_k$ converges weakly in $L^2(Q_{13/14})$ to $\nabla^2 v$. This implies

$$\left\| \nabla^2 v - \left[\nabla^2 v\right]_{Q_{13/14}} \right\|_{L^2(Q_{13/14})}^2 \leq \left\|\nabla^2 v\right\|_{L^2(Q_{13/14})}^2 \leq \liminf_{k\to\infty} \left\|\nabla^2 v_k\right\|_{L^2(Q_{13/14})}^2 \leq C\,.$$

In summary, we have proved that there is a constant C_1 independent of ρ such that

$$\left\| \nabla^2 v - \left[\nabla^2 v\right]_{Q_\rho} \right\|_{L^2(Q_\rho)}^2 \geq \frac{\rho^{\mathsf{d}+1}}{C_1} \left\| \nabla^2 v - \left[\nabla^2 v\right]_{Q_{13/14}} \right\|_{L^2(Q_{13/14})}^2\,. \tag{2.5.7}$$

On the other hand, $\Delta^2 v = 0$ in $Q_{13/14}$, and thus Lemma 2.5.3 implies that

$$\left\| \nabla^2 v - \left[\nabla^2 v\right]_{Q_\rho} \right\|_{L^2(Q_\rho)}^2 \leq C_2 \left(\frac{\rho}{\frac{13}{14}} \right)^{\mathsf{d}+\frac{3}{2}} \left\| \nabla^2 v - \left[\nabla^2 v\right]_{Q_{13/14}} \right\|_{L^2(Q_{13/14})}^2$$

for a constant C_2 independent of ρ.

This is a contradiction to (2.5.7) provided that we choose ρ small enough, namely $\rho < \frac{1}{C_1^2 C_2^2} \left(\frac{13}{14}\right)^{2\mathsf{d}+3}$. So we finally fix a ρ satisfying this condition, and proved that falsity of the claim leads to a contradiction. □

Now we can return to Lemma 2.5.1.

Proof of Lemma 2.5.1. We can assume w.l.o.g. that $x = 0$.

We claim that for any $0 < s' \leq s \leq r$ we have

$$\left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_{s'}} \right\|^2_{L^2(Q_{s'})} \leq C \left(\frac{s'}{s}\right)^{\mathsf{d}+1} \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_s} \right\|^2_{L^2(Q_s)} . \tag{2.5.8}$$

To prove this estimate, observe first that we can assume $s' \geq \frac{h}{2}$, as otherwise the left-hand side is 0. We can also assume $\frac{s}{s'} \geq 2M$ (where M is the constant from Lemma 2.5.2), as otherwise we can trivially estimate

$$\begin{aligned} \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_{s'}} \right\|^2_{L^2(Q_{s'})} &\leq \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_s} \right\|^2_{L^2(Q_s)} \\ &\leq C \left(\frac{s'}{s}\right)^{\mathsf{d}+1} \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_s} \right\|^2_{L^2(Q_s)} , \end{aligned}$$

which holds for $C \geq (2M)^{\mathsf{d}+1}$.

So we assume $s' \geq \frac{h}{2}$ and $\frac{s}{s'} \geq 2M$. Then in particular $s \geq Mh$. Consider the ρ from Lemma 2.5.2 and let κ be the largest integer such that $\rho^\kappa s \geq \max(s', Mh)$. We can then apply Lemma 2.5.2 repeatedly with radii $s, \rho s, \ldots, \rho^\kappa s$ to find

$$\left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_{\rho^\kappa s}} \right\|^2_{L^2(Q_{\rho^\kappa s})} \leq \rho^{\kappa(\mathsf{d}+1)} \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_s} \right\|^2_{L^2(Q_s)} .$$

Because $s' \leq \rho^\kappa s$, we also have

$$\left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_{s'}} \right\|^2_{L^2(Q_{s'})} \leq \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_{\rho^\kappa s}} \right\|^2_{L^2(Q_{\rho^\kappa s})} .$$

Here we have used the fact that $\|f - [f]_\Omega\|_{L^2(\Omega)}$ is monotone in Ω. If we combine the last two estimates and observe that $\rho^{\kappa+1} s < \max(s', Mh) \leq 2Ms'$, i.e. $\rho^\kappa \leq \frac{2M}{\rho}\frac{s'}{s}$, we indeed obtain (2.5.8) with $C = \left(\frac{2M}{\rho}\right)^{\mathsf{d}+1}$.

Now using (2.5.8) to prove the lemma is a standard iteration argument as e.g. in [Gia93, Theorem 3.1]. For the sake of completeness we sketch the proof.

If we apply (2.5.8) with $s = r$ and $s' = \frac{r}{2^\lambda}$ or $s' = \frac{r}{2^{\lambda+1}}$, we can estimate

$$\begin{aligned} &\left\| \left[\nabla_h^2 u_h\right]_{Q_{r/2^{\lambda+1}}} - \left[\nabla_h^2 u_h\right]_{Q_{r/2^\lambda}} \right\|^2_{L^2(Q_{r/2^{\lambda+1}})} \\ &\quad \leq 2 \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_{r/2^\lambda}} \right\|^2_{L^2(Q_{r/2^\lambda})} + 2 \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_{r/2^{\lambda+1}}} \right\|^2_{L^2(Q_{r/2^{\lambda+1}})} \\ &\quad \leq \frac{C}{2^{\lambda(\mathsf{d}+1)}} \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_r} \right\|^2_{L^2(Q_r)} \end{aligned}$$

and hence

$$\left| \left[\nabla_h^2 u_h\right]_{Q_{r/2^{\lambda+1}}} - \left[\nabla_h^2 u_h\right]_{Q_{r/2^\lambda}} \right| \leq \frac{C}{r^{\frac{\mathsf{d}}{2}} 2^{\frac{\lambda}{2}}} \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_r} \right\|_{L^2(Q_r)} .$$

If we sum this for $\lambda = 0, 1, \ldots$ and observe that for λ small enough $\left[\nabla_h^2 u_h\right]_{Q_{r/2^\lambda}} = \nabla_h^2 u_h(0)$ we obtain

$$\left| \nabla_h^2 u_h(0) - \left[\nabla_h^2 u_h\right]_{Q_r} \right| \leq \frac{C}{r^{\frac{\mathsf{d}}{2}}} \left\| \nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_r} \right\|_{L^2(Q_r)} .$$

Now we can estimate

$$\begin{aligned}\left|\nabla_h^2 u_h(0)\right|^2 &\leq 2\left|\nabla_h^2 u_h(0) - \left[\nabla_h^2 u_h\right]_{Q_r}\right|^2 + 2\left|\left[\nabla_h^2 u_h\right]_{Q_r}\right|^2 \\ &\leq \frac{C}{r^{\mathsf{d}}}\left(\left\|\nabla_h^2 u_h - \left[\nabla_h^2 u_h\right]_{Q_r}\right\|_{L^2(Q_r)}^2 + \left\|\left[\nabla_h^2 u_h\right]_{Q_r}\right\|_{L^2(Q_r)}^2\right) \\ &= \frac{C}{r^{\mathsf{d}}}\|\nabla_h^2 u_h\|_{L^2(Q_r)}^2,\end{aligned}$$

which proves the claim. □

2.5.2 Half-space

In the half-space we want to prove the following statement, which is a slightly weaker analogue of Lemma 2.5.1:

Lemma 2.5.5. *Let* $\mathsf{d} \in \mathbb{N}$, $u_h\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, *let* $x \in (h\mathbb{Z})^{\mathsf{d}}$, $r > 0$, $\nu \in \{e_1, -e_1, \dots, e_n, -e_n\}$. *Suppose that* $u_h(y) = 0$ *for all* $y \in Q_r^h(x)$ *such that* $(y-x)\cdot\nu \leq 0$, *and* $\Delta_h^2 u_h(y) = 0$ *for all* $y \in Q_{r-h}^h(x)$ *such that* $(y-x)\cdot\nu > 0$. *Then, for any* $s \leq r$,

$$\|\nabla_h^2 u_h\|_{L^2(Q_s(x))} \leq C\left(\frac{s}{r}\right)^{\frac{\mathsf{d}}{2}}\|\nabla_h^2 u_h\|_{L^2(Q_r(x))}.$$

The proof is mostly similar to that of Lemma 2.5.1, so we only give details where a new idea is required.

For $r > 0$ let $Q_{r,+} = Q_r \cap \{x_1 > 0\}$. The main step in the proof of Lemma 2.5.5 will be to prove the following estimate.

Lemma 2.5.6. *Let* $\mathsf{d} \in \mathbb{N}$. *There exist constants* $M \in \mathbb{N}$, $0 < \rho < \frac{1}{2}$ *with the following property: Let* $u_h\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, $r > 0$ *be such that* $u_h(y) = 0$ *whenever* $y \in Q_r^h$ *and* $y_1 \leq 0$, *and* $\Delta_h^2 u_h(y) = 0$ *for all* $y \in Q_{r-h}^h$ *such that* $y_1 > 0$. *Assume that* $\rho r \geq Mh$. *Then we have*

$$\begin{aligned}&\left\|\nabla_h^2 u_h - \left[D_{-1}^h D_1^h u_h\right]_{Q_{\rho r,+}} e_1\otimes e_1\right\|_{L^2(Q_{\rho r,+})}^2 \\ &\quad\leq \rho^{\mathsf{d}+1}\left\|\nabla_h^2 u_h - \left[D_{-1}^h D_1^h u_h\right]_{Q_{r,+}} e_1\otimes e_1\right\|_{L^2(Q_{r,+})}^2.\end{aligned}$$

Using a compactness argument, we will deduce this estimate from the following continuous estimate.

Lemma 2.5.7. *Let* $\mathsf{d} \in \mathbb{N}$, $0 < s \leq \frac{r}{2}$, $u \in W^{2,2}(Q_{r,+})$. *Assume that* $\Delta^2 u = 0$ *weakly in* $Q_{r,+}$ *and that* $u = 0$, $D_1 u = 0$ *on* $\partial Q_{r,+} \cap \{x_1 = 0\}$ *in the sense of traces. Then we have*

$$\left\|\nabla^2 u - \left[D_1^2 u\right]_{Q_{s,+}} e_1\otimes e_1\right\|_{L^2(Q_{s,+})}^2 \leq C\left(\frac{s}{r}\right)^{\mathsf{d}+\frac{3}{2}}\left\|\nabla^2 u - \left[D_1^2 u\right]_{Q_{r,+}} e_1\otimes e_1\right\|_{L^2(Q_{r,+})}^2.$$

Proof. This follows like Lemma 2.5.3 from Schauder estimates up to the boundary (cf. [Mor66, Theorem 6.4.8]). □

Proof of Lemma 2.5.6.
Step 1: Preparations
We follow the same strategy as in the proof of Lemma 2.5.2. That is, we assume that the

claim is wrong for some fixed ρ, and consider a sequence of counterexamples u_{h_k} and their interpolations $v_k = I_{h_k} u_{h_k}$. We can assume that $r_k = 1$.

Next observe that for $\omega_h(x) := \begin{cases} \frac{x_1(x_1+h)}{2} & x_1 \geq 0 \\ 0 & x_1 < 0 \end{cases}$ we have $\omega_h(x) = 0$ if $x_1 \leq 0$ and $D^h_{-1} D^h_1 \omega_h(x) = \begin{cases} 1 & x_1 \geq 0 \\ 0 & x_1 < 0 \end{cases}$. So by replacing u_h with $u_h - \left[D^h_{-1} D^h_1 u_h\right]_{Q_{1,+}} \omega_h$ we can also assume $\left[D^h_{-1} D^h_1 u_h\right]_{Q_{1,+}} = 0$. Having normalized u_h on $Q_{1,+}$ in this way, we now consider Q_1 again. We can assume

$$\left\| \nabla^2_{h_k} u_{h_k} \right\|_{L^2(Q_1)} = 1 \,. \tag{2.5.9}$$

Note that

$$\left\| \nabla^2_{h_k} u_{h_k} \right\|^2_{L^2(Q_1)} = \left\| \nabla^2_{h_k} u_{h_k} \right\|^2_{L^2(Q_{1,+})} + \left\| \nabla^2_{h_k} u_{h_k} \right\|^2_{L^2((-h/2,0)\times(-1,1)^{\mathrm{d}-1})}$$

and

$$\left\| \nabla^2_{h_k} u_{h_k} \right\|^2_{L^2((-h/2,0)\times(-1,1)^{\mathrm{d}-1})} = \left\| \nabla^2_{h_k} u_{h_k} \right\|^2_{L^2((0,h/2)\times(-1,1)^{\mathrm{d}-1})} \,.$$

Now (2.5.9) combined with the last two equalities implies that $\left\| \nabla^2_{h_k} u_{h_k} \right\|_{L^2(Q_{1,+})} \geq \frac{1}{2}$, so that

$$\left\| \nabla^2_{h_k} u_{h_k} - \left[D^h_{-1} D^h_1 u_{h_k}\right]_{Q_{\rho,+}} e_1 \otimes e_1 \right\|^2_{L^2(Q_{\rho,+})} > \frac{\rho^{\mathrm{d}+1}}{2} \,. \tag{2.5.10}$$

By (2.5.9), Proposition 2.4.2 and the Poincaré inequality with 0 boundary values (v_k) is bounded in $W^{2,2}(Q_{3/4})$, and so a non-relabeled subsequence converges weakly to some v in $W^{2,2}(Q_{3/4})$.

As in step 2 of the proof of Lemma 2.5.2 we can show that $\Delta^2 v = 0$ weakly in $Q_{3/4,+}$. We have $u_{h_k} = 0$ in $Q_1 \cap \{x_1 < 0\}$ and hence $v_k = 0$ in $Q_{3/4} \cap \{x_1 < -3h_k\}$. Since v_k converges to v strongly in $L^2(Q_{3/4})$, $v = 0$ in $\{x_1 < 0\}$, and because $v \in W^{2,2}(Q_{3/4})$, we obtain that $v = 0$ and $D_1 v = 0$ on $Q_{3/4} \cap \{x_1 = 0\}$ in the sense of traces.

We define $w_k = I^{pc}_{h_k} \nabla^2_{h_k} u_{h_k}$ and want to show next that $\nabla^2 v_k$ and w_k converge to $\nabla^2 v$ strongly in $L^2(Q_{1/2})$. We cannot directly reuse the argument in Step 3 of the proof of Lemma 2.5.2, as we now have to deal with boundary values. However, we can use that argument on any cube $Q_{\tilde{r}}(\tilde{x}) \subset Q_{5/8} \cap \{x_1 > 0\}$ to conclude that $\nabla^2 v_k$ and w_k converge to $\nabla^2 v$ strongly in $L^2(Q_{\tilde{r}/2})$. Since we can do this for any such cube, we conclude that $\nabla^2 v_k$ and w_k converge to $\nabla^2 v$ strongly in $L^2_{loc}(Q_{5/8,+})$.

Because $u_{h_k} = 0$ in $Q_{5/8} \cap \{x_1 < -3h_k\}$, we also have that $\nabla^2 v_k$ and w_k converge to 0 strongly in $L^2_{loc}(Q_{5/8} \cap \{x_1 < 0\})$. In summary, we have proved that $\nabla^2 v_k$ and w_k converge to $\nabla^2 v$ strongly in $L^2_{loc}(Q_{5/8} \setminus \{x_1 = 0\})$.

We still have to deal with $\{x_1 = 0\}$, and for this we need a new idea.

Step 2: Nonconcentration at the boundary

We claim that for any $y \in Q_{1/2} \cap \{x_1 = 0\}$ we have

$$\lim_{\tilde{r}\to 0} \limsup_{k\to\infty} \left\| \nabla^2_{h_k} u_{h_k} \right\|_{L^2(Q_{\tilde{r}}(y))} = 0 \,. \tag{2.5.11}$$

To see this, let $\tilde{r} > 0$. For h_k small enough Lemma 2.3.1 and Proposition 2.4.2 imply that

$$\left\| \nabla^2_{h_k} u_{h_k} \right\|^2_{L^2(Q_{\tilde{r}}(y))} \leq \frac{C}{\tilde{r}^2} \left\| \nabla_{h_k} u_{h_k} \right\|^2_{L^2(Q_{2\tilde{r}}(y))} + \frac{C}{\tilde{r}^4} \left\| u_{h_k} \right\|^2_{L^2(Q_{2\tilde{r}}(y))}$$

$$\leq \frac{C}{\tilde{r}^2} \|\nabla v_k\|^2_{L^2(Q_{4\tilde{r}}(y))} + \frac{C}{\tilde{r}^4} \|v_k\|^2_{L^2(Q_{4\tilde{r}}(y))} .$$

Now v_k converges to v weakly in $W^{2,2}(Q_{3/4})$, so v_k and ∇v_k converge strongly in $L^2(Q_{3/4})$. Hence we can pass to the limit in the above inequality and find

$$\limsup_{k\to\infty} \left\|\nabla^2_{h_k} u_{h_k}\right\|^2_{L^2(Q_{\tilde{r}}(y))} \leq \frac{C}{\tilde{r}^2} \|\nabla v\|^2_{L^2(Q_{4\tilde{r}}(y))} + \frac{C}{\tilde{r}^4} \|v\|^2_{L^2(Q_{4\tilde{r}}(y))} .$$

Furthermore v is 0 in $Q_{4\tilde{r}}(y) \cap \{x_1 < 0\}$, so we can apply the Poincaré inequality to conclude

$$\limsup_{k\to\infty} \left\|\nabla^2_{h_k} u_{h_k}\right\|^2_{L^2(Q_{\tilde{r}}(y))} \leq C \left\|\nabla^2 v\right\|^2_{L^2(Q_{4\tilde{r}}(y))} .$$

Now $\nabla^2 v$ is a fixed L^2-function, so if we pass to the limit $\tilde{r} \to 0$ here, we indeed obtain (2.5.11).

It is easy to see that (2.5.11) together with the fact that $w_k = I^{pc}_{h_k} \nabla^2_h u_{h_k}$ converges to $\nabla^2 v$ strongly in $L^2_{loc}(Q_{5/8} \setminus \{x_1 = 0\})$ imply that w_k actually converges to $\nabla^2 v$ strongly in $L^2(Q_{1/2})$.

We have for any $y \in Q_{1/2} \cap \{x_1 = 0\}$ and $\tilde{r} > 0$ that

$$\limsup_{k\to\infty} \left\|\nabla^2 v_k\right\|_{L^2(Q_{\tilde{r}}(y))} \leq C \limsup_{k\to\infty} \left\|\nabla^2_{h_k} u_{h_k}\right\|_{L^2(Q_{2\tilde{r}}(y))}$$

and so from (2.5.11) we also conclude

$$\lim_{\tilde{r}\to 0} \limsup_{k\to\infty} \left\|\nabla^2 v_k\right\|_{L^2(Q_{\tilde{r}}(y))} = 0 .$$

This in turn implies that also $\nabla^2 v_k$ converges to $\nabla^2 v$ strongly in $L^2(Q_{1/2})$.

Step 3: Conclusion of the argument

We can now continue as in Step 4 of the proof of Lemma 2.5.2: The strong convergence of w_k to $\nabla^2 v$ allows us to conclude from (2.5.10) that

$$\left\|\nabla^2 v - \left[D^2_1 v\right]_{Q_{\rho,+}} e_1 \otimes e_1\right\|^2_{L^2(Q_{\rho,+})} \geq \frac{\rho^{\mathsf{d}+1}}{2} .$$

On the other hand, we have

$$\left\|\nabla^2 v - \left[D^2_1 v\right]_{Q_{3/4,+}} e_1 \otimes e_1\right\|^2_{L^2(Q_{3/4,+})} \leq C$$

and it is easy to check that we arrive at a contradiction to Lemma 2.5.7 once we choose ρ small enough. □

Proof of Lemma 2.5.5. The proof is similar to the first half of the proof of Lemma 2.5.1: One can assume that $x = 0$, $\nu = e_1$. Then one first proves that, for any $0 < s' \leq s \leq r$,

$$\begin{aligned}
&\left\|\nabla^2_h u_h - \left[D^h_{-1} D^h_1 u_h\right]_{Q_{s',+}} e_1 \otimes e_1\right\|^2_{L^2(Q_{s',+})} \\
&\quad \leq C \left(\frac{s'}{s}\right)^{\mathsf{d}+1} \left\|\nabla^2_h u_h - \left[D^h_{-1} D^h_1 u_h\right]_{Q_{s,+}} e_1 \otimes e_1\right\|^2_{L^2(Q_{s,+})} ,
\end{aligned}$$

which already looks similar to the claimed estimate. We can again use this with $s = r$ and $s' = \frac{r}{2^\lambda}$ or $s' = \frac{r}{2^{\lambda+1}}$ to conclude

$$\begin{aligned}&\left|\left[D^h_{-1}D^h_1 u_h\right]_{Q_{r/2^{\lambda+1},+}} - \left[D^h_{-1}D^h_1 u_h\right]_{Q_{r/2^{\lambda},+}}\right| \\ &\quad \leq \frac{C}{r^{\frac{\mathsf{d}}{2}}2^{\frac{\lambda}{2}}}\left\|\nabla^2_h u_h - \left[D^h_{-1}D^h_1 u_h\right]_{Q_{r,+}} e_1\otimes e_1\right\|_{L^2(Q_{r,+})}.\end{aligned}$$

Let λ_0 be the largest integer such that $\frac{r}{2^{\lambda_0}} \geq s$. We can apply this estimate with radii $r, \frac{r}{2}, \ldots, \frac{r}{2^{\lambda_0-1}}$ and sum to conclude

$$\left|\left[D^h_{-1}D^h_1 u_h\right]_{Q_{r/2^{\lambda_0},+}} - \left[D^h_{-1}D^h_1 u_h\right]_{Q_{r,+}}\right| \leq \frac{C}{r^{\frac{\mathsf{d}}{2}}}\left\|\nabla^2_h u_h - \left[D^h_{-1}D^h_1 u_h\right]_{Q_{r,+}} e_1\otimes e_1\right\|_{L^2(Q_{r,+})}.$$

Using all this, we can estimate

$$\begin{aligned}\left\|\nabla^2_h u_h\right\|^2_{L^2(Q_{s,+})} &\leq \left\|\nabla^2_h u_h\right\|^2_{L^2(Q_{r/2^{\lambda_0},+})} \\ &\leq 2\left\|\nabla^2_h u_h - \left[D^h_{-1}D^h_1 u_h\right]_{Q_{r/2^{\lambda_0},+}} e_1\otimes e_1\right\|^2_{L^2(Q_{r/2^{\lambda_0},+})} \\ &\quad + 2\left\|\left[D^h_{-1}D^h_1 u_h\right]_{Q_{r/2^{\lambda_0},+}}\right\|^2_{L^2(Q_{r/2^{\lambda_0},+})} \\ &\leq \left(\frac{C}{2^{\lambda_0(\mathsf{d}+1)}} + \frac{C}{2^{\lambda_0 n}}\right)\left\|\nabla^2_h u_h - \left[D^h_{-1}D^h_1 u_h\right]_{Q_{r,+}} e_1\otimes e_1\right\|^2_{L^2(Q_{r,+})} \\ &\quad + \frac{C}{2^{\lambda_0 n}}\left\|\left[D^h_{-1}D^h_1 u_h\right]_{Q_{r,+}}\right\|^2_{L^2(Q_{r,+})} \\ &\leq \frac{C}{2^{\lambda_0 n}}\left\|\nabla^2_h u_h\right\|^2_{L^2(Q_{r,+})},\end{aligned}$$

which implies

$$\begin{aligned}\left\|\nabla^2_h u_h\right\|^2_{L^2(Q_{s,+})} &\leq C\left(\frac{r}{s}\right)^{\mathsf{d}}\left\|\nabla^2_h u_h\right\|^2_{L^2(Q_{r,+})} \\ &\leq C\left(\frac{r}{s}\right)^{\mathsf{d}}\left\|\nabla^2_h u_h\right\|^2_{L^2(Q_{r})}.\end{aligned} \tag{2.5.12}$$

Now by the same argument as in Step 1 of the proof of Lemma 2.5.6 we have

$$\left\|\nabla^2_h u_h\right\|^2_{L^2(Q_s)} \leq 2\left\|\nabla^2_h u_h\right\|^2_{L^2(Q_{s+})}.$$

Combining this with (2.5.12) yields the result. □

2.5.3 Edges and vertices

It remains to prove the analogue of Lemma 2.5.5 near edges (in 3D) and vertices (in 2D and 3D). The actual compactness argument requires no new idea, so we will only give a very brief sketch of the proofs. However, this time the continuous estimate require a bit more work, so we will go into detail there. Let us first state the two results:

Lemma 2.5.8. *Let $u_h\colon (h\mathbb{Z})^3 \to \mathbb{R}$, let $x \in (h\mathbb{Z})^3$, $r > 0$, $\nu_1, \nu_2 \in \{e_1, -e_1, \dots, e_3, -e_3\}$ such that $\nu_1 \neq \pm\nu_2$. Suppose that $u_h(y) = 0$ for all $y \in Q_r^h(x)$ such that $(y-x)\cdot\nu_1 \leq 0$ or $(y-x)\cdot\nu_2 \leq 0$, and $\Delta_h^2 u_h(y) = 0$ for all $y \in Q_{r-h}^h(x)$ such that $(y-x)\cdot\nu_1 > 0$ and $(y-x)\cdot\nu_2 > 0$. Then, for any $s \leq r$,*

$$\|\nabla_h^2 u_h\|_{L^2(Q_s(x))} \leq C\left(\frac{s}{r}\right)^{\frac{3}{2}} \|\nabla_h^2 u_h\|_{L^2(Q_r(x))}.$$

Lemma 2.5.9. *Let $\mathsf{d} = 2$ or $\mathsf{d} = 3$, $u_h\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, let $x \in (h\mathbb{Z})^{\mathsf{d}}$, $r > 0$, $\nu_i \in \{e_i, -e_i\}$ for $i \in \{1, \dots, n\}$. Suppose that $u_h(y) = 0$ for all $y \in Q_r^h(x)$ such that $(y-x)\cdot\nu_i \leq 0$ for at least one i, and $\Delta_h^2 u_h(y) = 0$ for all $y \in Q_{r-h}^h(x)$ such that $(y-x)\cdot\nu_i > 0$ for all i. Then, for any $s \leq r$,*

$$\|\nabla_h^2 u_h\|_{L^2(Q_s(x))} \leq C\left(\frac{s}{r}\right)^{\frac{\mathsf{d}}{2}} \|\nabla_h^2 u_h\|_{L^2(Q_r(x))}.$$

Proof of Lemma 2.5.8 and Lemma 2.5.9. This follows easily from the following two lemmata. □

Lemma 2.5.10. *There are constants $M \in \mathbb{N}$, $0 < \rho < \frac{1}{2}$ with the following property: let $u_h\colon (h\mathbb{Z})^3 \to \mathbb{R}$, $r > 0$, such that $u_h(y) = 0$ for all $y \in Q_r^h$ such that $y_1 \leq 0$ or $y_2 \leq 0$, and $\Delta_h^2 u_h(y) = 0$ for all $y \in Q_{r-h}^h(x)$ such that $y_1 > 0$ and $y_2 > 0$. Then we have that*

$$\left\|\nabla_h^2 u_h\right\|_{L^2(Q_{\rho r})}^2 \leq \rho^{\mathsf{d}} \left\|\nabla_h^2 u_h\right\|_{L^2(Q_r)}^2.$$

Lemma 2.5.11. *There are constants $M \in \mathbb{N}$, $0 < \rho < \frac{1}{2}$ with the following property: let $\mathsf{d} = 2$ or $\mathsf{d} = 3$, $u_h\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$, $r > 0$, such that $u_h(y) = 0$ for all $y \in Q_r^h$ such that $y_i \leq 0$ for at least one $i \in \{1, \dots, n\}$, and $\Delta_h^2 u_h(y) = 0$ for all $y \in Q_{r-h}^h$ such that $y_i > 0$ for all i. Assume that $\rho r \geq Mh$. Then we have that*

$$\left\|\nabla_h^2 u_h\right\|_{L^2(Q_{\rho r})}^2 \leq \rho^{\mathsf{d}} \left\|\nabla_h^2 u_h\right\|_{L^2(Q_r)}^2.$$

We will deduce these two lemmata from the following continuous estimates. D_ν denotes the derivative in normal direction.

Lemma 2.5.12. *There is a constant $\theta > 0$ with the following property: let $\mathsf{d} = 3$, $0 < s \leq \frac{r}{2}$, $u \in W^{2,2}(Q_{r,++})$, where $Q_{r,++} = Q_r \cap \{x_1 > 0, x_2 > 0\}$. Assume that $\Delta^2 u = 0$ weakly in $Q_{r,++}$ and that $u = 0$, $D_\nu u = 0$ on $\partial Q_{r,++} \cap \{x_1 = 0 \vee x_2 = 0\}$ in the sense of traces. Assume that $\rho r \geq Mh$. Then we have*

$$\left\|\nabla^2 u\right\|_{L^2(Q_{s,++})}^2 \leq C\left(\frac{s}{r}\right)^{3+\theta} \left\|\nabla^2 u\right\|_{L^2(Q_{r,++})}^2.$$

Lemma 2.5.13. *There is a constant $\theta > 0$ with the following property: let $0 < s \leq \frac{r}{2}$, $u \in W^{2,2}(Q_{r,\mathsf{d}+})$, where $Q_{r,\mathsf{d}+} = Q_{r,++} = Q_r \cap \{x_1 > 0, x_2 > 0\}$ if $\mathsf{d} = 2$, and $Q_{r,\mathsf{d}+} = Q_{r,+++} = Q_r \cap \{x_1 > 0, x_2 > 0, x_3 > 0\}$ if $\mathsf{d} = 3$. Assume that $\Delta^2 u = 0$ weakly in $Q_{r,\mathsf{d}+}$ and that $u = 0$, $D_\nu u = 0$ on $\partial Q_{r,\mathsf{d}+} \cap \{x_i = 0 \text{ for some } i\}$ in the sense of traces. Then we have*

$$\left\|\nabla^2 u\right\|_{L^2(Q_{s,\mathsf{d}+})}^2 \leq C\left(\frac{s}{r}\right)^{\mathsf{d}+\theta} \left\|\nabla^2 u\right\|_{L^2(Q_{r,\mathsf{d}+})}^2.$$

The proof of Lemma 2.5.12 and Lemma 2.5.13 relies heavily on the theory of elliptic equations in domains with singularities. We use results from [KMR97] and [MR10] and refer the reader to these monographs for more background information.

Proof of Lemma 2.5.12. Let $\mathbb{R}^3_{++} = \mathbb{R}^3 \cap \{x_1 > 0, x_2 > 0\}$. For $x \in \mathbb{R}^3_{++}$ write $x = (x', x_3)$.

The statement is trivial if $s \geq \frac{r}{4}$, so assume $s < \frac{r}{4}$. Let $\eta \in C_c^\infty(Q_r)$ be a cut-off function that is 1 on $Q_{r/2,++}$ and such that $|\nabla^\kappa \eta| \leq \frac{C}{r^\kappa}$ for $\kappa \leq 4$. Then $\eta \Delta^2 u = 0$ in $\partial\mathbb{R}^3_{++}$, and we can calculate (as an identity in the sense of distributions) that

$$\Delta^2(\eta u) = (\Delta^2 \eta)u + 4\nabla\Delta\eta \cdot \nabla u + 2\Delta\eta\Delta u + 4\nabla^2\eta : \nabla^2 u + 4\nabla\eta \cdot \nabla\Delta u\,.$$

In order to avoid terms with too many derivatives of u we rewrite the last term as

$$\nabla\eta \cdot \nabla\Delta u = \operatorname{div}(\nabla\eta\Delta u) - \Delta\eta\Delta u$$

to obtain

$$\Delta^2(\eta u) = \Delta^2\eta u + 4\nabla\Delta\eta \cdot \nabla u - 2\Delta\eta\Delta u + 4\nabla^2\eta : \nabla^2 u + 4\operatorname{div}(\nabla\eta\Delta u) =: f\,.$$

Because $u \in W^{2,2}(Q_{r,\mathrm{d}+})$ with zero boundary values on $\partial\mathbb{R}^3_{++}$, the right-hand side f is an element of $W^{-2,2}(\mathbb{R}^3_{++})$, while ηu is in $W_0^{2,2}(\mathbb{R}^3_{++})$. Hence (cf. [MR10], Theorem 2.5.1) we can represent ηu via the Green's function of $\mathbb{R}^3_{++}$ as

$$(\eta u)(x) = \int_{\mathbb{R}^3_{++}} G(x,\xi) f(\xi)\,\mathrm{d}\xi\,.$$

For $x \in Q_{s,++} \subset Q_{r/4,++}$ this implies

$$\nabla^2 u(x) = \int_{\mathbb{R}^3_{++}} \nabla_x^2 G(x,\xi) f(\xi)\,\mathrm{d}\xi\,.$$

Now f is supported in $Q_{r,++} \setminus Q_{r/2,++}$, whereas $x \in Q_{s,++} \subset Q_{r/4,++}$. So a decay estimate for G will directly lead to a pointwise estimate for $\nabla^2 u$.

In fact, Theorem 2.5.4 in [MR10] states that if $|x-\xi| \geq \min(|x'|, |\xi'|)$ we have, for every $\varepsilon > 0$,

$$|D_{x'}^\alpha D_{x_3}^j D_{\xi'}^\beta D_{\xi_3}^k G(x,\xi)| \leq C_\varepsilon \frac{|x'|^{1+\delta_+-|\alpha|-\varepsilon}|\xi'|^{1+\delta_--|\beta|-\varepsilon}}{|x-\xi|^{1+\delta_++\delta_-+j+k-2\varepsilon}}\,. \tag{2.5.13}$$

Here δ_+ and δ_- are certain real parameters defined in terms of eigenvalue problems related to the Bilaplacian (see [MR10, Section 2.4] for the precise definition). According to [MR10, Section 4.3] we have that $\delta_+ = \delta_- \approx 2.73959$. In particular, $\delta_\pm > 1$, so we can choose $\theta > 0$ such that $1 + \frac{\theta}{2} < \delta_\pm$. Then let $\varepsilon = \delta_\pm - 1 - \frac{\theta}{2} > 0$.

We are interested in the case where $x \in Q_{s,++}$, $\xi \in Q_{r,++} \setminus Q_{r/2,++}$. In that case the inequality $|x-\xi| \geq \min(|x'|, |\xi'|)$ certainly holds, and we can estimate $|x'| \leq s$, $|\xi'| \leq r$, $|x-\xi| \geq \frac{r}{4}$, so that (2.5.13) turns into

$$\begin{aligned}|D_{x'}^\alpha D_{x_3}^j D_{\xi'}^\beta D_{\xi_3}^k G(x,\xi)| &\leq C_\varepsilon s^{1+\delta_+-|\alpha|-\varepsilon} r^{\varepsilon-\delta_+-|\beta|-j-k} \\ &= C\frac{s^{2+\frac{\theta}{2}-|\alpha|}}{r^{1+\frac{\theta}{2}+|\beta|+j+k}}\,.\end{aligned}$$

This estimate is sharp enough to allow us to estimate the terms of f. For example we can calculate using the Poincaré and Hölder inequality that

$$\left|\int_{\mathbb{R}^3_{++}} \nabla_x^2 G(x,\xi)\Delta^2\eta(\xi)u(\xi)\,\mathrm{d}\xi\right| \leq C\int_{Q_{r,++}} \frac{s^{2+\frac{\theta}{2}-2}}{r^{1+\frac{\theta}{2}+0+0+0}}\frac{1}{r^4}\,|u(\xi)|\,\mathrm{d}\xi$$

$$
\begin{aligned}
&= C\frac{s^{\frac{\theta}{2}}}{r^{5+\frac{\theta}{2}}}\int_{Q_{r,++}} |u| \, \mathrm{d}\xi \\
&\leq C\frac{s^{\frac{\theta}{2}}}{r^{5+\frac{\theta}{2}}} r^2 r^{\frac{3}{2}} \left(\int_{Q_{r,++}} \left|\nabla^2 u\right|^2 \mathrm{d}\xi\right)^{\frac{1}{2}} \\
&\leq C\frac{s^{\frac{\theta}{2}}}{r^{\frac{3}{2}+\frac{\theta}{2}}} \left(\int_{\mathbb{R}^3_{++}} \left|\nabla^2 u\right|^2 \mathrm{d}\xi\right)^{\frac{1}{2}}
\end{aligned}
$$

and that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^3_{++}} \nabla_x^2 G(x,\xi)\,\mathrm{div}(\nabla\eta\Delta u)(\xi)\,\mathrm{d}\xi\right| &= \left|\int_{\mathbb{R}^3_{++}} \nabla_x^2\nabla_\xi G(x,\xi)\cdot\nabla\eta(\xi)\Delta u(\xi)\,\mathrm{d}\xi\right| \\
&\leq C\int_{Q_{r,++}} \frac{s^{\frac{\theta}{2}}}{r^{2+\frac{\theta}{2}}}\frac{1}{r}\left|\Delta u(\xi)\right| \mathrm{d}\xi \\
&\leq C\frac{s^{\frac{\theta}{2}}}{r^{\frac{3}{2}+\frac{\theta}{2}}} \left(\int_{\mathbb{R}^3_{++}} \left|\nabla^2 u\right|^2 \mathrm{d}\xi\right)^{\frac{1}{2}} .
\end{aligned}
$$

We can estimate the other terms on f analogously. If we integrate the sum of the squares of all these inequalities with respect to x we immediately obtain the conclusion. □

Proof of Lemma 2.5.13. The proof in the case of a vertex is very similar. One can again deduce the representation

$$
\nabla^2 u(x) = \int_{\mathbb{R}^{\mathsf{d}}_{\mathsf{d}+}} \nabla_x^2 G(x,\xi) f(\xi)\,\mathrm{d}\xi \tag{2.5.14}
$$

for $x \in Q_{r/4,\mathsf{d}+}$, so that one only needs sharp estimates for the Green's function to complete the argument.

If $\mathsf{d} = 2$, we can use for this purpose Theorem 8.4.8 in combination with Theorem 6.1.2 in [KMR97]. Theorem 8.4.8 gives a Green's function for right-hand sides in L^2. However, according to Theorem 6.1.2, the solution operator has a continuous extension to right-hand sides in $W^{-2,2}$, so that (2.5.14) holds for this Green's function. Now Theorem 8.4.8 also gives asymptotics for G in terms of the eigenvalues of a certain eigenvalue problem. If we stay in the eigenvalue-free strip, this estimate reads

$$
|D_x^\alpha D_\xi^\beta G(x,\xi)| \leq C_\varepsilon |x|^{1+\delta_+-|\alpha|-\varepsilon}|\xi|^{1-\delta_+-|\beta|+\varepsilon}
$$

where $2|x| \leq |\xi|$ and $\varepsilon > 0$ is arbitrary. Using this estimate we can continue as in the proof of Lemma 2.5.12.

The case $\mathsf{d} = 3$ is slightly more complicated. We can use [MR10, Theorem 3.4.5], which states that if $2|x| \leq |\xi|$, then for any $\varepsilon > 0$

$$
\begin{aligned}
&|D_x^\alpha D_\xi^\beta G(x,\xi)| \\
&\quad\leq C_\varepsilon |x|^{\Lambda_+-|\alpha|-\varepsilon}|\xi|^{1-\Lambda_+-|\beta|+\varepsilon}\prod_{j=1}^{3}\left(\frac{r_j(x)}{|x|}\right)^{1+\delta_+-|\alpha|-\varepsilon}\prod_{k=1}^{3}\left(\frac{r_k(\xi)}{|\xi|}\right)^{1+\delta_--|\beta|-\varepsilon}
\end{aligned}
$$

where $\delta_\pm$ are as before, Λ_+ is another constant defined in terms of a certain eigenvalue problem (see [MR10, Section 3.4] for the precise definition) and $r_j(x)$ denotes the distance of x to the line $\{x_j = 0\}$. If we choose $\varepsilon \leq \delta_+ - 1 = \delta_- - 1$, then the exponents of the terms

$\frac{r_k(x)}{|x|}$ and $\frac{r_k(\xi)}{|\xi|}$ are non-negative whenever $|\alpha| \leq 2$ and $|\beta| \leq 2$. So we obtain under these assumptions

$$|D_x^\alpha D_\xi^\beta G(x,\xi)| \leq C_\varepsilon |x|^{\Lambda_+ - |\alpha| - \varepsilon} |\xi|^{1-\Lambda_+ - |\beta| + \varepsilon}.$$

In [MR10, Section 4.3] it is proved that $\Lambda_+ \geq 3$. This allows us to take $\theta > 0$ such that $2 + \frac{\theta}{2} \leq \Lambda_+$ and $1 + \frac{\theta}{2} < \delta_\pm$. By choosing $\varepsilon = \min\left(\Lambda_+ - 2 - \frac{\theta}{2}, \delta_\pm - 1\right)$ we conclude

$$|D_x^\alpha D_\xi^\beta G(x,\xi)| \leq C \frac{s^{2+\frac{\theta}{2}-|\alpha|}}{r^{1+\frac{\theta}{2}+|\beta|}}$$

for $|\alpha| \leq 2$ and $|\beta| \leq 2$. Now we can continue as in the proof of Lemma 2.5.12 (observe that in that proof we only needed estimates for $D_x^\alpha D_\xi^\beta G(x,\xi)$ with $|\alpha| \leq 2$ and $|\beta| \leq 1$). □

Proof of Lemma 2.5.10 and Lemma 2.5.11. We follow the proofs of Lemma 2.5.2 and Lemma 2.5.6. The proof is slightly easier than the proof of Lemma 2.5.6 because we no longer need to worry about the subtraction of the averages of u_h. We assume that the claim is wrong for some fixed ρ, and consider a sequence of counterexamples u_{h_k} and their interpolations $v_k = I_{h_k} u_{h_k}$. We can assume that $r_k = 1$ and $\left\|\nabla_{h_k}^2 u_{h_k}\right\|_{L^2(Q_1)} = 1$, and conclude that (v_k) is bounded in $W^{2,2}(Q_{3/4})$, and so a non-relabeled subsequence converges to some v in $W^{2,2}(Q_{3/4})$.

As before we see that $\Delta^2 v = 0$ in $Q_{3/4,+++}$ and $Q_{3/4,\mathsf{d}+}$ respectively and that v has 0 boundary values. Also we obtain strong convergence of $\nabla^2 v_k$ and $w_k := I_{pc}^{h_k} \nabla_{h_k}^2 u_{h_k}$ in $L^2_{loc}(Q_{5/8} \setminus \partial Q_{3/4,+++})$ and $L^2_{loc}(Q_{5/8} \setminus \partial Q_{3/4,\mathsf{d}+})$, respectively. Now, as in Step 2 of the proof of Lemma 2.5.6, we find that $\nabla_{h_k}^2 u_{h_k}$ does not concentrate at the boundary, so that $\nabla^2 v_k$ and w_k actually converge strongly in $L^2(Q_{1/2})$.

This convergence allows us to pass to the limit in

$$\left\|\nabla_{h_k}^2 u_{h_k}\right\|_{L^2(Q_\rho)}^2 > \rho^{\mathsf{d}}$$

so that we easily arrive at a contradiction to Lemma 2.5.12 or Lemma 2.5.13 once we choose ρ small enough. □

2.6 Inner and outer decay estimates for discrete biharmonic functions

2.6.1 Inner estimates

We can now combine the results from the previous section in one general decay estimate for biharmonic functions:

Theorem 2.6.1. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$, $u_h \in \Phi_h$. *Let* $x \in \Lambda_h$, $r > 0$ *and suppose that* $\Delta_h^2 u_h(y) = 0$ *for all* $y \in Q_{r-h}(x) \cap \operatorname{int} \Lambda_h^{\mathsf{d}}$. *Then, for all* $z \in Q_{r/2}^h(x) \cap \Lambda_h^{\mathsf{d}}$,

$$|\nabla_h^2 u_h(z)| \leq \frac{C}{r^{\frac{\mathsf{d}}{2}}} \|\nabla_h^2 u_h\|_{L^2(Q_r(x))}. \tag{2.6.1}$$

Observe that $\nabla_h^2 u_h = 0$ is zero in $(h\mathbb{Z})^{\mathsf{d}} \setminus \Lambda_h$. Therefore we could equivalently only integrate over $Q_r(x) \cap (\Lambda_h)_{pc}$ on the right-hand side.

Proof. The proofs for the cases $\mathsf{d} = 2$ and $\mathsf{d} = 3$ are similar, but the latter is somewhat more tedious. Therefore we give the proof for $\mathsf{d} = 2$ in detail and then describe how to adapt it to the case $\mathsf{d} = 3$. So let $\mathsf{d} = 2$.

We first prove the statement in the special case $z = x$. By rotating and reflecting Λ_h^2 we may assume $x_2 \leq x_1 \leq \frac{1}{2}$. We may also assume $r \geq \frac{h}{2}$, as otherwise we can replace r by $\frac{h}{2}$ without changing (2.6.1).

Let $x^* = (x_1, 0)$ be a point on $\partial\Lambda_h^2$ closest to x. We consider the three cases $r \leq x_2$, $x_2 < r \leq x_1$ and $r > x_1$.

Case 1: $r \leq x_2$
In this case the interior estimate Lemma 2.5.1 applied to $Q_r(x)$ directly implies

$$|\nabla_h^2 u_h(x)| \leq \frac{C}{r} \|\nabla_h^2 u_h\|_{L^2(Q_r(x))} .$$

Case 2: $x_2 < r \leq x_1$
Apply first Lemma 2.5.1 to $Q_{x_2+h/2}(x)$ to find

$$|\nabla_h^2 u_h(x)| \leq \frac{C}{x_2 + \frac{h}{2}} \|\nabla_h^2 u_h\|_{L^2(Q_{x_2+h/2}(x))} .$$

If $r < 3x_2$ then this already implies (2.6.1) once we increase C by a factor of 3. If $r \geq 3x_2$ we have $Q_{x_2+h/2}(x) \subset Q_{2x_2+h/2}(x^*) \subset Q_r(x^*) \subset Q_r(x)$ and so, by Lemma 2.5.5,

$$|\nabla_h^2 u_h\|_{L^2(Q_{x_2+h/2}(x))} \leq \|\nabla_h^2 u_h\|_{L^2(Q_{2x_2+h/2}(x^*))} \leq C \frac{2x_2 + \frac{h}{2}}{r} \|\nabla_h^2 u_h\|_{L^2(Q_r(x^*))} .$$

This together with the previous equation implies (2.6.1).

Case 3: $x_1 < r$
As in the previous case we obtain

$$|\nabla_h^2 u_h(x)| \leq \frac{C}{x_1 + \frac{h}{2}} \|\nabla_h^2 u_h\|_{L^2(Q_{x_1+h/2}(x^*))} . \tag{2.6.2}$$

Now either $r < 3x_1$ and we are done, or we can continue with Lemma 2.5.9 to find

$$\|\nabla_h^2 u_h\|_{L^2(Q_{x_1+h/2}(x^*))} \leq \|\nabla_h^2 u_h\|_{L^2(Q_{2x_1+h/2}(0))} \leq C \frac{2x_1 + \frac{h}{2}}{r} \|\nabla_h^2 u_h\|_{L^2(Q_r(0))} ,$$

which in combination with (2.6.2) implies (2.6.1).

This proves (2.6.1) in the case $z = x$. For general z, it suffices to observe that $Q_{r/2}(z) \subset Q_r(x)$ and apply the statement we have just proved to $Q_{r/2}(z)$.

The proof for $\mathsf{d} = 3$ is analogous. However there is one more case and hence we need one more intermediate step, where we deal with the case of an edge. So one applies Lemmata 2.5.1, 2.5.5, 2.5.8, 2.5.9 in order until one reaches a radius of order r. We omit the details. □

2.6.2 Outer estimates via duality

Theorem 2.6.1 states that if a discrete function is biharmonic in a subcube $Q_r(x)$ of Λ_h, then we have pointwise control over its second derivatives in a smaller subcube $Q_{r/2}(x)$. Remarkably, a dual statement is also true: If a discrete function is biharmonic outside a subcube $Q_r(x)$ of Λ_h, then we have control over its second derivatives outside of a larger subcube $Q_{2r}(x)$. The following lemma does not claim pointwise control, but only control in L^2. However we will combine it with Theorem 2.6.1 into Theorem 2.6.3 where we actually obtain pointwise control.

Lemma 2.6.2. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$*, let* $u_h \in \Phi_h$*. Let* $x \in \Lambda_h$*,* $r \geq d(x)$ *and suppose that* $\Delta_h^2 u_h(x) = 0$ *for all* $x \in \operatorname{int}\Lambda_h \setminus Q_r(x)$*. Then, for all* $s \geq r$*,*

$$\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^\mathsf{d}\setminus Q_s(x))} \leq C\left(\frac{r}{s}\right)^{\frac{\mathsf{d}}{2}} \|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^\mathsf{d}\setminus Q_r(x))}\,. \tag{2.6.3}$$

Proof. Consider first the case $r < h$. Then $d(x) = 0$, i.e. $x \in \partial\Lambda_h$, and the assumptions imply $\Delta_h^2 u_h = 0$ in $\operatorname{int}\Lambda_h$, i.e. $u_h = 0$ in $\operatorname{int}\Lambda_h$ by the uniqueness of the Bilaplacian equation. So both sides of (2.6.3) are zero and the inequality holds.

So we can assume $r \geq h$. The statement is trivial in the case that $s < 23r$, so we can also assume $s \geq 23r$. We can then replace r and s by $\tilde{r} = \lfloor r - \frac{h}{2}\rfloor_h + \frac{3h}{2}$ and $\tilde{s} = \lfloor s - \frac{h}{2}\rfloor_h + \frac{h}{2}$, respectively. It is easy to see that then $\tilde{r} \geq r$, $\tilde{s} \leq s$ and $\tilde{s} \geq 11\tilde{r}$, and it suffices to prove the theorem for $\tilde{r}, \tilde{s}$. So we will directly assume $r, s \in h\mathbb{N} + \frac{h}{2}$, $s \geq 11r$ and $r \geq \frac{3h}{2}$.

Let $f_h = \nabla_h^2 u_h \chi_{\Lambda_h\setminus Q_s(x)}$, where χ_A is the indicator function of a set A. Let $v_h \in \Phi_h$ be the unique solution of $\Delta_h^2 v_h = \operatorname{div}_{-h}\operatorname{div}_h f_h$. Then, for any $\varphi_h \in \Phi_h$,

$$(\nabla_h^2 v_h, \nabla_h^2 \varphi_h)_{L^2(\mathbb{R}^\mathsf{d})} = (f_h, \nabla_h^2 \varphi_h)_{L^2(\mathbb{R}^\mathsf{d})}\,. \tag{2.6.4}$$

Also let ζ_h and η_h be discrete cut-off functions such that ζ_h is 1 on $\Lambda_h \setminus Q_{5r}(x)$, 0 on $Q_{3r}(x) \cap \Lambda_h$, η_h is 1 on $Q_{7r}(x) \cap \Lambda_h$, 0 on $\Lambda_h \setminus Q_{9r}(x)$ and such that $|\nabla_h^\kappa \zeta_h| \leq \frac{C}{r^\kappa}$ and $|\nabla_h^\kappa \eta_h| \leq \frac{C}{r^k}$ for $\kappa \leq 2$.

These choices ensure that

$$\nabla_h^2(\zeta_h u_h) = \nabla_h^2 u_h \text{ on the support of } f_h \tag{2.6.5}$$

and that

$$\eta_h = 1 \text{ on the support of } \Delta_h^2(\zeta_h u_h)\,. \tag{2.6.6}$$

Indeed, for example the support of $\Delta_h^2(\zeta_h u_h)$ is contained in $Q_{5r+2h}(x) \setminus Q_{3r-2h}(x) \subset Q_{7r}(x)$.

This implies

$$\begin{aligned}
\|\nabla_h^2 u_h\|^2_{L^2(\mathbb{R}^\mathsf{d}\setminus Q_s(x))} &= (f_h, \nabla_h^2 u_h)_{L^2(\mathbb{R}^\mathsf{d})} \\
&\overset{(2.6.5)}{=} (f_h, \nabla_h^2(\zeta_h u_h))_{L^2(\mathbb{R}^\mathsf{d})} \\
&\overset{(2.6.4)}{=} (\nabla_h^2 v_h, \nabla_h^2(\zeta_h u_h))_{L^2(\mathbb{R}^\mathsf{d})} \\
&= (v_h, \Delta_h^2(\zeta_h u_h))_{L^2(\mathbb{R}^\mathsf{d})} \\
&\overset{(2.6.6)}{=} (\eta_h v_h, \Delta_h^2(\zeta_h u_h))_{L^2(\mathbb{R}^\mathsf{d})} \\
&= (\nabla_h^2(\eta_h v_h), \nabla_h^2(\zeta_h u_h))_{L^2(\mathbb{R}^\mathsf{d})} \\
&\leq \|\nabla_h^2(\eta_h v_h)\|_{L^2(\mathbb{R}^\mathsf{d})} \|\nabla_h^2(\zeta_h u_h)\|_{L^2(\mathbb{R}^\mathsf{d})}\,.
\end{aligned} \tag{2.6.7}$$

Now by the product rule

$$\nabla_h^2(\eta_h v_h) = \sum_{i,j=1}^{\mathsf{d}} D_{-i}^h D_j^h \eta_h v_h + \tau_j^h D_{-i}^h \eta_h D_j^h v_h + \tau_{-i}^h D_j^h \eta_h D_{-i}^h v_h + \tau_{-i}^h \tau_j^h \eta_h D_{-i}^h D_j^h v_h$$

and so, using the Poincaré inequality[2] on $Q_{9r}(x)$,

[2]Here we have used the assumption $r \geq d(x)$ (or rather $7r \geq d(x)$): It ensures that we have zero boundary data somewhere on $Q_{7r}^h(x) \setminus Q_r^h(x)$ so that we can indeed use the Poincaré inequality.

$$\begin{aligned}\|\nabla_h^2(\eta_h v_h)\|_{L^2(\mathbb{R}^d)} &\le \frac{C}{r^2}\|v_h\|_{L^2(Q_{9r}(x))} + \frac{C}{r}\|\nabla_h v_h\|_{L^2(Q_{9r}(x))} + C\|\nabla_h^2 v_h\|_{L^2(Q_{9r}(x))} \\ &\le C\|\nabla_h^2 v_h\|_{L^2(Q_{9r}(x))}\,.\end{aligned} \tag{2.6.8}$$

Similarly, by the Poincaré inequality on the annulus $Q_{7r}(x) \setminus Q_r(x)$,

$$\begin{aligned}\|\nabla_h^2(\zeta_h u_h)\|_{L^2(\mathbb{R}^d)} &\le \frac{C}{r^2}\|u_h\|_{L^2(Q_{7r}(x)\setminus Q_r(x))} + \frac{C}{r}\|\nabla_h u_h\|_{L^2(Q_{7r}(x)\setminus Q_r(x)} + C\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_r(x))} \\ &\le C\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_r(x))}\,.\end{aligned}$$

If we plug the last two estimates into (2.6.7) and then use Theorem 2.6.1 for v_h we obtain

$$\begin{aligned}\|\nabla_h^2 u_h\|^2_{L^2(\Lambda_h\setminus Q_s(x))} &\le C\|\nabla_h^2 v_h\|_{L^2(Q_{9r}(x))}\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_r(x))} \\ &\le C\left(\frac{9r}{s}\right)^{\frac{d}{2}}\|\nabla_h^2(v_h)\|_{L^2(\mathbb{R}^d)}\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_r(x))}\,.\end{aligned}$$

This implies (2.6.3) once we use the energy estimate

$$\|\nabla_h^2 v_h\|_{L^2(\mathbb{R}^d)} \le \|f_h\|_{L^2(\mathbb{R}^d)} = \|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_s(x))}\,.$$

□

Now we can combine this lemma with Theorem 2.6.1 to obtain a pointwise outer estimate.

Theorem 2.6.3. *Let* $d = 2$ *or* $d = 3$*, let* $u_h \in \Phi_h$*. Let* $x \in \Lambda_h$*,* $r > 0$ *and suppose that* $\Delta_h^2 u_h(x) = 0$ *for all* $x \in \operatorname{int}\Lambda_h \setminus Q_r(x)$*.*

Then, for all $y \in \Lambda_h \setminus Q_{2r}(x)$*,*

$$|\nabla_h^2 u_h(y)| \le C\frac{(\max(d(x),r))^{\frac{d}{2}}}{|x-y|^d}\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_r(x))}\,. \tag{2.6.9}$$

Proof. As in the proof of Lemma 2.6.2 we see that $d(x) = 0$ implies $u = 0$ everywhere and (2.6.9) holds. So assume $d(x) \ge h$.

Let $y \in \Lambda_h \setminus Q_{2r}(x)$. If $y \in Q_{2d(x)}(x)$ we use Theorem 2.6.1 on $Q_{d(x)}(y) \subset \mathbb{R}^d \setminus Q_{2r}(x)$ to obtain

$$|\nabla_h^2 u_h(y)| \le \frac{C}{d(x)^{\frac{d}{2}}}\|\nabla_h^2 u_h\|_{L^2(Q_{d(x)}(y))} \le \frac{C}{d(x)^{\frac{d}{2}}}\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_{2r}(x))}\,,$$

which implies (2.6.9) because $|x-y| \le \sqrt{n}|x-y|_\infty \le 2\sqrt{n}d(x)$ and hence $\frac{1}{d(x)} \le 4n\frac{d(x)}{|x-y|^2}$.

If, on the other hand, $y \in \Lambda_h \setminus Q_{2d(x)}(x)$ then we use Theorem 2.6.1 on $Q_{|x-y|_\infty/2}(y)$ and then Lemma 2.6.2 as follows:

$$\begin{aligned}|\nabla_h^2 u_h(y)| &\le \frac{C}{\left(\frac{|x-y|_\infty}{2}\right)^{\frac{d}{2}}}\|\nabla_h^2 u_h\|_{L^2(Q_{|x-y|_\infty/2}(y))} \\ &\le \frac{C}{|x-y|_\infty^{\frac{d}{2}}}\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_{|x-y|_\infty/2}(x))} \\ &\le \frac{C}{|x-y|_\infty^{\frac{d}{2}}}\left(\frac{\max(d(x),r)}{\frac{|x-y|_\infty}{2}}\right)^{\frac{d}{2}}\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_{\max(d(x),r)}(x))} \\ &\le C\frac{(\max(d(x),r))^{\frac{d}{2}}}{|x-y|_\infty^{d}}\|\nabla_h^2 u_h\|_{L^2(\mathbb{R}^d\setminus Q_{\max(d(x),r)}(x))}\,,\end{aligned}$$

which implies (2.6.9). □

2.7 The discrete full-space Green's function

In order to obtain estimates for G_h, we will compare G_h with a Green's function of $(h\mathbb{Z})^{\mathsf{d}}$. In the absence of boundary conditions such a Green's function is not uniquely defined. We will choose a normalization that is best suited for our application. The necessary asymptotics for the Green's function of $(h\mathbb{Z})^{\mathsf{d}}$ have been derived by Mangad [Man67] using Fourier-theoretic methods.

By $\mathcal{F}$ we denote the Fourier transform of tempered distributions (where we use the convention $(\mathcal{F}f)(x) = \int_{\mathbb{R}^{\mathsf{d}}} f(\xi)\mathrm{e}^{-2\pi i x\cdot\xi}\,\mathrm{d}\xi$).

Theorem 2.7.1 ([Man67], Section 4)**.** *Let* $\mathsf{d} \in \mathbb{N}$. *Define* $F\colon \mathbb{Z}^{\mathsf{d}} \times \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ *by*

$$F(x,y) = \mathcal{F}\left(\frac{V(\xi)}{\left(4\sum_{j=1}^{\mathsf{d}}\sin^2(\pi\xi_j)\right)^2}\right)(x-y)$$

where $V \in C_c^\infty([-1,1]^{\mathsf{d}})$ *is chosen such that* $V = 1$ *near 0 and* $\sum_{z\in\mathbb{Z}} V(x+z) = 1$ *for all* x, *and* $\frac{V(\xi)}{\left(4\sum_{j=1}^{\mathsf{d}}\sin^2(\pi\xi_j)^2\right)^2}$ *denotes the tempered distribution given by its finite part in the sense of Hadamard (see [Sch66, Chapitre II, §2 and §3]).*

Then F *is a Green's function for* Δ_1^2 *in the sense that* $\Delta_1^2 F(\cdot, y) = \delta_y$. *It satisfies the following asymptotic expansion: If* $\mathsf{d} = 2$ *and* $z = x - y$,

$$F(x,y) = \frac{|z|^2\log|z|}{8\pi} + \frac{(\gamma - 1 + \log\pi)|z|^2}{8\pi} - \frac{\log|z|}{16\pi} + \frac{4(z_1^4 + z_2^4)}{|z|^4} - 12\log\pi - 12\gamma - 3 + O\left(\frac{1}{|z|^2}\right)$$

where γ *is the Euler-Mascheroni constant, and if* $\mathsf{d} = 3$ *and* $z = x - y$,

$$F(x,y) = -\frac{|z|}{8\pi} + \frac{z_1^4 + z_2^4 + z_3^4}{64\pi|z|^5} + \frac{1}{64\pi|z|} + O\left(\frac{1}{|z|^3}\right).$$

Let us briefly sketch how to prove this theorem: Observe that $\sigma(\xi) := \left(4\sum_{j=1}^{\mathsf{d}}\sin^2(\pi\xi_j)\right)^2$ is the symbol of Δ_1^2, so that $\Delta_1^2 F(x,y) = \mathcal{F}(V)(x-y)$. On the other hand one easily checks that $\sum_{z\in\mathbb{Z}} V(x+z) = 1$ implies that $\mathcal{F}(V)(m) = \delta_0(m)$ for any $m \in \mathbb{Z}^{\mathsf{d}}$. This proves that F is a Green's function. To derive the asymptotic expansion, one develops a Laurent series

$$\frac{1}{\sigma(\xi)} = \frac{1}{16\pi^2|\xi|^4} + \frac{f_{-2}(\xi)}{|\xi|^2} + f_0(\xi) + \cdots + o(|\xi|^{\mathsf{d}}).$$

Then one can check using the explicit formulas for the Fourier transforms of $|\xi|^m$ (see [Sch66]) and the Riemann-Lebesgue lemma that

$$\mathcal{F}\left(\frac{V(\xi)}{\sigma(\xi)} - \frac{1}{16\pi^2|\xi|^4} + \frac{f_{-2}(\xi)}{|\xi|^2} + f_0(\xi) + \cdots\right) = o(|x|^{-\mathsf{d}-N})$$

so it suffices to compute the Fourier transform of $\frac{1}{16\pi^2|\xi|^4} + \frac{f_{-2}(\xi)}{|\xi|^2} + f_0(\xi) + \cdots$. This one can again do explicitly and thereby obtain an asymptotic expansion for F up to $O(|x|^{\mathsf{d}})$. For details we refer to [Man67].

By scaling the lattice we can deduce from this estimates for Green's functions on $(h\mathbb{Z})^{\mathsf{d}}$. We state the estimates that we will need.

Lemma 2.7.2. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$, $h > 0$, $r \geq 4h$. *There exists a function* $\tilde{G}_h \colon (h\mathbb{Z})^{\mathsf{d}} \times (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$ *such that* $\Delta_h^2 \tilde{G}_h(\cdot, y) = \delta_{h,y}$ *and such that the following estimates are satisfied:*

$$|\nabla_{h,y} \tilde{G}_h(x,y)| \leq C r^{3-\mathsf{d}} \qquad \text{if } |x-y|_\infty \leq \frac{r}{2}, \tag{2.7.1}$$

$$|\nabla_{h,x}^2 \nabla_{h,y} \tilde{G}_h(x,y)| \leq \frac{C}{(|x-y|+h)^{\mathsf{d}-1}} \qquad \text{if } |x-y|_\infty \leq \frac{r}{2}, \tag{2.7.2}$$

$$|\nabla_{h,x}^2 \nabla_{h,y}^2 \tilde{G}_h(x,y)| \leq \frac{C}{(|x-y|+h)^{\mathsf{d}}} \qquad \text{if } |x-y|_\infty \leq \frac{r}{2} \tag{2.7.3}$$

and

$$|D_{h,x}^\alpha D_{h,y}^\beta \tilde{G}_h(x,y)| \leq C r^{4-\mathsf{d}-|\alpha|-|\beta|} \qquad \text{if } \frac{r}{2} \leq |x-y|_\infty \leq r, |\alpha|+|\beta| \leq 4\,. \tag{2.7.4}$$

For $\mathsf{d} = 2$ the function $\tilde{G}_h$ depends on r, but we will suppress this dependence for ease of notation.

Proof. We begin with the slightly easier case $\mathsf{d} = 3$. The asymptotic expansion in Theorem 2.7.1 easily implies that

$$|D_{1,x}^\alpha D_{1,y}^\beta F(x,y)| \leq C|x-y|^{1-|\alpha|-|\beta|}$$

for $|\alpha| + |\beta| \leq 4$ and any x, y with $|x-y| \geq 10$, say (observe that $g = O(|x|^{-3})$ implies $D_{\pm i}^1 g(x) = O(|x|^{-3})$, so we do not need to care about the error term). On the other hand F is finite everywhere, so that

$$|D_{1,x}^\alpha D_{1,y}^\beta F(x,y)| \leq C$$

for $|\alpha| + |\beta| \leq 4$ and any x, y with $|x-y| < 10$. If we combine these two estimates we conclude that we have

$$|D_{1,x}^\alpha D_{1,y}^\beta F(x,y)| \leq C(|x-y|+1)^{1-|\alpha|-|\beta|}\,.$$

Now if we set $\tilde{G}_h(x,y) = hF\left(\frac{x}{h}, \frac{y}{h}\right)$ then $\tilde{G}_h$ satisfies

$$|D_{h,x}^\alpha D_{h,y}^\beta \tilde{G}_h(x,y)| \leq C(|x-y|+h)^{1-|\alpha|-|\beta|}\,,$$

which immediately implies the claimed estimates.

If $\mathsf{d} = 2$ we need to take care of the logarithmic terms. So we set

$$\tilde{F}(x,y) = F(x,y) + \frac{|x-y|^2 \log\left(\frac{h}{r}\right)}{8\pi}\,.$$

Then $\tilde{F}$ has the asymptotic expansion

$$\begin{aligned}\tilde{F}(z) = \frac{z|^2 \log|z|}{8\pi} + \frac{(\log\left(\frac{h}{r}\right) + \gamma - 1 + \log\pi)|z|^2}{8\pi} - \frac{\log|z|}{16\pi} \\ + \frac{4(z_1^4 + z_2^4)}{|z|^4} - 12\log\pi - 12\gamma - 3 + O\left(\frac{1}{|z|^2}\right)\end{aligned}$$

and this implies

$$|D_{1,x}^\alpha D_{1,y}^\beta \tilde{F}(x,y)| \leq C|x-y|^{2-|\alpha|-|\beta|}\left(\left|\log|x-y| + \log\left(\frac{h}{r}\right)\right| + 1\right)$$

for $|\alpha| + |\beta| \leq 2$ and any x, y with $|x - y| \geq 10$. Because $D^{\alpha}_{1,x} D^{\beta}_{1,y} \tilde{F}(x, y)$ is bounded by $C\left(1 + \left|\log\left(\frac{h}{r}\right)\right|\right)$ for $|x - y| < 10$, we conclude

$$|D^{\alpha}_{1,x} D^{\beta}_{1,y} \tilde{F}(x, y)| \leq C(|x - y| + 1)^{2-|\alpha|-|\beta|} \left|\log\left(\frac{h(|x - y| + 1)}{r}\right)\right| .$$

We now set $\tilde{G}_h(x, y) = h^2 \tilde{F}\left(\frac{x}{h}, \frac{y}{h}\right)$ and obtain

$$|D^{\alpha}_{h,x} D^{\beta}_{h,y} \tilde{G}_h(x, y)| \leq C(|x - y| + h)^{2-|\alpha|-|\beta|} \left|\log\left(\frac{|x - y| + h}{r}\right)\right| .$$

It is easy to check that this implies (2.7.1) and (2.7.4) for $|\alpha| + |\beta| \leq 2$. If $|\alpha| + |\beta| \geq 3$ we need to be slightly more careful: Observe that third discrete derivatives of $|x - y|^2$ vanish, so that we actually have

$$|\nabla^{\alpha}_{1,x} \nabla^{\beta}_{1,y} \tilde{F}(x - y)| \leq \frac{C}{|x - y|^{|\alpha|+|\beta|-2}}$$

if $|x - y| \geq 10$ from which we conclude

$$|\nabla^{\alpha}_{1,x} \nabla^{\beta}_{1,y} \tilde{F}(x - y)| \leq \frac{C}{(|x - y| + 1)^{|\alpha|+|\beta|-2}}$$

for any x, y. Recalling that $\tilde{G}_h(x, y) = h^2 \tilde{F}\left(\frac{x}{h}, \frac{y}{h}\right)$ we immediately obtain (2.7.2), (2.7.3) and (2.7.4) for $|\alpha| + |\beta| \geq 3$. □

2.8 Proof of the main theorem

We are now able to prove Theorem 2.1.3. We first give the straightforward proof of part ii) and then continue with part i).

2.8.1 Lower bounds for $G_h(x, x)$

The proof is rather short and based on the choice of an appropriate test function.

Proof of Theorem 2.1.3 ii).
We can assume $d(x) \geq h$, as otherwise $d(x) = 0$ and hence $G_h(x, x) = 0$. If we test the equation $\Delta^2_h G_{h,x} = \delta_{h,x}$ with $G_{h,x}$, we find

$$\|\nabla^2_h G_{h,x}\|^2_{L^2(\mathbb{R}^d)} = (\Delta^2_h G_{h,x}, G_{h,x})_{L^2(\mathbb{R}^d)} = (\delta_{h,x}, G_{h,x})_{L^2(\mathbb{R}^d)} = G_h(x, x) . \tag{2.8.1}$$

Now let $\varphi_h \in \Phi_h$. Then testing the equation $\Delta^2_h G_{h,x} = \delta_{h,x}$ with φ_h and using the Cauchy-Schwarz inequality we find

$$\begin{aligned} \varphi_h(x) &= (\nabla^2_h G_{h,x}, \nabla^2_h \varphi_h)_{L^2(\mathbb{R}^d)} \\ &\leq \|\nabla^2_h G_{h,x}\|_{L^2(\mathbb{R}^d)} \|\nabla^2_h \varphi_h\|_{L^2(\mathbb{R}^d)} \\ &= \sqrt{G_h(x, x)} \|\nabla^2_h \varphi_h\|_{L^2(\mathbb{R}^d)} . \end{aligned}$$

If φ_h is not identically zero this implies

$$G_h(x, x) \geq \frac{(\varphi_h(x))^2}{\|\nabla^2_h \varphi_h\|^2_{L^2(\mathbb{R}^d)}}$$

and so it remains to find a $\varphi_h(x)$ such that $\frac{\varphi_h(x)}{\|\nabla_h^2\varphi_h\|_{L^2(\mathbb{R}^d)}} \geq Cd(x)^{2-\frac{d}{2}}$. But this is easy: Take $\varphi_{h,x} \in \Phi_h$ supported in $Q_{d(x)}(x)$ such that $\varphi_{h,x}(x) = 1$ and such that $|\nabla_h^2\varphi_{h,x}| \leq \frac{C}{d(x)^2}$ and extend it by 0 to all of Λ_h^d. □

2.8.2 Upper bounds for $G_h(x,y)$

In this section we prove part i) of Theorem 2.1.3.

We begin with a rather weak estimate for $G_h(x,y)$.

Lemma 2.8.1. *Let $d = 2$ or $d = 3$ and G_h be the Green's function of Λ_h^d. Then we have*

$$0 \leq G_h(x,x) = \|\nabla_h^2 G_{h,x}\|_{L^2(\mathbb{R}^d)}^2 \leq Cd(x)^{4-d} \tag{2.8.2}$$

for any $x \in \Lambda_h$ and

$$|G_h(x,y)| \leq Cd(x)^{2-\frac{d}{2}}d(y)^{2-\frac{d}{2}} \tag{2.8.3}$$

for any $x, y \in \Lambda_h$.

Proof. We first prove (2.8.2). By (2.8.1) we have

$$\|\nabla_h^2 G_{h,x}\|_{L^2(\mathbb{R}^d)}^2 = G_h(x,x)\,. \tag{2.8.4}$$

If $x \in \partial\Lambda_h$ then $G_h(x,x) = 0$ and (2.8.2) holds. So assume $x \in \operatorname{int}\Lambda_h$, i.e. $d(x) \geq h$. The Sobolev-Poincaré inequality implies that

$$\begin{aligned} G_h(x,x) \leq \|G_{h,x}\|_{L^\infty(Q_{d(x)+h/2}(x))} &\leq C\left(d(x)+\frac{h}{2}\right)^{2-\frac{d}{2}} \|\nabla_h^2 G_{h,x}\|_{L^2(Q_{d(x)+h/2}(x))} \\ &\leq Cd(x)^{2-\frac{d}{2}}\|\nabla_h^2 G_{h,x}\|_{L^2(Q_{2d(x)+h/2}(x))}\,. \end{aligned}$$

If we combine this estimate with (2.8.4) we find that

$$\|\nabla_h^2 G_{h,x}\|_{L^2(\mathbb{R}^d)}^2 = G_h(x,x) \leq Cd(x)^{2-\frac{d}{2}}\|\nabla_h^2 G_{h,x}\|_{L^2(Q_{2d(x)+h/2}(x))} \leq Cd(x)^{2-\frac{d}{2}}\|\nabla_h^2 G_{h,y}\|_{L^2(\mathbb{R}^d)}$$

and hence

$$0 \leq G_h(x,x) = \|\nabla_h^2 G_{h,x}\|_{L^2(\mathbb{R}^d)}^2 \leq Cd(x)^{4-d}\,.$$

This proves (2.8.2). For (2.8.3), we test $\Delta_h^2 G_{h,x} = \delta_{h,x}$ with $G_{h,y}$ and use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} |G_h(x,y)| &= \left|(\delta_{h,x}, G_{h,y})_{L^2(\mathbb{R}^d)}\right| \\ &= \left|(\nabla_h^2 G_{h,x}, \nabla_h^2 G_{h,y})_{L^2(\mathbb{R}^d)}\right| \\ &\leq \|\nabla_h^2 G_{h,x}\|_{L^2(\mathbb{R}^d)}\|\nabla_h^2 G_{h,y}\|_{L^2(\mathbb{R}^d)} \\ &\overset{(2.8.2)}{\leq} Cd(x)^{2-\frac{d}{2}}d(y)^{2-\frac{d}{2}}\,. \end{aligned}$$

□

The next lemma gives estimates for G_h and its derivatives that are sharp when x and y are far apart. We first prove a pointwise estimate for $\nabla_{h,x}^2\nabla_{h,y}G_h$ by applying Theorem 2.6.3 to a cut-off version of $\nabla_{h,y}G_{h,y}$. Afterwards we integrate it along suitable paths to deduce the estimates in the lemma.

Lemma 2.8.2. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$ *and* G_h *be the Green's function of* Λ_h^{d}. *If* $x, y \in \Lambda_h$ *and* $|x-y|_\infty > \frac{d(y)}{8}$ *then*

$$|G_h(x,y)| \le C\frac{(d(x)+h)^2(d(y)+h)^2}{|x-y|^{\mathsf{d}}}, \tag{2.8.5}$$

$$|\nabla_{h,x} G_h(x,y)| \le C\frac{(d(x)+h)(d(y)+h)^2}{|x-y|^{\mathsf{d}}}, \tag{2.8.6}$$

$$|\nabla_{h,x}^2 G_h(x,y)| \le C\frac{(d(y)+h)^2}{|x-y|^{\mathsf{d}}}, \tag{2.8.7}$$

$$|\nabla_{h,x}\nabla_{h,y} G_h(x,y)| \le C\frac{(d(x)+h)(d(y)+h)}{|x-y|^{\mathsf{d}}}. \tag{2.8.8}$$

Proof.
Step 1: Pointwise estimate for $\nabla_{h,x}^2 \nabla_{h,y} G_h(x,y)$
We claim that if $x, y \in \Lambda_h$ and $|x-y|_\infty > \frac{d(y)}{8}$ then

$$|\nabla_{h,x}^2 \nabla_{h,y} G_h(x,y)| \le C\frac{d(y)+h}{|x-y|^{\mathsf{d}}}. \tag{2.8.9}$$

In the following all derivatives will be with respect to x unless we mark them with a sub- or superscript y.

If $d(y) < 160h$ we can use a trivial estimate: From Lemma 2.8.1 we know

$$\|\nabla_h^2 G_{h,y'}\|_{L^2(\mathbb{R}^{\mathsf{d}})} \le Cd(y')^{2-\frac{\mathsf{d}}{2}} \le Ch^{2-\frac{\mathsf{d}}{2}}$$

if $|y'-y|_\infty \le h$. If we now use

$$|D_i^h f_h(y)|^2 = \left(\frac{1}{h}(f_h(y+e_i) - f_h(y)\right)^2 \le \frac{2}{h^2}(f_h(y+e_i)^2 + f_h(y)^2)$$

with $f_h = \nabla_h^2 G_h$ we get that

$$\|\nabla_h^2 D_i^{h,y} G_h\|_{L^2(\mathbb{R}^{\mathsf{d}})}^2 \le \frac{2}{h^2}\left(\|\nabla_h^2 \tau_i^{h,y} G_h\|_{L^2(\mathbb{R}^{\mathsf{d}})}^2 + \|\nabla_h^2 G_h\|_{L^2(\mathbb{R}^{\mathsf{d}})}^2\right) \le Ch^{2-\mathsf{d}},$$

i.e.

$$\|\nabla_h^2 D_i^{h,y} G_h\|_{L^2(\mathbb{R}^{\mathsf{d}})} \le Ch^{1-\frac{\mathsf{d}}{2}}.$$

Then Theorem 2.6.3 with $r = h$ implies

$$|\nabla_h^2 D_i^{h,y} G_h(x,y)| \le C\frac{\max(d(y),h)^{\frac{\mathsf{d}}{2}}}{|x-y|^{\mathsf{d}}}\|\nabla_h^2 D_i^{h,y} G_h\|_{L^2(\mathbb{R}^{\mathsf{d}})} \le C\frac{h^{\frac{\mathsf{d}}{2}}}{|x-y|^{\mathsf{d}}}h^{1-\frac{\mathsf{d}}{2}} = C\frac{h}{|x-y|^{\mathsf{d}}},$$

which implies (2.8.9) if we choose C there large enough.

So assume $d(y) \ge 160h$. Let η_h be a discrete cut-off function that is 1 on $Q_{d(y)/32+2h}$, 0 on $(h\mathbb{Z})^{\mathsf{d}} \setminus Q_{d(y)/16-2h}(x)$, and such that $|\nabla^\kappa \eta_h| \le \frac{C}{d(y)^\kappa}$ for $\kappa \le 2$. Let $H_h(x,y) = G_h(x,y) - \eta_h(x)\tilde{G}_h(x,y)$, where $\tilde{G}_h$ is the function from Lemma 2.7.2 with $r = \frac{d(y)}{16}$. We write $H_{h,y}$ for $H_h(\cdot, y)$.

Then, for $i \in \{1,\dots,n\}$, $D_i^{h,y} H_{h,y} \in \Phi_h$. Also, the singularities near y cancel out, so that $\Delta_h^2 D_i^{h,y} H_{h,y} = 0$ in $Q_{d(y)/32}(y)$ and in $\operatorname{int}\Lambda_h \setminus Q_{d(y)/16}(y)$.

Next, we want to bound $\|\nabla_h^2 D_i^{h,y} H_{h,y}\|_{L^2(\mathbb{R}^d)}$. To do so, we introduce another cut-off function ζ_h that is 1 on $\operatorname{int}\Lambda_h \setminus Q_{d(y)/32}(y)$, 0 on $Q_{d(y)/64}(y)$ and such that $|\nabla^\kappa \zeta_h| \leq \frac{C}{d(y)^\kappa}$ for $\kappa \leq 2$. Then we have that

$$\Delta_h^2 D_i^{h,y} H_{h,y} = \zeta_h \Delta_h^2 D_i^{h,y} H_{h,y} = -\zeta_h \Delta_h^2 D_i^{h,y} \left(\eta_h \tilde{G}_{h,y}\right) = -\zeta_h \Delta_h^2 \left(\eta_h D_i^{h,y} \tilde{G}_{h,y}\right)$$

where we have used that η_h does not depend on y. Thus

$$\begin{aligned}
\|\nabla_h^2 D_i^{h,y} H_{h,y}\|_{L^2(\mathbb{R}^d)}^2 &= (\Delta_h^2 D_i^{h,y} H_{h,y}, D_i^{h,y} H_{h,y})_{L^2(\mathbb{R}^d)} \\
&= -(\zeta_h \Delta_h^2 (\eta_h D_i^{h,y} \tilde{G}_{h,y}), D_i^{h,y} H_{h,y})_{L^2(\mathbb{R}^d)} \\
&= -(\Delta_h^2 (\eta_h D_i^{h,y} \tilde{G}_{h,y}), \zeta_h D_i^{h,y} H_{h,y})_{L^2(\mathbb{R}^d)} \\
&= -(\nabla_h^2 (\eta_h D_i^{h,y} \tilde{G}_{h,y}), \nabla_h^2 (\zeta_h D_i^{h,y} H_{h,y}))_{L^2(\mathbb{R}^d)} \\
&\leq \|\nabla_h^2 (\eta_h D_i^{h,y} \tilde{G}_{h,y})\|_{L^2(\mathbb{R}^d)} \|\nabla_h^2 (\zeta_h D_i^{h,y} H_{h,y})\|_{L^2(\mathbb{R}^d)} .
\end{aligned} \tag{2.8.10}$$

If we use the pointwise estimates for $\tilde{G}_{h,y}$ from Lemma 2.7.2, we conclude

$$|\nabla_h^2 (\eta_h D_i^{h,y} \tilde{G}_{h,y})| \leq C d(y)^{1-d}$$

and hence

$$\|\nabla_h^2 (\eta_h D_i^{h,y} \tilde{G}_{h,y})\|_{L^2(\mathbb{R}^d)} \leq C d(y)^{1-\frac{d}{2}} .$$

Furthermore, as in (2.6.8), the Poincaré inequality on $Q_{d(y)+h/2}(y)$ and the pointwise estimates for ζ_h imply that

$$\begin{aligned}
&\|\nabla_h^2 (\zeta_h D_i^{h,y} H_{h,y})\|_{L^2(\mathbb{R}^d)} \\
&\quad \leq \frac{C}{d(y)^2} \|D_i^{h,y} H_{h,y}\|_{L^2(Q_{d(y)+h/2}(y))} + \frac{C}{d(y)} \|\nabla_h D_i^{h,y} H_{h,y}\|_{L^2(Q_{d(y)+h/2}(y))} \\
&\qquad + \|\nabla_h^2 D_i^{h,y} H_{h,y}\|_{L^2(\mathbb{R}^d)} \\
&\quad \leq C \|\nabla_h^2 D_i^{h,y} H_{h,y}\|_{L^2(\mathbb{R}^d)} .
\end{aligned}$$

If we combine the last two estimates with (2.8.10) we conclude that

$$\|\nabla_h^2 D_i^{h,y} H_{h,y}\|_{L^2(\mathbb{R}^d)} \leq C d(y)^{1-\frac{d}{2}} .$$

We recall that $\Delta_h^2 H_h = 0$ in $\operatorname{int}\Lambda_h \setminus Q_{d(y)/16}$ and use Theorem 2.6.3 to find that, for $x \in \Lambda_h \setminus Q_{d(y)/8}(y)$,

$$|\nabla_h^2 D_i^{h,y} H_h(x)| \leq C \frac{d(y)^{\frac{d}{2}}}{|x-y|^d} \|\nabla_h^2 D_i^{h,y} H_h\|_{L^2(\mathbb{R}^d)} \leq C \frac{d(y)^{\frac{d}{2}}}{|x-y|^d} d(y)^{1-\frac{d}{2}} = C \frac{d(y)}{|x-y|^d} .$$

This implies (2.8.9) because $D_i^{h,y} H_{h,y}$ is equal to $D_i^{h,y} G_{h,y}$ in $\Lambda_h \setminus Q_{d(y)/16}(y)$ and therefore $\nabla_h^2 D_i^{h,y} H_{h,y}$ is equal to $\nabla_h^2 D_i^{h,y} G_{h,y}$ in $\Lambda_h \setminus Q_{d(y)/8}(y)$.

Step 2: Proof of (2.8.8)

We can obtain (2.8.8) by integrating (2.8.9) along a well-chosen path in x. Let $(x^{(k)})_{k=0}^L$ be a path of length Lh from $x^{(0)} = x$ to $x^{(L)} \in (h\mathbb{Z})^d \setminus \Lambda_h$ such that $|x^{(k+1)} - x^{(k)}|_\infty = h$, $|x^{(k)} - y| \geq |x-y|_\infty$ for all k, and $L \leq 2(d(x)+h)$. To construct such a path begin with the straight path from x to a closest point $x^* \in (h\mathbb{Z})^d \setminus \Lambda_h$ (which will have length $d(x)+h$).

If this path does not intersect $Q^h_{|x-y|_\infty-h}(y)$, we are done. Else we modify the path by taking a (shortest-possible) detour around $Q^h_{|x-y|_\infty-h}(y)$. This detour lengthens the path by at most $|x-y|_\infty$, and it is easy to check that if it is necessary then $y \in Q^h_{d(x)}(x)$, so that $|x-y|_\infty \le d(x)$, and our path has length at most $d(x)+h+|x-y|_\infty \le 2(d(x)+h)$.

Now, by (2.8.9),

$$|\nabla^2_{h,x}\nabla_{h,y}G_h(x^{(k)},y)| \le C\frac{d(y)+h}{(|x^{(k)}-y|+h)^{\mathsf{d}}} \le C\frac{d(y)+h}{(|x^{(k)}-y|_\infty+h)^{\mathsf{d}}} \le C\frac{d(y)+h}{(|x-y|_\infty+h)^{\mathsf{d}}}\,.$$

Now we can perform discrete integration along $(x^{(k)})_{k=0}^L$. Note that $\nabla_{h,x}\nabla_{h,y}G_h(x^{(L)},y)=0$ and so

$$\begin{aligned}|\nabla_{h,x}\nabla_{h,y}G_h(x,y)| &\le \sum_{k=0}^{L-1}|\nabla_{h,x}\nabla_{h,y}G_h(x^{(k+1)},y)-\nabla_{h,x}\nabla_{h,y}G_h(x^{(k)},y)|\\ &\le \sum_{k=0}^{L-1} h|\nabla^2_{h,x}\nabla_{h,y}G_h(x^{(k)},y)|\\ &\le L\frac{d(y)+h}{(|x-y|_\infty+h)^{\mathsf{d}}}\,,\end{aligned}$$

which implies (2.8.8).

Step 3: Proof of (2.8.7)

We proceed as in the previous step with the only difference that this time we integrate in y along a path that avoids x. Let $(y^{(k)})_{k=0}^L$ be a path of length Lh from $y^{(0)}=y$ to $y^{(L)} \in (h\mathbb{Z})^{\mathsf{d}}\setminus\Lambda_h$ such that $|y^{(k+1)}-y^{(k)}|_\infty = h$, $|y^{(k)}-x|_\infty \ge |y-x|_\infty$ for all k, and $L \le 2(d(y)+h)$. If we construct this path as in the previous step, we can in addition ensure that $d(y^{(k)}) \le d(y)$ for all k (then in particular $|y^{(k)}-x|_\infty \ge \frac{d(y^{(k)})}{8}$, so that (2.8.9) is applicable for all $y^{(k)}$).

Now by (2.8.9)

$$|\nabla^2_{h,x}\nabla_{h,y}G_h(x,y^{(k)})| \le C\frac{d(y^{(k)})+h}{(|x-y^{(k)}|+h)^{\mathsf{d}}} \le C\frac{d(y^{(k)})+h}{(|x-y^{(k)}|_\infty+h)^{\mathsf{d}}} \le C\frac{d(y)+h}{(|x-y|_\infty+h)^{\mathsf{d}}}$$

and if we integrate this along $(y^{(k)})_{k=0}^L$, we obtain (2.8.7).

Step 4: Proof of (2.8.6) *and* (2.8.5)

We proceed as in the previous two steps. If we integrate (2.8.7) along a path $(x^{(k)})_{k=0}^L$ that avoids y once, we obtain (2.8.6), and if we integrate once more, we obtain (2.8.5). □

Now we complement this lemma with an estimate when x and y are close:

Lemma 2.8.3. *Let* $\mathsf{d}=2$ *or* $\mathsf{d}=3$ *and* G_h *be the Green's function of* Λ_h^{d}. *If* $x,y\in\Lambda_h$ *and* $|x-y|_\infty \le \frac{d(y)}{8}$ *then*

$$|G_h(x,y)| \le C(d(x)+h)^{2-\frac{\mathsf{d}}{2}}(d(y)+h)^{2-\frac{\mathsf{d}}{2}}\,, \tag{2.8.11}$$

$$|\nabla_{h,x}G_h(x,y)| \le C(d(y)+h)^{3-\mathsf{d}}\,, \tag{2.8.12}$$

$$|\nabla^2_{h,x}G_h(x,y)| \le \begin{cases} C\log\left(\frac{d(y)+h}{|x-y|+h}\right) & \mathsf{d}=2\\ \frac{C}{|x-y|+h} & \mathsf{d}=3\end{cases}\,, \tag{2.8.13}$$

$$|\nabla_{h,x}\nabla_{h,y}G_h(x,y)| \le \begin{cases} C\log\left(\frac{(d(x)+h)(d(y)+h)}{(|x-y|+h)^2}\right) & \mathsf{d}=2\\ \frac{C}{|x-y|+h} & \mathsf{d}=3\end{cases}\,. \tag{2.8.14}$$

Proof.
Step 1: Pointwise estimate for $\nabla^2_{h,x}\nabla_{h,y}G_h(x,y)$
We claim that if $x, y \in \Lambda_h$ and $|x-y|_\infty \leq \frac{d(y)}{4}$ then

$$|\nabla^2_{h,x}\nabla_{h,y}G_h(x,y)| \leq \frac{C}{(|x-y|+h)^{\mathsf{d}-1}}\,. \tag{2.8.15}$$

The fact that we prove this for $|x-y|_\infty \leq \frac{d(y)}{4}$ will give us a bit of space to wiggle around in the following steps where we integrate (2.8.15). The proof of (2.8.15) is similar to the proof of (2.8.9). The main difference is that this time we choose the cut-off function further away from the singularity.

If $d(y) < 10h$ we can again use a trivial estimate: By Lemma 2.8.1, $G_h(x',y')$ is bounded by $Cd(x')^{2-\frac{\mathsf{d}}{2}}d(y')^{2-\frac{\mathsf{d}}{2}} \leq Ch^{4-\mathsf{d}}$ if $|x'-x|_\infty \leq h$ and $|y'-y|_\infty \leq h$, so that

$$|\nabla^2_{h,x}\nabla_{h,y}G_h(x,y)| \leq C\frac{1}{h^3}h^{4-\mathsf{d}} = Ch^{1-\mathsf{d}}\,.$$

Therefore (2.8.15) holds if we choose C sufficiently large.

So assume that $d(y) \geq 10h$. Let η_h be a discrete cut-off function that is 1 on $Q_{d(y)/2+2h}(y)$ and 0 on $(h\mathbb{Z})^{\mathsf{d}} \setminus Q_{d(y)-2h}(y)$ and such that $|\nabla^\kappa \eta_h| \leq \frac{C}{d(x)^k}$ for $\kappa \leq 2$ and let $H_h(x,y) = G_h(x,y) - \eta_h(x)\tilde{G}_h(x,y)$, where $\tilde{G}_h$ is the function from Lemma 2.7.2 with $r = d(y)$.

Then, for $i \in \{1,\dots,n\}$, $D_i^{h,y}H_{h,y} \in \Phi_h$ and $\Delta_h^2 D_i^{h,y}H_{h,y} = 0$ in $Q_{d(y)/2}(y)$ and in $\operatorname{int}\Lambda_h \setminus Q_{d(y)}(y)$. We can estimate $\|\nabla_h^2 D_i^{h,y}H_{h,y}\|_{L^2(\mathbb{R}^{\mathsf{d}})}$ just as in Step 1 of the proof of Lemma 2.8.2 and obtain that

$$\|\nabla_h^2 D_i^{h,y}H_{h,y}\|_{L^2(\mathbb{R}^{\mathsf{d}})} \leq Cd(y)^{1-\frac{\mathsf{d}}{2}}\,. \tag{2.8.16}$$

Now recall that H_h is biharmonic in $Q_{d(y)/2}(y)$. So Theorem 2.6.1 implies for $x \in Q^h_{d(y)/4}(y)$

$$|\nabla_h^2 D_i^{h,y}H_{h,y}(x)| \leq \frac{C}{d(y)^{\frac{\mathsf{d}}{2}}}\|\nabla_h^2 D_i^{h,y}H_{h,y}\|_{L^2(\mathbb{R}^{\mathsf{d}})} \leq Cd(y)^{1-\mathsf{d}}\,.$$

Because $\nabla_h^2 D_i^{h,y}H_{h,y} = \nabla_h^2 D_i^{h,y}G_{h,y} - \nabla_h^2 D_i^{h,y}\tilde{G}_{h,y}$ in $Q_{d(y)/2}(y)$ we can use (2.7.2) and obtain

$$|\nabla_h^2 D_i^{h,y}G_{h,y}(x)| \leq |\nabla_h^2 D_i^{h,y}H_{h,y}(x)| + |\nabla_h^2 D_i^{h,y}\tilde{G}_{h,y}(x)| \leq C\left(\frac{1}{d(y)^{\mathsf{d}-1}} + \frac{1}{(|x-y|+h)^{\mathsf{d}-1}}\right)\,.$$

This implies (2.8.15) if we use that $|x-y|_\infty \leq \frac{d(y)}{4}$ and $d(y) \geq 10h$ so that $|x-y|+h \leq Cd(y)$.

Step 2: Proof of (2.8.14)
If $d(y) < 9h$ we can repeat the trivial estimate from the previous step, so assume $d(y) \geq 9h$.

We want to integrate (2.8.15) along a suitable path. So let $(x^{(k)})_{k=0}^L$ be a straight path from $x^{(0)} = x$ to a closest point $x^{(L)} \in Q_{d(y)/4}(y) \setminus Q_{d(y)/4-h}(y)$. This path will have length $Lh = \left\lfloor \frac{d(y)}{4} \right\rfloor_h - |x-y|_\infty$ and we have $|y - x^{(L)}|_\infty \geq \frac{d(y)}{4} - h > \frac{d(y)}{8}$. By Lemma 2.8.2 we have

$$|\nabla_{h,x}\nabla_{h,y}G_h(x^{(L)},y)| \leq C\frac{(d(x^{(L)})+h)(d(y)+h)}{|x^{(L)}-y|^{\mathsf{d}}} \leq C\frac{(d(y)+h)^2}{|d(y)+h|^{\mathsf{d}}} \leq C\frac{1}{|d(y)+h|^{\mathsf{d}-2}}\,. \tag{2.8.17}$$

Furthermore (2.8.15) implies that

$$|\nabla_{h,x}^2 \nabla_{h,y} G_h(x^{(k)}, y)| \leq \frac{C}{(|x^{(k)} - y| + h)^{\mathsf{d}-1}} \leq \frac{C}{(|x-y| + (k+1)h)^{\mathsf{d}-1}}. \tag{2.8.18}$$

Now we can integrate (2.8.18) along $(x^{(k)})_{k=0}^L$ and use (2.8.17), and after a short calculation we arrive at (2.8.14).

Step 3: Proof of (2.8.13)

If $d(y) < 79h$ we can again use the trivial estimate from Step 1, so assume $d(y) \geq 79h$.

This is similar to the previous step: We choose a shortest-possible path $(y^{(k)})_{k=0}^L$ from $y^{(0)} = y$ to a point $y^{(L)} \in Q_{d(x)/6}(x) \setminus Q_{d(x)/6-h}(y)$. Then $|y^{(k)} - x|_\infty \leq \frac{d(x)}{6}$, so that $\frac{5}{6}d(x) \leq d(y^{(k)}) \leq \frac{7}{6}d(x)$ and hence

$$|y^{(k)} - x|_\infty \leq \frac{d(x)}{6} \leq \frac{d(y^{(k)})}{5}.$$

Therefore we can apply (2.8.15) at the point $(x, y^{(k)})$ for each k and conclude

$$|\nabla_{h,x}^2 \nabla_{h,y} G_h(x, y^{(k)})| \leq \frac{C}{(|x - y^{(k)}| + h)^{\mathsf{d}-1}} \leq \frac{C}{(|x-y| + (k+1)h)^{\mathsf{d}-1}}. \tag{2.8.19}$$

On the other hand,

$$d(y^{(L)}) \geq \frac{5}{6}d(x) \geq \frac{5}{6}\frac{6}{7}d(y) \geq 56h$$

so that

$$|y^{(L)} - x|_\infty \geq \frac{d(x)}{6} - h \geq \frac{d(y^{(L)})}{7} - h > \frac{d(y^{(L)})}{8}.$$

This means that we can apply (2.8.8) at the point $(x, y^{(L)})$ and conclude

$$|\nabla_{h,x}^2 G_h(x, y^{(L)})| \leq C\frac{(d(y^{(L)}) + h)^2}{|x - y^{(L)}|^{\mathsf{d}}} \leq C\frac{(d(y) + h)^2}{|d(y) + h|^{\mathsf{d}}} \leq C\frac{1}{|d(y) + h|^{\mathsf{d}-2}}. \tag{2.8.20}$$

Now we can integrate (2.8.19) along the path $(y^{(k)})_{k=0}^L$ and use the estimate (2.8.20) for the one endpoint to obtain (2.8.13).

Step 4: Proof of (2.8.12)

We could try to prove this by integrating (2.8.13) along a path. However, this turns out to be not sharp enough at least if $\mathsf{d} = 3$ (we would get a logarithmic term instead of a constant term). Instead we will use the Sobolev inequality on the function $H_{h,y}$ from Step 1. Thereby we get a bound for $\nabla_{h,y} G_h(x, y)$ if x, y are close. By the symmetry of G_h we can turn this into a bound for $\nabla_{h,x} G_h(x, y)$.

If $d(y) < 10h$ we can again use the trivial estimate from Step 1, so assume $d(y) \geq 10h$. Recall the function $H_{h,y}$ from Step 1. If we use the Sobolev and Poincaré inequality on $Q_{d(y)+h/2}(y)$ and the estimate (2.8.16) we obtain

$$\begin{aligned}
\|D_i^{h,y} H_{h,y}\|_{L^\infty(Q_{d(y)+h/2}(y))} &\leq C(d(y) + h/2)^{2-\frac{\mathsf{d}}{2}} \|\nabla_h^2 D_i^{h,y} H_{h,y}\|_{L^2(Q_{d(y)+h/2}(y))} \\
&\leq C d(y)^{2-\frac{\mathsf{d}}{2}} \|\nabla_h^2 D_i^{h,y} H_{h,y}\|_{L^2(\mathbb{R}^{\mathsf{d}})} \\
&\leq C d(y)^{3-\mathsf{d}}
\end{aligned}$$

and therefore

$$|\nabla_{h,y} H_{h,y}(x)| \leq C d(y)^{3-\mathsf{d}}$$

for any $x \in Q_{d(y)}(y)$. Now we can use (2.7.2) and the fact that $D_i^{h,y} H_{h,y} = D_i^{h,y} G_{h,y} - D_i^{h,y} \tilde{G}_{h,y}$ in $Q_{d(y)/2}(y)$ and obtain

$$|D_i^{h,y} G_{h,y}(x)| \leq |D_i^{h,y} H_{h,y}(x)| + |D_i^{h,y} \tilde{G}_{h,y}(x)| \leq C d(y)^{3-\mathsf{d}}$$

for any $x \in Q_{d(y)/2}(y)$ and any $i \in \{1, \dots, n\}$. By the symmetry of G_h in x and y we conclude that also

$$|D_i^{h,x} G_{h,x}(y)| \leq C d(x)^{3-\mathsf{d}} \tag{2.8.21}$$

for any $y \in Q_{d(x)/2}(x)$.

Now in the setting of (2.8.12) we are given x, y with $|y - x|_\infty \leq \frac{d(y)}{4}$. These satisfy $\frac{3}{4} d(y) \leq d(x) \leq \frac{5}{4} d(y)$, so that $|y - x|_\infty \leq \frac{1}{3} d(x)$ and in particular $y \in Q_{d(x)/2}(x)$. Thus we can apply (2.8.21) and obtain

$$|D_i^{h,x} G_{h,x}(y)| \leq C d(x)^{3-\mathsf{d}} \leq C d(y)^{3-\mathsf{d}},$$

which implies (2.8.12).

Step 5: Proof of (2.8.11)

This follows immediately from (2.8.3). □

Proof of Theorem 2.1.3 i). Now that we have proved Lemma 2.8.3 and Lemma 2.8.2 the proof is straightforward. First observe that it suffices to consider $x, y \in \Lambda_h$ as otherwise G_h and its relevant derivatives are trivially 0.

We claim that we can combine (2.8.8) and (2.8.14) to obtain (2.1.11). Indeed, if $|x - y|_\infty \leq \frac{d(y)}{8}$ we have $d(y) \leq \frac{8}{7} d(x)$ and $|x - y| + h \leq \sqrt{n}|x - y|_\infty + h < d(y) + h$ which implies

$$1 \leq \frac{(d(x) + h)(d(y) + h)}{(|x - y| + h)^2}$$

and we are done by (2.8.14).

If however $|x - y|_\infty > \frac{d(y)}{8}$, then we have in particular $|x - y| \geq h$, so that $|x - y| + h \leq 2|x - y|$. We also have $d(y) \leq 8|x - y|$ and $d(x) \leq 9|x - y|$ and we easily see that

$$\frac{(d(x) + h)(d(y) + h)}{|x - y|^{\mathsf{d}}} \leq \frac{C}{(|x - y| + h)^{\mathsf{d}-2}}$$

so we are done by (2.8.8).

Similarly, we can combine (2.8.7) and (2.8.13) into the estimate

$$|\nabla_{h,x}^2 G_h(x, y)| \leq \begin{cases} C \log\left(1 + \frac{(d(y)+h)^2}{(|x-y|+h)^2}\right) & \mathsf{d} = 2 \\ C \min\left(\frac{1}{|x-y|+h}, \frac{(d(y)+h)^2}{(|x-y|+h)^3}\right) & \mathsf{d} = 3 \end{cases}.$$

This is not quite (2.1.10), but it implies (2.1.10) unless $d(y) = 0$. On the other hand, if $d(y) = 0$ then $y \in \partial\Lambda_h$. Therefore $G_{h,y}$ is identically 0, so that $\nabla_{h,x^2} G_h(x, y) = 0$ and (2.1.10) holds as well.

Similarly we can combine (2.8.6) and (2.8.12), and (2.8.5) and (2.8.11) into

$$|\nabla_{h,x} G_h(x, y)| \leq C \min\left((d(y) + h)^{3-\mathsf{d}}, \frac{(d(x) + h) d(y)^2}{(|x - y| + h)^{\mathsf{d}}}\right),$$

$$|G_h(x, y)| \leq C \min\left((d(x) + h)^{2-\frac{\mathsf{d}}{2}} (d(y) + h)^{2-\frac{\mathsf{d}}{2}}, \frac{(d(x) + h)^2 (d(y) + h)^2}{(|x - y| + h)^{\mathsf{d}}}\right)$$

respectively. These estimates imply (2.1.9) and (2.8.11), except in the cases $d(x) = 0$ or $d(y) = 0$, which are again trivial. □

Remark 2.8.4. As a byproduct of the proofs of Lemma 2.8.3 and Lemma 2.8.2 we proved the estimates (2.8.9) and (2.8.15) which can easily be combined into the estimate

$$|\nabla^2_{h,x}\nabla_{h,y}G_h(x,y)| \le C\min\left(\frac{1}{(|x-y|+h)^{\mathsf{d}-1}}, \frac{d(y)+h}{(|x-y|+h)^{\mathsf{d}}}\right) \tag{2.8.22}$$

for any $x, y \in (h\mathbb{Z})^{\mathsf{d}}$.

With the same method of proof it is possible to prove an estimate for $\nabla^2_{h,x}\nabla^2_{h,y}G_h$ as well. One again considers $H_{h,y} = G_{h,y} - \eta_h\tilde{G}_{h,y}$ in Lemma 2.8.3 and Lemma 2.8.2 and derives estimates for $\|\nabla^2_{h,x}\nabla^2_{h,y}H_{h,y}\|_{L^2(\mathbb{R}^{\mathsf{d}})}$. In combination with the pointwise estimates for $\tilde{G}_h$ (in particular (2.7.3)) these again yield estimates for $\nabla^2_{h,x}\nabla^2_{h,y}G_h$ in the two regimes where x and y are far away and close together, respectively. The final result is

$$|\nabla^2_{h,x}\nabla^2_{h,y}G_h(x,y)| \le \frac{C}{(|x-y|+h)^{\mathsf{d}}} \tag{2.8.23}$$

for any $x, y \in (h\mathbb{Z})^{\mathsf{d}}$.

Actually it is even possible to derive estimates for higher derivatives $\nabla^a_{h,x}\nabla^b_{h,y}G_h$, at least when $a \le 2$ or $b \le 2$. However we cannot expect these estimates to be optimal any more, because high derivatives are increasingly divergent near the singular boundary points, and our approach does not really capture this behaviour.

2.8.3 Convergence of Green's functions

Proof of Corollary 2.1.4. We begin with the proof of assertion i). We can assume that $h \le \frac{1}{3}$. There exists a unique $y_h \in \Lambda_h$ such that $y \in y_h + [-\frac{h}{2}, \frac{h}{2})^2$. Set $u_h(x) = G_h(x, y_h)$. We extend u_h by zero to $(h\mathbb{Z})^{\mathsf{d}} \setminus \operatorname{int}\Lambda_h$.

To prove (i) we have to show that u_h converges uniformly to $G(\cdot, y)$. Testing the equation for $\Delta^2_h u_h$ with u_h we get (see Lemma 2.8.1)

$$\|\nabla^2_h u_h\|_{L^2(\mathbb{R}^{\mathsf{d}})} \le Cd^{2-\frac{\mathsf{d}}{2}}(y_h) \le C\,.$$

The discrete Sobolev-Poincaré inequality implies in particular that the u_h are uniformly Hölder continuous

$$[u_h]_{C^{0,\frac{1}{4}}_h(\mathbb{R}^{\mathsf{d}})} \le C. \tag{2.8.24}$$

Denote by J_h the interpolation operator introduced in Section 2.4. From Proposition 2.4.2 vi) and the Poincaré inequality we deduce that the sequence $J_h u_h$ is bounded in $W^{2,2}(\mathbb{R}^{\mathsf{d}})$ and $J_h u_h = 0$ in $\mathbb{R}^{\mathsf{d}} \setminus (-3h, 1+3h)^{\mathsf{d}}$. It follows that for a subsequence

$$J_{h_k}u_{h_k} \rightharpoonup u \quad \text{in } W^{2,2}(\mathbb{R}^{\mathsf{d}})\,, \qquad u = 0 \text{ in } \mathbb{R}^{\mathsf{d}} \setminus (0,1)^{\mathsf{d}}\,.$$

From the uniform Hölder continuity (2.8.24) and Proposition 2.4.2 iii), iv) and vi) we deduce that, for any $x \in (-3h, 1+3h)^{\mathsf{d}}$,

$$|J_{h_k}u_{h_k}(x) - I^{pc}_{h_k}u_{h_k}(x)| = |J_{h_k}(u_{h_k}(\cdot) - u_{h_k}(x))(x)| \le C\|u_{h_k} - u_{h_k}(x)\|_{L^\infty(Q_{3h_k}(x))} \le Ch_k^{\frac{1}{4}}$$

and therefore

$$\sup_{x\in(-1,2)^{\mathsf{d}}} |J_{h_k}u_{h_k}(x) - I^{pc}_{h_k}u_{h_k}(x)| \le Ch_k^{\frac{1}{4}}\,.$$

In connection with the compact embedding from $W_0^{2,2}((-1,2)^{\mathsf{d}})$ to $C^0((-1,2)^{\mathsf{d}})$ we conclude that

$$I_{h_k}^{pc} u_{h_k} \to u \quad \text{uniformly}\,. \tag{2.8.25}$$

If we can show that $u(x) = G(x,y)$ then by uniqueness of the limit it follows that the convergences above do not only hold along a particular subsequence $h_k \to 0$ but for every subsequence $h_k \to 0$ and we are done.

To show that $u(x) = G(x,y)$ we use that by definition of $G_h(\cdot, y_h)$ we have for each $\varphi \in C_c^5((0,1)^{\mathsf{d}})$

$$\begin{aligned}\varphi(y_{h_k}) &= \sum_{x \in \operatorname{int} \Lambda_h} \Delta_{h_k}^2 u_{h_k}(x)\varphi(x) h_k^{\mathsf{d}} = \sum_{x \in \operatorname{int} \Lambda_h} u_{h_k}(x) \Delta_{h_k}^2 \varphi(x) h_k^{\mathsf{d}} \\ &= \int_{(0,1)^{\mathsf{d}}} I_{h_k}^{pc} u_{h_k} I_{h_k}^{pc} \Delta_{h_k}^2 \varphi(x)\,\mathrm{d}x\,.\end{aligned}$$

Now by Taylor expansion $|I_{h_k}^{pc} \Delta_{h_k}^2 \varphi - \Delta^2 \varphi| \le C h_k$. Together with (2.8.25) we get

$$\varphi(y) = \lim_{k\to\infty} \varphi(y_{h_k}) = \lim_{k\to\infty} \int_{(0,1)^{\mathsf{d}}} I_{h_k}^{pc} u_{h_k} I_{h_k}^{pc} \Delta_{h_k}^2 \varphi_h \,\mathrm{d}x = \int_{(0,1)^{\mathsf{d}}} u \Delta^2 \varphi \,\mathrm{d}x\,.$$

Thus $\Delta^2 u = \delta_y$ in the sense of distributions. Since we also know that $u \in W_0^{2,2}((0,1)^{\mathsf{d}})$ we conclude that $u(x) = G(x,y)$ as desired.

To prove ii) note that the estimates in Theorem 2.1.3 show that the second discrete derivatives are bounded in L^p for all $p < \infty$. Hence by the discrete Sobolev embedding theorem the discrete first derivatives are bounded in $C^{0,\alpha}$ for all $\alpha < 1$. This implies that

$$|I_h^{pc} \nabla_h u - \nabla J_h u_h| \le C h^{\alpha}. \tag{2.8.26}$$

Moreover the L^p bound on the discrete second derivatives and (2.4.3) give a bound of $J u_h$ in $W^{2,p}$. Hence a subsequence of $J_h u_h$ converges in $C^{1,\alpha}$ to $G(\cdot, y)$. Since the limit is unique, the whole sequence converges in $C^{1,\alpha}$ to G. Together with (2.8.26) this yields uniform convergence of the discrete first derivatives.

The local compactness argument in Section 2.5 (and a diagonalisation argument) shows that a subsequence of $I_h^{pc} \nabla_h^2 u_h$ converges in $L^2_{\mathrm{loc}}((0,1)^2 \setminus \{y\})$ to a function v. Since $I_h^{pc} \nabla_h^2 u_h$ is also bounded in L^q for some $q > 2$ we get strong convergence in $L^2((0,1)^2)$. Using again the L^q bound we get strong convergence in all L^p with $p < q$. Since we have L^q bounds for all $q < \infty$ we get strong convergence for all $p < \infty$. It remains to show that $v = \nabla^2 G(\cdot, y)$. To obtain this identity we can use discrete integration by parts and pass to the limit on both sides, as in the proof that $\Delta^2 u = \delta_y$.

The proof of (iii) is similar. Uniform boundedness of the discrete derivatives follows directly from Theorem 2.1.3. This theorem also shows that the second discrete derivatives are uniformly bounded on the complement of any cube $Q_r(y)$. It follows that the functions u_h are uniformly Lipschitz on the complement of any cube $Q_r(y)$ and we obtain locally uniform convergence of $I_h^{pc} \nabla_h u_h$ in the complement of those cubes as in the proof of (ii). Combined with the uniform boundedness we immediately conclude convergence of $I_h^{pc} \nabla_h u_h$ in L^p for all $p < \infty$.

The proof of L^p convergence of $I_h^{pc} \nabla_h^2 u_h$ for $p < 3$ is again analogous to the argument for $\mathsf{d} = 2$.

□

3 Probability to be positive for the membrane model in dimensions two and three

This chapter is based on the paper [BDKS19], written jointly by Simon Buchholz, Jean-Dominique Deuschel, Noemi Kurt and the author, with only minor changes. A small part of the content of this chapter has already appeared in the author's M.Sc. thesis [Sch16], where the upper bound on the probability to be positive is shown using a very similar approach.

3.1 Introduction

In this section we consider entropic repulsion for the subcritical membrane model, as discussed in Section 1.3.3. Actually, for convenience we make a small change in comparison to the results mentioned there: instead of $\Lambda_N = \{0, \dots, N\}^{\mathsf{d}}$ we consider $\Lambda_N = \{-N, \dots, N\}^{\mathsf{d}}$. This means that we only consider boxes with odd sidelengths, and this has the small advantage that the centre of Λ_N is a lattice point. However, it is clear that our proof would also apply to boxes with even sidelengths. As we are only concerned with the membrane model, we drop the subscripts Δ.

3.1.1 Main results

Let $\Lambda = [-1, 1]^{\mathsf{d}}$ and $\Lambda_N = N\Lambda \cap \mathbb{Z}^{\mathsf{d}}$ with $\mathsf{d} \in \mathbb{N}^+$ and $N \in \mathbb{N}^+$. We are interested in the event $\Omega_{D_N,+} = \{\psi \colon \psi_x \geq 0 \ \forall x \in D_N\}$, where $D_N \subset \Lambda_N$, as well as the behaviour of ψ conditioned on $\Omega_{D_N,+}$.

Our main result is the following.

Theorem 3.1.1. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$. *There are constants* C, c *such that for all* $N \in \mathbb{N}^+$, $0 \leq L \leq N$,

$$e^{-C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}} \leq \mathbb{P}_N\left(\Omega_{\Lambda_{N-L},+}\right) \leq e^{-c\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}} . \tag{3.1.1}$$

A first result in this direction was already established by Sakagawa [Sak16] who proved that for every $x \in \Lambda$ there is a small neighbourhood B_x such that $\mathbb{P}_N(\Omega_{NB_x,+}) > c$ for some (universal) constant c.

Let us emphasize the two important special cases of our theorem that were already mentioned in Section 1.3.3. We first consider the case $D_N = \Lambda_{\delta N}$ for $\delta \in (0, 1)$, where the hard wall stays away from the boundary. In that case the fact that the membrane model is Hölder continuous suggests that the field has a decent chance to be positive if it is uniformly positive on a sufficiently dense set of lattice points of bounded cardinality. Thus the probability that ψ is positive on $D_N = \Lambda_{\delta N}$ should be comparable to the probability of uniform positivity on that dense set, and hence bounded away from zero. Indeed, Theorem 3.1.1 implies the following corollary.

Corollary 3.1.2. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$. *For* $\delta \in (0,1)$ *there is a constant* $c_\delta > 0$ *such that*

$$c_\delta \leq \mathbb{P}_N(\Omega_{\Lambda_{\delta N},+}) \leq \frac{1}{2}\,.$$

When $D_N = V_N$, the situation is somewhat different. While the Hölder continuity holds up to the boundary, the ψ_x for x near the boundary are only weakly correlated and behave almost like independent random variables. This suggests that the probability to be positive on all of V_N can at best scale like $e^{-cN^{\mathsf{d}-1}}$ (note that the number of points of distance 1 to the boundary is of the order $N^{\mathsf{d}-1}$). On the other hand, if the field is positive at all near-boundary points it gets pushed up in the interior quite a bit, and so the probability to be positive everywhere should be of the same order.

Indeed, another particular case of Theorem 3.1.1 is an estimate for $\mathbb{P}_N(\Omega_{\Lambda_N,+})$.

Corollary 3.1.3. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$. *There are constants* C, c *such that*

$$e^{-CN^{\mathsf{d}-1}} \leq \mathbb{P}_N\left(\Omega_{\Lambda_N,+}\right) \leq e^{-cN^{\mathsf{d}-1}}\,.$$

We expect this result to be true for the membrane model and the gradient model in any dimension $\mathsf{d} \geq 2$. For the gradient model a stronger result has been shown for $\mathsf{d} \geq 3$ in [Deu96, Theorem 4.1]. Note that the behaviour for general $L \geq 1$ in Theorem 3.1.1 is different for the gradient model in dimension $\mathsf{d} \geq 3$.

We give a proof of the lower and upper bound in Theorem 3.1.1 in Section 3.3 and 3.4, respectively.

3.1.2 Implications for entropic repulsion

Corollary 3.1.2 has some easy implications on the behaviour of the field when conditioned on $\Omega_{\Lambda_{\delta N},+}$. To state them precisely we need some preparation.

We define the interpolation $I_N\colon \mathbb{R}^{\Lambda_N} \to C^{0,1}([-1,1]^{\mathsf{d}})$ by $I_N f(x) = N^{-\frac{4-\mathsf{d}}{2}} f(Nx)$ for $x \in \left(\frac{1}{N}\mathbb{Z}\right)^{\mathsf{d}} \cap [-1,1]^{\mathsf{d}}$, and interpolated piecewise affinely on simplices for other values of x. As the proof of [CDH19, Theorem 2.1] shows, the pushforward measures $I_N\#\mathbb{P}_N$ converge weakly in $C^{0,\alpha}([-1,1]^{\mathsf{d}})$ for any $\alpha < \frac{4-\mathsf{d}}{2}$ to a limit law $\mathbb{P}_\infty$. The limit $\mathbb{P}_\infty$ is the continuum Bilaplace field, i.e., the centred Gaussian field whose covariance is the Green's function of the continuum Bilaplace operator on Λ. Now Corollary 3.1.2 implies that the laws $I_N\#\mathbb{P}_N$ still converge when one conditions on $\Omega_{\Lambda_{\delta N},+}$. Indeed, if we introduce the event $\Omega^*_{D,+} = \{u \in C^{0,\alpha}([-1,1]^{\mathsf{d}})\colon u(x) \geq 0\ \forall x \in D\}$ for $D \subset [-1,1]^{\mathsf{d}}$, we have the following result.

Corollary 3.1.4. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$, *and* $\delta \in (0,1)$. *Then* $I_N\#\mathbb{P}_N(\cdot \mid \Omega_{\Lambda_{\delta N},+})$ *converges weakly in* $C^{0,\alpha}([-1,1]^{\mathsf{d}})$ *for any* $\alpha < \frac{4-\mathsf{d}}{2}$ *to* $\mathbb{P}_\infty\left(\cdot \,\middle|\, \Omega^*_{\delta\Lambda,+}\right)$. *In particular, we have*

$$\lim_{N\to\infty} \mathbb{E}_N\left(N^{-\frac{4-\mathsf{d}}{2}} \max_{x\in\Lambda_N} \psi_x \,\middle|\, \Omega_{\Lambda_{\delta N},+}\right) < \infty\,.$$

The corollary follows from the facts that $\mathbb{P}_\infty\left(\Omega^*_{\delta\Lambda,+}\right)$ is a continuous function of δ and that $\Omega^\infty_{\delta\Lambda,+}$ is a continuity set for $\mathbb{P}_\infty$ (these both follow from [Bog98, Corollary 4.4.2]). Note that the second point combined with the convergence of $I_N\#\mathbb{P}_N \to \mathbb{P}_\infty$ and Corollary 3.1.2 implies that $\mathbb{P}_\infty\left(\Omega^*_{\delta\Lambda,+}\right) > 0$ so that the conditioned measure $\mathbb{P}_\infty\left(\cdot \,\middle|\, \Omega^*_{\delta\Lambda,+}\right)$ is well-defined.

This corollary shows that there is no entropic repulsion when conditioning on $\Omega_{\Lambda_{\delta N},+}$.

We conjecture that a similar result remains true if we condition on $\Omega_{\Lambda_N,+}$. However, due to the fact that the probability of $\Omega_{\Lambda_N,+}$ is exponentially small this is a difficult problem even in dimension one.

Conjecture 3.1.5. *For* $\mathsf{d} = 2$ *and* $\mathsf{d} = 3$ *the measures* $I_N\#\mathbb{P}_N(\cdot \mid \Omega_{\Lambda_N,+})$ *converge weakly in* $C^{0,\alpha}([-1,1]^{\mathsf{d}})$ *for any* $\alpha < \frac{4-\mathsf{d}}{2}$ *to some limiting measure. In particular,*

$$\lim_{N\to\infty} \mathbb{E}_N\left(N^{-\frac{4-\mathsf{d}}{2}} \max_{x\in\Lambda_N} \psi_x \,\middle|\, \Omega_{\Lambda_N,+}\right) < \infty\,.$$

As an analogue to this conjecture one can consider the gradient model in one dimension (i.e. the random walk on $\{-N, -N+1, \dots, N\}$ with i.i.d. Gaussian increments conditioned to be zero at its endpoints). It is well-known that this model, suitably rescaled, converges weakly in $C^{0,\alpha}([-1,1])$ for $\alpha < \frac{1}{2}$ to a Brownian bridge. Moreover, if one conditions the walk to be non-negative it converges weakly in $C^0([-1,1])$ to a Brownian excursion (see [CC13] and the references therein). Similar results (in particular a local limit theorem for the conditioned field) have also been shown for the membrane model in one dimension (at least if one only considers zero boundary data on one end of the interval), see [DW15].

3.1.3 Notation

For $x \in \mathbb{Z}^{\mathsf{d}}$ let $d_N(x) = \mathrm{dist}_\infty(x, \mathbb{Z}^{\mathsf{d}} \setminus \Lambda_N)$ be the distance to the boundary of Λ_N.

In the following c, C and C' denote constants that may change from line to line, but are always independent of N and L.

3.2 Preliminaries

Let us recall the relevant results that will be used in the proof of the main theorems. Let G_N be the Green's function of Δ^2 on Λ_N with 0 boundary data outside Λ_N, i.e. $G_N(\cdot, y) = 0$ if $y \notin \Lambda_N$ and

$$\begin{aligned} \Delta^2 G_N(\cdot, y) &= \delta_y \quad \text{in } \Lambda_N,, \\ G_N(\cdot, y) &= 0 \quad \text{outside } \Lambda_N \end{aligned}$$

if $y \in \Lambda_N$. The Green's function G_N agrees with the covariance matrix of ψ, i.e. we have that $\mathrm{Cov}_N(\psi_x, \psi_y) = G_N(x,y)$, see also [Kur09]. Our proofs are based on the estimates for the Green's function G_N from Chapter 2.

Theorem 3.2.1. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$. *Then we have for any* $x, y \in \Lambda_N$

$$c d_N(x)^{4-\mathsf{d}} \le G_N(x,x) \le C d_N(x)^{4-\mathsf{d}}\,, \tag{3.2.1}$$

$$|\nabla_{1,x} G_N(x,y)| \le C d_N(x)^{3-\mathsf{d}}\,, \tag{3.2.2}$$

$$|G_N(x,x) - G_N(x,y)| \le C d_N(x)^{3-\mathsf{d}} |x-y|_\infty\,, \tag{3.2.3}$$

$$|G_N(x,y)| \le C \frac{d_N(x)^2 d_N(y)^2}{(|x-y|_\infty + 1)^{\mathsf{d}}}\,, \tag{3.2.4}$$

where $\nabla_{1,x}$ *denotes the discrete gradient with respect to* x.

Proof. The estimates (3.2.1), (3.2.2) and (3.2.4) are taken from Theorem 2.1.1, while (3.2.3) follows from (3.2.2) by discrete integration along a path from x to y. □

The lower bound relies on Dudley's inequality proved in [Dud67]. To state the inequality we introduce the following two notions. For a Gaussian process $(X_t)_{t\in T}$ we define the pseudometric d_X by

$$d_X(s,t) = \sqrt{\mathbb{E}(|X_s - X_t|^2)}\,. \tag{3.2.5}$$

The entropy number $\mathcal{N}(T, d_X, r)$ is the minimal number of open balls of radius r in the d_X metric that are needed to cover T.

Theorem 3.2.2. *Let $(X_t)_{t\in T}$ be a centred Gaussian process. Then*

$$\mathbb{E}\left(\sup_{t\in T} X_t\right) \leq 24 \int_0^\infty \sqrt{\ln \mathcal{N}(T, d_X, r)}\,\mathrm{d}r\,.$$

Remark 3.2.3. The theorem is true for arbitrary sets T if one defines the supremum appropriately, see e.g. [Tal96]. Since we only apply it to finite index sets we do not discuss this issue here any further.

We also use the Gaussian correlation inequality due to Royen [Roy14] (see also [LM17]).

Theorem 3.2.4. *Let ν be a centred Gaussian measure on $\mathbb{R}^m$ and $K, L \subset \mathbb{R}^m$ be closed, symmetric and convex. Then*

$$\nu(K \cap L) \geq \nu(K)\nu(L)\,. \tag{3.2.6}$$

Finally, we recall a Gaussian correlation inequality do to Li and Shao [LS04, Lemma 5.1] that will be used in the proof of the upper bound

Lemma 3.2.5. *Let $m \in \mathbb{N}$, and $X = (X_1, \dots X_m)$, $Y = (Y_1, \dots Y_m)$ be Gaussian random vectors with mean 0 and positive definite covariance matrices Σ_X, Σ_Y, and let $\mathbb{P}$ denote their joint measure. If $\Sigma_Y \geq \Sigma_X$ (in the sense of symmetric matrices, i.e., $\Sigma_Y - \Sigma_X$ is positive semidefinite) then for every Borel set $F \subset \mathbb{R}^m$*

$$\mathbb{P}(Y \in F) \geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} P(X \in F)\,.$$

For the convenience of the reader we repeat the short proof.

Proof. Let f_X, f_Y be the densities of X and Y. The assumption $\Sigma_Y \geq \Sigma_X$ implies that $\Sigma_X^{-1} \geq \Sigma_Y^{-1}$ and hence $(x, \Sigma_X^{-1}x) \geq (x, \Sigma_Y^{-1}x)$ for all $x \in \mathbb{R}^m$. Therefore:

$$\begin{aligned} f_Y(x) &= \frac{1}{(2\pi)^{\frac{m}{2}}(\det \Sigma_Y)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(x, \Sigma_Y^{-1}x)\right) \\ &\geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \frac{1}{(2\pi)^{\frac{m}{2}}(\det \Sigma_X)^{\frac{1}{2}}} \exp\left(-\tfrac{1}{2}(x, \Sigma_X^{-1}x)\right) \\ &= \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} f_X(x)\,. \end{aligned}$$

Then

$$\mathbb{P}(Y \in F) = \int_F f_Y(x)\,\mathrm{d}x \geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \int_F f_X(x)\,\mathrm{d}x = \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \mathbb{P}(X \in F)\,.$$

□

3.3 Lower bounds

Let

$$\Omega_{\Lambda_{N-L},\infty} := \left\{\psi\colon |\psi_x| \le d_N(x)^{\frac{4-\mathsf{d}}{2}}\ \forall x \in \Lambda_{N-L}\right\}$$

be the event that ψ is uniformly small on Λ_{N-L}.

If ψ was $C^{0,\frac{4-\mathsf{d}}{2}}$-Hölder continuous with Hölder constant ≤ 1 with probability bounded below uniformly in N, this event would have a positive probability uniformly in N and L. Now ψ is only $C^{0,\frac{4-\mathsf{d}}{2}-\varepsilon}$-Hölder continuous (see Chapter 2 and [CDH19]), so we cannot expect a lower bound independent of N. Instead, we prove in Subsection 3.3.2 that the probability of $\Omega_{\Lambda_{N-L},\infty}$ is bounded below by $e^{-c\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}}$. Then, using a change of measure argument, we show in Subsection 3.3.3 that, given $f\colon \Lambda_N \to \mathbb{R}$, we have

$$\mathbb{P}_N(f + \Omega_{\Lambda_{N-L},\infty}) \ge e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2}\mathbb{P}_N(\Omega_{\Lambda_{N-L},\infty})\,. \tag{3.3.1}$$

Suppose now that we can find a function f such that $f(x) \ge d_N(x)^{\frac{4-\mathsf{d}}{2}}$ for $x \in \Lambda_{N-L}$ and such that $\|\Delta f\|_{L^2}^2 \le C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}$. Then $\Omega_{\Lambda_{N-L},+} \supset f + \Omega_{\Lambda_{N-L},\infty}$ and thus (3.3.1) will imply that

$$\begin{aligned}\mathbb{P}_N(\Omega_{\Lambda_{N-L},+}) &\ge \mathbb{P}_N(f + \Omega_{\Lambda_{N-L},\infty})\\ &\ge e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2}\mathbb{P}_N(\Omega_{\Lambda_{N-L},\infty})\\ &\ge e^{-C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}}\mathbb{P}_N(\Omega_{\Lambda_{N-L},\infty})\,.\end{aligned}$$

Combined with a lower bound on $\mathbb{P}_N(\Omega_{\Lambda_{N-L},\infty})$ this implies the lower bound in Theorem 3.1.1. In Lemma 3.3.4 we construct an f with the desired properties.

3.3.1 Local smallness of the field

We first prove that locally the field is small with a positive probability. For $x_0 \in \Lambda_N$ and $\gamma > 0$ we define the set

$$A_{x_0,\gamma} := \{x \in \Lambda_N\colon |x - x_0|_\infty \le \gamma d_N(x_0)\}\,. \tag{3.3.2}$$

Lemma 3.3.1. *Let* $\mathsf{d} = 2$ *or* $\mathsf{d} = 3$. *There is a pair of constants* $\gamma, \delta > 0$ *with the following property: For all* $x_0 \in \Lambda_N$ *the following estimate holds*

$$\mathbb{P}_N\left(\psi\colon |\psi_x| \le d_N(x)^{\frac{4-\mathsf{d}}{2}}\ \forall x \in A_{x_0,\gamma}\right) \ge \delta\,. \tag{3.3.3}$$

Proof. We apply Theorem 3.2.2 to the Gaussian process ψ distributed according to $\mathbb{P}_N$. We assume $\gamma < \frac{1}{2}$ so that $x \in A_{x_0,\gamma}$ implies

$$\frac{d_N(x_0)}{2} \le d_N(x) \le \frac{3d_N(x_0)}{2}\,.$$

Therefore we will always estimate distances to the boundary for $x \in A_{x_0,\gamma}$ by $d_N(x_0)$ in the following. The bound (3.2.3) implies

$$\mathbb{E}_N(\psi_x - \psi_y)^2 \le |G_N(x,x) - G_N(x,y)| + |G_N(y,y) - G_N(y,x)| \le \Theta d_N(x_0)^{3-\mathsf{d}}|x-y|_\infty \tag{3.3.4}$$

for $x, y \in A_{x_0,\gamma}$ and some $\Theta > 0$. Therefore we estimate the Gaussian pseudometric defined in (3.2.5) by

$$d_\psi(x,y) \le \sqrt{\Theta d_N(x_0)^{3-\mathsf{d}}|x-y|_\infty}\,.$$

This implies that for $r > 0$ and $x, y \in A_{x_0,\gamma}$ such that $|x-y|_\infty \le \frac{r^2}{\Theta d_N(x_0)^{3-\mathsf{d}}}$ we have

$$d_\psi(x,y) \le r\,.$$

In particular $B_\infty\left(x, \frac{r^2}{\Theta d_N(x_0)^{3-\mathsf{d}}}\right) \subset B_{d_\psi}(x,r)$ and therefore

$$\mathcal{N}(A_{x_0,\gamma}, d_\psi, r) \le \left\lceil \frac{\gamma d_N(x_0)}{\frac{r^2}{\Theta d_N(x_0)^{3-\mathsf{d}}}} \right\rceil^{\mathsf{d}} \le 1 \vee \left(\frac{2\gamma\Theta d_N(x_0)^{4-\mathsf{d}}}{r^2}\right)^{\mathsf{d}}.$$

Then Theorem 3.2.2 implies

$$\begin{aligned}\mathbb{E}_N\left(\sup_{x\in A_{x_0,\gamma}} \psi_x\right) &\le 24\int_0^{\sqrt{2\gamma\Theta d_N(x_0)^{4-\mathsf{d}}}} \sqrt{\ln\left(\frac{2\gamma\Theta d_N(x_0)^{4-\mathsf{d}}}{r^2}\right)^{\mathsf{d}}}\,\mathrm{d}r \\ &\le 24 d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\sqrt{2\gamma\Theta n}\int_0^1 \sqrt{-2\ln r}\,\mathrm{d}r \le \lambda\sqrt{\gamma} d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\end{aligned} \tag{3.3.5}$$

where λ only depends on d.

If we take $\gamma = (16\lambda)^{-2}$ we obtain

$$\mathbb{E}_N\left(\sup_{x\in A_{x_0,\gamma}} \psi_x\right) \le \frac{1}{16} d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\,. \tag{3.3.6}$$

Define the oscillation of a function f on a set T as usual by

$$\operatorname{osc}_T f = \sup_T f - \inf_T f\,.$$

Since ψ_x is a centred process (3.3.5) implies

$$\mathbb{E}_N\left(\operatorname{osc}_{A_{x_0,\gamma}} \psi_x\right) \le \frac{1}{8} d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\,.$$

This implies that

$$\mathbb{P}_N\left(\operatorname{osc}_{A_{x_0,\gamma}} \psi_x \le \tfrac{1}{4} d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\right) \ge \frac{1}{2}\,.$$

Note that we have the inclusions

$$\begin{aligned}\left\{\psi : |\psi_x| \le d_N(x)^{\frac{4-\mathsf{d}}{2}}\ \forall x \in A_{x_0,\gamma}\right\} &\supset \left\{\psi : |\psi_x| \le \tfrac{1}{2} d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\ \forall x \in A_{x_0,\gamma}\right\} \\ &\supset \left\{\psi : \operatorname{osc}_{A_{x_0,\gamma}} \psi_x \le \tfrac{1}{4} d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\right\} \cap \left\{\psi : |\psi_{x_0}| \le \tfrac{1}{4} d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\right\}.\end{aligned}$$

Now the Gaussian correlation inequality (3.2.6) together with (3.2.1) imply that

$$\mathbb{P}_N\left(\left\{\psi : |\psi_x| \le d_N(x)^{\frac{4-\mathsf{d}}{2}}\ \forall x \in A_{x_0,\gamma}\right\}\right) \ge \frac{1}{2}\mathbb{P}_N\left(|\psi_{x_0}| \le \tfrac{1}{4} d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\right) \ge \delta$$

for some fixed $\delta > 0$. □

Remark 3.3.2. The use of the Gaussian correlation inequality could be avoided here: from (3.3.6) and (3.2.1) one easily obtains

$$\mathbb{E}_N\Big(\sup_{x\in A_{x_0,\gamma}}|\psi_x|\Big)\le \mathbb{E}_N\Big(\sup_{x\in A_{x_0,\gamma}}|\psi_x-\psi_{x_0}|\Big)+\mathbb{E}_N(|\psi_{x_0}|)\le \Xi d_N(x_0)^{\frac{4-\mathsf{d}}{2}}$$

for some $\Xi>0$ and therefore

$$\mathbb{P}_N\left(\psi\colon |\psi_x|\le 4\Xi d_N(x)^{\frac{4-\mathsf{d}}{2}}\ \forall x\in A_{x_0,\gamma}\right)\ge \mathbb{P}_N\left(\psi\colon |\psi_x|\le 2\Xi d_N(x_0)^{\frac{4-\mathsf{d}}{2}}\ \forall x\in A_{x_0,\gamma}\right)\ge \frac{1}{2}\,.$$

We could work with this estimate instead of (3.3.3) by using

$$\tilde{\Omega}_{\Lambda_{N-L},\infty}:=\left\{\psi\colon |\psi_x|\le 4\Xi d_N(x)^{\frac{4-\mathsf{d}}{2}}\ \forall x\in\Lambda_{N-L}\right\}$$

instead of $\Omega_{\Lambda_{N-L},\infty}$ in the following.

3.3.2 Global smallness of the field

From the previous we know that on small boxes the field is small with probability bounded away from zero. We can cover Λ_{N-L} with these small boxes, and then use the Gaussian correlation inequality to obtain a bound on the probability that the field is globally small.

Lemma 3.3.3. *Let* $\mathsf{d}=2$ *or* $\mathsf{d}=3$, *let* $\Omega_{\Lambda_{N-L},\infty}$ *be as before. Then we have*

$$\mathbb{P}_N(\Omega_{\Lambda_{N-L},\infty})\ge e^{-C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}}\,.$$

Proof. Recall the definition of $A_{x,\gamma}$ in (3.3.2). Fix γ such that the conclusion of Lemma 3.3.1 holds and use the shorter notation $A_x:=A_{x,\gamma}$.

We want to construct a subset B_N of Λ_N such that $|B_N|\le C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}$ and such that

$$\Lambda_{N-L}\subset\bigcup_{x\in B_N}A_x\,.$$

If we have found such a set, then the Gaussian correlation inequality (Theorem 3.2.4) and Lemma 3.3.1 imply that

$$\begin{aligned}\mathbb{P}_N(\Omega_{\Lambda_{N-L},\infty})&\ge \mathbb{P}_N\Big(\bigcap_{x\in B_N}\{\psi\colon |\psi_y|<d_N(y)^{\frac{4-\mathsf{d}}{2}}\ \forall y\in A_x\}\Big)\\&\ge\prod_{x\in B_N}\mathbb{P}_N\left(\psi\colon |\psi_y|<d_N(y)^{\frac{4-\mathsf{d}}{2}}\ \forall y\in A_x\right)\\&\ge\prod_{x\in B_N}\delta=\delta^{|B_N|}\ge e^{-C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}}.\end{aligned}$$

It remains to prove the existence of B_N. The size of the boxes A_x depends on the distance to the boundary, so in order to construct B_N it is convenient to split Λ_N into the dyadic annuli $W_{N,k}=\{x\in\Lambda_N\colon 2^k\le d_N(x)<2^{k+1}\}$ for $k=0,1,\dots,\lfloor\log_2 N\rfloor$. For $x\in W_{N,k}$ the cube A_x has diameter $2\gamma d_N(x)\ge\gamma 2^{k+1}$. Because $W_{N,k}$ has outer sidelength $2(N-2^k)\le 2N$ and thickness 2^k, we can cover it by at most

$$2n\left(2\frac{2N}{\gamma 2^{k+1}}\right)^{\mathsf{d}-1}2\frac{2^k}{\gamma 2^{k+1}}\le C\frac{N^{\mathsf{d}-1}}{2^{k(\mathsf{d}-1)}}$$

cubes A_x, i.e. we find a set $B_{N,k}$ of at most $C\frac{N^{\mathsf{d}-1}}{2^{k(\mathsf{d}-1)}}$ points in Λ_N such that

$$W_{N,k} \subset \bigcup_{x \in B_{N,k}} A_x \,.$$

Let $k_0 = \lfloor \log_2(L+1) \rfloor$ which implies that $\Lambda_{N-L} \subset \bigcup_{k \geq k_0} W_{N,k}$.

Consider $B_N = \bigcup_{k=k_0}^{\log_2 N} B_{N,k}$. Then $\Lambda_{N-L} \subset \bigcup_{x \in B_N} A_x$, and we have

$$|B_N| \leq \sum_{k=k_0}^{\lfloor \log_2 N \rfloor} |B_{N,k}| \leq C \sum_{k=k_0}^{\infty} \frac{N^{\mathsf{d}-1}}{2^{k(\mathsf{d}-1)}} \leq C \frac{N^{\mathsf{d}-1}}{2^{k_0(\mathsf{d}-1)}} \leq C \frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}} \,.$$

□

3.3.3 Change of measure

We can now prove the lower bound in Theorem 3.1.1. The idea is simple: We use an explicit calculation with densities to prove that the probability of the event $\mathbb{P}_N(f + \Omega_{\Lambda_{N-L},\infty})$ is bounded below by $e^{-\|\Delta f\|_{L^2}^2} \mathbb{P}_N(\Omega_{\Lambda_{N-L},\infty})$. Then it remains to make a good choice of f.

Proof of Theorem 3.1.1, lower bound. Let $f \colon \Lambda_N \to \mathbb{R}$ be a function to be specified later, and extend it by 0 to all of $\mathbb{Z}^{\mathsf{d}}$. We want to estimate the probability of the event $f + \Omega_{\Lambda_{N-L},\infty} = \{f + \psi : \psi \in \Omega_{\Lambda_{N-L},\infty}\}$. To do so, we calculate

$$\begin{aligned}
\mathbb{P}_N(f + \Omega_{\Lambda_{N-L},\infty}) &= \int_{f+\Omega_{\Lambda_{N-L},\infty}} \frac{1}{Z_N} \exp\left(-\frac{1}{2}\|\Delta\psi\|_{L^2}^2\right) \mathrm{d}\psi \\
&= \int_{\Omega_{\Lambda_{N-L},\infty}} \frac{1}{Z_N} \exp\left(-\frac{1}{2}\|\Delta(f+\psi)\|_{L^2}^2\right) \mathrm{d}\psi \\
&= \int_{\Omega_{\Lambda_{N-L},\infty}} \frac{1}{Z_N} \exp\left(-\frac{1}{2}\|\Delta f\|_{L^2}^2 - \frac{1}{2}\|\Delta\psi\|_{L^2}^2 - (\Delta f, \Delta\psi)_{L^2}\right) \mathrm{d}\psi \,.
\end{aligned} \tag{3.3.7}$$

Because $\Omega_{\Lambda_{N-L},\infty}$ is symmetric around the origin, we can replace ψ by $-\psi$ and obtain that

$$\mathbb{P}_N(f + \Omega_{\Lambda_{N-L},\infty}) = \int_{\Omega_{\Lambda_{N-L},\infty}} \frac{1}{Z_N} \exp\left(-\frac{1}{2}\|\Delta f\|_{L^2}^2 - \frac{1}{2}\|\Delta\psi\|_{L^2}^2 + (\Delta f, \Delta\psi)_{L^2}\right) \mathrm{d}\psi \,. \tag{3.3.8}$$

If we add (3.3.7) and (3.3.8) and use the estimate $e^t + e^{-t} \geq 2$, we conclude

$$\begin{aligned}
\mathbb{P}_N(f + \Omega_{\Lambda_{N-L},\infty}) &= \frac{1}{2} \int_{\Omega_{\Lambda_{N-L},\infty}} \frac{e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2 - \frac{1}{2}\|\Delta\psi\|_{L^2}^2} \left(e^{(\Delta f, \Delta\psi)_{L^2}} + e^{-(\Delta f, \Delta\psi)_{L^2}}\right)}{Z_N} \mathrm{d}\psi \\
&\geq e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2} \int_{\Omega_{\Lambda_{N-L},\infty}} \frac{e^{-\frac{1}{2}\|\Delta\psi\|_{L^2}^2}}{Z_N} \mathrm{d}\psi \\
&= e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2} \mathbb{P}_N(\Omega_{\Lambda_{N-L},\infty}) \,.
\end{aligned} \tag{3.3.9}$$

Note that the conclusion in (3.3.9) could also be derived from (3.3.7) using Jensen's inequality.

We now choose f as in Lemma 3.3.4 below. Then

$$\|\Delta f\|_{L^2}^2 \leq C \frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}} \,. \tag{3.3.10}$$

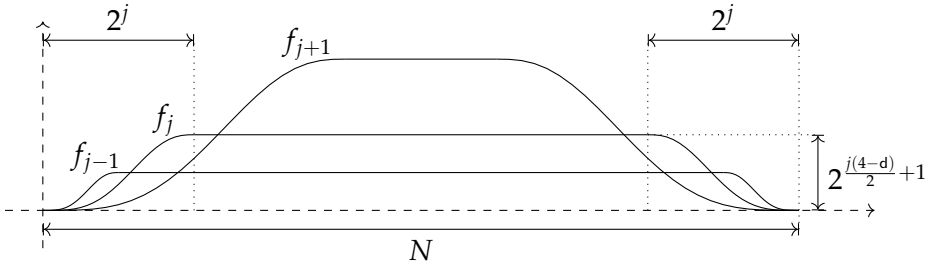


Figure 3.1: The functions f_j.

Moreover this choice of f ensures that $\Omega_{\Lambda_{N-L},+} \supset f + \Omega_{\Lambda_{N-L},\infty}$, and so (3.3.9), (3.3.10) and Lemma 3.3.3 imply that

$$\begin{aligned}\mathbb{P}_N(\Omega_{\Lambda_{N-L},+}) &\geq \mathbb{P}_N(f + \Omega_{\Lambda_{N-L},\infty}) \geq e^{-\frac{1}{2}\|\Delta f\|_{L^2}^2}\mathbb{P}_N(\Omega_{\Lambda_{N-L},\infty}) \\ &\geq e^{-C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}}e^{-C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}} = e^{-C'\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}}.\end{aligned}$$

□

Lemma 3.3.4. *There is a constant $C > 0$ such that for every N and $0 \leq L \leq N$ there is a function $f : \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ such that* supp $f \subset \Lambda_N$, $f(x) \geq d_N(x)^{\frac{4-\mathsf{d}}{2}}$ *for all $x \in \Lambda_{N-L}$ and*

$$\sum_{x\in\mathbb{Z}^{\mathsf{d}}} |\Delta f(x)|^2 \leq C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}\,.$$

Proof. We again use a dyadic construction. Recall $W_{N,k} = \{x \in \Lambda_N \colon 2^k \leq d_N(x) < 2^{k+1}\}$ for $k = 0, 1, \ldots, \lfloor \log_2 N \rfloor$. Let in addition $W_{N,-1} = \mathbb{Z}^{\mathsf{d}} \setminus \Lambda_N$.

Fix a smooth function $\eta \colon \mathbb{R} \to \mathbb{R}$ such that $\eta \geq 0$, $\eta = 1$ on $[1, \infty)$ and $\eta = 0$ on $(-\infty, 0]$. For $i \in \{1, 2, \ldots, n\}$ and $x \in \mathbb{Z}^{\mathsf{d}}$ we introduce the distance $d_i(x) = \operatorname{dist}(x, \mathbb{Z}^{\mathsf{d}} \setminus (\mathbb{Z}^{i-1} \times \{-N, \ldots, N\} \times \mathbb{Z}^{\mathsf{d}-i}))$ of x to the boundary in direction x_i.

For $j = 0, 1, \ldots \lfloor \log_2 N \rfloor - 1$ consider the function

$$f_j(x) = 2^{\frac{j(4-\mathsf{d})}{2}+1}\prod_{i=1}^{\mathsf{d}}\eta\left(\frac{d_i(x)}{2^j}\right)$$

(cf. Figure 3.1). Note that

$$f_j(x) = 2^{\frac{j(4-\mathsf{d})}{2}+1} \tag{3.3.11}$$

for all $x \in \Lambda_N$ such that $d_N(x) \geq 2^j$. Moreover

$$|\Delta f_j(x)| \leq C2^{\frac{j(4-\mathsf{d})}{2}+1}\|\eta''\|_{L^\infty}\frac{1}{2^{2j}} \leq \frac{C\|\eta''\|_{L^\infty}}{2^{\frac{j\mathsf{d}}{2}}}\,. \tag{3.3.12}$$

In fact $\Delta f_j(x) = 0$ if $d_N(x) > 2^j$ because f_k is constant on Λ_{N-2^j}. We define the function

$$f = \sum_{j=\lfloor \log_2(L+1)\rfloor}^{\lfloor \log_2 N\rfloor} f_j\,.$$

For $x \in \Lambda_{N-L}$ let now k be such that $x \in W_{N,k}$, and observe that $\lfloor \log_2(L+1)\rfloor \leq k \leq \lfloor \log_2 N\rfloor$. The estimate (3.3.11) implies

$$f(x) \geq f_k(x) \geq 2^{\frac{k(4-\mathsf{d})}{2}+1} \geq \left(2\cdot 2^k\right)^{\frac{4-\mathsf{d}}{2}} \geq d_N(x)^{\frac{4-\mathsf{d}}{2}}\,.$$

For an arbitrary $x \in \mathbb{Z}^{\mathsf{d}}$ let again $k \in \{-1, 0, 1, \dots\}$ be such that $x \in W_{N,k}$. Then (3.3.12) implies that

$$|\Delta f(x)| \leq \sum_{j=k\vee\lfloor\log_2(L+1)\rfloor}^{\lfloor\log_2 N\rfloor} |\Delta f_j| \leq \sum_{j=k\vee\lfloor\log_2(L+1)\rfloor}^{\infty} \frac{C\|\eta''\|_{L^\infty}}{2^{\frac{j\mathsf{d}}{2}}} \leq \frac{C'}{2^{\frac{(k\vee\lfloor\log_2(L+1)\rfloor)\mathsf{d}}{2}}}.$$

Using that $|W_{N,k}| \leq C2^k N^{\mathsf{d}-1}$ for $k \geq 0$ and that $\Delta f(x)$ is zero on $W_{N,-1}$ except possibly on the set $\Lambda_{N+1} \setminus \Lambda_N$ of cardinality $CN^{\mathsf{d}-1} \leq C'2^{-1}N^{\mathsf{d}-1}$, the previous estimate implies that

$$\begin{aligned}
\sum_{x\in\mathbb{Z}^{\mathsf{d}}} |\Delta f(x)|^2 &\leq \sum_{k=-1}^{\lfloor\log_2 N\rfloor} \sum_{x\in W_{N,k}} |\Delta f(x)|^2 \leq \sum_{k=-1}^{\infty} \frac{C2^k N^{\mathsf{d}-1}}{2^{(k\vee\lfloor\log_2(L+1)\rfloor)\mathsf{d}}} \\
&\leq \sum_{k=-1}^{\lfloor\log_2(L+1)\rfloor} \frac{C2^k N^{\mathsf{d}-1}}{2^{\lfloor\log_2(L+1)\rfloor\mathsf{d}}} + \sum_{k=\lfloor\log_2(L+1)\rfloor+1}^{\infty} \frac{C2^k N^{\mathsf{d}-1}}{2^{k\mathsf{d}}} \\
&\leq C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}} + C\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}} = C'\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}.
\end{aligned}$$

□

3.4 Upper bounds

In order to prove the upper bound in Theorem 3.1.1, we will find a suitably sparse set $E_{N,L}$ of points at the boundary such that the random variables $\{\psi_x : x \in E_{N,L}\}$ are almost independent in the sense that their covariance matrix is diagonally dominant. We can then use Lemma 3.2.5 to compare them to actually independent random variables. The following argument is taken from [Sch16, Section 6.2.1].

Proof of Theorem 3.1.1, upper bound. Note that for $N \geq L > \frac{N}{2}$ the upper bound is trivial. Indeed, Λ_{N-L} is nonempty and so the symmetry of the field implies $\mathbb{P}_N(\Omega_{\Lambda_{N-L},+}) \leq \frac{1}{2}$, while the right hand side of (3.1.1) exceeds $\frac{1}{2}$ if $L > \frac{N}{2}$ and $c < 2^{-\mathsf{d}}$. We assume $L \leq \frac{N}{2}$ in the following. Let $E_{N,L} = \Lambda_{N-L} \cap ((\lceil\alpha(L+1)\rceil\mathbb{Z})^{\mathsf{d}-1} \times \{N-L\})$ where $\alpha \geq 1$ is a constant to be chosen later. This is a set of points on one face of $[-N+L, N-L]^{\mathsf{d}}$ such that any two points have distance at least αL. Its cardinality satisfies

$$\begin{aligned}
|E_{N,L}| &= \left(2\left\lfloor\frac{N-L}{\lceil\alpha(L+1)\rceil}\right\rfloor + 1\right)^{\mathsf{d}-1} \geq \left(\left\lfloor\frac{N-\frac{N}{2}}{\alpha(L+1)+1}\right\rfloor + 1\right)^{\mathsf{d}-1} \\
&\geq \left(\frac{\frac{N}{2}}{\alpha(L+1)+1}\right)^{\mathsf{d}-1} \geq c\frac{N^{\mathsf{d}-1}}{\alpha^{\mathsf{d}-1}(L+1)^{\mathsf{d}-1}}.
\end{aligned} \tag{3.4.1}$$

Clearly $d_N(x) = L+1$ for any $x \in E_{N,L}$. Therefore according to (3.2.4) for $x \neq y$

$$|G_N(x,y)| \leq C\frac{(L+1)^4}{(|x-y|_\infty+1)^{\mathsf{d}}} \leq C\frac{(L+1)^4}{|x-y|_\infty^{\mathsf{d}}}.$$

If we combine this with (3.2.1) we obtain for any $x \in E_{N,L}$

$$\sum_{\substack{y\in E_{N,L}\\ y\neq x}} \frac{|G_N(x,y)|}{\sqrt{G_N(x,x)G_N(y,y)}} \leq C \sum_{\substack{y\in E_{N,L}\\ y\neq x}} \frac{(L+1)^4}{(L+1)^{4-\mathsf{d}}|x-y|_\infty^{\mathsf{d}}}$$

$$= C \sum_{j=1}^{\infty} |\{y \in E_{N,L} \colon |y-x|_\infty = j\lceil \alpha(L+1)\rceil\}| \frac{(L+1)^{\mathsf{d}}}{(j\lceil \alpha(L+1)\rceil)^{\mathsf{d}}}$$
$$\leq \frac{C}{\alpha^{\mathsf{d}}} \sum_{j=1}^{\infty} \frac{a_j}{j^{\mathsf{d}}}$$

where $a_j = 2$ for $\mathsf{d} = 2$ and $a_j = 8j$ for $\mathsf{d} = 3$. Thus $\sum_{j=1}^{\infty} \frac{a_j}{j^{\mathsf{d}}} < \infty$ and hence

$$\sum_{\substack{y \in E_{N,L} \\ y \neq x}} \frac{|G_N(x,y)|}{\sqrt{G_N(x,x)G_N(y,y)}} \leq \frac{C}{\alpha^{\mathsf{d}}} \,. \tag{3.4.2}$$

We now choose α large enough that the right hand side of (3.4.2) becomes less than $\frac{1}{4}$.

We define the Gaussian random vector $(X_x)_{x \in E_{N,L}}$ by $X_x = \frac{\psi_x}{\sqrt{G_N(x,x)}}$. Let Σ_X be its covariance matrix. Then $(\Sigma_X)_{x,x} = 1$ for all x and (3.4.2) implies that

$$\sum_{\substack{y \in E_{N,L} \\ y \neq x}} |(\Sigma_X)_{x,y}| \leq \frac{1}{4} \,. \tag{3.4.3}$$

Let $\{Y_x\}_{x \in E_{N,L}}$ be i.i.d. normal variables distributed according to $\mathcal{N}\left(0, \frac{3}{2}\right)$ and let $\Sigma_Y = \frac{3}{2}\mathbb{1}_{E_{N,L}}$ be their joint covariance matrix, where $\mathbb{1}_{E_{N,L}}$ is a unit matrix indexed by $E_{N,L}$.

Because of (3.4.3) the matrix $\Sigma_Y - \Sigma_X$ then satisfies

$$(\Sigma_Y - \Sigma_X)_{x,x} = \frac{3}{2} - 1 = \frac{1}{2} > \sum_{\substack{y \in E_{N,L} \\ y \neq x}} (\Sigma_X)_{x,y}$$

This means that $\Sigma_Y - \Sigma_X$ is strictly diagonally dominant and hence positive definite. Hence we can apply Lemma 3.2.5 and obtain

$$\begin{aligned}
\left(\frac{1}{2}\right)^{|E_{N,L}|} &= \mathbb{P}(Y \in (0,\infty)^{E_{N,L}}) \\
&\geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \mathbb{P}(X \in (0,\infty)^{E_{N,L}}) \\
&= \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \mathbb{P}_N(\psi_x \geq 0 \; \forall x \in E_{N,L}) \\
&\geq \left(\frac{\det \Sigma_X}{\det \Sigma_Y}\right)^{\frac{1}{2}} \mathbb{P}_N(\Omega_{\Lambda_{N-L},+}) \,.
\end{aligned}$$

It remains to estimate $\frac{\det \Sigma_X}{\det \Sigma_Y}$. Since Σ_Y is diagonal, $\det \Sigma_Y = \left(\frac{3}{2}\right)^{|E_{N,L}|}$. On the other hand, by (3.4.3) the matrix $\Sigma_X - \frac{3}{4}\mathbb{1}_{E_{N,L}}$ is still diagonally dominant and hence positive semidefinite. Hence all eigenvalues of Σ_X must be at least $\frac{3}{4}$. Therefore $\det \Sigma_X \geq \left(\frac{3}{4}\right)^{|E_{N,L}|}$.

We conclude

$$\mathbb{P}_N(\Omega_{\Lambda_{N-L},+}) \leq \left(\frac{1}{2}\right)^{|E_{N,L}|} \left(\frac{\det \Sigma_Y}{\det \Sigma_X}\right)^{\frac{1}{2}} \leq \left(\frac{1}{2}\right)^{|E_{N,L}|} \left(\frac{3/2}{3/4}\right)^{\frac{|E_{N,L}|}{2}} = \left(\frac{1}{\sqrt{2}}\right)^{|E_{N,L}|} \,. \tag{3.4.4}$$

Recall that by (3.4.1) we have $|E_{N,L}| \geq c\frac{N^{\mathsf{d}-1}}{\alpha^{\mathsf{d}-1}(L+1)^{\mathsf{d}-1}}$. Thus we finally obtain

$$\mathbb{P}_N(\Omega_{\Lambda_{N-L},+}) \leq \exp\left(-c\frac{N^{\mathsf{d}-1}}{(L+1)^{\mathsf{d}-1}}\right)$$

for $c = \frac{1}{2\alpha^{\mathsf{d}-1}}\log 2$. □

4 The maximum of the four-dimensional membrane model

This chapter is based on the author's paper [Sch20a], with only minor changes.

4.1 Introduction

As discussed in Section 1.3.2, in this chapter we will study the maximum of the four-dimensional membrane model. The main part of the proof are estimates for the Green's function of the discrete Bilaplacian in dimension 4. These were described in Section 1.4.3.

4.1.1 Main result for the membrane model

Recall that ψ_N denotes a sample of $\mathbb{P}_{\Delta,\Lambda_N}$, where $\Lambda_N = [0,N]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}}$ and that $M_N^\Delta = \max_{x\in\Lambda_N} \psi_{N,x}$.

Our main result is the following.

Theorem 4.1.1. *Let* $\mathsf{d} = 4$. *The random variable*

$$M_N^\Delta - m_N^\Delta := M_N^\Delta - \frac{1}{\pi}\log N + \frac{3}{16\pi}\log\log N$$

converges in distribution. The limit law is a randomly shifted Gumbel distribution μ_∞, *given by*

$$\mu_\infty((-\infty,x]) = \mathbb{E}e^{-\gamma^*\mathcal{Z}e^{-8\pi x}}\ \forall x$$

where γ^* *is a constant and* $\mathcal{Z}$ *is a positive random variable that is the limit in law of*

$$\mathcal{Z}_N = \sqrt{8}\sum_{v\in\Lambda_N}(\log N - \pi\psi_{N,v})e^{-8(\log N - \pi\psi_{N,v})}\,.$$

Before we discuss our proof strategy, let us point out a generalization.

Remark 4.1.2. Our approach is not limited to the membrane model. In fact, consider for $l \in \mathbb{N}^+$ the ∇^l-model, given by the probability measure

$$\mathbb{P}_A^{(l)}(\mathrm{d}\psi) = \begin{cases} \frac{1}{Z_A^{(l)}}\exp\left(-\frac{1}{2}\sum_{v\in\mathbb{Z}^{\mathsf{d}}}|\Delta_1^{\frac{l}{2}}\psi_v|^2\right)\prod_{v\in A}\mathrm{d}\psi_v\prod_{v\in\mathbb{Z}^{\mathsf{d}}\setminus A}\delta_0(\mathrm{d}\psi_v) & l \text{ even} \\ \frac{1}{Z_A^{(l)}}\exp\left(-\frac{1}{2}\sum_{v\in\mathbb{Z}^{\mathsf{d}}}|\nabla_1\Delta_1^{\frac{l-1}{2}}\varphi_v|^2\right)\prod_{v\in A}\mathrm{d}\psi_v\prod_{v\in\mathbb{Z}^{\mathsf{d}}\setminus A}\delta_0(\mathrm{d}\psi_v) & l \text{ odd} \end{cases}$$

(note that $l = 1$ corresponds to the gradient model and $l = 2$ to the membrane model) in the critical dimension $\mathsf{d} = 2l$ on the cube $A = [0,N]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}}$. Then Theorem 4.1.1 generalizes to this setting, and the maximum of the field, appropriately centred, converges in law to a randomly shifted Gumbel distribution. Our proof in the following would only require minor modifications to yield this more general result. However, since the case $l = 1$ is covered by [BDZ16], while the ∇^l-model for $l > 2$ is rarely studied, we choose to focus on the case $l = 2$ in the following. This allows us to avoid more complicated notation.

4.1.2 Log-correlated fields

As discussed in Section 1.3.2, one needs to consider this result in the context of log-correlated Gaussian fields, where one expects certain universality properties of features such as the maximum of the field. In particular, in [DRZ17] Ding, Roy and Zeitouni gave a set of four assumptions that ensure that the maximum of a field converges in distribution. Let us recall their result, slightly reformulated (we have changed the domain from $[0, N-1]^{\mathsf{d}}$ to $[0, N]^{\mathsf{d}}$, and replaced $\log_+ |a|$ with $\log(1+|a|)$ in (A.0) and (A.1), but it is straightforward to check that the theorem stated here is equivalent to the theorem as stated in [DRZ17]). We write $d_N(v) := \operatorname{dist}(v, \partial[0, N]^{\mathsf{d}})$ for the distance of v to the boundary of $[0, N]^{\mathsf{d}}$ and $d(x) := d_1(x)$.

Theorem 4.1.3 ([DRZ17, Theorem 1.3 and Theorem 1.4])**.** *Let $\Lambda_N = [0, N]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}}$, and let $\varphi_N = \{\varphi_{N,v} \colon v \in \Lambda_N\}$ be a centred Gaussian field. Assume that*

(A.0) ***(Logarithmically bounded fields)*** *There is a constant $\alpha_0 > 0$ such that for all $u, v \in \Lambda_N$,*

$$\operatorname{Var} \varphi_{N,v} \leq \log N + \alpha_0$$

and

$$\mathbb{E}(\varphi_{N,v} - \varphi_{N,u})^2 \leq 2\log(1+|u-v|) - |\operatorname{Var} \varphi_{N,v} - \operatorname{Var} \varphi_{N,u}| + 4\alpha_0 \,.$$

(A.1) ***(Logarithmically correlated fields)*** *For any $\delta > 0$ there is a constant $\alpha^{(\delta)} > 0$ such that for all $u, v \in \Lambda_N$ with $\min(d_N(u), d_N(v)) \geq \delta N$*

$$|\operatorname{Cov}(\varphi_{N,v}, \varphi_{N,u}) - (\log N - \log(1+|u-v|))| \leq \alpha^{(\delta)} \,.$$

(A.2) ***(Near diagonal behaviour)*** *There are both a continuous function $f_1 \colon (0,1)^{\mathsf{d}} \to \mathbb{R}$ and a function $f_2 \colon \mathbb{Z}^{\mathsf{d}} \times \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ such that the following holds. For all $L, \varepsilon, \delta > 0$, there exists $N_0 = N_0(L, \varepsilon, \delta)$ such that for all $x \in [0,1]^{\mathsf{d}}$, $N \geq N_0$ such that $Nx \in \mathbb{Z}^{\mathsf{d}}$ and $d(x) \geq \delta$, and for all $u, v \in [0, L]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}}$ we have*

$$|\operatorname{Cov}(\varphi_{N,Nx+v}, \varphi_{N,Nx+u}) - \log N - f_1(x) - f_2(u,v)| < \varepsilon \,.$$

(A.3) ***(Off diagonal behaviour)*** *There is a continuous function $f_3 \colon \mathcal{D}^{\mathsf{d}} \to \mathbb{R}$, where $\mathcal{D}^{\mathsf{d}} = \{(x,y) : x, y \in (0,1)^{\mathsf{d}}, x \neq y\}$ such that the following holds. For all $L, \varepsilon, \delta > 0$ there exists $N_1 = N_1(L, \varepsilon, \delta) > 0$ such that for all $x, y \in [0,1]^{\mathsf{d}}$, $N \geq N_1$ such that $Nx, Ny \in \mathbb{Z}^{\mathsf{d}}$, $\min(d(x), d(y)) \geq \delta$ and $|x-y| \geq \frac{1}{L}$ we have*

$$|\operatorname{Cov}(\varphi_{N,Nx}, \varphi_{N,Ny}) - f_3(x,y)| < \varepsilon \,.$$

Let $M_N = \max_{v \in \Lambda_N} \varphi_{N,v}$ and

$$m_N = \sqrt{2\mathsf{d}} \log N - \frac{3}{2\sqrt{2\mathsf{d}}} \log\log N \,.$$

Then the sequence $M_N - m_N$ converges in distribution to a randomly shifted Gumbel distribution μ_∞. The limit distribution is given by

$$\mu_\infty((-\infty, x]) = \mathbb{E} e^{-\gamma^* \mathcal{Z} e^{-\sqrt{2\mathsf{d}} x}} \ \forall x$$

where γ^* is a constant and $\mathcal{Z}$ is a positive random variable that is the limit in law of

$$\mathcal{Z}_N = \sum_{v \in \Lambda_N} (\sqrt{2\mathsf{d}} \log N - \varphi_{N,v}) e^{-\sqrt{2\mathsf{d}}(\sqrt{2\mathsf{d}} \log N - \varphi_{N,v})} \,.$$

This theorem easily implies Theorem 4.1.1 once we show that ψ_N^Δ (or rather $\sqrt{8}\pi\psi_N^\Delta$) satisfies assumptions (A.0), (A.1), (A.2), (A.3). In fact we can prove even slightly stronger statements than these. Let us state the precise results that we will prove. We abbreviate $\lambda = \sqrt{8}\pi$.

Theorem 4.1.4. *The field $\varphi_N := \lambda\psi_N^\Delta$ in dimension $\mathsf{d} = 4$ satisfies*

(A.0′) There is a constant $\alpha_0' > 0$ such that for all $u, v \in \Lambda_N$,

$$\operatorname{Var}\varphi_{N,v} \le \min\big(\log N + \alpha_0', \alpha_0' \log(2 + d_N(v))\big)$$

and

$$\operatorname{Var}\varphi_{N,v} - \operatorname{Cov}(\varphi_{N,v}, \varphi_{N,u}) \le \log(1 + |u - v|) + 2\alpha_0'.$$

(A.1′) There is a constant $\alpha_0'' > 0$ such that for all $u, v \in \Lambda_N$

$$\left|\operatorname{Cov}(\varphi_{N,v}, \varphi_{N,u}) - \log\left(2 + \frac{\max(d_N(u), d_N(v))}{1 + |u - v|}\right)\right| \le \alpha_0''.$$

(A.2′) There are a constant $\theta_0 > 0$, a continuous function $f_1\colon (0,1)^4 \to \mathbb{R}$ and a function $f_2\colon \mathbb{Z}^4 \times \mathbb{Z}^4 \to \mathbb{R}$ such that the following holds. For all $L, \varepsilon > 0$, $\theta > \theta_0$ there exists $N_0' = N_0'(L, \varepsilon, \theta)$ such that for all $x \in [0,1]^4$, $N \ge N_0'$ such that $Nx \in \mathbb{Z}^4$ and $d(x) \ge \frac{(\log N)^\theta}{N}$, and for all $u, v \in [0, L]^4 \cap \mathbb{Z}^4$ we have

$$|\operatorname{Cov}(\varphi_{N,Nx+v}, \varphi_{N,Nx+u}) - \log N - f_1(x) - f_2(u, v)| < \varepsilon.$$

(A.3′) There are a constant $\theta_1 > 0$ and a continuous function $f_3\colon \mathcal{D}^4 \to \mathbb{R}$, where $\mathcal{D}^4 = \{(x, y) : x, y \in (0,1)^4, x \ne y\}$ such that the following holds. For all $L, \varepsilon > 0$, $\theta > \theta_1$ there exists $N_1' = N_1'(L, \varepsilon, \theta)$ such that for all $x, y \in V$, $N \ge N_1'$ such that $Nx, Ny \in \mathbb{Z}^4$, $\min(d(x), d(y)) \ge \frac{(\log N)^\theta}{N}$ and $|x - y| \ge \frac{1}{L}$ we have

$$|\operatorname{Cov}(\varphi_{N,Nx}, \varphi_{N,Ny}) - f_3(x, y)| < \varepsilon.$$

It is not hard to check that the assumptions (A.0′), (A.1′), (A.2′), (A.3′) imply (A.0), (A.1), (A.2), (A.3) respectively, so that Theorem 4.1.1 is a straightforward corollary of Theorem 4.1.4. We give a few more details in Section 4.4.

The proof of Theorem 4.1.4 is the main contribution of this chapter. In the next section we will describe our approach.

4.1.3 Green's function estimates

The covariance function of the membrane model is the Green's function G_N^Δ of the discrete Bilaplacian on the grid $[0, N]^\mathsf{d}$ with zero boundary data, and the assumptions (A.0′), (A.1′), (A.2′), (A.3′) all correspond to certain estimates for this Green's function. Therefore our goal is to understand this Green's function. We are going to apply tools from PDE theory and numerical analysis, so before proceeding further it is convenient to rescale our domain to a unit box. Let $h = \frac{1}{N}$, let $\Lambda_h = [0,1]^4 \cap (h\mathbb{Z})^4$, and let $\psi_{h,x}^\Delta := \psi_{N,\frac{x}{h}}^\Delta$. Let G_N^Δ and G_h^Δ be the covariance functions of ψ_N^Δ and ψ_h^Δ. Then also $G_h^\Delta(x, y) = G_N^\Delta\left(\frac{x}{h}, \frac{y}{h}\right)$.

Using G_N^Δ and G_h^Δ, Λ_N and Λ_h, and ψ_N and ψ_h simultaneously is a slight abuse of notation. It should, however, always be clear from the context which object we are referring to. Let

us also remark that from a PDE point of view it would arguably be more natural to choose $h = \frac{1}{N+2}$ and rescale $[0,N]^4$ to $[h, 1-h]^4$, as this would give our domain a natural boundary layer of zeros, matching the continuous Dirichlet boundary data. Our choice of rescaling, however, is in line with [DRZ17].

Observation 4.1.5. *Under the aforementioned rescaling, each statement (A.0′), (A.1′), (A.2′), (A.3′) from Theorem 4.1.4 for $\lambda\psi_N^\Delta$ in dimension $\mathsf{d} = 4$ is equivalent to the corresponding following statement for G_h^Δ.*

(B.0′) There is a constant $\alpha_0' > 0$ such that for all $x, y \in \Lambda_h$,

$$\lambda^2 G_h^\Delta(x,x) \le \min\left(-\log h + \alpha_0', \alpha_0' \log\left(2 + \frac{d(x)}{h}\right)\right)$$

and

$$\lambda^2\left(G_h^\Delta(x,x) - G_h^\Delta(x,y)\right) \le \log\left(1 + \frac{|x-y|}{h}\right) + 2\alpha_0'.$$

(B.1′) There is a constant $\alpha_0'' > 0$ such that for all $x, y \in \Lambda_h$

$$\left|\lambda^2 G_h^\Delta(x,y) - \log\left(2 + \frac{\max(d(x), d(y))}{h + |x-y|}\right)\right| \le \alpha_0''.$$

(B.2′) There are a constant $\theta_0 > 0$, a continuous function $f_1\colon (0,1)^4 \to \mathbb{R}$ and a function $f_2\colon \mathbb{Z}^4 \times \mathbb{Z}^4 \to \mathbb{R}$ such that the following holds. For all $L, \varepsilon > 0$, $\theta > \theta_0$ there exists $N_0' = N_0'(L, \varepsilon, \theta)$ such that for all $h \le \frac{1}{N_0'}$ with $\frac{1}{h} \in \mathbb{N}$, all $x \in \Lambda_h$ such that $d(x) \ge h|\log h|^\theta$ and for all $u, v \in [0,L]^4 \cap \mathbb{Z}^4$ we have

$$\left|\lambda^2 G_h^\Delta(x+hu, x+hv) + \log h - f_1(x) - f_2(u,v)\right| < \varepsilon.$$

(B.3′) There are a constant $\theta_1 > 0$ and a continuous function $f_3\colon \mathcal{D}^4 \to \mathbb{R}$, where $\mathcal{D}^4 = \{(x,y) : x, y \in (0,1)^4, x \ne y\}$ such that the following holds. For all $L, \varepsilon > 0$, $\theta > \theta_1$ there exists $N_1' = N_1'(L, \varepsilon, \theta)$ such that for all $h \le \frac{1}{N_1'}$ with $\frac{1}{h} \in \mathbb{N}$ and for $x, y \in \Lambda_h$ such that $\min(d(x), d(y)) \ge h|\log h|^\theta$ and $|x-y| \ge \frac{1}{L}$ we have

$$\left|\lambda^2 G_h^\Delta(x,y) - f_3(x,y)\right| < \varepsilon.$$

Let us discuss how one might prove Theorem 4.1.4, or rather the statements (B.0′), (B.1′), (B.2′), (B.3′). We write $\Gamma_h = (h\mathbb{Z})^4 \cap \left([-h, 1+h]^4 \setminus [0,h]^4\right)$. The function G_h^Δ is the Green's function associated to the discrete boundary value problem

$$\begin{aligned} \Delta_h^2 u_h &= f_h && \text{in } \Lambda_h \\ u_h &= 0 && \text{on } \Gamma_h \\ D_\nu^h u_h &= 0 && \text{on } \Gamma_h \end{aligned} \tag{4.1.1}$$

(where $D_\nu^h u(x) = \frac{u(x+h\nu) - u(x)}{h}$ and ν is an outward unit normal vector). That is, for $y \in \Lambda_h$ the function $G_h(\cdot, y)$ is the unique solution of that equation with right hand side $f_h = \delta_h(y)$, defined as $\delta_{h,y}(x) = \begin{cases} \frac{1}{h^4} & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$.

One previous strategy to prove estimates for G_h^Δ, introduced in [Kur09] and used as well in [Cip13], was to compare G_h^Δ to $\overline{G}_h^\Delta$, the Green's function associated to the discrete boundary value problem

$$\begin{aligned} \Delta_h^2 u_h &= f_h && \text{in } \Lambda_h \\ u_h &= 0 && \text{on } \Gamma_h' \\ \Delta_h u &= 0 && \text{on } \Gamma_h \end{aligned} \tag{4.1.2}$$

where $\Gamma_h' = (h\mathbb{Z})^4 \cap \big([-2h, 1+2h]^4 \setminus [-h, 1+h]^4\big)$. The problem (4.1.2) can be seen as an iterated version of the discrete Poisson problem, and so many of the analytic and probabilistic tools available for the latter also have a version for (4.1.2). In particular, there are random walk representations for $\overline{G}_h^\Delta$ that allow to control it well. The strategy in [Kur09] then was to use PDE techniques to compare solutions of (4.1.1) and (4.1.2). This allows to estimate the difference between G_h and $\overline{G}_h$ uniformly in compact subsets of $(0,1)^4$. For our purposes, this is not good enough, as for (B.2′) and (B.3′) an error term that is only bounded is already too much. Note however that results similar to (B.0′), (B.1′) can be proved using these methods. In fact, [Kur09, Proposition 1.1] and [Cip13, Lemma 2.1] are already weaker versions of (B.0′) and (B.1′).

In Chapter 2 we considered G_h^Δ in dimensions 2 and 3, and used a very different strategy, namely a compactness argument to transfer estimates for the continuous Green's function in domains with singularities to the discrete setting. This allowed us the prove discrete Caccioppoli inequalities (i.e. L^2-based decay estimates on balls of various sizes) and to conclude from these estimates for G_h^Δ. In principle, this strategy can also be applied in our four-dimensional setting. One obstacle to this is that, unlike the two- or three-dimensional case, the relevant continuous estimates cannot be found in the literature. Even more importantly, the estimates in Chapter 2 are all up to a possibly large constant, and so the argument would have to be modified significantly to obtain estimates such as (B.2′) and (B.3′).

Instead of the aforementioned approaches to derive estimates for G_h^Δ we will use estimates for the approximation quality of finite difference schemes for the Bilaplacian. This idea is not completely new, as for example in [CDH19] estimates for finite difference schemes from [Tho64] were used to prove convergence of the rescaled four-dimensional membrane model in some negative Sobolev space. However, we would like to obtain a much stronger conclusion, namely pointwise estimates for the difference of the discrete and continuous Green's function. The result from [Tho64] is very general, but because of its generality it requires in our specific case very strong assumptions on the solution of the continuous Bilaplace equation to be approximated (being C^5) to yield estimates useful for us (the $W_h^{2,2}$-approximation error decaying like $h^{\frac{1}{2}}$).

We will use a rather different estimate for the approximation quality of finite difference schemes. We will discuss the details in Section 4.2.2. Roughly speaking, the result is the following: Let $2 < s < \frac{5}{2}$, let $u \in W^{s,2} \cap W_0^{2,2}((0,1)^4)$ extended by 0 to $\mathbb{R}^4$, and assume that $\Delta^2 u = f$ in $(0,1)^4$, so that u satisfies

$$\begin{aligned} \Delta^2 u &= f && \text{in } (0,1)^4 \\ u &= 0 && \text{on } \partial(0,1)^4 \\ \partial_\nu u &= 0 && \text{on } \partial(0,1)^4 . \end{aligned} \tag{4.1.3}$$

Furthermore, let $u_h\colon (h\mathbb{Z})^4 \to \mathbb{R}$ be the solution of

$$\begin{aligned} \Delta_h^2 u_h &= T^{h,3,3,3,3} f && \text{in } \Lambda_h \\ u_h &= 0 && \text{on } (h\mathbb{Z})^4 \setminus \Lambda_h \end{aligned}$$

where $T^{h,3,3,3,3}$ is a certain regularization operator. Then

$$\|u - u_h\|_{W_h^{2,2}(\Lambda_h)} \leq Ch^{s-2}\|u\|_{W^{s,2}((0,1)^4)}$$

where $\|\cdot\|_{W_h^{2,2}(\Lambda_h)}$ is a discrete Sobolev norm.

This result is inspired by closely related results in Chapter 5. However, in that chapter the focus is on obtaining estimates as above for s as large as possible. In the case of interest to us, $s < \frac{5}{2}$, the result can essentially be shown using the methods from [GMP83, IĬS86, JS14].

We will use this result to compare solutions of (4.1.1) with solutions of (4.1.3). In particular, we will use it when u is the regular part of the continuous Green's function on $[0,1]^4$. To do so, we need regularity estimates for solutions of (4.1.3). As already mentioned, optimal estimates for higher order elliptic problems on four-dimensional polyhedral domains are not yet in the literature. Instead we will use much weaker estimates (similar to ones in [MM13, MM14]) which are nonetheless sharp enough for our purposes. These estimates will allow us to place the regular part of the Green's function in $W^{2+\kappa_0,2}$ for some small $\kappa_0 > 0$, and this is good enough to apply the estimate above.

We will also need to have good estimates for the discrete Green's function on the full space $(h\mathbb{Z})^4$. These were derived in [Man67] using Fourier analysis. Furthermore, Theorem 4.2.3 gives us control over the $W_h^{2,2}$-norm of the difference of u and u_h, while we are actually interested in the L_h^∞-norm and want it to decay. To achieve this, we will use a discrete Sobolev-inequality that allows us to control the L_h^∞-norm by the $W_h^{2,2}$-norm at the cost of a term logarithmic in h. The presence of this term is the reason why we can prove (B.2′) and (B.3′) only up to distance $|\log h|^\theta$ to the boundary. For (B.0′) and (B.1′) we do not need a decaying but only a bounded error term and so we can prove these estimates on the whole domain.

We will give the details of the argument that we sketched here in the following sections. In Section 4.2 we gather various useful results: The aforementioned result on finite difference schemes, as well as some discrete inequality of Poincaré-Sobolev-type. These tools will allow us to compare G_h^Δ with various other Green's functions: the discrete Green's function of the full space (that we discuss in Section 4.3.1) and the continuous Green's functions of the box $[0,1]^4$ and of the full space (that we both discuss in Section 4.3.2). After all these preparations we can then turn to the proof of Theorem 4.1.4 in Section 4.4. We first prove a crucial lemma, Lemma 4.4.1 that shows that the regular part of the discrete and continuous Green's functions on the box are uniformly close, and then we use this Lemma and the results of the preceding sections to establish Theorem 4.1.4. Finally we use Theorem 4.1.3 to conclude Theorem 4.1.1 as well.

4.1.4 Notation

From now on we will only consider the membrane and not the gradient model, so there is no risk of confusion when we drop all superscripts Δ.

Occasionally we write $r = s + O(t)$ to express $|r - s| \leq Ct$.

We use the Sobolev space $W^{k,2}(\Omega)$ with the norm $\|u\|^2_{W^{k,2}(\Omega)} = \sum_{|\alpha|\le k} \|\partial^\alpha u\|^2_{L^2(\Omega)}$. For $s > 0$ not an integer (i.e. $s = k + t$ where $k \in \mathbb{N}$, $0 < t < 1$) we will also encounter the fractional Sobolev space $W^{s,2}(\Omega)$ with norm $\|u\|^2_{W^{s,2}(\Omega)} = \|u\|^2_{W^{k,2}(\Omega)} + [u]^2_{W^{s,2}(\Omega)}$ and the seminorm $[u]^2_{W^{s,2}(\Omega)} = \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{4+2t}} \,\mathrm{d}x\,\mathrm{d}y$. For any $s < 0$ we define $W^{s,2}(\Omega)$ as the dual of $W_0^{-s,2}(\Omega)$. We extend these definitions to vector-valued functions by taking the Euclidean norm of the norms of the components.

For $A \subset (h\mathbb{Z})^4$ and $u_h \colon A \to \mathbb{R}$, we define $\|u_h\|^2_{L^2_h(A)} = \sum_{x\in A} h^4 |u_h(x)|^2$, and $\|u_h\|_{L^\infty_h(A)} = \sup_{x\in A} |u_h(x)|$. We will also use the discrete Sobolev-norm $\|u_h\|^2_{W^{2,2}_h(A)} = \|u_h\|^2_{L^2_h(A)} + \|\nabla_h u_h\|^2_{L^2_h(A)} + \|\nabla_h^2 u_h\|^2_{L^2_h(A)}$, where we extend the definitions to vector-valued functions as before.

Let us also fix once and for all a smooth function $\eta \colon \mathbb{R}^4 \to \mathbb{R}$ that is equal to 1 on $B_{\frac{1}{2}}(0)$ and 0 outside $B_1(0)$. We define $\eta^{(r)}(x) = \eta(rx)$, $\eta^{(r)}_y(x) = \eta^{(r)}(x-y)$ and let $\eta^{(r)}_{h,y}$ be the restriction of $\eta^{(r)}_y$ to $(h\mathbb{Z})^4$. Thus $\eta^{(r)}_y$ and $\eta^{(r)}_{h,y}$ are cut-off functions at scale r around y.

4.2 Preliminaries

4.2.1 Discrete Inequalities

We collect here two discrete inequalities that we will use several times in the following. We begin with a Poincaré inequality.

Lemma 4.2.1. *Let $x_* \in (h\mathbb{Z})^4$, $r \ge 0$. Let $u_h \colon (h\mathbb{Z})^4 \to \mathbb{R}$ and suppose that u_h vanishes on at least one of the faces of $Q_r(x_*)$. Let this face be contained in a plane $x_i = c$. Then*

$$\|u_h\|^2_{L^2_h(Q^h_r(x_*))} \le Cr^2 \sum_{x\colon \{x, x+he_i\} \subset Q^h_r(x_*)} h^4 |D^h_i u_h(x)|^2 \le Cr^2 \|\nabla_h u_h\|^2_{L^2_h(Q^h_r(x_*))}\,. \tag{4.2.1}$$

Proof. This is a particular case of Lemma 2.2.1, but let us give a direct proof for the case at hand. The second inequality is obvious, so we only prove the first. By translating and reflecting the lattice and renaming the coordinates, we can assume $i = 4$, $Q^h_r(x_*) = [0, 2r]^4 \cap (h\mathbb{Z})^4$. We write $x = (x', x_4)$ where $x' \in \mathbb{R}^3$, $x_4 \in \mathbb{R}$, $u_h = 0$ if $x_4 = 0$. We will prove the one-dimensional estimate

$$\sum_{x_4 \in [0,2r]\cap h\mathbb{Z}} |u_h(x', x_4)|^2 \le Cr^2 \sum_{x_4 \in [0,2r-h]\cap h\mathbb{Z}} |D^h_4 u_h(x', x_4)|^2\,. \tag{4.2.2}$$

Once we have established this, (4.2.1) follows by multiplying (4.2.2) by h^4 and summing over all $x' \in [0, 2r]^3 \cap (h\mathbb{Z})^3$. To prove (4.2.2), we use $u(x', 0) = 0$ and write

$$\begin{aligned}
|u_h(x', x_4)| &= \left| \sum_{y_4 \in [0, x_4 - h]\cap h\mathbb{Z}} u_h(x', y_4 + h) - u_h(x', y_4) \right| \\
&= \left| \sum_{y_4 \in [0, x_4 - h]\cap h\mathbb{Z}} h D^h_4 u_h(x', y_4) \right| \\
&\le h \left(\frac{x_4}{h}\right)^{\frac{1}{2}} \left(\sum_{y_4 \in [0, x_4 - h]\cap h\mathbb{Z}} |D^h_4 u_h(x', y_4)|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \sqrt{2hr}\left(\sum_{y_4\in[0,2r-h]\cap h\mathbb{Z}} |D_4^h u_h(x',y_4)|^2\right)^{\frac{1}{2}}$$

and therefore

$$\begin{aligned}\sum_{x_4\in[0,2r]\cap h\mathbb{Z}} |u_h(x',x_4)|^2 &\leq \frac{2r}{h}2hr \sum_{y_4\in[0,2r-h]\cap h\mathbb{Z}} |D_4^h u_h(x',y_4)|^2 \\ &\leq 4r^2 \sum_{y_4\in[0,2r-h]\cap h\mathbb{Z}} |D_4^h u_h(x',y_4)|^2 .\end{aligned}$$

This shows (4.2.2). □

Next we give an inequality of Poincaré-Sobolev type. Given $u_h\colon (h\mathbb{Z})^4 \to \mathbb{R}$ that vanishes outside of Λ_h we would like to estimate its pointwise values by the $\|u_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}$-norm. We cannot hope for such an estimate to hold with a constant independent of h, as the (continuous) Sobolev space $W^{2,2}((0,1)^4)$ does not embed into $L^\infty((0,1)^4)$. However, by Strichartz's [Str72] version of the Moser-Trudinger inequality any $u \in W^{2,2}((0,1)^4)$ with $\|u\|_{W^{2,2}((0,1)^4)} = 1$ satisfies $\int_{(0,1)^4} e^{c|u(x)|^2}\,\mathrm{d}x \leq C$, and this suggests that u can diverge at worst like $\sqrt{|\log|x||}$. So back in the discrete setting we can hope for an estimate with a factor scaling like $\sqrt{|\log h|}$. Indeed we have the following result:

Lemma 4.2.2. *Assume that $u_h\colon (h\mathbb{Z})^4 \to \mathbb{R}$ vanishes outside of Λ_h. Then for any $x \in \Lambda_h$ we have*

$$|u_h(x)| \leq C\sqrt{\log\left(2+\frac{d(x)}{h}\right)}\|u_h\|_{W_h^{2,2}((h\mathbb{Z})^4)} .$$

This lemma in combination with Theorem 4.2.3 will allow us to control the distance between the solution of a continuous Bilaplace equation and its discrete approximation at the cost of a logarithmic divergence (which we will be able to absorb in the applications in Section 4.4).

Proof of Lemma 4.2.2. We first want to localize to a ball around x. Let $v_h = \eta_{h,x}^{(d(x)+h)} u_h$. Then $v_h(x) = u_h(x)$. Furthermore v_h is supported on $Q_{d(x)+h}^h(x)$. The discrete chain rule implies that

$$\begin{aligned}|D_i^h v_h(y)| &\leq C \sup_{z\in Q_h^h(y)} \left|D_i^h \eta_{h,x}^{(d(x)+h)}(z)\right| \sup_{z\in Q_h^h(y)} |u_h(z)| \\ &\quad + C \sup_{z\in Q_h^h(y)} \left|\eta_{h,x}^{(d(x)+h)}(z)\right| \sup_{z\in Q_h^h(y)} |D_i^h u_h(z)| \\ &\leq C \sup_{z\in Q_h^h(y)} \left|D_i^h \eta_{h,x}^{(d(x)+h)}(z)\right| \left(\sum_{z\in Q_h^h(y)} |u_h(z)|^2\right)^{\frac{1}{2}} \\ &\quad + C \sup_{z\in Q_h^h(y)} \left|\eta_{h,x}^{(d(x)+h)}(z)\right| \left(\sum_{z\in Q_h^h(y)} |D_i^h u_h(z)|^2\right)^{\frac{1}{2}}\end{aligned}$$

and a similar expression for $|D_i^h D_{-j}^h v_h(y)|$. If we sum the squares of these eximates over y, we see that

$$
\begin{aligned}
\|v_h\|_{W_h^{2,2}((h\mathbb{Z})^4)} &\leq C\|\eta_{h,x}^{(d(x)+h)}\|_{L_h^\infty((h\mathbb{Z})^4)}\|\nabla_h^2 u_h\|_{L_h^2(Q_{d(x)+2h}^h(x))} \\
&\quad + C\|\nabla_h \eta_{h,x}^{(d(x)+h)}\|_{L_h^\infty((h\mathbb{Z})^4)}\|\nabla_h u_h\|_{L_h^2(Q_{d(x)+2h}^h(x))} \\
&\quad + C\|\nabla_h^2 \eta_{h,x}^{(d(x)+h)}\|_{L_h^\infty((h\mathbb{Z})^4)}\|u_h\|_{L_h^2(Q_{d(x)+2h}^h(x))} \\
&\leq C\|\nabla_h^2 u_h\|_{L_h^2(Q_{d(x)+2h}^h(x))} + \frac{C}{d(x)+h}\|\nabla_h u_h\|_{L_h^2(Q_{d(x)+2h}^h(x))} \\
&\quad + \frac{C}{(d(x)+h)^2}\|u_h\|_{L_h^2(Q_{d(x)+2h}^h(x))}\,.
\end{aligned}
\tag{4.2.3}
$$

We can apply Lemma 4.2.1 to u_h and $D_i^h u_h$ for any $i \in \{1,\dots,4\}$, because these vanish on $Q_{d(x)+2h}^h(x) \setminus [-h, 1+h]^4$ and hence in particular on a face of $Q_{d(x)+2h}^h(x)$. Thus we obtain

$$
\begin{aligned}
\|u_h\|_{L_h^2(Q_{d(x)+2h}^h(x))} &\leq C(d(x)+2h)\|\nabla_h u_h\|_{L_h^2(Q_{d(x)+2h}^h(x))} \\
&\leq C(d(x)+2h)^2\|\nabla_h^2 u_h\|_{L_h^2(Q_{d(x)+2h}^h(x))}\,.
\end{aligned}
\tag{4.2.4}
$$

If we combine this with (4.2.3) and note that $d(x)+2h \leq 2(d(x)+h)$, we obtain

$$
\|v_h\|_{W_h^{2,2}((h\mathbb{Z})^4)} \leq C\|u_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}\,. \tag{4.2.5}
$$

Furthermore, an argument analogous to the one that led to (4.2.4) shows that

$$
\|v_h\|_{L_h^2((h\mathbb{Z})^4)} \leq C(d(x)+h)^2\|\nabla_h^2 v_h\|_{L_h^2((h\mathbb{Z})^4)}\,. \tag{4.2.6}
$$

Now we are in a position to apply discrete Fourier analysis, similar to the proof of [Kur09, Proposition B.1]. Let

$$
\widehat{v_h}(\xi) = h^4 \sum_{y\in(h\mathbb{Z})^4} v_h(y)e^{iy\cdot\xi}
$$

for any $\xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^4$ be the Fourier transform of v_h. Then we also have the inverse formula

$$
v_h(z) = \frac{1}{(2\pi)^4}\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4} \widehat{v_h}(\xi)e^{-iz\cdot\xi}\,\mathrm{d}\xi
$$

for any $z \in (h\mathbb{Z})^4$, and Plancherel's formula in the form

$$
\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4} |\widehat{v_h}(\xi)|^2\,\mathrm{d}\xi = (2\pi h)^4 \sum_{y\in(h\mathbb{Z})^4} |v_h(y)|^2 = (2\pi)^4\|v_h(y)\|_{L_h^2((h\mathbb{Z})^4)}^2\,.
$$

We have

$$
\widehat{D_\alpha^h v_h}(\xi) = (e^{-ih\xi_1}-1)^{\alpha_1}\dots(e^{-ih\xi_4}-1)^{\alpha_4}\widehat{v_h}(\xi)
$$

for any $\alpha \in \mathbb{N}^4$. This implies

$$
\left|\widehat{D_\alpha^h v_h}(\xi)\right| \geq \frac{1}{C}|\xi_1|^{\alpha_1}\dots|\xi_4|^{\alpha_4}|\widehat{v_h}(\xi)|
$$

for any $\xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^4$. In combination with Plancherel's formula and (4.2.6) we conclude

$$
\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4} |\xi|^4|\widehat{v_h}(\xi)|^2 \leq C\|\nabla_h^2 v_h\|_{L_h^2((h\mathbb{Z})^4)}^2 \leq C\|v_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}^2\,, \tag{4.2.7}
$$

$$\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4} |\widehat{v_h}(\xi)|^2 \leq C\|v_h\|^2_{L^2_h((h\mathbb{Z})^4)} \leq C(d(x)+h)^4\|v_h\|^2_{W^{2,2}_h((h\mathbb{Z})^4)}\,. \tag{4.2.8}$$

Next, we estimate

$$\begin{aligned}
|v_h(x)| &= \frac{1}{(2\pi)^4}\left|\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4} \widehat{v_h}(\xi)e^{-ix\cdot\xi}\,\mathrm{d}\xi\right| \\
&\leq C\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4} |\widehat{v_h}(\xi)|\,\mathrm{d}\xi \\
&\leq C\left(\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4}\left(|\xi|^4+\frac{1}{(d(x)+h)^4}\right)|\widehat{v_h}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4}\left(|\xi|^4+\frac{1}{(d(x)+h)^4}\right)^{-1}\mathrm{d}\xi\right)^{\frac{1}{2}}.
\end{aligned}$$

Using (4.2.7) and (4.2.8) we see that

$$\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4}\left(|\xi|^4+\frac{1}{(d(x)+h)^4}\right)|\widehat{v_h}(\xi)|^2\,\mathrm{d}\xi \leq C\|v_h\|^2_{W^{2,2}_h((h\mathbb{Z})^4)}\,.$$

Furthermore we can compute using polar coordinates that

$$\begin{aligned}
\int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4}\left(|\xi|^4+\frac{1}{(d(x)+h)^4}\right)^{-1}\mathrm{d}\xi &= \int_{\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^4}\frac{(d(x)+h)^4}{1+(d(x)+h)^4|\xi|^4}\,\mathrm{d}\xi \\
&\leq C\int_0^{\frac{2\pi}{h}}\frac{(d(x)+h)^4s^3}{1+(d(x)+h)^4s^4}\,\mathrm{d}s \\
&\leq C\log\left(1+(d(x)+h)^4\left(\frac{2\pi}{h}\right)^4\right) \\
&\leq C\log\left(2+\frac{d(x)}{h}\right).
\end{aligned}$$

Putting everything together we indeed arrive at

$$\begin{aligned}
|u_h(x)| &= |v_h(x)| \\
&\leq C\sqrt{\log\left(2+\frac{d(x)}{h}\right)}\|v_h\|_{W^{2,2}_h((h\mathbb{Z})^4)} \\
&\leq C\sqrt{\log\left(2+\frac{d(x)}{h}\right)}\|u_h\|_{W^{2,2}_h((h\mathbb{Z})^4)}\,.
\end{aligned}$$

□

4.2.2 Estimates for finite difference schemes

Let us discuss next the estimate for the approximation order of finite difference schemes that was already mentioned in the introduction.

To state it we need some definitions. These definition will be discussed in more detail in Chapter 5. For $j \geq 1$ let θ_j be the standard univariate centred B-spline of degree $j-1$ (cf.

[JS14, Section 1.9.4]). Of interest to us are

$$\theta_3(z) := \begin{cases} \frac{3}{4} - z^2 & |z| \leq \frac{1}{2} \\ \frac{1}{2}\left(|z| - \frac{3}{2}\right)^2 & \frac{1}{2} < |z| \leq \frac{3}{2} \\ 0 & \text{else} \end{cases},$$

$$\theta_1(z) := \begin{cases} 1 & |z| \leq \frac{1}{2} \\ 0 & \text{else} \end{cases}.$$

Using this, we can define the smoothing operator $T_i^{h,j}$ for $1 \leq i \leq 4$ as

$$T_i^{h,j} f(x) := \frac{1}{h} \int_{\mathbb{R}} f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_4) \theta_j \left(\frac{x_i - y_i}{h} \right) \mathrm{d}y_i$$

extended to distributions on $\mathbb{R}^4$ in the obvious way. Furthermore, we set

$$T^{h,j,\ldots,j} f := T_1^{h,j} \circ \cdots \circ T_4^{h,j} f .$$

It is important for us that $T_i^{h,j}$ maps constant functions to themselves and that

$$T_i^{h,j} \partial_i^2 f = D_i^h D_{-i}^h T_i^{h,j-2} f .$$

If we define the shorthand

$$T^{h,3,3,3,3-2e_i} := T_1^{h,3} \circ \ldots \circ T_{i-1}^{h,3} \circ T_i^{h,1} \circ T_{i+1}^{h,3} \circ \ldots \circ T_4^{h,3}$$

we also have

$$T^{h,3,3,3,3} \partial_i^2 f = D_i^h D_{-i}^h T^{h,3,3,3,3-2e_i} f . \tag{4.2.9}$$

Theorem 4.2.3. *Let $2 < s < \frac{5}{2}$, let $u \in W_0^{s,2}((0,1)^4)$, extended by 0 to $\tilde{u} \in W^{s,2}(\mathbb{R}^4)$. Let $\Delta^2 \tilde{u} = f$ as distributions, so that in particular*

$$\Delta^2 u = f \qquad \text{in } (0,1)^4 .$$

Furthermore, let $u_h \colon (h\mathbb{Z})^4 \to \mathbb{R}$ be the solution of

$$\begin{aligned} \Delta_h^2 u_h &= T^{h,3,3,3,3} f && \text{in } \Lambda_h \\ u_h &= 0 && \text{on } (h\mathbb{Z})^4 \setminus \Lambda_h . \end{aligned}$$

Then we have

$$\|u_h - \tilde{u}\|_{W_h^{2,2}((h\mathbb{Z})^4)} \leq C_s h^{s-2} \|u\|_{W^{s,2}((0,1)^4)} .$$

Note that $f = \Delta^2 \tilde{u} \in W^{s-4,2}(\mathbb{R}^4)$ is in a negative Sobolev space. The operator $T^{h,3,3,3,3}$ maps $W^{t,2}(\mathbb{R}^4)$ to $C(\mathbb{R}^4)$ for any $t > -\frac{5}{2}$ (see [JS14, Section 1.9.4]). So in particular $T^{h,3,3,3,3} f$ has pointwise values and the difference scheme in Theorem 4.2.3 makes sense.

This theorem is closely related to Theorem 5.1.2 in Chapter 5. In that theorem one takes $\frac{5}{2} < s \leq 3$, and $T^{h,3,3,3,3}$ is replaced by $T^{h,2,2,2,2}$. The novelty of that chapter lies in choosing a good extension $\tilde{u}$ and dealing with its boundary values. In our case we can just extend u by 0 and thereby avoid many of these subtleties. In fact, all the ideas for the proof of Theorem 4.2.3 are already for example in [JS14].

To make this chapter more self-contained we give some details for a proof of Theorem 4.2.3.

Proof of Theorem 4.2.3. First of all, $s < \frac{5}{2}$ and $u \in W_0^{s,2}((0,1)^4)$ imply that $\tilde{u}$ is actually in $W^{s,2}(\mathbb{R}^4)$ and $\|\tilde{u}\|_{W^{s,2}(\mathbb{R}^4)} = \|u\|_{W^{s,2}((0,1)^4)}$.

Let $e_h \colon (h\mathbb{Z})^4 \to \mathbb{R}$ be given by $e_h = \tilde{u} - u_h$. Then,

$$\begin{aligned} \Delta_h^2 e_h &= \Delta_h^2 \tilde{u} - \Delta_h^2 u_h = \Delta_h^2 \tilde{u} - T^{h,3,3,3,3} \Delta^2 \tilde{u} && \text{on } \Lambda_h \\ e_h &= 0 && \text{on } (h\mathbb{Z})^4 \setminus \Lambda_h \end{aligned}$$

and by summation by parts we have

$$\|\nabla_h^2 e_h\|^2_{L_h^2((h\mathbb{Z})^4)} = (e_h, \Delta_h^2 e_h)_{L_h^2((h\mathbb{Z})^4)} = (e_h, \Delta_h^2 \tilde{u} - T^{h,3,3,3,3} \Delta^2 \tilde{u})_{L_h^2((h\mathbb{Z})^4)} \,. \tag{4.2.10}$$

We can rewrite $\Delta_h^2 \tilde{u} - T^{h,3,3,3,3} \Delta^2 \tilde{u}$ using (4.2.9) as

$$\begin{aligned} \Delta_h^2 \tilde{u} - T^{h,3,3,3,3} \Delta^2 \tilde{u} &= \sum_{i=1}^4 D_i^h D_{-i}^h \Delta_h \tilde{u} - T^{h,3,3,3,3} \partial_i^2 \Delta \tilde{u} \\ &= \sum_{i=1}^4 D_i^h D_{-i}^h \Delta_h \tilde{u} - D_i^h D_{-i}^h T^{h,3,3,3,3-2e_i} \Delta \tilde{u} \\ &= \sum_{i=1}^4 D_i^h D_{-i}^h g_i \end{aligned}$$

where

$$g_i := \Delta_h \tilde{u} - T^{h,3,3,3,3-2e_i} \Delta \tilde{u} \,.$$

We can insert this into (4.2.10) and use summation-by-parts once again to obtain

$$\begin{aligned} \|\nabla_h^2 e_h\|^2_{L_h^2((h\mathbb{Z})^4)} &= \sum_{i=1}^4 (e_h, D_i^h D_{-i}^h g_i)_{L_h^2((h\mathbb{Z})^4)} \\ &= \sum_{i=1}^4 (D_i^h D_{-i}^h e_h, g_i)_{L_h^2((h\mathbb{Z})^4)} \\ &\le \sum_{i=1}^4 \|g_i\|_{L_h^2((h\mathbb{Z})^4)} \|\nabla_h^2 e_h\|_{L_h^2((h\mathbb{Z})^4)} \end{aligned}$$

and thus

$$\|\nabla_h^2 e_h\|_{L_h^2((h\mathbb{Z})^4)} \le \sum_{i=1}^4 \|g_i\|_{L_h^2((h\mathbb{Z})^4)} \,. \tag{4.2.11}$$

The summands on the right hand side can be bounded using the Bramble–Hilbert lemma (see e.g. [JS14, Theorem 2.28]): As $s > 2$,

$$|\Delta_h \tilde{u}(x)| \le C_h \|\tilde{u}\|_{L^\infty(x+(-3h/2,3h/2)^4)} \le C_{h,s} \|\tilde{u}\|_{H^s(x+(-3h/2,3h/2)^4)} \,.$$

Because $s > \frac{3}{2}$ and $T^{h,3,3,3,3-2e_i} f(x)$ only depends on $f|_{x+(-3h/2,3h/2)^4}$ we can conclude from [JS14, Theorem 1.67] and the locality of $T^{h,3,3,3,3-2e_i}$ that

$$|T^{h,3,3,3,3-2e_i} \Delta \tilde{u}(x)| \le C_{h,s} \|\tilde{u}\|_{H^s(x+(-3h/2,3h/2)^4)} \,.$$

Thus $g_i(x)$ is a bounded linear functional of $\tilde{u} \in W^{s,2}(x + (-3h/2, 3h/2)^4)$. This functional vanishes when $\tilde{u}|_{x+(-3h/2,3h/2)^4}$ is a polynomial of degree at most 2. Indeed, if that is the

case then $\Delta\tilde{u}|_{x+(-3h/2,3h/2)^4}$ is a constant function, and $\Delta_h\tilde{u}(x)$ is equal to the same constant, and the claim follows from the fact that $T_1^{h,3}\dots T_{i-1}^{h,3}T_i^{h,1}T_{i+1}^{h,3}\dots T_4^{h,3}$ maps constant functions to themselves.

We have shown that $g_i(x)$ is a bounded linear functional of $\tilde{u}\in W^{s,2}(x+(-3h/2,3h/2)^4)$ that vanishes on polynomials of degree at most 2. By the Bramble–Hilbert lemma it is bounded by $C_{h,s}[\tilde{u}]_{W^{s,2}(x+(-3h/2,3h/2)^4)}$ for $s\le 3$. Using a scaling argument to determine the correct prefactor of h, we obtain

$$|g_i(x)|\le C_s h^{s-4}[\tilde{u}]_{W^{s,2}(x+(-3h/2,3h/2)^4)}$$

and hence

$$\begin{aligned}\|g_i\|^2_{L^2_h((h\mathbb{Z})^4)} &\le Ch^4\sum_{x\in(h\mathbb{Z})^4} h^{2(s-4)}[\tilde{u}]^2_{W^{s,2}(x+(-3h/2,3h/2)^4)}\\ &\le C_s h^{2(s-2)}[\tilde{u}]^2_{W^{s,2}(\mathbb{R}^4)}\le C_s h^{2(s-2)}\|u\|^2_{W^{s,2}((0,1)^4)}\end{aligned}\tag{4.2.12}$$

for those s. Now we can plug (4.2.12) into (4.2.11) and obtain

$$\|\nabla_h^2 e_h\|_{L^2_h((h\mathbb{Z})^4)}\le C_s h^{s-2}\|u\|_{W^{s,2}((0,1)^4)}$$

for $s<\frac{5}{2}$. Using the discrete Poincaré inequality completes the proof. □

4.3 Estimates for other Green's functions

4.3.1 Estimates for the discrete Green's function of the full space

Our strategy will be to compare G_h with several other Green's functions, so let us introduce these first.

Recall that $\lambda=\sqrt{8}\pi$. Let G be the Green's function of the continuous Bilaplacian on $[0,1]^4$ with Dirichlet boundary data (i.e. of the problem (4.1.3)). We also need Green's functions on the full space. Let $\hat{G}(x,y):=-\frac{1}{\lambda^2}\log|x-y|$. It is easy to check that this is a fundamental solution of the Bilaplacian (i.e. that $\Delta^2\left(-\frac{1}{\lambda^2}\log|\cdot-y|\right)=\delta_y$ in the sense of distributions). We also define $\hat{G}_h\colon (h\mathbb{Z})^4\times(h\mathbb{Z})^4\to\mathbb{R}$ by $\hat{G}_h(x,y)=F\left(\frac{x-y}{h}\right)-\frac{1}{\lambda^2}\log h$ where F is the function introduced in the following lemma. We added the summand $-\frac{1}{\lambda^2}\log h$ here to ensure that $\hat{G}_h$ has the same asymptotic behaviour as $\hat{G}$. We also define shifted versions of $\hat{G}_h$ and $\hat{G}$, namely for $r>0$ we let $\hat{G}^{(r)}=\hat{G}+\frac{\log r}{\lambda^2}$, and $\hat{G}_h^{(r)}=\hat{G}_h+\frac{\log r}{\lambda^2}$. We occasionally write G_y for $G(\cdot,y)$, and define $G_{h,y}$, $\hat{G}_y$, $\hat{G}_{h,y}$, $\hat{G}_y^{(r)}$ and $\hat{G}_{h,y}^{(r)}$ analogously.

Lemma 4.3.1 ([Man67, pp. 96-97]). *There is a function $F\colon\mathbb{Z}^4\to\mathbb{R}$ such that $\Delta_1^2F(x)=\begin{cases}1 & x=0\\ 0 & \text{else}\end{cases}$, satisfying the asymptotics*

$$F(x)=-\frac{1}{8\pi^2}\log|x|+\frac{1}{24\pi^2}\frac{x_1^4+x_2^4+x_3^4+x_4^4}{|x|^6}+O\left(\frac{1}{|x|^4}\right)$$

for $x\neq 0$.

In [Man67], F is defined using the discrete Fourier multiplier associated to Δ_1^2. By expanding that multiplier into a Laurent series and computing the Fourier transform termwise it is possible to give asymptotic expansions to arbitrary high order. This technique also applies to other discrete polyharmonic Green's functions. For our purposes the first two terms quoted above are sufficient.

Lemma 4.3.1 immediately gives us an asymptotic expansion of $\hat{G}_h$, and so we can easily obtain estimates for $\hat{G}_h$ and $\hat{G}_h^{(r)}$.

Lemma 4.3.2. *Let $h > 0$, and $r \geq 192h$. Let $\alpha \in \mathbb{N}^4$ with $|\alpha| \leq 2$. Then for any $x, y \in (h\mathbb{Z})^4$ with $\frac{r}{64} \leq |x-y|_\infty \leq 16r$ we have*

$$\left| \hat{G}_h^{(r)}(x,y) - \frac{1}{\lambda^2} \log\left(\frac{r}{|x-y|+h}\right) \right| \leq C\,, \tag{4.3.1}$$

$$\left| D_\alpha^h \hat{G}_{h,y}^{(r)}(x) \right| \leq \frac{C}{r^{|\alpha|}}\,, \tag{4.3.2}$$

$$\left| D_\alpha^h \hat{G}_{h,y}^{(r)}(x) - \partial^\alpha \hat{G}_y^{(r)}(x) \right| \leq C \frac{h}{r^{|\alpha|+1}}\,. \tag{4.3.3}$$

Proof. By translation invariance we may assume $y = 0$. The definition of $\hat{G}_h^{(r)}$ implies that

$$\begin{aligned}
\hat{G}_h^{(r)}(x,0) &= F\left(\frac{x}{h}\right) - \frac{1}{\lambda^2}\log h + \frac{1}{\lambda^2}\log r \\
&= -\frac{1}{\lambda^2}\log\frac{|x|}{h} + \frac{h^2}{24\pi^2}\frac{x_1^4+x_2^4+x_3^4+x_4^4}{|x|^6} + O\left(\frac{h^4}{|x|^4}\right) - \frac{1}{\lambda^2}\log h + \frac{1}{\lambda^2}\log r \\
&= \frac{1}{\lambda^2}\log\frac{r}{|x|} + \frac{h^2}{24\pi^2}\frac{x_1^4+x_2^4+x_3^4+x_4^4}{|x|^6} + O\left(\frac{h^4}{|x|^4}\right).
\end{aligned} \tag{4.3.4}$$

From this we immediately conclude (4.3.1) in the case $x \neq 0$. In case $x = 0$ we can directly use

$$\hat{G}_h^{(r)}(0,0) = F(0) + \frac{1}{\lambda^2}\log\frac{r}{h}$$

to obtain (4.3.1).

The explicit formula for $\hat{G}$ reveals that

$$\left|\partial^\alpha \hat{G}_0^{(r)}(x)\right| = \left|\partial^\alpha \frac{1}{\lambda^2}\log\frac{r}{|x|}\right| \leq \frac{C}{r^{|\alpha|}}$$

if $\frac{r}{64} \leq |x|_\infty$, and thus (4.3.2) easily follows from (4.3.3).

For (4.3.3) we want to take discrete derivatives of each summand in (4.3.4) separately. If $g = O\left(\frac{h^4}{|\cdot|^4}\right)$ then $|D_\alpha^h g(x)| \leq \frac{C}{h^{|\alpha|}}\frac{h^4}{|x|^4} = C\frac{h^{4-|\alpha|}}{|x|^4}$ so for $|\alpha| \leq 2$ we can neglect the error term. Using Taylor's theorem we can see that

$$D_\alpha^h\left(\frac{1}{\lambda^2}\log\frac{r}{|x|} + \frac{h^2}{24\pi^2}\frac{x_1^4+x_2^4+x_3^4+x_4^4}{|x|^6}\right) = \partial^\alpha\frac{1}{\lambda^2}\log\frac{r}{|x|} + O\left(\frac{h}{|x|^{|\alpha|+1}}\right).$$

Note that we can avoid the singularity here because $|x| \geq \frac{r}{64} \geq 3h$. This easily implies (4.3.3). □

4.3.2 Estimates for continuous Green's functions

We want to compare G and G_h. This is only useful if we also have estimates for G to begin with. We will derive such estimates in this section. The following estimates are far from optimal, but sufficient for our purposes.

We obviously have a well-posedness result for the Bilaplace equation in the energy space $W^{2,2}$. The following result states that the same holds true if we raise the regularity slightly.

Theorem 4.3.3. *There exists $\kappa_0 > 0$ with the following property: Let $0 \le \kappa \le \kappa_0$. Then for each $f \in W^{-2+\kappa,2}((0,1)^4)$ there is a unique $u \in W^{2+\kappa,2} \cap W_0^{2,2}((0,1)^4)$ such that $\Delta^2 u = f$ in the sense of distributions, and we have the estimate*

$$\|u\|_{W^{2+\kappa,2}((0,1)^4)} \le C_\kappa \|f\|_{W^{-2+\kappa,2}((0,1)^4)} \tag{4.3.5}$$

for a constant C_κ depending only on κ.

For convenience we will assume in the following that $\kappa_0 < \frac{1}{2}$, and fix such a κ_0. Note that $W^{2+\kappa,2} \cap W_0^{2,2}((0,1)^4) = W_0^{2+\kappa,2}((0,1)^4)$ if $\kappa < \frac{1}{2}$.

Proof of Theorem 4.3.3. This is a special case e.g. of [MM13, Theorem 6.32], but for the convenience of the reader we give the short argument.

We begin with the case $\kappa = 0$. In that case we can test the weak form of $\Delta^2 u = f$ with u and obtain

$$\|\nabla^2 u\|^2_{L^2((0,1)^4)} = (u, \Delta^2 u)_{L^2((0,1)^4)} = (u, f)_{L^2((0,1)^4)} \le \|u\|_{W^{2,2}((0,1)^4)} \|f\|_{W^{-2,2}((0,1)^4)} .$$

The Poincaré inequality implies $\|u\|_{W^{2,2}((0,1)^4)} \le C\|\nabla^2 u\|_{L^2((0,1)^4)}$ and so we obtain (4.3.5).

For the general case we can use a stability result for analytic families of operators on Banach spaces: The spaces $W^{s,2}((0,1)^4)$ and $W_0^{s,2}((0,1)^4)$ each form an interpolation family with respect to complex interpolation, and so by [TVV88, Proposition 4.1] the set of those s for which $\Delta^2 \colon W_0^{s,2}((0,1)^4) \to W^{s-4,2}((0,1)^4)$ has a bounded inverse is open. We know that this set contains 2, so the existence of κ_0 as in the theorem follows. □

Next we state some estimates for G. We begin by estimating the regular part of G in certain Sobolev norms. Recall that $\hat{G}^{(r)}(x,y) = \hat{G}(x,y) + \frac{\log r}{\lambda^2}$ for any $r > 0$.

Lemma 4.3.4. *Let κ_0 be as in Theorem 4.3.3, and let $0 \le \kappa \le \kappa_0$. Let $K \ge 2$, $r > 0$, $y \in (0,1)^4$ be such that $\frac{d(y)}{K} \le r \le \frac{d(y)}{2}$. Then*

$$\left\| G_y - \eta_y^{(r)} \hat{G}_y^{(r)} \right\|_{W^{2+\kappa,2}((0,1)^4)} \le \frac{C_{K,\kappa}}{r^\kappa} \tag{4.3.6}$$

for a constant $C_{K,\kappa}$ depending only on K and κ.

Proof. Let $H^{(r)} = G_y - \eta_y^{(r)} \hat{G}_y^{(r)}$. By Theorem 4.3.3 it suffices to show

$$\|\Delta^2 H^{(r)}\|_{W^{-2+\kappa,2}((0,1)^4)} \le \frac{C_{K,\kappa}}{r^\kappa} . \tag{4.3.7}$$

By standard interpolation theory and our assumption $\kappa \in [0, \kappa_0] \subset [0,2]$ it suffices to establish this for $\kappa \in \{0, 2\}$.

Observe that $\Delta^2 H^{(r)}$ is zero in $(0,1)^4 \setminus B_r(y)$ as well as in $B_{r/2}(y)$ (as the two singularities cancel out). This means that $\Delta^2 H^{(r)}$ is supported in $B_r(y) \setminus B_{r/2}(y)$ and there it is equal to

$-\Delta^2\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)$. We have an explicit formula for $\hat{G}_y^{(r)}$, and so it is straightforward to check that $\left|\Delta^2\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)\right|$ is bounded by $\frac{C_K}{r^4}$ on $B_r(y)\setminus B_{r/2}(y)$. This easily implies (4.3.7) for $\kappa=2$.

For the case $\kappa=0$ we need to be slightly more careful: Let $\chi_y^{(r)}$ be a cut-off function that is 1 on $B_r(y)\setminus B_{r/2}(y)$ and zero outside $B_{2r}(y)\setminus B_{r/4}(y)$ (e.g. $\chi_y^{(r)}=\eta_y^{(2r)}-\eta_y^{(r/2)}$). Then we have $\Delta^2 H^{(r)}=-\chi_y^{(r)}\Delta^2\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)$ and thus we can calculate

$$\begin{aligned}
\|\Delta^2 H^{(r)}\|_{W^{-2,2}((0,1)^4)} &= \sup_{\|\varphi\|_{W_0^{2,2}((0,1)^4)}=1}\int \Delta^2 H^{(r)}\varphi \\
&= \sup_{\|\varphi\|_{W_0^{2,2}((0,1)^4)}=1}\int -\Delta^2\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)\chi_y^{(r)}\varphi \\
&= \sup_{\|\varphi\|_{W_0^{2,2}((0,1)^4)}=1}\int -\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)\Delta(\chi_y^{(r)}\varphi) \\
&\le C\left\|\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)\right\|_{L^2(B_{2r}(y)\setminus B_{r/4}(y))}\sup_{\|\varphi\|_{W_0^{2,2}((0,1)^4)}=1}\|\Delta(\chi_y^{(r)}\varphi)\|_{L^2((0,1)^4)}\,.
\end{aligned}$$

To estimate the second factor we proceed as in the calculation that led to (4.2.5). We have a Poincaré inequality

$$\|u\|_{L^2(z+(-s,s)^4)}\le Cs\|\nabla u\|_{L^2(z+(-s,s)^4)} \tag{4.3.8}$$

for any $u\in W^{1,2}(z+(-s,s)^4)$ that is zero (in the sense of traces) on one of the faces of $z+(-s,s)^4$. This is the continuous analogue to Lemma 4.2.1, and the proof is very similar. Using (4.3.8) we can estimate

$$\begin{aligned}
\|\Delta(\chi_y^{(r)}\varphi)\|_{L^2((0,1)^4)} &\le C\|\nabla^2\varphi\|_{L^2(B_{d(y)}(y))}+\frac{C}{r}\|\nabla\varphi\|_{L^2(B_{d(y)}(y))}+\frac{C}{r^2}\|\varphi\|_{L^2(B_{d(y)}(y))} \\
&\le C\left(1+\frac{d(y)}{r}+\frac{d(y)^2}{r^2}\right)\|\nabla^2\varphi\|_{L^2(y+(-d(y),d(y))^4)} \\
&\le C_K\|\varphi\|_{W_0^{2,2}((0,1)^4)}\,.
\end{aligned}$$

We also have that $\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)$ is bounded by $\frac{C}{r^2}$ on $B_{2r}(y)\setminus B_{r/4}(y)$ and hence

$$\left\|\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)\right\|_{L^2(B_{2r}(y)\setminus B_{r/4}(y))}\le Cr^2\cdot\frac{1}{r^2}=C\,.$$

Using this we obtain (4.3.7) for $\kappa=0$. □

Next we give some estimates on the local behaviour of G. The first two allow us to control G far from and close to the singularity, respectively, while the last one expresses the Hölder continuity of $G-\hat{G}$ near the diagonal.

Lemma 4.3.5. *Let κ_0 be as in Theorem 4.3.3. Let $y\in(0,1)^4$. The function G_y is smooth on $(0,1)^4\setminus\{y\}$, and $G-\hat{G}$ is symmetric and smooth on $(0,1)^4\times(0,1)^4\setminus\{(x,x)\colon x\in(0,1)^4\}$ and can be extended continuously to $(0,1)^4\times(0,1)^4$. Slightly abusing notation, we write*

$$G(y,y)-\hat{G}(y,y):=\lim_{\substack{(y',y'')\to(y,y)\\ y'\neq y''}}G(y',y'')-\hat{G}(y',y'')\,.$$

Let $K \geq 1$. We have the following estimates, where $\frac{d(y)}{K} \leq r \leq \frac{d(y)}{2}$:

$$|G(x,y)| \leq C \quad \text{if } |x-y| \geq \frac{d(y)}{4}\,, \tag{4.3.9}$$

$$|G(x,y) - \hat{G}^{(r)}(x,y)| \leq C_K \quad \text{if } |x-y| \leq d(y)\,. \tag{4.3.10}$$

Furthermore if $r > 0$ is arbitrary, $|y'-y| \leq \frac{d(y)}{8}$ and $|y''-y| \leq \frac{d(y)}{8}$ we have the estimate

$$\left|G(y',y'') - \hat{G}^{(r)}(y',y'') - \left(G(y,y) - \hat{G}^{(r)}(y,y)\right)\right| \leq C\frac{|y'-y|^{\kappa_0} + |y''-y|^{\kappa_0}}{d(y)^{\kappa_0}}\,. \tag{4.3.11}$$

Proof. The smoothness of G and $G - \hat{G}$ follows from standard regularity theory for higher order elliptic equations. The estimate (4.3.9) is given in [MM14, Theorem 8.1]. There also a variant of (4.3.10) (without the correction $\frac{\log r}{\lambda^2}$ and with slightly worse error term) is given. The results in [MM14] however are in a far more general setting, so we prefer to give an elementary proof of the specific estimates we need.

We use a standard Caccioppoli inequality (see e.g. [Cam80, Capitolo II, Teorema 3.II or Teorema 6.I]): If $u \in W^{2,2}(B_s(z))$ and $\Delta^2 u = 0$ in $B_s(z)$ then

$$\|\nabla^2 u\|_{L^\infty(B_{s/2}(z))} \leq \frac{C}{s^2}\|\nabla^2 u\|_{L^2(B_s(z))}\,. \tag{4.3.12}$$

We will also need a special case of the Gagliardo-Nirenberg interpolation inequality, namely

$$\|u\|_{L^\infty(B_s(z))} \leq C\left(s^2\|\nabla^2 u\|_{L^\infty(B_s(z))} + \frac{1}{s^2}\|u\|_{L^2(B_s(z))}\right)\,. \tag{4.3.13}$$

To see this, observe first that by scaling we can assume $s = 1$. The Poincaré inequality implies that $\|u - a - b\cdot(\cdot - z)\|_{L^\infty(B_1(z))} \leq C\|\nabla^2 u\|_{L^\infty(B_1(z))}$, where $a = \frac{1}{|B_1|}\int u$ and $b = \frac{1}{|B_1|}\int \nabla u$, and so we only have to bound a and b. We have $|a| \leq C\|u\|_{L^2(B_1(z))}$, and the estimate $\|u-a\|_{L^2(B_1(z))} \leq \|u\|_{L^2(B_1(z))}$ implies

$$\begin{aligned}
|b| &\leq C\|b\cdot(\cdot - z)\|_{L^2(B_1(z))} \\
&\leq C\left(\|u - a - b\cdot(\cdot - z)\|_{L^2(B_1(z))} + \|u-a\|_{L^2(B_1(z))}\right) \\
&\leq C\left(\|\nabla^2 u\|_{L^\infty(B_1(z))} + \|u\|_{L^2(B_1(z))}\right)\,.
\end{aligned}$$

This completes the proof of (4.3.13).

After these preparations we can now begin with the proof of (4.3.9). We first assume that $d(x) \leq 2d(y)$. Let $H^{(d(y)/8)} = G_y - \eta_y^{(d(y)/8)}\hat{G}_y^{(d(y)/8)}$. Lemma 4.3.4 with $\kappa = 0$ implies that

$$\|\nabla^2 H^{(d(y)/8)}\|_{L^2((0,1)^4)} \leq C\,. \tag{4.3.14}$$

The function $H^{(d(y)/8)}$ agrees with G_y on $(0,1)^4 \setminus B_{d(y)/8}(y)$. Because $\frac{d(x)}{16} + \frac{d(y)}{8} \leq \frac{d(y)}{4} \leq |x-y|_\infty$ we have $B_{d(x)/16}(x) \cap B_{d(y)/8}(y) = \varnothing$ and thus (4.3.14) implies

$$\|\nabla^2 G_y\|_{L^2(B_{d(x)/16}(x))} \leq C\,.$$

Using the Caccioppoli inequality (4.3.12) we conclude

$$\|\nabla^2 G_y\|_{L^\infty(B_{d(x)/32}(x))} \leq \frac{C}{d(x)^2}\,. \tag{4.3.15}$$

Next, note that the Poincaré inequality (4.3.8) applied on $x + (-d(x), d(x))^4$ and (4.3.14) imply that

$$\|H^{(d(y)/8)}\|_{L^2(B_{d(x)}(x))} \leq Cd(x)^2 \|\nabla^2 H^{(d(y)/8)}\|_{L^2(x+(-d(x),d(x))^4)} \leq Cd(x)^2$$

and therefore

$$\|G_y\|_{L^2(B_{d(x)/32}(x))} \leq Cd(x)^2 .$$

Recalling (4.3.15), an application of (4.3.13) concludes the proof.

It remains to consider the case $d(x) > 2d(y)$. In that case $|x-y| \geq d(x) - d(y) \geq \frac{d(x)}{2}$, so we can interchange the roles of x and y and repeat the above proof (using that $G(x,y) = G(y,x)$).

Next we give a proof of (4.3.10). This is quite similar to the preceding argument. Because $G^{(r)}$ differs from $G^{(d(y))}$ only by at most $\frac{1}{\lambda^2}\log K \leq C_K$ we can assume $r = d(y)$. Let again $H^{(d(y))} = G_y - \eta_y^{(d(y))}\hat{G}_y^{(d(y))}$. Observe first that if $|x-y| \geq \frac{d(y)}{4}$ then (4.3.9) implies (4.3.10). Therefore we can restrict our attention to the case $|x-y| \leq \frac{d(y)}{4}$. By Lemma 4.3.4 we have that

$$\|\nabla^2 H^{(d(y))}\|_{L^2((0,1)^4)} \leq C .$$

The function $H^{(d(y))}$ agrees with $G_y - \hat{G}_y^{(d(y))}$ on $B_{d(y)/2}(y)$. Thus, as before, the Caccioppoli inequality implies that

$$\|\nabla^2(G_y - \hat{G}_y)\|_{L^\infty(B_{d(y)/4}(y))} \leq \frac{C}{d(y)^2}$$

and the Poincaré inequality implies

$$\|G_y - \hat{G}_y\|_{L^2(B_{d(y)/4}(y))} \leq \|H^{(d(y))}\|_{L^2(B_{d(y)}(y))} \leq Cd(y)^2$$

so that the conclusion follows from the interpolation inequality (4.3.13).

For (4.3.11) observe that by Lemma 4.3.4 we control the $W^{2+\kappa_0,2}$-norm of $G_y - \eta_y^{(d(y))}\hat{G}_y^{(d(y))}$. That Sobolev space embeds into the Hölder space C^{0,κ_0} and so we have

$$\left[G_y - \eta_y^{(d(y))}\hat{G}_y^{(d(y))}\right]_{C^{0,\kappa_0}((0,1)^4)} \leq C\left\|G_y - \eta_y^{(d(y))}\hat{G}_y^{(d(y))}\right\|_{W^{2+\kappa_0,2}((0,1)^4)} \leq \frac{C}{d(y)^{\kappa_0}} .$$

Because $G_y - \eta_y^{(d(y))}\hat{G}_y^{(d(y))}$ agrees with $G_y - \hat{G}_y^{(d(y))}$ on $B_{d(y)/2}(y)$ this implies

$$\left|G(y',y) - \hat{G}^{(d(y))}(y',y) - \left(G(y,y) - \hat{G}^{(d(y))}(y,y)\right)\right| \leq C\frac{|y'-y|^{\kappa_0}}{d(y)^{\kappa_0}} .$$

If we add and subtract $\frac{\log r - \log d(y)}{\lambda^2}$ on the left-hand side we obtain

$$\left|G(y',y) - \hat{G}^{(r)}(y',y) - \left(G(y,y) - \hat{G}^{(r)}(y,y)\right)\right| \leq C\frac{|y'-y|^{\kappa_0}}{d(y)^{\kappa_0}} .$$

Similarly we obtain

$$\left|G(y'',y') - \hat{G}^{(r)}(y'',y') - \left(G(y,y') - \hat{G}^{(r)}(y,y')\right)\right| \leq C\frac{|y''-y|^{\kappa_0}}{d(y')^{\kappa_0}}$$

where we used that $d(y') \geq \frac{7}{8}d(y)$ so that $y, y'' \in B_{d(y')/2}(y')$. If we add the last two estimates and use once again that $d(y') \geq \frac{7}{8}d(y)$ we arrive at (4.3.11). □

4.4 Proof of the main theorems

In this section we will finally prove that G_h satisfies (B.0′), (B.1′), (B.2′), (B.3′), which according to Observation 4.1.5 implies Theorem 4.1.4.

Recall that G_h is the Green's function of the discrete Bilaplacian on Λ_h with zero boundary data outside Λ_h, G is the Green's function of the continuous Bilaplacian on $(0,1)^4$ with zero Dirichlet boundary data, and $\hat{G}_h^{(r)}$ and $\hat{G}^{(r)}$ are shifted versions of the discrete and continuous full space Green's function.

The main technical statement used in the proof of Theorem 4.1.4 will be the following.

Lemma 4.4.1. *Let κ_0 be as in Theorem 4.3.3. Let $K \geq 2$, and $r \geq 192h$. Then for all $x, y \in \Lambda_h$ with $\frac{d(y)}{K} \leq r \leq \frac{d(y)}{2}$ we have*

$$\begin{aligned}&\left|\left(G_h(x,y) - \eta_{h,y}^{(r)}(x)\hat{G}_h^{(r)}(x,y)\right) - \left(G(x,y) - \eta_y^{(r)}(x)\hat{G}^{(r)}(x,y)\right)\right| \\ &\leq C_K \frac{h^{\kappa_0}}{r^{\kappa_0}} \sqrt{\log\left(2 + \frac{d(x)}{h}\right)}\,.\end{aligned}$$

This lemma is so useful because it simultaneously provides control over the difference between the discrete and continuous Green's function when x, y are far apart and over the difference of the regular part of the discrete and continuous Green's function when x, y are close.

Proof of Lemma 4.4.1. We define $H_h = G_{h,y} - \eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}$ and $H = G_y - \eta_y^{(r)}\hat{G}_y^{(r)}$. Let $\tilde{H}_h$ be the solution of

$$\begin{aligned}\Delta_h^2 \tilde{H}_h &= T^{h,3,3,3,3}\Delta^2 H \quad \text{in } \Lambda_h \\ \tilde{H}_h &= 0 \quad \text{on } (h\mathbb{Z})^4 \setminus \Lambda_h\,.\end{aligned}$$

Our goal is to estimate $|H_h(x) - H(x)|$. We will estimate $H_h - \tilde{H}_h$ and $\tilde{H}_h - H$ separately.

The estimate of the latter term is straightforward: Using Theorem 4.2.3 and Lemma 4.3.4, we obtain

$$\|\tilde{H}_h - H\|_{W_h^{2,2}((h\mathbb{Z})^4)} \leq C_K h^{\kappa_0} \|H\|_{W^{2+\kappa_0,2}((0,1)^4)} \leq C_K \frac{h^{\kappa_0}}{r^{\kappa_0}}\,.$$

Estimating $H_h - \tilde{H}_h$ is more tedious. Similarly as in the proof of Lemma 4.3.4 we let $\chi_y^{(r)} = \eta_y^{(4r)} - \eta_y^{(r/4)}$ and $\chi_{h,y}^{(r)}$ be the restriction of $\chi_y^{(r)}$ to $(h\mathbb{Z})^4$. Then we have

$$\begin{aligned}\Delta_h^2(H_h - \tilde{H}_h) &= \Delta_h^2\left(G_{h,y} - \eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right) - T^{h,3,3,3,3}\Delta^2\left(G_y - \eta_y^{(r)}\hat{G}_y^{(r)}\right) \\ &= \chi_{h,y}^{(r)}\Delta_h^2\left(G_{h,y} - \eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right) - \chi_{h,y}^{(r)}T^{h,3,3,3,3}\Delta^2\left(G_y - \eta_y^{(r)}\hat{G}_y^{(r)}\right) \\ &= -\chi_{h,y}^{(r)}\Delta_h^2\left(\eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right) + \chi_{h,y}^{(r)}T^{h,3,3,3,3}\Delta^2\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right) \\ &= -\chi_{h,y}^{(r)}\Delta_h^2\left(\eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right) + \sum_{i=1}^{4}\chi_{h,y}^{(r)}T^{h,3,3,3,3}\partial_i^2\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right) \\ &= -\chi_{h,y}^{(r)}\sum_{i=1}^{4}D_i^h D_{-i}^h\left(\Delta_h\left(\eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right) + T^{h,3,3,3,3-2e_i}\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)})\right)\right)\,.\end{aligned} \tag{4.4.1}$$

Because $H_h - \tilde{H}_h$ is supported in Λ_h we have

$$\|\nabla_h^2(H_h - \tilde{H}_h)\|_{L_h^2((h\mathbb{Z})^4)}^2 = \left(\Delta_h^2(H_h - \tilde{H}_h), H_h - \tilde{H}_h\right)_{L_h^2((h\mathbb{Z})^4)}$$

$$\leq \sup_{\substack{\varphi_h=0 \text{ on } (h\mathbb{Z})^4\setminus\Lambda_h \\ \|\varphi_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}=1}} \left(\Delta_h^2(H_h-\tilde{H}_h),\varphi_h\right)_{L_h^2((h\mathbb{Z})^4))} \|H_h-\tilde{H}_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}$$

which together with the Poincaré inequality implies that

$$\|H_h-\tilde{H}_h\|_{W_h^{2,2}((h\mathbb{Z})^4)} \leq C \sup_{\substack{\varphi_h=0 \text{ on } (h\mathbb{Z})^4\setminus\Lambda_h \\ \|\varphi_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}=1}} \left(\Delta_h^2(H_h-\tilde{H}_h),\varphi_h\right)_{L_h^2((h\mathbb{Z})^4))} .$$

Combining this with (4.4.1), and abbreviating $T_i^* := T^{h,3,3,3,3-2e_i}$ we see that

$$\begin{aligned}
&\|H_h-\tilde{H}_h\|_{W_h^{2,2}((h\mathbb{Z})^4)} \\
&\leq C \sup_{\substack{\varphi_h=0 \text{ on } (h\mathbb{Z})^4\setminus\Lambda_h \\ \|\varphi_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}=1}} \sum_{i=1}^4 \left(D_i^h D_{-i}^h \left(-\Delta_h\left(\eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right)+T_i^*\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)\right),\chi_{h,y}^{(r)}\varphi_h\right)_{L_h^2((h\mathbb{Z})^4))} \\
&\leq C \sup_{\substack{\varphi_h=0 \text{ on } (h\mathbb{Z})^4\setminus\Lambda_h \\ \|\varphi_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}=1}} \sum_{i=1}^4 \left(-\Delta_h\left(\eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right)+T_i^*\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right),D_i^h D_{-i}^h\chi_{h,y}^{(r)}\varphi_h\right)_{L_h^2((h\mathbb{Z})^4))} \\
&\leq C\sum_{i=1}^4 \left\|-\Delta_h\left(\eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right)+T_i^*\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)\right\|_{L_h^2(Q_{8r}^h(y)\setminus Q_{r/32}^h(y))} \\
&\quad\times \sup_{\substack{\varphi_h=0 \text{ on } (h\mathbb{Z})^4\setminus\Lambda_h \\ \|\varphi_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}=1}} \left\|\nabla_h^2\left(\chi_{h,y}^{(r)}\varphi_h\right)\right\|_{L_h^2((h\mathbb{Z})^4))} ,
\end{aligned} \tag{4.4.2}$$

where we used that $\chi_{h,y}^{(r)}$ is supported in $B_{4r}(y)\setminus B_{r/8}(y)$ so that the support of $\Delta_h\left(\chi_{h,y}^{(r)}\varphi_h\right)$ is certainly contained in $Q_{8r}^h(y)\setminus Q_{r/32}^h(y)$. The discrete product rule and the Poincaré inequality imply that

$$\begin{aligned}
&\left\|\nabla_h^2\left(\chi_{h,y}^{(r)}\varphi_h\right)\right\|_{L_h^2((h\mathbb{Z})^4))} \\
&\leq C\|\nabla_h^2\varphi_h\|_{L_h^2(Q_{d(y)+h}^h(y))}+\frac{C}{r}\|\nabla_h\varphi_h\|_{L_h^2(Q_{d(y)+h}^h(y))}+\frac{C}{r^2}\|\varphi_h\|_{L_h^2(Q_{d(y)+h}^h(y))} \\
&\leq C\left(1+\frac{d(y)+h}{r}+\frac{(d(y)+h)^2}{r^2}\right)\|\varphi_h\|_{W_h^{2,2}((h\mathbb{Z})^4)} \\
&\leq C_K\|\varphi_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}
\end{aligned}$$

and hence

$$\sup_{\substack{\varphi_h=0 \text{ on } (h\mathbb{Z})^4\setminus\Lambda_h \\ \|\varphi_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}=1}} \left\|\nabla_h^2\left(\chi_{h,y}^{(r)}\varphi_h\right)\right\|_{L_h^2((h\mathbb{Z})^4))} \leq C_K . \tag{4.4.3}$$

Let us now also estimate the first factor in (4.4.2). The operator $T^{h,3,3,3,3-2e_i}$ preserves constant functions. Therefore for any z with $|z-y|_\infty \geq \frac{r}{32}$

$$
\begin{aligned}
&\left(T^{h,3,3,3,3-2e_i}\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)\right)(z)\\
&\quad=\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)(z)+\left(T^{h,3,3,3,3-2e_i}\left(\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)(\cdot)-\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)(x)\right)\right)(z)\\
&\quad=\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)(z)+O\left(h\sup_{z+(-\frac{3h}{2},\frac{3h}{2})}\left|\nabla^3\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)\right|\right)\\
&\quad=\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)(z)+O\left(\frac{h}{r^3}\right),
\end{aligned}
\tag{4.4.4}
$$

where we have used that $\left|T^{h,3,3,3,3-2e_i}f(z)\right|\leq C\sup_{z+(-3h/2,3h/2)}|f|$ in the second step as well as the explicit formula for $\hat{G}^{(r)}(z,y)$ in the third step. From Lemma 4.3.2 and Taylor's theorem we know that for $\frac{r}{64}\leq|z-y|_\infty\leq 16r$

$$
\begin{aligned}
D_\alpha^h\hat{G}_{h,y}^{(r)}(z)&=\partial^\alpha\hat{G}_y^{(r)}(z)+O\left(\frac{h}{r^{|\alpha|+1}}\right),\\
D_\alpha^h\hat{G}_{h,y}^{(r)}(z)&=O\left(\frac{1}{r^{|\alpha|}}\right),\\
D_\alpha^h\eta_{h,y}^{(r)}(z)&=\partial^\alpha\eta_y^{(r)}(z)+O\left(\frac{h}{r^{|\alpha|+1}}\right),\\
D_\alpha^h\eta_{h,y}^{(r)}(z)&=O\left(\frac{1}{r^{|\alpha|}}\right).
\end{aligned}
$$

If we combine these estimates with the discrete product rule we obtain that for any z with $\frac{r}{32}\leq|z-y|_\infty\leq 8r$

$$
\Delta_h\left(\eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right)(z)=\Delta\left(\eta_y^{(r)}\hat{G}_y^{(r)}\right)(z)+O\left(\frac{h}{r^3}\right). \tag{4.4.5}
$$

Combining (4.4.4) and (4.4.5) we find that

$$
\left|-\Delta_h\left(\eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right)+\sum_{i=1}^4 T^{h,3,3,3,3-2e_i}\Delta\left(\eta_y^{(r)}\hat{G}_y\right)\right|\leq C\frac{h}{r^3}
$$

on $Q_{8r}^h(y)\setminus Q_{r/32}^h(y)$ and therefore

$$
\left\|-\Delta_h\left(\eta_{h,y}^{(r)}\hat{G}_{h,y}^{(r)}\right)+\sum_{i=1}^4 T^{h,3,3,3,3-2e_i}\Delta\left(\eta_y^{(r)}\hat{G}_y\right)\right\|_{L_h^2(Q_{8r}^h(y)\setminus Q_{r/32}^h(y))}\leq C\frac{h}{r}.
$$

If we use this result and (4.4.3) in (4.4.2) we see that

$$
\|H_h-\tilde{H}_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}\leq C_K\frac{h}{r}.
$$

In summary,

$$
\|H_h-H\|_{W_h^{2,2}((h\mathbb{Z})^4)}\leq\|H_h-\tilde{H}_h\|_{W_h^{2,2}((h\mathbb{Z})^4)}+\|\tilde{H}_h-H\|_{W_h^{2,2}((h\mathbb{Z})^4)}
$$

$$\leq C_K \left(\frac{h^{\kappa_0}}{r^{\kappa_0}} + \frac{h}{r} \right) \leq C_K \frac{h^{\kappa_0}}{r^{\kappa_0}}$$

because $\frac{h}{r} \leq 1$. Finally, Lemma 4.2.2 allows us to conclude that for any $x \in (h\mathbb{Z})^4$

$$|H_h(x) - H(x)| \leq C_K \sqrt{\log\left(2 + \frac{d(x)}{h}\right)} \|H_h - H\|_{W_h^{2,2}((h\mathbb{Z})^4)} \leq C_K \frac{h^{\kappa_0}}{r^{\kappa_0}} \sqrt{\log\left(2 + \frac{d(x)}{h}\right)}.$$

This completes the proof. □

Before we turn to the proof of Theorem 4.1.4 let us observe that Lemma 4.2.2 already implies an upper bound on $G_h(x, y)$.

Lemma 4.4.2. *For any x, y we have that*

$$|G_h(x, y)| \leq C \sqrt{\log\left(2 + \frac{d(x)}{h}\right) \log\left(2 + \frac{d(y)}{h}\right)}. \tag{4.4.6}$$

Proof. The idea is the same as in the proof of Lemma 2.8.1. We have

$$G_h(x, y) = (G_{h,x}, \delta_{h,y})_{L_h^2((h\mathbb{Z})^4)} = (G_{h,x}, \Delta_h^2 G_{h,y})_{L_h^2((h\mathbb{Z})^4)} = (\nabla_h^2 G_{h,x}, \nabla_h^2 G_{h,y})_{L_h^2((h\mathbb{Z})^4)}.$$

This implies on the one hand

$$|G_h(x, y)| \leq \|\nabla_h^2 G_{h,x}\|_{L_h^2((h\mathbb{Z})^4)} \|\nabla_h^2 G_{h,y}\|_{L_h^2((h\mathbb{Z})^4)} \tag{4.4.7}$$

and on the other hand (by choosing $y = x$) that

$$|G_h(x, x)| = \|\nabla_h^2 G_{h,x}\|^2_{L_h^2((h\mathbb{Z})^4)}.$$

From Lemma 4.2.2 we know that

$$|G_h(x, x)| \leq \sqrt{\log\left(2 + \frac{d(x)}{h}\right)} \|\nabla_h^2 G_{h,x}\|_{L_h^2((h\mathbb{Z})^4)}.$$

Combining the last two estimates we obtain

$$|G_h(x, x)| \leq C \log\left(2 + \frac{d(x)}{h}\right)$$

which is (4.4.6) in the special case $x = y$. For the general case we can use (4.4.7) to see that

$$|G_h(x, y)| \leq \|\nabla_h^2 G_{h,x}\|_{L_h^2((h\mathbb{Z})^4)} \|\nabla_h^2 G_{h,y}\|_{L_h^2((h\mathbb{Z})^4)} \leq C \sqrt{\log\left(2 + \frac{d(x)}{h}\right) \log\left(2 + \frac{d(y)}{h}\right)}.$$

□

Now we can turn to the proof of the main technical result of this chapter, Theorem 4.1.4.

Proof of Theorem 4.1.4. Recall that according to Observation 4.1.5 we actually have to verify (B.0′), (B.1′), (B.2′) and (B.3′).

Step 1: Proof of (B.1′)

Let $x, y \in (h\mathbb{Z})^4$. We can assume w.l.o.g. that $d(x) \leq d(y)$ (else interchange x and

y). If $d(y) < 768h$ we have that $\left|\log\left(2+\frac{\max(d(x),d(y))}{h+|x-y|}\right)\right| \leq C$, and by Lemma 4.4.2 also $|G_h(x,y)| \leq C$, so that (B.1′) holds trivially. Thus we can assume $d(y) \geq 768h$.

Consider first the case $|x-y| \leq \frac{d(y)}{4}$. Then Lemma 4.4.1 with $K=2$, i.e. $r = \frac{d(y)}{2} \geq 192h$ implies

$$\begin{aligned}&\left|G_h(x,y) - \eta_{h,y}^{(d(y)/2)}(x)\hat{G}_h^{(d(y)/2)}(x,y) - G(x,y) + \eta_y^{(d(y)/2)}(x)\hat{G}^{(d(y)/2)}(x,y)\right| \\ &\quad \leq C\frac{h^{\kappa_0}}{r^{\kappa_0}}\sqrt{\log\left(2+\frac{d(x)}{h}\right)}\end{aligned}$$

which implies that

$$\left|G_h(x,y) - \hat{G}_h^{(d(y)/2)}(x,y) - G(x,y) + \hat{G}^{(d(y)/2)}(x,y)\right| \leq C\frac{h^{\kappa_0}}{r^{\kappa_0}}\sqrt{\log\left(2+\frac{2r}{h}\right)}\,.$$

The function $s \mapsto \frac{1}{s^{\kappa_0}}\sqrt{\log(2+2s)}$ is bounded on $[1,\infty)$, so that we actually obtain

$$\left|G_h(x,y) - \hat{G}_h^{(d(y)/2)}(x,y) - G(x,y) + \hat{G}^{(d(y)/2)}(x,y)\right| \leq C\,. \tag{4.4.8}$$

From Lemma 4.3.2 we know

$$\left|\hat{G}_h^{(d(y)/2)}(x,y) - \frac{1}{\lambda^2}\log\left(\frac{d(y)}{|x-y|+h}\right)\right| \leq C$$

(where we have absorbed a term $\frac{1}{\lambda^2}\log 2$ into the constant). Furthermore by Lemma 4.3.5

$$\left|G(x,y) - \hat{G}^{(d(y)/2)}(x,y)\right| \leq C\,.$$

If we use these estimates in (4.4.8) we obtain

$$\left|G_h(x,y) - \frac{1}{\lambda^2}\log\left(\frac{d(y)}{|x-y|+h}\right)\right| \leq C\,.$$

Because $|x-y| \leq \frac{d(y)}{4}$, $\frac{d(y)}{|x-y|+h}$ is bounded away from 1 by a constant, and so

$$\left|\frac{1}{\lambda^2}\log\left(\frac{d(y)}{|x-y|+h}\right) - \frac{1}{\lambda^2}\log\left(2+\frac{d(y)}{|x-y|+h}\right)\right| \leq C\,.$$

Combining this with the preceding inequality we arrive at (B.1′).

If $|x-y| \geq \frac{d(y)}{4}$ we argue similarly. We use Lemma 4.4.1 with $r = \frac{d(y)}{4} \geq 192h$ and conclude

$$|G_h(x,y) - G(x,y)| \leq C\,.$$

This combined with Lemma 4.3.5 implies again (B.1′), as now $\frac{d(y)}{|x-y|+h}$ is bounded above.

Step 2: Proof of (B.2′)

Recall from Lemma 4.3.5 that the term $a(x) := \lambda^2 \lim_{\substack{(x',x'')\to(x,x)\\ x'\neq x''}}(G(x',x'') - \hat{G}(x',x''))$ is well-defined for each $x \in (0,1)^4$ and that $a\colon (0,1)^4 \to \mathbb{R}$ is continuous.

After this remark we can proceed similarly as in the first step. We choose $f_1(x) = a(x)$, $f_2(u,v) = \lambda^2 F(u-v)$ with the F from Lemma 4.3.1. Furthermore we choose $\theta_0 = \frac{1}{2\kappa_0}$. Given L and $\theta > \theta_0$ we take N_0' so large that $768L \leq |\log h|^\theta$ when $h \leq \frac{1}{N_0'}$. Then $d(x) \geq$

$h|\log h|^\theta \geq 768Lh$. We want to apply Lemma 4.4.1 with $K = 8$ and $r = \frac{d(x)}{4}$ at the point $(x+hu, x+hv)$. We have that $r = \frac{d(x)}{4} \leq \frac{d(x+hv)+Lh}{4} \leq \frac{d(x+hv)}{2}$ and similarly $r \geq \frac{d(x+hv)}{8}$, and also $r = \frac{d(x)}{4} \geq 192Lh \geq 192h$ so that all assumptions of the lemma are satisfied. We obtain

$$\begin{aligned}
&\Big| G_h(x+hu, x+hv) - \hat{G}_h^{(d(x)/4)}(x+hu, x+hv) \\
&\qquad - G(x+hu, x+hv) + \hat{G}^{(d(x)/4)}(x+hu, x+hv) \Big| \\
&\leq C\frac{h^{\kappa_0}}{r^{\kappa_0}}\sqrt{\log\left(2+\frac{d(x+hu)}{h}\right)} \leq C\frac{h^{\kappa_0}\sqrt{|\log h|}}{r^{\kappa_0}} \leq C\frac{h^{\kappa_0}\sqrt{|\log h|}}{(h|\log h|^\theta)^{\kappa_0}} \leq C|\log h|^{\frac{1}{2}-\theta\kappa_0}.
\end{aligned} \tag{4.4.9}$$

Here we could omit the cut-off functions $\eta_h^{(d(x)/4)}$ and $\eta^{(d(x)/4)}$ because $|x+hu-(x+hv)| \leq 4Lh \leq \frac{d(x)}{8}$. Since $\theta\kappa_0 > \theta_0\kappa_0 = \frac{1}{2}$, for N_0' large enough the term on the right hand side will be less than $\frac{\varepsilon}{2\lambda^2}$ whenever $h \leq \frac{1}{N_0'}$.

By (4.3.11) in Lemma 4.3.5 we have for $u, v \in [0, L]^4$

$$\begin{aligned}
&\left| G(x+hu, x+hv) - \hat{G}^{(d(x)/4)}(x+hu, x+hv) - \frac{a(x)}{\lambda^2} - \frac{1}{\lambda^2}\log\frac{d(x)}{4} \right| \\
&\leq C\left(\frac{|hu|^{\kappa_0}+|hv|^{\kappa_0}}{d(x)^{\kappa_0}}\right) \leq C_L\frac{h^{\kappa_0}}{d(x)^{\kappa_0}} \leq C_L|\log h|^{-\theta\kappa_0}.
\end{aligned}$$

Thus we can choose N_0' large enough such that for $h \leq \frac{1}{N_0'}$ we have

$$\sup_{u,v\in[0,L]^4\cap\mathbb{Z}^4} \left| G(x+hu, x+hv) - \hat{G}^{(d(x)/4)}(x+hu, x+hv) - \frac{a(x)}{\lambda^2} - \frac{1}{\lambda^2}\log\frac{d(x)}{4} \right| \leq \frac{\varepsilon}{2\lambda^2}$$

uniformly in x. Our definition of $G_h^{(d(x)/4)}$ implies that

$$\begin{aligned}
\hat{G}_h^{(d(x)/4)}(x+hu, x+hv) &= F\left(\frac{x+hu}{h} - \frac{x+hv}{h}\right) - \frac{1}{\lambda^2}\log h + \frac{1}{\lambda^2}\log\frac{d(x)}{4} \\
&= F(u-v) - \frac{1}{\lambda^2}\log h + \frac{1}{\lambda^2}\log\frac{d(x)}{4}.
\end{aligned}$$

Using these results in (4.4.9) we arrive at

$$\left| G_h(x+hu, x+hv) - F(u-v) + \frac{1}{\lambda^2}\log h - \frac{a(x)}{\lambda^2} \right| \leq \frac{\varepsilon}{\lambda^2}$$

for $h \leq \frac{1}{N_0'}$, which implies (B.2′).

Step 3: Proof of (B.3′)

This is very similar to Step 2. We set $f_3(x,y) = \lambda^2 G(x,y)$, which is continuous away from the diagonal according to Lemma 4.3.5.

We use Lemma 4.4.1 with $K = L$ and $r = \frac{d(y)}{L} \leq \frac{1}{L} \leq |x-y|$. For N_1' large enough we have $r \geq 192h$, and the lemma implies

$$|G_h(x,y) - G(x,y)| \leq C_L\frac{h^{\kappa_0}\sqrt{|\log h|}}{r^{\kappa_0}} \leq C_L|\log h|^{\frac{1}{2}-\theta\kappa_0}$$

and it suffices to take N_1' so large that the right hand side is less than $\frac{\varepsilon}{\lambda^2}$ for any $h \leq \frac{1}{N_1'}$.

Step 4: Proof of (B.0′)

Here we actually need to prove three estimates, namely

$$\lambda^2 G_h(x,x) \leq |\log h| + C \tag{4.4.10}$$

$$\lambda^2 G_h(x,x) \leq C \log\left(2 + \frac{d(x)}{h}\right) \tag{4.4.11}$$

$$\lambda^2 (G_h(x,x) - G_h(x,y)) \leq \log\left(1 + \frac{|x-y|}{h}\right) + C\,. \tag{4.4.12}$$

Now (4.4.10) follows immediately from (B.1′), and (4.4.11) is a special case of Lemma 4.4.2. Finally, (4.4.12) can be obtained from (B.1′) as follows. We know that

$$\begin{aligned}\lambda^2 (G_h(x,x) - G_h(x,y)) &\leq \log\left(2 + \frac{d(x)}{h}\right) - \log\left(2 + \frac{\max(d(x),d(y))}{h + |x-y|}\right) + C \\ &= \log\left(\frac{(d(x)+2h)(|x-y|+h)}{h(h+|x-y|+2\max(d(x),d(y)))}\right) + C\end{aligned}$$

so one only has to observe that

$$\frac{d(x)+2h}{h+|x-y|+2\max(d(x),d(y))} \leq C\,.$$

□

Finally we give the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. Because of Theorem 4.1.3 and Observation 4.1.5 all we have to check is that each of the statements (A.0′), (A.1′), (A.2′), (A.3′) implies its counterpart without the prime.

We begin with (A.0′) $\Longrightarrow$ (A.0). We know that

$$\operatorname{Var} \varphi_{N,v} \leq \min\left(\log N + \alpha_0', \alpha_0' \log(2 + d_N(v))\right)$$

and this implies in particular that

$$\operatorname{Var} \varphi_{N,v} \leq \log N + \alpha_0'\,.$$

Furthermore, if we know

$$\operatorname{Var} \varphi_{N,v} - \operatorname{Cov}(\varphi_{N,v}, \varphi_{N,u}) \leq \log_+ |u-v| + 2\alpha_0'$$

then by symmetry this also holds with u, v interchanged, so that we actually have

$$\max\left(\operatorname{Var} \varphi_{N,v} - \operatorname{Cov}(\varphi_{N,v}, \varphi_{N,u}), \operatorname{Var} \varphi_{N,u} - \operatorname{Cov}(\varphi_{N,v}, \varphi_{N,u})\right) \leq \log_+ |u-v| + 2\alpha_0'$$

and a short calculation shows that this is the same as

$$\mathbb{E}(\varphi_{N,v} - \varphi_{N,u})^2 \leq 2\log_+ |u-v| - |\operatorname{Var} \varphi_{N,v} - \operatorname{Var} \varphi_{N,u}| + C\,.$$

For (A.1′) $\Longrightarrow$ (A.1) one has to verify that $\min(d(u), d(v)) \geq \delta N$ implies

$$\left|\log\left(2 + \frac{\max(d_N(u), d_N(v))}{1+|u-v|}\right) - \log\left(\frac{N}{1+|u-v|}\right)\right| \leq C_\delta\,,$$

which is straightforward.

For (A.2′) $\Longrightarrow$ (A.2) we fix some $\theta > \theta_0$. Given L, ε, δ, we choose $N_0 \geq N_0'(L, \varepsilon, \theta)$ large enough such that $|\log N|^\theta \leq \delta N$ for all $N \geq N_0$ and conclude (A.2). Analogously one sees that (A.3′) $\Longrightarrow$ (A.3).

□

5 Optimal order finite difference approximation of generalized solutions to the biharmonic equation in a cube

This chapter is based on the paper [MSS20], written jointly by Stefan Müller, Endre Süli and the author, with only minor changes.

5.1 Introduction

In this chapter we show some error estimates for finite difference schemes for the Bilaplacian. In Section 1.4.4 we gave some background and described some of the results. We begin by stating our results in detail.

5.1.1 Main results

We mostly follow the notation from the introduction (see, however, Section 5.1.3 for the precise definitions). Let $\mathsf{d} \in \mathbb{N}^+$, $\Omega := (0,1)^{\mathsf{d}}$, $\Gamma := \partial\Omega$. For $h \in \mathbb{R}^+$ such that $\frac{1}{h} \in \mathbb{N}$, let $\Lambda_h := \Omega \cap (h\mathbb{Z})^{\mathsf{d}}$, $\Gamma_h := \Gamma \cap (h\mathbb{Z})^{\mathsf{d}}$, and

$$\tilde{\Lambda}_h := [-h, 1+h]^{\mathsf{d}} \cap (h\mathbb{Z})^{\mathsf{d}} \setminus \{-h, 1+h\}^{\mathsf{d}} .$$

Consider the elliptic boundary-value problem

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } \Omega , \\ u &= 0 \quad \text{on } \Gamma , \\ \partial_\nu u &= 0 \quad \text{on } \Gamma , \end{aligned} \tag{5.1.1}$$

where ∂_ν denotes the derivative in the normal direction (ν is a unit outward normal vector to Γ). We approximate the solution of this problem by the finite difference scheme (compare [JS14, Section 1.9.4])

$$\begin{aligned} \Delta_h^2 U &= T^{h,2,\dots,2} f \quad \text{in } \Lambda_h , \\ U &= 0 \quad \text{on } \Gamma_h , \\ D_{0,\nu}^h U &= 0 \quad \text{on } \Gamma_h . \end{aligned} \tag{5.1.2}$$

Here U is defined on $\tilde{\Lambda}_h$, $D_{0,\nu}^h U(x) := \frac{1}{2h}\big(U(x+h\nu) - U(x-h\nu)\big)$ [1], and $T^{h,2,\dots,2} f$ is a smoothing operator acting on f, defined by convolving f with a B-spline on the scale h (see below for the precise definition).

[1] At the singular points (i.e., at the vertices and points on the faces/edges) of Γ_h there are up to d possible boundary normal vectors. For (5.1.2) we consider all of them. Because $U = 0$ on Γ_h by assumption, this corresponds to setting $U = 0$ at all points of $\tilde{\Lambda}_h \setminus (\Lambda_h \cup \Gamma_h)$ that have distance h to a singular point of Γ_h.

The finite difference scheme (5.1.2) makes sense in any dimension d, as the smoothing operator $T^{h,2,\dots,2}$ maps $f \in H^{s-4}$ into a continuous function whenever $s > \frac{5}{2}$ (cf. [JS14, Theorem 1.69]).

Our objective is to prove an error bound in the discrete Sobolev norm $\|\cdot\|_{H_h^2(\Lambda_h)}$ (which is denoted by $\|\cdot\|_{W_2^2(\Lambda_h)}$ in [JS14, Section 2.2.4]).

Theorem 5.1.1. *Suppose that $\frac{1}{2}\max(5,\mathsf{d}) < s \le 4$, and let $u \in H^s(\Omega) \cap H_0^2(\Omega)$; then, there exists a positive constant $C = C(n,s)$, independent of h, such that*

$$\|u - U\|_{H_h^2(\Lambda_h)} \le Ch^{s-2}\|u\|_{H^s(\Omega)}\,. \tag{5.1.3}$$

This improves [JS14, Theorem 2.69] (where the above result was proved for $\mathsf{d} = 2$ and $\frac{5}{2} < s \le 4$ in the more general setting of fourth-order elliptic equations with nonsmooth variable coefficients, but the order of convergence $O(h^{\min\{s-2,3/2\}}|\log h|^{1-|\mathrm{sgn}(s-7/2)|})$ established there was optimal only in the case of $\frac{5}{2} < s < \frac{7}{2}$, and is reduced to the suboptimal rate of $O(h^{\frac{3}{2}})$, instead of the optimal rate of $O(h^{s-2})$, for $\frac{7}{2} < s \le 4$) as well as the main result in [GMP83] (where the theorem was proved for $\mathsf{d} = 2$ under the additional assumption that the third normal derivative of u vanishes at the boundary).

Our method also yields estimates for other discretizations of the boundary conditions. Consider, for instance, the finite difference scheme

$$\begin{aligned} \Delta_h^2 U^* &= T^{h,2,\dots,2} f && \text{in } \Lambda_h\,,\\ U^* &= 0 && \text{on } \Gamma_h\,,\\ D_\nu^h U^* &= 0 && \text{on } \Gamma_h\,. \end{aligned} \tag{5.1.4}$$

Here again U^* is defined on $\tilde{\Lambda}_h$, and $D_\nu^h U^*(x) := \frac{1}{h}\big(U^*(x+h\nu) - U^*(x)\big)$. The conditions $U^* = 0$ and $D_\nu^h U^* = 0$ on Γ_h are equivalent to $U^* = 0$ on $\tilde{\Lambda}_h \setminus \Lambda_h$, so that we could equivalently consider the finite difference scheme

$$\begin{aligned} \Delta_h^2 U^* &= T^{h,2,\dots,2} f && \text{in } \Lambda_h\,,\\ U^* &= 0 && \text{on } \tilde{\Lambda}_h \setminus \Lambda_h\,. \end{aligned} \tag{5.1.5}$$

For this difference scheme we can show the following error bound.

Theorem 5.1.2. *Suppose that $\frac{1}{2}\max(5,\mathsf{d}) < s \le 3$, and let $u \in H^s(\Omega) \cap H_0^2(\Omega)$; then, there exists a positive constant $C = C(n,s)$, independent of h, such that*

$$\|u - U^*\|_{H_h^2(\Lambda_h)} \le Ch^{s-2}\|u\|_{H^s(\Omega)}\,. \tag{5.1.6}$$

In Theorems 5.1.1 and 5.1.2 we made the assumption $\frac{1}{2}\max(5,\mathsf{d}) < s$. In view of the fact that the problem (5.1.1) makes sense already for $s > \frac{3}{2}$, the requirement $\frac{1}{2}\max(5,\mathsf{d}) < s$ might seem surprising. The condition $s > \frac{n}{2}$ ensures that u is continuous. Otherwise, $\|u - U\|_{H_h^2(\Lambda_h)}$ and $\|u - U^*\|_{H_h^2(\Lambda_h)}$ would be undefined. The condition $s > \frac{5}{2}$ implies that $T^{h,2,\dots,2} f$ is continuous so that its pointwise values are defined and the finite difference schemes (5.1.2) and (5.1.4) make sense. It should be possible to relax the assumption $s > \frac{n}{2}$ by replacing u in the expressions $\|u - U\|_{H_h^2(\Lambda_h)}$ and $\|u - U^*\|_{H_h^2(\Lambda_h)}$ with a suitably mollified version of u. Similarly, one can relax the assumption $s > \frac{5}{2}$ by replacing $T^{h,2,\dots,2}$ with a stronger mollification operator; see also Remark 5.4.3 for additional comments in this direction.

Our results extend to more general fourth-order elliptic elliptic operators with variable coefficients, such as those treated in [JS14, Section 2.7], with similar proofs. The main difference compared to the analysis here is that in addition to terms appearing in our error bounds one encounters a variety of mixed terms. On can deal with these as in the proof of Theorem 2.68 in [JS14], using the bilinear Bramble–Hilbert lemma. It should also be possible to extend our results to other (higher order) elliptic operators, such as the polyharmonic operator Δ^k for $k \geq 3$, but the study of that question is beyond the scope of this chapter.

5.1.2 Outline of the proof

We discuss the proof of Theorem 5.1.1 only; the proof of Theorem 5.1.2 is very similar. We proceed similarly to the proof of [JS14, Theorem 2.69]. In fact, when $s < \frac{7}{2}$ we could directly use the argument in [JS14] with only minor notational changes. Let us review that argument here briefly. We begin by extending u symmetrically across Γ to a H^s-function $\hat{u}$ on $(-1,2)^{\mathsf{d}}$ such that $\|\hat{u}\|_{H^s((-1,2)^{\mathsf{d}})} \leq C\|u\|_{H^s(\Omega)}$. Here and henceforth C signifies a generic positive constant, which may depend on the Sobolev index s and on the number of space dimensions d, but is independent of the discretization parameter h. Let $E := \hat{u} - U$. Then, E satisfies

$$\begin{aligned} E &= 0 \qquad \text{on } \Gamma_h\,, \\ D^h_{0,\nu} E &= 0 \qquad \text{on } \Gamma_h\,, \end{aligned}$$

and we calculate (compare [JS14, Equation (2.209)])

$$\Delta_h^2 E = \Delta_h^2 \hat{u} - \Delta_h^2 U = \Delta_h^2 \hat{u} - T^{h,2,\dots,2} f = \Delta_h^2 \hat{u} - T^{h,2,\dots,2} \Delta^2 \hat{u}\,.$$

Using summations by parts we obtain

$$\|\nabla_h^2 E\|_{L^2_h(\Lambda_h)} \leq \|\Delta_h^2 \hat{u} - T^{h,2,\dots,2} \Delta^2 \hat{u}\|_{H_h^{-2}(\Lambda_h)}\,,$$

where ∇_h^2 is the discrete Hessian. Now one can use the Bramble–Hilbert lemma (cf. [JS14]) to deduce that the right-hand side is bounded by $Ch^{s-2}\|u\|_{H^s(\Omega)}$, which directly implies (5.1.3).

When $s \geq \frac{7}{2}$ one can no longer extend u symmetrically across the boundary while preserving its Sobolev regularity. This means that we cannot make $D^h_{0,\nu} u$ equal to 0 on Γ, and therefore the above argument based on summation by parts no longer works.

Our alternative approach is as follows. Although we cannot ensure that the boundary values of $D^h_{0,\nu} E$ are exactly zero, we will show that they can nevertheless be made small in an appropriate norm. To this end, we will first show (in Section 5.2.1) that we can take a slightly different extension $\tilde{u}$ with $\|\tilde{u}\|_{H^s((-1,2)^{\mathsf{d}})} \leq C\|u\|_{H^s(\Omega)}$, $\frac{7}{2} \leq s \leq 4$, such that $\tilde{u}$ and its derivatives vanish on the hyperplanes supporting the faces of Γ.

This will allow us to estimate the boundary values in an optimal space. In fact, in Section 5.2.2 we prove that

$$\|D^h_{0,\nu} \tilde{u}\|_{H_h^{\frac{1}{2}}(\Gamma_h)} \leq Ch^{s-2}\|\tilde{u}\|_{H^s(\Omega)}\,. \tag{5.1.7}$$

Actually, we only control the $H_h^{\frac{1}{2}}$-norm on each of the faces of Γ_h, but we ignore this issue here for the sake of simplicity and refer the reader to Section 5.2.2 for precise statements.

Then, in Section 5.2.3, we show that (5.1.7) implies the existence of a function $\hat{E}$ that agrees with $\tilde{u}$ on $\tilde{\Lambda}_h \setminus \Lambda_h$ and such that $\|\nabla_h^2 \hat{E}\|_{L^2(\Lambda_h)} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}$. We shall construct $\hat{E}$ by giving an explicit extension using the Fourier series representation of the boundary values,

which we then carefully cut off to comply with the boundary conditions. This is a special case of an inverse trace theorem in the following sense: for any function ψ on the boundary there is a lattice function w such that

$$\begin{aligned} w &= 0 \quad \text{on } \Gamma_h\,, \\ D^h_{0,\nu} w &= \psi \quad \text{on } \Gamma_h\,, \end{aligned}$$

and

$$\|\nabla_h^2 w\|_{L_h^2(\Lambda_h)} \leq C\|\psi\|_{H_h^{\frac{1}{2}}(\Gamma_h)}\,. \tag{5.1.8}$$

Now that we have $\hat{E}$ at our disposal, we can apply the argument formulated at the beginning of this subsection to $E - \hat{E}$ (which has zero boundary values) and find that

$$\|\nabla_h^2(E-\hat{E})\|_{L^2(\Lambda_h)} \leq C\left(h^{s-2}\|u\|_{H^s(\Omega)} + \|\Delta_h^2\hat{E}\|_{H_h^{-2}(\Lambda_h)}\right)\,.$$

Thus, by observing that

$$\|\Delta_h^2\hat{E}\|_{H_h^{-2}(\Lambda_h)} \leq C\|\nabla_h^2\hat{E}\|_{L^2(\Lambda_h)} \leq C\|D^h_{0,\nu}\tilde{u}\|_{H_h^{\frac{1}{2}}(\Gamma_h)} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}\,,$$

we directly deduce (5.1.3). The details for this argument are given in Section 5.3.2.

The heart of the matter, resulting in our main result stated in Theorem 5.1.1, are the trace estimate (5.1.7) and the inverse trace estimate (5.1.8), established in Section 5.2.2 and Section 5.2.3, respectively.

5.1.3 Notation and preliminaries

Our notation is based on that in [JS14], however we made some changes that we will review in the following.

For $s \geq 0$ and $\Xi \subset \mathbb{R}^{\mathsf{d}}$ open with Lipschitz boundary we define the Sobolev space $H^s(\Xi)$ as the space of restrictions of $H^s(\mathbb{R}^{\mathsf{d}})$-functions to Ξ. By $H_0^s(\Xi)$ we denote the closure of the set of all $C_c^\infty(\Xi)$-functions in the $\|\cdot\|_{H^s(\Xi)}$-norm.

Assume that $\Xi := I_1 \times \cdots \times I_{\mathsf{d}}$, where $I_j \subset \mathbb{R}$ are (possibly unbounded) open intervals. This assumption ensures that we have $\mathcal{H}^{\mathsf{d}-1}$-almost everywhere on $\partial\Xi$ an axiparallel normal vector. Given a $k \in \mathbb{N}_0$ with $k + \frac{1}{2} < s$, we denote by $H^s_{(k)}(\Xi)$ the space of all functions $u \in H^s(\Xi)$ such that the traces of $\partial_\nu^i u$ for $0 \leq i \leq k$ vanish on each face of $\partial\Xi$. We extend this definition to $k > s - \frac{1}{2}$, provided $s \notin \mathbb{N} + \frac{1}{2}$, by setting $H^s_{(k)}(\Xi) = H^s_{(\lfloor s-1/2 \rfloor)}(\Xi)$.

There are several other equivalent definitions of $H^s_{(k)}(\Xi)$. Let $C_c^\infty(\overline{\Xi})$ denote the space of functions on Ξ, which are in $C^\infty(\Xi)$, for which all derivatives admit continuous extensions to $\overline{\Xi}$, and which are supported in $K \cap \Xi$ for some $K \subset \mathbb{R}^{\mathsf{d}}$ compact. In other words, $C_c^\infty(\overline{\Xi})$ denotes the set of restrictions of $C_c^\infty(\mathbb{R}^{\mathsf{d}})$-functions to Ξ, where the equivalence follows from Whitney's extension theorem [Whi34]. Then, $H^s_{(k)}(\Xi)$ is also the closure in the $H^s(\Xi)$-norm of the set of all functions in $C_c^\infty(\overline{\Xi})$ whose derivatives up to order k vanish on $\partial\Xi$ Furthermore, $H^s_{(k)}(\Xi)$ is equal to $H^s(\Xi) \cap H_0^{k+1}(\Xi)$ if $s \geq k+1$, and equal to $H_0^s(\Xi)$ if $s \leq k+1$. In particular, the space $H^s(\Omega) \cap H_0^2(\Omega)$ from the main theorems can now be written as $H^s_{(1)}(\Omega)$.

The fact that these definitions are equivalent should not be surprising. Nonetheless we could not locate a reference for this precise equivalence result, and so we present its proof in Appendix 5.4.2.

Given a $j \in \mathbb{N}$, we let θ_j be the standard univariate centered B-spline of degree $j-1$, defined, for example, as the indicator function of the closed interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ convolved with itself $j-1$ times (cf. [JS14, Section 1.9.4]). Using this, we define the smoothing operator $T_i^{h,j}$ for $1 \leq i \leq \mathsf{d}$ as

$$T_i^{h,j} f := \frac{1}{h} f *_i \theta_j \left(\frac{\cdot}{h}\right),$$

where $*_i$ means convolution in the variable x_i. This is a well-defined operator on distributions on $\mathbb{R}^{\mathsf{d}}$. Furthermore, we set

$$T^{h,j,\dots,j} f := T_1^{h,j} \circ \cdots \circ T_{\mathsf{d}}^{h,j} f .$$

Each θ_j is in $H^t(\mathbb{R})$ for any $t < j - \frac{1}{2}$. Using this, one can verify (cf. [JS14, Section 1.9.4]) that $T^{h,j,\dots,j}$ is a bounded linear operator from $H^t(\mathbb{R}^{\mathsf{d}})$ to $C_b(\mathbb{R}^{\mathsf{d}})$ whenever $t > -j + \frac{1}{2}$.

We define the discrete Sobolev norm $\|v\|_{H_h^2(\Lambda_h)}$ of $v\colon \tilde{\Lambda}_h \to \mathbb{R}$ as the sum of the L_h^2-norms of v, $\nabla_h v$ and $\nabla_h^2 v$, wherever they are defined; more precisely,

$$\begin{aligned}\|v\|_{H_h^2}^2 := \sum_{x \in \tilde{\Omega}_h} h^{\mathsf{d}} v(x)^2 + \sum_{i=1}^{\mathsf{d}} \sum_{\substack{x \in \tilde{\Omega}_h: \\ x + he_i \in \tilde{\Omega}_h}} h^{\mathsf{d}} (D_i^h v(x))^2 \\ + \sum_{i,j=1}^{\mathsf{d}} \sum_{\substack{x \in \tilde{\Omega}_h: \\ x+he_i, x-he_j, x+he_i-he_j \in \tilde{\Omega}_h}} h^{\mathsf{d}} (D_i^h D_{-j}^h v(x))^2 .\end{aligned}$$

Note that we have the crucial property

$$D_i^h D_{-i}^h T_i^{h,j-2} f = T_i^{h,j} \partial_i^2 f \tag{5.1.9}$$

for any i and any $j \geq 2$.

5.2 Discrete trace and inverse trace theorems

5.2.1 Construction of a good extension

Recall that $H_{(1)}^s(\Omega)$ denotes the space of functions $u \in H^s(\Omega)$ for which u and ∇u vanish on $\partial\Omega$. Our first goal is to construct an extension $\tilde{u}$ of u that preserves its Sobolev regularity and has the additional property that $\tilde{u}$ and $\nabla \tilde{u}$ vanish on the hyperplanes supporting the faces of Ω.

Later in our argument it will be necessary to localize the functions concerned in order to deal with the 2^{d} corners of $\Omega = (0,1)^{\mathsf{d}}$ separately. Actually, it is most convenient to do so right from the start. Thus we shall use a partition of unity, which allows us to split u into 2^{d} parts localized near the corners. These parts can all be dealt with in a similar way, so we focus on one of them and assume that u is supported in $\left[0, \frac{2}{3}\right)^{\mathsf{d}}$.

Lemma 5.2.1. *Let $\frac{3}{2} < s \leq 4$, let $u \in H_{(1)}^s(\Omega)$ be supported in $\left[0, \frac{2}{3}\right)^{\mathsf{d}}$. Then, there exists a function $\tilde{u} \in H_0^s((-1,1)^{\mathsf{d}})$ such that $\|\tilde{u}\|_{H^s((-1,1)^{\mathsf{d}})} \leq C\|u\|_{H^s(\Omega)}$, $\tilde{u}_{|\Omega} = u$, and $\tilde{u} = 0$, $\nabla \tilde{u} = 0$ on the $(\mathsf{d}-1)$-dimensional hyperplanes $x_i = 0$ for $i \in \{1, \dots, n\}$ in the sense of traces.*

Because $\tilde{u} \in H_0^s((-1,1)^{\mathsf{d}})$, we can extend $\tilde{u}$ outside $(-1,1)^{\mathsf{d}}$ by zero to a function in $H^s(\mathbb{R}^{\mathsf{d}})$ (that we continue to call $\tilde{u}$).

The construction of the extension is classical; see, e.g., [LM72a, Section 11.5]. Nonetheless we give some details, in particular because a similar construction will be used in Section 5.2.3.

Proof. We proceed by applying an extension operator similar to the one in [LM72a, Section 2.2] once across every hyperplane (or in other words by applying a tensorized version of that extension operator). There exist $\lambda_{-1}, \lambda_{-2} \in \mathbb{R}$ such that

$$\lambda_{-1} + \lambda_{-2} 2^k = (-1)^k \quad \text{for } k \in \{2,3\}\,.$$

We also let $\lambda_1 = 1$. Then we define the extension $\tilde{u}$ of u by

$$\tilde{u}(x_1,\dots,x_{\mathsf{d}}) = \sum_{\substack{\varepsilon_1=1 \text{ if } x_1\geq 0\\ \varepsilon_1\in\{-1,-2\} \text{ if } x_1<0}} \cdots \sum_{\substack{\varepsilon_{\mathsf{d}}=1 \text{ if } x_{\mathsf{d}}\geq 0\\ \varepsilon_{\mathsf{d}}\in\{-1,-2\} \text{ if } x_{\mathsf{d}}<0}} \lambda_{\varepsilon_1}\cdot\ldots\cdot\lambda_{\varepsilon_{\mathsf{d}}} u(\varepsilon_1 x_1,\dots,\varepsilon_{\mathsf{d}} x_{\mathsf{d}})\,.$$

For example, for $\mathsf{d}=2$ we have

$$\tilde{u}(x_1,x_2) = \begin{cases} u(x_1,x_2) & \text{for } x_1\geq 0, x_2\geq 0\,,\\ \lambda_{-1}u(-x_1,x_2)+\lambda_{-2}u(-2x_1,x_2) & \text{for } x_1<0, x_2\geq 0\,,\\ \lambda_{-1}u(x_1,-x_2)+\lambda_{-2}u(x_1,-2x_2) & \text{for } x_1\geq 0, x_2<0\,,\\ (\lambda_{-1})^2u(-x_1,-x_2)+\lambda_{-1}\lambda_{-2}u(-x_1,-2x_2) & \\ \quad +\lambda_{-1}\lambda_{-2}u(-2x_1,-x_2)+(\lambda_{-2})^2u(-2x_1,-2x_2) & \text{for } x_1<0, x_2<0\,. \end{cases}$$

One easily checks that both $\tilde{u}=0$ and $\nabla\tilde{u}=0$ on the face $x_i=0$ for $i\in\{1,\dots,\mathsf{d}\}$. In addition, the support of $\tilde{u}$ is contained in $\left(-\frac{2}{3},\frac{2}{3}\right)^{\mathsf{d}} \subset (-1,1)^{\mathsf{d}}$.

It remains to show that $\tilde{u}\in H^s((-1,1)^{\mathsf{d}})$ and

$$\|\tilde{u}\|_{H^s((-1,1)^{\mathsf{d}})} \leq C\|u\|_{H^s(\Omega)}\,. \tag{5.2.1}$$

For this we use interpolation. If $s=4$, and $u\in H^4_{(1)}(\Omega)$ observe that by the construction of $\tilde{u}$ for $k\in\{0,1,2,3\}$ the traces of $\partial_i^k\tilde{u}$ from the two sides of $\{x_i=0\}$ agree. This implies that $\tilde{u}\in H^4((-1,1)^{\mathsf{d}})$ and $\|\tilde{u}\|_{H^4((-1,1)^{\mathsf{d}})}\leq C\|u\|_{H^4(\Omega)}$. If $s=1$ and $u\in H^1_{(1)}(\Omega)=H^1_0(\Omega)$ we can use the same argument to obtain (5.2.1) once again. Now, by Lemma 5.4.9 from the Appendix, for any $\frac{3}{2}<s\leq 4$ the interpolation space $\left[H^4_{(1)}(\Omega), H^1_{(1)}(\Omega)\right]_{\frac{1}{3}(4-s)}$ is equal to $H^s_{(1)}(\Omega)$. Thus (5.2.1) follows by standard function space interpolation theory. □

5.2.2 Estimate of the boundary values

In this section we prove the estimate (5.1.7), i.e., that the discrete normal derivatives of $\tilde{u}$ at the boundary can be estimated in the fractional discrete Sobolev space $H_h^{\frac{1}{2}}$. One can think of this result, stated in Lemma 5.2.2, as a discrete trace theorem. Before giving the precise statement we define the appropriate (semi-)norms.

Let S be a subset of $\mathbb{R}^{\mathsf{d}}$ that is contained in an axiparallel $(\mathsf{d}-1)$-dimensional affine subspace of $\mathbb{R}^{\mathsf{d}}$ such that $S\cap(h\mathbb{Z})^{\mathsf{d}}\neq\varnothing$, and let $w\colon S\cap(h\mathbb{Z})^{\mathsf{d}}\to\mathbb{R}$. We then define

$$[w]^2_{H_h^{\frac{1}{2}}(S\cap(h\mathbb{Z})^{\mathsf{d}})} := \sum_{\substack{x,y\in S\cap(h\mathbb{Z})^{\mathsf{d}}\\ x\neq y}} \frac{|w(x)-w(y)|^2}{|x-y|^{\mathsf{d}}} h^{2\mathsf{d}-2}$$

and

$$|w\|^2_{H_h^{\frac{1}{2}}(S\cap(h\mathbb{Z})^{\mathsf{d}})} := [w]^2_{H_h^{\frac{1}{2}}(S\cap(h\mathbb{Z})^{\mathsf{d}})} + \sum_{x\in S\cap(h\mathbb{Z})^{\mathsf{d}}} h^{\mathsf{d}-1}|w(x)|^2 .$$

For the discrete inverse trace theorem stated in Section 5.2.3 we will need to use the extension by zero of $D^h_{0,\nu}\tilde{u}$ and $D^h_{\nu}\tilde{u}$. Therefore we directly estimate the $H_h^{\frac{1}{2}}$-seminorm of that extension in the following lemma.[2] At the first glance it might seem problematic that we are extending $D^h_{0,\nu}\tilde{u}$ by zero, because for $s \geq \frac{5}{2}$ this extension does not preserve the H^s-regularity of $\tilde{u}$. However it turns out that it is possible to estimate $[D^h_{0,\nu}\tilde{u}]_{H_h^{\frac{1}{2}}}$ by expressions that involve several derivatives in the direction e_{d}, but at most one derivative in the directions e_i for $1 \leq i \leq \mathsf{d}-1$, so our assumptions on the boundary values are sufficient.

Lemma 5.2.2. *Let $s > \frac{1}{2}\max(3,\mathsf{d})$ and let $\tilde{u}$ be as in Lemma 5.2.1. For $i \in \{1,\dots,\mathsf{d}\}$ let $g_{h,i}$ and $g^*_{h,i}$ be the extension by zero of $D^h_{0,i}\tilde{u}$ and $D^h_{-i}\tilde{u}$ in the hyperplane $(h\mathbb{Z})^{i-1}\times\{0\}\times(h\mathbb{Z})^{\mathsf{d}-i}$, respectively, i.e., $g_{h,i}\colon (h\mathbb{Z})^{i-1}\times\{0\}\times(h\mathbb{Z})^{\mathsf{d}-i}\to\mathbb{R}$ and $g^*_{h,i}\colon (h\mathbb{Z})^{i-1}\times\{0\}\times(h\mathbb{Z})^{\mathsf{d}-i}\to\mathbb{R}$ satisfy*

$$g_{h,i}(x) = \begin{cases} D^h_{0,i}\tilde{u}(x) & \text{when } x\in(0,\infty)^{i-1}\times\{0\}\times[0,\infty)^{\mathsf{d}-i},\\ 0 & \text{otherwise}, \end{cases}$$

$$g^*_{h,i}(0) = \begin{cases} D^h_{-i}\tilde{u}(x) & \text{when } x\in(0,\infty)^{i-1}\times\{0\}\times[0,\infty)^{\mathsf{d}-i},\\ 0 & \text{otherwise}. \end{cases}$$

We have that, if $s \leq 4$, then

$$\|g_{h,i}\|_{H_h^{\frac{1}{2}}((h\mathbb{Z})^{i-1}\times\{0\}\times(h\mathbb{Z})^{\mathsf{d}-i})} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}, \tag{5.2.2}$$

and, if $s \leq 3$, then

$$\|g^*_{h,i}\|_{H_h^{\frac{1}{2}}((h\mathbb{Z})^{i-1}\times\{0\}\times(h\mathbb{Z})^{\mathsf{d}-i})} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}. \tag{5.2.3}$$

We can assume that $i = \mathsf{d}$, the other cases being analogous. For simplicity we identify $\mathbb{R}^{\mathsf{d}-1}$ with the hyperplane $\mathbb{R}^{\mathsf{d}-1}\times\{0\}\subset\mathbb{R}^{\mathsf{d}}$, and write $x = (x', x_{\mathsf{d}})$, with $x' := (x_1,\dots,x_{\mathsf{d}-1})$.

Before embarking on the proof of our main result, we state and prove two estimates that we will need.

Lemma 5.2.3. *Let $s > \frac{1}{2}\max(3,\mathsf{d})$ and $v \in H^s(\mathbb{R}^{\mathsf{d}})$ such that $v = 0$ and $\partial_{\mathsf{d}}v = 0$ on $\mathbb{R}^{\mathsf{d}-1}$ in the sense of trace. Let $h > 0$, let $x' \in (h\mathbb{Z})^{\mathsf{d}-1}$, $\hat{x}' \in \mathbb{R}^{\mathsf{d}-1}\times\{0\}$ and suppose that $|x'-\hat{x}'|_\infty < \frac{h}{2}$. Let further $Q_{h/2}(x') := x' + (-h/2, h/2)^{\mathsf{d}-1}$ be the $(\mathsf{d}-1)$-dimensional axiparallel cube of edge-length h centered at x'. If $s \leq 4$, we have that*

$$|v(x',h) - v(x',-h) - v(\hat{x}',h) + v(\hat{x}',-h)| \leq Ch^{s-\frac{\mathsf{d}}{2}}\|v\|_{H^s(Q_{h/2}(x')\times\mathbb{R})}, \tag{5.2.4}$$

and if $s \leq 3$ we have that

$$|v(x',0) - v(x',-h) - v(\hat{x}',0) + v(\hat{x}',-h)| \leq Ch^{s-\frac{\mathsf{d}}{2}}\|v\|_{H^s(Q_{h/2}(x')\times\mathbb{R})}. \tag{5.2.5}$$

[2] Alternatively one could define a discrete analogue of the $H^{\frac{1}{2}}_{00}$-norm from [LM72a, Section 11.5]; that however leads to unnecessary technicalities in the present context.

Proof. We begin with (5.2.4). By scaling and translating we can assume that without loss of generality that $h = 1$ and $x' = 0$. Because $s > \frac{\mathsf{d}}{2}$ the left-hand side of (5.2.4) is bounded by $C\|v\|_{H^s(Q_{1/2}(0)\times(-2,2))}$. Furthermore it vanishes when v is a polynomial of degree at most 3. Indeed the boundary condition ensures that each monomial of degree at most 3 has degree at least 2 in x_{d} and the left-hand side vanishes for such monomials. So (5.2.4) follows from the Bramble–Hilbert lemma (applied in $H^s(Q_{1/2}(0) \times (-2,2))$). The estimate (5.2.5) can be proved analogously. □

Lemma 5.2.4. *Let $s > \frac{3}{2}$ and $v \in H^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R})$. Suppose that for all $i \in \{1,\dots,\mathsf{d}-1\}$ we have $v = 0$ on $\{x_i = 0\}$ in the sense of trace, and that furthermore we have $\partial_{\mathsf{d}} v = 0$ on $\{x_{\mathsf{d}} = 0\}$ in the sense of trace. Let $\hat{v}$ be the extension by zero in the first $\mathsf{d}-1$ variables of v to $\mathbb{R}^{\mathsf{d}}$, i.e.,*

$$\hat{v}(x) := \begin{cases} v(x) & x \in (0,\infty)^{\mathsf{d}-1} \times \mathbb{R}, \\ 0 & \textit{otherwise}, \end{cases}$$

and let $h > 0$. If $s \le 4$, then we have that

$$\|\hat{v}(\cdot,h) - \hat{v}(\cdot,-h)\|_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})} \le Ch^{s-1}\|v\|_{H^s((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})}, \tag{5.2.6}$$

and if $s \le 3$, then we have that

$$\|\hat{v}(\cdot,0) - \hat{v}(\cdot,-h)\|_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})} \le Ch^{s-1}\|v\|_{H^s((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})}. \tag{5.2.7}$$

Proof. Let us define the function spaces $G^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R})$ for $s \in [0,\infty) \setminus \{\frac{1}{2}, \frac{3}{2}\}$ as follows. When $s > \frac{3}{2}$, G^s is the space that is mentioned in the statement of the lemma, i.e.,

$$\begin{aligned} G^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}) := H^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}) &\cap \{u\colon u = 0 \text{ on } \partial((0,\infty)^{\mathsf{d}-1} \times \mathbb{R})\} \\ &\cap \{u\colon \partial_{\mathsf{d}} u = 0 \text{ on } (0,\infty)^{\mathsf{d}-1} \times \{0\}\}. \end{aligned}$$

When $\frac{1}{2} < s < \frac{3}{2}$,

$$G^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}) := H^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}) \cap \{u\colon u = 0 \text{ on } \partial((0,\infty)^{\mathsf{d}-1} \times \mathbb{R})\},$$

and if $s < \frac{1}{2}$,

$$G^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}) := H^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}).$$

According to Lemma 5.4.10 from the Appendix we have that, for $s \notin \{\frac{1}{2}, \frac{3}{2}\}$,

$$\begin{aligned} G^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}) &= \left[G^4((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}), G^1((0,\infty)^{\mathsf{d}-1} \times \mathbb{R})\right]_{\frac{4-s}{3}}, \\ G^s((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}) &= \left[G^3((0,\infty)^{\mathsf{d}-1} \times \mathbb{R}), G^1((0,\infty)^{\mathsf{d}-1} \times \mathbb{R})\right]_{\frac{3-s}{2}}. \end{aligned}$$

Thus it suffices to prove (5.2.6) for $s = 4$ and $s = 1$ and (5.2.7) for $s = 3$ and $s = 1$, and then the result follows by interpolation. We prove the former two statements; the proofs of the latter two are completely analogous.

If $s = 1$, the condition that $v = 0$ on $\{x_i = 0\}$ in the sense of trace ensures that $\hat{v} \in H^1(\mathbb{R}^{\mathsf{d}})$ and $\|\hat{v}\|_{H^1(\mathbb{R}^{\mathsf{d}})} \le \|v\|_{H^1((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})}$. Now we can use standard trace theorems to bound

$$\begin{aligned} \|\hat{v}(\cdot,h) - \hat{v}(\cdot,-h)\|_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})} &\le \|\hat{v}(\cdot,h)\|_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})} + \|\hat{v}(\cdot,-h)\|_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})} \\ &\le 2\|v\|_{H^1((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})}. \end{aligned}$$

If $s = 4$ the proof is less straightforward. The main difficulty is that $\hat{v}$ is in general not in $H^4(\mathbb{R}^{\mathsf{d}})$. Instead we write

$$v(\cdot, h) = v(\cdot, 0) + h\partial_{\mathsf{d}} v(\cdot, 0) + \frac{h^2}{2}\partial_{\mathsf{d}}^2 v(\cdot, 0) + \int_0^h \frac{(h-s)^2}{2}\partial_{\mathsf{d}}^3 v(\cdot, s)\,\mathrm{d}s\,.$$

This does not make sense as a pointwise equality, but we can interpret it as an equality in $H^{\frac{1}{2}}((0,\infty)^{\mathsf{d}-1})$, with the integral on the right-hand side being understood as a Bochner integral. Similarly, we have

$$v(\cdot, -h) = v(\cdot, 0) - h\partial_{\mathsf{d}} v(\cdot, 0) + \frac{h^2}{2}\partial_{\mathsf{d}}^2 v(\cdot, 0) - \int_{-h}^0 \frac{(h+s)^2}{2}\partial_{\mathsf{d}}^3 v(\cdot, s)\,\mathrm{d}s\,.$$

Because we know that $v(\cdot, 0) = 0$ and $\partial_{\mathsf{d}} v(\cdot, 0) = 0$ in $H^{\frac{1}{2}}((0,\infty)^{\mathsf{d}-1})$, we deduce from this that

$$v(\cdot, h) - v(\cdot, -h) = h^2 \int_{-h}^h m\left(\frac{s}{h}\right)\partial_{\mathsf{d}}^3 v(\cdot, s)\,\mathrm{d}s, \tag{5.2.8}$$

as an identity in $H^{\frac{1}{2}}((0,\infty)^{\mathsf{d}-1})$, where $m(t) := \begin{cases} \frac{1}{2}(1-t)^2 & \text{for } t \geq 0, \\ \frac{1}{2}(1+t)^2 & \text{for } t \leq 0\,. \end{cases}$

Let $\hat{w}$ be the extension by zero in the first $\mathsf{d}-1$ variables of $\partial_{\mathsf{d}}^3 v$ to $\mathbb{R}^{\mathsf{d}}$, i.e.,

$$\hat{w}(x) := \begin{cases} \partial_{\mathsf{d}}^3 v(x) & \text{for } x \in (0,\infty)^{\mathsf{d}-1} \times \mathbb{R}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Our assumptions on v imply that $\partial_{\mathsf{d}}^3 v$ belongs to $H_0^1((0,\infty)^{\mathsf{d}-1} \times \mathbb{R})$. Therefore $\hat{w} \in H^1(\mathbb{R}^{\mathsf{d}})$ and $\|\hat{w}\|_{H^1(\mathbb{R}^{\mathsf{d}})} = \|\partial_{\mathsf{d}}^3 v\|_{H_0^1((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})}$. Furthermore (5.2.8) continues to hold for the extensions by zero of both sides, so that we also have

$$\hat{v}(\cdot, h) - \hat{v}(\cdot, -h) = h^2 \int_{-h}^h m\left(\frac{s}{h}\right)\hat{w}(\cdot, s)\,\mathrm{d}s$$

as an identity in $H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})$, and hence

$$\begin{aligned} \|\hat{v}(\cdot, h) - \hat{v}(\cdot, -h)\|_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})} &\leq h^2 \int_{-h}^h m\left(\frac{s}{h}\right)\|\hat{w}(\cdot, s)\|_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})}\,\mathrm{d}s \\ &\leq h^2 \int_{-h}^h \frac{1}{2}\|\hat{w}\|_{H^1(\mathbb{R}^{\mathsf{d}})}\,\mathrm{d}s \\ &\leq h^3 \|\partial_{\mathsf{d}}^3 v\|_{H^1((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})} \\ &\leq h^3 \|v\|_{H^4((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})}\,, \end{aligned}$$

which is (5.2.6). □

Proof of Lemma 5.2.2. We begin with (5.2.2). As before, we shall assume without loss of generality that $i = \mathsf{d}$, and we identify $\mathbb{R}^{\mathsf{d}-1}$ with $\mathbb{R}^{\mathsf{d}-1} \times \{0\} \subset \mathbb{R}^{\mathsf{d}}$ and write $x = (x', x_{\mathsf{d}})$.

Note that $D_{0,\mathsf{d}}^h \tilde{u}(x)$ makes sense for any $x \in [0,1)^{\mathsf{d}-1} \times \{0\}$, not only for those in $(h\mathbb{Z})^{\mathsf{d}}$. We denote by g_{d} the extension by zero of $D_{0,\mathsf{d}}^h \tilde{u}$ in the hyperplane $\mathbb{R}^{\mathsf{d}-1} \times \{0\}$, i.e., $g_{\mathsf{d}} \colon \mathbb{R}^{\mathsf{d}-1} \times \{0\} \to \mathbb{R}$ satisfies

$$g_{\mathsf{d}}(x) = \begin{cases} D_{0,\mathsf{d}}^h \tilde{u}(x) & \text{for } x \in (0,\infty)^{\mathsf{d}-1} \times \{0\}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, $g_{h,\mathsf{d}}$ is the restriction of g_d to $(h\mathbb{Z})^\mathsf{d}$, and our goal will be to relate the discrete $H_h^{1/2}$-norm of $g_{h,\mathsf{d}}$ and the continuous $H^{1/2}$-norm of g_d. We begin by estimating the latter.

Applying Lemma 5.2.4 to $hg_{h,\mathsf{d}}$ we obtain

$$\|g_\mathsf{d}\|_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})} = \|D^h_{0,\mathsf{d}}\tilde{u}\|_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})} \leq Ch^{s-2}\|\tilde{u}\|_{H^s(\mathbb{R}^\mathsf{d})}\,. \tag{5.2.9}$$

Next, let $x' \in (h\mathbb{Z})^{\mathsf{d}-1}$, $\hat{x}' \in \mathbb{R}^{\mathsf{d}-1}$, and suppose that $|x' - \hat{x}'|_\infty < \frac{h}{2}$. Recall that $Q_{h/2}(x') = x' + (-h/2, h/2)^{\mathsf{d}-1}$ is the $(\mathsf{d}-1)$-dimensional axiparallel cube of edge-length h centered at x'. Then, Lemma 5.2.3 implies that

$$|\tilde{u}(x',h) - \tilde{u}(x',-h) - \tilde{u}(\hat{x}',h) + \tilde{u}(\hat{x}',-h)| \leq Ch^{s-\frac{\mathsf{d}}{2}}\|\tilde{u}\|_{H^s(Q_{h/2}(x')\times\mathbb{R})}\,.$$

If $x'_i > 0$ for all $i = 1,\dots,\mathsf{d}-1$, then

$$|g_\mathsf{d}(x') - g_\mathsf{d}(\hat{x}')| = \frac{1}{2h}|\tilde{u}(x',h) - \tilde{u}(x',-h) - \tilde{u}(\hat{x}',h) + \tilde{u}(\hat{x}',-h)|\,. \tag{5.2.10}$$

On the other hand, if $x_i \leq 0$ for some $i \in \{1,\dots,\mathsf{d}-1\}$, then $\tilde{u}(x',h) = \tilde{u}(x',-h) = g_\mathsf{d}(x',0) = 0$ and

$$|g_\mathsf{d}(\hat{x}')| = \begin{cases} \frac{1}{2h}|\tilde{u}(\hat{x}',h) - \tilde{u}(\hat{x}',-h)| & \text{for } x' \in (0,\infty)^{\mathsf{d}-1}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

This, together with (5.2.10), implies that we have in any case

$$|g_\mathsf{d}(x') - g_\mathsf{d}(\hat{x}')| \leq \frac{1}{2h}|\tilde{u}(x',h) - \tilde{u}(x',-h) - \tilde{u}(\hat{x}',h) + \tilde{u}(\hat{x}',-h)|\,.$$

Thus we get that

$$|g_\mathsf{d}(x') - g_\mathsf{d}(\hat{x}')| \leq Ch^{s-1-\frac{\mathsf{d}}{2}}[\tilde{u}]_{H^s(Q_{h/2}(x')\times\mathbb{R})} \leq Ch^{s-1-\frac{\mathsf{d}}{2}}\|\tilde{u}\|_{H^s(Q_{h/2}(x')\times\mathbb{R})}\,. \tag{5.2.11}$$

Now let $x', y' \in (h\mathbb{Z})^{\mathsf{d}-1}$, $\hat{x}' \in Q_{h/2}(x')$ and $\hat{y}' \in Q_{h/2}(y')$. We then have that

$$|g_\mathsf{d}(x') - g_\mathsf{d}(y')| \leq |g_\mathsf{d}(\hat{x}') - g(\hat{y}')| + |g(x') - g(\hat{x}')| + |g(y') - g(\hat{y}')|\,.$$

This implies that $|g_\mathsf{d}(x') - g_\mathsf{d}(y')|^2 \leq 3\big(|g_\mathsf{d}(\hat{x}') - g_\mathsf{d}(\hat{y}')|^2 + |g_\mathsf{d}(x') - g_\mathsf{d}(\hat{x}')|^2 + |g_\mathsf{d}(y') - g_\mathsf{d}(\hat{y}')|^2\big)$, and, using (5.2.11), we deduce that

$$\begin{aligned} &|g_\mathsf{d}(x') - g_\mathsf{d}(y')|^2 \\ &\quad \leq 3|g_\mathsf{d}(\hat{x}') - g_\mathsf{d}(\hat{y}')|^2 + Ch^{2s-2-\mathsf{d}}\|\tilde{u}\|^2_{H^s(Q_h(x')\times(-2h,2h))} + Ch^{2s-2-\mathsf{d}}\|\tilde{u}\|^2_{H^s(Q_h(y')\times(-2h,2h))}\,. \end{aligned} \tag{5.2.12}$$

Thus, taking the average of (5.2.12) over all $\hat{x}' \in Q_{h/2}(x')$ and $\hat{y}' \in Q_{h/2}(y')$, we obtain

$$\begin{aligned} |g_\mathsf{d}(x') - g_\mathsf{d}(y')|^2 \leq{}& 3h^{2-2\mathsf{d}} \int_{Q_{h/2}(x')}\int_{Q_{h/2}(y')} |g_\mathsf{d}(\hat{x}') - g_\mathsf{d}(\hat{y}')|^2 \,\mathrm{d}\hat{x}'\,\mathrm{d}\hat{y}' \\ &+ Ch^{2s-2-\mathsf{d}}\left(\|\tilde{u}\|^2_{H^s(Q_h(x')\times\mathbb{R})} + \|\tilde{u}\|^2_{H^s(Q_h(y')\times\mathbb{R})}\right)\,. \end{aligned}$$

Observe that for $|x' - y'| \geq h$ we have

$$\begin{aligned} |\hat{x}' - \hat{y}'| &= |x' - y' - (x' - \hat{x}') + (y' - \hat{y}')| \\ &\leq |x' - y'| + |x' - \hat{x}'| + |y' - \hat{y}'| \end{aligned}$$

$$\leq |x'-y'| + h \leq 2|x'-y'|\,.$$

Using this, we deduce that

$$\begin{aligned}
[g_{h,\mathsf{d}}]^2_{H^{\frac{1}{2}}_h((h\mathbb{Z})^{\mathsf{d}-1})} &= \sum_{\substack{x',y'\in(h\mathbb{Z})^{\mathsf{d}-1}\\ x'\neq y'}} \frac{|g_{\mathsf{d}}(x')-g_{\mathsf{d}}(y')|^2}{|x'-y'|^{\mathsf{d}}} h^{2\mathsf{d}-2} \\
&\leq 3\cdot 2^{\mathsf{d}} \int_{\mathbb{R}^{\mathsf{d}-1}}\int_{\mathbb{R}^{\mathsf{d}-1}} \frac{|g_{\mathsf{d}}(\hat{x}')-g_{\mathsf{d}}(\hat{y}')|^2}{|\hat{x}'-\hat{y}'|^{\mathsf{d}}}\,\mathrm{d}\hat{x}'\,\mathrm{d}\hat{y}' \\
&\quad + Ch^{\mathsf{d}+2s-4} \sum_{\substack{x',y'\in(h\mathbb{Z})^{\mathsf{d}-1}\\ x'\neq y'}} \frac{1}{|y'-x'|^{\mathsf{d}}} \|\tilde{u}\|^2_{H^s(Q_h(x')\times\mathbb{R})} \\
&\quad + Ch^{\mathsf{d}+2s-4} \sum_{\substack{x',y'\in(h\mathbb{Z})^{\mathsf{d}-1}\\ x'\neq y'}} \frac{1}{|x'-y'|^{\mathsf{d}}} \|\tilde{u}\|^2_{H^s(Q_h(y')\times\mathbb{R})}\,.
\end{aligned} \tag{5.2.13}$$

The first term on the right-hand side is a constant times $[g_{\mathsf{d}}]^2_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})}$. To estimate the second term, notice that

$$\begin{aligned}
\sum_{y'\in(h\mathbb{Z})^{\mathsf{d}-1},y'\neq x'} \frac{1}{|y'-x'|^{\mathsf{d}}} &= \frac{1}{h^{\mathsf{d}-1}} \sum_{y'\in(h\mathbb{Z})^{\mathsf{d}-1},y'\neq x'} \frac{1}{|y'-x'|^{\mathsf{d}}} h^{\mathsf{d}-1} \\
&\leq \frac{C}{h^{\mathsf{d}-1}} \int_{|y'-x'|\geq h} \frac{1}{|y'-x'|^{\mathsf{d}}}\,\mathrm{d}y' \leq \frac{C}{h^{\mathsf{d}}}
\end{aligned}$$

and

$$\sum_{x'\in(h\mathbb{Z})^{\mathsf{d}-1}} \|\tilde{u}\|^2_{H^s(Q_h(x')\times\mathbb{R})} \leq \|\tilde{u}\|^2_{H^s(\mathbb{R}^{\mathsf{d}})}\,,$$

by superadditivity of the fractional Sobolev norm.

Together with the analogous estimate for the third term and (5.2.9) we arrive at

$$\begin{aligned}
[g_{h,\mathsf{d}}]^2_{H^{\frac{1}{2}}_h((h\mathbb{Z})^{\mathsf{d}-1})} &\leq C[g_{\mathsf{d}}]^2_{H^{\frac{1}{2}}(\mathbb{R}^{\mathsf{d}-1})} + Ch^{2s-4}\|\tilde{u}\|^2_{H^s(\mathbb{R}^{\mathsf{d}})} \\
&\leq Ch^{2s-4}\|\tilde{u}\|^2_{H^s(\mathbb{R}^{\mathsf{d}})} \\
&\leq Ch^{2s-4}\|u\|^2_{H^s(\Omega)}\,.
\end{aligned} \tag{5.2.14}$$

It remains to estimate $\|g_{h,\mathsf{d}}\|_{L^2_h((h\mathbb{Z})^{\mathsf{d}-1})}$. A simple way to do so is to observe that we have a Poincaré-type inequality. Indeed, $g_{h,\mathsf{d}}$ is supported in $\left[0,\frac{2}{3}\right]^{\mathsf{d}-1}\cap(h\mathbb{Z})^{\mathsf{d}-1}$ and therefore

$$\begin{aligned}
[g_{h,\mathsf{d}}]^2_{H^{\frac{1}{2}}_h((h\mathbb{Z})^{\mathsf{d}-1})} &= \sum_{\substack{x',y'\in(h\mathbb{Z})^{\mathsf{d}-1}\\ x'\neq y'}} \frac{|g_{\mathsf{d}}(x')-g_{\mathsf{d}}(y')|^2}{|x'-y'|^{\mathsf{d}}} h^{2\mathsf{d}-2} \\
&\geq \sum_{x'\in[0,1)^{\mathsf{d}-1}\cap(h\mathbb{Z})^{\mathsf{d}-1}} \quad \sum_{y'\in[-2,-1)^{\mathsf{d}-1}\cap(h\mathbb{Z})^{\mathsf{d}-1}} \frac{|g_{\mathsf{d}}(x')-g_{\mathsf{d}}(y')|^2}{|x'-y'|^{\mathsf{d}}} h^{2\mathsf{d}-2} \\
&\geq \sum_{x'\in[0,1)^{\mathsf{d}-1}\cap(h\mathbb{Z})^{\mathsf{d}-1}} \quad \sum_{y'\in[-2,-1)^{\mathsf{d}-1}\cap(h\mathbb{Z})^{\mathsf{d}-1}} \frac{|g_{\mathsf{d}}(x')|^2}{(3\sqrt{\mathsf{d}})^{\mathsf{d}}} h^{2\mathsf{d}-2} \\
&\geq \frac{1}{(3\sqrt{n})^{\mathsf{d}}} \sum_{x'\in[0,1)^{\mathsf{d}-1}\cap(h\mathbb{Z})^{\mathsf{d}-1}} h^{\mathsf{d}-1}|g_{h,\mathsf{d}}(x')|^2\,.
\end{aligned}$$

Combining this with (5.2.14) we obtain (5.2.2). The proof of (5.2.3) is similar, with the only difference that we use (5.2.5) and (5.2.7) instead of (5.2.4) and (5.2.6). □

5.2.3 A discrete inverse trace theorem on the cube

In the previous section we proved that the discrete normal derivative of $\tilde{u}$ has small trace in $H_h^{\frac{1}{2}}$, and thus the same holds true for $E = \tilde{u} - U$ and $E^* = \tilde{u} - U^*$. We now want to construct a function $\hat{E}$ such that $\hat{E}$ and E agree on Γ_h and such that the H_h^2-norm of $\hat{E}$ is small (and similarly for $\hat{E}^*$). The existence of $\hat{E}$ and $\hat{E}^*$ follows from a discrete inverse trace theorem, as in (5.1.8). However we will not prove (or even state precisely) a general result, as we did in (5.1.8); instead, we shall state the result when applied directly to $\hat{E}$ and $\hat{E}^*$. The following two lemmas concern the boundary conditions appearing in Theorems 5.1.1 and 5.1.2, respectively.

Lemma 5.2.5. *Let $\frac{1}{2}\max(3,\mathsf{d}) < s \leq 4$ and let $\tilde{u}$ be as in Lemma 5.2.1. Then, there is a function $\hat{E}$ on Λ_h such that*

$$\begin{aligned} \hat{E} &= 0 && \text{on } \Gamma_h\,, \\ D_{0,\nu}^h \hat{E} &= D_{0,\nu}^h \tilde{u} && \text{on } \Gamma_h\,, \end{aligned}$$

and such that $\|\nabla_h^2 \hat{E}\|_{L^2(\Lambda_h)} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}$.

Lemma 5.2.6. *Let $\frac{1}{2}\max(3,\mathsf{d}) < s \leq 3$ and let $\tilde{u}$ be as in Lemma 5.2.1. Then, there is a function $\hat{E}^*$ on Λ_h such that*

$$\begin{aligned} \hat{E}^* &= 0 && \text{on } \Gamma_h\,, \\ D_{\nu}^h \hat{E}^* &= D_{\nu}^h \tilde{u} && \text{on } \Gamma_h\,, \end{aligned}$$

and such that $\|\nabla_h^2 \hat{E}^*\|_{L^2(\Lambda_h)} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}$.

The strategy for the proof of both lemmas is the following. It suffices to consider the case when $D_\nu^h \tilde{u}$ is nonzero only on one face, say $\{x_\mathsf{d} = 0\}$. We construct an extension of the boundary values there in Fourier space. This extension is constructed in such a way that we can control its H_h^2-norm by the $H_h^{\frac{1}{2}}$-norm of the boundary values (at least after localizing to a bounded set). However this extension does not yet have the appropriate boundary values at $\{x_i = 0\}$ for $i < \mathsf{d}$. To fix this we use a projection operator $H_h^2 \to H_{h,0}^2$ on each fixed slice $\{x_\mathsf{d} = c\}$ and show that we retain control of the H_h^2-norm.

Proof of Lemma 5.2.5. Step 1: Preliminaries
Because we have localized $\tilde{u}$, $D_{0,\nu}^h \tilde{u}$ has nonzero boundary values only on the faces $\Gamma_h \cap \{x_i = 0\}$. We can deal with the faces separately. In fact we will construct functions $\hat{E}_i$ for $i \in \{1,\ldots,\mathsf{d}\}$ such that $\hat{E}_i = 0$ on Γ_h, $D_{0,\nu}^h \hat{E}_i = D_{0,\nu}^h \tilde{u}$ on $x_i = 0$ while $D_{0,\nu}^h \hat{E}_i = 0$ on $\Gamma_h \setminus \{x_i = 0\}$, satisfying the estimate $\|\nabla_h^2 \hat{E}_i\|_{L^2(\Lambda_h)} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}$. Then we can choose $\hat{E} = \sum_i \hat{E}_i$, which will have the desired properties. As the $\hat{E}_i$ can be constructed analogously, we shall focus on $\hat{E}_\mathsf{d}$ only.

Step 2: Construction of an extension in Fourier space
Recall the function $g_{h,\mathsf{d}}$, the extension by zero of $D_{0,\mathsf{d}}^h \hat{u}$. Thanks to our assumption, $g_{h,\mathsf{d}}$ is supported in $\left[0, \frac{2}{3}\right]^{\mathsf{d}-1} \cap (h\mathbb{Z})^{\mathsf{d}-1}$. We can extend this function periodically with period 2 and represent it by its discrete Fourier series

$$g_{h,\mathsf{d}}(x') = \sum_{k' \in \left\{-\frac{1}{h}+1,\ldots,\frac{1}{h}\right\}^{\mathsf{d}-1}} \gamma_{k'} \mathrm{e}^{i\pi(k'\cdot x')}\,,$$

where

$$\gamma_{k'} = \left(\frac{h}{2}\right)^{\mathsf{d}-1} \sum_{\xi' \in [-1,1)^{\mathsf{d}-1} \cap (h\mathbb{Z})^{\mathsf{d}-1}} g_{h,\mathsf{d}}(\xi')\, \mathrm{e}^{-i\pi k' \cdot \xi'}$$

and $k' := (k'_1, \dots, k'_{\mathsf{d}-1}) \in \mathbb{Z}^{\mathsf{d}-1}$.

It is easy to verify that

$$\sum_{k' \in \left\{-\frac{1}{h}+1, \dots, \frac{1}{h}\right\}^{\mathsf{d}-1}} (1 + |k'|)\gamma_{k'}^2 \le C \|g_{h,\mathsf{d}}\|^2_{H_h^{\frac{1}{2}}((h\mathbb{Z})^{\mathsf{d}-1})} . \tag{5.2.15}$$

Indeed, the Fourier norm on the left-hand side is controlled by the $H_h^{\frac{1}{2}}$-norm on the torus $([-1,1]^{\mathsf{d}-1} \cap (h\mathbb{Z})^{\mathsf{d}-1})/_\sim$ (compare, e.g., [Hac81, Section 2.3]) and the latter is bounded by the $H_h^{\frac{1}{2}}$-norm on $(h\mathbb{Z})^{\mathsf{d}-1}$ because the support of $g_{h,\mathsf{d}}$ is bounded away from $\partial[-1,1]^{\mathsf{d}-1}$.

Define

$$a(x', x_\mathsf{d}) := \sum_{k' \in \left\{-\frac{1}{h}+1, \dots, \frac{1}{h}\right\}^{\mathsf{d}-1}} \frac{\gamma_{k'}}{\cosh(|k'|h)} x_\mathsf{d}\, \mathrm{e}^{-|k'|x_\mathsf{d}}\, \mathrm{e}^{i\pi k' \cdot x'} .$$

It is then easy to check that $a(x', 0) = 0$ and $D^h_{0,\mathsf{d}} a(x', 0) = g_{h,\mathsf{d}}(x')$ for $x' \in (-1,1)^{\mathsf{d}-1} \cap (h\mathbb{Z})^{\mathsf{d}-1}$. Furthermore, the H_h^2-norm of a is controlled. Indeed, we have that

$$\nabla_h^2 a(x', x_\mathsf{d}) = \sum_{k' \in \left\{-\frac{1}{h}+1, \dots, \frac{1}{h}\right\}^{\mathsf{d}-1}} \sigma(k', h, x_\mathsf{d})\gamma_{k'}\, \mathrm{e}^{i\pi k' \cdot x'} ,$$

where the coefficients $\sigma(k', h, x_\mathsf{d})$ satisfy $|\sigma(k', h, x_\mathsf{d})| \le C|k'|(|k'|x_\mathsf{d} + 1)\mathrm{e}^{-|k'|x_\mathsf{d}}$. This can be seen using Taylor's theorem in the form $\nabla_h^2 v(x) = \nabla^2 v(x) + O\left(h \sup_{|\hat{x}-x|_\infty \le h} |\nabla^3 v(\hat{x})|\right)$. For example,

$$\begin{aligned} &D^h_{-\mathsf{d}} D^h_\mathsf{d} \left(x_\mathsf{d} \mathrm{e}^{-|k'|x_\mathsf{d}} \mathrm{e}^{i\pi k' \cdot x'}\right) \\ &= (|k'|^2 x_\mathsf{d} - 2|k'|)\mathrm{e}^{-|k'|x_\mathsf{d}} \mathrm{e}^{i\pi k' \cdot x'} + O(h(|k'|^2 + |k'|^3(x_\mathsf{d} + h))\mathrm{e}^{-|k'|(x_\mathsf{d}-h)}) , \end{aligned}$$

and therefore $\left| D^h_{-\mathsf{d}} D^h_\mathsf{d} \left(x_\mathsf{d} \mathrm{e}^{-|k'|x_\mathsf{d}} \mathrm{e}^{i\pi k' \cdot x'}\right)\right| \le C|k'|(|k'|x_\mathsf{d} + 1)\mathrm{e}^{-|k'|x_\mathsf{d}}$.

Now, using orthogonality in x' we get, for $x_\mathsf{d} \ge 0$,

$$\begin{aligned} \sum_{x' \in [-1,1)^{\mathsf{d}-1} \cap (h\mathbb{Z})^{\mathsf{d}-1}} h^{\mathsf{d}-1} |\nabla_h^2 a(x', x_\mathsf{d})|^2 &= 2^{\mathsf{d}-1} \sum_{k' \in \left\{-\frac{1}{h}+1, \dots, \frac{1}{h}\right\}^{\mathsf{d}-1}} |\sigma(k', h, x_\mathsf{d})|^2 |\gamma_{k'}|^2 \\ &\le C \sum_{k' \in \left\{-\frac{1}{h}+1, \dots, \frac{1}{h}\right\}^{\mathsf{d}-1}} |k'|^2 (|k'|^2 x_\mathsf{d}^2 + 1)\, \mathrm{e}^{-2|k'|x_\mathsf{d}} |\gamma_{k'}|^2 , \end{aligned}$$

and hence

$$\begin{aligned} &\sum_{x \in [-1,1)^{\mathsf{d}-1} \times [0,2] \cap (h\mathbb{Z})^\mathsf{d}} h^\mathsf{d} |\nabla_h^2 a(x', x_\mathsf{d})|^2 \\ &\le Ch \sum_{x_\mathsf{d} \in [0,2] \times h\mathbb{Z}} \sum_{k' \in \left\{-\frac{1}{h}+1, \dots, \frac{1}{h}\right\}^{\mathsf{d}-1}} |k'|^2 (|k'|^2 x_\mathsf{d}^2 + 1)\, \mathrm{e}^{-2|k'|x_\mathsf{d}} |\gamma_{k'}|^2 . \end{aligned}$$

Next, we use the estimate

$$\sum_{x_\mathsf{d} \in [0,2] \cap h\mathbb{Z}} h x_\mathsf{d}^\alpha\, \mathrm{e}^{-2|k'|x_\mathsf{d}} \le C_\alpha \int_0^\infty \xi^\alpha \mathrm{e}^{-2|k'|\xi}\, \mathrm{d}\xi = C_\alpha \frac{1}{|k'|^{1+\alpha}} \int_0^\infty \theta^2\, \mathrm{e}^{-2\theta}\, \mathrm{d}\theta \le \frac{C_\alpha}{|k'|^{1+\alpha}}$$

for $\alpha = 2$ and $\alpha = 0$ to deduce that

$$\sum_{x\in[-1,1)^{\mathsf{d}-1}\times[0,2]\cap(h\mathbb{Z})^{\mathsf{d}}} h^{\mathsf{d}}|\nabla_h^2 a(x',x_{\mathsf{d}})|^2 \le C \sum_{k'\in\left\{-\frac{1}{h}+1,\dots,\frac{1}{h}\right\}^{\mathsf{d}-1}} |k'||\gamma_{k'}|^2 ,$$

and thus, taking into account (5.2.15) and (5.2.2),

$$\|\nabla_h^2 a\|_{L_h^2((-1,1)^{\mathsf{d}-1}\times[0,2]\cap(h\mathbb{Z})^{\mathsf{d}})} \le C[g_{h,\mathsf{d}}]_{H_h^{\frac{1}{2}}((h\mathbb{Z})^{\mathsf{d}-1})} \le Ch^{s-2}\|u\|_{H^s(\Omega)} . \tag{5.2.16}$$

Similarly, we estimate

$$\|\nabla_h a\|_{L_h^2((-1,1)^{\mathsf{d}-1}\times[0,2]\cap(h\mathbb{Z})^{\mathsf{d}})} \le Ch^{s-2}\|u\|_{H^s(\Omega)} , \tag{5.2.17}$$

$$\|a\|_{L_h^2((-1,1)^{\mathsf{d}-1}\times[0,2]\cap(h\mathbb{Z})^{\mathsf{d}})} \le Ch^{s-2}\|u\|_{H^s(\Omega)} \tag{5.2.18}$$

(note that for these estimates we actually need control of $\|g_{h,\mathsf{d}}\|_{H_h^{\frac{1}{2}}((h\mathbb{Z})^{\mathsf{d}-1})}$, not just of $[g_{h,\mathsf{d}}]_{H_h^{\frac{1}{2}}((h\mathbb{Z})^{\mathsf{d}-1})}$).

Step 3: Localization

Let $\eta \in \mathbb{C}_c^\infty(\mathbb{R})$ be such that $\eta = 1$ in $\left[-\frac{3}{4},\frac{3}{4}\right]$, $\eta = 0$ in $\mathbb{R}\setminus[-1,1]$, and let

$$\tilde{a}(x) := \eta(x_1)\cdot\ldots\cdot\eta(x_{\mathsf{d}})a(x) .$$

Because $a = 0$ on $\{x_i = 0\}$ for all i, we have that $\tilde{a} = 0$ on Γ_h. Furthermore, $D_{0,\mathsf{d}}^h a = 0$ except possibly in $\left[-\frac{2}{3},\frac{2}{3}\right]^{\mathsf{d}-1}\times\{0\}$, and the product $\eta(x_1)\cdots\eta(x_{\mathsf{d}})$ is equal to the constant 1 in a neighborhood of that set. Therefore, $D_{0,\mathsf{d}}^h\tilde{a} = D_{0,\mathsf{d}}^h a = g_{h,\mathsf{d}}$ on $\{x_{\mathsf{d}} = 0\}$.

Using the estimates (5.2.16), (5.2.17), (5.2.18) and the discrete product rule, we also obtain

$$\begin{aligned}\|\nabla_h^2\tilde{a}\|_{L_h^2(\Lambda_h)} &\le C\Big(\|\nabla_h^2 a\|_{L_h^2((-1,1)^{\mathsf{d}-1}\times[0,2]\cap(h\mathbb{Z})^{\mathsf{d}})} + \|\nabla_h a\|_{L_h^2((-1,1)^{\mathsf{d}-1}\times[0,2]\cap(h\mathbb{Z})^{\mathsf{d}})} \\ &\qquad + \|a\|_{L_h^2((-1,1)^{\mathsf{d}-1}\times[0,2]\cap(h\mathbb{Z})^{\mathsf{d}})}\Big) \\ &\le Ch^{s-2}\|u\|_{H^s(\Omega)} .\end{aligned} \tag{5.2.19}$$

Step 4: Correction of the boundary values

Unfortunately, $\tilde{a}$ does not yet have the correct boundary values at $\{x_i = 0\}$ for $1 \le i \le \mathsf{d}-1$. To rectify this we use a discrete projection from H^2 to H_0^2. First we define the corresponding continuous projection. It is defined in a similar way as the extension we used in the proof of Lemma 5.2.1, namely by tensorizing the restriction operator from [LM72a, Section 11.5]. Thus we choose $\lambda_{-1},\lambda_{-2} \in \mathbb{R}$ such that

$$\lambda_{-1} + \lambda_{-2}2^k = (-1)^{k+1} \quad \text{for } k \in \{0,1\} .$$

(i.e., $\lambda_{-1} = -3$, $\lambda_{-2} = 2$); we let $\lambda_1 = 1$ and define a restriction operator $R\colon H^2(\mathbb{R}^{\mathsf{d}-1}) \to H_0^2((0,\infty)^{\mathsf{d}-1})$ by

$$Rv(x) := \sum_{\varepsilon_1\in\{1,-1,-2\}} \dots \sum_{\varepsilon_{\mathsf{d}}\in\{1,-1,-2\}} \lambda_{\varepsilon_1}\cdot\ldots\cdot\lambda_{\varepsilon_{\mathsf{d}-1}} v(\varepsilon_1 x_1,\dots,\varepsilon_{\mathsf{d}-1}x_{\mathsf{d}-1}).$$

One can check that we indeed have $Rv \in H_0^2((0,\infty)^{\mathsf{d}-1})$ and $\|Rv\|_{H^2((0,\infty)^{\mathsf{d}-1})} \le C\|v\|_{H^2(\mathbb{R}^{\mathsf{d}-1})}$. If we extend Rv by zero to $\mathbb{R}^{\mathsf{d}-1}$ we can also consider R as an operator mapping $H^2(\mathbb{R}^{\mathsf{d}-1})$

to itself. Note that if $x' \in (h\mathbb{Z})^{\mathsf{d}-1}$, then $Rv(x')$ depends only on $v|_{(h\mathbb{Z})^{\mathsf{d}-1}}$. Thus we can define $R_h \colon H^2_h((h\mathbb{Z})^{\mathsf{d}-1}) \to H^2_h((h\mathbb{Z})^{\mathsf{d}-1})$ by

$$R_h v(x') := \begin{cases} Rv(x') & \text{for } x' \in [0,\infty)^{\mathsf{d}-1}\,, \\ Rv(x'+2he_i) & \text{for } x' \in [0,\infty)^{i-1} \times \{-h\} \times [0,\infty)^{\mathsf{d}-1-i}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

We claim that

$$\|R_h v\|_{L^2_h([0,\infty)^{\mathsf{d}-1}\cap(h\mathbb{Z})^{\mathsf{d}-1})} \le C\|v\|_{L^2_h((h\mathbb{Z})^{\mathsf{d}-1})}\,, \tag{5.2.20}$$

$$\|\nabla_h R_h v\|_{L^2_h([0,\infty)^{\mathsf{d}-1}\cap(h\mathbb{Z})^{\mathsf{d}-1})} \le C\|\nabla_h v\|_{L^2_h((h\mathbb{Z})^{\mathsf{d}-1})}\,, \tag{5.2.21}$$

$$\|\nabla_h^2 R_h v\|_{L^2_h([0,\infty)^{\mathsf{d}-1}\cap(h\mathbb{Z})^{\mathsf{d}-1})} \le C\|\nabla_h^2 v\|_{L^2_h((h\mathbb{Z})^{\mathsf{d}-1})}\,. \tag{5.2.22}$$

Indeed, these estimates follow from the discrete chain rule. The only exception are the terms $D_i^h D_{-i}^h R_h v(x')$ in (5.2.22), which are not, a priori, controlled on $\{x_i = 0\}$. However an explicit calculation shows that for such x' one has

$$\begin{aligned} D_i^h D_{-i}^h R_h v(x') &= 2\frac{R_h v(x'+he_i)}{h^2} \\ &= 2\frac{v(x'+he_i) - 3v(x'-he_i) + 2v(x'-2he_i)}{h^2} \\ &= 2\frac{v(x'+he_i) - 2v(x') + v(x'-he_i)}{h^2} + 4\frac{v(x') - 2v(x'-he_i) + v(x'-2he_i)}{h^2} \\ &= 2D_i^h D_{-i}^h v(x') + 4D_i^h D_{-i}^h v(x'-he_i)\,, \end{aligned}$$

so that these terms, which are 'crossing the boundary', are still controlled.[3]

We now apply R_h along every slice $(h\mathbb{Z})^{\mathsf{d}-1} \times \{x_{\mathsf{d}}\}$, i.e., we set

$$b(x) := R_h \tilde{a}(\cdot, x_{\mathsf{d}})(x')\,.$$

Then by construction of R_h we have $b = 0$ and $D_{0,i}^h b(x) = 0$ on $\{x_i = 0\}$. Furthermore, b is supported in $\left[-h, \frac{3}{4}\right]^{\mathsf{d}}$ and we have $b = 0$ on $\{x_{\mathsf{d}} = 0\}$. We know that $D_{0,\mathsf{d}}^h \tilde{a} = g_{h,\mathsf{d}}$ on $\{x_{\mathsf{d}} = 0\}$. In addition, $R_h g_{h,\mathsf{d}} = g_{h,\mathsf{d}}$ on $[0,\infty)^{\mathsf{d}-1} \times \{0\}$, and so $D_{0,\mathsf{d}}^h b = g_{h,\mathsf{d}}$ on $[0,\infty)^{\mathsf{d}-1} \cap (h\mathbb{Z})^{\mathsf{d}-1}$ follows from the fact that R_h and $D_{0,\mathsf{d}}^h$ commute.

We next estimate $\|\nabla_h^2 b\|_{L^2(\Lambda_h)} = \|\nabla_h^2 R_h \tilde{a}\|_{L^2(\Lambda_h)}$. If $i, j \le \mathsf{d}-1$ then (5.2.22) implies that

$$\|D_i^h D_{-j}^h R_h \tilde{a}\|_{L^2(\Lambda_h)} \le C\|\nabla_h^2 \tilde{a}\|_{L^2_h((h\mathbb{Z})^{\mathsf{d}-1})}\,.$$

When taking derivatives in the direction e_{d} we use (5.2.21) and the fact that $D_{\pm n}^h$ and R_h commute, to obtain (for $i < \mathsf{d}$) that

$$\begin{aligned} \|D_i^h D_{-\mathsf{d}}^h R_h \tilde{a}\|_{L^2(\Lambda_h)} &= \|D_i^h R_h D_{-\mathsf{d}}^h \tilde{a}\|_{L^2(\Lambda_h)} \\ &\le C\|D_i^h D_{-\mathsf{d}}^h \tilde{a}\|_{L^2(\Lambda_h)}\,, \end{aligned}$$

and similarly, using (5.2.20),

$$\|D_{\mathsf{d}}^h D_{-\mathsf{d}}^h R_h \tilde{a}\|_{L^2(\Lambda_h)} \le C\|D_{\mathsf{d}}^h D_{-\mathsf{d}}^h \tilde{a}\|_{L^2(\Lambda_h)}\,.$$

[3] It is of course no coincidence that we have such an identity. In fact, $\nabla^2(Ru)$ is bounded in the L^2 norm thanks to the construction of R, and one can therefore also expect R_h to be well-behaved at the boundary.

If we combine the last three estimates and use (5.2.19) we deduce that

$$\|\nabla_h^2 b\|_{L^2(\Lambda_h)} = \|\nabla_h^2 \tilde{a}\|_{L^2(\Lambda_h)} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}\,.$$

Thus we can set $b = \hat{E}_{\mathsf{d}}$, and have shown that $\hat{E}_{\mathsf{d}}$ has all of the desired properties. □

Proof of Lemma 5.2.6. The proof is quite similar to the proof of Lemma 5.2.5. Let us outline the differences. In Step 2 we use a different extension operator, namely

$$a^*(x', x_{\mathsf{d}}) := \sum_{k' \in \left\{-\frac{1}{h}+1, \dots, \frac{1}{h}\right\}^{\mathsf{d}-1}} \frac{\gamma_{k'}}{\mathrm{e}^{|k'|h}} x_{\mathsf{d}}\, \mathrm{e}^{-|k'|x_{\mathsf{d}}}\, \mathrm{e}^{i\pi k' \cdot x'}\,,$$

so that $a^*(x', 0) = 0$ and $D^h_{-\mathsf{d}} a(x', 0) = g^*_{h,\mathsf{d}}(x')$ for $x' \in (-1,1)^{\mathsf{d}-1} \cap (h\mathbb{Z})^{\mathsf{d}-1}$. Using Lemma 5.2.2 we then again obtain

$$\|\nabla_h^2 a^*\|_{L^2_h((-1,1)^{\mathsf{d}-1}\times[0,2]\cap(h\mathbb{Z})^{\mathsf{d}})} \leq C[g_{h,\mathsf{d}}]_{H^{\frac{1}{2}}_h(\mathbb{R}^{\mathsf{d}-1})} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}$$

for $s \leq 3$. The localization step remains unchanged. To correct the boundary values we use

$$R^*_h v(x') := \begin{cases} Rv(x') & \text{for } x' \in [0,\infty)^{\mathsf{d}-1}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

instead of R_h. By using this projection operator we can then proceed as before. □

5.3 Estimates for the finite difference schemes

5.3.1 Summation-by-parts formulae and Poincaré inequalities

For the sake of completeness we record some summation-by-parts formulae that we will use in the following. These formulae are adapted to the two boundary conditions that we encounter in (5.1.2) and (5.1.4). Zero boundary conditions are easier to deal with, so we begin with those.

Lemma 5.3.1. *Let $v, \varphi\colon \tilde{\Lambda}_h \to \mathbb{R}$, and assume that $\varphi = D^h_\nu \varphi = 0$ on Γ_h.*
We have that

$$\sum_{z\in\Lambda_h\cup\Gamma_h} h^{\mathsf{d}} \Delta_h^2 v(z)\varphi(z) = \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in\Lambda_h\cup\Gamma_h} h^{\mathsf{d}} D^h_i D^h_{-j} v(z) D^h_i D^h_{-j}\varphi(z)\,. \tag{5.3.1}$$

So, if we define the scalar product $(f,g)_{L^2_{h,*}(\Lambda_h\cup\Gamma_h)}$ on functions $f, g\colon \tilde{\Lambda}_h \to \mathbb{R}^{n\times n}$ by

$$(f,g)_{L^2_{h,*}(\Lambda_h\cup\Gamma_h)} := \sum_{z\in\Lambda_h} \sum_{i,j=1}^{\mathsf{d}} h^{\mathsf{d}} f_{i,j}(z) g_{i,j}(z)\,,$$

we have

$$(\Delta_h^2 v, \varphi)_{L^2_h(\Lambda_h\cup\Gamma_h)} = (\nabla_h^2 v, \nabla_h^2 \varphi)_{L^2_{h,*}(\Lambda_h\cup\Gamma_h)}\,. \tag{5.3.2}$$

Furthermore, we have, for any $i \in \{1, \dots, \mathsf{d}\}$, that

$$(D^h_i D^h_{-i} v, \varphi)_{L^2_h(\Lambda_h\cup\Gamma_h)} = (v, D^h_i D^h_{-i}\varphi)_{L^2_h(\Lambda_h\cup\Gamma_h)}\,. \tag{5.3.3}$$

Proof. Observe that we have the summation-by-parts identity

$$\sum_{z\in(h\mathbb{Z})^{\mathsf{d}}} D^h_{\pm i} f(z) g(z) = \sum_{z\in(h\mathbb{Z})^{\mathsf{d}}} f(z) D^h_{\mp i} g(z) \tag{5.3.4}$$

for $f, g\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$ such that at least one of f, g has compact support, and $i \in \{1, \dots, \mathsf{d}\}$ (this follows from the one-dimensional case, where it can be easily checked). This immediately implies (5.3.3).

Next, observe that none of the terms in (5.3.1) depends on values of v or φ outside of $\tilde{\Lambda}_h$. Thus we can extend v and φ by 0 to all of $(h\mathbb{Z})^{\mathsf{d}}$ and prove equivalently that

$$\sum_{z\in(h\mathbb{Z})^{\mathsf{d}}} h^{\mathsf{d}} \Delta_h^2 v(z)\varphi(z) = \sum_{z\in(h\mathbb{Z})^{\mathsf{d}}} h^{\mathsf{d}} \Delta_h v(z) \Delta_h \varphi(z) = \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in(h\mathbb{Z})^{\mathsf{d}}} h^{\mathsf{d}} D_i^h D_{-j}^h v(z) D_i^h D_{-j}^h \varphi(z)\,.$$

This follows from repeated application of (5.3.4). □

For the case of the boundary conditions in (5.1.2), the situation is slightly more involved. We define, for $i, j \in \{1, \dots, \mathsf{d}\}$ with $i \neq j$, the set

$$\Gamma_h^{ij} := \{z \in \Gamma_h : z + hA_{ij} \subset [0,1]^{\mathsf{d}}\}\,,$$

where A_{ij} is the discrete square

$$A_{ij} := \{0, e_i, -e_j, e_i - e_j\}\,,$$

and note that

$$z \in \Gamma_h \setminus \Gamma_h^{ij} \quad \Longrightarrow \quad (z + hA_{ij}) \cap \Lambda_h = \varnothing \quad \text{if } i \neq j\,.$$

Lemma 5.3.2. *Let $v, \varphi\colon \tilde{\Lambda}_h \to \mathbb{R}$, and assume that $\varphi = D^h_{0,\nu}\varphi = 0$ on Γ_h. We then have that*

$$\begin{aligned}\sum_{z\in\Lambda_h\cup\Gamma_h} h^{\mathsf{d}} \Delta_h^2 v(z)\varphi(z) &= \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in\Lambda_h} h^{\mathsf{d}} D_i^h D_{-j}^h v(z) D_i^h D_{-j}^h \tilde{\varphi}(z) \\ &+ \frac{1}{2}\sum_{i=1}^{\mathsf{d}} \sum_{z\in\Gamma_h} h^{\mathsf{d}} D_i^h D_{-i}^h v(z) D_i^h D_{-i}^h \tilde{\varphi}(z) + \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in\Gamma_h^{ij}} h^{\mathsf{d}} D_i^h D_{-j}^h v(z) D_i^h D_{-j}^h \tilde{\varphi}(z)\,.\end{aligned} \tag{5.3.5}$$

So, if we define the scalar product $(f,g)_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)}$ on functions $f, g\colon \tilde{\Lambda}_h \to \mathbb{R}^{n\times n}$ by

$$(f,g)_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)} := \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in\Lambda_h} h^{\mathsf{d}} f_{ij}(z) g_{ij}(z) + \frac{1}{2}\sum_{i=1}^{\mathsf{d}} \sum_{z\in\Gamma_h} h^{\mathsf{d}} f_{ii}(z) g_{ii}(z) + \sum_{\substack{i,j=1\\ i\neq j}}^{\mathsf{d}} \sum_{z\in\Gamma_h^{ij}} h^{\mathsf{d}} f_{ij}(z) g_{ij}(z)\,,$$

then we have that

$$(\Delta_h^2 v, \varphi)_{L^2_h(\Lambda_h\cup\Gamma_h)} = (\nabla_h^2 v, \nabla_h^2 \varphi)_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)}\,. \tag{5.3.6}$$

In addition, if we also define for $f, g\colon \tilde{\Lambda}_h \to \mathbb{R}$ the scalar product

$$(f,g)_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)} := \sum_{z\in\Lambda_h} h^{\mathsf{d}} f_{ij}(z) g_{ij}(z) + \frac{1}{2}\sum_{z\in\Lambda_h} h^{\mathsf{d}} f_{ii}(z) g_{ii}(z)\,,$$

then we have, for any $i \in \{1, \dots, \mathsf{d}\}$, that

$$(D_i^h D_{-i}^h v, \varphi)_{L^2_h(\Lambda_h\cup\Gamma_h)} = (D_i^h D_{-i}^h v, \varphi)_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)} = (v, D_i^h D_{-i}^h \varphi)_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)}\,. \tag{5.3.7}$$

Proof. Define $\tilde{\varphi}\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$ as $\tilde{\varphi}(z) := \begin{cases} \varphi(z) & \text{for } z \in \Lambda_h \\ 0 & \text{otherwise} \end{cases}$. Then we can apply Lemma 5.3.1 to v, $\tilde{\varphi}$ and obtain

$$\sum_{z\in\Lambda_h\cup\Gamma_h} h^{\mathsf{d}}\Delta_h^2 v(z)\tilde{\varphi}(z) = \sum_{z\in\Lambda_h\cup\Gamma_h} h^{\mathsf{d}}\Delta_h v(z)\Delta_h\tilde{\varphi}(z) = \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in\Lambda_h\cup\Gamma_h} h^{\mathsf{d}} D_i^h D_{-j}^h v(z) D_i^h D_{-j}^h \tilde{\varphi}(z)\,.$$

We trivially have

$$\sum_{z\in\Lambda_h\cup\Gamma_h} h^{\mathsf{d}}\Delta_h^2 v(z)\tilde{\varphi}(z) = \sum_{z\in\Lambda_h\cup\Gamma_h} h^{\mathsf{d}}\Delta_h^2 v(z)\varphi(z)\,.$$

Furthermore, $D_i^h D_{-j}^h\tilde{\varphi}(z)$ is equal to $D_i^h D_{-j}^h\varphi(z)$ if $z \in \Lambda_h$. If $z \in \Gamma_h$ we have $D_i^h D_{-i}^h\tilde{\varphi}(z) = \frac{1}{2}D_i^h D_{-i}^h\varphi(z)$ and $D_i^h D_{-j}^h\tilde{\varphi}(z) = \begin{cases} D_i^h D_{-j}^h\varphi(z) & \text{for } z \in \Gamma_h^{ij} \\ 0 & \text{otherwise} \end{cases}$. Therefore,

$$\begin{aligned}\sum_{i,j=1}^{\mathsf{d}} \sum_{z\in\Lambda_h\cup\Gamma_h} h^{\mathsf{d}} D_i^h D_{-j}^h v(z) D_i^h D_{-j}^h \tilde{\varphi}(z) &= \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in\Lambda_h} h^{\mathsf{d}} D_i^h D_{-j}^h v(z) D_i^h D_{-j}^h \tilde{\varphi}(z) \\ &+ \frac{1}{2}\sum_{i=1}^{\mathsf{d}} \sum_{z\in\Gamma_h} h^{\mathsf{d}} D_i^h D_{-i}^h v(z) D_i^h D_{-i}^h \tilde{\varphi}(z) + \sum_{i,j=1}^{\mathsf{d}} \sum_{z\in\Gamma_h^{ij}} h^{\mathsf{d}} D_i^h D_{-j}^h v(z) D_i^h D_{-j}^h \tilde{\varphi}(z)\,.\end{aligned}$$

By combining the last three displayed equalities we deduce (5.3.5). With a similar argument we can obtain (5.3.7) from (5.3.3). □

Next, we state Poincaré-type inequalities for the two sets of boundary conditions considered.

Lemma 5.3.3. *Let $v\colon \tilde{\Lambda}_h \to \mathbb{R}$, and suppose that $\varphi = D_\nu^h\varphi = 0$ on Γ_h. Then,*

$$\|v\|_{H_h^2(\Lambda_h)} \le C\|\nabla_h^2 v\|_{L_{h,*}^2(\Lambda_h)}\,. \tag{5.3.8}$$

Lemma 5.3.4. *Let $v\colon \tilde{\Lambda}_h \to \mathbb{R}$, and suppose that $\varphi = D_{0,\nu}^h\varphi = 0$ on Γ_h. Then,*

$$\|v\|_{H_h^2(\Lambda_h)} \le C\|\nabla_h^2 v\|_{L_{h,\sim}^2(\Lambda_h)}\,. \tag{5.3.9}$$

Proof of Lemma 5.3.3. We can extend v by 0 to $(h\mathbb{Z})^{\mathsf{d}}$ without changing the statement of the lemma. Now observe that for $f\colon (h\mathbb{Z})^{\mathsf{d}} \to \mathbb{R}$ with support contained in a cube of side-length L, and $i \in \{1,\dots,\mathsf{d}\}$, we have the Poincaré inequality

$$\|f\|_{L_h^2((h\mathbb{Z})^{\mathsf{d}})} \le CL\|D_{\pm i}^h f\|_{L_h^2((h\mathbb{Z})^{\mathsf{d}})}\,.$$

Indeed this follows from the one-dimensional case, which can be proved by a straightforward summation by parts. If we apply this inequality to v and ∇v, we easily deduce (5.3.8). □

Proof of Lemma 5.3.4. Let $\tilde{v}(z) := \begin{cases} v(z) & \text{for } z \in \Lambda_h \\ 0 & \text{otherwise} \end{cases}$. Then, $\tilde{v}$ satisfies the assumptions of Lemma 5.3.3, so that

$$\|\tilde{v}\|_{H_h^2(\Lambda_h)} \le C\|\nabla_h^2\tilde{v}\|_{L_{h,*}^2(\Lambda_h)}\,.$$

Furthermore it is easy to check that

$$\|\nabla_h^2 \tilde{v}\|_{L^2_{h,*}(\Lambda_h)} \leq \|\nabla_h^2 v\|_{L^2_{h,\sim}(\Lambda_h)}$$

and

$$\|v\|_{H^2_h(\Lambda_h)} \leq 2\|\tilde{v}\|_{H^2_h(\Lambda_h)}\,,$$

and hence we directly deduce (5.3.9). □

5.3.2 Proofs of the main theorems

We have already sketched the proof of Theorem 5.1.1 in the introduction. We now provide additional details. Parts of the following argument already appeared in the proof of Theorem 4.2.3.

Proof of Theorem 5.1.1. As was mentioned at the start of Section 5.2.1, we can assume that u is supported in $\left[0, \frac{2}{3}\right)^2$. Let $E\colon \tilde{\Lambda}_h \to \mathbb{R}$ be defined by $E := u - U$. Then,

$$\begin{aligned} E &= 0 && \text{on } \Gamma_h\,, \\ D^h_{0,\nu} E &= D^h_{0,\nu} \tilde{u} && \text{on } \Gamma_h\,. \end{aligned}$$

Let $\hat{E}$ be the function from Lemma 5.2.5. Then,

$$\begin{aligned} E - \hat{E} &= 0 \quad \text{on } \Gamma_h\,, \\ D^h_{0,\nu}(E - \hat{E}) &= 0 \quad \text{on } \Gamma_h\,. \end{aligned}$$

Therefore, using the results from Section 5.3.1 we deduce that

$$\begin{aligned} \|\nabla_h^2 (E - \hat{E})\|^2_{L^2_{h,\sim}(\Lambda_h \cup \Gamma_h)} &= (\Delta_h^2 (E - \hat{E}), E - \hat{E})_{L^2_h(\Lambda_h \cup \Gamma_h)} \\ &= (\Delta_h^2 E, E - \hat{E})_{L^2_h(\Lambda_h \cup \Gamma_h)} - (\nabla_h^2 \hat{E}, \nabla_h^2 (E - \hat{E}))_{L^2_{h,\sim}(\Lambda_h \cup \Gamma_h)}\,. \end{aligned} \tag{5.3.10}$$

Using (5.1.9) we can rewrite $\Delta_h^2 E$ as follows

$$\begin{aligned} \Delta_h^2 E = \Delta_h^2 \tilde{u} - \Delta_h^2 U &= \Delta_h^2 \tilde{u} - T^{2,\dots,2} f = \Delta_h^2 \tilde{u} - T^{h,2,\dots,2} \Delta^2 \tilde{u} \\ &= \sum_{i=1}^{\mathsf{d}} D_i^h D_{-i}^h \Delta_h \tilde{u} - T^{h,2,\dots,2} \partial_i^2 \Delta \tilde{u} \\ &= \sum_{i=1}^{\mathsf{d}} D_i^h D_{-i}^h \Delta_h \tilde{u} - D_i^h D_{-i}^h T_1^{h,2} \dots T_{i-1}^{h,2} T_{i+1}^{h,2} \dots T_{\mathsf{d}}^{h,2} \Delta \tilde{u} \\ &= \sum_{i=1}^{\mathsf{d}} D_i^h D_{-i}^h \varphi_i\,, \end{aligned}$$

where we have abbreviated

$$\varphi_i := \Delta_h \tilde{u} - T_1^{h,2} \dots T_{i-1}^{h,2} T_{i+1}^{h,2} \dots T_{\mathsf{d}}^{h,2} \Delta \tilde{u}\,.$$

If we insert this into (5.3.10) and use the summation-by-parts formula (5.3.7) we arrive at

$$\|\nabla_h^2 (E - \hat{E})\|^2_{L^2_{h,\sim}(\Lambda_h \cup \Gamma_h)} = \sum_{i=1}^{\mathsf{d}} (\varphi_i, D_i^h D_{-i}^h (E - \hat{E}))_{L^2_h(\Lambda_h \cup \Gamma_h)} - (\nabla_h^2 \hat{E}, \nabla_h^2 (E - \hat{E}))_{L^2_{h,\sim}(\Lambda_h \cup \Gamma_h)}$$

$$\leq \left(\sum_{i=1}^{\mathsf{d}} \|\varphi_i\|_{L^2_h(\Lambda_h \cup \Gamma_h)} + \|\nabla_h^2 \hat{E}\|_{L^2_h(\Lambda_h \cup \Gamma_h)} \right) \|\nabla_h^2 (E - \hat{E})\|_{L^2_{h,\sim}(\Lambda_h \cup \Gamma_h)},$$

and thus

$$\|\nabla_h^2 E\|_{L^2_{h,\sim}(\Lambda_h \cup \Gamma_h)} \leq \|\nabla_h^2 \hat{E}\|_{L^2_h(\Lambda_h \cup \Gamma_h)} + \sum_{i=1}^{\mathsf{d}} \|\varphi_i\|_{L^2_h(\Lambda_h \cup \Gamma_h)}. \tag{5.3.11}$$

The first term on the right-hand side here is bounded by $Ch^{s-2}\|u\|_{H^s(\Omega)}$ by construction of $\hat{E}$. The summands of the sum can be bounded using the Bramble–Hilbert lemma as in the proof of [JS14, Theorem 2.68]. Let us sketch the argument for completeness:

Recall that

$$\varphi_i(x) = \Delta_h \tilde{u}(x) - T_1^{h,2} \dots T_{i-1}^{h,2} T_{i+1}^{h,2} \dots T_{\mathsf{d}}^{h,2} \Delta \tilde{u}(x).$$

Because $s > \frac{\mathsf{d}}{2}$,

$$|\Delta_h \tilde{u}(x)| \leq C(h) \|\tilde{u}\|_{L^\infty(x+(-h,h)^{\mathsf{d}})} \leq C(h) \|\tilde{u}\|_{H^s(x+(-h,h)^{\mathsf{d}})}.$$

In addition $s > \frac{5}{2}$ implies according to [JS14, Theorem 1.67] that

$$|T_1^{h,2} \dots T_{i-1}^{h,2} T_{i+1}^{h,2} \dots T_{\mathsf{d}}^{h,2} \Delta \tilde{u}(x)| \leq C(h) \|\tilde{u}\|_{H^s(x+(-h,h)^{\mathsf{d}})}.$$

Thus $\varphi_i(x)$ is a bounded linear functional of $\tilde{u} \in H^s(x + (-h,h)^{\mathsf{d}})$. This functional vanishes when $\tilde{u}|_{x+(-h,h)^{\mathsf{d}}}$ is a polynomial of degree at most 3. Indeed, then $\Delta \tilde{u}(y)$ is equal to some affine function $a(y)$, and $\Delta_h \tilde{u}(x) = a(x)$. On the other hand, the smoothing operators $T_j^{h,2}$ map affine functions to themselves, so that $\varphi_i(x) = 0$.

To summarize $\varphi_i(x)$ is a bounded linear functional of $\tilde{u} \in H^s(x + (-h,h)^{\mathsf{d}})$ that vanishes on polynomials of degree at most 3. Hence by the Bramble–Hilbert lemma it is bounded by $C(h)[\tilde{u}]_{H^s(x+(-h,h)^{\mathsf{d}})}$ for the range of s as in the statement of the theorem. Using a scaling argument to determine the correct prefactor of h, we obtain

$$\|\varphi_i\|_{L^2_{h,\sim}(\Lambda_h \cup \Gamma_h)} \leq Ch^{s-2}[\tilde{u}]_{H^s(\mathbb{R}^{\mathsf{d}})} \leq Ch^{s-2}\|u\|_{H^s(\Omega)} \tag{5.3.12}$$

for those s.

Now we substitute (5.3.12) into (5.3.11) and obtain the bound

$$\|\nabla_h^2 E\|_{L^2_{h,\sim}(\Lambda_h \cup \Gamma_h)} \leq Ch^{s-2}\|u\|_{H^s(\Omega)}$$

for the range of s as in the statement of the theorem. The discrete Poincaré inequality, Lemma 5.3.4, immediately implies the asserted error bound. □

Proof of Theorem 5.1.2. The proof is the same as that of Theorem 5.1.1. The only differences are that we work with the inner product $(\cdot,\cdot)_{L^2_{h,*}(\Lambda_h \cup \Gamma_h)}$ instead of $(\cdot,\cdot)_{L^2_{h,\sim}(\Lambda_h \cup \Gamma_h)}$, use $\hat{E}^*$ instead of $\hat{E}$, and Lemma 5.3.3 instead of Lemma 5.3.4. □

5.4 Further remarks

5.4.1 Variants and extensions

Let us finally collect some miscellaneous remarks on possible variations of our results and their proofs.

Remark 5.4.1. By Section 5.2.3 we know that there are extensions of the boundary values of $\tilde{u}$ with controlled $\|\cdot\|_\sim$-norm and $\|\cdot\|_*$-norm respectively. In fact, the optimal such extension is in both cases the biharmonic extension of the boundary values, i.e., the unique function V with the given boundary values that satisfies $\Delta_h^2 V = 0$ in Ω. Indeed, if ψ is a function such that $\psi = 0$, $D_{0,\nu}^h \psi = 0$ on Γ, then

$$\|\nabla_h^2(V+\psi)\|^2_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)} = \|\nabla_h^2 V\|^2_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)} + \|\nabla_h^2\psi\|^2_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)} + 2(\nabla_h^2 V, \nabla_h^2\psi)_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)}$$

and $(\nabla_h^2 V, \nabla_h^2\psi)_{L^2_{h,\sim}(\Lambda_h\cup\Gamma_h)} = 0$, and similarly for $\|\cdot\|_{L^2_{h,*}(\Lambda_h\cup\Gamma_h)}$. This means that we could assume $\hat{E}$ to be discretely biharmonic, and this would simplify the proof of Theorem 5.1.1 slightly. However, for more general fourth-order elliptic operators one cannot use this fact, so we chose to avoid it here.

Remark 5.4.2. Using function space interpolation as in Lemma 5.4.9 it is possible to deduce the intermediate cases of Theorem 5.1.1 and 5.1.2 from the borderline cases $s = 4$ (or $s = 3$) and $s = \frac{5}{2} + \varepsilon$. Our method of proof for $s = 4$ (or $s = 3$) however directly yields the desired bounds for all relevant s, without the need to resort to function space interpolation here.

Remark 5.4.3. Our smoothing operator $T^{h,2,\dots,2}$ has the advantage that it is given by convolution with a kernel with support in $[-h,h]^{\mathsf{d}}$ so that, when it is applied to f, the values in Λ_h of the resulting function do not involve values of f outside Ω. However, one might want to use stronger mollification operators, as in [JIS85], for example. In particular using $T^{h,3,\dots,3}$ would allow one to weaken the assumptions on s to $s \geq \frac{1}{2}\max(3,\mathsf{d})$. This is possible if one extends f to a function in $H^{s-4}(\mathbb{R}^{\mathsf{d}})$ in some way or redefines the finite difference scheme appropriately near the boundary. Apart from this issue, our proof applies equally well to regularization by $T^{h,3,\dots,3}$. See Theorem 4.2.3 in Chapter 4 for a result of this kind.

Remark 5.4.4. In (5.1.2) and (5.1.4) we regularized the right-hand side by applying $T^{h,2,\dots,2}$. One might wonder whether some other choice of a regularizing operator, T_h' say, would have been equally appropriate here.

While we do not have a full answer to this question, we shall present a few necessary conditions on T_h' that will clarify why $T^{h,2,\dots,2}$ is a natural choice. We only consider T_h' defined by convolution with some kernel Θ_h, where $\Theta_h = \Theta\left(\frac{\cdot}{h}\right)$ for some $\Theta\colon \mathbb{R}^{\mathsf{d}} \to \mathbb{R}$. As discussed in the previous remark, Θ should have support in $[-1,1]^{\mathsf{d}}$. We want $T_h' f$ to be continuous for each $f \in H^{s-4}(\Omega)$, where $s > \frac{5}{2}$, and this requires $\Theta \in \bigcap_{s>\frac{5}{2}} H^{4-s}(\Omega)$. Furthermore, $T_h' f$ should approximate f in some sense, and thus we require $T_h' f \to f$ pointwise as $h \to 0$ for any $f \in C_c^\infty(\Omega)$, say.

Suppose now that the analogues of Theorem 5.1.1 and Theorem 5.1.2 hold with T_h' in place of $T^{h,2,\dots,2}$. This means that $u \mapsto U'$ is uniformly bounded in h as a map from $H_0^s(\Omega)$ into $H_h^2(\Lambda_h)$ for any $\frac{5}{2} < s \leq 4$; note that one can easily verify that $\|u\|_{H_h^2} \leq C\|u\|_{H^s(\Omega)}$, uniformly in $h \leq 1$. This means that $f \mapsto T_h' f$ is uniformly bounded in h as a map from $H^{s-4}(\Omega)$ into $H_h^{-2}(\Lambda_h)$ [4]. and therefore

$$(T_h' f, \varphi)_{L_h^2(\Omega_h)} \leq C\|f\|_{H^{s-4}(\Omega)}\|\varphi\|_{H_h^2(\Omega_h)} \quad \forall \varphi \in H^2_{h,0}(\Lambda_h)\,,$$

uniformly in $h \leq 1$. After a short calculation one sees that this implies that

$$\left\| \sum_{x\in\Lambda_h} \Theta\left(\frac{x-\cdot}{h}\right)\varphi(x) \right\|_{H_0^{4-s}(\Omega)} \leq C\|\varphi\|_{H_h^2(\Lambda_h)} \quad \forall\, \varphi \in H^2_{h,0}(\Omega_h)\,, \tag{5.4.1}$$

[4] Here the norm on $H_h^{-2}(\Lambda_h)$ is given as the dual of the norm on $H^2_{h,0}(\Lambda_h)$, the subspace of $H_h^2(\Lambda_h)$ consisting of those functions which are 0 outside of Λ_h.

uniformly in $h \leq 1$. In particular the H^{4-s}-seminorm of the term on the left stays bounded as $h \to 0$. Choosing $s > 1$ and using test functions φ of the form $\varphi(x) = (a \cdot x + b)\eta(x)$ for some cut-off function η that is equal to 1 on some open set, one can show that (5.4.1) implies that

$$\sum_{x \in \mathbb{Z}^{\mathsf{d}}} (a \cdot x + b)\Theta(x - \cdot) \quad \text{is affine for each } a \in \mathbb{R}^{\mathsf{d}} \text{ and } b \in \mathbb{R}.$$

This affine function needs to be the same function $y \mapsto a \cdot y + b$ as otherwise $T'_h f$ does not approximate f for functions f that are locally equal to $y \mapsto a \cdot y + b$. Therefore we actually need that

$$\sum_{x \in \mathbb{Z}^{\mathsf{d}}} (a \cdot x + b)\Theta(x - \cdot) = a \cdot y + b \text{ is affine for each } a \in \mathbb{R}^{\mathsf{d}} \text{ and } b \in \mathbb{R}\,. \tag{5.4.2}$$

This is a rather strong condition on Θ. If we also recall the requirements $\operatorname{supp}\Theta \subset [-1,1]^{\mathsf{d}}$ and $\Theta \in \bigcap_{s > \frac{5}{2}} H^{4-s}(\Omega)$, then in dimension $\mathsf{d} = 1$ the only remaining Θ is given by $\Theta(x) = \theta_2(x)$. In dimension $\mathsf{d} \geq 2$ there are other choices beyond $\Theta(x) = \theta_2(x_1) \cdot \ldots \cdot \theta_2(x_{\mathsf{d}})$, but that kernel is the unique one if we also demand that it factorizes into functions of the d coordinates. For further results on mollifiers in Sobolev spaces the reader is referred to [JS14, Section 1.9].

5.4.2 Density results

This section is concerned with the various definitions of the space $H^s_{(k)}$ in the introduction. Let us recall what we want to prove.

Lemma 5.4.5. *Let Let $\Xi = I_1 \times \cdots \times I_{\mathsf{d}}$, where $I_j \subset \mathbb{R}$ are (possibly unbounded) open intervals, $s \in \mathbb{R}$, $s \geq 0$, and $k \in \mathbb{N}_0$ such that $k + \frac{1}{2} < s$. Then, the following spaces are equal:*

i) $H^s_{(k)}(\Xi)$, the space of all $u \in H^s(\Xi)$ such that the traces of $\partial^i_\nu u$ for $0 \leq i \leq k$ vanish on $\partial\Xi$;

ii) $\overline{\{u \in C^\infty(\overline{\Xi}) \colon \partial^i_\nu u = 0 \text{ on } \partial\Xi \ \forall\, i \leq k\}}^{\|\cdot\|_{H^s(\Xi)}}$, the closure in the $H^s(\Xi)$-norm of the set of all functions in $C^\infty(\overline{\Xi})$ whose derivatives up to order k vanish on $\partial\Xi$;

iii) $H^s(\Xi) \cap H_0^{\min(k+1,s)}(\Xi)$.

Remark 5.4.6. This result actually holds in far more generality (with basically the same proof): on the one hand one can replace the condition $\partial^i_\nu u = 0$ for $0 \leq i \leq k$ by the more general condition $\partial^i_\nu u = 0$ for $i \in K$, where $K \subset \mathbb{N}$, as long as $s - \frac{1}{2} \notin K$. On the other hand one can take Ξ to be any domain with Lipschitz boundary. The only additional difficulty then is to define $\partial^i_\nu u$ in view of the fact that ν is in general only a measurable function. However if one defines $\partial^i_\nu u$ as the appropriate linear combination of the traces of $\partial^\alpha u$ for $|\alpha| = i$ (cf. [MM13, p. 156]) the results still hold.

Proof of Lemma 5.4.5. As was already remarked in Section 5.1.3, for the Lipschitz domain Ξ, every function in $C_c^\infty(\overline{\Xi})$ is the restriction of a function in $C_c^\infty(\mathbb{R}^{\mathsf{d}})$ to Ξ. In particular, we have that $C_c^\infty(\overline{\Xi}) \subset H^s(\Xi)$.

We will prove the inclusions

$$\overline{\{u \in C_c^\infty(\overline{\Xi}) \colon \partial^i_\nu u = 0 \text{ on } \partial\Xi \ \forall\, i \leq k\}}^{\|\cdot\|_{H^s(\Xi)}} \subset H^s(\Xi) \cap H_0^{\min(k+1,s)}(\Xi)\,, \tag{5.4.3}$$

$$H^s(\Xi) \cap H_0^{\min(k+1,s)}(\Xi) \subset H^s_{(k)}(\Xi)\,, \tag{5.4.4}$$

$$H^s_{(k)}(\Xi) \subset \overline{\left\{u \in C_c^\infty(\overline{\Xi})\colon \partial_\nu^i u = 0 \text{ on } \partial\Xi \ \forall\, i \le k\right\}}^{\|\cdot\|_{H^s(\Xi)}}. \tag{5.4.5}$$

The inclusion (5.4.4) follows immediately from the definitions and standard trace theorems.

Next observe that trivially

$$\overline{\left\{u \in C_c^\infty(\overline{\Xi})\colon \partial_\nu^i u = 0 \text{ on } \partial\Xi \ \forall\, i \le k\right\}}^{\|\cdot\|_{H^s(\Xi)}} \subset H^s(\Xi)\,,$$

so in order to prove (5.4.3) we only need to verify that

$$\overline{\left\{u \in C_c^\infty(\overline{\Xi})\colon \partial_\nu^i u = 0 \text{ on } \partial\Xi \ \forall\, i \le k\right\}}^{\|\cdot\|_{H^s(\Xi)}} \subset H_0^{\min(k+1,s)}(\Xi)\,.$$

To see this, it suffices to prove that

$$\overline{\left\{u \in C_c^\infty(\overline{\Xi})\colon \partial_\nu^i u = 0 \text{ on } \partial\Xi \ \forall\, i \le k\right\}}^{\|\cdot\|_{\mathbf{H}^{\min(k+1,s)}(\Xi)}} \subset H_0^{\min(k+1,s)}(\Xi)\,.$$

This follows from general theory (e.g. [MM13, Theorem 3.18]), but it is also easy to verify by direct calculations: we need to check that we can approximate any function $v \in \left\{u \in C_c^\infty(\overline{\Xi})\colon \partial_\nu^i u = 0 \text{ on } \partial\Xi \ \forall\, i \le k\right\}$ with $C_c^\infty(\Xi)$-functions in the $H^{\min(k+1,s)}$-norm.

The proof of this assertion proceeds as follows. The assumptions on v imply that the extension $\bar{v}$ of v by 0 to $\mathbb{R}^\mathsf{d}$ is in $C^k(\mathbb{R}^\mathsf{d})$. In addition, $\bar{v} \in H^{k+1}(\mathbb{R}^\mathsf{d})$. To verify this one can use that all derivatives of v of order k are continuous, have zero trace, and are in $H^1(\Xi)$. Hence, their extensions by zero belong to $H^1(\mathbb{R}^\mathsf{d})$. This is well known for general Lipschitz domains (and is easily seen by a partition of unity argument and transformation to the half-space situation by composition with a bi-Lipschitz map). Now dilation is continuous in $H^{k+1}(\mathbb{R}^\mathsf{d})$, and hence v can be approximated by $H_0^{k+1}(\Xi)$ functions in the H^{k+1} norm. Thus, $v \in H_0^{k+1}(\Xi)$. Consequently, v can be approximated in the $\|\cdot\|_{H^{k+1}}$ norm, and in particular in the possibly weaker norm $\|\cdot\|_{H^{\min(s,k+1)}}$ by $C_c^\infty(\overline{\Xi})$ functions.

It remains to prove (5.4.5). We first observe that

$$\left\{u \in C_c^\infty(\overline{\Xi})\colon \partial_\nu^i u = 0 \text{ on } \partial\Xi \ \forall\, i \le k\right\} = C_c^\infty(\overline{\Xi}) \cap H^s_{(k)}(\Xi)\,.$$

Taking this into account, we need to verify that $C_c^\infty(\overline{\Xi}) \cap H^s_{(k)}(\Xi)$ is dense in $H^s_{(k)}(\Xi)$. It is easy to see that $C_c^\infty(\overline{\Xi}) \cap H^s_{(k)}(\Xi)$ is dense in $C^\infty(\overline{\Xi}) \cap H^s_{(k)}(\Xi)$, so it remains to prove that the latter space is dense in $H^s_{(k)}(\Xi)$. To see this we apply the criterion of Burenkov [Bur98, Theorem 2 on p. 49]. The first three assumptions of that theorem are obviously satisfied, and for the fourth we need to check that every $u \in H^s_{(k)}(\Xi)$ of compact support is continuous under translations, which is once again clear. □

5.4.3 Remarks on Interpolation

In this section we shall collect and discuss various results on interpolation spaces that were used in our work. As we only consider Hilbert spaces, we do not need the theory of interpolation spaces in its full generality and can make some simplifications.

We consider two separable Hilbert spaces X and Y such that $X \subset Y$ is dense and the injection is continuous. Then, given $\theta \in [0,1]$, we can consider the associated interpolation spaces $[X,Y]_\theta := D(\Lambda^{1-\theta})$ equipped with the graph norm, where Λ is a self-adjoint positive operator on Y with domain X (see [LM72a, Section 2] for details, starting with the nontrivial

fact that such a Λ always exists). Because we are considering Hilbert spaces, this definition yields up to equivalence of norms the same space as the complex interpolation space $[X, Y]_{[\theta]}$ or the real interpolation space $[X, Y]_{\theta,2}$ (see [LM72a, Section 14.2 and Section 15] for proofs). Thus we will be able to freely use results for either of these interpolation techniques from the literature.

Our first task is to study whether the spaces $H^s_{(k)}(\Xi)$ form an interpolation scale, where $\Xi \subset \mathbb{R}^d$ is open and connected. If Ξ has a smooth boundary this was shown in [Gri67] with an alternative proof in [Löf92]. However we are interested in the cases $\Xi = (0,1)^d$ or $\Xi = (0,\infty)^d$, and the two aforementioned proofs do not easily extend to that case. On the other hand, if Ξ has Lipschitz boundary then there are results concerning the interpolation scales $H^s(\Xi)$ and $H^s_0(\Xi)$ (see e.g. [Bra95]), but not for our mixed case.

Fortunately, in our case it is possible to use the fact that our domain is a cartesian product in combination with results from [LM72a] to give a proof of the desired result by induction on the dimension.

We begin by stating a one-dimensional but vector-valued result that we will need in the proof of the following lemmas. In addition to the notation from the introduction, we define $H^s_\#(I)$, where $I \subset \mathbb{R}$ is an open interval such that $0 \in I$, as the closure of the linear space of functions u contained in $C^\infty(I) \cap H^s(I)$ with $u'(0) = 0$ in the $\|\cdot\|_{H^s}$-norm.

Lemma 5.4.7. *Let E be a separable Hilbert space, $I \subset \mathbb{R}$ a (possibly unbounded) open interval, and $k \in \mathbb{N}$. Let $s \geq t \geq 1$, and let $\theta \in (0,1)$. If $s - \frac{1}{2} \notin \{0,1,\dots,k\}$ and $(1-\theta)s - \frac{1}{2} \notin \{0,1,\dots,k\}$, then we have that*

$$\left[H^s_{(k)}(I,E), L^2(I,E)\right]_\theta = H^{(1-\theta)s}_{(k)}(I,E)\,. \tag{5.4.6}$$

Furthermore, if $0 \in I$, $s \neq \frac{3}{2}$ and $(1-\theta)s \neq \frac{3}{2}$, then

$$\left[H^s_\#(I,E), L^2(I,E)\right]_\theta = H^{(1-\theta)s}_\#(I,E)\,. \tag{5.4.7}$$

Proof. If $E = \mathbb{R}$ then (5.4.6) is a special case of [Gri67, Théorème 8.1]. The Hilbert-space-valued case follows from a simple general tensorization argument, see Lemma 5.4.8 below.

For (5.4.7) it suffices again to consider the case $E = \mathbb{R}$. The inclusion "$\subset$" is straightforward. For the converse inclusion we adapt the strategy from [Gri67]. Our goal is to construct for any given $f \in H^{(1-\theta)s}_{(k)}(I)$ some $u \in L^2(\mathbb{R}^+, H^s_{(k)}(I) \cap H^{\frac{1}{2\theta}}(\mathbb{R}^+, L^2(I))$ with $u(\cdot,0) = f$ (cf. [Gri67, Definition 2.2]).

If $s < \frac{3}{2}$ then $H^s_\#(I) = H^s(I)$ and the assertion follows by standard results. Thus we may assume $s > \frac{3}{2}$.

We first assume that $s - \frac{1}{2} \notin \mathbb{N}$ and $(1-\theta)s - \frac{1}{2} \notin \mathbb{N}$. We first define the extension u of f on $I \cap \mathbb{R}^+$ and $I \cap R^-$ separately. Let $\eta \in C^\infty([0,1))$ with $\eta = 1$ on $[0, \frac{1}{2}]$ and set, for $x \in I \cap \mathbb{R}^+$,

$$\begin{aligned}
f_0^\pm(x) &= f(x)\,,\\
f_k &= 0 \quad \text{for } 1 \leq k < \frac{1}{2\theta} - \frac{1}{2}\,,\\
g_1^\pm(0,y) &= 0\,,\\
g_j^\pm &= (\pm 1)^j \frac{\partial^j f}{\partial \nu^j}(0)\eta(y) \quad \text{for } j < (1-\theta)s - \frac{1}{2} \text{ and } j \neq 1,\\
g_j^\pm(0,y) &= 0 \qquad \text{for } (1-\theta)s - \frac{1}{2} \leq j < s - \frac{1}{2}\,.
\end{aligned}$$

Then, the compatibility conditions in [Gri67, Théorème 7.2] are satisfied and thus there exist

$$u^{\pm} \in L^2(\mathbb{R}^+, H^s(I \cap \mathbb{R}^{\pm})) \cap H^{\frac{1}{2\theta}}(\mathbb{R}^+, L^2(I \cap \mathbb{R}^{\pm}))$$

such that

$$\begin{aligned} u^{\pm}(0,x) &= f(x)\,, \\ \frac{\partial^j}{\partial \nu^j} u^{\pm}(y,0) &= g_j^{\pm}(y)\,. \end{aligned}$$

Set $u(y,x) = u^{\pm}(y,x)$ for $\pm x > 0$. Then, in particular $u'(y,0) = 0$. The condition $f \in H^{(1-\theta)s}$ implies that $(-1)^j g_j^- = g_j^+$ for $j < (1-\theta)s - \frac{1}{2}$ and hence this condition holds for all j. Because $s \notin \mathbb{N}_0 + \frac{1}{2}$ it follows that $u(\cdot, y) \in H^s(I)$ for all $y > 0$ and in fact $u(\cdot, y) \in H^s_{\#}(I)$. Thus,

$$u \in L^2(\mathbb{R}^+, H^s_{\#}(I)) \cap H^{\frac{1}{2\theta}}(\mathbb{R}^+, L^2(I))$$

and $u(0,x) = f(x)$. By [Gri67, Definition 2.2] we deduce that $f \in \left[H^s_{\#}(I), L^2(I)\right]_\theta$, and this concludes the proof of (5.4.7).

It remains to remove the assumptions $(1-\theta)s - \frac{1}{2} \notin \mathbb{N}$ and $s - \frac{1}{2} \notin \mathbb{N}$. This can easily be handled by using [LM72a, Theorem 13.3] and reiteration. For the convenience of the reader we give the details.

Consider the case $s - \frac{1}{2} \in \mathbb{N}$, but $(1-\theta)s - \frac{1}{2} \notin \mathbb{N}$. Let $s_* > s > \frac{3}{2}$ be such that $s_* - \frac{1}{2} \notin \mathbb{N}$ and let θ_* be such that $s = (1-\theta_*)s_*$. By the reiteration theorem [LM72a, Theorem 6.1] we have

$$\left[H^s_{\#}(I), L^2(I)\right]_\theta = \left[\left[H^{s_*}_{\#}(I), L^2(I)\right]_{\theta_*}, L^2(I)\right]_\theta = \left[H^{s_*}_{\#}(I), L^2(I)\right]_{\theta+\theta_*-\theta\theta_*}$$

and the right-hand side equals $H^{(1-\theta)s}_{\#}(I)$ by what we have already shown (note that $(1-(\theta+\theta_*-\theta\theta_*))s_* = (1-\theta)s$).

Next consider the remaining case that $(1-\theta)s - \frac{1}{2} \in \mathbb{N}$. Choose $\theta_- < \theta < \theta_+$ close enough to θ such that $(1-\theta_\pm)s - \frac{1}{2} \notin \mathbb{N}$ and $\frac{3}{2} \notin [(1-\theta_+)s, (1-\theta_-)s]$. Let $\tilde{\theta}$ be such that $\theta = (1-\tilde{\theta})\theta_- + \tilde{\theta}\theta_+$. Again by reiteration and the previous results we have

$$\left[H^s_{\#}(I), L^2(I)\right]_\theta = \left[\left[H^s_{\#}(I), L^2(I)\right]_{\theta_-}, \left[H^s_{\#}(I), L^2(I)\right]_{\theta_+}\right]_{\tilde{\theta}} = \left[H^{(1-\theta_-)s}_{\#}(I), H^{(1-\theta_+)s}_{\#}(I)\right]_{\tilde{\theta}}$$

and it suffices to show that the right-hand side equals $H^{(1-\tilde{\theta})(1-\theta_-)s+\tilde{\theta}(1-\theta_+)s}_{\#}(I) = H^{(1-\theta)s}_{\#}(I)$. To that end, observe that

$$H^t_{\#}(I) = \begin{cases} \{f \in H^t(I) \colon f'(0) = 0\} & \text{for } t > \frac{3}{2}, \\ H^t(I) & \text{for } t < \frac{3}{2}, \end{cases}$$

is a closed subspace of $H^t(I)$ of finite codimension for any $t \neq \frac{3}{2}$. Now [LM72a, Theorem 13.3] implies that for $t < t' < \frac{3}{2}$ or $\frac{3}{2} < t < t'$ and $\hat{\theta} \in [0,1]$ we have

$$\left[H^{t'}_{\#}(I), H^t_{\#}(I)\right]_{\hat{\theta}} = H^{(1-\hat{\theta})t'+\hat{\theta}t}_{\#}(I)\,.$$

In particular,

$$\left[H^{(1-\theta_-)s}_{\#}(I), H^{(1-\theta_+)s}_{\#}(I)\right]_{\tilde{\theta}} = H^{(1-\tilde{\theta})(1-\theta_-)s+\tilde{\theta}(1-\theta_+)s}_{\#}(I)\,.$$

This completes the proof.

□

Lemma 5.4.8. *Let $X \subset Y$ be Hilbert spaces of real-valued functions, as above. Let E be a separable Hilbert space and denote by $X \otimes E$ and $Y \otimes E$ the corresponding spaces of E-valued functions. Then, for all $\theta \in (0,1)$,*

$$[X \otimes E, Y \otimes E]_\theta = [X, Y]_\theta \otimes E\,,$$

with equivalent norms. Here for a Hilbert space Z of real-valued functions the scalar product on $Z \otimes E$ is defined as follows. If $(e_m)_{m=1}^\infty$ is an orthonormal basis of E and $f = \sum_{m=1}^\infty f_m e_m$, $g = \sum_{m=1}^\infty g_m e_m$, then

$$(f, g)_{X\otimes E} = \sum_{m=1}^\infty (f_m, g_m)_X\,.$$

Proof. To show the inclusion "$\supset$" let $a \in [X, Y]_\theta \otimes E$ and $\delta > 0$. Then $a_m \in [X, Y]_\theta$ and by [Gri67, Definition 2.2]) there exist $u_m \in L^2(\mathbb{R}^+, X) \cap H^{\frac{1}{2\theta}}(\mathbb{R}^+, Y)$ with $a_m = u_m(0)$ and

$$\|u_m\|_{L^2(\mathbb{R}^+,X)} + \|u_m\|_{H^{\frac{1}{2\theta}}(\mathbb{R}^+,Y)} \le (1+\delta)\|a_m\|_{[X,Y]_\theta}, \qquad m = 1, 2, \dots\,.$$

Taking the square and summing over m we see that

$$S := \sum_{m=1}^\infty \|u_m\|^2_{L^2(\mathbb{R}^+,X)} + \|u_m\|^2_{H^{\frac{1}{2\theta}}(\mathbb{R}^+,Y)} \le (1+\delta)^2\|a\|^2_{[X,Y]_\theta\otimes E}\,.$$

Set $u := \sum_{m=1}^\infty u_m$. Since $S < \infty$ we see that $u \in L^2(\mathbb{R}^+, X \otimes E) \cap H^{\frac{1}{2\theta}}(\mathbb{R}^+, Y \otimes E)$. Thus $a = u(0) \in [X \otimes E, Y \otimes E]_\theta$ and

$$\|a\|^2_{[X\otimes E, Y\otimes E]_\theta} \le 2S \le 2(1+\delta)^2\|a\|^2_{[X,Y]_\theta\otimes E}\,.$$

The proof of the converse inclusion, "$\subset$", is similar: Let $a \in [X \otimes E, Y \otimes E]_\theta$. Then there is a $u \in L^2(\mathbb{R}^+, X \otimes E) \cap H^{\frac{1}{2\theta}}(\mathbb{R}^+, Y \otimes E)$ with $u(0) = a$ and

$$\|u\|_{L^2(\mathbb{R}^+,X\otimes E)} + \|u\|_{H^{\frac{1}{2\theta}}(\mathbb{R}^+,Y\otimes E)} \le (1+\delta)\|a\|_{[X\otimes E, Y\otimes E]_\theta}\,.$$

In particular $u_m = (u, e_m)_E$ satisfies $u_m \in L^2(\mathbb{R}^+, X) \cap H^{\frac{1}{2\theta}}(\mathbb{R}^+, Y)$, $m = 1, 2, \dots$. Thus $a_m = u_m(0) \in [X, Y]_\theta$, and we have

$$\begin{aligned}
\|a\|^2_{[X,Y]_\theta\otimes E} &= \sum_{m=1}^\infty \|a_m\|^2_{[X,Y]_\theta} \\
&\le 2\sum_{m=1}^\infty \|u_m\|^2_{L^2(\mathbb{R}^+,X)} + \|u_m\|^2_{H^{\frac{1}{2\theta}}(R^+,Y)} \\
&= 2\|u\|^2_{L^2(\mathbb{R}^+,X)} + 2\|u\|^2_{H^{\frac{1}{2\theta}}(R^+,Y)} \\
&\le 2(1+\delta)^2\|a\|^2_{[X\otimes E, Y\otimes E]_\theta}\,.
\end{aligned}$$

That completes the proof of the lemma. □

Now we can establish the desired interpolation results in higher dimensions. For the following lemma, we are interested in the cases $\Xi = \Omega$ and $k = 1$ or $\Xi = (0,\infty)^{\mathsf{d}}$ and $k = 0$.

Lemma 5.4.9. *Let $\Xi = I_1 \times \cdots \times I_{\mathsf{d}}$, where $I_j \subset \mathbb{R}$ are (possibly unbounded) open intervals. Let $K = \{k_1, \dots, k_m\} \subset \mathbb{N}_0$. Let $s \ge t \ge 0$, and let $\theta \in [0,1]$. If none of $s - \frac{1}{2}$, $t - \frac{1}{2}$ and $(1-\theta)s + \theta t - \frac{1}{2}$ are in $\{0, 1, \dots k\}$, then*

$$\left[H^s_{(k)}(\Xi), H^t_{(k)}(\Xi)\right]_\theta = H^{(1-\theta)s+\theta t}_{(k)}(\Xi)\,, \tag{5.4.8}$$

and, in particular, if $s \notin \{0, 1, \dots k\}$, $(1-\theta)s - \frac{1}{2} \notin \{0, 1, \dots k\}$, *then*

$$\left[H^s_{(k)}(\Xi), L^2(\Xi)\right]_\theta = H^{(1-\theta)s}_{(k)}(\Xi)\,. \tag{5.4.9}$$

Proof. The identity (5.4.8) immediately follows from (5.4.9) and reiteration, so it suffices to establish (5.4.9).

We proceed by induction on d. The case $\mathsf{d} = 1$ was established in Lemma 5.4.7. Now assume that the theorem holds for $\mathsf{d} - 1$ dimensions. The following argument is similar to the one in Section 2.1 in [LM72b].

Let $\Xi' = I_1 \times \cdots \times I_{\mathsf{d}-1}$, and write $x = (x', x_\mathsf{d})$. If we interpret a function $\Xi \to \mathbb{R}$ as a function $I_\mathsf{d} \to (\Xi' \to \mathbb{R})$, we claim that

$$L^2(\Xi) = L^2(I_\mathsf{d}, L^2(\Xi')) \tag{5.4.10}$$

and

$$H^s_{(k)}(\Xi) = L^2(I_\mathsf{d}, H^s_{(k)}(\Xi')) \cap H^s_{(k)}(I_\mathsf{d}, L^2(\Xi'))\,. \tag{5.4.11}$$

Indeed, (5.4.10) is obvious. For (5.4.11) one can argue as follows. It is well-known (and can be proved using the Fourier transform, for example) that

$$H^s(\mathbb{R}^\mathsf{d}) = L^2(\mathbb{R}, H^s(\mathbb{R}^{\mathsf{d}-1})) \cap H^s(\mathbb{R}, L^2(\mathbb{R}^{\mathsf{d}-1}))\,. \tag{5.4.12}$$

The sets Ξ' and I_d have Lipschitz boundary, and so there exists an extension operator E, mapping $H^t(\Xi)$ continuously to $H^t(\mathbb{R}^\mathsf{d})$ for $t \in \{0, s\}$, that also maps $H^t(\Xi' \times \{x_\mathsf{d}\})$ continuously to $H^t(\mathbb{R}^{\mathsf{d}-1} \times \{x_\mathsf{d}\})$ for any $x_\mathsf{d} \in I_\mathsf{d}$. One can construct such an E by first applying an appropriate extension operator on each slice $\Xi' \times \{x_\mathsf{d}\} \subset \mathbb{R}^{\mathsf{d}-1} \times \{x_\mathsf{d}\}$ and then extending in the direction e_d. Using this extension operator, one can easily check that (5.4.12) implies also that

$$H^s(\Xi) = L^2(I_\mathsf{d}, H^s(\Xi')) \cap H^s(I_\mathsf{d}, L^2(\Xi'))\,.$$

From this we want to deduce (5.4.11) by considering the faces of Ξ separately. We begin with "$\subset$" in (5.4.11). Let $u \in H^s_{(k)}(\Xi)$, and take $j \le \mathsf{d} - 1$. Let $\Xi'_{j,\pm} \times I_\mathsf{d}$ be the two faces of Ξ orthogonal to e_j. By assumption the trace of $\partial_j^i u$ for $i \le k$ vanishes on $\Xi'_{j,\pm} \times I_\mathsf{d}$ as an element of $H^{s-j-1/2}(\Xi'_{j,\pm} \times I_\mathsf{d})$ and thus also as an element of $L^2(I_\mathsf{d}, H^{s-j-1/2}(\Xi'_{j,\pm})$. In particular, for almost every x_d the trace of $\partial_j^i u(\cdot, x_\mathsf{d})$ vanishes on $\Xi'_{j,\pm} \times \{x_\mathsf{d}\}$. This holds for all $j \le \mathsf{d} - 1$ and all $i \le k$, and so $u \in L^2(I_\mathsf{d}, H^s_{(k)}(\Xi'))$. We can argue similarly for the case $j = \mathsf{d}$ to deduce that $u \in H^s_{(k)}(I_\mathsf{d}, L^2(\Xi'))$ and have thus shown "$\subset$" in (5.4.11). The argument for "$\supset$" is analogous. Thus we have established (5.4.11).

We have that $L^2(I_\mathsf{d}, H^s_{(k)}(\Xi'))$ is the domain of an unbounded positive operator Λ_1 on $L^2(I_\mathsf{d}, L^2(\Xi'))$, and Λ_1 is an operator in x', independent of x_d. Similarly, $H^s_{(k)}(I_\mathsf{d}, L^2(\Xi'))$ is the domain of an unbounded positive operator Λ_2 on $L^2(I_\mathsf{d}, L^2(\Xi'))$, and Λ_2 is an operator in x', independent of x_d. In particular, Λ_1 and Λ_2 commute. Thus we can apply the criterion for the interpolation space of an intersection [LM72a, Theorem 13.1] and obtain that

$$\begin{aligned}
&\left[H^s_{(k)}(\Xi), L^2(\Xi)\right]_\theta \\
&\quad = \left[L^2(I_\mathsf{d}, H^s_{(k)}(\Xi')) \cap H^s_{(k)}(I_\mathsf{d}, L^2(\Xi')), L^2(I_\mathsf{d}, L^2(\Xi'))\right]_\theta \\
&\quad = \left[L^2(I_\mathsf{d}, (H^s_{(k)}(\Xi')), L^2(I_\mathsf{d}, L^2(\Xi'))\right]_\theta \cap \left[H^s_{(k)}(I_\mathsf{d}, L^2(\Xi')), L^2(I_\mathsf{d}, L^2(\Xi'))\right]_\theta .
\end{aligned}$$

Now, according to [LM72a, Remark 14.4] and using the induction hypothesis (5.4.9) for $\mathsf{d}-1$, we have

$$\begin{aligned}\left[L^2(I_{\mathsf{d}}, H^s_{(k)}(\Xi')), L^2(I_{\mathsf{d}}, L^2(\Xi'))\right]_\theta &= L^2\left(I_{\mathsf{d}}, \left[H^s_{(k)}(\Xi'), L^2(\Xi')\right]_\theta\right)\\ &= L^2(I_{\mathsf{d}}, H^{(1-\theta)s}_{(k)}(\Xi'))\,.\end{aligned}$$

Similarly, using (5.4.6), we find

$$\left[H^s_{(k)}(I_{\mathsf{d}}, L^2(\Xi')), L^2(I_{\mathsf{d}}, L^2(\Xi'))\right]_\theta = H^{(1-\theta)s}_{(k)}(I_{\mathsf{d}}, L^2(\Xi'))\,.$$

If we combine the last three equalities, we deduce that

$$\begin{aligned}\left[H^s_{(k)}(\Xi), L^2(\Xi)\right]_\theta &= L^2(I_{\mathsf{d}}, H^{(1-\theta)s}_{(k)}(\Xi')) \cap H^{(1-\theta)s}_{(k)}(I_{\mathsf{d}}, L^2(\Xi'))\\ &= H^{(1-\theta)s}_{(k)}(\Xi)\,.\end{aligned}$$

That completes the proof of the lemma. □

For the next lemma recall the definition of G^s from the proof of Lemma 5.2.4.

Lemma 5.4.10. *Let $s \geq t \geq 0$, and let $\theta \in (0,1)$. Then, if none of s, t and $(1-\theta)s+\theta t$ are in $\left\{\frac{1}{2}, \frac{3}{2}\right\}$, we have*

$$\left[G^s((0,\infty)^{\mathsf{d}-1}\times\mathbb{R}), G^t((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})\right]_\theta = G^{(1-\theta)s+\theta t}((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})\,, \tag{5.4.13}$$

and in particular, if $s \notin \left\{\frac{1}{2}, \frac{3}{2}\right\}$, $(1-\theta)s \notin \left\{\frac{1}{2}, \frac{3}{2}\right\}$, then

$$\left[G^s((0,\infty)^{\mathsf{d}-1}\times\mathbb{R}), L^2((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})\right]_\theta = G^{(1-\theta)s}((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})\,. \tag{5.4.14}$$

Proof. As in the previous lemma, (5.4.13) follows from (5.4.14) and reiteration, so we will only prove (5.4.14). Observe that

$$G^s((0,\infty)^{\mathsf{d}-1}\times\mathbb{R}) = L^2(\mathbb{R}, H^s_{(0)}((0,\infty)^{\mathsf{d}-1})) \cap H^s_{\#}(\mathbb{R}, L^2((0,\infty)^{\mathsf{d}-1})$$

and

$$L^2((0,\infty)^{\mathsf{d}-1}\times\mathbb{R}) = L^2(\mathbb{R}, L^2((0,\infty)^{\mathsf{d}-1}))\,.$$

Intersection and interpolation commute by the same argument as in the proof of Lemma 5.4.9, and so we have

$$\begin{aligned}\left[G^s((0,\infty)^{\mathsf{d}-1}\times\mathbb{R}), L^2((0,\infty)^{\mathsf{d}-1}\times\mathbb{R})\right]_\theta &= \left[L^2(\mathbb{R}, H^s_{(0)}((0,\infty)^{\mathsf{d}-1}), L^2(\mathbb{R}, L^2((0,\infty)^{\mathsf{d}-1}))\right]_\theta\\ &\quad\cap \left[H^s_{\#}(\mathbb{R}, L^2((0,\infty)^{\mathsf{d}-1}), L^2(\mathbb{R}, L^2((0,\infty)^{\mathsf{d}-1}))\right]_\theta\,.\end{aligned}$$

Now, by Lemma 5.4.9 we have

$$\left[L^2(\mathbb{R}, H^s_{(0)}((0,\infty)^{\mathsf{d}-1}), L^2(\mathbb{R}, L^2((0,\infty)^{\mathsf{d}-1}))\right]_\theta = L^2(\mathbb{R}, H^{(1-\theta)s}_{(0)}((0,\infty)^{\mathsf{d}-1})\,,$$

and Lemma 5.4.7 implies that

$$\left[H^s_{\#}(\mathbb{R}, L^2((0,\infty)^{\mathsf{d}-1}), L^2(\mathbb{R}, L^2((0,\infty)^{\mathsf{d}-1}))\right]_\theta = H^{(1-\theta)s}_{\#}(\mathbb{R}, L^2((0,\infty)^{\mathsf{d}-1})\,.$$

The last three equalities combined imply (5.4.14). □

6 Pinning for the membrane model in dimension four and above

This chapter is based on the author's preprint [Sch20b], with only minor changes.

6.1 Introduction

In this chapter we study the pinned membrane model in dimension four and above. The problem and our main results were described in Section 1.3.4. In this chapter we are focussed on the membrane model, and so we drop all subscripts Δ.

Recall that we consider the membrane model

$$\mathbb{P}_\Lambda(\mathrm{d}\psi) = \frac{1}{Z_\Lambda} \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d} |\Delta_1\psi_x|^2\right) \prod_{x\in\Lambda} \mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^d\setminus\Lambda} \delta_0(\mathrm{d}\psi_x)\,. \tag{6.1.1}$$

on some $\Lambda \Subset \mathbb{Z}^d$ as well as the membrane model with δ-pinning of strength $\varepsilon > 0$,

$$\mathbb{P}^\varepsilon_\Lambda(\mathrm{d}\psi) = \frac{1}{Z^\varepsilon_\Lambda} \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d} |\Delta_1\psi_x|^2\right) \prod_{x\in\Lambda} (\mathrm{d}\psi_x + \varepsilon\delta_0(\mathrm{d}\psi_x)) \prod_{x\in\mathbb{Z}^d\setminus\Lambda} \delta_0(\mathrm{d}\psi_x)\,. \tag{6.1.2}$$

6.1.1 Main results

Let us describe our results in detail. First of all, expanding the bracket in (6.1.2), we see that for $f\colon \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ we have

$$\begin{aligned}
\mathbb{E}^\varepsilon_\Lambda(f) &= \frac{1}{Z^\varepsilon_\Lambda} \int \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d} |\Delta_1\psi_x|^2\right) f(\psi) \prod_{x\in\Lambda} (\mathrm{d}\psi_x + \varepsilon\delta_0(\mathrm{d}\psi_x)) \prod_{x\in\mathbb{Z}^d\setminus\Lambda} \delta_0(\mathrm{d}\psi_x) \\
&= \frac{1}{Z^\varepsilon_\Lambda} \sum_{A\subset\Lambda} \int \exp\left(-\frac{1}{2}\sum_{v\in\mathbb{Z}^d} |\Delta_1\psi_x|^2\right) f(\psi)\varepsilon^{|A|} \prod_{x\in\Lambda\setminus A} \mathrm{d}\psi_v \prod_{x\in\mathbb{Z}^d\setminus(\Lambda\setminus A)} \delta_0(\mathrm{d}\psi_x) \qquad (6.1.3)\\
&= \sum_{A\subset\Lambda} \frac{\varepsilon^{|A|} Z_{\Lambda\setminus A}}{Z^\varepsilon_\Lambda} \mathbb{E}_{\Lambda\setminus A}(f)\,.
\end{aligned}$$

where $\mathbb{E}^\varepsilon_\Lambda$ and $\mathbb{E}_{\Lambda\setminus A}$ denote the expectation with respect to $\mathbb{P}^\varepsilon_\Lambda$ and $\mathbb{P}_{\Lambda\setminus A}$, respectively. Thus, we have

$$\mathbb{P}^\varepsilon_\Lambda(\mathrm{d}\psi) = \sum_{A\subset\Lambda} \zeta^\varepsilon_\Lambda(A)\mathbb{P}_{\Lambda\setminus A}(\mathrm{d}\psi) \tag{6.1.4}$$

where

$$\zeta^\varepsilon_\Lambda(A) = \frac{\varepsilon^{|A|} Z_{\Lambda\setminus A}}{Z^\varepsilon_\Lambda}$$

so that $\zeta^\varepsilon_\Lambda$ is a probability measure on $\mathfrak{P}(\Lambda)$, the powerset of Λ. It describes the set of pinned points. In fact, one easily sees that for any $A \subset \Lambda$ we have

$$A = \{x \in \Lambda \colon \psi_x = 0\} \quad \mathbb{P}_{\Lambda\setminus A}\text{-almost surely}\,.$$

By (6.1.4), $\mathbb{P}^\varepsilon_\Lambda$ is a mixture of the Gaussian measures $\mathbb{P}_{\Lambda\setminus A}$ for $A \subset \Lambda$. Our first goal will therefore be to understand the weights of this mixture, i.e. the measure $\zeta^\varepsilon_\Lambda$. We write $\zeta^\varepsilon_\Lambda(f)$ for $\sum_{A\subset\Lambda} f(A)\zeta^\varepsilon_\Lambda(A)$. The first result is that the measures $\zeta^\varepsilon_\Lambda$ satisfies a correlation inequality.

Theorem 6.1.1. *The measure $\zeta^\varepsilon_\Lambda$ satisfies the FKG inequality, i.e.*

$$\zeta^\varepsilon_\Lambda(fg) \geq \zeta^\varepsilon_\Lambda(f)\zeta^\varepsilon_\Lambda(g)$$

for any pair of increasing functions $f, g \colon \mathfrak{P}(\Lambda) \to \mathbb{R}$.

This FKG inequality allows us to prove directly that a thermodynamic limit of the $\zeta^\varepsilon_\Lambda$ exists. We can also prove that a thermodynamic limit of the $\mathbb{P}^\varepsilon_\Lambda$ exists. That result relies on the estimates for the Green's function which we state in Theorem 6.1.5 below.

Theorem 6.1.2. *If $\mathsf{d} \geq 4$, the thermodynamic limit*

$$\zeta^\varepsilon := \lim_{\Lambda\nearrow\mathbb{Z}^\mathsf{d}} \zeta^\varepsilon_\Lambda$$

exists and is translation invariant.

Furthermore, there is a constant ε_d depending on d only such that for any $\varepsilon < \varepsilon_\mathsf{d}$ the thermodynamic limit

$$\mathbb{P}^\varepsilon := \lim_{\Lambda\nearrow\mathbb{Z}^\mathsf{d}} \mathbb{P}^\varepsilon_\Lambda$$

exists and is translation invariant.

The convergence here is meant as weak convergence of measures on $\mathbb{R}^{\mathbb{Z}^\mathsf{d}}$ equipped with the cylinder σ-algebra, i.e. the measures integrated against any bounded local function converge.

It is easy to see that $\mathbb{P}^\varepsilon$ is an infinite volume Gibbs measure for the interaction (6.1.2) (with appropriate boundary conditions). We write $\mathbb{E}^\varepsilon$ for the expectation with respect to $\mathbb{P}^\varepsilon$.

We will now state a few results on $\zeta^\varepsilon_\Lambda$ and $\mathbb{P}^\varepsilon_\Lambda$ that hold uniformly in Λ. Theorem 6.1.2 then implies that they hold for ζ^ε and $\mathbb{P}^\varepsilon$ as well.

We begin with precise estimates on the pinned set. The heuristic is that this set behaves like a Bernoulli point process with density p_d depending on ε. It turns out that this is true in a rather strong sense if $\mathsf{d} \geq 5$. In $\mathsf{d} = 4$ this no longer holds, but fortunately we can still compare the probabilities that large sets are free of pinned points, and this is sufficient to continue with our argument. The precise result is the following. For the definition of strong stochastic domination see Definition 6.2.2. We denote by $\mathcal{A}$ a random variable distributed according to $\zeta^\varepsilon_\Lambda$.

Theorem 6.1.3. *Let $\mathsf{d} \geq 4$. There are constants $c_\mathsf{d}, C_\mathsf{d}, \varepsilon_{\mathsf{d},*}$ depending on d only with the following properties.*

a) *If* $\mathsf{d} \geq 5$ *and* $p_{\mathsf{d},-} = c_{\mathsf{d}}\varepsilon$, *then for any* $\Lambda \Subset \mathbb{Z}^d$ *and any* $\varepsilon < \varepsilon_{\mathsf{d},*}$ *the measure* $\zeta^\varepsilon_\Lambda$ *strongly dominates the Bernoulli measure on* $\mathfrak{P}(\Lambda)$ *with parameter* $p_{\mathsf{d},-}$. *In particular for any* $E \subset \Lambda$

$$(1 - p_{\mathsf{d},-})^{|E|} \geq \zeta^\varepsilon_\Lambda(\mathcal{A} \cap E = \varnothing)\,. \tag{6.1.5}$$

b) *If* $\mathsf{d} \geq 5$ *and* $p_{\mathsf{d},+} = C_{\mathsf{d}}\varepsilon$, *then for any* $\Lambda \Subset \mathbb{Z}^d$ *and any* $\varepsilon < \varepsilon_{\mathsf{d},*}$ *the measure* $\zeta^\varepsilon_\Lambda$ *is strongly dominated by the Bernoulli measure on* $\mathfrak{P}(\Lambda)$ *with parameter* $p_{\mathsf{d},+}$. *In particular for any* $E \subset \Lambda$

$$(1 - p_{\mathsf{d},+})^{|E|} \leq \zeta^\varepsilon_\Lambda(\mathcal{A} \cap E = \varnothing)\,. \tag{6.1.6}$$

c) *If* $\mathsf{d} = 4$ *and* $p_{4,-} = c_4 \frac{\varepsilon}{|\log \varepsilon|^{1/2}}$, *then for any* $E \subset \Lambda$ *and any* $\varepsilon < \varepsilon_{4,*}$ *we have*

$$(1 - p_{4,-})^{|E|} \geq \zeta^\varepsilon_\Lambda(\mathcal{A} \cap E = \varnothing)\,. \tag{6.1.7}$$

d) *If* $\mathsf{d} = 4$, *then there is for any* $\alpha > 0$ *a constant* $C_{4,\alpha}$ *depending on* d *and* α *such that with* $p_{4,+,\alpha} = C_{4,\alpha} \frac{\varepsilon}{|\log \varepsilon|^{1/2}}$ *for any* $E \subset \Lambda$ *with* $d(E, \mathbb{Z}^{\mathsf{d}} \setminus \Lambda) \geq \varepsilon^{-\alpha}$ *and any* $\varepsilon < \varepsilon_{4,*}$ *we have*

$$(1 - p_{4,+,\alpha})^{|E|} \leq \zeta^\varepsilon_\Lambda(\mathcal{A} \cap E = \varnothing)\,. \tag{6.1.8}$$

All estimates also hold with ζ^ε *in place of* $\zeta^\varepsilon_\Lambda$.

Let us warn the reader that we use the notation $p_{\mathsf{d},\pm}$ in the opposite way as in [BV01]. Our convention here follows [BCK17].

Note carefully that we do not claim any domination result in case $\mathsf{d} = 4$. In fact, the same argument as in [BV01, Section 2] shows that neither $\zeta^\varepsilon_\Lambda$ is strongly dominated by the Bernoulli measure on $\mathfrak{P}(\Lambda)$ with parameter $p_{4,+}$, nor that $\zeta^\varepsilon_\Lambda$ strongly dominates the Bernoulli measure on $\mathfrak{P}(\Lambda)$ with parameter $p_{4,-}$.

In the subcritical dimensions $\mathsf{d} < 4$ the set of pinned points is too correlated for any meaningful comparison with a Bernoulli measure. This is the reason why new techniques would be necessary to study the pinned membrane model in dimensions 2 and 3.

From Theorems 6.1.2 and 6.1.3 one immediately obtains the following corollary, which strengthens Sakagawa's result [Sak12] that the density of pinned points is positive for any $\varepsilon > 0$.

Corollary 6.1.4. *Let* $\mathsf{d} \geq 4$. *Consider the density of pinned points*

$$\rho_\varepsilon = \liminf_{\Lambda \nearrow \mathbb{Z}^{\mathsf{d}}} \frac{1}{|\Lambda|} \zeta^\varepsilon_\Lambda(|\mathcal{A}|) = \liminf_{\Lambda \nearrow \mathbb{Z}^{\mathsf{d}}} \frac{1}{|\Lambda|} \sum_{A \subset \Lambda} |A| \zeta^\varepsilon_\Lambda(A)\,.$$

For each $\varepsilon > 0$ *we have* $\rho_\varepsilon \geq c_{\mathsf{d}} p_{\mathsf{d},-} > 0$.

It is unclear whether the limit here exists in general. However, using Theorem 6.1.1 and a subadditivity argument one can show that it exists along the sequence $\Lambda_n = [-n, n]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}}$, say.

Using the knowledge about $\zeta^\varepsilon_\Lambda$ from Theorem 6.1.3 we can now establish some more precise results on $\mathbb{P}^\varepsilon$. For any vector $\theta \in \mathbb{S}^{\mathsf{d}-1}$ (where $\mathbb{S}^{\mathsf{d}-1}$ is the unit sphere in $\mathbb{R}^{\mathsf{d}}$) define the mass

$$m_\varepsilon(\theta) := -\limsup_{k \to \infty} \frac{1}{k} \log |\mathbb{E}^\varepsilon(\psi_0 \psi_{\lfloor k\theta \rfloor})|\,. \tag{6.1.9}$$

where we set $\log 0 = -\infty$. Note that we take the absolute value of $\mathbb{E}^\varepsilon(\psi_0\psi_{\lfloor k\theta\rfloor})$ in this definition. This is because for the membrane model correlations can be negative. In fact, the heuristic in Section 6.1.2 below suggests that $\mathbb{E}^\varepsilon(\psi_0\psi_{\lfloor k\theta\rfloor})$ behaves like an underdamped harmonic oscillator. In particular, we expect that the limit in (6.1.9) does not exist. In contrast, for the gradient model the limit in (6.1.9) exists, even without the absolute values, cf. [BV01, Appendix A].

We can show the following results on the variance, covariance and mass. Here $d(x, E)$ denotes the distance from x to the set E.

Theorem 6.1.5. *Let $\mathsf{d} \geq 4$, and $\Lambda \Subset \mathbb{Z}^{\mathsf{d}}$. There are constants $c_{\mathsf{d}}, C_{\mathsf{d}}, \varepsilon_{\mathsf{d},**}$ depending on d only with the following property.*

a) Let $x \in \Lambda$. Then for $\varepsilon < \varepsilon_{\mathsf{d},**}$ we have the following estimates on the variance: if $\mathsf{d} \geq 5$, then

$$c_{\mathsf{d}} \leq \mathbb{E}^\varepsilon_\Lambda(\psi_x^2) \leq C_{\mathsf{d}}\,, \tag{6.1.10}$$

while if $\mathsf{d} = 4$ and $\alpha > 0$, then

$$\frac{|\log\varepsilon|}{32\pi^2} - C_{4,\alpha}\log|\log\varepsilon| \leq \mathbb{E}^\varepsilon_\Lambda(\psi_x^2) \leq \frac{|\log\varepsilon|}{16\pi^2} + C_4\log|\log\varepsilon|\,, \tag{6.1.11}$$

where the lower bound only holds if $d(x, \mathbb{Z}^{\mathsf{d}} \setminus \Lambda) \geq \varepsilon^{-\alpha} + \varepsilon^{-1/4}$. The same estimates hold for $\mathbb{E}^\varepsilon$ instead of $\mathbb{E}^\varepsilon_\Lambda$ (with the condition on $d(x, \mathbb{Z}^d \setminus \Lambda)$ becoming vacuous).

b) Let $x, y \in \Lambda$. Then for $\varepsilon < \varepsilon_{\mathsf{d},**}$ we have the following estimates on the covariance: if $\mathsf{d} \geq 5$, then

$$|\mathbb{E}^\varepsilon_\Lambda(\psi_x\psi_y)| \leq \frac{C_{\mathsf{d}}}{\varepsilon^{1/2}}\exp\left(-c_{\mathsf{d}}\varepsilon^{1/4}|x-y|\right)\,, \tag{6.1.12}$$

while if $\mathsf{d} = 4$, then

$$|\mathbb{E}^\varepsilon_\Lambda(\psi_x\psi_y)| \leq C_4\left(\frac{|\log\varepsilon|^{5/4}}{\varepsilon^{1/2}} + \log(1+|x-y|)\right)\exp\left(-\frac{c_4\varepsilon^{1/4}|x-y|}{|\log\varepsilon|^{3/8}}\right)\,. \tag{6.1.13}$$

The same estimates hold for $\mathbb{E}^\varepsilon$ instead of $\mathbb{E}^\varepsilon_\Lambda$.

In particular, we have the following estimates on the mass: if $\mathsf{d} \geq 5$, then

$$c_{\mathsf{d}}\varepsilon^{1/4} \leq m_\varepsilon(\theta) \qquad \forall\theta \in \mathbb{S}^{\mathsf{d}-1}\,, \tag{6.1.14}$$

while if $\mathsf{d} = 4$, then

$$c_4\frac{\varepsilon^{1/4}}{|\log\varepsilon|^{3/8}} \leq m_\varepsilon(\theta) \qquad \forall\theta \in \mathbb{S}^3\,. \tag{6.1.15}$$

The estimates in Theorem 6.1.5 are only valid for sufficiently small $\varepsilon > 0$. However, a calculation similar to (6.1.3) reveals that for $\varepsilon' < \varepsilon$ the measure $\mathbb{P}^{(\varepsilon)}_\Lambda$ is a mixture of the measures $\mathbb{P}^{(\varepsilon')}_{\Lambda\setminus A}$, and so the theorem also implies that for any $\varepsilon > 0$ the measure $\mathbb{P}^{(\varepsilon)}_\Lambda$ has bounded variances and exponentially decaying covariances.

In the next section we describe some heuristics for the exponential decay of the correlations. These heuristics suggest that the exponential rates in (6.1.12) and (6.1.13) are optimal, but the prefactors are not. In fact, we have made no real effort to optimise these prefactors, as this would require further technicalities. Nonetheless, as we believe that the exponential rates in (6.1.12) and (6.1.13) are optimal, the same holds for the rates in (6.1.14) and (6.1.15).

The results of Theorem 6.1.5 are a far-reaching generalization of the results in [BCK16, BCK17]. In [BCK17] it is shown that for $\mathsf{d} \geq 5$ the mass is positive for each fixed $\varepsilon > 0$. No explicit lower bound on the mass is given, but if one keeps track of the constants in their argument one can check that their proof gives the estimate

$$c_{\mathsf{d}} \varepsilon^{\frac{(2\mathsf{d}+4)(2\mathsf{d}+1)}{\mathsf{d}}} \leq m_\varepsilon(\theta)\,.$$

For $\mathsf{d} = 4$ in [BCK16] stretched-exponential decay of the covariances is shown, and it was unknown whether the decay is actually exponential.

6.1.2 Heuristics: The continuous Bilaplace equation in a perforated domain

Before we describe the proofs of our results in more detail, let us discuss a related problem that provides some heuristics. Namely we consider the continuous Bilaplace equation in a domain perforated by small holes. This is a well-studied problem, and the analogous problem for the Laplacian even more so, cf. [CM97, MK06][1]. If one lets the size of the holes tend to zero while keeping their capacity density constant, the problem converges (in an appropriate sense) to a Bilaplace equation with a mass term on the whole domain. The Green's function of the associated operator decays exponentially, and so it is unsurprising that the same holds true already for the Green's function in the perforated domain.

In our context, this connection gives a hint how to deduce Theorem 6.1.5 if one assumes Theorem 6.1.3. Let us explain this in detail: fix $\varepsilon > 0$, and consider a fixed large, but bounded domain $\Omega \subset \mathbb{R}^{\mathsf{d}}$ with smooth boundary. Let $N \in \mathbb{N}$ be a large parameter to be chosen later. We perforate the domain $N\Omega$ with small holes of radius $r > 0$, centred at a subset of $\mathbb{Z}^{\mathsf{d}}$. Theorem 6.1.3 suggests that we choose a fraction $p_{\mathsf{d},-}$ of the points in $\mathbb{Z}^{\mathsf{d}}$ as the centres of these holes. For now we consider the simplest case of equally-spaced holes, i.e. we place them at $(\lambda_{\mathrm{mic}}\mathbb{Z})^{\mathsf{d}}$, where $\lambda_{\mathrm{mic}} \approx (p_{\mathsf{d},-})^{-\frac{1}{\mathsf{d}}}$ is an integer. That is, we consider the equation

$$\begin{aligned} \Delta^2 u &= f \quad \text{in } N\Omega \setminus \bigcup_{x \in (\lambda_{\mathrm{mic}}\mathbb{Z})^{\mathsf{d}}} \overline{B_r(x)}\,, \\ u &= 0 \quad \text{else}\,. \end{aligned} \tag{6.1.16}$$

The Green's function G of this problem should predict the behaviour of the covariances in Theorem 6.1.5.

We can rescale (6.1.16) back to a unit domain by letting $\hat{f}(y) = \frac{1}{N^{\mathsf{d}-4}} f(Ny)$, $\hat{u}(y) = \frac{1}{N^{\mathsf{d}}} u(Ny)$, so that $\hat{u}$ and $\hat{f}$ solve

$$\begin{aligned} \Delta^2 \hat{u} &= \hat{f} \quad \text{in } \Omega \setminus \bigcup_{x \in ((\lambda_{\mathrm{mic}}/N)\mathbb{Z})^{\mathsf{d}}} \overline{B_{r/N}(x)}\,, \\ \hat{u} &= 0 \quad \text{else}\,. \end{aligned} \tag{6.1.17}$$

In order to apply now results from [CM97], we need to treat $\mathsf{d} \geq 5$ and $\mathsf{d} = 4$ separately. We begin with the former case. The collection of balls $\bigcup_{x \in ((\lambda_{\mathrm{mic}}/N)\mathbb{Z})^{\mathsf{d}}} B_{r/N}(x)$ has capacity density

$$\mu = C_{\mathsf{d}} \left(\frac{N}{\lambda_{\mathrm{mic}}}\right)^{\mathsf{d}} \left(\frac{r}{N}\right)^{\mathsf{d}-4} = C_{\mathsf{d}} N^4 r^{\mathsf{d}-4} \lambda_{\mathrm{mic}}^{-\mathsf{d}} \approx C_{\mathsf{d}} N^4 r^{\mathsf{d}-4} p_{\mathsf{d},-} = C_{\mathsf{d}} N^4 r^{\mathsf{d}-4} \varepsilon\,.$$

[1] Note that the original French and Russian works date back to the 70s and 80s.

We want to consider a limit of dense small holes where μ is constant, and so we choose $N = \varepsilon^{-1/4}$. Then, according to [CM97, Example 2.14], the solution of (6.1.17) in the limit $\varepsilon \to 0$ behaves like the solution of

$$\begin{aligned} \Delta^2 \hat{u} + \mu \hat{u} &= \hat{f} \quad \text{in } \Omega\,, \\ \hat{u} &= 0 \quad \text{else}\,. \end{aligned} \tag{6.1.18}$$

This is a Bilaplace equation with a mass term. Its Green's function $\hat{G}$ behaves like the Green's function $\hat{G}_{\mathbb{R}^d}$ of the same equation in the full space $\mathbb{R}^d$ (at least when we stay away from the boundary of Ω). The latter Green's function can be computed quite explicitly using separation of variables in spherical coordinates. One finds that $\hat{G}_{\mathbb{R}^d}(\hat{x}, \hat{y}) = F(\mu^{1/4}|x-y|)$, where $F(r)$ is a linear combination of $\operatorname{Re}\left((\zeta_8 r)^{-(d-2)/2} H^{(1)}_{(d-2)/2}(\zeta_8 r)\right)$. Here $H^{(1)}_\nu$ is the Hankel function of the first kind, and ζ_8 runs through the primitive eighth roots of unity. A short calculation using the asymptotic expansion for these functions (cf. [AS64, Equation 9.7.2]) and the fact that $\hat{G}_{\mathbb{R}^d}$ needs to decay at infinity reveals that

$$\begin{aligned} \hat{G}_{\mathbb{R}^d}(\hat{x}, \hat{y}) = C_d \left(\mu^{1/4}|\hat{x}-\hat{y}|\right)^{-(d-1)/2} &\left(\sin\left(\frac{\mu^{1/4}|\hat{x}-\hat{y}|}{2^{1/2}} - \omega_d\right) + O(\mu^{-1/4}|\hat{x}-\hat{y}|^{-1})\right) \\ &\times \exp\left(-\frac{\mu^{1/4}|\hat{x}-\hat{y}|}{2^{1/2}}\right) \end{aligned}$$

where ω_d is a phase shift depending only on d, and we used the standard Landau notation. Neglecting the error term altogether, we thus expect

$$\hat{G}(\hat{x}, \hat{y}) \approx C_d \left(\mu^{1/4}|\hat{x}-\hat{y}|\right)^{-(d-1)/2} \sin\left(\frac{\mu^{1/4}|\hat{x}-\hat{y}|}{2^{1/2}} - \omega_d\right) \exp\left(-\frac{\mu^{1/4}|\hat{x}-\hat{y}|}{2^{1/2}}\right) \tag{6.1.19}$$

when $|\hat{x}-\hat{y}| \gg \mu^{-1/4}$, and the Green's function of (6.1.17) should behave similarly (at least if Ω is large enough, i.e. $\operatorname{diam}\Omega \gg \mu^{-1/4}$). Rescaling back, we thus expect for the Green's function G of (6.1.16) that

$$\begin{aligned} G(x,y) &= \frac{1}{N^{d-4}} \hat{G}\left(\frac{x}{N}, \frac{y}{N}\right) \\ &\approx \frac{1}{N^{d-4}} C_d \left(\frac{\mu^{1/4}|x-y|}{2^{1/2}N}\right)^{-(d-1)/2} \sin\left(-\frac{\mu^{1/4}|x-y|}{2^{1/2}N} - \omega_d\right) \exp\left(-\frac{\mu^{1/4}|x-y|}{2^{1/2}N}\right) \\ &\approx \frac{C_d \varepsilon^{(d-7)/8}}{r^{(d-1)(d-4)/8}|x-y|^{(d-1)/2}} \sin\left(-C_d \varepsilon^{1/4} r^{(d-4)/4}|x-y| - \omega_d\right) \\ &\quad \times \exp\left(-C_d \varepsilon^{1/4} r^{(d-4)/4}|x-y|\right) \end{aligned}$$

when $|x-y| \gg N\mu^{-1/4} = \varepsilon^{-1/4} r^{-(d-4)/4}$. Thus, G decays exponentially, with polynomial corrections and an oscillatory term that makes G change sign. While the polynomial corrections and the oscillatory term are not captured in (6.1.12), the exponential decay rates in both estimates are the same (up to constant factors).

If $d = 4$, the argument is in principle the same, but we need to use extra care when defining the capacity density. Following [CM97, Example 2.14] we define

$$\mu = C\left(\frac{N}{\lambda_{\text{mic}}}\right)^4 \frac{1}{|\log \frac{r}{N}|} \approx C N^4 p_{4,-} \frac{1}{\log N - \log r} \approx C \frac{N^4 \varepsilon}{(\log N - \log r)|\log \varepsilon|^{\frac{1}{2}}}\,.$$

We want $\varepsilon \to 0$ while μ is constant, and so we choose $N = \frac{|\log\varepsilon|^{3/8}}{\varepsilon^{1/4}} + O_r\left(\frac{|\log\varepsilon|^{1/8}\log|\log\varepsilon|}{\varepsilon^{1/4}}\right) = \frac{|\log\varepsilon|^{3/8}+o_r(|\log\varepsilon|^{3/8})}{\varepsilon^{1/4}}$ accordingly. Then we conclude from [CM97, Example 2.14] that the solutions of (6.1.17) and (6.1.18) are close. The Green's function of (6.1.18) in $\mathsf{d} = 4$ still behaves like (6.1.19), and so, rescaling back, we find again

$$\begin{aligned}
G(x,y) &= \hat{G}\left(\frac{x}{N},\frac{y}{N}\right) \\
&\approx C\left(\frac{\mu^{1/4}|x-y|}{N}\right)^{-3/2}\sin\left(-\frac{\mu^{1/4}|x-y|}{2^{1/2}N}-\omega_4\right)\exp\left(-\frac{\mu^{1/4}|x-y|}{2^{1/2}N}\right) \\
&\approx \frac{C|\log\varepsilon|^{9/16}}{\varepsilon^{3/8}|x-y|^{3/2}}\sin\left(-\frac{C\varepsilon^{1/4}|x-y|}{|\log\varepsilon|^{3/8}+o_r(|\log\varepsilon|^{3/8}))}-\omega_4\right) \\
&\qquad\times\exp\left(-\frac{C\varepsilon^{1/4}|x-y|}{|\log\varepsilon|^{3/8}+o_r(|\log\varepsilon|^{3/8}))}\right).
\end{aligned}$$

when $|x-y| \gg_r N\mu^{-1/4} \approx \varepsilon^{-1/4}|\log\varepsilon|^{3/8}$. This is again exponential decay with polynomial corrections and an oscillatory term. The exponential decay rate is again the same as in (6.1.13) (up to constants).

In summary, our heuristic predicts the same exponential decay rates as in Theorem 6.1.5. The heuristic we used is rather simplistic, though. One problem is that in the context of the membrane model a single pinned point forces the field to be zero there, but does not pose any restrictions on the gradient of the field. In contrast, in (6.1.16) we force the field and all its derivatives to be zero at the pinned balls. One way to improve the heuristics would thus be to only prescribe that u has average zero over each $B_r(x)$ for $x \in (\lambda_{\mathrm{mic}}\mathbb{Z})^{\mathsf{d}}$, instead of it being identically zero there. This is not a serious change, though, as a modification of [CM97, Example 2.14] or an application of the general framework in [MK06] show that the convergence of (6.1.17) to (6.1.18) still holds, albeit with a different constant prefactor in μ.

A more serious problem is that the pinned points are not distributed on a lattice, but following the probability distribution $\zeta^{\varepsilon}_{\Lambda}$. If this distribution were, say, a Poisson point process, then the framework from [MK06] would still apply. Our actual $\zeta^{\varepsilon}_{\Lambda}$ is possibly quite correlated (at least if $\mathsf{d} = 4$), though, and so it is not clear that the heuristic still applies. On the other hand, we are not actually interested in "quenched" estimates that hold for all realizations of the sets of pinned points, but rather in "annealed" estimates where we average over the randomness of the pinned points. So there is hope to retain the heuristic.

A further question is how to rigorously show that the convergence of the boundary value problem (6.1.17) to the boundary value problem (6.1.18) implies that the Green's function of (6.1.17) already has the predicted behaviour. There are very few results on this in the literature. One exception is [NV06], where this is proved rigorously for the case of the Laplace equation in $\mathsf{d} = 3$. However, that approach relies on the maximum principle, and so one cannot extend it to our situation. Instead, in [HV18] a more robust approach is used: There (in another context) exponential decay of the L^2-norm of harmonic functions on perforated large annuli is shown, using Widman's hole filler technique [Wid71] in combination with the fact that one has a local Poincaré inequality. A similar argument is also used in [BCK17], and the authors describe that they learned it from Vladimir Maz'ya. The decay rates in [HV18] are not optimal, but a small modification of their argument leads to the optimal decay rate. These arguments are the inspiration for our proof of Theorem 6.1.5 from Theorem 6.1.3. We shall explain this in more detail in the next subsection, where we outline the proofs of our results.

6.1.3 Main ideas of the proofs

This chapter consists of two main parts. We first establish the various results on the pinned set, and then deduce from them the results on the variances and covariances. We will discuss these parts separately. Before doing so, let us remark that there are two natural lengthscales occuring. There are the average distance between pinned points

$$\lambda_{\mathrm{mic}} \approx \begin{cases} \frac{1}{\varepsilon^{1/\mathsf{d}}} & \mathsf{d} \geq 5 \\ \frac{|\log \varepsilon|^{1/8}}{\varepsilon^{1/4}} & \mathsf{d} = 4 \end{cases},$$

and the lengthscale on which the correlations decay

$$\lambda_{\mathrm{mac}} \approx \begin{cases} \frac{1}{\varepsilon^{1/4}} & \mathsf{d} \geq 5 \\ \frac{|\log \varepsilon|^{3/8}}{\varepsilon^{1/4}} & \mathsf{d} = 4 \end{cases}.$$

Note that we have $1 \ll \lambda_{\mathrm{mic}} \ll \lambda_{\mathrm{mac}}$ as $\varepsilon \to 0$ for any $\mathsf{d} \geq 4$.

Estimates on the pinned set

The first novel result of this chapter is the FKG inequality for the pinned set, Theorem 6.1.1. As already mentioned, it follows rather directly from the Gaussian correlation inequality [Roy14], and it is standard to deduce from the FKG inequality the existence of the thermodynamic limit of the set of pinned points, i.e. the first part of Theorem 6.1.2. We give these proofs in Section 6.2.1. Note, however, that our proof of Theorem 6.1.1 is specific to the case of δ-pinning, and we conjecture that the result is not true for other pinning potentials such as a square-well potential. The point is that conditioning a Gaussian vector on being 0 at some coordinates yields another Gaussian vector, but that is no longer true if we condition on some coordinates being small instead. We give a more detailed explanation in Remark 6.2.1.

For the proof of Theorem 6.1.3 in Section 6.2.2 we follow [BV01] rather closely. The domination results in Theorem 6.1.3 a) and b) are actually already in [BCK17]. They follow via a short calculation from the boundedness of the Green's function in $\mathsf{d} \geq 5$. Part d) is a little more difficult. It could be proven as in [BV01, Section 3.2], but we give a slightly simpler proof. The idea is that if $x \in E$ is quite far from the pinned points we have already found, the fluctuations of ψ_x are quite big, and so the chance that x is pinned is low.

By far the most difficult part of Theorem 6.1.3 is part c), where we again mostly follow [BV01]. There we want to control the probability that $E \subset \Lambda$ is free of pinned points from above. To do so, we need to find for any configuration of pinned points that avoids E many others that intersect E. This is done using a two-scale argument. We first consider the case that E is a union of boxes of sidelength $C\lambda_{\mathrm{mic}}$, and prove (6.1.7) in this case by carefully tracking how a pinned point in one of these boxes makes it likely that there are pinned points in the neighbouring boxes. Next, we pass to the larger lengthscale $C\lambda_{\mathrm{mac}}$ and deduce from the first step that for an arbitrary E, most points of E are at a distance $\leq C\lambda_{\mathrm{mac}}$ from a pinned point. Finally we use this knowledge together with an argument similar to the first step to construct many configurations of pinned points that intersect E.

The main difference to [BV01] is that one cannot use random walk estimates to see how pinning at some $x \in \Lambda$ influences the variance at $y \neq x$. Instead we use an explicit variance estimate (Lemma 6.2.4) that follows from the monotonicity of the variance in the set of pinned points. We also streamline the argument from [BV01] at some points and correct a minor mistake there.

Asymptotics for the variances and covariances

The remainder of the chapter is then concerned with proving Theorem 6.1.5 and the second part of Theorem 6.1.2. In [BV01] the random walk representation of the Green's function of the Laplacian is used for that purpose. In our case there is no such representation, so we need a completely new argument.

It turns out that the estimates for the variance follow quite easily from Theorem 6.1.3 and the variance estimate in Lemma 6.2.4. We give details in Sections 6.5.1 and 6.5.2.

The estimates on the covariance in Theorem 6.1.3 are much more difficult. The general strategy is the same as in [BCK17]. That is, we show that the L^2-norm of the second derivative of the "quenched" Green's function decays exponentially on large annuli. These annuli have to be chosen adapted to the set of pinned points, and so we do not get an estimate valid for all realizations of $\mathcal{A}$. But our estimates hold up to an exponentially small probability, so that we control $G_{\Lambda\setminus A}$ for all but exponentially few A. For these we can use a rather crude estimate. Finally, we can average these quenched estimates for the Green's function over A to deduce "annealed" bounds for the covariance.

The existence of the thermodynamic limit of the field in Theorem 6.1.2 follows then from the existence of the thermodynamic limit of the set of pinned points and the quenched decay estimates on the Green's function. The somewhat technical proof is given in Section 6.5.3.

For the remainder of this section, let us describe in more detail how we prove the quenched estimates on the covariance. Our main technical result used for that purpose is, roughly speaking, the following (see Theorem 6.4.1 for the precise statement): There is a constant $\hat{N}_{\mathsf{d}}$ such that if $k \in \mathbb{N}$ and ε is sufficiently small there is an event $\Omega_{U,k}$ with $\zeta^{\varepsilon}_{\Lambda}(\Omega_{U,k}) \geq 1 - \frac{C_U}{2^k}$ such that if $A \in \Omega_{U,k}$, and if $u\colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ is a function such that $u = 0$ on $A \setminus U$ and $u\Delta_1^2 u = 0$ on $\mathbb{Z}^{\mathsf{d}} \setminus U$, we have the estimate

$$\left\|\nabla_1^2 u\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} \leq \frac{1}{2^k}\left\|\nabla_1^2 u\right\|^2_{L^2((U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)\setminus U)}\,. \tag{6.1.20}$$

Here $Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)$ denotes a cube of halfdiameter $2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}$ centred at 0. This is an exterior decay estimate for biharmonic functions that holds up to exponentially small probability (and we state and prove in Theorem 6.4.1 also the analogous interior decay estimate). Applying (6.1.20) with $u = G_{\Lambda\setminus\mathcal{A}}(\cdot, y)$ it is a bit tedious but not difficult to deduce the aforementioned quenched estimates on the Green's function, and we do so in Sections 6.5.1 and 6.5.2.

Let us describe how to prove (6.1.20). We first outline the basic strategy that was used in [BCK17] and (in another context) in [HV18], and then describe our novel ideas. For convenience we pretend in the following that u is a continuous function. Adapting the argument to the discrete setting will be somewhat technical but not hard.

We try to iterate a Widman hole filler argument [Wid71] (see, e.g., [GM12, Section 4.4] for a modern presentation). That is, given $U \subset \mathbb{R}^{\mathsf{d}}$, we want to find $U' \supset U$, so that the L^2-norm of $\nabla^2 u$ on $\mathbb{R}^{\mathsf{d}} \setminus U$ is controlled by a constant less than 1 times the L^2-norm of $\nabla^2 u$ on $U' \setminus U$. We also want $\operatorname{dist}(U, \mathbb{R}^{\mathsf{d}} \setminus U') \leq C\lambda_{\mathrm{mac}}$. Once we have such an estimate, we can iterate it to deduce exponential decay at rate $\frac{1}{C\lambda_{\mathrm{mac}}}$, at least on the L^2-level.

So suppose that $U \subset U'$ are open sets and η is a smooth cut-off function such that

$$\{\Delta^2 u \neq 0\} \subset U \subset \{\eta = 0\} \subset \{\eta \neq 1\} \subset U'\,.$$

Then we have

$$0 = (\Delta^2 u, \eta u) = (\nabla^2 u, \nabla^2(\eta u)) = \int \eta |\nabla^2 u|^2 + 2\int \nabla^2 u : \nabla u \otimes \nabla \eta + \int u \nabla^2 u : \nabla^2 \eta \tag{6.1.21}$$

and one can rewrite this using the Cauchy-Schwarz inequality as

$$\begin{aligned}
&\int_{\mathbb{R}^d \setminus U'} |\nabla^2 u|^2 \\
&\quad \leq \int \eta |\nabla^2 u|^2 \\
&\quad = -2\int \nabla^2 u : \nabla u \otimes \nabla \eta - \int u \nabla^2 u : \nabla^2 \eta \\
&\quad \leq \frac{1}{5}\int_{U' \setminus U} |\nabla^2 u|^2 + 5\int_{U' \setminus U} |\nabla u|^2 |\nabla \eta|^2 + \frac{1}{5}\int_{U' \setminus U} |\nabla^2 u|^2 + \frac{5}{4}\int_{U' \setminus U} |u|^2 |\nabla^2 \eta|^2 .
\end{aligned} \tag{6.1.22}$$

Now if we could choose η in such a way that the second and fourth summand here are both bounded by $\frac{1}{5}\int_{U' \setminus U} |\nabla^2 u|^2$ we would obtain the desired decay estimate. In fact, this is what was done in [BCK17]. However, in order to bound both the second and fourth summand, one needs to impose strong pointwise conditions on $\nabla \eta$ and $\nabla^2 \eta$, and, in particular, both need to be near zero on mesoscopic holes in the pinned set. These conditions do not allow growth of η at the optimal rate, and so using this argument one cannot obtain the optimal estimate for the decay rate (but is it comparably easy to construct an η that satisfies these conditions and grows at a non-optimal rate, cf. [BCK17]).

To solve this problem we first rewrite the right hand side of (6.1.21) so that there are no longer any terms containing $\nabla \eta$. An integration by parts shows that

$$\int \nabla^2 u : \nabla u \otimes \nabla \eta = -\int \nabla u \cdot (\nabla \cdot (\nabla u \otimes \nabla \eta)) = -\int \nabla^2 u : \nabla u \otimes \nabla \eta - \int |\nabla u|^2 \Delta \eta$$

and hence

$$\int \nabla^2 u : \nabla u \otimes \nabla \eta = -\frac{1}{2}\int |\nabla u|^2 \Delta \eta .$$

Plugging this into (6.1.21) we see that

$$\int \eta |\nabla^2 u|^2 = \frac{1}{2}\int |\nabla u|^2 \Delta \eta - \int u \nabla^2 u : \nabla^2 \eta . \tag{6.1.23}$$

Using the assumptions on η and the Cauchy-Schwarz inequality we can now estimate

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus U'} |\nabla^2 u|^2 &\leq \int \eta |\nabla^2 u|^2 \\
&= \frac{1}{2}\int |\nabla u|^2 \Delta \eta - \int_{\mathbb{R}^d \setminus U} u \nabla^2 u : \nabla^2 \eta \\
&\leq \frac{1}{2}\int |\nabla u|^2 \Delta \eta + \int |u|^2 |\nabla^2 \eta|^2 + \frac{1}{4}\int_{U' \setminus U} |\nabla^2 u|^2 .
\end{aligned} \tag{6.1.24}$$

If we can now arrange things in such a way that the first two summands here are each bounded by $\frac{1}{4}\int_{U' \setminus U} |\nabla^2 u|^2$, we see that

$$\int_{\mathbb{R}^d \setminus U'} |\nabla^2 u|^2 \leq \frac{3}{4}\int_{U' \setminus U} |\nabla^2 u|^2 \tag{6.1.25}$$

and now we can try to iterate this estimate to obtain exponential decay of the L^2-norm of $\nabla^2 u$. Note that unlike (6.1.22) we now only need to impose conditions on $\nabla^2\eta$.

As it turns out, the first summand in (6.1.24) can be controlled by the second and third summand using an interpolation inequality on lengthscale λ_{mic} that we discuss in 6.3.2.

The remaining task is thus to choose η in such a way that it grows fast enough, but we nonetheless can bound the term $\int |u|^2|\nabla^2\eta|^2$. For that purpose we need some sort of local Poincaré inequality on scale λ_{mic}. Of course, such an estimate can only hold if there are enough pinned points close to the point of interest. In [BCK17, Lemma 4.1] this was done provided there is a nearby cube of 3^{d} points, which are all pinned. On that small cube we then have $u = \nabla u = 0$, and some version of the Hardy-Rellich inequality forces u to be small near that cube as well. However, this is not optimal, as cubes of 3^{d} points that are all pinned are very rare. We show that it is sufficient if there are $\mathsf{d}+1$ pinned points somewhere nearby that are well-spread out. The number $\mathsf{d}+1$ arises from the fact that we need to eliminate nonzero affine functions on $\mathbb{R}^{\mathsf{d}}$. Thus, in some sense we use a multipolar Hardy-Rellich inequality instead of a unipolar one. For multipolar Hardy inequalities cf. e.g. [CZ13]; we could not find a detailed discussion of multipolar Hardy-Rellich inequalities in the literature.

The local Poincaré inequality result is, roughly speaking, the following (see Theorem 6.3.1 for the precise result): Let $A \subset \Lambda$, and $V \subset \Lambda$ be an arbitrary subset. Let $R \in \mathbb{N}$, $R \geq 2$ be a parameter. Then

$$\|u\mathbb{1}_{\cdot\in X_R}\|^2_{L^2(V)} \leq C_{\mathsf{d}} R^{\mathsf{d}}(1 + \mathbb{1}_{\mathsf{d}=4}\log R)\left\|\nabla^2 u\right\|^2_{L^2(V+Q_R(0))}$$

where X_R is the set of those points that have $\mathsf{d}+1$ well-separated pinned points at distance $\leq R\lambda_{\text{mic}}$ around them, and we write $\mathbb{1}_{\cdot\in X_R}$ for the indicator function of that set.

This result makes it clear what we need to require of η. Namely we want $|\nabla^2\eta| \leq C\mathbb{1}_{\cdot\in X_R}$ for some R. If we have this relation, then our multipolar Hardy-Rellich inequality allows us to control the second term on the right hand side in (6.1.24), and we can close the argument for the exponential decay estimate.

It thus remains to choose R and construct η such that $|\nabla^2\eta| \leq C\mathbb{1}_{\cdot\in X_R}$ in such a way that η grows fast enough, and the construction should work up to an exponentially small probability. This is the content of Sections 6.4.1 and 6.4.3. This is the technical heart of the present chapter, and the arguments are novel. We can think of X_R as the good set, and its complement as the bad set, and we need to construct η such that it is locally affine on the bad set, but still grows quadratically. To execute this construction, we start with one η_* that grows quadratically, and then try to modify it so that it becomes affine on the bad set. For such modifications it is necessary that the components of the bad set are well-separated from each other. In general this will not be the case, but we make a multiscale composition of the bad set into parts that live on lengthscale ℓ_j and are well-separated on that lengthscale, and then we change η_* to be affine on those parts separately. The correct choice of the lengthscales ℓ_j turns out to be the rather strange looking $\ell_j = CM^{j^3}\lambda_{\text{mic}}$, where M is some large integer. The construction can be carried out provided the multiscale decomposition vanishes beyond some large lengthscale. Using the results from Theorem 6.1.3 we show that this is the case up to a probability that can be made arbitrarily small.

Unfortunately "arbitrarily small" is not quite good enough, as that means that there are still exceptional pairs (U, U') on which we cannot deduce (6.1.25). But such exceptional pairs are rare, and when we iterate (6.1.25) to conclude (6.1.20) it is sufficient if we can apply

(6.1.25) on at least half of the possible (U, U'), which is possible up to an exponentially small probability.

This completes the construction of η. Once we have η at our disposal, we can complete the proof of (6.1.20). We refer to Section 6.4 for a more detailed exposition of the argument.

6.1.4 Notation and preliminaries

Recall the conventions from Section 1.5.

We will freely use various summation by part identities as in Chapter 2.

For $r > 0$ and $x \in \mathbb{Z}^4$ we let $Q_r(x) = x + [-r, r]^4 \cap \mathbb{Z}^4$ be the cube of diameter $2r$ around x. Note that this deviates from the definitions in Section 1.5. We will frequently use the Minkowski-sum of sets E, E' defined by $E + E' = \{e + e' \colon e \in E, e' \in E'\}$. In particular, $E + Q_r(0)$ is the set of all points at distance $\leq r$ from E.

For measures μ on $\mathfrak{P}(\Lambda)$ we write $\mu(f)$ for $\int f \, \mathrm{d}\mu = \sum_{A \subset \Lambda} f(A)\mu(A)$. We denote a sample from $\zeta_\Lambda^\varepsilon$ by $\mathcal{A}$. We define $\tilde{A} = A \cup (\mathbb{Z}^\mathsf{d} \setminus \Lambda)$ for $A \subset \Lambda$ and analogously $\tilde{\mathcal{A}} = \mathcal{A} \cup (\mathbb{Z}^\mathsf{d} \setminus \Lambda)$. We let $G_{\Lambda \setminus A}$ be the discrete Green's function of Δ_1^2 on $\Lambda \setminus A$, i.e. $G_{\Lambda \setminus A}(x, y) := \mathbb{P}_{\Lambda \setminus A}(\psi_x \psi_y)$.

We use these distances d with respect to $|\cdot|_1$ and $|\cdot|_\infty$ instead of $|\cdot|$, and in that case we write d_1 or d_∞ instead of d.

We will use two different length scales λ_{mic} and λ_{mac}. The former describes the typical distance between two pinned points, which according to Theorem 6.1.3 is of the order $\frac{1}{\varepsilon^{1/\mathsf{d}}}$ if $\mathsf{d} \geq 5$ and $\frac{|\log \varepsilon|^{1/8}}{\varepsilon^{1/4}}$ if $\mathsf{d} = 4$. We hence define

$$\lambda_{\mathrm{mic}} = \begin{cases} \frac{1}{\varepsilon^{1/\mathsf{d}}} + \alpha_{\mathrm{mic},5}(\varepsilon) & \mathsf{d} \geq 5 \\ \frac{|\log \varepsilon|^{1/8}}{\varepsilon^{1/4}} + \alpha_{\mathrm{mic},4}(\varepsilon) & \mathsf{d} = 4 \end{cases}.$$

Here $\alpha_{\mathrm{mic},\mathsf{d}}(\varepsilon) \in [0, 2)$ is chosen in such a way that λ_{mic} is an odd integer.

The latter corresponds to the length scale on which correlations decay, and so, in line with Theorem 6.1.5 we set

$$\lambda_{\mathrm{mac}} = \begin{cases} \frac{1}{\varepsilon^{1/4}} + \alpha_{\mathrm{mac},5}(\varepsilon) & \mathsf{d} \geq 5 \\ \frac{|\log \varepsilon|^{3/8}}{\varepsilon^{1/4}} + \alpha_{\mathrm{mac},4}(\varepsilon) & \mathsf{d} = 4 \end{cases}$$

where $\alpha_{\mathrm{mac},\mathsf{d}}(\varepsilon) \in [0, 2\lambda_{\mathrm{mic}})$ is chosen such that λ_{mac} is an odd multiple of λ_{mic}. Note that for any $\mathsf{d} \geq 4$ we have $1 \ll \lambda_{\mathrm{mic}} \ll \lambda_{\mathrm{mac}}$ as $\varepsilon \to 0$.

Given an odd integer $l > 0$, we consider the set of l-boxes

$$\mathcal{Q}_l = \left\{ Q_{l/2}(x) \colon x \in (l\mathbb{Z})^\mathsf{d} \right\} = \left\{ x + \left[-\frac{l}{2}, \frac{l}{2} \right]^\mathsf{d} \cap \mathbb{Z}^\mathsf{d} \colon x \in (l\mathbb{Z})^\mathsf{d} \right\}$$

and the set

$$\mathcal{P}_l = \left\{ \bigcup_{Q \in I} Q \colon I \subset \mathcal{Q}_l \right\}$$

of l-polymers. We identify each box with the polymer consisting just of that box. We call polymers connected if they are connected as subgraphs of $\mathbb{Z}^\mathsf{d}$ with nearest-neighbour edges. We say that two polymers touch if they are disjoint but their union is connected.

The boxes in $\mathcal{Q}_l$ form a partition of $\mathbb{Z}^{\mathsf{d}}$. Later on we will also need boxes with some overlap. Thus if $l > 0$ is an odd multiple of 3 we define

$$\mathcal{Q}_l^{\#} = \left\{ Q_{l/2}(x) \colon x \in \left(\frac{l}{3}\mathbb{Z}\right)^{\mathsf{d}} \right\} = \left\{ x + \left[-\frac{l}{2}, \frac{l}{2}\right]^{\mathsf{d}} \cap \mathbb{Z}^{\mathsf{d}} \colon x \in \left(\frac{l}{3}\mathbb{Z}\right)^{\mathsf{d}} \right\}.$$

Then every point of $\mathbb{Z}^{\mathsf{d}}$ is contained in precisely 3^{d} boxes in $\mathcal{Q}_l^{\#}$.

For some statement s we let $\mathbb{1}_s$ be the indicator function of s, that is $\mathbb{1}_s = 1$ if s is true, and $\mathbb{1}_s = 0$ else.

6.2 Structure of the pinned set

In this section we prove our results on the distribution of the pinned set, i.e. Theorem 6.1.1, the first part of Theorem 6.1.2, as well as Theorem 6.1.3.

6.2.1 Correlation inequalities

We want to establish the FKG inequality for the set of pinned points in Theorem 6.1.1. We begin with a useful calculation. Let $A \subset A' \subset \Lambda \Subset \mathbb{Z}^{\mathsf{d}}$. Then, using that $\delta_0(\mathrm{d}\psi)$ is a weak limit of the measures $\frac{1}{2t}\mathbb{1}_{\psi \in (-t,t)}\mathrm{d}\psi$ as $t \to 0$, we have

$$\begin{aligned}
\frac{Z_{\Lambda\setminus A'}}{Z_{\Lambda\setminus A}} &= \frac{1}{Z_{\Lambda\setminus A}} \int \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^{\mathsf{d}}} |\Delta_1\psi_x|^2\right) \prod_{x\in\Lambda\setminus A'} \mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus(\Lambda\setminus A')} \delta_0(\mathrm{d}\psi_x) \\
&= \frac{1}{Z_{\Lambda\setminus A}} \lim_{t\to 0} \int \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^{\mathsf{d}}} |\Delta_1\psi_v|^2\right) \\
&\qquad \times \prod_{x\in\Lambda\setminus A'} \mathrm{d}\psi_x \prod_{x\in A'\setminus A} \frac{1}{2t}\mathbb{1}_{\psi_x\in(-t,t)}\mathrm{d}\psi_x \prod_{x\in\mathbb{Z}^{\mathsf{d}}\setminus(\Lambda\setminus A)} \delta_0(\mathrm{d}\psi_x) \\
&= \lim_{t\to 0} \frac{1}{(2t)^{|A'\setminus A|}} \mathbb{P}_{\Lambda\setminus A}(|\psi_x| < t \ \forall x \in A' \setminus A)\,.
\end{aligned} \tag{6.2.1}$$

We can also interpret the right hand side as the density at zero of the Gaussian vector $(\psi_x)_{x\in A'\setminus A}$ under $\mathbb{P}_{\Lambda\setminus A}$ (this observation was essentially already made in [Vel06, p. 143]). If $A' \setminus A = \{x\}$ is a singleton, the density of ψ_x at 0 is equal to $\frac{1}{\sqrt{2\pi}}$ times the inverse of its standard deviation. We thus obtain the formula

$$\frac{Z_{\Lambda\setminus(A\cup\{x\})}}{Z_{\Lambda\setminus A}} = \frac{1}{\sqrt{2\pi G_{\Lambda\setminus A}(x,x)}}\,. \tag{6.2.2}$$

Proof of Theorem 6.1.1. We will prove the FKG lattice condition

$$\zeta_\Lambda^\varepsilon(A\cup A')\zeta_\Lambda^\varepsilon(A\cap A') \geq \zeta_\Lambda^\varepsilon(A)\zeta_\Lambda^\varepsilon(A') \quad \forall A, A' \subset \Lambda\,. \tag{6.2.3}$$

It is well-known that this is a sufficient condition for the validity of the FKG inequality.

Now (6.2.3) is an easy consequence of the Gaussian correlation inequality [Roy14, LM17]. Indeed, note that by definition of $\zeta_\Lambda^\varepsilon$ the estimate (6.2.3) is equivalent to

$$Z_{\Lambda\setminus(A\cup A')}Z_{\Lambda\setminus(A\cap A')} \geq Z_{\Lambda\setminus A}Z_{\Lambda\setminus A'}$$

(here we used $|A \cup A'| + |A \cap A'| = |A| + |A'|$). Dividing both sides by $(Z_{\Lambda\setminus(A\cap A')})^2$ and using (6.2.1) we only have to verify

$$\begin{aligned}&\lim_{t\to 0} \mathbb{P}_{\Lambda\setminus(A\cap A')}(|\psi_x| < t\ \forall x \in (A' \setminus A) \cup (A \setminus A'))\\ &\quad \geq \lim_{t\to 0} \mathbb{P}_{\Lambda\setminus(A\cap A')}(|\psi_x| < t\ \forall x \in A' \setminus A)\mathbb{P}_{\Lambda\setminus(A\cap A')}(|\psi_x| < t\ \forall x \in A \setminus A')\,.\end{aligned}$$

The sets $\{\psi\colon |\psi_x| < t\ \forall x \in A' \setminus A\}$ and $\{\psi\colon |\psi_x| < t\ \forall x \in A \setminus A'\}$ are convex and symmetric around the origin, and the measure $\mathbb{P}_{\Lambda\setminus(A\cap A')}$ is Gaussian. Thus, the claim follows from the Gaussian correlation inequality, applied for each $t > 0$. □

Remark 6.2.1. In [BV01] it is shown that the set of pinned points for the gradient model satisfies a FKG inequality not only in the case of δ-pinning, but also in the case of pinning by a square-well potential $b\mathbb{1}_{|\cdot|<a}$. The proof in [BV01] uses the Ginibre (or GKS) inequality (as described in detail e.g. in [DMRR92, Appendix A]), and thus requires that the measure describing the field is an even fermionic measure. This is certainly not the case for the membrane model, and so that proof cannot be applied in our setting.

Our proof of Theorem 6.1.1 only used that $\mathbb{P}_\Lambda$ is a non-degenerate Gaussian measure. However, this proof would not work if we considered pinning by a square-well potential $b\mathbb{1}_{|\psi|<a}$ instead of δ-pinning. Namely, in this case we would need to consider $\mathbb{P}_\Lambda(\cdot \mid |\psi_x| < a\ \forall x \in A \cap A')$ instead of $\mathbb{P}_{\Lambda\setminus(A\cap A')}$, and the former measure is not Gaussian, so that we cannot apply the Gaussian correlation inequality.

This is not a shortcoming of our proof. Namely, we conjecture that the analogue of (6.2.3) in the case of pinning by a square-well potential is false. We do not have a counterexample for the case of the membrane model. However, we can give an example of a Gaussian measure where the set of pinned points with respect to a square-well potential does not satisfy (6.2.3).

For this example, let X_1, X_2 be independent standard Gaussians, and $N > 0$ a large parameter, and define

$$\begin{pmatrix} Y_1\\ Y_2\\ Y_3\\ Y_4\\ Y_5\\ Y_6\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1\\ N & 0\\ 0 & N\\ 1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix} X_1\\ X_2\end{pmatrix}.$$

Then Y is a multivariate Gaussian vector. It is degenerate, but one can fix this later by adding some small Gaussian noise to it, so we will ignore that point. Let also $A = \{1,3,5,6\}$ and $A' = \{2,4,5,6\}$.

In this setting (6.2.3) would correspond to

$$\mathbb{P}\left(|Y_i| \leq t\ \forall t \in A \cup A'\right)\mathbb{P}\left(|Y_i| \leq t\ \forall t \in A \cap A'\right) \geq \mathbb{P}\left(|Y_i| \leq t\ \forall t \in A\right)\mathbb{P}\left(|Y_i| \leq t\ \forall t \in A'\right) \tag{6.2.4}$$

for any $t > 0$. The probabilities here are equal to the Gaussian measure of certain sets in $\mathbb{R}^2$ (cf. Figure 6.1). As $t \to 0$, we can approximate this Gaussian measure by the Lebesgue measure, and thereby compute that

$$\begin{aligned}\lim_{t\to 0} \frac{2\pi}{t^2}\mathbb{P}\left(|Y_i| \leq t\ \forall t \in A \cup A'\right) &= \frac{4}{N^2}\,,\\ \lim_{t\to 0} \frac{2\pi}{t^2}\mathbb{P}\left(|Y_i| \leq t\ \forall t \in A \cap A'\right) &= 2\,,\end{aligned}$$

$$\lim_{t\to 0}\frac{2\pi}{t^2}\mathbb{P}\left(|Y_i|\le t\ \forall t\in A\right)=\frac{4}{N}-\frac{2}{N^2}\,,$$
$$\lim_{t\to 0}\frac{2\pi}{t^2}\mathbb{P}\left(|Y_i|\le t\ \forall t\in A'\right)=\frac{4}{N}-\frac{2}{N^2}\,.$$

In particular, for N large and t small (6.2.4) is wrong by a factor arbitrarily close to 2.

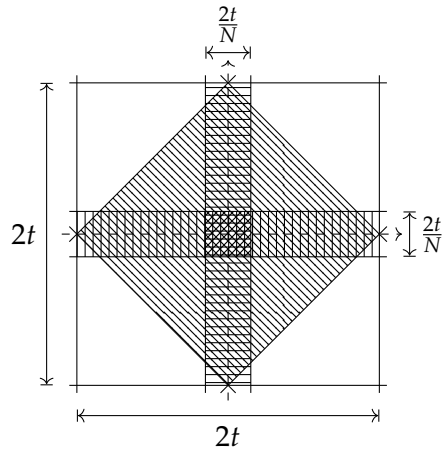


Figure 6.1: The sets associated to the probabilities in (6.2.4). The product of the areas of the large and small square is about half the product of the two areas of the thin rectangles.

Theorem 6.1.1 directly implies the existence of a thermodynamic limit of the $\zeta^\varepsilon_\Lambda$:

Proof of Theorem 6.1.2, first part. It suffices to check that the limit $\lim_{\Lambda\nearrow\mathbb{Z}^d}\zeta^\varepsilon_\Lambda(f)$ exists for each bounded $f\colon\mathfrak{P}(\mathbb{Z}^d)\to\mathbb{R}$ that is a local function (i.e. depends only on finitely many points). Each such f is a linear combination of increasing functions, and so it actually suffices to check that $\lim_{\Lambda\nearrow\mathbb{Z}^d}\zeta^\varepsilon_\Lambda(f)$ exists for each local increasing f.

For that purpose note that Theorem 6.1.1 implies that for any $\Lambda\subset\Lambda'\Subset\mathbb{Z}^d$ large enough so that f only depends on the points in Λ, we have $\zeta^\varepsilon_\Lambda(f)\ge\zeta^\varepsilon_{\Lambda'}(f)$. Thus, $\lim_{R\to\infty}\zeta^\varepsilon_{Q_R(0)}(f)$ exists as a limit of a bounded decreasing sequence. Furthermore, for any $\Lambda\Subset\mathbb{Z}^d$ with $Q_r(0)\subset\Lambda\subset Q_R(0)$ we have

$$\zeta^\varepsilon_{Q_r(0)}(f)\ge\zeta^\varepsilon_\Lambda(f)\ge\zeta^\varepsilon_{Q_R(0)}(f)\ge\lim_{R\to\infty}\zeta^\varepsilon_{Q_R(0)}(f)\,.$$

Since $\Lambda\nearrow\mathbb{Z}^d$ allows us to take $r\to\infty$, we see from this that indeed $\lim_{\Lambda\nearrow\mathbb{Z}^d}\zeta^\varepsilon_\Lambda(f)$ exists and is equal to $\lim_{R\to\infty}\zeta^\varepsilon_{Q_R(0)}(f)$.

Thus, the unique weak limit ζ^ε exists. Its translation invariance follows from the fact that

$$\zeta^\varepsilon(f)=\lim_{\Lambda\nearrow\mathbb{Z}^d}\zeta^\varepsilon_\Lambda(f)=\lim_{\Lambda\nearrow\mathbb{Z}^d}\zeta^\varepsilon_{\Lambda+x}(f(\cdot-x))=\zeta^\varepsilon(f(\cdot-x))$$

for any $x\in\mathbb{Z}^d$. □

6.2.2 Estimates on the pinned set

We will prove the various domination results of Theorem 6.1.3. We first show some estimates on the variance of the membrane model. We begin with the straightforward proofs of part a) and b), then show part d), and finally part c). See Section 6.1.3 for an outline of the proofs.

Let us first give the precise definition of (strong) domination, as in [BV01].

Definition 6.2.2. Let Λ be a finite set, and let ν, ν' be two probability measures on $\mathfrak{P}(\Lambda)$. We say that ν dominates ν' if we have

$$\nu(f) \geq \nu'(f)$$

for all increasing functions $f\colon \mathfrak{P}(\Lambda) \to \mathbb{R}$. We say that ν strongly dominates ν', if for all $x \in \Lambda$ and for all $E \subset \Lambda \setminus \{x\}$ we have

$$\nu(A \ni x \mid A \setminus \{x\} = E) \geq \nu'(A \ni x \mid A \setminus \{x\} = E)\,.$$

It is easy to see that strong stochastic domination implies stochastic domination, and the latter implies

$$\nu(A \cap E = \varnothing) \leq \nu'(A \cap E = \varnothing) \quad \forall E \subset \Lambda\,.$$

Our proof of Theorem 6.1.3 is based on the proof of the corresponding result for the gradient model in [BV01]. We begin with some useful estimates on the variance of the membrane model.

The first one states the fact that the variance is non-increasing in the size of the pinned set.

Lemma 6.2.3. *Let $A \subset A' \subset \Lambda \Subset \mathbb{Z}^{\mathsf{d}}$, and let $x \in \Lambda$. Then $G_{\Lambda\setminus A'}(x,x) \leq G_{\Lambda\setminus A}(x,x)$.*

Proof. This follows easily from the Markov property of the field. See e.g. [BCK17, Corollary 3.2]. □

The preceding lemma allows us to conclude bounds on the variances.

Lemma 6.2.4. *Let $\varnothing \neq A \subset \Lambda \Subset \mathbb{Z}^{\mathsf{d}}$, and let $x \in \Lambda$. If $\mathsf{d} \geq 5$, we have*

$$c_{\mathsf{d}} \leq G_{\Lambda\setminus A}(x,x) \leq C_{\mathsf{d}}\,. \tag{6.2.5}$$

If $\mathsf{d} = 4$, we have

$$\frac{1}{8\pi^2}\log(1+d(x,A)) - C \leq G_{\Lambda\setminus A}(x,x) \leq \frac{1}{4\pi^2}\log(1+d(x,A)) + C\,. \tag{6.2.6}$$

Proof. We begin with the upper bound in (6.2.6). Let $a \in A$ be such that $|x-a| = d(x,A)$. Let $N \in \mathbb{N}$. For large enough N we have $\Lambda \subset Q_N(x)$. Now Lemma 6.2.3 implies that

$$G_{\Lambda\setminus A}(x,x) \leq G_{Q_N(x)\setminus\{a\}}(x,x) \tag{6.2.7}$$

for all N large enough. The right hand side can be computed quite explicitly: We have

$$G_{Q_N(x)\setminus\{a\}}(x,x) = G_{Q_N(x)}(x,x) - \frac{G_{Q_N(x)}(a,x)^2}{G_{Q_N(x)}(a,a)}$$

and by Theorem 4.1.4 we have

$$\left| G_{Q_N(x)}(y,y') - \frac{1}{8\pi^2}\log\left(\frac{N}{1+|y-y'|}\right)\right| \leq C$$

for all $y, y' \in Q_N(x)$ with $d(y, \partial\Lambda_N) \geq cN$, $d(y', \partial\Lambda_N) \geq cN$.

Using this in (6.2.7) we find for N large enough

$$G_{Q_N(x)\setminus\{a\}}(x,x) \leq \frac{1}{8\pi^2}\left(\log N + C - \frac{\left(\log\left(\frac{N}{1+|x-a|}\right) - C\right)^2}{\log N - C}\right)$$

$$\leq \frac{1}{4\pi^2}\log(1+|x-a|)+C$$

and this implies the upper bound in (6.2.6). The lower bound is similar: This time we compare $G_{\Lambda\setminus A}(x,x)$ with $G_{Q_{d(x,A)-1}(x)}(x,x)$.

Finally, the proof of (6.2.5) is similar, using that $G_{Q_N(x)}(x,x)$ is bounded above and below if $\mathsf{d}\geq 5$. □

Proof of Theorem 6.1.3 a) and b). The two results are already proven in [BCK17, Lemma 3.4]. Nonetheless, we repeat the short argument: For $x\in\Lambda$, $E\subset\Lambda\setminus\{x\}$ we have

$$\begin{aligned}\zeta^\varepsilon_\Lambda(\mathcal{A}\ni x\mid \mathcal{A}\setminus\{x\}=E) &= \frac{\zeta^\varepsilon_\Lambda(E\cup\{x\})}{\zeta^\varepsilon_\Lambda(E\subset\mathcal{A}\subset E\cup\{x\})}\\ &= \frac{\zeta^\varepsilon_\Lambda(E\cup\{x\})}{\zeta^\varepsilon_\Lambda(E)+\zeta^\varepsilon_\Lambda(E\cup\{x\})}\\ &= \left(1+\frac{Z_{\Lambda\setminus E}}{\varepsilon Z_{\Lambda\setminus(E\cup\{x\})}}\right)^{-1}\\ &= \left(1+\frac{\sqrt{2\pi G_{\Lambda\setminus E}(x,x)}}{\varepsilon}\right)^{-1}\end{aligned}\tag{6.2.8}$$

where the last step follows from (6.2.2). Now in dimension $\mathsf{d}\geq 5$ we have $c_\mathsf{d}\leq G_{\Lambda\setminus E}(x,x)\leq C_\mathsf{d}$ by Lemma 6.2.4, and this implies

$$c_\mathsf{d}\varepsilon\leq\zeta^\varepsilon_\Lambda(\mathcal{A}\ni x\mid\mathcal{A}\setminus\{x\}=E)\leq C_\mathsf{d}\varepsilon$$

for all ε small enough. From this we immediately conclude the strong domination results from both sides, and these easily imply (6.1.5) and (6.1.6). □

Remark 6.2.5. When $\mathsf{d}=4$ the calculation (6.2.8) is still valid, but we do no longer have a uniform upper bound on $G_{\Lambda\setminus E}(x,x)$. Let us point out for future use though that (6.2.8) and Lemma 6.2.4 imply that

$$\zeta^\varepsilon_\Lambda(\mathcal{A}\ni x\mid\mathcal{A}\setminus\{x\}=E)\leq C\varepsilon$$

and thus the measure $\zeta^\varepsilon_\Lambda$ is strongly dominated by the Bernoulli measure on $\mathfrak{P}(\Lambda)$ with parameter $p'_{4,+}:=C\varepsilon$.

Proof of Theorem 6.1.3 d). One could prove (6.1.8) analogously as in [BV01, Section 3.2]. We, however, give a slightly different proof in the following.

The events $\mathcal{A}\ni x$ for $x\in E$ are decreasing, and so by the FKG property of $\zeta^\varepsilon_\Lambda$ we have

$$\zeta^\varepsilon_\Lambda(\mathcal{A}\cap E=\varnothing)=\zeta^\varepsilon_\Lambda\left(\bigcap_{x\in E}\{\mathcal{A}\not\ni x\}\right)\geq\prod_{x\in E}\zeta^\varepsilon_\Lambda(\mathcal{A}\not\ni x)\,.$$

Thus, to establish (6.1.8) it suffices to show

$$\zeta^\varepsilon_\Lambda(\mathcal{A}\not\ni x)\geq 1-C_\alpha\frac{\varepsilon}{|\log\varepsilon|^{1/2}}$$

or equivalently

$$\zeta^\varepsilon_\Lambda(\mathcal{A}\ni x)\leq C_\alpha\frac{\varepsilon}{|\log\varepsilon|^{1/2}}\tag{6.2.9}$$

where the constant C_α depends only on α.

For this we consider the box $Q := Q_{\min(\varepsilon^{-\alpha},\varepsilon^{-1/5})}(x)$. We can write

$$\begin{aligned}\zeta_\Lambda^\varepsilon(\mathcal{A} \ni x) &= \zeta_\Lambda^\varepsilon(\mathcal{A} \cap Q = \{x\}) + \zeta_\Lambda^\varepsilon(\mathcal{A} \cap Q \supsetneq \{x\}) \\ &\leq \zeta_\Lambda^\varepsilon(\mathcal{A} \ni x \mid \mathcal{A} \cap (Q \setminus \{x\}) = \varnothing) + \zeta_\Lambda^\varepsilon(\mathcal{A} \cap Q \supsetneq \{x\})\,.\end{aligned} \tag{6.2.10}$$

By Remark 6.2.5 the second summand can be estimated as

$$\begin{aligned}\zeta_\Lambda^\varepsilon(\mathcal{A} \cap Q \supsetneq \{x\}) &\leq p'_{4,+} - p'_{4,+}(1 - p'_{4,+})^{|Q|} \\ &= C\varepsilon(1 - (1 - C\varepsilon)^{|Q|-1}) \\ &\leq C\varepsilon^2|Q| \\ &\leq C\varepsilon^2 \left(\varepsilon^{-1/5}\right)^4 \\ &= C\varepsilon^{6/5}\end{aligned} \tag{6.2.11}$$

whenever ε is small enough. For the first summand we can use the FKG property once more and then proceed as in (6.2.8) to see that

$$\begin{aligned}\zeta_\Lambda^\varepsilon(\mathcal{A} \ni x \mid \mathcal{A} \cap (Q \setminus \{x\}) = \varnothing) &\leq \zeta_\Lambda^\varepsilon(\mathcal{A} \ni x \mid \mathcal{A} \cap (Q \setminus \{x\}) = \varnothing, \mathcal{A} \supset \Lambda \setminus Q) \\ &= \zeta_Q^\varepsilon(\mathcal{A} \ni x \mid \mathcal{A} \subset \{x\}) \\ &= \frac{\zeta_Q^\varepsilon(\{x\})}{\zeta_Q^\varepsilon(\varnothing) + \zeta_Q^\varepsilon(\{x\})} \\ &= \left(1 + \frac{Z_Q}{\varepsilon Z_{Q\setminus\{x\}}}\right)^{-1} \\ &= \left(1 + \frac{\sqrt{2\pi G_Q(x,x)}}{\varepsilon}\right)^{-1}.\end{aligned}$$

From Lemma 6.2.4 we know

$$G_Q(x,x) \geq \frac{1}{C}\log\left(1 + \min(\varepsilon^{-\alpha}, \varepsilon^{-1/5}\right) \geq \frac{1}{C_\alpha}|\log\varepsilon|$$

and thus

$$\zeta_\Lambda^\varepsilon(\mathcal{A} \ni x \mid \mathcal{A} \cap (Q \setminus \{x\}) \leq C_\alpha \frac{\varepsilon}{|\log\varepsilon|^{1/2}}\,.$$

When we combine this with (6.2.10) and (6.2.11) we obtain (6.2.9). This completes the proof. □

In this proof the choice of $\varepsilon^{-1/5}$ for the halfdiameter of Q might seem arbitrary. Indeed, one could also choose $\varepsilon^{-1/4}|\log\varepsilon|^{-1/8}$ and obtain the same result. This is still smaller than λ_{mic} which is the actual length scale that one expects here. However, because we have to use Remark 6.2.5 instead of a comparison with $p_{4,+}$, we lose some logarithmic factor and hence cannot use the natural length scale for the size of Q. Fortunately, this does not affect the proof as the estimate (6.2.11) shows that the second summand in (6.2.10) is of lower order.

Proof of Theorem 6.1.3 c). The following proof is based on the proof in [BV01, Section 3.3], which itself is based on [DV00, IV00]. However, that proof is a bit hard to follow as one has to refer to all three references. Furthermore, there is a small mistake in [IV00] that needs to be fixed (cf. Remark 6.2.6 below). Thus, we give a complete proof for the case at hand.

Step 1: Growing microscopic polymers
For reasons that will become clear in the next step we need a procedure to grow microscopic polymers in a controlled way. Thus, we begin with the necessary definitions.

Let K be an odd integer to be fixed later (in (6.2.18)). We consider the polymers in $\mathcal{P}_{K\lambda_{\text{mic}}}$. Let $E \in \mathcal{P}_{K\lambda_{\text{mic}}}$ be such a polymer. Suppose that it has n connected components . We want to define for any multiindex $\underline{k} \in \mathbb{N}^n$ an enlarged polymer $E^{(\underline{k})} \in \mathcal{P}_{K\lambda_{\text{mic}}}$ in such a way that we add k_i boxes to the i-th connected component.

To be precise, fix some enumeration of the boxes in $\mathcal{Q}_{K\lambda_{\text{mic}}}$ by the natural numbers. Let the connected components of E be $E_1, \dots, E_n$, named in such a way that the minimal label of a box in E_i increases with i.

For $i \in \{1, \dots, n\}$, $j \in \{0, \dots, k_i\}$ we define inductively a polymer $E^{(i,j)} \supset E$ as follows. If $j = 0$, we let $E^{(i,j)} = E^{(i-1,k_{i-1})}$ (and $E^{(1,0)} = E$). If $j > 0$, let $\tilde{E}_j$ be the connected component of $E^{(i,j-1)}$ that contains E_j, let $Q^{(i,j)} \in \mathcal{Q}_{K\lambda_{\text{mic}}}$ be the box of smallest index that touches $\tilde{E}_j$, and let $E^{(i,j)} = E^{(i,j-1)} \cup Q^{(i,j)}$. Finally we let $E^{\underline{k}} = E^{(n,k_n)}$.

Let us note some properties of $E^{\underline{k}}$. First of all, it contains precisely $|\underline{k}|_1 := k_1 + \ldots + k_n$ boxes of $\mathcal{Q}_{K\lambda_{\text{mic}}}$ more than E. In other words,

$$\left|E^{\underline{k}}\right| = |E| + |\underline{k}|_1 K^4 \lambda_{\text{mic}}^4 \,. \tag{6.2.12}$$

Furthermore, $E^{\underline{k}}$ has at most n connected components. Each E_i is contained in one of the connected components of $E^{\underline{k}}$, and the latter has grown by at least k_i boxes. Also each fixed box in $\mathcal{Q}_{K\lambda_{\text{mic}}}$ is eventually contained in $E^{\underline{k}}$ whenever $|\underline{k}|_1$ is large enough. Let us also note that each connected component of E consists of at least one box. Therefore we have the estimate

$$n \leq \frac{|E|}{K^4 \lambda_{\text{mic}}^4} \,. \tag{6.2.13}$$

Step 2: Estimate for microscopic polymers
We first prove (6.1.7) for the special case that E is a polymer in $\mathcal{P}_{K\lambda_{\text{mic}}}$, where K is a constant as in Step 1. That is, we claim that there is $\varepsilon_{4,*}$ such that for any $E \subset \Lambda$ such that $E \in \mathcal{P}_{K\lambda_{\text{mic}}}$ and any $\varepsilon < \varepsilon_{4,*}$ we have

$$\left(1 - C\frac{\varepsilon}{|\log \varepsilon|^{1/2}}\right)^{|E|} \geq \zeta_\Lambda^\varepsilon(\mathcal{A} \cap E = \varnothing) \,. \tag{6.2.14}$$

Suppose that E has n connected components, and consider for $\underline{k} \in \mathbb{N}^n$ the polymers $E^{(\underline{k})}$ constructed in the previous section. For $\underline{l} \in \mathbb{N}^n$ we write $\underline{l} > \underline{k}$ to denote $l_i \geq k_i$ for all i and $l_i > k_i$ for at least one i. Recall that $\tilde{\mathcal{A}} = \mathcal{A} \cup (\mathbb{Z}^{\mathsf{d}} \setminus \Lambda)$.

For $|\underline{k}|_1$ large enough we have $E^{(\underline{k})} \not\subset \Lambda$ and therefore $\tilde{\mathcal{A}} \cap E^{(\underline{k})} \neq \varnothing$ almost surely. Thus

$$\zeta_\Lambda^\varepsilon(\mathcal{A} \cap E = \varnothing) = \zeta_\Lambda^\varepsilon(\tilde{\mathcal{A}} \cap E = \varnothing) = \zeta_\Lambda^\varepsilon\left(\exists \underline{k} \in \mathbb{N}^n \colon \tilde{\mathcal{A}} \cap E^{(\underline{k})} = \varnothing, \tilde{\mathcal{A}} \cap E^{(\underline{l})} \neq \varnothing \ \forall \underline{l} > \underline{k}\right)$$

and so in particular

$$\begin{aligned} \zeta_\Lambda^\varepsilon(\mathcal{A} \cap E = \varnothing) &\leq \sum_{\underline{k} \in \mathbb{N}^n} \zeta_\Lambda^\varepsilon\left(\tilde{\mathcal{A}} \cap E^{(\underline{k})} = \varnothing, \tilde{\mathcal{A}} \cap E^{(\underline{l})} \neq \varnothing \ \forall \underline{l} > \underline{k}\right) \\ &\leq \sum_{\underline{k} \in \mathbb{N}^n} \zeta_\Lambda^\varepsilon\left(\tilde{\mathcal{A}} \cap E^{(\underline{k})} = \varnothing \middle| \tilde{\mathcal{A}} \cap E^{(\underline{l})} \neq \varnothing \ \forall \underline{l} > \underline{k}\right) \,. \end{aligned} \tag{6.2.15}$$

Note that this sum is actually a finite sum as for large enough $|\underline{k}|_1$ the conditional probability is equal to 0. Let us estimate the summands in (6.2.15) separately. We have

$$\begin{aligned}
\zeta_\Lambda^\varepsilon\left(\tilde{A}\cap E^{(\underline{k})}=\varnothing \middle| \tilde{A}\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}\right) &= \frac{\zeta_\Lambda^\varepsilon\left(\tilde{A}\cap E^{(\underline{k})}=\varnothing, \tilde{A}\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}\right)}{\zeta_\Lambda^\varepsilon\left(\tilde{A}\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}\right)} \\
&= \frac{\sum\limits_{\substack{A\subset\Lambda\setminus E^{(\underline{k})}\\ \tilde{A}\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}}} \varepsilon^{|A|}\frac{Z_{\Lambda\setminus A}}{Z_\Lambda^\varepsilon}}{\sum\limits_{\substack{A'\subset\Lambda\\ \tilde{A}'\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}}} \varepsilon^{|A'|}\frac{Z_{\Lambda\setminus A'}}{Z_\Lambda^\varepsilon}} \\
&= \frac{\sum\limits_{\substack{A\subset\Lambda\setminus E^{(\underline{k})}\\ \tilde{A}\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}}} \varepsilon^{|A|}Z_{\Lambda\setminus A}}{\sum\limits_{B\subset E^{(\underline{k})}}\sum\limits_{\substack{A\subset\Lambda\setminus E^{(\underline{k})}\\ \tilde{A}\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}}} \varepsilon^{|A|+|B|}Z_{\Lambda\setminus(A\cup B)}} \\
&= \left(\sum_{B\subset E^{(\underline{k})}} \varepsilon^{|B|}\frac{\sum\limits_{\substack{A\subset\Lambda\setminus E^{(\underline{k})}\\ \tilde{A}\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}}} \varepsilon^{|A|}Z_{\Lambda\setminus(A\cup B)}}{\sum\limits_{\substack{A\subset\Lambda\setminus E^{(\underline{k})}\\ \tilde{A}\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}}} \varepsilon^{|A|}Z_{\Lambda\setminus A}}\right)^{-1} \\
&\leq \left(\sum_{B\subset E^{(\underline{k})}} \varepsilon^{|B|}\min_{\substack{A\subset\Lambda\setminus E^{(\underline{k})}\\ \tilde{A}\cap E^{(\underline{l})}\neq\varnothing\ \forall \underline{l}>\underline{k}}}\frac{Z_{\Lambda\setminus(A\cup B)}}{Z_{\Lambda\setminus A}}\right)^{-1}
\end{aligned} \tag{6.2.16}$$

where we have used $\frac{\sum_{i\in I}x_i}{\sum_{i\in I}y_i}\geq \min_{i\in I}\frac{x_i}{y_i}$ in the last step.

Next, we estimate this minimum from below, at least for sufficiently many sets B. Let $m=\frac{|E^{(\underline{k})}|}{K^4\lambda_{\text{mic}}^4}$ be the number of boxes in $E^{(\underline{k})}$. We will consider the class of sets B that contain exactly one point in each box of $E^{(\underline{k})}$.

Consider some $A\subset\Lambda\setminus E^{(\underline{k})}$ such that $\tilde{A}\cap E^{(\underline{l})}\neq\varnothing$ for all $\underline{l}>\underline{k}$. The properties of A imply that each connected component of $E^{(\underline{k})}$ touches a box that contains a point of $\tilde{A}$, as otherwise we could still grow one of the components (by choosing a larger multiindex) without intersecting $\tilde{A}$. Therefore we can enumerate the boxes of $E^{(\underline{k})}$ as $D_1,\dots,D_m$ in such a way that each D_i touches a box that contains a point of $\tilde{A}$ or a box D_j with $j<i$. As mentioned, we consider sets $B=\{b_1,\dots,b_m\}$ that contain one point b_i in each box D_i. Let $B_i=\{b_1,\dots,b_i\}$ (and $B_0=\varnothing$). We have that

$$\frac{Z_{\Lambda\setminus(A\cup B)}}{Z_{\Lambda\setminus A}}=\prod_{i=1}^m\frac{Z_{\Lambda\setminus(A\cup B_i)}}{Z_{\Lambda\setminus(A\cup B_{i-1})}}\,.$$

Pick some $i\in\{1,\dots,m\}$. Our construction of the D_j ensures that D_i touches a box containing a point of $\tilde{A}\cup B_{i-1}$. In particular, $b_i\in D_i$ has distance at most $\sqrt{2^2+1^2+1^2+1^2}K\lambda_{\text{mic}}=\sqrt{7}K\lambda_{\text{mic}}$ from a point in $\tilde{A}\cup B_{i-1}$. Now, (6.2.2) and Lemma 6.2.4 imply that

$$\frac{Z_{\Lambda\setminus(A\cup B_i)}}{Z_{\Lambda\setminus(A\cup B_{i-1})}}=\frac{1}{\sqrt{2\pi G_{\Lambda\setminus(A\cup B_{i-1})}(b_i,b_i)}}$$

$$\geq \frac{1}{C\sqrt{\log\left(1+\sqrt{7}K\lambda_{\text{mic}}\right)}}$$
$$\geq \frac{1}{C|\log\varepsilon|^{1/2}}$$

as soon as ε is small enough (depending on K). Thus,

$$\frac{Z_{\Lambda\setminus(A\cup B)}}{Z_{\Lambda\setminus A}} \geq \left(\frac{1}{C|\log\varepsilon|^{1/2}}\right)^m .$$

This estimate holds for all $A \subset \Lambda \setminus E^{(\underline{k})}$ such that $\tilde{A} \cap E^{(\underline{l})} \neq \varnothing$ for all $\underline{l} > \underline{k}$, and all B that contain exactly one point in each box of $E^{(\underline{k})}$. The number of such sets B is $(K^4\lambda_{\text{mic}}^4)^m$, and so (6.2.16) implies that

$$\begin{aligned}\zeta_\Lambda^\varepsilon\left(\tilde{\mathcal{A}} \cap E^{(\underline{k})} = \varnothing \middle| \tilde{\mathcal{A}} \cap E^{(\underline{l})} \neq \varnothing\ \forall \underline{l} > \underline{k}\right) &\leq \left((K^4\lambda_{\text{mic}}^4)^m \varepsilon^m \left(\frac{1}{C|\log\varepsilon|^{1/2}}\right)^m\right)^{-1} \\ &\leq \left(\left(2K\frac{|\log\varepsilon|^{1/8}}{\varepsilon^{1/4}}\right)^4 \frac{\varepsilon}{C|\log\varepsilon|^{1/2}}\right)^{-m} \\ &= \left(\frac{K^4}{\gamma}\right)^{-m}\end{aligned} \tag{6.2.17}$$

for a certain constant γ. We can now choose K as an odd integer such that

$$K \geq (e\gamma)^{1/4} . \tag{6.2.18}$$

Then (6.2.17) in combination with (6.2.12) implies

$$\begin{aligned}\zeta_\Lambda^\varepsilon\left(\tilde{\mathcal{A}} \cap E^{(\underline{k})} = \varnothing \middle| \tilde{\mathcal{A}} \cap E^{(\underline{l})} \neq \varnothing\ \forall \underline{l} > \underline{k}\right) &\leq \exp(-m) \\ &= \exp\left(-\frac{|E^{(\underline{k})}|}{K^4\lambda_{\text{mic}}^4}\right) \\ &= \exp\left(-\frac{|E|}{K^4\lambda_{\text{mic}}^4} - |\underline{k}|_1\right) .\end{aligned}$$

Now we can use this result in (6.2.15) and obtain

$$\begin{aligned}\zeta_\Lambda^\varepsilon(\mathcal{A} \cap E = \varnothing) &\leq \sum_{\underline{k}\in\mathbb{N}^n} \exp\left(-\frac{|E|}{K^4\lambda_{\text{mic}}^4} - |\underline{k}|_1\right) \\ &= \exp\left(-\frac{|E|}{K^4\lambda_{\text{mic}}^4}\right)\left(\sum_{k_1=0}^\infty \exp(-k_1)\right)\cdot\ldots\cdot\left(\sum_{k_n=0}^\infty \exp(-k_1)\right) \\ &= \exp\left(-\frac{|E|}{K^4\lambda_{\text{mic}}^4}\right)\left(\frac{e}{e-1}\right)^n \\ &= \exp\left(-\frac{|E|}{K^4\lambda_{\text{mic}}^4} + n(1-\log(e-1))\right) .\end{aligned}$$

Finally, we can recall (6.2.13) and conclude

$$\zeta_\Lambda^\varepsilon(\mathcal{A} \cap E = \varnothing) \leq \exp\left(-\frac{|E|}{K^4\lambda_{\text{mic}}^4} + \frac{|E|}{K^4\lambda_{\text{mic}}^4}(1-\log(e-1))\right)$$

$$
\begin{aligned}
&= \exp\left(-\log(e-1)\frac{|E|}{K^4\lambda_{\mathrm{mic}}^4}\right)\\
&\le \exp\left(-C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}|E|\right)\\
&\le \left(1-C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}\right)^{|E|}
\end{aligned}
$$

whenever ε is small enough, and the K (that is now fixed) has been absorbed into the constant. This completes the proof of (6.2.14).

Step 3: Density of pinned points on macroscopic scales

We now show that on the length scale λ_{mac} most points of a set $E \subset \Lambda$ are close to a point in $\tilde{\mathcal{A}}$. To make this precise, we need to make a few definitions. Let L be an odd integer to be fixed later (in (6.2.20) and (6.2.22)). We consider polymers in $\mathcal{P}_{KL\lambda_{\mathrm{mac}}}$. Observe that $KL\lambda_{\mathrm{mac}}$ is an odd multiple of $K\lambda_{\mathrm{mic}}$, the lengthscale from Step 2. For $E \subset \Lambda$ let

$$S_E = \{Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}} \colon Q \cap E \neq \varnothing\}$$

and

$$S_{E,\mathrm{bad}}(\mathcal{A}) = \{Q \in S_E \colon Q \cap \tilde{A} = \varnothing\}\,.$$

We think of the boxes in $S_{E,\mathrm{bad}}(\mathcal{A})$ as bad boxes, as they contain points of E but no pinned point. We will show that not too many boxes are bad. Note that $|S_E| \ge \frac{|E|}{(KL)^4\lambda_{\mathrm{mac}}^4}$. Our claim now is that there is $\varepsilon_{4,*}$ such that for any $E \subset \Lambda$ and any $\varepsilon < \varepsilon_{4,*}$ we have

$$\zeta_\Lambda^\varepsilon\left(|S_{E,\mathrm{bad}}(\mathcal{A})| > \frac{|E|}{2(KL)^4\lambda_{\mathrm{mac}}^4}\right) \le \left(1-C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}\right)^{|E|}. \tag{6.2.19}$$

To see this, we use the result from the previous step to estimate

$$
\begin{aligned}
&\zeta_\Lambda^\varepsilon\left(|S_{E,\mathrm{bad}}(\mathcal{A})| > \frac{|E|}{2(KL)^4\lambda_{\mathrm{mac}}^4}\right)\\
&= \sum_{\substack{T\subset S_E\\ |T|\ge |E|/(2(KL)^4\lambda_{\mathrm{mac}}^4)}} \zeta_\Lambda^\varepsilon(S_{E,\mathrm{bad}}(\mathcal{A}) = T)\\
&\le \sum_{\substack{T\subset S_E\\ |T|\ge |E|/(2(KL)^4\lambda_{\mathrm{mac}}^4)}} \zeta_\Lambda^\varepsilon(S_{E,\mathrm{bad}}(\mathcal{A}) \supset T)\\
&= \sum_{\substack{T\subset S_E\\ |T|\ge |E|/(2(KL)^4\lambda_{\mathrm{mac}}^4)}} \zeta_\Lambda^\varepsilon\left(\mathcal{A}\cap\bigcup_{Q\in T} Q = \varnothing\right)\\
&\le \sum_{\substack{T\subset S_E\\ |T|\ge |E|/(2(KL)^4\lambda_{\mathrm{mac}}^4)}} \left(1-C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}\right)^{|T|(KL)^4\lambda_{\mathrm{mac}}^4}\\
&= \sum_{j=\lceil |E|/(2(KL)^4\lambda_{\mathrm{mac}}^4)\rceil}^{|S_E|} \binom{|S_E|}{j}\left(1-C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}\right)^{j(KL)^4\lambda_{\mathrm{mac}}^4}\\
&\le \sum_{j=\lceil |E|/(2(KL)^4\lambda_{\mathrm{mac}}^4)\rceil}^{|S_E|} \binom{|S_E|}{j}\exp\left(-C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}(KL)^4\lambda_{\mathrm{mac}}^4 j\right)
\end{aligned}
$$

$$\leq \sum_{j=\lceil |E|/(2(KL)^4\lambda_{\text{mac}}^4)\rceil}^{|S_E|} \binom{|S_E|}{j} \exp\left(-C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}(KL)^4\frac{|\log\varepsilon|^{3/2}}{\varepsilon}j\right)$$
$$= \sum_{j=\lceil |E|/(2(KL)^4\lambda_{\text{mac}}^4)\rceil}^{|S_E|} \binom{|S_E|}{j} \varepsilon^{(KL)^4\gamma' j}$$

for a certain constant γ'. We now want to apply the estimate for binomial sums that is stated in Lemma 6.2.7 below with $N = |S_E|$, $p = \varepsilon^{(KL)^4\gamma' j}$ and $r = \frac{|E|}{2(KL)^4\lambda_{\text{mac}}^4|S_E|}$. To do so, we need $p \leq r \leq \frac{1}{2}$. Because $1 \leq \frac{|E|}{|S_E|} \leq (KL)^4\lambda_{\text{mac}}^4$ we always have $r \leq \frac{1}{2}$, and for $p \leq r$ it suffices that $\varepsilon^{(KL)^4\gamma' j} \leq \frac{\varepsilon}{2(KL)^4|\log\varepsilon|^{3/2}}$. To ensure the latter we choose L such that

$$L > \frac{\gamma'^{1/4}}{K} \tag{6.2.20}$$

and ε is small enough. Using Lemma 6.2.7 we then obtain

$$\zeta_\Lambda^\varepsilon\left(|S_{E,\text{bad}}(\mathcal{A})| > \frac{|E|}{2(KL)^4\lambda_{\text{mac}}^4}\right) \leq \left(\frac{p}{r^2}\right)^{r|S_E|}. \tag{6.2.21}$$

We can estimate that

$$\begin{aligned}\frac{p}{r^2} &= \exp\left(-(KL)^4\gamma'|\log\varepsilon| - 2\log\frac{|E|}{2(KL)^4\lambda_{\text{mac}}^4|S_E|}\right)\\ &\leq \exp\left(-(KL)^4\gamma'|\log\varepsilon| + 2\log\frac{1}{2(KL)^4\lambda_{\text{mac}}^4}\right)\\ &\leq \exp\left(-(KL)^4\gamma'|\log\varepsilon| + 2|\log\varepsilon| + 2\log(2(KL)^4) + \log(|\log\varepsilon|^{3/2})\right).\end{aligned}$$

Provided that we choose

$$L > \frac{(2\gamma')^{1/4}}{K} \tag{6.2.22}$$

we can estimate this as

$$\frac{p}{r^2} \leq \exp(-C|\log\varepsilon|)$$

whenever ε is small enough (depending on K, L that are now fixed). Returning to (6.2.21), we see that

$$\begin{aligned}\zeta_\Lambda^\varepsilon\left(|S_{E,\text{bad}}(\mathcal{A})| > \frac{|E|}{2(KL)^4\lambda_{\text{mac}}^4}\right) &\leq \exp\left(-C|\log\varepsilon|\frac{|E|}{2(KL)^4\lambda_{\text{mac}}^4|S_E|}|S_E|\right)\\ &\leq \exp\left(-C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}|E|\right)\end{aligned}$$

which implies (6.2.19).

Step 4: Estimate for arbitrary sets

We now can prove the actual result (6.1.7). So let $E \subset \Lambda$. Using the notation from the previous step, we let

$$E_{\text{bad}}(\mathcal{A}) = E \cap \bigcup_{Q\in S_{E,\text{bad}}(\mathcal{A})} Q$$

be the set of bad points (those which are far from a pinned point). We have the estimate $|E_{\text{bad}}(\mathcal{A})| \leq (KL)^4\lambda_{\text{mac}}^4|S_{E,\text{bad}}(\mathcal{A})|$ and so the previous step implies that

$$\zeta_\Lambda^\varepsilon\left(|E_{\text{bad}}(\mathcal{A})| > \frac{|E|}{2}\right) \leq \zeta_\Lambda^\varepsilon\left(|S_{E,\text{bad}}(\mathcal{A})| > \frac{|E|}{2(KL)^4\lambda_{\text{mac}}^4}\right) \leq \left(1 - C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}\right)^{|E|}.$$

We can now write

$$\begin{aligned}\zeta_\Lambda^\varepsilon(\mathcal{A}\cap E=\varnothing) &\le \zeta_\Lambda^\varepsilon\left(\mathcal{A}\cap E=\varnothing, |E_{\text{bad}}(\mathcal{A})|\le \frac{|E|}{2}\right)+\zeta_\Lambda^\varepsilon\left(|E_{\text{bad}}(\mathcal{A})|>\frac{|E|}{2}\right)\\ &\le \zeta_\Lambda^\varepsilon\left(\mathcal{A}\cap E=\varnothing, |E_{\text{bad}}(\mathcal{A})|\le \frac{|E|}{2}\right)+\left(1-C\frac{\varepsilon}{|\log\varepsilon|^{1/2}}\right)^{|E|}\end{aligned}$$

and so we only need to estimate the first term to establish (6.1.7). If $\zeta_\Lambda^\varepsilon\left(|E_{\text{bad}}(\mathcal{A})|\le\frac{|E|}{2}\right)=0$, that term is equal to 0 and we are trivially done. So we can assume otherwise, and estimate

$$\zeta_\Lambda^\varepsilon\left(\mathcal{A}\cap E=\varnothing, |E_{\text{bad}}(\mathcal{A})|\le \frac{|E|}{2}\right)\le \zeta_\Lambda^\varepsilon\left(\mathcal{A}\cap E=\varnothing\middle||E_{\text{bad}}(\mathcal{A})|\le \frac{|E|}{2}\right).$$

Next, we can apply a similar argument as in (6.2.16) to see that

$$\begin{aligned}\zeta_\Lambda^\varepsilon\left(\mathcal{A}\cap E=\varnothing\middle||E_{\text{bad}}(\mathcal{A})|\le \frac{|E|}{2}\right) &= \frac{\sum\limits_{\substack{A\subset\Lambda\setminus E\\ |E_{\text{bad}}(A)|\le|E|/2}}\varepsilon^{|A|}Z_{\Lambda\setminus A}}{\sum\limits_{B\subset E}\sum\limits_{\substack{A\subset\Lambda\setminus E\\ |E_{\text{bad}}(A)|\le|E|/2}}\varepsilon^{|A|+|B|}Z_{\Lambda\setminus(A\cup B)}}\\ &= \left(\frac{\sum\limits_{\substack{A\subset\Lambda\setminus E\\ |E_{\text{bad}}(A)|\le|E|/2}}\sum\limits_{B\subset E}\varepsilon^{|A|+|B|}Z_{\Lambda\setminus(A\cup B)}}{\sum\limits_{\substack{A\subset\Lambda\setminus E\\ |E_{\text{bad}}(A)|\le|E|/2}}\varepsilon^{|A|}Z_{\Lambda\setminus A}}\right)^{-1}\\ &\le \left(\min_{\substack{A\subset\Lambda\setminus E\\ |E_{\text{bad}}(A)|\le|E|/2}}\sum_{B\subset E}\varepsilon^{|B|}\frac{Z_{\Lambda\setminus(A\cup B)}}{Z_{\Lambda\setminus A}}\right)^{-1}.\end{aligned}\tag{6.2.23}$$

Note that unlike in (6.2.16) we interchanged the summations over A and B in an intermediate step, which allows us to have $\min_A\sum_B$ instead of $\sum_B\min_A$ in the result of this calculation.

We can estimate this further by only allowing good points for B, that is by estimating

$$\zeta_\Lambda^\varepsilon\left(\mathcal{A}\cap E=\varnothing\middle||E_{\text{bad}}(\mathcal{A})|\le \frac{|E|}{2}\right)\le\left(\min_{\substack{A\subset\Lambda\setminus E\\ |E_{\text{bad}}(A)|\le|E|/2}}\sum_{B\subset E\setminus E_{\text{bad}}(A)}\varepsilon^{|B|}\frac{Z_{\Lambda\setminus(A\cup B)}}{Z_{\Lambda\setminus A}}\right)^{-1}.\tag{6.2.24}$$

Consider some $A\subset\Lambda\setminus E$, and some $B\subset E\setminus E_{\text{bad}}(A)$. By definition of $E_{\text{bad}}(A)$, each point in B is in the same macroscopic box as a point of $\tilde{A}$. In particular, each point in B has distance at most $\sqrt{7}KL\lambda_{\text{mac}}$ to a point of $\tilde{A}$. Thus, if we let $B=\{b_1,\dots,b_{|B|}\}$, and $B_i=\{b_1,\dots,b_i\}$ we see as in Step 2 that

$$\begin{aligned}\frac{Z_{\Lambda\setminus(A\cup B)}}{Z_{\Lambda\setminus A}} &= \prod_{i=1}^{|B|}\frac{Z_{\Lambda\setminus(A\cup B_i)}}{Z_{\Lambda\setminus(A\cup B_{i-1})}}\\ &= \prod_{i=1}^{|B|}\frac{1}{\sqrt{2\pi G_{\Lambda\setminus(A\cup B_{i-1})}}(b_i,b_i)}\end{aligned}$$

$$\geq \prod_{i=1}^{|B|} \frac{1}{C\sqrt{\log\left(1+\sqrt{7}KL\lambda_{\mathrm{mac}}\right)}}$$
$$\geq \left(\frac{1}{C|\log\varepsilon|^{1/2}}\right)^{|B|}$$

where we used (6.2.2) and Lemma 6.2.4. Returning to (6.2.23) and (6.2.24), we obtain

$$\begin{aligned}
&\zeta_\Lambda^\varepsilon\left(\mathcal{A}\cap E=\varnothing \,\middle|\, |E_{\mathrm{bad}}(\mathcal{A})|\leq\frac{|E|}{2}\right)\\
&\leq\left(\min_{\substack{A\subset\Lambda\setminus E\\ |E_{\mathrm{bad}}(A)|\leq|E|/2}}\sum_{B\subset E\setminus E_{\mathrm{bad}}(A)}\varepsilon^{|B|}\left(\frac{1}{C|\log\varepsilon|^{1/2}}\right)^{|B|}\right)^{-1}\\
&=\left(\min_{\substack{A\subset\Lambda\setminus E\\ |E_{\mathrm{bad}}(A)|\leq|E|/2}}\sum_{j=0}^{|E\setminus E_{\mathrm{bad}}(A)|}\binom{|E\setminus E_{\mathrm{bad}}(A)|}{j}\left(\frac{\varepsilon}{C|\log\varepsilon|^{1/2}}\right)^{j}\right)^{-1}\\
&=\left(\min_{\substack{A\subset\Lambda\setminus E\\ |E_{\mathrm{bad}}(A)|\leq|E|/2}}\left(1+\frac{\varepsilon}{C|\log\varepsilon|^{1/2}}\right)^{|E\setminus E_{\mathrm{bad}}(A)|}\right)^{-1}\\
&\leq\left(\left(1+\frac{\varepsilon}{C|\log\varepsilon|^{1/2}}\right)^{|E|/2}\right)^{-1}\\
&\leq\left(1-\frac{\varepsilon}{C|\log\varepsilon|^{1/2}}\right)^{|E|}.
\end{aligned}$$

This finally completes the proof. □

Remark 6.2.6. In [BV01] a similar argument is used. However, for growing the polymers [BV01] refers to [IV00], where a construction that is different from ours is used. Unfortunately, the argument from [IV00] contains a small gap.

The problem is as follows: Take $\mathsf{d}\geq 2$. In [IV00] the grown polymer $\tilde{E}^{\underline{k}}$ is only defined for certain admissible $\underline{k}$. Using our notation, one defines $\tilde{E}^{\underline{k}}$ by adding k_i layers of microscopic cubes to E_i, i.e. one replaces E by

$$\tilde{E}^{\underline{k}} := \bigcup_{i=1}^{n} E_i + Q_{k_iK\lambda_{\mathrm{mic}}}(0)\,.$$

However, this is only done if for each $i\in\{1,\dots,n\}$ we have that $E_i+Q_{k_iK\lambda_{\mathrm{mic}}}(0)$ and $\bigcup_{j=1}^{i-1}E_i+Q_{k_iK\lambda_{\mathrm{mic}}}$ are disjoint or $k_i=0$ (and the $\underline{k}$ with this property are called admissible). Now in [IV00, p. 398] it is claimed that this construction satisfies

$$|\tilde{E}^{\underline{k}}|\geq|E|+|\underline{k}|_1K^{\mathsf{d}}\lambda_{\mathrm{mic}}^{\mathsf{d}}\,, \tag{6.2.25}$$

or in other words that we have added at least $|\underline{k}|_1$ boxes. This is not true in general: For example if L is a large odd number and

$$E_1=\left[-\frac{K\lambda_{\mathrm{mic}}}{2},\frac{K\lambda_{\mathrm{mic}}}{2}\right]\cap\mathbb{Z}^{\mathsf{d}}$$

$$E_2 = \left(\left[-\frac{KL\lambda_{\mathrm{mic}}}{2}, \frac{KL\lambda_{\mathrm{mic}}}{2} \right] \setminus \left[-\frac{3K\lambda_{\mathrm{mic}}}{2}, \frac{3K\lambda_{\mathrm{mic}}}{2} \right] \right) \cap \mathbb{Z}^{\mathsf{d}}$$

and $E = E_1 \cup E_2$, then for any $k_1 \in \left\{1, \frac{L-1}{2}\right\}$ the multiindex $\underline{k} = (k_1, 0)$ is admissible, but to obtain $\tilde{E}^{\underline{k}}$ we only add the $3^{\mathsf{d}} - 1$ cubes that form the gap between E_1 and E_2. If L is large enough, we can take $k_1 \geq 3^{\mathsf{d}}$, and we arrive at a contradiction to (6.2.25).

Note that this problem is not present in the construction that we used in Step 1 of the proof of Theorem 6.1.3 c), as our construction directly ensures that (6.2.12) holds. The same construction could also be used in [IV00] to fix the gap there.

Alternatively (as pointed out to the author by Yvan Velenik) one can also fix the gap in [IV00] by first ordering the E_k in such a way that no E_i completely surrounds an E_j with $i < j$.

In our proof of Theorem 6.1.3 c) we used a tail bound for certain binomial sums. We will use this estimate a few more times in Section 6.4.3, so we state and prove it separately.

Lemma 6.2.7. *Let $N \in \mathbb{N}$, and $\frac{1}{2} \geq r \geq p \geq 0$. Then*

$$\sum_{j=\lceil rN \rceil}^{N} \binom{N}{j} p^j \leq \left(\frac{p}{r^2}\right)^{rN} . \tag{6.2.26}$$

This estimate is very similar to standard Chernoff tail bounds for the binomial distribution. A special case was used in [BV01, Section 3.3.2]. For the proof we will follow the proof of the Chernoff tail bound.

Proof. For any $t \geq 0$ we have the estimate

$$\begin{aligned} \sum_{j=\lceil rN \rceil}^{N} \binom{N}{j} p^j &\leq e^{-trN} \sum_{j=0}^{N} \binom{N}{j} e^{tj} p^j \\ &\leq e^{-trN} (1 + e^t p)^N . \end{aligned}$$

The optimal choice for t is $t = \log\left(\frac{r}{(1-r)p}\right)$, and this yields

$$\sum_{j=\lceil rN \rceil}^{N} \binom{N}{j} p^j \leq \left(\frac{(1-r)^{r-1}}{r^r} \right)^N p^{rN} .$$

It remains to observe that for $0 < r \leq \frac{1}{2}$ one has

$$\frac{(1-r)^{r-1}}{r^r} \leq \frac{1}{r^{2r}} .$$

□

6.3 Some inequalities

In this section we provide some tools that will be used in the next two sections to establish Theorem 6.1.5, namely a discrete multipolar Hardy-Rellich inequality as well as an interpolation inequality. We begin with the former.

6.3.1 A discrete multipolar Hardy-Rellich inequality

We want to give a quantitative estimate on the strength of the pinning effect on $x \in \Lambda$. More precisely, consider a function $u\colon \Lambda \to \mathbb{R}$ such that $u = 0$ on $\tilde{A} = A \cup (\mathbb{Z}^{\mathsf{d}} \setminus \Lambda)$. We want to control a weighted L^2-norm of u by the L^2-norm of $\nabla_1^2 u$. The weight at $x \in \Lambda$ will have to depend on the location of x with respect to $\tilde{A}$. If $\nabla_1^2 u$ is small, then u is (locally) close to an affine function. We need to ensure that this affine function is close to zero near x, and for this purpose we need that u is close to 0 at $\mathsf{d}+1$ points that are well-spread out, i.e. we need that x is close to $\mathsf{d}+1$ pinned points.

To state our precise result we need some definitions. First we construct $\mathsf{d}+1$ cones of directions that are well-spread out: Let $\theta_1, \dots, \theta_{\mathsf{d}+1} \in \mathbb{S}^{\mathsf{d}-1}$ be such that $\theta_i \cdot \theta_j = -\frac{1}{\mathsf{d}}$ for $i \neq j$ (e.g. take $(\theta_i)_{i=1}^{\mathsf{d}+1}$ to be the vertices of a regular d-dimensional simplex with circumsphere $\mathbb{S}^{\mathsf{d}-1}$).

For $\kappa > 0$ let $\Theta_i = B_\kappa(\theta_i) \cap \mathbb{S}^{\mathsf{d}-1}$. For κ small enough we have $\theta_i' \cdot \theta_j' < 0$ for all $\theta_i' \in \Theta_i$, $\theta_j' \in \Theta_j$ for $i \neq j$. Fix one such choice of κ. Finally let $\Xi_i = \left\{ y \in \mathbb{R}^{\mathsf{d}} \setminus \{0\} \colon \frac{y}{|y|} \in \Theta_i \right\}$ (cf. Figure 6.2).

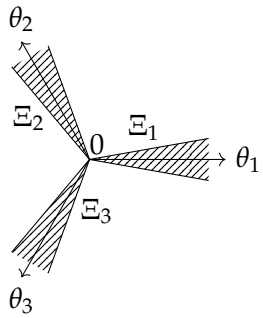


Figure 6.2: The sets Ξ_i for $\mathsf{d} = 2$

For $x \in \Lambda$ let

$$d^{(i)}(x, \tilde{A}) = \inf_{a \in \tilde{A} \cap (x + \Xi_i)} |x - a|_1 ,$$
$$d_*(x, \tilde{A}) = \max_{i \in \{1, \dots, \mathsf{d}+1\}} d^{(i)}(x, \tilde{A})$$

with the convention that $\inf \varnothing = +\infty$. Thus, for each x there are $\mathsf{d}+1$ points in $\tilde{A}$ which are well-spread out around A with distance at most $d_*(x, \tilde{A})$.

Then we have the following statement.

Theorem 6.3.1. *Let $A \subset \Lambda$ be arbitrary. Let $V \subset \Lambda$ be an arbitrary subset. Let $R \in \mathbb{N}$, $R \geq 2$ be a parameter. Suppose that $u\colon \Lambda \to \mathbb{R}$ is such that $u = 0$ on $\tilde{A}$. Then*

$$\left\| u \mathbb{1}_{d_*(\cdot, \tilde{A}) \leq R} \right\|_{L^2(V)}^2 \leq C_{\mathsf{d}} R^{\mathsf{d}} (1 + \mathbb{1}_{\mathsf{d}=4} \log R) \left\| \nabla_1^2 u \right\|_{L^2(V + Q_R(0))}^2 \tag{6.3.1}$$

where the constant is independent of R, V and A.

Proof. We begin with the case $\mathsf{d} \geq 5$. We fix an enumeration of the points in $\mathbb{Z}^{\mathsf{d}}$.

We first establish a pointwise bound for $u \mathbb{1}_{d_*(\cdot, \tilde{A}) \leq R}$. Let $x \in V$ such that $d_*(x, \tilde{A}) \leq R$. For $i \in \{1, \dots, \mathsf{d}+1\}$ consider the points in $\tilde{A} \cap (x + \Xi_i)$ of minimal l^1-distance to x, and

let $a_x^{(i)}$ be the one among those that comes first with respect to our fixed enumeration. By assumption $|x - a_x^{(i)}|_1 = d^{(i)}(x, \tilde{A}) \leq R$.

We first claim

$$\begin{aligned} |u(x)| &\leq \max_{i \in \{1,\dots,\mathsf{d}+1\}} \left| u(a_x^{(i)}) - u(x) - \nabla_1 u(x) \cdot \left(a_x^{(i)} - x \right) \right| \\ &= \max_{i \in \{1,\dots,\mathsf{d}+1\}} \left| u(x) + \nabla_1 u(x) \cdot \left(a_x^{(i)} - x \right) \right| . \end{aligned} \tag{6.3.2}$$

Indeed, we can assume $u(x) \geq 0$ (the other case is analogous). There is an index i such that $\nabla_1 u(x) \cdot \left(a_x^{(i)} - x \right) \geq 0$, as otherwise the $\mathsf{d}+2$ vectors

$$\nabla_1 u(x), a_x^{(1)} - x, \dots, a_x^{(\mathsf{d}+1)} - x$$

would have pairwise negative scalar products, while it is easy to see that this is possible in $\mathbb{R}^{\mathsf{d}}$ for at most $\mathsf{d}+1$ vectors. In particular, we have

$$\max_{i \in \{1,\dots,\mathsf{d}+1\}} \nabla_1 u(x) \cdot \left(a_x^{(i)} - x \right) \geq 0 .$$

By assumption $u(a_x^{(i)}) = 0$ and so

$$\begin{aligned} u(x) &\leq u(x) + \max_{i \in \{1,\dots,\mathsf{d}+1\}} \nabla_1 u(x) \cdot \left(a_x^{(i)} - x \right) \\ &= \max_{i \in \{1,\dots,\mathsf{d}+1\}} u(x) - u(a_x^{(i)}) + \nabla_1 u(x) \cdot \left(a_x^{(i)} - x \right) \\ &\leq \max_{i \in \{1,\dots,\mathsf{d}+1\}} \left| u(x) - u(a_x^{(i)}) + \nabla_1 u(x) \cdot \left(a_x^{(i)} - x \right) \right| \end{aligned}$$

which implies (6.3.2).

We now want to pick a nearest neighbour path $\Psi_x^{(i)} = \left(\Psi_x^{(i)}(0), \dots, \Psi_x^{(i)}(d^{(i)}(x, \tilde{A})) \right)$ such that $\Psi_x^{(i)}(0) = x$, $\Psi_x^{(i)}(d^{(i)}(x, \tilde{A})) = a_x^{(i)}$. We can pick this path in such a way that all of its points have distance at most $\sqrt{\mathsf{d}}$ from the straight line connecting x and $a_x^{(i)}$, and such that all but possible the first $\tilde{\alpha}_{\mathsf{d}}$ of its vertices lie inside the widening cone $x + \Xi_i$ (cf. Figure 6.3). Here $\tilde{\alpha}_{\mathsf{d}}$ is a constant depending only on d and the Ξ_i.

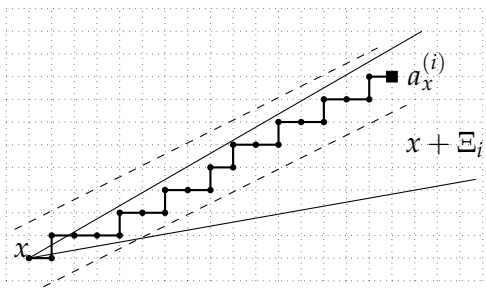


Figure 6.3: Choice of the path $\Psi_x^{(i)}$. We require all points to have distance at most $\sqrt{\mathsf{d}}$ from the straight line between x and $a_x^{(i)}$ (i.e. to be in the dashed strip), and all but the first $\tilde{\alpha}_{\mathsf{d}}$ to be inside the cone $x + \Xi_i$.

We can now apply a discrete version of the fundamental theorem of calculus along the paths $\Psi_x^{(i)}$ to the function $v := u(\cdot) - u(x) - \nabla_1 u(x) \cdot (\cdot - x)$. Namely, we know $v(x) = 0$

and $\nabla_1 v(x) = 0$. The point $\Psi_x^{(i)}(1)$ is one of the 2d neighbours of x. If it happens that $\Psi_x^{(i)}(1) \in \{x + e_1, \dots, x + e_{\mathsf{d}}\}$, then $v(\Psi_x^{(i)}(1)) = 0$ and we can write

$$v(a_x^{(i)}) = \sum_{t=0}^{d^{(i)}(x,A)-1} \sum_{s=1}^{t-1} v(\Psi_x^{(i)}(s+1)) - 2v(\Psi_x^{(i)}(s)) + v(\Psi_x^{(i)}(s-1)) \,.$$

On the other hand, if $\Psi_x^{(i)}(1) \in \{x - e_1, \dots, x - e_{\mathsf{d}}\}$, we can temporarily add a point $\Psi_x^{(i)}(-1) = 2x - \Psi_x^{(i)}(1)$ to our path, so that $v(\Psi_x^{(i)}(-1)) = 0$, and then write

$$v(a_x^{(i)}) = \sum_{t=0}^{d^{(i)}(x,\tilde{A})-1} \sum_{s=0}^{t-1} v(\Psi_x^{(i)}(s+1)) - 2v(\Psi_x^{(i)}(s)) + v(\Psi_x^{(i)}(s-1)) \,.$$

In both cases we can conclude that

$$\begin{aligned}
\left| u(a_x^{(i)}) - u(x) - \nabla_1 u(x) \cdot \left(a_x^{(i)} - x \right) \right| &= \left| v(a_x^{(i)}) \right| \\
&\leq \sum_{t=0}^{d^{(i)}(x,\tilde{A})-1} \sum_{s=0}^{t-1} \left| \nabla_1^2 v(\Psi_x^{(i)}(s)) \right| \\
&= \sum_{t=0}^{d^{(i)}(x,\tilde{A})-1} \sum_{s=0}^{t-1} \left| \nabla_1^2 u(\Psi_x^{(i)}(s)) \right| \\
&= \sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} (d^{(i)}(x,\tilde{A}) - s) \left| \nabla_1^2 u(\Psi_x^{(i)}(s)) \right|
\end{aligned}$$

where we have changed the order of summation in the last step. Thus, (6.3.2) implies that

$$\begin{aligned}
|u(x)| &\leq \max_{i \in \{1,\dots,\mathsf{d}+1\}} \left| u(a_x^{(i)}) - u(x) - \nabla_1 u(x) \cdot \left(a_x^{(i)} - x \right) \right| \\
&= \max_{i \in \{1,\dots,\mathsf{d}+1\}} \sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} (d^{(i)}(x,\tilde{A}) - s) \left| \nabla_1^2 u(\Psi_x^{(i)}(s)) \right| \,.
\end{aligned} \tag{6.3.3}$$

We have this estimate for all x such that $d_*(x, \tilde{A}) \leq R$. Defining Ψ arbitrarily for the other x and summing the square of (6.3.3) over x, we find

$$\begin{aligned}
\sum_{x \in V} |u(x)|^2 \mathbb{1}_{d_*(x,\tilde{A}) \leq R} &\leq \sum_{x \in V} \mathbb{1}_{d_*(x,\tilde{A}) \leq R} \left(\max_{i \in \{1,\dots,\mathsf{d}\}} \sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} (d^{(i)}(x,\tilde{A}) - s) \left| \nabla_1^2 u(\Psi_x^{(i)}(s)) \right| \right)^2 \\
&\leq \sum_{i=1}^{\mathsf{d}+1} \sum_{x \in V} \mathbb{1}_{d_*(x,\tilde{A}) \leq R} \left(\sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} (d^{(i)}(x,\tilde{A}) - s) \left| \nabla_1^2 u(\Psi_x^{(i)}(s)) \right| \right)^2 .
\end{aligned} \tag{6.3.4}$$

Consider a nonzero summand of the outer two sums. Then $d_*(x, \tilde{A}) \leq R$, and (recalling that

$\mathsf{d} \geq 5$) we can apply Hölder's inequality to the innermost sum to obtain

$$\begin{aligned}
&\left(\sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} (d^{(i)}(x,\tilde{A})-s)\left|\nabla_1^2 u(\Psi_x^{(i)}(s))\right|\right)^2 \\
&\leq \left(\sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} \frac{1}{(d^{(i)}(x,\tilde{A})-s)^{\mathsf{d}-3}}\right)\left(\sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} (d^{(i)}(x,\tilde{A})-s)^{\mathsf{d}-1}\left|\nabla_1^2 u(\Psi_x^{(i)}(s))\right|^2\right) \\
&\leq C_{\mathsf{d}} \sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} (d^{(i)}(x,\tilde{A})-s)^{\mathsf{d}-1}\left|\nabla_1^2 u(\Psi_x^{(i)}(s))\right|^2 .
\end{aligned} \tag{6.3.5}$$

If $s > \tilde{\alpha}_{\mathsf{d}}$, we know $\Psi_x^{(i)}(s) \in x + \Xi_i$, and hence $\Psi_x^{(i)}(s) + \Xi_i \subset x + \Xi_i$, which implies $d^{(i)}(\Psi_x^{(i)}(s), \tilde{A}) \geq d^{(i)}(x,\tilde{A}) - s$. If $s \leq \tilde{\alpha}_{\mathsf{d}}$, we can just use the estimate $(d^{(i)}(x,\tilde{A})-s)^{\mathsf{d}-1} \leq R^{\mathsf{d}-1}$.

Using this in (6.3.5) we obtain

$$\begin{aligned}
&\left(\sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} (d^{(i)}(x,\tilde{A})-s)\left|\nabla_1^2 u(\Psi_x^{(i)}(s))\right|\right)^2 \\
&\leq C_{\mathsf{d}} \sum_{s=\tilde{\alpha}_{\mathsf{d}}+1}^{d^{(i)}(x,\tilde{A})-1} d^{(i)}(\Psi_x^{(i)}(s),\tilde{A})^{\mathsf{d}-1}\left|\nabla_1^2 u(\Psi_x^{(i)}(s))\right|^2 + C_{\mathsf{d}} \sum_{s=0}^{\tilde{\alpha}_{\mathsf{d}}} R^{\mathsf{d}-1}\left|\nabla_1^2 u(\Psi_x^{(i)}(s))\right|^2 \\
&\leq C_{\mathsf{d}} \sum_{y\in\Psi_x^{(i)}} \left(\mathbb{1}_{|y-x|_1>\tilde{\alpha}_{\mathsf{d}}} d^{(i)}(y,\tilde{A})^{\mathsf{d}-1} + \mathbb{1}_{|y-x|_1\leq\tilde{\alpha}_{\mathsf{d}}} R^{\mathsf{d}-1}\right)\left|\nabla_1^2 u(y)\right|^2 \\
&\leq C_{\mathsf{d}} \sum_{y\in\Psi_x^{(i)}} \left(\mathbb{1}_{|y-x|_1>\tilde{\alpha}_{\mathsf{d}}, d^{(i)}(y,\tilde{A})>\tilde{\alpha}_{\mathsf{d}}} d^{(i)}(y,\tilde{A})^{\mathsf{d}-1} + \mathbb{1}_{d^{(i)}(y,\tilde{A})\leq\tilde{\alpha}_{\mathsf{d}}} + \mathbb{1}_{|y-x|_1\leq\tilde{\alpha}_{\mathsf{d}}} R^{\mathsf{d}-1}\right)\left|\nabla_1^2 u(y)\right|^2 \\
&\leq C_{\mathsf{d}} \sum_{y\in\Psi_x^{(i)}} \left(\mathbb{1}_{|y-x|_1>\tilde{\alpha}_{\mathsf{d}}, d^{(i)}(y,\tilde{A})>\tilde{\alpha}_{\mathsf{d}}} d^{(i)}(y,\tilde{A})^{\mathsf{d}-1} + \mathbb{1}_{|y-x|_1\leq\tilde{\alpha}_{\mathsf{d}}} R^{\mathsf{d}-1} + 1\right)\left|\nabla_1^2 u(y)\right|^2 .
\end{aligned}$$

We can insert this into (6.3.4) and change the order of summation once more to obtain that

$$\begin{aligned}
&\sum_{x\in V} |u(x)|^2 \mathbb{1}_{d_*(x,\tilde{A})\leq R} \\
&\leq C_{\mathsf{d}} \sum_{i=1}^{\mathsf{d}+1} \sum_{x\in V} \mathbb{1}_{d_*(x,\tilde{A})\leq R} \\
&\qquad \sum_{y\in\Psi_x^{(i)}} \left(\mathbb{1}_{|y-x|_1>\tilde{\alpha}_{\mathsf{d}}, d^{(i)}(y,\tilde{A})>\tilde{\alpha}_{\mathsf{d}}} d^{(i)}(y,\tilde{A})^{\mathsf{d}-1} + \mathbb{1}_{|y-x|_1\leq\tilde{\alpha}_{\mathsf{d}}} R^{\mathsf{d}-1} + 1\right)\left|\nabla_1^2 u(y)\right|^2 \\
&\leq C_{\mathsf{d}} \sum_{i=1}^{\mathsf{d}+1} \sum_{y\in V+Q_R(0)} \left|\nabla_1^2 u(y)\right|^2 \\
&\qquad \sum_{x:\, y\in\Psi_x^{(i)}} \mathbb{1}_{d_*(x,\tilde{A})\leq R}\left(\mathbb{1}_{|y-x|_1>\tilde{\alpha}_{\mathsf{d}}, d^{(i)}(y,\tilde{A})>\tilde{\alpha}_{\mathsf{d}}} d^{(i)}(y,\tilde{A})^{\mathsf{d}-1} + \mathbb{1}_{|y-x|_1\leq\tilde{\alpha}_{\mathsf{d}}} R^{\mathsf{d}-1} + 1\right)
\end{aligned}$$

$$\leq C_{\mathsf{d}} \sum_{i=1}^{\mathsf{d}+1} \sum_{y \in V+Q_R(0)} \left|\nabla_1^2 u(y)\right|^2 \left(\left|\left\{x\colon y \in \Psi_x^{(i)}, |y-x|_1 > \tilde{\alpha}_{\mathsf{d}}, d^{(i)}(y,\tilde{A}) > \tilde{\alpha}_{\mathsf{d}}, d_*(x,\tilde{A}) \leq R\right\}\right| d^{(i)}(y,\tilde{A})^{\mathsf{d}-1} + \left|\left\{x\colon y \in \Psi_x^{(i)}, |y-x|_1 \leq \tilde{\alpha}_{\mathsf{d}}\right\}\right| R^{\mathsf{d}-1} + \left|\left\{x\colon y \in \Psi_x^{(i)}, d_*(x,\tilde{A}) \leq R\right\}\right| \right). \tag{6.3.6}$$

The cardinality of the second set here is trivial to estimate and we find $|\{x\colon y \in \Psi_x^{(i)}, |y-x|_1 \leq \tilde{\alpha}_{\mathsf{d}}\}| \leq C_{\mathsf{d}}\tilde{\alpha}_{\mathsf{d}}^{\mathsf{d}}$. Similarly, $y \in \Psi_x^{(i)}$ and $d_*(x,\tilde{A}) \leq R$ imply $|x-y|_1 \leq R$, and so the cardinality of the third set can be estimated as $|\{x\colon y \in \Psi_x^{(i)}, d_*(x,\tilde{A}) \leq R\}| \leq C_{\mathsf{d}}R^{\mathsf{d}}$.

To estimate the cardinality of the first set we need to work a bit. The heuristic here is that the paths $\Psi_x^{(i)}$ are close to straight lines with the same endpoint passing through y, so there cannot be too many of them. To make this precise, fix y with $d^{(i)}(y,\tilde{A}) > \tilde{\alpha}_{\mathsf{d}}$ and consider some x such that $y \in \Psi_x^{(i)}$, $|y-x|_1 > \tilde{\alpha}_{\mathsf{d}}$ and $d_*(x,\tilde{A}) \leq R$. Because $|y-x|_1 > \tilde{\alpha}_{\mathsf{d}}$ and $d^{(i)}(y,\tilde{A}) > \tilde{\alpha}_{\mathsf{d}}$, we know that $a_y^{(i)} \in y + \Xi_i$ and $y \in x + \Xi_i$, and hence $a_y^{(i)} \subset x + \Xi_i$. Thus, $a_y^{(i)}$ is one of the candidates for the endpoint of the path $\Psi_x^{(i)}$, and our definition of the paths ensures that we actually have $a_x^{(i)} = a_y^{(i)}$. Because $y \in \Psi_x^{(i)}$, the point y has distance at most $\sqrt{\mathsf{d}}$ from the straight line connecting x and $a_x^{(i)} = a_y^{(i)}$. Therefore x is contained in some fixed cone with tip $a_y^{(i)}$ and opening angle $\leq \frac{C_{\mathsf{d}}}{|a_y^{(i)}-y|_1} = \frac{C_{\mathsf{d}}}{d^{(i)}(y,\tilde{A})}$. The point x is also contained in the cube around $a_y^{(i)}$ with diameter $2R$, as otherwise $d_*(x,\tilde{A}) \leq d^{(i)}(x,\tilde{A}) = |x-a_y^{(i)}|_1 \geq |x-a_y^{(i)}|_\infty > R$. Thus, x is contained in the intersection of the aforementioned cone with that cube. This intersection contains at most $C_{\mathsf{d}}R\left(\frac{R}{d^{(i)}(y,\tilde{A})}\right)^{\mathsf{d}-1} = C_{\mathsf{d}}\frac{R^{\mathsf{d}}}{(d^{(i)}(y,\tilde{A}))^{\mathsf{d}-1}}$ lattice points, and so

$$\left|\left\{x\colon y \in \Psi_x^{(i)}, |y-x|_1 > \tilde{\alpha}_{\mathsf{d}}, d^{(i)}(y,\tilde{A}) > \tilde{\alpha}_{\mathsf{d}}, d_*(x,\tilde{A}) \leq R\right\}\right| \leq C_{\mathsf{d}}\frac{R^{\mathsf{d}}}{(d^{(i)}(y,\tilde{A}))^{\mathsf{d}-1}}\,.$$

Returning now to (6.3.6), we find

$$\begin{aligned}
&\sum_{x \in V} |u(x)|^2 \mathbb{1}_{d_*(x,\tilde{A}) \leq R} \\
&\quad \leq C_{\mathsf{d}} \sum_{i=1}^{\mathsf{d}+1} \sum_{y \in V+Q_R(0)} \left|\nabla_1^2 u(y)\right|^2 \left(\frac{R^{\mathsf{d}}}{d^{(i)}(y,\tilde{A})^{\mathsf{d}-1}} d^{(i)}(y,\tilde{A})^{\mathsf{d}-1} + \tilde{\alpha}_{\mathsf{d}} R^{\mathsf{d}-1} + R^{\mathsf{d}} \right) \\
&\quad \leq C_{\mathsf{d}} R^{\mathsf{d}} \sum_{i=1}^{\mathsf{d}+1} \sum_{y \in V+Q_R(0)} \left|\nabla_1^2 u(y)\right|^2 \\
&\quad \leq C_{\mathsf{d}} R^{\mathsf{d}} \sum_{y \in V+Q_R(0)} \left|\nabla_1^2 u(y)\right|^2 .
\end{aligned}$$

This completes the proof in the case $\mathsf{d} \geq 5$. The case $\mathsf{d} = 4$ is very similar. The only difference is that in the estimate (6.3.5) we no longer have

$$\sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} \frac{1}{(d^{(i)}(x,\tilde{A})-s)^{\mathsf{d}-3}} \leq C_{\mathsf{d}}$$

but instead

$$\sum_{s=0}^{d^{(i)}(x,\tilde{A})-1} \frac{1}{d^{(i)}(x,\tilde{A})-s} \le C\log(2+d^{(i)}(x,\tilde{A})) \le C\log R\,.$$

This is the additional factor $\log R$ that appears on the right hand side in (6.3.1). □

Later we will also use a probabilistic quenched version of this estimate.

Lemma 6.3.2. *Let $\mathsf{d} \ge 4$. There is an odd integer N_{d} with the following property: Let $\Lambda \Subset \mathbb{Z}^{\mathsf{d}}$, and let $x \in \Lambda$, and $k \in \mathbb{N}$. Then if ε is sufficiently small (depending on d) there is an event $\Omega_{x,k}$ such that $\zeta_\Lambda^\varepsilon(\Omega_{x,k}) \ge 1 - \frac{1}{2^{k^{\mathsf{d}}}}$ and such that whenever $A \in \Omega_{x,k}$ the following estimate holds: if $u\colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ is a function such that $u = 0$ on $\tilde{A}$, then*

$$|u(x)| \le C_{\mathsf{d}} \frac{k^{\mathsf{d}/2}\left(1+\mathbb{1}_{\mathsf{d}=4}((\log k)^{1/2}+|\log\varepsilon|^{3/4})\right)}{\varepsilon^{1/2}} \|\nabla_1^2 u\|_{L^2(Q_{kN_{\mathsf{d}}\lambda_{\mathrm{mic}}}(x))}\,. \tag{6.3.7}$$

Proof. We want to apply Theorem 6.3.1 with $V = \{x\}$ and $R = kN\lambda_{\mathrm{mic}}$. Then $R^{\mathsf{d}}(1+\mathbb{1}_{\mathsf{d}=4}\log R) \le C_{\mathsf{d},N}\frac{k^{\mathsf{d}}(1+\mathbb{1}_{\mathsf{d}=4}(\log k+|\log\varepsilon|^{3/2})}{\varepsilon}$, and so (6.3.7) follows with the choice $N_{\mathsf{d}} = N$ provided that $d_*(x,\tilde{\mathcal{A}}) \le kN\lambda_{\mathrm{mic}}$. Thus, if we define

$$\Omega_{x,k} = \left\{A \subset \Lambda\colon d_*(x,\tilde{A}) \le kN\lambda_{\mathrm{mic}}\right\}$$

it remains to choose N in such a way that we can show that $\zeta_\Lambda^\varepsilon(\Omega_{x,k}) \ge 1 - \frac{1}{2^k}$.

For an odd integer N, let $\Xi_{i,kN\lambda_{\mathrm{mic}}}(x) = (x+\Xi_i) \cap Q_{kN\lambda_{\mathrm{mic}}}(x)$. When $kN \ge N'_{\mathsf{d}}$ for some dimensional constant N'_{d} (and so in particular when $N \ge N'_{\mathsf{d}}$) the fraction of points in $Q_{kN\lambda_{\mathrm{mic}}}(x)$ that are in $\Xi_{i,kN\lambda_{\mathrm{mic}}}(x)$ is bounded below, i.e. we have $|\Xi_{i,kN\lambda_{\mathrm{mic}}}(x)| \ge \frac{(kN)^{\mathsf{d}}\lambda_{\mathrm{mic}}^{\mathsf{d}}}{C_{\mathsf{d}}}$. On the other hand, $d_*(x,\tilde{A}) \le kN\lambda_{\mathrm{mic}}$ holds if and only if all $\Xi_{i,kN\lambda_{\mathrm{mic}}}(x)$ contain some point in $\tilde{A}$. Therefore, using Theorem 6.1.3 c) we see

$$\begin{aligned}
1-\zeta_\Lambda^\varepsilon(\Omega_{x,k}) &= \zeta_\Lambda^\varepsilon(d_*(x,\tilde{\mathcal{A}}) > kN\lambda_{\mathrm{mic}}) \\
&\le \zeta_\Lambda^\varepsilon(d_*(x,\mathcal{A}) > kN\lambda_{\mathrm{mic}}) \\
&= \zeta_\Lambda^\varepsilon\left(\exists i \in \{1,\dots,\mathsf{d}+1\}\colon \mathcal{A} \cap \Xi_{i,kN\lambda_{\mathrm{mic}}}(x) = \varnothing\right) \\
&= \sum_{i=1}^{\mathsf{d}+1} \zeta_\Lambda^\varepsilon\left(\mathcal{A} \cap \Xi_{i,kN\lambda_{\mathrm{mic}}}(x) = \varnothing\right) \\
&\le \sum_{i=1}^{\mathsf{d}+1} (1-p_{\mathsf{d},-})^{|\Xi_{i,kN\lambda_{\mathrm{mic}}}(x)|} \\
&\le \sum_{i=1}^{\mathsf{d}+1} \exp\left(-p_{\mathsf{d},-}\frac{(kN)^{\mathsf{d}}\lambda_{\mathrm{mic}}^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\,.
\end{aligned}$$

For any $\mathsf{d} \ge 4$ we have $p_{\mathsf{d},-}\lambda_{\mathrm{mic}}^{\mathsf{d}} \ge \frac{1}{C_{\mathsf{d}}}$, and so

$$\begin{aligned}
1-\zeta_\Lambda^\varepsilon(\Omega_{x,k}) &\le (\mathsf{d}+1)\exp\left(-\frac{(kN)^{\mathsf{d}}}{C_{\mathsf{d}}}\right) \\
&\le \left((\mathsf{d}+1)\exp\left(-\frac{N^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{k^{\mathsf{d}}}
\end{aligned}$$

and it suffices to choose $N_{\mathsf{d}} \ge N'_{\mathsf{d}}$ in such a way that the right hand side is less than $\frac{1}{2^{k^{\mathsf{d}}}}$ when $N \ge N_{\mathsf{d}}$.

□

6.3.2 An interpolation inequality

Let $Q \subset \Lambda$ be a discrete cube of sidelength R. In the following section we will need to control $\|\nabla_1 u\|_{L^2(Q)}$ by terms involving only u and $\nabla_1^2 u$. Usually one would expect to do this by an interpolation inequality of the form

$$\|\nabla_1 u\|_{L^2(Q)}^2 \leq C_{\mathsf{d}} \left(R^2 \|\nabla_1^2 u\|_{L^2(Q)}^2 + \frac{1}{R^2} \|u\|_{L^2(Q)}^2 \right)$$

where the factors $R^{\pm 2}$ are due to scaling. For our purposes this is not good enough, however, as we do not control u on all of Q. So it is crucial for us that a similar inequality still holds when we only control u on a large enough subset of Q. Indeed we have the following result.

Lemma 6.3.3. *Let $\mathsf{d} \in \mathbb{N}$. Let R be an odd integer and let $Q \subset \Lambda$ be a discrete cube of sidelength R (i.e. $Q = Q_{R/2}(x_*)$ for some $x_* \in \mathbb{Z}^{\mathsf{d}}$), and assume $R \geq 12(\sqrt{\mathsf{d}})^{\mathsf{d}-1}\sqrt{\mathsf{d}}$. Let $B \subset Q$ such that $|B| \geq \frac{1}{2}|Q|$. Let $u \colon \Lambda \to \mathbb{R}$. Then we have the estimate*

$$\|\nabla_1 u\|_{L^2(Q)}^2 \leq C_{\mathsf{d}} \left(R^2 \|\nabla_1^2 u\|_{L^2(Q)}^2 + \frac{1}{R^2} \|\mathbb{1}_{\cdot \in B} u\|_{L^2(Q)}^2 \right) . \tag{6.3.8}$$

Proof. By translation invariance we can assume that Q is centred at 0.

We first prove (6.3.8) with u replaced by an affine function v, where $v(x) = b \cdot x + a$ for some $a \in \mathbb{R}$, $b \in \mathbb{R}^{\mathsf{d}}$. That is, we want to show

$$|b|^2 \leq C \frac{1}{R^{\mathsf{d}+2}} \|\mathbb{1}_{\cdot \in B} v\|_{L^2(Q)}^2 . \tag{6.3.9}$$

To see this, note first that we can assume $b \neq 0$ (else there is nothing to show). Let $\theta = \frac{b}{|b|} \in \mathbb{S}^{\mathsf{d}-1}$. For $\lambda > 0$ consider the set $E = \left\{ x \in Q \colon \left| \theta \cdot x + \frac{a}{|b|} \right| \leq \lambda \right\}$. The set E is the intersection of Q with a slab of width 2λ, and so for each point $x \in E$ the cube $x + \left[-\frac{1}{2}, \frac{1}{2}\right]^{\mathsf{d}}$ is contained in $\left[-\frac{R}{2}, \frac{R}{2}\right]^{\mathsf{d}}$ intersected with a slab of width $2\lambda + \sqrt{\mathsf{d}}$. Thus, we can estimate the number of points in E as

$$|E| \leq (2\lambda + \sqrt{\mathsf{d}})(R\sqrt{\mathsf{d}})^{\mathsf{d}-1} .$$

If we assume $\lambda \geq \sqrt{\mathsf{d}}$, we can bound this by

$$|E| \leq 3\lambda (R\sqrt{\mathsf{d}})^{\mathsf{d}-1} .$$

We want to pick $\lambda = \frac{R}{12(\sqrt{\mathsf{d}})^{\mathsf{d}-1}}$. This is possible, as $\frac{R}{12(\sqrt{\mathsf{d}})^{\mathsf{d}-1}} \geq \sqrt{\mathsf{d}}$ by our assumption on R. Then for our choice of λ we see that $|E| \leq \frac{1}{4} R^{\mathsf{d}} = \frac{1}{4}|Q|$.

On the other hand, we know $|B| \geq \frac{1}{2}|Q|$, and therefore $|B \setminus E| \geq \frac{1}{4}|Q|$. Now for each $x \in B \setminus E \subset Q \setminus E$ we have $\left| \theta \cdot x + \frac{a}{|b|} \right| > \lambda$ and hence

$$|v(x)| = |b| \left| \theta \cdot x + \frac{a}{|b|} \right| \geq \lambda |b| \geq \frac{R|b|}{C_{\mathsf{d}}}$$

Summing the square of this estimate over all $x \in B \setminus E$, we see that

$$\sum_{x \in B \setminus E} |v(x)|^2 \geq \sum_{x \in B \setminus E} \frac{R^2 |b|^2}{C_{\mathsf{d}}} \geq \frac{1}{4} |Q| \frac{R^2 |b|^2}{C_{\mathsf{d}}}$$

which immediately implies (6.3.9).

Let now $(u)_Q := \frac{1}{|Q|}\sum_{x\in Q} u(x)$ and $(\nabla_1 u)_Q := \frac{1}{|Q|}\sum_{x\in Q}\nabla_1 u(x)$, and define $v(x) = (u)_Q + (\nabla_1 u)_Q \cdot x$. Then v is an affine function to which we will be able to apply (6.3.9), while $u - v$ and $\nabla_1(u-v)$ have average zero over Q, which allows using the discrete Poincaré inequality with zero mean. We can thus write

$$\begin{aligned}
\|\nabla_1 u\|^2_{L^2(Q)} &\le 2\|\nabla_1 v\|^2_{L^2(Q)} + 2\|\nabla_1(u-v)\|^2_{L^2(Q)} \\
&\le \frac{2}{R^2}\|\mathbb{1}_{\cdot\in B} v\|^2_{L^2(Q)} + C_{\mathsf{d}} R^2 \|\nabla_1^2(u-v)\|^2_{L^2(Q)} \\
&= \frac{2}{R^2}\|\mathbb{1}_{\cdot\in B} v\|^2_{L^2(Q)} + C_{\mathsf{d}} R^2 \|\nabla_1^2 u\|^2_{L^2(Q)} \\
&\le \frac{C_{\mathsf{d}}}{R^2}\|\mathbb{1}_{\cdot\in B}(u-v)\|^2_{L^2(Q)} + \frac{C_{\mathsf{d}}}{R^2}\|\mathbb{1}_{\cdot\in B} u\|^2_{L^2(Q)} + C_{\mathsf{d}} R^2 \|\nabla_1^2 u\|^2_{L^2(Q)} \\
&\le \frac{C_{\mathsf{d}}}{R^2}\|u-v\|^2_{L^2(Q)} + \frac{C_{\mathsf{d}}}{R^2}\|\mathbb{1}_{\cdot\in B} u\|^2_{L^2(Q)} + C_{\mathsf{d}} R^2 \|\nabla_1^2 u\|^2_{L^2(Q)} \\
&\le \frac{C_{\mathsf{d}}}{R^2} R^4 \|\nabla_1^2 u\|^2_{L^2(Q)} + \frac{C_{\mathsf{d}}}{R^2}\|\mathbb{1}_{\cdot\in B} u\|^2_{L^2(Q)} + C_{\mathsf{d}} R^2 \|\nabla_1^2 u\|^2_{L^2(Q)} \\
&\le \frac{C_{\mathsf{d}}}{R^2}\|\mathbb{1}_{\cdot\in B} u\|^2_{L^2(Q)} + C_{\mathsf{d}} R^2 \|\nabla_1^2 u\|^2_{L^2(Q)} .
\end{aligned}$$

This is what we wanted to show. □

6.4 Probabilistic decay of the L^2-norm for biharmonic functions

In this section we will prove a decay estimate for the L^2-norm of the Hessian of a discrete biharmonic function. This estimate does not hold for all realizations of $\mathcal{A}$, but we prove that it holds for all but an exceptional set of realizations whose probability decays exponentially.

The precise result is the following. Recall that $\tilde{\mathcal{A}} = \mathcal{A} \cup (\mathbb{Z}^{\mathsf{d}} \setminus \Lambda)$.

Theorem 6.4.1. *Let $\mathsf{d} \ge 4$. There is an odd integer $\hat{N}_{\mathsf{d}}$ with the following property: Let $\Lambda \Subset \mathbb{Z}^{\mathsf{d}}$, Let $U \in \mathcal{P}_{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}$ be a polymer consisting of $n = \frac{|U|}{\hat{N}_{\mathsf{d}}^{\mathsf{d}}\lambda_{\mathrm{mac}}^{\mathsf{d}}}$ boxes, and $k \in \mathbb{N}$. Then if ε is sufficiently small (depending on d only) there is an event $\Omega_{U,k}$ such that $\zeta_\Lambda^\varepsilon(\Omega_{U,k}) \ge 1 - \frac{n}{2^k}$, and such that whenever $A \in \Omega_{U,k}$ the following estimates hold:*

a) If $u\colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ is a function such that $u = 0$ on $\tilde{A} \setminus U$ and $u\Delta_1^2 u = 0$ on $\mathbb{Z}^{\mathsf{d}} \setminus U$, we have the estimate

$$\left\|\nabla_1^2 u\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} \le \frac{1}{2^k}\left\|\nabla_1^2 u\right\|^2_{L^2((U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0))\setminus U)} . \tag{6.4.1}$$

b) If $u\colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ is a function such that $u = 0$ on $(U + Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)) \cap \tilde{A}$ and $u\Delta_1^2 u = 0$ on $U + Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)$, we have the estimate

$$\left\|\nabla_1^2 u\right\|^2_{L^2(U)} \le \frac{1}{2^k}\left\|\nabla_1^2 u\right\|^2_{L^2((U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0))\setminus U)} . \tag{6.4.2}$$

We have already outlined the strategy of the proof in Section 6.1.3. Namely, to prove (6.4.1) and (6.4.2) we want to iterate the Widman hole filler argument k times, that is we need to find k pairs U_j, U_j' on which we can apply it.

In order to make the Widman hole filler argument work, we need to be able to apply Theorem 6.3.1 and Lemma 6.3.3. We can ensure this by finding a cut-off function η_j that grows from 0 to 1 in such a way that $\nabla_1^2 \eta_j = 0$ on those points on which Theorem 6.3.1 or Lemma 6.3.3 cannot be applied.

For that purpose pick an odd integer N to be fixed later and decompose $\mathbb{Z}^d$ into the boxes in $\mathcal{Q}_{N\lambda_{\mathrm{mac}}}$. We will declare some of these boxes to be bad in such a way that on the good (i.e. non-bad) boxes we can construct an η growing from 0 to 1 on that box and satisfying the conditions on $\nabla_1^2 \eta$. If we can show that bad boxes are rare, then with high probability we can find at least k annuli consisting only of good boxes inbetween U and $\mathbb{Z}^d \setminus (U + Q_{2kN\lambda_{\mathrm{mac}}}(0))$, and then we can iterate the Widman hole filler argument on these annuli.

In Section 6.4.1 we describe in detail how we choose the bad boxes, and we prove that on the good boxes there exist cut-off functions as required. In Section 6.4.2 we carry out the Widman hole filler argument provided all relevant boxes are good. Finally, in Section 6.4.3 we show that the bad boxes are sparse enough that with sufficiently high probability we can find enough annuli to use the hole filler argument on. Using this result we will complete the proof of Theorem 6.4.1.

6.4.1 Bad boxes and cut-off functions

The definition of the bad boxes depends on three odd integers K, L, M, where K is always a multiple of 3, and $M \geq 12$. Eventually (in Section 6.4.3), we will choose them large enough in the order M, K, L to close the argument. For now we will track all dependencies on K, L, M. The parameters K and L will play similar roles as in the proof of Theorem 6.1.3 c), albeit not quite the same. We hope this will not confuse the reader.

There will be two reasons that lead to a cube $Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$ being bad. The first one is related to Lemma 6.3.3. We want to be able to apply that Lemma on each subcube $Q' \in \mathcal{Q}_{K\lambda_{\mathrm{mac}}}$ such that $Q' \subset Q$ with $\mathbb{1}_{\cdot \in B}$ being the indicator function from Theorem 6.3.1. So we will define Q to be bad of type II if there is a subcube $Q' \subset Q$ on which the indicator function from Theorem 6.3.1 is equal to 0 too often.

The second reason is more complicated. We want to be able to modify an initial guess η_* for the cut-off function in such a way that $\nabla_1^2 \eta = 0$ on those sets on which the indicator function from Theorem 6.3.1 is equal to 0 (and we think of those points as bad as well). This is easy if the bad points are very isolated and sparse, as we then can make local adjustments to η_* that do not interfere with each other. So we start at scale $\ell_0 = K\lambda_{\mathrm{mic}}$ and consider the bad points (or actually the bad boxes in $\mathcal{Q}_{K\lambda_{\mathrm{mic}}}$ that contain at least a bad point) and split them into the isolated and the clustered ones. The former ones we ignore for the moment, and the latter ones can be covered by cubes in $\mathcal{Q}^{\#}_{\ell_1}$ such that each cube covers at least two of the small cubes. These are the bad cubes on scale ℓ_1. Now we can once again split those cubes into the isolated ones (that we ignore for the moment) and those that are clustered and can be covered by cubes on scale ℓ_2, and we continue like this. This process terminates once at some scale all bad cubes are isolated. Then we can adjust η_* on those isolated cubes, and then proceed backwards and apply our adjustment also on the isolated cubes on the smaller scales. We thus call Q bad of type I if this process terminates too late.

We have not mentioned yet how to choose the scales ℓ_j. There is a trade-off here: on the one hand, the lengthscale should grow fast so that we have enough space around each isolated bad cube on scale ℓ_j to adjust η_* there. On the other hand we want many of the cubes to be isolated, so that our process terminates soon, and this we can achieve by letting

ℓ_j not grow too fast. It turns out that a good compromise is

$$\ell_j := M^{j^3} K\lambda_{\text{mic}}$$

for each $j \geq 0$. We give a more detailed explanation of this choice later, in Remark 6.4.9.

Let us now give the precise definitions. Recall that $\mathcal{Q}_l = \left\{ Q_{l/2}(x) \colon x \in (l\mathbb{Z})^{\mathsf{d}} \right\}$ and $\mathcal{Q}_l^{\#} = \left\{ Q_{l/2}(x) \colon x \in \left(\frac{l}{3}\mathbb{Z}\right)^{\mathsf{d}} \right\}$. We consider the microscopic cubes in $\mathcal{Q}_{K\lambda_{\text{mic}}}$, the macroscopic cubes in $\mathcal{Q}_{KL\lambda_{\text{mac}}}$, and the further sets of boxes $\mathcal{Q}_{\ell_0}$ and $\mathcal{Q}_{\ell_j}^{\#}$ for $j \geq 1$, associated to $\ell_j = M^{j^3} K\lambda_{\text{mic}}$ for $j \geq 0$. The reason that we use $\mathcal{Q}_{\ell_j}^{\#}$ and not $\mathcal{Q}_{\ell_j}$ for $j \geq 1$ is that we want to ensure that for any two cubes on lengthscale ℓ_{j-1} that are sufficiently close there is a cube in $\mathcal{Q}_{\ell_j}^{\#}$ that contains both of them. We assumed $M \geq 12$, and so $\ell_{j-1} \leq \frac{1}{12}\ell_j$ for each $j \geq 1$. We fix for later use an enumeration of the boxes in each $\mathcal{Q}_{\ell_j}^{\#}$.

We define

$$S_{K,M,\text{bad}}^{(0)}(A) = \left\{ Q \in \mathcal{Q}_{K\lambda_{\text{mic}}} \colon \exists x \in Q \text{ with } d_*(x, \tilde{A}) > K\lambda_{\text{mic}} \right\} .$$

Note that $S_{K,M,\text{bad}}^{(0)}(A) \subset \mathcal{Q}_{K\lambda_{\text{mic}}} = \mathcal{Q}_{\ell_0} \subset \mathcal{Q}_{\ell_0}^{\#}$. For M large enough the set $S_{K,M,\text{bad}}^{(0)}(A)$ is finite (as cubes far outside of Λ will not be bad).

For $j \geq 1$ we define $S_{K,M,\text{bad}}^{(j)}(A) \subset \mathcal{Q}_{\ell_j}^{\#}$ inductively as follows: Given $S_{K,M,\text{bad}}^{(j-1)}(A) \subset \mathcal{Q}_{\ell_{j-1}}^{\#}$ such that the set $S_{K,M,\text{bad}}^{(j-1)}(A)$ is finite, we want to split it into two sets of boxes: $S_{K,M,\text{bad,clust}}^{(j-1)}(A)$ will contain those boxes that are clustered in the sense that there is another bad box at distance $\leq \frac{\ell_j}{2}$ that is disjoint from the original box, and $S_{K,M,\text{bad,isol}}^{(j-1)}(A)$ will contain the other boxes. These other boxes are isolated in the sense that all bad boxes that are disjoint from them are far away. Let us make this precise: we define

$$\begin{aligned} &S_{K,M,\text{bad,clust}}^{(j-1)}(A) \\ &\quad = \left\{ Q \in S_{K,M,\text{bad}}^{(j-1)}(A) \colon \exists Q' \in S_{K,M,\text{bad}}^{(j-1)}(A) \text{ with } Q' \cap Q = \varnothing, d_\infty(Q, Q') \leq \frac{\ell_j}{2} \right\} , \\ &S_{K,M,\text{bad,isol}}^{(j-1)}(A) = S_{K,M,\text{bad}}^{(j-1)}(A) \setminus S_{K,M,\text{bad,clust}}^{(j-1)}(A) . \end{aligned}$$

If $S_{K,M,\text{bad,clust}}^{(j-1)}(A) = \varnothing$, we define $S_{K,M,\text{bad}}^{(j)}(A) = \varnothing$. Otherwise, if $Q \in S_{K,M,\text{bad,clust}}^{(j-1)}(A)$ and $Q' \in S_{K,M,\text{bad}}^{(j-1)}(A)$ is a witness for this in the sense that $Q' \cap Q = \varnothing$ and $d_\infty(Q, Q') \leq \frac{\ell_j}{2}$ then also $Q' \in S_{K,M,\text{bad,clust}}^{(j-1)}(A)$, as Q then is a witness for Q'. In particular,

$$\begin{aligned} &S_{K,M,\text{bad,clust}}^{(j-1)}(A) \\ &\quad = \left\{ Q \in S_{K,M,\text{bad}}^{(j-1)}(A) \colon \exists Q' \in S_{K,M,\text{bad,clust}}^{(j-1)}(A) \text{ with } Q' \cap Q = \varnothing, d_\infty(Q, Q') \leq \frac{\ell_j}{2} \right\} . \end{aligned}$$

Furthermore, if $Q, Q' \in S_{K,M,\text{bad,clust}}^{(j-1)}(A)$ with $Q' \cap Q = \varnothing$ and $d_\infty(Q, Q') \leq \frac{\ell_j}{2}$ then Q and Q' are contained in a common box of sidelength at most $\frac{\ell_j}{2} + 2\ell_{j-1} \leq \frac{2\ell_j}{3}$ (as $\ell_{j-1} \leq \frac{1}{12}\ell_j$), and so there is a cube in $\mathcal{Q}_{\ell_j}^{\#}$ containing both of them. This means that we can cover $S_{K,M,\text{bad,clust}}^{(j-1)}(A)$ by finitely many cubes from $\mathcal{Q}_{\ell_j}^{\#}$ in such a way that each cube from the cover contains at least

two small cubes from $S^{(j-1)}_{K,M,\mathrm{bad,clust}}(A)$ that are disjoint. We can consider the set of subsets of $\mathcal{Q}^{\#}_{\ell_j}$ with that property,

$$\mathcal{S}^{(j)}_{K,M,\mathrm{bad}}(A) = \Big\{ S \subset \mathcal{Q}^{\#}_{\ell_j} \colon \bigcup_{Q' \in S^{(j-1)}_{K,M,\mathrm{bad,clust}}(A)} Q' \subset \bigcup_{Q \in S} Q, \\ \forall Q \in S \; \exists Q', Q'' \in S^{(j-1)}_{K,M,\mathrm{bad,clust}}(A) \text{ with } Q' \cap Q'' = \varnothing, Q' \cup Q'' \subset Q \Big\}.$$

The previous discussion implies that if $S^{(j-1)}_{K,M,\mathrm{bad,clust}}(A)$ is non-empty, then also $\mathcal{S}^{(j)}_{K,M,\mathrm{bad}}(A)$ is non-empty. Now we consider the elements of $\mathcal{S}^{(j)}_{K,M,\mathrm{bad}}(A)$ of minimum cardinality, and among those we define $S^{(j)}_{K,M,\mathrm{bad}}(A)$ to be that element that is lexicographically first (with respect to the enumeration of $\mathcal{Q}^{\#}_{\ell_j}$ that we had fixed).

To summarize, $S^{(j)}_{K,M,\mathrm{bad}}(A)$ is a finite subset of $\mathcal{Q}^{\#}_{\ell_j}$ that covers $S^{(j-1)}_{K,M,\mathrm{bad,clust}}(A)$ in such a way that each of its boxes contains two disjoint elements from $S^{(j-1)}_{K,M,\mathrm{bad,clust}}(A)$.

Finally we can define macroscopic bad boxes. Let $j_*(\varepsilon)$ be the largest integer j such that $\ell_j \leq \frac{KL\lambda_{\mathrm{mac}}}{8}$ (we assume that ε is small enough so that $j_*(\varepsilon) \geq 1$). We consider the macroscopic boxes in $\mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$, and call a macroscopic box bad of type I if it contains at least one box in $S^{(j_*(\varepsilon))}_{K,M,\mathrm{bad}}(A)$, bad of type II if one of its $K\lambda_{\mathrm{mac}}$-subboxes contains many boxes in $S^{(0)}_{K,M,\mathrm{bad}}(A)$, and bad if it is bad of type I or type II. More precisely

$$S^{*,I}_{K,L,M,\mathrm{bad}}(A) = \Big\{ Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}} \colon \exists q \in S^{(j_*(\varepsilon))}_{K,M,\mathrm{bad}}(A) \text{ with } q \cap Q \neq \varnothing \Big\},$$

$$S^{*,II}_{K,L,M,\mathrm{bad}}(A) = \Big\{ Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}} \colon \exists Q' \in \mathcal{Q}_{K\lambda_{\mathrm{mac}}} \text{ with } Q' \subset Q, \\ \Big| \Big\{ q \in S^{(0)}_{K,M,\mathrm{bad}}(A) \colon q \subset Q' \Big\} \Big| \geq \frac{1}{4} \left(\frac{\lambda_{\mathrm{mac}}}{\lambda_{\mathrm{mic}}} \right)^{\mathsf{d}} \Big\},$$

$$S^{*}_{K,L,M,\mathrm{bad}}(A) = S^{*,I}_{K,L,M,\mathrm{bad}}(A) \cup S^{*,II}_{K,L,M,\mathrm{bad}}(A).$$

The point of these definitions is that we can apply our construction of a cut-off function on all cubes in $\mathcal{Q}_{KL\lambda_{\mathrm{mac}}} \setminus S^{*,I}_{K,L,M,\mathrm{bad}}(A)$, while we can apply Lemma 6.3.3 on every $K\lambda_{\mathrm{mac}}$-subcube of the cubes in $\mathcal{Q}_{KL\lambda_{\mathrm{mac}}} \setminus S^{*,II}_{K,L,M,\mathrm{bad}}(A)$. Of course, this is only useful if we show that bad cubes are rare. This will be established in Section 6.4.3. For now we show that our definition of good boxes fulfils its purpose in the sense that we can construct a cut-off function growing from 0 to 1 on them in such a way that their second derivatives are 0 on the set of microscopic bad boxes.

Lemma 6.4.2. *Let $\mathsf{d} \geq 1$. Then there is a constant $M_{\mathsf{d}} \geq 12$ with the following property: Let K, L, M be odd integers such that K is a multiple of 3, and $M \geq M_{\mathsf{d}}$. Then for all ε sufficiently small (depending on d, K) the following holds: Let $U \in \mathcal{P}_{KL\lambda_{\mathrm{mac}}}$ be a polymer. Suppose that none of the $KL\lambda_{\mathrm{mac}}$-boxes touching its boundary (in the l^∞-sense) are bad of type I, i.e.*

$$\{ Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}} \colon Q \subset (U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U \} \cap S^{*,I}_{K,L,M,\mathrm{bad}}(A) = \varnothing$$

Then there is a function $\eta \colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ such that

i) $\eta(x) = 0$ *for* $x \in U + Q_{2K\lambda_{\mathrm{mic}}}(0)$,

ii) $\eta(x) = 1$ *for* $x \in \mathbb{Z}^{\mathsf{d}} \setminus (U + Q_{KL\lambda_{\mathrm{mac}} - 2K\lambda_{\mathrm{mic}}}(0))$,

iii) $|\nabla_1^2 \eta(x)| \leq \frac{C_{\mathsf{d}}}{(KL)^2 \lambda_{\mathrm{mac}}^2} \mathbb{1}_{Q_1(x) \cap \bigcup_{Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q = \varnothing}$.

Here the constant C_{d} *depends only on* d.

Morally, property iii) is $|\nabla_1^2 \eta(x)| \leq \frac{C_{\mathsf{d}}}{(KL)^2 \lambda_{\mathrm{mac}}^2} \mathbb{1}_{x \notin \bigcup_{Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q}$. However, the discrete product rule lets translation operators arise, and this is why we require the slightly stronger condition iii) above.

To prove Lemma 6.4.2, we will begin with a function η_* which satisfies i) and ii), but only $|\nabla_1^2 \eta(x)| \leq \frac{C_{\mathsf{d}}}{(KL)^2 \lambda_{\mathrm{mac}}^2}$ instead of iii). Then we will modify η_* iteratively to make it affine on larger and larger subsets of $S^{(0)}_{K,M,\mathrm{bad}}(A)$, so that eventually iii) is satisfied as well. The following lemma gives details on how to carry out a single of these modification steps.

Lemma 6.4.3. *Let* $\mathsf{d} \geq 1$. *There is a constant* $\gamma_{\mathsf{d}} > 0$ *with the following property: Let* $x \in \mathbb{Z}^{\mathsf{d}}$, *let* r, R *be positive integers such that* $R \geq 16r$. *Let* $v \colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ *be a function. Then there is a function* $w \colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ *with the following properties:*

i) $w = 0$ *on* $\mathbb{Z}^{\mathsf{d}} \setminus Q_{R-1}(x)$,

ii) $\nabla_1^2 (v + w) = 0$ *on* $Q_r(x)$,

iii) $\|\nabla_1^2 (v + w)\|_{L^\infty(Q_R(x))} \leq \left(1 + \frac{\gamma_{\mathsf{d}}}{\log R - \log r}\right) \|\nabla_1^2 v\|_{L^\infty(Q_R(x))}$.

Note that condition i) ensures that $\nabla_1^2 w = 0$ on $\mathbb{Z}^{\mathsf{d}} \setminus Q_R(x)$.

Proof. By translation invariance we can assume $x = 0$. Suppose for the moment that there is a function $\xi \colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ such that $\xi = 1$ on $Q_{r+1}(0)$, $\xi = 0$ on $\mathbb{Z}^{\mathsf{d}} \setminus Q_{R-1}(0)$ and

$$
\begin{aligned}
|\xi(y)| &\leq 1\,, \\
|\nabla_1 \xi(y)| &\leq \frac{C_{\mathsf{d}}}{|y|(\log R - \log r)}\,, \\
|\nabla_1^2 \xi(y)| &\leq \frac{C_{\mathsf{d}}}{|y|^2(\log R - \log r)}
\end{aligned}
$$

for $y \in Q_R(0)$.

Let u be the affine function $u(y) = v(0) + y \cdot \nabla_1 v(0)$. Then we can set $w(y) = \xi(y)(u(y) - v(y))$. This choice of w clearly satisfies i) and ii), and so we only have to check iii). We know that $u(0) - v(0) = 0$ and $\nabla_1 u(0) - \nabla_1 v(0) = 0$, and so by discrete Taylor expansion, using that u is affine, we have

$$
\begin{aligned}
|\nabla_1 u(y) - \nabla_1 v(y)| &\leq C_{\mathsf{d}} |y| \|\nabla_1^2 v\|_{L^\infty(Q_R(0))} \\
|u(y) - v(y)| &\leq C_{\mathsf{d}} |y|^2 \|\nabla_1^2 v\|_{L^\infty(Q_R(0))}\,.
\end{aligned}
$$

Note that $v + w = \xi u + (1 - \xi) v$. The discrete product rule allows us to write

$$
\begin{aligned}
D_i^1 D_{-j}^1 (v + w)(y) = {} & D_i^1 D_{-j}^1 v(y) \tau_i^1 \tau_{-j}^1 (1 - \xi)(y) + D_i^1 (u - v)(y) \tau_i^1 D_{-j}^1 \xi(y) \\
& + D_{-j}^1 (u - v)(y) \tau_{-j}^1 D_i^1 \xi(y) + (u - v)(y) D_i^1 D_{-j}^1 \xi(y)
\end{aligned}
$$

and thus, using the estimates on ξ and $v-u$,

$$\begin{aligned}
|D_i^1 D_{-j}^1(v+w)(y)| &\le |D_i^1 D_{-j}^1 v(y)| + |D_i^1(u-v)(y)| \max_{|z-y|_\infty \le 1} |\nabla_1 \xi(z)| \\
&\quad + |D_{-j}^1(u-v)(y)| \max_{|z-y|_\infty \le 1} |\nabla_1 \xi(z)| + |(u-v)(y)| \max_{|z-y|_\infty \le 1} |\nabla_1^2 \xi(z)| \\
&\le |D_i^1 D_{-j}^1 v(y)| + C_{\mathsf{d}} |y| \|\nabla_1^2 v\|_{L^\infty(Q_R(0))} \frac{1}{|y|(\log R - \log r)} \\
&\quad + C_{\mathsf{d}} |y|^2 \|\nabla_1^2 v\|_{L^\infty(Q_R(0))} \frac{1}{|y|^2(\log R - \log r)} \\
&\le |D_i^1 D_{-j}^1 v(y)| + C_{\mathsf{d}} \|\nabla_1^2 v\|_{L^\infty(Q_R(0))} \frac{1}{(\log R - \log r)} .
\end{aligned}$$

This immediately implies that v satisfies iii).

It remains to show the existence of ξ with the desired properties. To do so, we choose a function $\chi \in C^\infty(\mathbb{R}^{\mathsf{d}})$ that is 1 on $[-1,1]^{\mathsf{d}}$, 0 outside of $[-2,2]^{\mathsf{d}}$ such that $0 \le \chi \le 1$, and for $\rho > 0$ define $\chi_\rho = \chi\left(\frac{\cdot}{\rho}\right)$. We define $\tilde{\xi}\colon \mathbb{R}^{\mathsf{d}} \to \mathbb{R}$ by

$$\tilde{\xi}(y) = \chi_{2r}(y) + (\chi_{R/4}(y) - \chi_{2r}(y)) \frac{\log R - \log |y|}{\log R - \log r} .$$

One can check that $0 \le \tilde{\xi} \le 1$, $\tilde{\xi} = 1$ on $[-2r, 2r]^{\mathsf{d}} \supset Q_{r+1}(0)$, $\tilde{\xi} = 0$ on $\mathbb{R}^{\mathsf{d}} \setminus \left[-\frac{R}{2}, \frac{R}{2}\right]^{\mathsf{d}} \supset \mathbb{Z}^{\mathsf{d}} \setminus Q_{R-1}R(0)$ as well as

$$|\nabla^k \xi(y)| \le \frac{C_{\mathsf{d},k}}{|y|^k(\log R - \log r)}$$

for $k \ge 1$. We can now let ξ be the restriction of $\tilde{\xi}$ to $\mathbb{Z}^{\mathsf{d}}$. The estimates on $\tilde{\xi}$ together with Taylor's theorem then imply the corresponding estimates on ξ. □

Before we turn to the proof of Lemma 6.4.2, let us investigate the structure of the $S^{(j)}_{K,M,\mathrm{bad}}(A)$ in more detail. Note first that

$$\left|S^{(j)}_{K,M,\mathrm{bad}}(A)\right| \ge \left|S^{(j)}_{K,M,\mathrm{bad,clust}}(A)\right| \ge \left|S^{(j+1)}_{K,M,\mathrm{bad}}(A)\right| + 1$$

as each cube of $S^{(j+1)}_{K,M,\mathrm{bad}}(A)$ covers at least two cubes of $S^{(j)}_{K,M,\mathrm{bad,clust}}(A)$, and $S^{(j+1)}_{K,M,\mathrm{bad}}(A)$ is chosen with the smallest possible cardinality. We have already noted that $S^{(j)}_{K,M,\mathrm{bad}}(A)$ is a finite set. This implies that $S^{(j)}_{K,M,\mathrm{bad}}(A) = \varnothing$ for j sufficiently large.

If $Q \in S^{(j)}_{K,M,\mathrm{bad}}(A)$ for some $j \ge 0$, then either $Q \in S^{(j)}_{K,M,\mathrm{bad,isol}}(A)$ or $Q \in S^{(j)}_{K,M,\mathrm{bad,clust}}(A)$. In the latter case there is at least one $Q' \in S^{(j+1)}_{K,M,\mathrm{bad}}(A)$ such that $Q \subset Q'$. If there is more than one such Q', we choose the one that comes first with respect to the enumeration of $\mathcal{Q}^{\#}_{\ell_j}$ that we had fixed, and call it the parent of Q. In this manner, given $Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)$ we can find a sequence

$$Q \subset Q^{(1)} \subset \ldots \subset Q^{(j)}$$

such that each cube is the parent of the preceding cube. This sequence is necessarily finite as $S^{(j)}_{K,M,\mathrm{bad}}(A) = \varnothing$ for j sufficiently large. It terminates as soon as we reach a cube $Q^{(j)} \in S^{(j)}_{K,M,\mathrm{bad,isol}}(A)$. We denote that value j by $j_{\mathrm{isol},Q}$. In summary, we have a sequence

$$Q = Q^{(0)} \subset Q^{(1)} \subset \ldots \subset Q^{(j_{\mathrm{isol},Q})} \tag{6.4.3}$$

where each cube is the parent of the preceding cube, the cubes $Q^{(j)}$ for $j < j_{\mathrm{isol},Q}$ are in $S^{(j)}_{K,M,\mathrm{bad,clust}}(A)$, while $Q^{(j_{\mathrm{isol},Q})} \in S^{(j_{\mathrm{isol},Q})}_{K,M,\mathrm{bad,isol}}(A)$. This allows us to find for each cube $Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)$ a lengthscale $\ell_{j_{\mathrm{isol},Q}}$ on which its parents become isolated. It will be that lengthscale on which we will ensure that η is locally affine on Q.

After these preparations we can turn to the proof of the main result of this section, the construction of a cut-off function.

Proof of Lemma 6.4.2.

Step 1: Construction of a function satisfying a weaker version of iii)

We assume that ε is small enough so that $\lambda_{\mathrm{mic}} \geq 4$, say. Then also $\ell_j \geq 4$ for all $j \geq 0$. We first claim that there is a function η_* satisfying i) and ii) and such that $|\nabla_1^2 \eta_*(x)| \leq \frac{C_{\mathrm{d}}}{(KL)^2\lambda_{\mathrm{mac}}^2}$. This should be intuitively clear, as we want to interpolate from 0 to 1 on scale $KL\lambda_{\mathrm{mac}}$. One way to make this rigorous is as follows. Let $\tilde{U} = U + \left[-\frac{1}{2}, \frac{1}{2}\right]^{\mathrm{d}}$. Choose a function $\hat{\chi} \in C^\infty(\mathbb{R}^{\mathrm{d}})$ that is 1 on $[-1,1]^{\mathrm{d}}$, 0 outside of $\left[-\frac{9}{7}, \frac{9}{7}\right]^{\mathrm{d}}$ and such that $0 \leq \chi \leq 1$. For each $Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$ such that $Q \subset U$ let x_Q be its centre, and define $\tilde{\eta}_* \colon \mathbb{R}^{\mathrm{d}} \to \mathbb{R}$ by

$$\tilde{\eta}_*(y) = \prod_{\substack{Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}} \\ Q \subset U}} \left(1 - \hat{\chi}\left(\frac{y - x_Q}{\frac{7}{8}KL\lambda_{\mathrm{mac}}}\right)\right).$$

This function is then equal to 0 on $\tilde{U} + \left[-\frac{3KL\lambda_{\mathrm{mac}}}{8}, \frac{3KL\lambda_{\mathrm{mac}}}{8}\right]^{\mathrm{d}}$ and also equal to 1 on $\mathbb{R}^{\mathrm{d}} \setminus \left(\tilde{U} + \left[-\frac{5KL\lambda_{\mathrm{mac}}}{8}, \frac{5KL\lambda_{\mathrm{mac}}}{8}\right]^{\mathrm{d}}\right)$. Each of the factors in the definition of $\tilde{\eta}_*$ satisfies

$$\left\| \nabla^k \left(1 - \hat{\chi}\left(\frac{\cdot - x_Q}{\frac{7}{8}KL\lambda_{\mathrm{mac}}}\right)\right)\right\|_{L^\infty(\mathbb{R}^{\mathrm{d}})} \leq \frac{C_{\mathrm{d},k}}{(KL)^k\lambda_{\mathrm{mac}}^k}$$

for $k \geq 0$. Furthermore, for each fixed $y \in \mathbb{R}^{\mathrm{d}}$ there is a neighbourhood in which at most 3^{d} of the factors are non-constant. Thus, if we compute $\nabla^k \tilde{\eta}_*(y)$ we get contributions only from these at most 3^{d} factors. Therefore,

$$\|\nabla^k \tilde{\eta}_*\|_{L^\infty(\mathbb{R}^{\mathrm{d}})} \leq \frac{C_{\mathrm{d},k}}{(KL)^k\lambda_{\mathrm{mac}}^k}$$

for $k \geq 0$. We can now let η_* be the restriction of $\tilde{\eta}_*$ to $\mathbb{Z}^{\mathrm{d}}$. Then Taylor's theorem implies easily that $|\nabla_1^2 \eta(x)| \leq \frac{C_{\mathrm{d}}}{(KL)^2\lambda_{\mathrm{mac}}^2}$.

Note also that η_* is equal to 0 on $U + \left[-\frac{3KL\lambda_{\mathrm{mac}}}{8}, \frac{3KL\lambda_{\mathrm{mac}}}{8}\right]$ and equal to 1 on $\mathbb{Z}^{\mathrm{d}} \setminus (U + Q_{5KL\lambda_{\mathrm{mac}}/8}(0))$. Therefore, $\nabla_1^2 \eta_*$ is equal to 0 except possibly on $V := (U + Q_{5KL\lambda_{\mathrm{mac}}/8+1}(0) \setminus (U + Q_{3KL\lambda_{\mathrm{mac}}/8-1}(0))$.

Step 2: Modification of η_*

Let $q \in S^{(0)}_{K,M,\mathrm{bad}}(A)$ with $q \subset (U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U$, and consider the sequence of cubes (6.4.3) with $Q = q$. By our assumption none of the macroscopic cubes in $(U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U$ are bad of type I, i.e. there is no $Q' \in S^{(j_*(\varepsilon))}_{K,M,\mathrm{bad}}(A)$ intersecting $(U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U$. This means that the sequence (6.4.3) necessarily terminates before $j = j_*(\varepsilon)$. In particular, we have $j_{\mathrm{isol},q} < j_*(\varepsilon)$. We can now partition the cubes in $S^{(0)}_{K,M,\mathrm{bad}}(A)$ according to the value of $j_{\mathrm{isol},q}$, and define for $j \in \{0, \ldots, j_*(\varepsilon) - 1\}$ the set

$$T_{K,M,j}(A) = \left\{q \in S^{(0)}_{K,M,\mathrm{bad}}(A) \colon q \subset (U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U, j_{\mathrm{isol},q} = j\right\}.$$

We will now construct a sequence of functions $\eta^{(j)}$ by reverse induction in such a way that $\nabla_1^2\eta^{(j)} = 0$ on $\bigcup_{k\geq j}\bigcup_{q\in T_{K,M,k}(A)}(q+Q_1(0))$. Eventually, we will show that the choice $\eta = \eta^{(0)}$ satisfies the properties claimed in the Lemma.

Thus, we start with $\eta^{(j_*(\varepsilon))} = \eta_*$. Let also $V^{(j_*(\varepsilon))} = V$, and note that $\operatorname{supp}\nabla_1^2\eta^{(j_*(\varepsilon))}$ is a subset of $V^{(j_*(\varepsilon))} = V$. Suppose now that for some $j \in \{1,\dots,j_*(\varepsilon)\}$ we have defined $\eta^{(j)}$ and $V^{(j)}$ with $\operatorname{supp}\nabla_1^2\eta^{(j)} \subset V^{(j)}$ and $\eta^{(j)} = \eta_*$ on $\mathbb{Z}^{\mathsf{d}} \setminus V^{(j)}$ such that $\nabla_1^2\eta^{(j)} = 0$ on $\bigcup_{k\geq j}\bigcup_{q\in T_{K,M,k}(A)}(q+Q_1(0))$, and let us define $\eta^{(j-1)}$ and $V^{(j-1)}$.

Since $\operatorname{supp}\nabla_1^2\eta^{(j)} \subset V^{(j)}$, we trivially have $\nabla_1^2\eta^{(j)} = 0$ on those cubes that do not intersect $V^{(j)}$, and so there is no need to change $\eta^{(j)}$ there. Let

$$Y^{(j-1)}_{K,M,V}(A) = \left\{Q \in S^{(j-1)}_{K,M,\mathrm{bad,isol}}(A)\colon (Q+Q_1(0)) \cap V^{(j)} \neq \varnothing\right\}$$

be the set of cubes where we will adjust $\eta^{(j)}$. By definition, this is a set of cubes on scale ℓ_{j-1} that either overlap or are far apart. That is, if $Q \in Y^{(j-1)}_{K,M,V}(A)$, then all $Q' \in S^{(j-1)}_{K,M,\mathrm{bad}}(A)$ either satisfy $Q \cap Q' \neq \varnothing$ or $d_\infty(Q,Q') > \frac{\ell_j}{2}$. Let $\tilde{Y}^{(j-1)}_{K,M,V}(A)$ be a subset of $Y^{(j-1)}_{K,M,V}(A)$ of maximum cardinality such that the cubes in $\tilde{Y}^{(j-1)}_{K,M,V}(A)$ are pairwise disjoint. By construction for each $Q \in Y^{(j-1)}_{K,M,V}(A)$ there is a $Q' \in \tilde{Y}^{(j-1)}_{K,M,V}(A)$ (possibly equal to Q) with $Q \cap Q' \neq \varnothing$. In particular,

$$\bigcup_{Q\in Y^{(j-1)}_{K,M,V}(A)} Q \subset \bigcup_{Q\in \tilde{Y}^{(j-1)}_{K,M,V}(A)} (Q+Q_{\ell_{j-1}}(0))\,.$$

Let now $Q \in \tilde{Y}^{(j-1)}_{K,M,V}(A)$. We know that there is no cube $Q' \in S^{(j-1)}_{K,M,\mathrm{bad}}(A)$ that intersects $(Q+Q_{\ell_j/2}(0)) \setminus (Q+Q_{\ell_{j-1}}(0))$. This has several implications. An obvious one is that the cubes $Q+Q_{\ell_j/4}(0)$ for $Q \in \tilde{Y}^{(j-1)}_{K,M,V}(A)$ are pairwise disjoint. Slightly less obviously, we claim that for $Q \in \tilde{Y}^{(j-1)}_{K,M,V}(A)$ there is no $q \in \bigcup_{k\geq j} T_{K,M,k}(A)$ such that $q \cap ((Q+Q_{\ell_j/4+1}(0)) \setminus (Q+Q_{\ell_{j-1}}(0))) \neq \varnothing$. Indeed, if $q \in \bigcup_{k\geq j} T_{K,M,k}(A)$ then we have $j_{\mathrm{isol},q} \geq j$, so the sequence (6.4.3) contains a parent $q \subset q^{(j-1)} \in S^{(j-1)}_{K,M,\mathrm{bad}}(A)$. Then $q^{(j-1)}$ cannot intersect $(Q+Q_{\ell_j/2}(0)) \setminus (Q+Q_{\ell_{j-1}}(0))$, and so the same holds true for q.

We would now like to apply Lemma 6.4.3 to the function $v = \eta^{(j)}$ and the cubes $Q + Q_{\ell_{j-1}+1}(0) \subset Q + Q_{\ell_j/4}(0)$. For that purpose we fix $M_{\mathsf{d}} = 8\max(16, \exp(\gamma_{\mathsf{d}}))$, where γ_{d} is the constant from Lemma 6.4.3. If $M \geq M_{\mathsf{d}}$, then

$$\frac{\left\lfloor \frac{\ell_{j-1}}{2} + \frac{\ell_j}{4} \right\rfloor}{\left\lceil \frac{\ell_{j-1}}{2} + \ell_{j-1} + 1 \right\rceil} \geq \frac{\frac{\ell_j}{4}}{2\ell_{j-1}} = \frac{1}{8}\frac{M^{j^3}K\lambda_{\mathrm{mic}}}{M^{(j-1)^3}K\lambda_{\mathrm{mic}}} = \frac{1}{8}M^{3j^2-3j+1}\,. \tag{6.4.4}$$

The term on the right hand side is in particular bounded below by $\frac{M}{8} \geq 16$, and thus we can indeed apply the lemma. We obtain that there is a function w_Q such that $w_Q = 0$ on

$\mathbb{Z}^{\mathsf{d}} \setminus (Q + Q_{\ell_j/4-1}(0))$, $\nabla_1^2 \left(w_Q + \eta^{(j)}\right)$ is zero on $Q + Q_{\ell_{j-1}+1}(0)$, and such that

$$\begin{aligned}
&\left\|\nabla_1^2 \left(w_Q + \eta^{(j)}\right)\right\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})} \\
&\leq \left(1 + \frac{\gamma_{\mathsf{d}}}{\log\left(\frac{\ell_{j-1}}{2} + \frac{\ell_j}{4}\right) - \log\left(\frac{\ell_{j-1}}{2} + \ell_{j-1} + 1\right)}\right) \left\|\nabla_1^2 \eta^{(j)}\right\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})} \\
&\leq \left(1 + \frac{\gamma_{\mathsf{d}}}{\log\left(\frac{1}{8} M^{3j^2-3j+1}\right)}\right) \left\|\nabla_1^2 \eta^{(j)}\right\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})} \\
&\leq \left(1 + \frac{1}{3j^2 - 3j + 1}\right) \left\|\nabla_1^2 \eta^{(j)}\right\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})}
\end{aligned} \tag{6.4.5}$$

where we have used (6.4.4) and the fact that $\frac{\gamma_{\mathsf{d}}}{\log\left(\frac{M}{8}\right)} \leq \frac{\gamma_{\mathsf{d}}}{\log \exp(\gamma_{\mathsf{d}})} = 1$. We set

$$\eta^{(j-1)} = \eta^{(j)} + \sum_{Q \in \tilde{Y}^{(j)}_{K,M,V}(A)} w_Q$$

and

$$V^{(j-1)} = V^{(j)} + Q_{\ell_j}(0)$$

and finally we set $\eta = \eta^{(0)}$.

Step 3: Proof that the $\eta^{(j)}$ are locally affine on the bad cubes with $j_{isol,Q} \geq j$
We prove by reverse induction that $\operatorname{supp} \eta^{(j)} \subset V^{(j)}$, $\eta^{(j)} = \eta_*$ on $\mathbb{Z}^{\mathsf{d}} \setminus V^{(j)}$ and $\nabla_1^2 \eta^{(j)} = 0$ on $\bigcup_{k \geq j} \bigcup_{q \in T_{K,M,k}(A)} (q + Q_1(0))$. This is obvious for $j = j_*(\varepsilon)$, so assume that it holds for some $j \in \{1, \ldots, j_*(\varepsilon)\}$.

We claim that then also $\nabla_1^2 \eta^{(j-1)} = 0$ on $\bigcup_{k \geq j-1} \bigcup_{q \in T_{K,M,k}(A)} (q + Q_1(0))$. To see this, let $q \in T_{K,M,k}(A)$ for $k = j_{\mathrm{isol},q} \geq j - 1$. We distinguish the two cases $j_{\mathrm{isol},q} \geq j$ and $j_{\mathrm{isol},q} = j - 1$.

In the former case by our inductive assumption already $\nabla_1^2 \eta^{(j)} = 0$ on $q + Q_1(0)$. Furthermore, we have argued that q does not intersect $(Q + Q_{\ell_j/4+1}(0)) \setminus (Q + Q_{\ell_{j-1}}(0))$ for any $Q \in \tilde{Y}^{(j)}_{K,M,V}(A)$. So either q does not intersect $Q + Q_{\ell_j/4+1}(0)$ for any $Q \in \tilde{Y}^{(j)}_{K,M,V}(A)$, or q is contained in $Q + Q_{\ell_{j-1}}(0)$ for exactly one $Q \in \tilde{Y}^{(j)}_{K,M,V}(A)$. In the former case, all $\nabla_1^2 w_Q$ are equal to 0 on $q + Q_1(0)$, and thus $\nabla_1^2 \eta^{(j-1)} = \nabla_1^2 \eta^{(j)} = 0$ on $q + Q_1(0)$, while in the latter case it holds that $\nabla_1^2 \eta^{(j-1)} = \nabla_1^2 (\eta^{(j)} + w_Q)$ on $q + Q_1(0)$, and thus by construction of w_Q we have $\nabla_1^2 \eta^{(j-1)} = 0$ on $q + Q_1(0)$.

It remains to consider the case that $j_{\mathrm{isol},q} = j - 1$. In that case the sequence (6.4.3) contains a parent $q \subset q^{(j-1)} \in S^{(j-1)}_{K,M,\mathrm{bad}}(A)$. If $(q^{(j-1)} + Q_1(0)) \cap V^{(j)} = \varnothing$, then $\nabla_1^2 \eta^{(j)} = 0$ on $q^{(j-1)} + Q_1(0)$. Furthermore, by the definition of $Y^{(j-1)}_{K,M,V}(A)$, $q^{(j-1)}$ does not intersect $(Q + Q_{\ell_j/4+1}(0)) \setminus (Q + Q_{\ell_{j-1}}(0))$ for any $Q \in \tilde{Y}^{(j-1)}_{K,M,V}(A) \subset Y^{(j-1)}_{K,M,V}(A)$, and so neither does q. Arguing as in the previous case, we find that $\nabla_1^2 \eta^{(j-1)} = \nabla_1^2 \eta^{(j)}$ on $q + Q_1(0)$. On the other hand, it could be that $(q^{(j-1)} + Q_1(0)) \cap V^{(j)} \neq \varnothing$. Then $q^{(j-1)} \in Y^{(j-1)}_{K,M,V}(A)$, and so there is some $Q \in \tilde{Y}^{(j-1)}_{K,M,V}(A)$ with $q \subset q^{(j-1)} \subset Q + Q_{\ell_{j-1}}(0)$. Then it holds once again that $\nabla_1^2 \eta^{(j-1)} = \nabla_1^2 (\eta^{(j)} + w_Q) = 0$ on $q + Q_1(0)$.

This proves that $\nabla_1^2 \eta^{(j-1)} = 0$ on $\bigcup_{k \geq j-1} \bigcup_{q \in T_{K,M,k}(A)} (q + Q_1(0))$. Furthermore, each $Q \in \tilde{Y}^{(j-1)}_{K,M,V}(A)$ is contained in $V^{(j)} + Q_{2(\ell_{j-1}+2)}(0)$, so the support of the associated w_Q is contained in $V^{(j)} + Q_{2(\ell_{j-1}+2)+\ell_j/4-1}(0)$. So, $\nabla_1^2 \eta^{(j-1)}$ is supported in $V^{(j)} + Q_{2(\ell_{j-1}+2)+\ell_j/4}(0) \subset V^{(j)} + Q_{\ell_j}(0) = V^{(j-1)}$, and $\eta^{(j)} = \eta_*$ on $\mathbb{Z}^{\mathsf{d}} \setminus V^{(j-1)}$. This completes the induction.

Step 4: Proof that η satisfies i), ii) and iii)
We define $\eta = \eta^{(0)}$. In the previous step we have shown that $\nabla_1^2\eta$ is supported in $V^{(0)}$ and $\eta = \eta_*$ on $\mathbb{Z}^{\mathsf{d}} \setminus V^{(0)}$. We have

$$\begin{aligned} V^{(0)} &= V + Q_{\ell_0}(0) + \ldots + Q_{\ell_{j_*(\varepsilon)}}(0) \\ &\subset V + Q_{2\ell_{j_*(\varepsilon)}}(0) \subset V + Q_{KL\lambda_{\mathrm{mac}}/4}(0) \subset (U + Q_{7KL\lambda_{\mathrm{mac}}/8+1}(0)) \setminus (U + Q_{KL\lambda_{\mathrm{mac}}/8-1}(0)) \,. \end{aligned}$$

Thus, $\eta = \eta_* = 0$ on $U + Q_{KL\lambda_{\mathrm{mac}}/8-1}(0)$, $\eta = \eta_* = 1$ on $\mathbb{Z}^{\mathsf{d}} \setminus (U + Q_{7KL\lambda_{\mathrm{mac}}/8+1}(0))$. This means that η satisfies i) and ii) as soon as ε is small enough (depending on d and K).

In Step 3 we have also seen that $\eta = \eta^{(0)}$ satisfies $\nabla_1^2\eta = 0$ on $\bigcup_{k\geq 0}\bigcup_{q\in T_{K,M,k}(A)}(q + Q_1(0)) = \bigcup_{q\in S^{(0)}_{K,M,\mathrm{bad}}(A)}(q + Q_1(0))$. Thus, to show that η also satisfies iii) we only have to check that $\|\nabla_1^2\eta\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})} \leq \frac{C_{\mathsf{d}}}{(KL)^2\lambda_{\mathrm{mac}}^2}$.

To see this, note the supports of the functions $\nabla_1^2 w_Q$ for $Q \in \tilde{Y}^{(j-1)}_{K,M,V}(A)$ are disjoint. Thus, (6.4.5) implies the bound

$$\left\|\nabla_1^2\eta^{(j-1)}\right\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})} \leq \left(1 + \frac{1}{3j^2 - 3j + 1}\right)\left\|\nabla_1^2\eta^{(j)}\right\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})}$$

for $j \geq 1$. Iterating this, we find that

$$\begin{aligned} \|\nabla_1^2\eta\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})} &\leq \left(\prod_{j=1}^{j_*(\varepsilon)}\left(1 + \frac{1}{3j^2 - 3j + 1}\right)\right)\|\nabla_1^2\eta_*\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})} \\ &\leq \left(\prod_{j=1}^{\infty}\left(1 + \frac{1}{3j^2 - 3j + 1}\right)\right)\frac{C_{\mathsf{d}}}{(KL)^2\lambda_{\mathrm{mac}}^2} \\ &\leq \frac{C_{\mathsf{d}}}{(KL)^2\lambda_{\mathrm{mac}}^2} \end{aligned} \tag{6.4.6}$$

where in the last step we used that $\frac{1}{3j^2-3j+1}$ is summable and therefore $\prod_{j=1}^{\infty}\left(1 + \frac{1}{3j^2-3j+1}\right) < \infty$. This completes the proof. □

6.4.2 Decay estimates on good domains

Now that we have a construction of a cut-off function at our disposal, we can execute the Widman hole filler argument.

Lemma 6.4.4. *Let $\mathsf{d} \geq 1$ and let M_{d} be the constant from Lemma 6.4.2. Let K, L, M be odd integers such that K is a multiple of 3 and $M \geq M_{\mathsf{d}}$. Then for all ε sufficiently small (depending on d, K) the following holds. Let $U \in \mathcal{P}_{KL\lambda_{\mathrm{mac}}}$ be a polymer. Suppose that none of the $KL\lambda_{\mathrm{mac}}$-boxes touching U (in the l_∞-sense) are bad of type I or II, i.e.*

$$\{Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}} \colon Q \subset (U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U\} \cap S^*_{K,L,M,\mathrm{bad}}(A) = \varnothing \,.$$

Then the following holds.

a) If $u \colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ is a function such that $u = 0$ on $\tilde{A} \setminus U$ and $u\Delta_1^2 u = 0$ on $\mathbb{Z}^{\mathsf{d}} \setminus U$, we have the estimate

$$\begin{aligned} &\left\|\nabla_1^2 u\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{KL\lambda_{\mathrm{mac}}}(0)))} \\ &\quad\leq \left(\frac{C_{\mathsf{d}}K^{\mathsf{d}-4}\left(1 + \mathbb{1}_{\mathsf{d}=4}\log K\right)}{L^2} + \frac{1}{4}\right)\left\|\nabla_1^2 u\right\|^2_{L^2((U+Q_{KL\lambda_{\mathrm{mac}}}(0))\setminus U))} \,. \end{aligned} \tag{6.4.7}$$

b) If $u\colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ is a function such that $u = 0$ on $(U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \cap \tilde{A}$ and $u\Delta_1^2 u = 0$ on $U + Q_{KL\lambda_{\mathrm{mac}}}(0)$, we have the estimate

$$\left\|\nabla_1^2 u\right\|_{L^2(U)}^2 \leq \left(\frac{C_{\mathsf{d}} K^{\mathsf{d}-4}\left(1 + \mathbb{1}_{\mathsf{d}=4}\log K\right)}{L^2} + \frac{1}{4}\right)\left\|\nabla_1^2 u\right\|_{L^2((U+Q_{KL\lambda_{\mathrm{mac}}}(0))\setminus U))}^2 . \tag{6.4.8}$$

Proof. The proof proceeds as outlined in Section 6.1.3. We begin with the proof of part a); the proof of part b) will be very similar.

Step 1: Discrete integration by parts

We know that none of the cubes in $(U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U$ is bad of type I. Thus, we can apply Lemma 6.4.2. Let η be the cut-off function that we obtain from that Lemma.

We now carry out the discrete analogue of the calculation that lead to (6.1.24). Namely we see that

$$\begin{aligned}
0 &= (\Delta_1^2 u, \eta u)_{L^2(\mathbb{Z}^{\mathsf{d}})} \\
&= \sum_{i,j=1}^{\mathsf{d}} \left(D_i^1 D_{-j}^1 u, D_i^1 D_{-j}^1 (\eta u)\right)_{L^2(\mathbb{Z}^{\mathsf{d}})} \\
&= \sum_{i,j=1}^{\mathsf{d}} \left(D_i^1 D_{-j}^1 u, \left(u D_i^1 D_{-j}^1 \eta + D_i^1 u \tau_i^1 D_{-j}^1 \eta + D_{-j}^1 u \tau_{-j}^1 D_i^1 \eta + D_i^1 D_{-j}^1 u \tau_i^1 \tau_{-j}^1 \eta\right)\right)_{L^2(\mathbb{Z}^{\mathsf{d}})} \\
&= \sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} \left|D_i^1 D_{-j}^1 u(x)\right|^2 \tau_i^1 \tau_{-j}^1 \eta(x) + \sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} u(x) D_i^1 D_{-j}^1 u(x) D_i^1 D_{-j}^1 \eta(x) \\
&\quad + \sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} D_i^1 D_{-j}^1 u(x) D_i^1 u(x) \tau_i^1 D_{-j}^1 \eta(x) + \sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} D_i^1 D_{-j}^1 u(x) D_{-j}^1 u(x) \tau_{-j}^1 D_i^1 \eta(x) .
\end{aligned} \tag{6.4.9}$$

Consider the third term in this sum. We can apply summation by parts here and obtain

$$\begin{aligned}
&\sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} D_i^1 D_{-j}^1 u(x) D_i^1 u(x) \tau_i^1 D_{-j}^1 \eta(x) \\
&\quad = -\sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} D_i^1 u(x) D_j^1 \left(D_i^1 u(x) \tau_i^1 D_{-j}^1 \eta(x)\right) \\
&\quad = -\sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} D_i^1 u(x) D_i^1 u(x) \tau_i^1 D_j^1 D_{-j}^1 \eta(x) - \sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} D_i^1 u(x) D_i^1 D_j^1 u(x) \tau_i^1 \tau_j^1 D_{-j}^1 \eta(x) \\
&\quad = -\sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} |D_i^1 u(x)|^2 \tau_i^1 D_j^1 D_{-j}^1 \eta(x) - \sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} D_i^1 u(x) D_i^1 D_{-j}^1 u(x) \tau_i^1 D_{-j}^1 \eta(x)
\end{aligned}$$

where in the last step we changed the index of summation from x to $\tau_{-j}^1 x$. This implies

$$\sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} D_i^1 D_{-j}^1 u(x) D_i^1 u(x) \tau_i^1 D_{-j}^1 \eta(x) = -\frac{1}{2} \sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} |D_i^1 u(x)|^2 \tau_i^1 D_j^1 D_{-j}^1 \eta(x) .$$

Similarly, we find for the fourth summand in (6.4.9) that

$$\sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} D_i^1 D_{-j}^1 u(x) D_{-j}^1 u(x) \tau_{-j}^1 D_i^1 \eta(x) = -\frac{1}{2} \sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}} |D_{-j}^1 u(x)|^2 \tau_{-j}^1 D_i^1 D_{-i}^1 \eta(x) .$$

If we use the last two equalities in (6.4.9), we arrive at

$$\begin{aligned}
&\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}\left|D_i^1D_{-j}^1u(x)\right|^2\tau_i^1\tau_{-j}^1\eta(x)\\
&\quad=\frac{1}{2}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}|D_i^1u(x)|^2\tau_i^1D_j^1D_{-j}^1\eta(x)+\frac{1}{2}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}|D_{-j}^1u(x)|^2\tau_{-j}^1D_i^1D_{-i}^1\eta(x)\\
&\qquad-\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}u(x)D_i^1D_{-j}^1u(x)D_i^1D_{-j}^1\eta(x)\,.
\end{aligned}$$

Here, in the second summand on the right-hand side, we can shift the summation from x to $x+e_j$ and then interchange the indices i and j to see that

$$\begin{aligned}
&\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}\left|D_i^1D_{-j}^1u(x)\right|^2\tau_i^1\tau_{-j}^1\eta(x)\\
&\quad=\frac{1}{2}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}|D_i^1u(x)|^2\tau_i^1D_j^1D_{-j}^1\eta(x)+\frac{1}{2}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}|D_i^1u(x)|^2D_j^1D_{-j}^1\eta(x)\\
&\qquad-\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}u(x)D_i^1D_{-j}^1u(x)D_i^1D_{-j}^1\eta(x)\\
&\quad=\frac{1}{2}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}|D_i^1u(x)|^2\left(\tau_i^1D_j^1D_{-j}^1\eta(x)+D_j^1D_{-j}^1\eta(x)\right)\\
&\qquad-\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}u(x)D_i^1D_{-j}^1u(x)D_i^1D_{-j}^1\eta(x)\,.
\end{aligned}\tag{6.4.10}$$

This is the discrete analogue of (6.1.23). To continue, we can use the Cauchy-Schwarz inequality on the second term on the right hand side of (6.4.10) and obtain

$$\begin{aligned}
&\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}\left|D_i^1D_{-j}^1u(x)\right|^2\tau_i^1\tau_{-j}^1\eta(x)\\
&\quad\le\frac{1}{2}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}|D_i^1u(x)|^2\left(\tau_i^1D_j^1D_{-j}^1\eta(x)+D_j^1D_{-j}^1\eta(x)\right)+\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}|u(x)|^2\left|D_i^1D_{-j}^1\eta(x)\right|^2\\
&\qquad+\frac{1}{4}\sum_{i,j=1}^{d}\sum_{x\in\mathbb{Z}^d}\left|D_i^1D_{-j}^1u(x)\right|^2\mathbb{1}_{\nabla_1^2\eta(x)\neq0}
\end{aligned}\tag{6.4.11}$$

which is the discrete analogue of (6.1.24). Next, we use the properties of η. First, recall that $\eta=1$ on $\mathbb{Z}^d\setminus(U+Q_{KL\lambda_{\mathrm{mac}}-2K\lambda_{\mathrm{mic}}}(0))$, and $\eta=0$ on $U+Q_{2K\lambda_{\mathrm{mic}}}(0)$. Therefore certainly $\tau_i^1\tau_{-j}^1\eta=1$ on $\mathbb{Z}^d\setminus(U+Q_{KL\lambda_{\mathrm{mac}}}(0))$, and $\nabla_1^2\eta=0$ on $\mathbb{Z}^d\setminus V$, where $V:=(U+Q_{KL\lambda_{\mathrm{mac}}-K\lambda_{\mathrm{mic}}}(0))\setminus(U+Q_{K\lambda_{\mathrm{mic}}}(0))$, which we can use to bound the left-hand side and the third term on the right-hand side of (6.4.11). We also know that

$$|\nabla_1^2\eta(x)|\le\frac{C_d}{(KL)^2\lambda_{\mathrm{mac}}^2}\mathbb{1}_{Q_1(x)\cap\bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)}Q=\varnothing}$$

which implies that

$$|\tau_{\pm k}^1D_i^1D_{-j}^1\eta(x)|\le\frac{C_d}{(KL)^2\lambda_{\mathrm{mac}}^2}\mathbb{1}_{x\notin\bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)}Q}$$

for any $i, j, k \in \{1, \dots, \mathsf{d}\}$. We can use this for the first and second term on the right-hand side of (6.4.11). Putting everything together, we see that

$$\begin{aligned}
&\left\|\nabla_1^2 u\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{KL\lambda_{\mathrm{mac}}}(0)))} \\
&= \sum_{i,j=1}^{\mathsf{d}} \sum_{x\in\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{KL\lambda_{\mathrm{mac}}}(0))} \left|D_i^1 D_{-j}^1 u(x)\right|^2 \\
&\leq \frac{C_{\mathsf{d}}}{(KL)^2\lambda_{\mathrm{mac}}^2} \sum_{i=1}^{\mathsf{d}} \sum_{x\in V} |D_i^1 u(x)|^2 \mathbb{1}_{x\notin\bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q} \\
&\quad + \frac{C_{\mathsf{d}}}{(KL)^4\lambda_{\mathrm{mac}}^4} \sum_{x\in V} |u(x)|^2 \mathbb{1}_{x\notin\bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q} + \frac{1}{4}\sum_{i,j=1}^{\mathsf{d}} \sum_{x\in V} \left|D_i^1 D_{-j}^1 u(x)\right|^2 \\
&= \frac{C_{\mathsf{d}}}{(KL)^2\lambda_{\mathrm{mac}}^2} \left\|\nabla_1 u \mathbb{1}_{\cdot\notin\bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q}\right\|^2_{L^2(V)} + \frac{C_{\mathsf{d}}}{(KL)^4\lambda_{\mathrm{mac}}^4} \left\|u \mathbb{1}_{\cdot\notin\bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q}\right\|^2_{L^2(V)} \\
&\quad + \frac{1}{4}\left\|\nabla_1^2 u\right\|^2_{L^2(V)} .
\end{aligned} \tag{6.4.12}$$

Step 2: Use of Poincaré and interpolation inequalities

We continue by estimating the first term on the right hand side of (6.4.12). To do so, we want to apply Lemma 6.3.3 on each of the $K\lambda_{\mathrm{mac}}$-boxes in $(U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U$. Note that $(U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U \in \mathcal{P}_{KL\lambda_{\mathrm{mac}}}$, i.e. it is the disjoint union of some cubes in $\mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$. Let Q' be one such cube. It is the disjoint union of L^{d} cubes in $\mathcal{Q}_{K\lambda_{\mathrm{mac}}}$. Let q be one of them, and let $B_q := q \cap V \setminus \bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q$. We claim that $|B_q| \geq \frac{1}{2}|q| = \frac{1}{2}K^{\mathsf{d}}\lambda_{\mathrm{mac}}^{\mathsf{d}}$. To see this, note that

$$|q \setminus V| \leq 2\mathsf{d}K^{\mathsf{d}-1}\lambda_{\mathrm{mac}}^{\mathsf{d}-1}K\lambda_{\mathrm{mic}} = 2\mathsf{d}K^{\mathsf{d}}\lambda_{\mathrm{mac}}^{\mathsf{d}-1}\lambda_{\mathrm{mic}} .$$

Furthermore, by assumption Q' is not bad of type II. Therefore, each of its $K\lambda_{\mathrm{mac}}$-subcubes contains at most $\frac{1}{4}\left(\frac{\lambda_{\mathrm{mac}}}{\lambda_{\mathrm{mic}}}\right)^{\mathsf{d}}$ cubes in $S^{(0)}_{K,M,\mathrm{bad}}(A)$, i.e.

$$\left|q \cap \bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q\right| \leq \frac{1}{4}\left(\frac{\lambda_{\mathrm{mac}}}{\lambda_{\mathrm{mic}}}\right)^{\mathsf{d}} K^{\mathsf{d}}\lambda_{\mathrm{mic}}^{\mathsf{d}} = \frac{1}{4}K^{\mathsf{d}}\lambda_{\mathrm{mac}}^{\mathsf{d}} = \frac{1}{4}|q| .$$

Therefore,

$$|B_q| \geq |q| - |q \setminus V| - \left|q \cap \bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q\right| \geq |q| - \frac{1}{4}|q| - 2\mathsf{d}K^{\mathsf{d}}\lambda_{\mathrm{mac}}^{\mathsf{d}-1}\lambda_{\mathrm{mic}} \geq \frac{1}{2}|q|$$

whenever ε is small enough (in terms of d, K) so that $2K^{\mathsf{d}}\lambda_{\mathrm{mic}} \leq \frac{1}{4}\lambda_{\mathrm{mac}}$. Thus, we can apply Lemma 6.3.3 on q with $B = B_q$ and obtain

$$\begin{aligned}
&\left\|\nabla_1 u \mathbb{1}_{\cdot\notin\bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q}\right\|^2_{L^2(V\cap q)} \\
&= \left\|\nabla_1 u \mathbb{1}_{\cdot\in V\setminus\bigcup_{Q\in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q}\right\|^2_{L^2(q)}
\end{aligned}$$

$$\leq C_{\mathsf{d}} K^2 \lambda_{\mathrm{mac}}^2 \left\| \nabla_1^2 u \right\|_{L^2(V\cap q)}^2 + \frac{C_{\mathsf{d}}}{K^2 \lambda_{\mathrm{mac}}^2} \left\| u \mathbb{1}_{\cdot \in V \setminus \bigcup_{Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q} \right\|_{L^2(q)}^2 ,$$

and summing this over all Q' and all $q \subset Q'$ we see that

$$\begin{aligned} &\left\| \nabla_1 u \mathbb{1}_{\cdot \notin \bigcup_{Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q} \right\|_{L^2(V)}^2 \\ &\quad \leq C_{\mathsf{d}} K^2 \lambda_{\mathrm{mac}}^2 \left\| \nabla_1^2 u \right\|_{L^2(V)}^2 + \frac{C_{\mathsf{d}}}{K^2 \lambda_{\mathrm{mac}}^2} \left\| u \mathbb{1}_{\cdot \notin \bigcup_{Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q} \right\|_{L^2(V)}^2 . \end{aligned}$$

Using this estimate in (6.4.12) we arrive at

$$\begin{aligned} &\left\| \nabla_1^2 u \right\|_{L^2(\mathbb{Z}^{\mathsf{d}} \setminus (U + Q_{KL\lambda_{\mathrm{mac}}}(0)))}^2 \\ &\quad \leq \frac{C_{\mathsf{d}}}{L^2} \left\| \nabla_1^2 u \right\|_{L^2(V)}^2 + \frac{C_{\mathsf{d}}}{K^4 L^2 \lambda_{\mathrm{mac}}^4} \left\| u \mathbb{1}_{\cdot \notin \bigcup_{Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q} \right\|_{L^2(V)}^2 \\ &\qquad + \frac{C_{\mathsf{d}}}{(KL)^4 \lambda_{\mathrm{mac}}^4} \left\| u \mathbb{1}_{\cdot \notin \bigcup_{Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q} \right\|_{L^2(V)}^2 + \frac{1}{4} \left\| \nabla_1^2 u \right\|_{L^2(V)}^2 \\ &\quad \leq \left(\frac{C_{\mathsf{d}}}{L^2} + \frac{1}{4} \right) \left\| \nabla_1^2 u \right\|_{L^2(V)}^2 + \frac{C_{\mathsf{d}}}{K^4 L^2 \lambda_{\mathrm{mac}}^4} \left\| u \mathbb{1}_{\cdot \notin \bigcup_{Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q} \right\|_{L^2(V)}^2 \end{aligned} \tag{6.4.13}$$

where we used $L \geq 1$ in the last step.

Theorem 6.3.1 with $R = K\lambda_{\mathrm{mic}}$ allows to bound

$$\begin{aligned} \left\| u \mathbb{1}_{\cdot \notin \bigcup_{Q \in S^{(0)}_{K,M,\mathrm{bad}}(A)} Q} \right\|_{L^2(V)}^2 &\leq C_{\mathsf{d}} K^{\mathsf{d}} \lambda_{\mathrm{mic}}^{\mathsf{d}} \left(1 + \mathbb{1}_{\mathsf{d}=4} \log(K\lambda_{\mathrm{mic}})\right) \left\| \nabla_1^2 u \right\|_{L^2(V + Q_{K\lambda_{\mathrm{mic}}}(0))}^2 \\ &\leq C_{\mathsf{d}} K^{\mathsf{d}} \lambda_{\mathrm{mic}}^{\mathsf{d}} \left(1 + \mathbb{1}_{\mathsf{d}=4} \log(K\lambda_{\mathrm{mic}})\right) \left\| \nabla_1^2 u \right\|_{L^2((U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U)}^2 . \end{aligned} \tag{6.4.14}$$

Now one easily checks that for any $\mathsf{d} \geq 4$ we have $\lambda_{\mathrm{mic}}^{\mathsf{d}} \left(1 + \mathbb{1}_{\mathsf{d}=4} \log \lambda_{\mathrm{mic}}\right) \leq C_{\mathsf{d}} \lambda_{\mathrm{mac}}^4$. Thus, combining (6.4.13) and (6.4.14) we see that

$$\begin{aligned} &\left\| \nabla_1^2 u \right\|_{L^2(\mathbb{Z}^{\mathsf{d}} \setminus (U + Q_{KL\lambda_{\mathrm{mac}}}(0)))}^2 \\ &\quad \leq \left(\frac{C_{\mathsf{d}}}{L^2} + \frac{1}{4} \right) \left\| \nabla_1^2 u \right\|_{L^2((U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U)}^2 \\ &\qquad + \frac{C_{\mathsf{d}} K^{\mathsf{d}-4} \left(1 + \mathbb{1}_{\mathsf{d}=4} \log K\right)}{L^2} \left\| \nabla_1^2 u \right\|_{L^2((U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U)}^2 \\ &\quad \leq \left(\frac{C_{\mathsf{d}} K^{\mathsf{d}-4} \left(1 + \mathbb{1}_{\mathsf{d}=4} \log K\right)}{L^2} + \frac{1}{4} \right) \left\| \nabla_1^2 u \right\|_{L^2((U + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U)}^2 \end{aligned}$$

which is (6.4.8).

Step 3: Proof of part b)

We proceed completely analogously as in Step 1 and 2. The only difference is that we work with $\hat{\eta} := 1 - \eta$ instead of η. The assumptions for $\nabla_1^2 \eta$ carry over to $\nabla_1^2 \hat{\eta}$, and so the proof carries over.

□

Lemma 6.4.4 has the following straightforward corollary:

Lemma 6.4.5. *Let* $\mathsf{d} \geq 1$ *and let* M_{d} *be the constant from Lemma 6.4.2. Let* K *be an odd multiple of 3 and* $M \geq M_{\mathsf{d}}$*. Then there is a constant* $L_{\mathsf{d},K}$ *depending on* d *and* K *only such that for any odd integers* $L \geq L_{\mathsf{d},K}$*,* $M \geq M_{\mathsf{d}}$ *and for all* ε *sufficiently small (depending on* d*,* K*) the following holds: Let* $U_0, \dots, U_k \in \mathcal{P}_{KL\lambda_{\mathrm{mac}}}$ *be polymers. Suppose that for each* $j \in \{0, \dots, k-1\}$ *we have* $U_j + Q_{KL\lambda_{\mathrm{mac}}}(0) \subset U_{j+1}$ *and*

$$\{Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}} \colon Q \subset (U_j + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U_j\} \cap S^*_{K,L,M,\mathsf{bad}}(A) = \varnothing\,.$$

Then the following holds:

a) If $u \colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ *is a function such that* $u = 0$ *on* $\tilde{\mathcal{A}} \setminus U_0$ *and* $u\Delta_1^2 u = 0$ *on* $\mathbb{Z}^{\mathsf{d}} \setminus U_0$*, we have the estimate*

$$\left\|\nabla_1^2 u\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}} \setminus U_k)} \leq \frac{1}{2^k} \left\|\nabla_1^2 u\right\|^2_{L^2((U_0 + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U_0)}\,. \tag{6.4.15}$$

b) If $u \colon \mathbb{Z}^{\mathsf{d}} \to \mathbb{R}$ *is a function such that* $u = 0$ *on* $U_k \cap \tilde{\mathcal{A}}$ *and* $u\Delta_1^2 u = 0$ *on* U_k*, we have the estimate*

$$\left\|\nabla_1^2 u\right\|^2_{L^2(U_0)} \leq \frac{1}{2^k} \left\|\nabla_1^2 u\right\|^2_{L^2((U_{k-1} + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U_{k-1})}\,. \tag{6.4.16}$$

Proof. We choose L large enough so that the prefactors on the right-hand side in (6.4.7) and (6.4.8) become less than $\frac{1}{2}$, and then apply Lemma 6.4.4 iteratively on each U_j. □

6.4.3 Sparsity of bad boxes

In order to conclude Theorem 6.4.1 from Lemma 6.4.5 it remains to show that with sufficiently high probability we can find sets U_j as in that Lemma. For that purpose we need to show that bad cubes are sparse enough.

In fact, we will show that we can make the probability of a cube in $Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$ being bad arbitrarily small. We even have a slightly stronger result, namely that for each finite $T^* \subset \mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$ we can control the probability that all cubes in T^* are bad.

Lemma 6.4.6. *Let* $\mathsf{d} \geq 4$*, let* $p > 0$ *be arbitrary. Let* $M \geq 12$ *be an odd integer. Then there is* $K_{\mathsf{d},M,p}$ *depending on* d*,* M *and* p *only with the following property: let* $K \geq K_{\mathsf{d},M,p}$ *be an odd multiple of 3, let* L *be an odd integer, let* ε *be small enough (depending on* d*,* L*,* M *and* p*), and let* T^* *be an arbitrary finite subset of* $\mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$*. Then*

$$\zeta^\varepsilon_\Lambda\left(T^* \subset S^{*,I}_{K,L,M,\mathsf{bad}}(\mathcal{A})\right) \leq p^{|T^*|}\,, \tag{6.4.17}$$

$$\zeta^\varepsilon_\Lambda\left(T^* \subset S^{*,II}_{K,L,M,\mathsf{bad}}(\mathcal{A})\right) \leq p^{|T^*|}\,, \tag{6.4.18}$$

$$\zeta^\varepsilon_\Lambda\left(T^* \subset S^{*}_{K,L,M,\mathsf{bad}}(\mathcal{A})\right) \leq 2(4p)^{\frac{|T^*|}{2}}\,. \tag{6.4.19}$$

In order to prove this Lemma, we will have go to into the definition of the bad cubes of type I and II, and, in particular, we will have to understand how rare cubes in $S^{(j)}_{K,M,\mathsf{bad}}(\mathcal{A})$ are. This is quantified in the following Lemma.

Lemma 6.4.7. *Let* $\mathsf{d} \geq 4$*. There is a constant* K'_{d} *with the following property: Let* ε *be small enough (depending on* d *only) so that the conclusion of Theorem 6.1.3 holds. Let* $M \geq 12$ *be an odd integer and assume that* $K \geq K'_{\mathsf{d}}$ *is an odd multiple of 3. Let* $j \geq 0$*. Then we have the following estimates.*

a) If $j = 0$ and $T^{(0)} \subset \mathcal{Q}_{\ell_0}$ is a finite subset, then

$$\zeta_\Lambda^\varepsilon \left(T^{(0)} \subset S^{(0)}_{K,M,\mathrm{bad}}(\mathcal{A}) \right) \leq \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}} (\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}} \right) \right)^{|T^{(0)}|}. \tag{6.4.20}$$

b) If $j > 0$ and $T^{(j)} \subset \mathcal{Q}^{\#}_{\ell_j}$ is a finite subset such that the elements of $T^{(j)}$ are pairwise disjoint, then

$$\zeta_\Lambda^\varepsilon \left(T^{(j)} \subset S^{(j)}_{K,M,\mathrm{bad}}(\mathcal{A}) \right) \leq \left(3^{(2^j-2)\mathsf{d}} M^{s_j \mathsf{d}} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}} (\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}} \right) \right)^{2^j} \right)^{|T^{(j)}|} \tag{6.4.21}$$

where $s_j := j^3 + \sum_{m=0}^{j} m^3 2^{j-m}$.

Here part a) is rather easy to show. If $Q \in S^{(0)}_{K,M,\mathrm{bad}}(\mathcal{A})$, then there are too few pinned points around Q, and the probability for that can be estimated using Theorem 6.1.5. The crucial point is that by choosing K large this probability can be made arbitrarily small.

Part b) then follows by induction. Each cube in $S^{(j)}_{K,M,\mathrm{bad}}(\mathcal{A})$ contains at least two cubes in $S^{(j-1)}_{K,M,\mathrm{bad}}(\mathcal{A})$, and so if the latter cubes are rare, an union bound will show that the former cubes will be rare as well.

Proof. We show first part a), then part b) for $j = 1$, and then use that result to start an induction that will yield part b) for $j > 1$ as well.

Step 1: Proof of part a)

This is similar to the proof of Lemma 6.3.2. However, we want a uniform estimate over the cubes in $T^{(0)}$, and so we need to be more careful.

Let $Q \in T^{(0)}$, and let $q \in \mathcal{Q}_{\lambda_{\mathrm{mic}}}$ be such that $q \subset Q$. Suppose that q has centre $x \in \mathbb{Z}^{\mathsf{d}}$. For $i \in \{1, \ldots, \mathsf{d}+1\}$ consider the sets $\Xi_i(q) = \bigcap_{y \in q} (y + \Xi_i)$ and $\Xi_{i,K\lambda_{\mathrm{mic}}/2}(q) = \Xi_i(q) \cap Q_{K\lambda_{\mathrm{mic}}/2}(x)$.

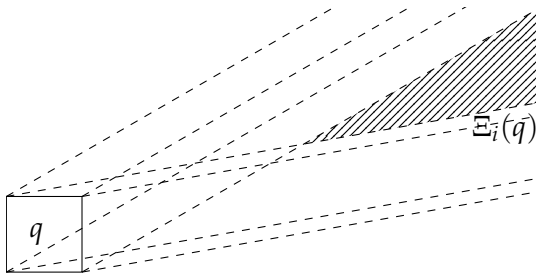


Figure 6.4: A set $\Xi_i(q)$, given as the intersection of $x + \Xi_i$ for $x \in q$.

The set $\Xi_i(q)$ is an intersection of translates of the same cone, where the tips of the cone range over the set q of diameter $\leq \sqrt{\mathsf{d}}\lambda_{\mathrm{mic}}$ (cf. Figure 6.4). As soon as $K \geq K'_{\mathsf{d}}$ for some dimensional constant K'_{d} the fraction of points in $Q_{K\lambda_{\mathrm{mic}}/2}(x)$ that are in $\Xi_{i,K\lambda_{\mathrm{mic}}/2}(q)$ is bounded below. We fix such a K'_{d}. In other words, we have the estimate

$$|\Xi_{i,K\lambda_{\mathrm{mic}}/2}(q)| \geq \frac{1}{C_{\mathsf{d}}} (K\lambda_{\mathrm{mic}})^{\mathsf{d}} \tag{6.4.22}$$

for $K \geq K'_{\mathsf{d}}$. Furthermore, if $\Xi_{i,K\lambda_{\mathrm{mic}}/2}(q) \cap \tilde{\mathcal{A}} \neq \varnothing$ then $d^{(i)}(y, \tilde{\mathcal{A}}) \leq \frac{K}{2}\lambda_{\mathrm{mic}} + \frac{1}{2}\lambda_{\mathrm{mic}} \leq K\lambda_{\mathrm{mic}}$ for all $y \in q$.

The preceding discussion implies that if $\Xi_{i,K\lambda_{\text{mic}}/2}(q) \cap \tilde{\mathcal{A}} \neq \varnothing$ for all i then $d_*(y, \mathcal{A}) \leq K\lambda_{\text{mic}}$ for all $y \in q$. Thus, if $\Xi_{i,K\lambda_{\text{mic}}/2}(q) \cap \tilde{\mathcal{A}} \neq \varnothing$ holds for all i and all $q \subset Q$ then $Q \notin S^{(0)}_{K,M,\text{bad}}(\mathcal{A})$. Using this we can write

$$\begin{aligned}
&\zeta^{\varepsilon}_{\Lambda}\left(T^{(0)} \subset S^{(0)}_{K,M,\text{bad}}(\mathcal{A})\right) \\
&= \zeta^{\varepsilon}_{\Lambda}\left(\forall Q \in T^{(0)}\ \exists q_Q \in \mathcal{Q}_{\lambda_{\text{mic}}}\ \exists i_Q \in \{1,\dots,\mathsf{d}+1\} \text{ with } q_Q \subset Q, \Xi_{i_Q,K\lambda_{\text{mic}}/2}(q_Q) \cap \tilde{\mathcal{A}} = \varnothing\right) \\
&= \zeta^{\varepsilon}_{\Lambda}\Bigg(\forall Q \in T^{(0)}\ \exists q_Q \in \mathcal{Q}_{\lambda_{\text{mic}}}\ \exists i_Q \in \{1,\dots,\mathsf{d}+1\} \text{ with } q_Q \subset Q \\
&\qquad\qquad \text{such that } \bigcup_{Q \in T^{(0)}} \Xi_{i_Q,K\lambda_{\text{mic}}/2}(q_Q) \cap \tilde{\mathcal{A}} = \varnothing\Bigg).
\end{aligned}$$

We can estimate this probability by summing over all choices $\underline{q} = (q_Q)_{Q \in T^{(0)}} \in (\mathcal{Q}_{\lambda_{\text{mic}}})^{T^{(0)}}$ and $\underline{i} = (i_Q)_{Q \in T^{(0)}} \in \{1,\dots,\mathsf{d}+1\}^{T^{(0)}}$ to find

$$\zeta^{\varepsilon}_{\Lambda}\left(T^{(0)} \subset S^{(0)}_{K,M,\text{bad}}(\mathcal{A})\right) \leq \sum_{\substack{\underline{q} \in (\mathcal{Q}_{\lambda_{\text{mic}}})^{T^{(0)}} \\ q_Q \subset Q\ \forall Q \in T^{(0)}}} \sum_{\underline{i} \in \{1,\dots,\mathsf{d}+1\}^{T^{(0)}}} \zeta^{\varepsilon}_{\Lambda}\left(\bigcup_{Q \in T^{(0)}} \Xi_{i_Q,K\lambda_{\text{mic}}/2}(q_Q) \cap \tilde{\mathcal{A}} = \varnothing\right).$$

Assume for the moment that the elements of $T^{(0)}$ are well-separated in the sense that for any $Q, Q' \in T^{(0)}$ with $Q \neq Q'$ we have $d_\infty(Q, Q') \geq K\lambda_{\text{mic}}$. Because $\Xi_{i_Q,K\lambda_{\text{mic}}/2}(q_Q)$ is a subset of $q_Q + Q_{K\lambda_{\text{mic}}/2}(0)$, it is a subset of the cube with sidelength $2K\lambda_{\text{mic}}$ concentric to Q. In particular, by our temporary assumption, if $Q \neq Q'$, then $\Xi_{i_Q,K\lambda_{\text{mic}}/2}(q_Q)$ and $\Xi_{i_{Q'},K\lambda_{\text{mic}}/2}(q_{Q'})$ are disjoint. Thus, (6.4.22) implies

$$\left|\bigcup_{Q \in T^{(0)}} \Xi_{i_Q,K\lambda_{\text{mic}}/2}(q_Q)\right| \geq \frac{|T^{(0)}|}{C_{\mathsf{d}}}(K\lambda_{\text{mic}})^{\mathsf{d}}.$$

Using Theorem 6.1.3 we now see

$$\begin{aligned}
\zeta^{\varepsilon}_{\Lambda}\left(T^{(0)} \subset S^{(0)}_{K,M,\text{bad}}(\mathcal{A})\right) &\leq \sum_{\substack{\underline{q} \in (\mathcal{Q}_{\lambda_{\text{mic}}})^{T^{(0)}} \\ q_Q \subset Q\ \forall Q \in T^{(0)}}} \sum_{\underline{i} \in \{1,\dots,\mathsf{d}+1\}^{T^{(0)}}} (1 - p_{\mathsf{d},-})^{\left|\bigcup_{Q \in T^{(0)}} \Xi_{i_Q,K\lambda_{\text{mic}}/2}(q_Q)\right|} \\
&\leq \sum_{\substack{\underline{q} \in (\mathcal{Q}_{\lambda_{\text{mic}}})^{T^{(0)}} \\ q_Q \subset Q\ \forall Q \in T^{(0)}}} \sum_{\underline{i} \in \{1,\dots,\mathsf{d}+1\}^{T^{(0)}}} \exp\left(-p_{\mathsf{d},-}\frac{|T^{(0)}|(K\lambda_{\text{mic}})^{\mathsf{d}}}{C_{\mathsf{d}}}\right) \\
&\leq \sum_{\substack{\underline{q} \in (\mathcal{Q}_{\lambda_{\text{mic}}})^{T^{(0)}} \\ q_Q \subset Q\ \forall Q \in T^{(0)}}} \sum_{\underline{i} \in \{1,\dots,\mathsf{d}+1\}^{T^{(0)}}} \exp\left(-p_{\mathsf{d},-}\frac{|T^{(0)}|(K\lambda_{\text{mic}})^{\mathsf{d}}}{C_{\mathsf{d}}}\right).
\end{aligned}$$

In any dimension $\mathsf{d} \geq 4$ we have $p_{\mathsf{d},-}\lambda^{\mathsf{d}}_{\text{mic}} \geq \frac{1}{C_{\mathsf{d}}}$. We also have $(K^{\mathsf{d}})^{|T^{(0)}|}$ choices for $\underline{q}$, and $(\mathsf{d}+1)^{|T^{(0)}|}$ choices for $\underline{i}$, and so

$$\zeta^{\varepsilon}_{\Lambda}\left(T^{(0)} \subset S^{(0)}_{K,M,\text{bad}}(\mathcal{A})\right) \leq (K^{\mathsf{d}})^{|T^{(0)}|}(\mathsf{d}+1)^{|T^{(0)}|}\exp\left(-\frac{|T^{(0)}|K^{\mathsf{d}}}{C_{\mathsf{d}}}\right). \tag{6.4.23}$$

This estimate was derived under the assumption that $T^{(0)}$ is such that for any $Q, Q' \in T^{(0)}$ with $Q \neq Q'$ we have $d_\infty(Q, Q') \geq 2K\lambda_{\text{mic}}$. In general, this will not be the case. However, we can partition $T^{(0)}$ into 2^{d} subsets $T_i^{(0)}$ for $i \in \{1, \dots, 2^{\mathsf{d}}\}$ such that for any i and any $Q, Q' \in T_i^{(0)}$ with $Q \neq Q'$ we have $d_\infty(Q, Q') \geq K\lambda_{\text{mic}}$. At least one of these subsets, say $T_{i_*}^{(0)}$, will contain at least $\frac{|T^{(0)}|}{2^{\mathsf{d}}}$ boxes. Then we can apply the estimate (6.4.23) to $T_{i_*}^{(0)}$ and obtain

$$\begin{aligned}
\zeta_\Lambda^\varepsilon\left(T^{(0)} \subset S_{K,M,\text{bad}}^{(0)}(\mathcal{A})\right) &\leq \zeta_\Lambda^\varepsilon\left(T_{i_*}^{(0)} \subset S_{K,M,\text{bad}}^{(0)}(\mathcal{A})\right) \\
&\leq (K^{\mathsf{d}})^{\frac{|T^{(0)}|}{2^{\mathsf{d}}}} (\mathsf{d}+1)^{\frac{|T^{(0)}|}{2^{\mathsf{d}}}} \exp\left(-\frac{|T^{(0)}|K^{\mathsf{d}}}{2^{\mathsf{d}} C_{\mathsf{d}}}\right) \\
&\leq \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}} (\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{|T^{(0)}|}
\end{aligned}$$

which is (6.4.20).

Step 2: Proof of part b) for $j = 1$

We want to prove (6.4.21) by induction on j. In principle, we would want to use (6.4.20) as the base case. However, that statement is for $\mathcal{Q}_{\ell_0}$ instead of $\mathcal{Q}_{\ell_0}^{\#}$, and so we first derive (6.4.21) for $j = 1$ from (6.4.20) and then use this assertion to start our induction.

Let $Q \in T^{(1)}$. By construction $Q \in S_{K,M,\text{bad}}^{(1)}(\mathcal{A})$ if and only if there are at least two disjoint cubes $q, q' \in S_{K,M,\text{bad,clust}}^{(0)}(\mathcal{A}) \subset S_{K,M,\text{bad}}^{(0)}(\mathcal{A})$ such that $q, q' \subset Q$, and so

$$\begin{aligned}
&\zeta_\Lambda^\varepsilon\left(T^{(1)} \subset S_{K,M,\text{bad}}^{(1)}(\mathcal{A})\right) \\
&= \zeta_\Lambda^\varepsilon\left(\forall Q \in T^{(1)}\ \exists q_Q, q'_Q \in \mathcal{Q}_{\ell_0} \text{ with } q_Q \cap q'_Q = \varnothing, q_Q \cup q'_Q \subset Q, \{q_Q, q'_Q\} \subset S_{K,M,\text{bad}}^{(0)}(\mathcal{A})\right) \\
&= \zeta_\Lambda^\varepsilon\Bigg(\forall Q \in T^{(1)}\ \exists q_Q, q'_Q \in \mathcal{Q}_{\ell_0} \text{ with } q_Q \cap q'_Q = \varnothing, q_Q \cup q'_Q \subset Q \\
&\qquad\qquad \text{such that } \bigcup_{Q \in T^{(1)}} \{q_Q, q'_Q\} \subset S_{K,M,\text{bad}}^{(0)}(\mathcal{A})\Bigg).
\end{aligned}$$

As in Step 1 we can estimate this expression by the sum over all possibilities for q_Q, q'_Q to find

$$\zeta_\Lambda^\varepsilon\left(T^{(1)} \subset S_{K,M,\text{bad}}^{(1)}(\mathcal{A})\right) \leq \sum_{\substack{\underline{q}, \underline{q'} \in (\mathcal{Q}_{\ell_0})^{T^{(1)}} \\ q_Q \cap q'_Q = \varnothing, q_Q \cup q'_Q \subset Q\ \forall Q \in T^{(1)}}} \zeta_\Lambda^\varepsilon\left(\bigcup_{Q \in T^{(1)}} \{q_Q, q'_Q\} \subset S_{K,M,\text{bad}}^{(0)}(\mathcal{A})\right).$$

By assumption the elements of $T^{(1)}$ are pairwise disjoint. Therefore, all q_Q and q'_Q are pairwise distinct. Hence, the set $\bigcup_{Q \in T^{(1)}} \{q_Q, q'_Q\}$ has cardinality $2|T^{(1)}|$, and so (6.4.20) implies

$$\zeta_\Lambda^\varepsilon\left(T^{(1)} \subset S_{K,M,\text{bad}}^{(1)}(\mathcal{A})\right) \leq \sum_{\substack{\underline{q}, \underline{q'} \in (\mathcal{Q}_{\ell_0})^{T^{(1)}} \\ q_Q \cap q'_Q = \varnothing, q_Q \cup q'_Q \subset Q\ \forall Q \in T^{(1)}}} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}} (\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2|T^{(1)}|}$$

$$
\begin{aligned}
&\leq \left(\left(\frac{\ell_1}{\ell_0}\right)^{2\mathsf{d}}\right)^{|T^{(1)}|} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2|T^{(1)}|} \\
&= M^{2\mathsf{d}|T^{(1)}|} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2|T^{(1)}|} \\
&= \left(M^{2\mathsf{d}} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2}\right)^{|T^{(1)}|}
\end{aligned}
$$

which is (6.4.21) (as $s_1 = 2$).

Step 3: Proof of part b) for $j > 1$

We proceed by induction on j, using the result from Step 2 as the base case. That is, we assume that (6.4.21) holds for $j-1$, and we want to conclude that it also holds for j. The argument for this is analogous to the previous step. The only difference is that now the smaller cubes q, q' live in $\mathcal{Q}^{\#}_{\ell_{j-1}}$ instead of $\mathcal{Q}_{\ell_{j-1}}$, and so the number of possible $\underline{q}, \underline{q'}$ is now larger. Arguing as in Step 2, we obtain, using the assumption that the elements of $|T^{(j)}|$ are pairwise disjoint, that

$$
\begin{aligned}
&\zeta^{\varepsilon}_{\Lambda}\left(T^{(j)} \subset S^{(j)}_{K,M,\mathrm{bad}}(\mathcal{A})\right) \\
&\leq \sum_{\substack{\underline{q},\underline{q'} \in (\mathcal{Q}^{\#}_{\ell_{j-1}})^{T^{(j)}} \\ q_Q \cap q'_Q = \varnothing, q_Q \cup q'_Q \subset Q \ \forall Q \in T^{(j)}}} \zeta^{\varepsilon}_{\Lambda}\left(\bigcup_{Q \in T^{(j)}} \{q_Q, q'_Q\} \subset S^{(j-1)}_{K,M,\mathrm{bad}}(\mathcal{A})\right) \\
&\leq \left(\left(\frac{3\ell_j}{\ell_{j-1}}\right)^{2\mathsf{d}}\right)^{|T^{(j)}|} \left(3^{(2^{j-1}-2)\mathsf{d}} M^{s_{j-1}\mathsf{d}} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2^{j-1}}\right)^{2|T^{(j)}|} \\
&= \left(3^{2\mathsf{d}+2(2^{j-1}-2)\mathsf{d}} M^{2(j^3-(j-1)^3)\mathsf{d}+2s_{j-1}\mathsf{d}} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{M^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2^j}\right)^{|T^{(j)}|}.
\end{aligned}
\tag{6.4.24}
$$

It remains to observe that $2\mathsf{d} + 2(2^{j-1}-2)\mathsf{d} = (2^j - 2)\mathsf{d}$ and

$$
\begin{aligned}
2(j^3-(j-1)^3)\mathsf{d} + 2s_{j-1}\mathsf{d} &= 2\mathsf{d}\left(j^3 - (j-1)^3 + (j-1)^3 + \sum_{m=0}^{j-1} m^3 2^{j-1-m}\right) \\
&= \mathsf{d}\left(2j^3 + \sum_{m=0}^{j-1} m^3 2^{j-m}\right) \\
&= \mathsf{d}\left(j^3 + \sum_{m=0}^{j} m^3 2^{j-m}\right) = s_j \mathsf{d}\,.
\end{aligned}
$$

□

Now we can turn to the proof of Lemma 6.4.6.

Proof of Lemma 6.4.6.

Step 1: Proof of (6.4.18)

Our main tool will be Lemma 6.4.7 a), and the argument is similar to the one in Step 3 of the proof of Theorem 6.1.3 c). We will choose $K_{\mathsf{d},M,p} \geq K'_{\mathsf{d}}$ so that Lemma 6.4.7 can be applied.

Proceeding as in the proof of Lemma 6.4.7 we can estimate

$$\begin{aligned}
&\zeta^\varepsilon_\Lambda\left(T^* \subset S^{*,II}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad= \zeta^\varepsilon_\Lambda\Bigg(\forall Q \in T^* \exists Q'_Q \in \mathcal{Q}_{K\lambda_{\mathrm{mac}}} \text{ with } Q'_Q \subset Q,\\
&\qquad\qquad\qquad \left|\left\{q \in S^{(0)}_{K,M,\mathrm{bad}}(\mathcal{A}) \colon q \subset Q'_Q\right\}\right| \geq \frac{1}{4}\left(\frac{\lambda_{\mathrm{mac}}}{\lambda_{\mathrm{mic}}}\right)^{\mathsf{d}}\Bigg)\\
&\quad\leq \sum_{\substack{\underline{Q'} \in (\mathcal{Q}_{\ell_{j*(\varepsilon)}})^{T^*}\\ Q'_Q \subset Q\ \forall Q \in T^*}} \zeta^\varepsilon_\Lambda\left(\forall Q \in T^* \ \left|\left\{q \in S^{(0)}_{K,M,\mathrm{bad}}(\mathcal{A}) \colon q \subset Q'_Q\right\}\right| \geq \frac{1}{4}\left(\frac{\lambda_{\mathrm{mac}}}{\lambda_{\mathrm{mic}}}\right)^{\mathsf{d}}\right)\\
&\quad\leq \sum_{\substack{\underline{Q'} \in (\mathcal{Q}_{\ell_{j*(\varepsilon)}})^{T^*}\\ Q'_Q \subset Q\ \forall Q \in T^*}} \zeta^\varepsilon_\Lambda\left(\left|\left\{q \in S^{(0)}_{K,M,\mathrm{bad}}(\mathcal{A}) \colon q \subset \bigcup_{Q \in T^*} Q'_Q\right\}\right| \geq \frac{1}{4}\left(\frac{\lambda_{\mathrm{mac}}}{\lambda_{\mathrm{mic}}}\right)^{\mathsf{d}} |T^*|\right).
\end{aligned}$$

Let

$$\tilde{T}_{\underline{Q'}} = \{q \in \mathcal{Q}_{\ell_0} \colon q \subset \bigcup_{Q \in T^*} Q'_Q\}$$

and note that $\left|\tilde{T}_{\underline{Q'}}\right| = \left(\frac{\lambda_{\mathrm{mac}}}{\lambda_{\mathrm{mic}}}\right)^{\mathsf{d}} |T^*|$. Using (6.4.20) we can now continue to estimate

$$\begin{aligned}
\zeta^\varepsilon_\Lambda\left(T^* \subset S^{*,II}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right) &\leq \sum_{\substack{\underline{Q'} \in (\mathcal{Q}_{\ell_{j*(\varepsilon)}})^{T^*}\\ Q'_Q \subset Q\ \forall Q \in T^*}} \zeta^\varepsilon_\Lambda\left(\left|\tilde{T}_{\underline{Q'}} \cap S^{(0)}_{K,M,\mathrm{bad}}(\mathcal{A})\right| \geq \frac{1}{4}\left|\tilde{T}_{\underline{Q'}}\right|\right)\\
&= \sum_{\substack{\underline{Q'} \in (\mathcal{Q}_{\ell_{j*(\varepsilon)}})^{T^*}\\ Q'_Q \subset Q\ \forall Q \in T^*}} \sum_{\substack{T \subset \tilde{T}_{\underline{Q'}}\\ |T| \geq |\tilde{T}_{\underline{Q'}}|/4}} \zeta^\varepsilon_\Lambda\left(\tilde{T}_{\underline{Q'}} \cap S^{(0)}_{K,M,\mathrm{bad}}(\mathcal{A}) = T\right)\\
&\leq \sum_{\substack{\underline{Q'} \in (\mathcal{Q}_{\ell_{j*(\varepsilon)}})^{T^*}\\ Q'_Q \subset Q\ \forall Q \in T^*}} \sum_{\substack{T \subset \tilde{T}_{\underline{Q'}}\\ |T| \geq |\tilde{T}_{\underline{Q'}}|/4}} \zeta^\varepsilon_\Lambda\left(T \subset S^{(0)}_{K,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\leq \sum_{\substack{\underline{Q'} \in (\mathcal{Q}_{\ell_{j*(\varepsilon)}})^{T^*}\\ Q'_Q \subset Q\ \forall Q \in T^*}} \sum_{\substack{T \subset \tilde{T}_{\underline{Q'}}\\ |T| \geq |\tilde{T}_{\underline{Q'}}|/4}} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{|T|}\\
&= \sum_{\substack{\underline{Q'} \in (\mathcal{Q}_{\ell_{j*(\varepsilon)}})^{T^*}\\ Q'_Q \subset Q\ \forall Q \in T^*}} \sum_{i=\lceil |\tilde{T}_{\underline{Q'}}|/4 \rceil}^{|\tilde{T}_{\underline{Q'}}|} \binom{|\tilde{T}_{\underline{Q'}}|}{i} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{i}\\
&= (L^{\mathsf{d}})^{|T^*|} \sum_{i=\lceil N/4 \rceil}^{N} \binom{N}{i} \left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{i}
\end{aligned}$$

where we have abbreviated $N := |\tilde{T}_{\underline{Q'}}| = \left(\frac{\lambda_{\mathrm{mac}}}{\lambda_{\mathrm{mic}}}\right)^{\mathsf{d}} |T^*|$. Let $p_{\mathsf{d},K} := K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}} \exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)$. Clearly $\lim_{K\to\infty} p_{\mathsf{d},K} = 0$, so we can pick $K_{\mathsf{d},M,p}$ (at this point independently of M) large enough such that for $K \geq K_{\mathsf{d},M,p}$ we have $p_{\mathsf{d},K} \leq \frac{1}{32}$. Then Lemma 6.2.7 with that choice of p

implies that

$$\zeta_\Lambda^\varepsilon\left(T^* \subset S^{*,II}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right) \le (L^{\mathsf{d}})^{|T^*|}\left(\frac{1}{2}\right)^{\frac{N}{4}} = \left(L^{\mathsf{d}}\left(\frac{1}{2}\right)^{\frac{1}{4}\left(\frac{\lambda_{\mathrm{mac}}}{\lambda_{\mathrm{mic}}}\right)^{\mathsf{d}}}\right)^{|T^*|}.$$

For ε small enough (depending on d, L, and p) the term in brackets is less than p, and we obtain (6.4.18).

Step 2: Proof of (6.4.17)

We can use Lemma 6.4.7 b). Given $Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$, there are at most

$$\left(\left\lceil 3\frac{KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}}\right\rceil + 3\right)^{\mathsf{d}} \le \left(\frac{4KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}}\right)^{\mathsf{d}}$$

cubes in $\mathcal{Q}^{\#}_{\ell_{j_*(\varepsilon)}}$ that intersect Q (we used that $\frac{KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}} \ge 8$). We can now proceed as in the proof of Lemma 6.4.7 and obtain

$$\begin{aligned}
&\zeta_\Lambda^\varepsilon\left(T^* \subset S^{*,I}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad= \zeta_\Lambda^\varepsilon\left(\forall Q \in T^*\ \exists q_Q \in \mathcal{Q}^{\#}_{\ell_{j_*(\varepsilon)}} \text{ with } q_Q \cap Q \neq \varnothing, q_Q \in S^{(j_*(\varepsilon))}_{K,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad= \zeta_\Lambda^\varepsilon\left(\forall Q \in T^*\ \exists q_Q \in \mathcal{Q}^{\#}_{\ell_{j_*(\varepsilon)}} \text{ with } q_Q \cap Q \neq \varnothing \text{ such that } \bigcup_{Q\in T^*}\{q_Q\} \subset S^{(j_*(\varepsilon))}_{K,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad\le \sum_{\substack{\underline{q}\in(\mathcal{Q}^{\#}_{\ell_{j_*(\varepsilon)}})^{T^*}\\ q_Q\cap Q\neq\varnothing\ \forall Q\in T^*}} \zeta_\Lambda^\varepsilon\left(\bigcup_{Q\in T^*}\{q_Q\} \subset S^{(j_*(\varepsilon))}_{K,M,\mathrm{bad}}(\mathcal{A})\right).
\end{aligned}$$

If we assume for the moment that none of the cubes in T^* are l^∞-neighbours, then the q_Q are pairwise distinct, and so $\bigcup_{Q\in T^*}\{q_Q\}$ has cardinality $|T^*|$. Now (6.4.21) implies

$$\begin{aligned}
&\zeta_\Lambda^\varepsilon\left(T^* \subset S^{*,I}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad\le \sum_{\substack{\underline{q}\in(\mathcal{Q}^{\#}_{\ell_{j_*(\varepsilon)}})^{T^*}\\ q_Q\cap Q\neq\varnothing\ \forall Q\in T^*}} \left(3^{(2^{j_*(\varepsilon)}-2)\mathsf{d}} M^{s_{j_*(\varepsilon)}\mathsf{d}}\left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}}\exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2^{j_*(\varepsilon)}}\right)^{|T^*|}\\
&\quad\le \left(\left(\frac{4KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}}\right)^{\mathsf{d}}\right)^{|T^*|}\left(3^{(2^{j_*(\varepsilon)}-2)\mathsf{d}} M^{s_{j_*(\varepsilon)}\mathsf{d}}\left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}}\exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2^{j_*(\varepsilon)}}\right)^{|T^*|}.
\end{aligned} \tag{6.4.25}$$

To remove the assumption that none of the the elements of T^* are neighbours we proceed as in Step 1 of the proof of Lemma 6.4.7 and find a subset $T^*_{i_*}$ of T^* of cardinality at least $\frac{|T^*|}{2^{\mathsf{d}}}$

for which this is the case. Using (6.4.25) for $T^*_{i_*}$ instead of T^* we arrive at

$$\begin{aligned}
&\zeta^\varepsilon_\Lambda\left(T^* \subset S^{*,I}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad\le \zeta^\varepsilon_\Lambda\left(T^*_{i_*} \subset S^{*,I}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad\le \left(\left(\frac{4KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}}\right)^{\mathsf{d}}\right)^{\frac{|T^*|}{2^{\mathsf{d}}}}\left(3^{(2^{j_*(\varepsilon)}-2)\mathsf{d}}M^{s_{j_*(\varepsilon)}\mathsf{d}}\left(K^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{2^{\mathsf{d}}}}\exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2^{j_*(\varepsilon)}}\right)^{\frac{|T^*|}{2^{\mathsf{d}}}}.
\end{aligned} \tag{6.4.26}$$

It remains to bound the right hand side in (6.4.26). We begin by bounding s_j from above. We have

$$s_j = j^3 + \sum_{m=0}^{j} m^3 2^{j-m} = 2^j\left(\frac{j^3}{2^j} + \sum_{m=0}^{j}\frac{m^3}{2^m}\right) \le 2^j\left(4 + \sum_{m=0}^{\infty}\frac{m^3}{2^m}\right) = 30\cdot 2^j$$

and so (6.4.26) implies that

$$\begin{aligned}
\zeta^\varepsilon_\Lambda\left(T^* \subset S^{*,I}_{K,M,\mathrm{bad}}(\mathcal{A})\right) &\le \left(\left(\frac{4KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}}\left(3M^{30}K^{\frac{1}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{d2^{\mathsf{d}}}}\exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)\right)^{2^{j_*(\varepsilon)}}\right)^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}\right)^{|T^*|}\\
&= \left(\left(\frac{4KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}}p_{\mathsf{d},K,M}^{2^{j_*(\varepsilon)}}\right)^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}\right)^{|T^*|}
\end{aligned}$$

where we have abbreviated $p_{\mathsf{d},K,M} = 3M^{30}K^{\frac{1}{2^{\mathsf{d}}}}(\mathsf{d}+1)^{\frac{1}{d2^{\mathsf{d}}}}\exp\left(-\frac{K^{\mathsf{d}}}{C_{\mathsf{d}}}\right)$. Note for each fixed M and d we have $\lim_{K\to\infty} p_{\mathsf{d},K,M} = 0$, and so we can pick $K_{\mathsf{d},M,p}$ such that $p_{\mathsf{d},K,M} \le \frac{1}{2}$ for $K \ge K_{\mathsf{d},M,p}$.

For these choices of K we then know that

$$\zeta^\varepsilon_\Lambda\left(T^* \subset S^{*,I}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right) \le \left(\left(\frac{4KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}}2^{-2^{j_*(\varepsilon)}}\right)^{\frac{\mathsf{d}}{2^{\mathsf{d}}}}\right)^{|T^*|}$$

and we only need to show that

$$\frac{4KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}}2^{-2^{j_*(\varepsilon)}} \le p^{\frac{2^{\mathsf{d}}}{\mathsf{d}}} \tag{6.4.27}$$

when ε is small enough. To show this, we need to bound $j_*(\varepsilon)$ from below. By definition, $j_*(\varepsilon)$ is equal to the largest integer j such that $\ell_j \le \frac{KL\lambda_{\mathrm{mac}}}{8}$. In particular, $\ell_{j_*(\varepsilon)+1} > \frac{KL\lambda_{\mathrm{mac}}}{8}$, i.e. $M^{(j_*(\varepsilon)+1)^3}K\lambda_{\mathrm{mic}} > \frac{KL\lambda_{\mathrm{mac}}}{8}$. Estimating $(j_*(\varepsilon)+1)^3 \le (2j_*(\varepsilon))^3$, we conclude

$$j_*(\varepsilon) \ge \frac{1}{2}\sqrt[3]{\log_M\frac{L\lambda_{\mathrm{mac}}}{8\lambda_{\mathrm{mic}}}}.$$

Let us also abbreviate $X_{\mathsf{d},\varepsilon,L} = \frac{L\lambda_{\mathrm{mac}}}{8\lambda_{\mathrm{mic}}}$, and observe that $\lim_{\varepsilon\to 0} X_{\mathsf{d},\varepsilon,L} = \infty$. For t sufficiently large we have $\frac{1}{2}\sqrt[3]{\log_M t} \ge \log_2\log_2(t^2)$. This means that for $X_{\mathsf{d},\varepsilon,L}$ sufficiently large (i.e. ε

sufficiently small) we have $j_*(\varepsilon) \geq \frac{1}{2}\sqrt[3]{\log_M X_{\mathsf{d},\varepsilon,L}} \geq \log_2\log_2(X^2_{\mathsf{d},\varepsilon,L})$. Using this and the rather crude estimate $\ell_{j_*(\varepsilon)} \geq \ell_{j_0} = K\lambda_{\mathrm{mic}}$ for the denominator, we find

$$\frac{4KL\lambda_{\mathrm{mac}}}{\ell_{j_*(\varepsilon)}}2^{-2^{j_*(\varepsilon)}} \leq 32X_{\mathsf{d},\varepsilon,L}2^{-2^{\log_2\log_2(X^2_{\mathsf{d},\varepsilon,L})}} \leq \frac{32X_{\mathsf{d},\varepsilon,L}}{X^2_{\mathsf{d},\varepsilon,L}} = \frac{32}{X_{\mathsf{d},\varepsilon,L}}$$

for ε sufficiently small. This clearly implies that (6.4.27) holds for ε sufficiently small, which is (6.4.18).

Step 3: Proof of (6.4.19)

We can assume without loss of generality that $p \leq \frac{1}{4}$, as otherwise the estimate is trivial. Using (6.4.17) and (6.4.18) we see that

$$\begin{aligned}
&\zeta^\varepsilon_\Lambda\left(T^* \subset S^*_{K,L,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad\leq \sum_{T^*_I\cup T^*_{II}=T^*} \zeta^\varepsilon_\Lambda\left(T^*_I \subset S^{*,I}_{K,L,M,\mathrm{bad}}(\mathcal{A}), T^*_{II} \subset S^{*,II}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad\leq \sum_{\substack{T^*_I\subset T^*\\ |T^*_I|\geq|T^*|/2}} \zeta^\varepsilon_\Lambda\left(T^*_I \subset S^{*,I}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right) + \sum_{\substack{T^*_{II}\subset T^*\\ |T^*_{II}|\geq|T^*|/2}} \zeta^\varepsilon_\Lambda\left(T^*_{II} \subset S^{*,II}_{K,L,M,\mathrm{bad}}(\mathcal{A})\right)\\
&\quad\leq 2\sum_{i=\lceil|T^*|/2\rceil}^{|T^*|}\binom{|T^*|}{i}p^i\\
&\quad\leq 2(4p)^{\frac{|T^*|}{2}}
\end{aligned}$$

where we have used Lemma 6.2.7 in the last step. □

Using Lemma 6.4.6 we can now estimate the probability that we find sets U_j as in Lemma 6.4.3.

Lemma 6.4.8. *Let* $\mathsf{d} \geq 4$, *and* $\Lambda \Subset \mathbb{Z}^{\mathsf{d}}$. *Let* $M \geq 12$ *be an odd integer. Then there is* $K_{\mathsf{d},M}$ *depending on* d, M *only with the following property: Let* $K \geq K_{\mathsf{d},M}$ *be an odd multiple of 3, let* L *be an odd integer. Let* $U \in \mathcal{P}_{KL\lambda_{\mathrm{mac}}}$ *be a polymer consisting of* $n = \frac{|U|}{(KL)^{\mathsf{d}}\lambda^{\mathsf{d}}_{\mathrm{mac}}}$ *boxes. Let* $k \geq 0$ *be an integer and let* $\Omega_{U,k}$ *be the event that there exist* $U_0,\dots,U_k \in \mathcal{P}_{KL\lambda_{\mathrm{mac}}}$ *such that* $U \subset U_0$, *for* $j \in \{0,\dots,k-1\}$ *we have* $U_j + Q_{KL\lambda_{\mathrm{mac}}}(0) \subset U_{j+1}$, $U_k \subset U + Q_{2kKL\lambda_{\mathrm{mac}}}(0)$, *and*

$$\{Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}} : Q \subset (U_j + Q_{KL\lambda_{\mathrm{mac}}}(0)) \setminus U_j\} \cap S^*_{K,L,M,\mathrm{bad}}(\mathcal{A}) = \varnothing\,.$$

Then, if ε *is small enough (depending on* K, L *and* d*), we have*

$$\zeta^\varepsilon_\Lambda(\Omega_{U,k}) \geq 1 - \frac{n}{2^k}\,. \tag{6.4.28}$$

Proof. Let $p > 0$ be a constant to be chosen later (depending on d only). We pick $K_{\mathsf{d},M} \geq K_{\mathsf{d},M,p}$ with the $K_{\mathsf{d},M,p}$ from Lemma 6.4.6 so that this lemma can be applied.

We try to define the U_j using a greedy algorithm. That is, we define U_j as the union of all cubes $Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$ that can be connected to U by a non-selfintersecting l^∞-path of cubes in $\mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$ that contains at most j non-bad cubes. More precisely, $Q \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$ is a subset of U if and only if there are $l \geq 0$ and $Q^{(0)} = Q, Q^{(1)},\dots,Q^{(l)} \subset U \in \mathcal{Q}_{KL\lambda_{\mathrm{mac}}}$ pairwise disjoint, with $d_\infty(Q^{(i)}, Q^{(i+1)}) \leq 1$ for all $i \in \{0,\dots,l-1\}$, such that at most j of $Q^{(0)}, Q^{(1)},\dots,Q^{(l-1)}$ are not in $S^*_{K,L,M,\mathrm{bad}}(\mathcal{A})$.

This definition ensures that all l^∞-neighbouring cubes to U_j are not in $S^*_{K,L,M,\mathrm{bad}}(\mathcal{A})$. So one sees that the U_j satisfies all the conditions from $\Omega_{U,k}$ except that we do not yet know whether $U_k \subset U + Q_{2kKL\lambda_{\mathrm{mac}}}(0)$. This means that

$$1 - \zeta^\varepsilon_\Lambda(\Omega_{U,k}) \leq \zeta^\varepsilon_\Lambda\left(U_k \not\subset U + Q_{2kKL\lambda_{\mathrm{mac}}}(0)\right) \tag{6.4.29}$$

and so it suffices to estimate the latter probability.

To do so, we define $\Pi_{U,k}$ to be the set of non-selfintersecting l^∞-nearest neighbour paths $\Psi = \left(Q^{(0)} = Q, Q^{(1)}, \dots, Q^{(l)}\right)$ of cubes, that connect a cube Q outside of $Q_{2kKL\lambda_{\mathrm{mac}}}(0)$ with $Q^{(l)} \subset U$. For $\Psi = \left(Q^{(0)} = Q, Q^{(1)}, \dots Q^{(l)}\right)$ let $\tilde{\Psi} = \left\{Q^{(0)}, Q^{(1)}, \dots Q^{(l)}\right\}$ be the set of cubes in Ψ, and let $|\Psi| = |\tilde{\Psi}| = l + 1$ be the number of cubes in it.

If $U_k \not\subset U + Q_{2kKL\lambda_{\mathrm{mac}}}(0)$, then there is some $\Psi \in \Pi_{U,k}$ that contains at most k cubes within $Q^{(0)}, \dots, Q^{(l-1)}$ (and thus at most $k+1$ cubes within the cubes in Ψ) that are not bad. Because Ψ connects U with a cube outside of $U + Q_{2kKL\lambda_{\mathrm{mac}}}(0)$, we have $|\Psi| \geq 2k+2$. We can now continue (6.4.29) by using a union bound over all $\Psi \in \Pi_{U,k}$, and later over all bad subsets of $\tilde{\Psi}$, and obtain using Lemma 6.4.6 that

$$\begin{aligned}
1 - \zeta^\varepsilon_\Lambda(\Omega_{U,k}) &\leq \zeta^\varepsilon_\Lambda\left(\exists \Psi \in \Pi_{U,k} \colon \left|\tilde{\Psi} \setminus S^*_{K,L,M,\mathrm{bad}}(\mathcal{A})\right| \leq k+1\right) \\
&\leq \sum_{\Psi \in \Pi_{U,k}} \zeta^\varepsilon_\Lambda\left(\left|\tilde{\Psi} \setminus S^*_{K,L,M,\mathrm{bad}}(\mathcal{A})\right| \leq k+1\right) \\
&= \sum_{\Psi \in \Pi_{U,k}} \zeta^\varepsilon_\Lambda\left(\exists T^*_\Psi \subset \tilde{\Psi} \colon T^*_\Psi \subset S^*_{K,L,M,\mathrm{bad}}(\mathcal{A}), |T^*_\Psi| \geq |\Psi| - k - 1\right) \\
&\leq \sum_{\Psi \in \Pi_{U,k}} \sum_{\substack{T^*_\Psi \subset \tilde{\Psi} \\ |T^*_\Psi| \geq |\Psi| - k - 1}} \zeta^\varepsilon_\Lambda\left(T^*_\Psi \subset S^*_{K,L,M,\mathrm{bad}}(\mathcal{A})\right) \\
&\leq \sum_{\Psi \in \Pi_{U,k}} \sum_{\substack{T^*_\Psi \subset \tilde{\Psi} \\ |T^*_\Psi| \geq |\Psi| - k - 1}} 2(4p)^{\frac{|T^*_\Psi|}{2}} \\
&\leq \sum_{\Psi \in \Pi_{U,k}} 2(4p)^{\frac{|\Psi| - k - 1}{2}} .
\end{aligned}$$

We can reorganize this expression by summing over the lengths of Ψ. Recall that this length needs to be at least $2k+2$, and note that there are at most $n(2\mathrm{d})^l$ paths in $\Pi_{U,k}$ of length $l+1$. Thus, we obtain

$$\begin{aligned}
1 - \zeta^\varepsilon_\Lambda(\Omega_{U,k}) &\leq \sum_{l=2k+1}^{\infty} \sum_{\substack{\Psi \in \Pi_{U,k} \\ |\tilde{\Psi}| = l+1}} 2(4p)^{\frac{l-k}{2}} \\
&\leq \sum_{l=2k+1}^{\infty} n(2\mathrm{d})^l 2(4p)^{\frac{l-k}{2}} \\
&= \frac{2n}{(2\sqrt{p})^k} \sum_{l=2k+1}^{\infty} (4\mathrm{d}\sqrt{p})^l .
\end{aligned}$$

We choose $p \leq \frac{1}{144\mathrm{d}^2}$, so that $4\mathrm{d}\sqrt{p} \leq \frac{1}{3}$. Then, in particular, the series on the right-hand side converges, and we can continue

$$1 - \zeta^\varepsilon_\Lambda(\Omega_{U,k}) \leq \frac{2n}{(2\sqrt{p})^k} \frac{(4\mathrm{d}\sqrt{p})^{2k+1}}{1 - 4\mathrm{d}\sqrt{p}}$$

$$\leq \frac{n(4\mathsf{d}\sqrt{p})^{2k}}{(2\sqrt{p})^k}$$
$$= n(8\mathsf{d}^2\sqrt{p})^k .$$

We finalize our choice of p as $p = \frac{1}{256\mathsf{d}^4}$. Then (6.4.28) follows. □

Proof of Theorem 6.4.1. We want to combine Lemma 6.4.5 and Lemma 6.4.8. That is, we first choose M so large that Lemma 6.4.5 can be applied. Then we choose K large enough that Lemma 6.4.8 can be applied. Then Lemma 6.4.5 applies for sufficiently large L. We choose $\hat{N}_\mathsf{d} = KL$, and note that Lemma 6.4.8 implies the bound on the probability of $\Omega_{U,k}$.

It remains to check that (6.4.15) and (6.4.16) imply (6.4.1) and (6.4.2) if $\Omega_{U,k}$ holds. This follows from the observation that $(U_0 + Q_{\hat{N}_\mathsf{d}\lambda_{\mathrm{mac}}}(0)) \setminus U_0 \subset (U + Q_{2k\hat{N}_\mathsf{d}\lambda_{\mathrm{mac}}}(0)) \setminus U$ and $(U_{k-1} + Q_{\hat{N}_\mathsf{d}\lambda_{\mathrm{mac}}}(0)) \setminus U_{k-1} \subset (U + Q_{2k\hat{N}_\mathsf{d}\lambda_{\mathrm{mac}}}(0)) \setminus U$. □

Remark 6.4.9. Let us comment on why the lengthscales $\ell_j = M^{j^3}K\lambda_{\mathrm{mic}}$ are a natural choice. For the construction in Step 2 of the proof of Lemma 6.4.2 we need that $\log \frac{\ell_j}{\ell_{j+1}}$ is summable as otherwise we could not bound $\|\nabla_1^2\eta\|_{L^\infty}$ in (6.4.6). This means that ℓ_j needs to grow rather fast (e.g., $\ell_j = M^{j^2}K\lambda_{\mathrm{mic}}$ would not be fast enough). On the other hand, for the estimate on the probability of bad cubes of type I in Lemma 6.4.6 we need that the exponent $s_j\mathsf{d}$ of M in (6.4.21) is at most $C_\mathsf{d}2^j$. This exponent arises from the combinatorial factors $\left(\frac{\ell_j}{\ell_{j-1}}\right)^{2\mathsf{d}}$ in (6.4.24). This means that ℓ_j cannot grow too fast (e.g., $\ell_j = M^{2^j}K\lambda_{\mathrm{mic}}$ would be too fast).

Fortunately, both requirements are compatible, and in fact, our choice $\ell_j = M^{j^3}K\lambda_{\mathrm{mic}}$ satisfies both of them.

6.5 Pathwise bounds on the field

We can now turn to the proof of Theorem 6.1.5 and of the second part of Theorem 6.1.2. Before we actually give the proofs, however, we state and prove various quenched estimates for $G_{\Lambda\setminus A}$ that hold for all A, or at least up to exponentially small probability in A. The main tool for that will be Theorem 6.4.1.

We prove those estimates in Section 6.5.1. Then, in Sections 6.5.2 and 6.5.3 we use them to deduce Theorem 6.1.5 and the second part of Theorem 6.1.2, respectively.

6.5.1 Quenched estimates on the Green's function

We write $G_{\Lambda,y}$ for $G_\Lambda(\cdot, y)$. We have the following straightforward result for G_Λ. This is essentially the same as Lemma 2.8.1 or Lemma 4.4.2.

Lemma 6.5.1. *Let $\Lambda \Subset \mathbb{Z}^\mathsf{d}$ and $x, y \in \mathbb{Z}^\mathsf{d}$. Then*

$$G_\Lambda(x,y) = \left(\nabla_1^2 G_{\Lambda,x}, \nabla_1^2 G_{\Lambda,y}\right)_{L^2(\mathbb{Z}^\mathsf{d})} . \tag{6.5.1}$$

Furthermore, we have

$$|G_\Lambda(x,y)| \leq \sqrt{G_\Lambda(x,x)G_\Lambda(y,y)} . \tag{6.5.2}$$

Proof. For (6.5.1) we calculate

$$G_\Lambda(x,y) = \left(\mathbb{1}_{\cdot=x}, G_{\Lambda,y}\right)_{L^2(\mathbb{Z}^\mathsf{d})} = \left(\Delta_1^2 G_{\Lambda,x}, G_{\Lambda,y}\right)_{L^2(\mathbb{Z}^\mathsf{d})} = \left(\nabla_1^2 G_{\Lambda,x}, \nabla_1^2 G_{\Lambda,y}\right)_{L^2(\mathbb{Z}^\mathsf{d})} .$$

The estimate (6.5.2) follows directly from the interpretation of G_Λ as a covariance. Alternatively, we can use (6.5.1) together with the Cauchy-Schwarz inequality to estimate

$$\begin{aligned}|G_\Lambda(x,y)| &= \left|\left(\nabla_1^2 G_{\Lambda,x}, \nabla_1^2 G_{\Lambda,y}\right)_{L^2(\mathbb{Z}^\mathsf{d})}\right| \\ &\leq \left\|\nabla_1^2 G_{\Lambda,x}\right\|_{L^2(\mathbb{Z}^\mathsf{d})} \left\|\nabla_1^2 G_{\Lambda,y}\right\|_{L^2(\mathbb{Z}^\mathsf{d})} \\ &= \sqrt{G_\Lambda(x,x)G_\Lambda(y,y)}\,.\end{aligned}$$

□

Next, we establish some quenched tail estimates on $G_{\Lambda\setminus\mathcal{A}}(x,x)$. If $\mathsf{d} \geq 5$, then there are deterministic bounds on $G_{\Lambda\setminus\mathcal{A}}(x,x)$ by (6.2.5), so this is only interesting if $\mathsf{d} = 4$.

Lemma 6.5.2. *If $\mathsf{d} = 4$, there is a constant $\tilde{\gamma} > 0$ such that if $\Lambda \Subset \mathbb{Z}^d$, $x \in \Lambda$ and ε is small enough (depending on d only) then for any $t \geq \tilde{\gamma}$ we have*

$$\zeta_\Lambda^\varepsilon\left(G_{\Lambda\setminus\mathcal{A}}(x,x) \leq t\right) \geq 1 - \exp\left(-\frac{\varepsilon\exp(16\pi^2(t-\tilde{\gamma}))}{C|\log\varepsilon|^{\frac{1}{2}}}\right), \tag{6.5.3}$$

and for $\alpha > 0$, $x \in \Lambda$ with $d(x, \mathbb{Z}^\mathsf{d}\setminus\Lambda) \geq \varepsilon^{-\alpha}$ and $0 \leq t \leq \frac{1}{8\pi^2}\log\left(1 + d(x,\mathbb{Z}^\mathsf{d}\setminus\Lambda) - \varepsilon^{-\alpha}\right) - \tilde{\gamma}$ we have

$$\zeta_\Lambda^\varepsilon\left(G_{\Lambda\setminus\mathcal{A}}(x,x) \leq t\right) \leq 1 - \exp\left(-\frac{C_\alpha\varepsilon\exp(32\pi^2(t+\tilde{\gamma}))}{|\log\varepsilon|^{\frac{1}{2}}}\right) \tag{6.5.4}$$

for some constant C.

Furthermore, if $\mathsf{d} \geq 4$, $k \in \mathbb{N}$, and $y \in \Lambda$ there are constants $\tilde{\gamma}_\mathsf{d}$ such that

$$\zeta_\Lambda^\varepsilon\left(G_{\Lambda\setminus\mathcal{A}}(y,y) \leq \mathbb{1}_{\mathsf{d}=4}\frac{\log k + |\log\varepsilon|}{16\pi^2} + \tilde{\gamma}_\mathsf{d}\right) \geq 1 - \frac{1}{2^k}\,. \tag{6.5.5}$$

Proof. We begin with (6.5.3). This follows easily from Lemma 6.2.4 and Theorem 6.1.3 c). Indeed, if $x \in \mathcal{A}$ then $G_{\Lambda\setminus\mathcal{A}}(x,x) = 0$, while if $x \notin \mathcal{A}$ we know from (6.2.5) that

$$G_{\Lambda\setminus\mathcal{A}}(x,x) \leq \frac{1}{4\pi^2}\log(1 + d(x,\tilde{\mathcal{A}})) + C \leq \frac{1}{4\pi^2}\log(d(x,\tilde{\mathcal{A}})) + C \leq \frac{1}{4\pi^2}\log(d(x,\mathcal{A})) + C\,.$$

So, there is a constant $\tilde{\gamma}'$ such that $G_{\Lambda\setminus\mathcal{A}}(x,x) > t$ for $t \geq \tilde{\gamma}'$ implies $d(x,\mathcal{A}) \geq \exp(4\pi^2(t - \tilde{\gamma}'))$. Using (6.1.7) we can estimate that

$$\begin{aligned}\zeta_\Lambda^\varepsilon\left(G_{\Lambda\setminus\mathcal{A}}(x,x) \leq t\right) &\geq \zeta_\Lambda^\varepsilon\left(d(x,\mathcal{A}) \leq \exp(4\pi^2(t-\tilde{\gamma}'))\right) \\ &= 1 - \zeta_\Lambda^\varepsilon\left(\mathcal{A} \cap Q_{\exp(4\pi^2(t-\tilde{\gamma}'))}(x) = \varnothing\right) \\ &\geq 1 - (1 - p_{4,-})^{|Q_{\exp(4\pi^2(t-\tilde{\gamma}'))}(x)|} \\ &\geq 1 - \exp\left(-\frac{p_{4,-}\exp(4\pi^2(t-\tilde{\gamma}'))^4}{C}\right) \\ &\geq 1 - \exp\left(-\frac{\varepsilon\exp(16\pi^2(t-\tilde{\gamma}'))}{C|\log\varepsilon|^{\frac{1}{2}}}\right)\end{aligned}$$

which is (6.5.3), if we choose $\tilde{\gamma} \geq \tilde{\gamma}'$.

The argument for (6.5.4) is similar. We have that

$$G_{\Lambda\setminus\mathcal{A}}(x,x) \geq \frac{1}{8\pi^2}\log(1+d(x,\tilde{\mathcal{A}})) - C \geq \frac{1}{8\pi^2}\log(d(x,\tilde{\mathcal{A}})) - C$$

if $x \notin \mathcal{A}$ and $G_{\Lambda\setminus\mathcal{A}}(x,x) = 0$ if $x \in \mathcal{A}$. So there is a constant $\tilde{\gamma}''$ such that $G_{\Lambda\setminus\mathcal{A}}(x,x) \leq t$ implies $d(x,\tilde{\mathcal{A}}) \leq \exp(8\pi^2(t+\tilde{\gamma}''))$. Our assumption $t \leq \frac{1}{8\pi^2}\log\left(1+d(x,\mathbb{Z}^{\mathsf{d}}\setminus\Lambda) - \varepsilon^{-\alpha}\right) - \tilde{\gamma}$ ensures that $\exp(8\pi^2(t+\gamma'')) \leq 1 + d(x,\mathbb{Z}^{\mathsf{d}}\setminus\Lambda) - \varepsilon^{-\alpha}$. This means that $Q_{\exp(8\pi^2(t+\tilde{\gamma}''))}(x)$ still has distance at least $\varepsilon^{-\alpha}$ from $\mathbb{Z}^{\mathsf{d}}\setminus\Lambda$ (and in particular $d(x,\tilde{\mathcal{A}}) < d(x,\mathbb{Z}^{\mathsf{d}}\setminus\Lambda)$, so that $d(x,\tilde{\mathcal{A}}) < d(x,\mathcal{A})$. Thus, we can apply (6.1.8) and obtain

$$\begin{aligned}
\zeta_\Lambda^\varepsilon\left(G_{\Lambda\setminus\mathcal{A}}(x,x) \leq t\right) &\leq \zeta_\Lambda^\varepsilon\left(d(x,\mathcal{A}) \leq \exp(8\pi^2(t+\tilde{\gamma}''))\right) \\
&= 1 - \zeta_\Lambda^\varepsilon\left(\mathcal{A}\cap Q_{\exp(8\pi^2(t+\tilde{\gamma}''))}(x) = \varnothing\right) \\
&\leq 1 - \exp\left(-\frac{C_\alpha\varepsilon\exp(32\pi^2(t+\tilde{\gamma}''))}{|\log\varepsilon|^{\frac{1}{2}}}\right)
\end{aligned}$$

This is (6.5.4), if we choose $\tilde{\gamma} \geq \tilde{\gamma}''$.

Regarding (6.5.5), note that if $\mathsf{d} \geq 5$ this is a trivial consequence of (6.2.5), while if $\mathsf{d} = 4$ we can consider the choice $t = \frac{\log k + |\log\varepsilon|}{16\pi^2} + \tilde{\gamma}$ in (6.5.3) to obtain

$$\begin{aligned}
\zeta_\Lambda^\varepsilon\left(G_{\Lambda\setminus\mathcal{A}}(y,y) \leq \frac{\log k + |\log\varepsilon|}{16\pi^2} + \tilde{\gamma}\right) &\geq 1 - \exp\left(-\frac{\varepsilon\exp(16\pi^2\frac{\log k+|\log\varepsilon|}{16\pi^2})}{C|\log\varepsilon|^{\frac{1}{2}}}\right) \\
&\leq 1 - \exp\left(-\frac{k}{C|\log\varepsilon|^{\frac{1}{2}}}\right)
\end{aligned}$$

and the right-hand side is at least $1 - \frac{1}{2^k}$ if ε is small enough. □

Next, we prove quenched bounds on the covariance.

Lemma 6.5.3. *Let* $\mathsf{d} \geq 4$, $\Lambda \Subset \mathbb{Z}^{\mathsf{d}}$, *and* $x, y \in \Lambda$. *Then, if* ε *is small enough (depending on* d*), we have*

$$\zeta_\Lambda^\varepsilon\left(|G_{\Lambda\setminus\mathcal{A}}(x,y)| \leq C_{\mathsf{d}}\exp\left(-\frac{|x-y|_\infty}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right)\frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/4}}{\varepsilon^{1/2}}\right) \geq 1 - \exp\left(-\frac{|x-y|_\infty}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right) \tag{6.5.6}$$

for some constant C_{d}.

Proof. By translating Λ and $\mathcal{A}$ we can assume $y = 0$. This ensures in particular that y is in the centre of a box in $\mathcal{Q}_l$ for any l. Let $U = Q_{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)$ with the $\hat{N}_{\mathsf{d}}$ from Theorem 6.4.1, and consider for now the case that $|x-y|_\infty \geq 8\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}$. Let $k = \left\lceil\frac{|x-y|_\infty}{8\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right\rceil$, and note that $k \leq \frac{|x-y|_\infty}{4\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}$.

Assume that $A \in \Omega_{U,k}$ with the $\Omega_{U,k}$ from Theorem 6.4.1. Then that theorem (applied to $G_{\Lambda\setminus A,y}$) and (6.5.1) imply that

$$\begin{aligned}
\left\|\nabla_1^2 G_{\Lambda\setminus A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} &\leq \frac{1}{2^k}\left\|\nabla_1^2 G_{\Lambda\setminus A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus U)} \\
&= \frac{1}{2^k}G_{\Lambda\setminus A}(y,y)\,.
\end{aligned} \tag{6.5.7}$$

Furthermore, suppose that $A \in \Omega_{x,k}$ with the $\Omega_{x,k}$ from Lemma 6.3.2. Then we can conclude

$$\begin{aligned} |G_{\Lambda\setminus A}(x,y)|^2 &= |G_{\Lambda\setminus A,y}(x)|^2 \\ &\leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{\varepsilon} \|\nabla_1^2 G_{\Lambda\setminus A,y}\|^2_{L^2(Q_{kN_{\mathsf{d}}\lambda_{\mathrm{mic}}}(x))} \,. \end{aligned} \tag{6.5.8}$$

For ε small enough (depending on d) we have $N_{\mathsf{d}}\lambda_{\mathrm{mic}} \leq \hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}$. Then $U + Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0) = Q_{(2k+1)\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)$ and $Q_{kN_{\mathsf{d}}\lambda_{\mathrm{mic}}}(x)$ are disjoint, and so we can combine (6.5.7) and (6.5.8) into

$$\begin{aligned} |G_{\Lambda\setminus A}(x,y)|^2 &\leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{\varepsilon} \|\nabla_1^2 G_{\Lambda\setminus A,y}\|^2_{L^2(Q_{kN_{\mathsf{d}}\lambda_{\mathrm{mic}}}(x))} \\ &\leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{\varepsilon} \left\|\nabla_1^2 G_{\Lambda\setminus A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} \\ &\leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{2^k\varepsilon} G_{\Lambda\setminus A}(y,y) \,. \end{aligned} \tag{6.5.9}$$

Next, let $\tilde{\Omega}_{y,k}$ be the event from (6.5.5). If $A \in \tilde{\Omega}_{y,k}$, then (6.5.5) and (6.5.9) imply

$$\begin{aligned} |G_{\Lambda\setminus A}(x,y)|^2 &\leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{2^k\varepsilon} \left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|)\right) \\ &\leq C_{\mathsf{d}} \left(\frac{3}{4}\right)^k \frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/2}}{\varepsilon} \\ &\leq C_{\mathsf{d}} \exp\left(-\log\frac{3}{4}\frac{|x-y|_\infty}{4\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right) \frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/2}}{\varepsilon} \\ &\leq C_{\mathsf{d}} \exp\left(-\frac{|x-y|_\infty}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right) \frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/2}}{\varepsilon} \,. \end{aligned} \tag{6.5.10}$$

This estimate holds if $A \in \Omega_{U,k} \cap \Omega_{x,k} \cap \tilde{\Omega}_{y,k}$. But that probability is easy to bound:

$$\zeta_\Lambda^\varepsilon\left(\Omega_{U,k} \cap \Omega_{x,k} \cap \tilde{\Omega}_{y,k}\right) \geq 1 - \frac{1}{2^k} - \frac{1}{2^{k^{\mathsf{d}}}} - \frac{1}{2^k} \geq 1 - \exp\left(\frac{k}{C_{\mathsf{d}}}\right) \geq 1 - \exp\left(-\frac{|x-y|_\infty}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right) \,.$$

Therefore we have shown that the set of A for which (6.5.10) holds has measure at least $1 - \exp\left(-\frac{|x-y|_\infty}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right)$, and this implies (6.5.6).

It remains to consider the case that $|x-y|_\infty < 8\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}$. In that case we need to show

$$\zeta_\Lambda^\varepsilon\left(|G_{\Lambda\setminus\mathcal{A}}(x,y)| \leq C_{\mathsf{d}} \frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/2}}{\varepsilon}\right) \geq c_{\mathsf{d}} \,.$$

This follows immediately from (6.5.5) and (6.5.2). □

We also need to quantify that for a large domain Λ the covariances far inside Λ depend only weakly on the precise shape of Λ.

Lemma 6.5.4. *Let $\mathsf{d} \geq 4$, $\Lambda' \subset \Lambda \Subset \mathbb{Z}^{\mathsf{d}}$. Let ε be small enough (depending on d only). Suppose that r, R are integers with $\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}} \leq r$, $8r \leq R$ and $Q_R(0) \subset \Lambda'$. we have*

$$\begin{aligned} &\zeta_\Lambda^\varepsilon\left(\max_{x,y\in Q_r(0)} \left|G_{\Lambda\setminus\mathcal{A}}(x,y) - G_{\Lambda'\setminus\mathcal{A}}(x,y)\right| \leq C_{\mathsf{d}} \exp\left(-\frac{R-r}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right) \frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/4}}{\varepsilon^{1/2}}\right) \\ &\quad \geq 1 - C_{\mathsf{d}} r^{\mathsf{d}} \exp\left(-\frac{R-r}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right) \,. \end{aligned} \tag{6.5.11}$$

Proof. The idea is that $H_y := G_{\Lambda\setminus\mathcal{A},y} - G_{\Lambda'\setminus\mathcal{A},y}$ is biharmonic in $Q_R(0)$. We will use Theorem 6.4.1 a) to conclude that the L^2-norm of $\nabla_1^2 H$ outside of $Q_{R/2}(0)$ is exponentially small, and then use Theorem 6.4.1 b) to conclude that the L^2-norm of $\nabla_1^2 H$ in $Q_r(0)$ is exponentially small. Of course these estimates hold not for all realizations of $\mathcal{A}$, but we will estimate that they hold for sufficiently many.

Let $\tilde{r} = \left\lceil \frac{r}{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}} \right\rceil \hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}$ and $\tilde{R} = \left\lfloor \frac{R}{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}} \right\rfloor \hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}$. Then $r \le \tilde{r} \le 2r$, $\frac{R}{2} \le \tilde{R} \le R$. We let $U = Q_{\tilde{r}}(0)$ and note that $U \in \mathcal{P}_{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}$ is a polymer consisting of $\left(\frac{\tilde{r}}{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right)^{\mathsf{d}}$ boxes in $\mathcal{Q}_{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}$. Let $k = \left\lfloor \frac{\tilde{R}-\tilde{r}}{4\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}} \right\rfloor$ and note that $k \ge \frac{R-r}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}$. Theorem 6.4.1 a) implies that on the event $\Omega_{U,k}$ we have

$$\begin{aligned}
\left\|\nabla_1^2 G_{\Lambda\setminus A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} &\le \frac{1}{2^k}\left\|\nabla_1^2 G_{\Lambda\setminus A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus U)} \\
&\le \frac{1}{2^k}\left\|\nabla_1^2 G_{\Lambda\setminus A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}})} \\
&= \frac{1}{2^k} G_{\Lambda\setminus A}(y,y)
\end{aligned}$$

as $G_{\Lambda\setminus A,y} = 0$ on $\tilde{A}\setminus U$ and $G_{\Lambda\setminus A,y}\Delta_1^2 G_{\Lambda\setminus A,y} = 0$ on $\mathbb{Z}^{\mathsf{d}}\setminus U$.

Analogously we have

$$\left\|\nabla_1^2 G_{\Lambda'\setminus A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} \le \frac{1}{2^k} G_{\Lambda'\setminus A}(y,y)$$

as $G_{\Lambda\setminus A,y} = 0$ on $\tilde{A}\setminus U$ and $G_{\Lambda'\setminus A,y}\Delta_1^2 G_{\Lambda'\setminus A,y} = 0$ on $\mathbb{Z}^{\mathsf{d}}\setminus U$ (even though $G_{\Lambda'\setminus A,y}$ is not biharmonic everywhere on $\Lambda\setminus(A\cup U)$).

If we define $H_{A,y} =: G_{\Lambda\setminus A,y} - G_{\Lambda'\setminus A,y}$, the preceding two estimates imply that

$$\begin{aligned}
&\left\|\nabla_1^2 H_{A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} \\
&\quad\le 2\left\|\nabla_1^2 G_{\Lambda\setminus A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} + 2\left\|\nabla_1^2 G_{\Lambda'\setminus A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} \\
&\quad\le \frac{1}{2^{k-1}}\left(G_{\Lambda\setminus A}(y,y) + G_{\Lambda'\setminus A}(y,y)\right) .
\end{aligned} \tag{6.5.12}$$

The polymer $U + Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)$ consists of $\left(\frac{\tilde{r}}{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}} + 2k\right)^{\mathsf{d}}$ boxes in $\mathcal{Q}_{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}$. The function H_y is biharmonic on $U + Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0) \subset Q_R(0)$ as the two singularities cancel out. So we can apply Theorem 6.4.1 b) and obtain on the event $\Omega_{U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0),k}$ that

$$\left\|\nabla_1^2 H_{A,y}\right\|^2_{L^2(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0))} \le \frac{1}{2^k}\left\|\nabla_1^2 H_{A,y}\right\|^2_{L^2((U+Q_{4k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0))\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} . \tag{6.5.13}$$

Furthermore, we can introduce the event $\tilde{\Omega}_{y,k}$ as in (6.5.5). By definition we have

$$G_{\Lambda\setminus A}(y,y) \le C_{\mathsf{d}}\left(1 + \mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|)\right) \tag{6.5.14}$$

on that event. We claim that on the event $\tilde{\Omega}_{y,k}$ we also have

$$G_{\Lambda'\setminus A}(y,y) \le C_{\mathsf{d}}\left(1 + \mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|)\right) . \tag{6.5.15}$$

Indeed, if $\mathsf{d} \geq 5$ this is once more a trivial consequence of (6.2.5), while if $\mathsf{d} = 4$ we can use (6.2.6) to estimate

$$\begin{aligned} G_{\Lambda'\setminus A}(y,y) &\leq \frac{1}{4\pi^2}\log(1+d(x,(A\cup(\mathbb{Z}^{\mathsf{d}}\setminus\Lambda'))) + C \\ &\leq \frac{1}{4\pi^2}\log(1+d(x,(A\cup(\mathbb{Z}^{\mathsf{d}}\setminus\Lambda))) + C \\ &\leq G_{\Lambda\setminus A}(y,y) + C \end{aligned}$$

so that (6.5.15) is a consequence of (6.5.14).

Finally, if $A \in \Omega_{x,k}$ with the event $\Omega_{x,k}$ from Lemma 6.3.2, we have

$$|H_{A,y}(x)|^2 \leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{\varepsilon} \|\nabla_1^2 H_{A,y}(x)\|^2_{L^2(Q_{kN_{\mathsf{d}}\lambda_{\mathrm{mic}}}(x))}. \tag{6.5.16}$$

We choose ε small enough so that $N_{\mathsf{d}}\lambda_{\mathrm{mic}} \leq 2\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}$. Then, in particular, $Q_{kN_{\mathsf{d}}\lambda_{\mathrm{mic}}}(x) \subset U + Q_{kN_{\mathsf{d}}\lambda_{\mathrm{mic}}}(0) \subset U + Q_{r+2kN_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)$.

Now we can combine the estimates we have just collected. More precisely, assume that $A \in \Omega_{U,k} \cap \Omega_{U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0),k} \cap \tilde{\Omega}_{y,k} \cap \Omega_{x,k}$. Then we can use (6.5.12), (6.5.13), (6.5.14), (6.5.15) and (6.5.16) to obtain

$$\begin{aligned} &\left|G_{\Lambda\setminus A}(x,y) - G_{\Lambda'\setminus A}(x,y)\right|^2 \\ &\quad = |H_{A,y}(x)|^2 \\ &\quad \leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{\varepsilon} \|\nabla_1^2 H_{A,y}\|^2_{L^2(Q_{kN_{\mathsf{d}}\lambda_{\mathrm{mic}}}(x))} \\ &\quad \leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{\varepsilon} \left\|\nabla_1^2 H_{A,y}\right\|^2_{L^2(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0))} \\ &\quad \leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{2^k\varepsilon} \left\|\nabla_1^2 H_{A,y}\right\|^2_{L^2((U+Q_{4k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}})\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} \\ &\quad \leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{2^k\varepsilon} \left\|\nabla_1^2 H_{A,y}\right\|^2_{L^2(\mathbb{Z}^{\mathsf{d}}\setminus(U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0)))} \\ &\quad \leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{2^{2k-1}\varepsilon} \left(G_{\Lambda\setminus A}(y,y) + G_{\Lambda'\setminus A}(y,y)\right) \\ &\quad \leq C_{\mathsf{d}} \frac{k^{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|^{3/2})\right)}{2^{2k-1}\varepsilon} \left(1+\mathbb{1}_{\mathsf{d}=4}(\log k + |\log\varepsilon|)\right) \\ &\quad \leq C_{\mathsf{d}} \left(\frac{1}{2}\right)^k \frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/2}}{\varepsilon} \\ &\quad \leq C_{\mathsf{d}} \exp\left(-\frac{R-r}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right) \frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/2}}{\varepsilon}. \end{aligned} \tag{6.5.17}$$

From (6.5.17) we see that on the event

$$\Omega := \Omega_{U,k} \cap \Omega_{U+Q_{2k\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}(0),k} \cap \bigcap_{y\in Q_r(0)} \tilde{\Omega}_{y,k} \cap \bigcap_{x\in Q_r(0)} \Omega_{x,k}$$

we have the desired estimate. So it only remains to bound the probability of Ω from below. For this we use a union bound to see

$$\zeta_\Lambda^\varepsilon(\Omega) \geq 1 - \left(\frac{\tilde{r}}{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right)^{\mathsf{d}} \frac{1}{2^k} - \left(\frac{\tilde{r}}{\hat{N}_{\mathsf{d}}\lambda_{\mathrm{mac}}} + 2k\right)^{\mathsf{d}} \frac{1}{2^k} - (2r+1)^{\mathsf{d}} \frac{1}{2^{k^{\mathsf{d}}}} - (2r+1)^{\mathsf{d}} \frac{1}{2^k}$$

$$\geq 1 - C_{\mathsf{d}} r^{\mathsf{d}} \exp\left(-\frac{k}{C_{\mathsf{d}}}\right)$$
$$\geq 1 - C_{\mathsf{d}} r^{\mathsf{d}} \exp\left(-\frac{R-r}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right).$$

This completes the proof. □

6.5.2 Estimates on variance and covariance

Proof of Theorem 6.1.5. We first prove part a) and then part b).

Step 1: Estimates on the variance

We have that

$$\mathbb{E}^{\varepsilon}_{\Lambda}(\psi_x^2) = \sum_{A\subset\Lambda} \zeta^{\varepsilon}_{\Lambda}(A)\mathbb{E}_{\Lambda\setminus A}(\psi_x^2) = \sum_{A\subset\Lambda} \zeta^{\varepsilon}_{\Lambda}(A) G_{\Lambda\setminus A}(x,x)\,. \tag{6.5.18}$$

Thus, (6.1.10) follows immediately from (6.2.5). For (6.1.11) we use Lemma 6.5.2. Indeed, using Fubini's theorem and (6.5.3) we can rewrite (6.5.18) as

$$\begin{aligned}
\mathbb{E}^{\varepsilon}_{\Lambda}(\psi_x^2) &= \int_0^\infty \zeta^{\varepsilon}_{\Lambda}\left(G_{\Lambda\setminus\mathcal{A}}(x,x)\geq t\right)\,\mathrm{d}t \\
&\leq \int_{\tilde\gamma}^\infty \zeta^{\varepsilon}_{\Lambda}\left(G_{\Lambda\setminus\mathcal{A}}(x,x)\geq t\right)\,\mathrm{d}t + \tilde\gamma \\
&\leq \int_{\tilde\gamma}^\infty \exp\left(-\frac{\varepsilon\exp(16\pi^2(t-\tilde\gamma))}{C|\log\varepsilon|^{\frac12}}\right)\,\mathrm{d}t + \tilde\gamma \\
&\leq \int_0^\infty \exp\left(-\frac{\varepsilon\exp(16\pi^2 t)}{C|\log\varepsilon|^{\frac12}}\right)\,\mathrm{d}t + C \\
&\leq \int_0^{\frac{|\log\varepsilon|}{16\pi^2}+\frac{\log|\log\varepsilon|}{32\pi^2}} 1\,\mathrm{d}t + \int_{\frac{|\log\varepsilon|}{16\pi^2}+\frac{\log|\log\varepsilon|}{32\pi^2}}^\infty \exp\left(-\frac{\varepsilon\exp(16\pi^2 t)}{C|\log\varepsilon|^{\frac12}}\right)\,\mathrm{d}t + C \\
&\leq \frac{|\log\varepsilon|}{16\pi^2} + \frac{\log|\log\varepsilon|}{32\pi^2} + \int_0^\infty \exp\left(-\frac{\exp(16\pi^2 t)}{C}\right)\,\mathrm{d}t + C \\
&\leq \frac{|\log\varepsilon|}{16\pi^2} + \frac{\log|\log\varepsilon|}{32\pi^2} + C \\
&\leq \frac{|\log\varepsilon|}{16\pi^2} + C\log|\log\varepsilon|
\end{aligned}$$

for ε small enough, which establishes the upper bound in (6.1.11). For the lower bound we argue similarly using (6.5.4) and obtain

$$\begin{aligned}
\mathbb{E}^{\varepsilon}_{\Lambda}(\psi_x^2) &= \int_0^\infty \zeta^{\varepsilon}_{\Lambda}\left(G_{\Lambda\setminus\mathcal{A}}(x,x)\geq t\right)\,\mathrm{d}t \\
&\geq \int_0^{\frac{1}{8\pi^2}\log\left(1+d(x,\mathbb{Z}^{\mathsf{d}}\setminus\Lambda)-\frac{1}{\varepsilon}\right)-\tilde\gamma} \exp\left(-\frac{C_\alpha\varepsilon\exp(32\pi^2(t+\tilde\gamma))}{|\log\varepsilon|^{\frac12}}\right)\,\mathrm{d}t - C \\
&\geq \int_0^{\min\left(\frac{1}{8\pi^2}\log\left(1+d(x,\mathbb{Z}^{\mathsf{d}}\setminus\Lambda)-\varepsilon^{-\alpha}\right),\frac{|\log\varepsilon|}{32\pi^2}-\frac{\log|\log\varepsilon|}{64\pi^2}\right)-\tilde\gamma} \exp\left(-\frac{C_\alpha}{|\log\varepsilon|}\right)\,\mathrm{d}t - C\,.
\end{aligned}$$

The assumption $d(x,\mathbb{Z}^{\mathsf{d}}\setminus\Lambda)\geq\varepsilon^{-\alpha}+\varepsilon^{-1/4}$ ensures that the second term in the minimum here is smaller than the first, and so we see that indeed

$$\mathbb{E}^{\varepsilon}_{\Lambda}(\psi_x^2) \geq \frac{|\log\varepsilon|}{32\pi^2} - \frac{\log|\log\varepsilon|}{64\pi^2}\left(1-\frac{C_\alpha}{|\log\varepsilon|}\right)$$

$$\geq \frac{|\log \varepsilon|}{32\pi^2} - C_\alpha \log|\log \varepsilon| \,.$$

Step 2: Estimates on the covariance
As in (6.5.19) we have

$$|\mathbb{E}^\varepsilon_\Lambda(\psi_x\psi_y)| = \left|\sum_{A\subset\Lambda} \zeta^\varepsilon_\Lambda(A)\mathbb{E}_{\Lambda\setminus A}(\psi_x\psi_y)\right| \leq \sum_{A\subset\Lambda} \zeta^\varepsilon_\Lambda(A)|G_{\Lambda\setminus A}(x,y)| \,. \tag{6.5.19}$$

From Lemma 6.5.3 we know

$$\zeta^\varepsilon_\Lambda\left(|G_{\Lambda\setminus\mathcal{A}}(x,y)| \leq C_\mathsf{d}\exp\left(-\frac{|x-y|_\infty}{C_\mathsf{d}\lambda_{\mathrm{mac}}}\right)\frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/4}}{\varepsilon^{1/2}}\right) \geq 1-\exp\left(-\frac{|x-y|_\infty}{C_\mathsf{d}\lambda_{\mathrm{mac}}}\right) .$$

Abbreviate the event described here by Ω. The decomposition (6.5.19) implies

$$\begin{aligned}|\mathbb{E}^\varepsilon_\Lambda(\psi_x\psi_y)| &\leq \sum_{\substack{A\subset\Lambda\\ A\in\Omega}} C_\mathsf{d}\exp\left(-\frac{|x-y|_\infty}{C_\mathsf{d}\lambda_{\mathrm{mac}}}\right)\frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/4}}{\varepsilon^{1/2}} + \sum_{\substack{A\subset\Lambda\\ A\notin\Omega}} \zeta^\varepsilon_\Lambda(A)|G_{\Lambda\setminus A}(x,y)| \\ &\leq C_\mathsf{d}\exp\left(-\frac{|x-y|_\infty}{C_\mathsf{d}\lambda_{\mathrm{mac}}}\right)\frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/4}}{\varepsilon^{1/2}} + \sum_{\substack{A\subset\Lambda\\ A\notin\Omega}} \zeta^\varepsilon_\Lambda(A)|G_{\Lambda\setminus A}(x,y)|\end{aligned} \tag{6.5.20}$$

and so we only need to bound $|G_{\Lambda\setminus A}(x,y)|$ on the rare event Ω^c.

If $\mathsf{d}\geq 5$, we can use the bound

$$G_{\Lambda\setminus A}(x,y) \leq \max\left(G_{\Lambda\setminus A}(x,x), G_{\Lambda\setminus A}(y,y)\right) \leq C_\mathsf{d}$$

that follows from (6.2.5) and (6.5.2) to conclude from (6.5.20) that

$$\begin{aligned}|\mathbb{E}^\varepsilon_\Lambda(\psi_x\psi_y)| &\leq C_\mathsf{d}\exp\left(-\frac{|x-y|_\infty}{C_\mathsf{d}\lambda_{\mathrm{mac}}}\right)\frac{1}{\varepsilon^{1/2}} + C_\mathsf{d}\zeta^\varepsilon_\Lambda(\Omega^c) \\ &\leq C_\mathsf{d}\exp\left(-\frac{|x-y|_\infty}{C_\mathsf{d}\lambda_{\mathrm{mac}}}\right)\frac{1}{\varepsilon^{1/2}} + C_\mathsf{d}\exp\left(-\frac{|x-y|_\infty}{C_\mathsf{d}\lambda_{\mathrm{mac}}}\right) \\ &\leq \frac{C_\mathsf{d}}{\varepsilon^{1/2}}\exp\left(-\frac{|x-y|_\infty}{C_\mathsf{d}\lambda_{\mathrm{mac}}}\right)\end{aligned}$$

which implies (6.1.12).

If $\mathsf{d}=4$, the estimate is slighty more complicated, as $G_{\Lambda\setminus A}(x,y)$ is no longer uniformly bounded. Instead we use Lemma 6.5.2 to deduce a tail bound on $G_{\Lambda\setminus A}(x,y)$. Note first that if $x=y$ then (6.1.13) follows (6.1.11), and so we can assume $x\neq y$. By (6.5.2) and (6.5.3) we have for any $t\geq\tilde{\gamma}$ that

$$\begin{aligned}\zeta^\varepsilon_\Lambda\left(|G_{\Lambda\setminus\mathcal{A}}(x,y)|\geq t\right) &\leq \zeta^\varepsilon_\Lambda\left(\max\left(G_{\Lambda\setminus\mathcal{A}}(x,x), G_{\Lambda\setminus\mathcal{A}}(y,y)\right)\geq t\right) \\ &\leq \zeta^\varepsilon_\Lambda\left(G_{\Lambda\setminus\mathcal{A}}(x,x)\geq t\right) + \zeta^\varepsilon_\Lambda\left(G_{\Lambda\setminus\mathcal{A}}(y,y)\geq t\right) \\ &\leq 2\exp\left(-\frac{\varepsilon\exp(16\pi^2(t-\tilde{\gamma}))}{C|\log\varepsilon|^{\frac{1}{2}}}\right) .\end{aligned}$$

We can now use Fubini's theorem to estimate the second summand in (6.5.20) as

$$\begin{aligned}
&\sum_{\substack{A\subset\Lambda\\ A\notin\Omega}} \zeta_\Lambda^\varepsilon(A)|G_{\Lambda\setminus A}(x,y)| \\
&\quad\le \int_0^\infty \zeta_\Lambda^\varepsilon\left(|G_{\Lambda\setminus\mathcal{A}}(x,y)|\ge t, \mathcal{A}\notin\Omega\right)\,\mathrm{d}t \\
&\quad\le \int_0^\infty \min\left(\zeta_\Lambda^\varepsilon\left(|G_{\Lambda\setminus\mathcal{A}}(x,y)|\ge t\right), \zeta_\Lambda^\varepsilon(\Omega^c)\right)\,\mathrm{d}t \\
&\quad\le \int_{\tilde\gamma}^\infty \min\left(\zeta_\Lambda^\varepsilon\left(|G_{\Lambda\setminus\mathcal{A}}(x,y)|\ge t\right), \zeta_\Lambda^\varepsilon(\Omega^c)\right)\,\mathrm{d}t + \int_0^{\tilde\gamma}\zeta_\Lambda^\varepsilon(\Omega^c)\,\mathrm{d}t \\
&\quad\le \int_0^\infty \min\left(2\exp\left(-\frac{\varepsilon\exp(16\pi^2 t)}{C|\log\varepsilon|^{\frac12}}\right), \exp\left(-\frac{|x-y|_\infty}{C_{\mathrm{d}}\lambda_{\mathrm{mac}}}\right)\right)\,\mathrm{d}t + \tilde\gamma\exp\left(-\frac{|x-y|_\infty}{C\lambda_{\mathrm{mac}}}\right).
\end{aligned} \tag{6.5.21}$$

To estimate the remaining integral, note that for $a,b<1$ we have $a=\exp(-b\exp(16\pi^2 t))$ for $t=t_*:=\frac{1}{16\pi^2}(\log|\log a|+|\log b|)$ and so

$$\begin{aligned}
&\int_0^\infty \min(a,\exp(-b\exp(16\pi^2 t)))\,\mathrm{d}t \\
&\quad= \int_0^{t_*} a\,\mathrm{d}t + \int_{t_*}^\infty \exp(-b\exp(16\pi^2 t))\,\mathrm{d}t \\
&\quad= t_* a + \int_0^\infty \exp(-b\exp(16\pi^2 t_*)\exp(16\pi^2 t))\,\mathrm{d}t \\
&\quad\le t_* a + \int_0^\infty \exp(-b\exp(16\pi^2 t_*)(1+16\pi^2 t))\,\mathrm{d}t \\
&\quad= t_* a + \exp(-b\exp(16\pi^2 t_*))\int_0^\infty \exp(-16\pi^2 t b\exp(16\pi^2 t_*))\,\mathrm{d}t \\
&\quad= t_* a + \frac{\exp(-b\exp(16\pi^2 t_*))}{16\pi^2 b\exp(16\pi^2 t_*)} \\
&\quad= t_* a + \frac{a}{16\pi^2|\log a|} \\
&\quad= \frac{1}{16\pi^2}\left(a\log|\log a| + a|\log b|\frac{1}{|\log a|}\right). \\
&\quad\le Ca(\log|\log a| + |\log b|)
\end{aligned}$$

With the choices $a=\exp\left(-\frac{|x-y|_\infty}{C_{\mathrm{d}}\lambda_{\mathrm{mac}}}\right)$ and $b=\frac{\varepsilon}{C|\log\varepsilon|^{1/2}}$ we then obtain from (6.5.21) that

$$\begin{aligned}
&\sum_{\substack{A\subset\Lambda\\ A\notin\Omega}} \zeta_\Lambda^\varepsilon(A)|G_{\Lambda\setminus A}(x,y)| \\
&\quad\le C\exp\left(-\frac{|x-y|_\infty}{C\lambda_{\mathrm{mac}}}\right)\left(\log\frac{|x-y|_\infty}{C\lambda_{\mathrm{mac}}} + \left|\log\frac{\varepsilon}{C|\log\varepsilon|^{1/2}}\right| + 1\right) \\
&\quad\le C\exp\left(-\frac{|x-y|_\infty}{C\lambda_{\mathrm{mac}}}\right)\left(\log\frac{|x-y|_\infty}{\lambda_{\mathrm{mac}}} + \log\frac{|\log\varepsilon|^{1/2}}{\varepsilon} + 1\right) \\
&\quad\le C\exp\left(-\frac{|x-y|_\infty}{C\lambda_{\mathrm{mac}}}\right)(\log|x-y|_\infty - \log|\log\varepsilon| + 1)
\end{aligned}$$

where we have used that $4\log\frac{1}{\lambda_{\mathrm{mac}}} + \log\frac{|\log\varepsilon|^{1/2}}{\varepsilon} = -\log|\log\varepsilon|$. Finally we can return to (6.5.20) and obtain

$$|\mathbb{E}_\Lambda^\varepsilon(\psi_x\psi_y)|$$

$$\leq C\exp\left(-\frac{|x-y|_\infty}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right)\frac{|\log\varepsilon|^{5/4}}{\varepsilon^{1/2}}+C\exp\left(-\frac{|x-y|_\infty}{C\lambda_{\mathrm{mac}}}\right)(\log|x-y|_\infty-\log|\log\varepsilon|+1)$$
$$\leq C\left(\frac{|\log\varepsilon|^{5/4}}{\varepsilon^{1/2}}+\log|x-y|_\infty\right)\exp\left(-\frac{|x-y|_\infty}{C\lambda_{\mathrm{mac}}}\right)$$

which implies (6.1.13).

Finally, the estimates (6.1.14) and (6.1.15) are straightforward consequences of (6.1.12) and (6.1.13), respectively. □

6.5.3 Existence of the thermodynamic limit of the field

It remains to prove the existence of the thermodynamic limit of the pinned field. This is significantly more difficult than the existence of the thermodynamic limit of the set of pinned points, as we do not have correlation inequalities for the field or a random walk representation. Instead we show by hand that the exponential decay of correlations implies convergence of $\mathbb{E}^\varepsilon_\Lambda(f)$ for any bounded local f.

Proof of Theorem 6.1.2, second part. As in the proof of the first part it suffices to check that the limit $\lim_{\Lambda\nearrow\mathbb{Z}^{\mathsf{d}}}\mathbb{E}^\varepsilon_\Lambda(f)$ exists for any bounded local function $f\colon\mathbb{R}^{\mathbb{Z}^{\mathsf{d}}}\to\mathbb{R}$. Our tail estimates on $\mathbb{P}^\varepsilon_\Lambda$ easily imply boundedness of $\mathbb{E}^\varepsilon_\Lambda(f)$, so if the limit exists it is finite.

So let a local function f be given. Suppose that f only depends on the values of ψ in $Q_r(0)$ for some r. We can assume that $r\geq\hat{N}_{\mathsf{d}}$. Let $R\in\mathbb{N}$ with $R\geq 8r$. We set $\Lambda'=Q_R(0)$. Let Ω be the event described in (6.5.11). Let also $k\in\mathbb{N}$ and consider the event $\tilde{\Omega}_{0,k}$ from (6.5.5) (with $y=0$). Note that if $A\in\tilde{\Omega}_{0,k}$ we have

$$G_{\Lambda\setminus A}(0,0)\leq C_{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log k+|\log\varepsilon|)\right).$$

Similarly as for (6.5.15), we see that this implies for any $x\in Q_r(0)$

$$G_{\Lambda\setminus A}(x,x)\leq C_{\mathsf{d}}\left(1+\mathbb{1}_{\mathsf{d}=4}(\log r+\log k+|\log\varepsilon|)\right)=:X_{\mathsf{d},\varepsilon,k,r}$$

and in combination with (6.5.2) also

$$\max_{x,y\in Q_r(0)}\left|G_{\Lambda\setminus A}(x,y)\right|\leq X_{\mathsf{d},\varepsilon,k,r}\,.$$

We can now write

$$\mathbb{E}^\varepsilon_\Lambda(f)=\sum_{A\subset\Lambda}\mathbb{E}_{\Lambda\setminus A}(f)\zeta^\varepsilon_\Lambda(A)=\sum_{\substack{A\subset\Lambda\\A\in\Omega\cap\tilde{\Omega}_{0,k}}}\mathbb{E}_{\Lambda\setminus A}(f)\zeta^\varepsilon_\Lambda(A)+\sum_{\substack{A\subset\Lambda\\A\notin\Omega\cap\tilde{\Omega}_{0,k}}}\mathbb{E}_{\Lambda\setminus A}(f)\zeta^\varepsilon_\Lambda(A)\,. \tag{6.5.22}$$

The second summand here is an error term that is easy to estimate. We have

$$\begin{aligned}\left|\sum_{\substack{A\subset\Lambda\\A\notin(\Omega\cap\tilde{\Omega}_{0,k})}}\mathbb{E}_{\Lambda\setminus A}(f)\zeta^\varepsilon_\Lambda(A)\right|&\leq\|f\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})}\sum_{\substack{A\subset\Lambda\\A\notin\Omega\cap\tilde{\Omega}_{0,k}}}\zeta^\varepsilon_\Lambda(A)\\&\leq\|f\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})}\left(\zeta^\varepsilon_\Lambda(\Omega^c)+\zeta^\varepsilon_\Lambda(\tilde{\Omega}^c_{0,k})\right)\\&\leq\left(C_{\mathsf{d}}r^{\mathsf{d}}\exp\left(-\frac{R-r}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right)+\frac{1}{2^k}\right)\|f\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})}\end{aligned} \tag{6.5.23}$$

and note that the right-hand side tends to 0 as $k, R \to \infty$, uniformly in $\Lambda \supset Q_R(0)$.

Next, we begin to analyse the main term in (6.5.22), i.e. the first summand. $\mathbb{P}_{\Lambda\setminus A}$ is the law of a multivariate Gaussian measure. Thus, $\mathbb{E}_{\Lambda\setminus A}(f)$ depends only on the variances and covariances of that measure. Because of the locality of f it depends only on those variances and covariances where the sites are in $Q_r(0)$. In particular, $\mathbb{E}(f)$ is a continuous function of $\left(G_{\Lambda\setminus A}(x,y)\right)_{x,y\in Q_r(0)} \in \mathbb{R}^{Q_r(0)\times Q_r(0)}$. If we restrict it to the compact set $[-X_{\mathsf{d},\varepsilon,k,r}, X_{\mathsf{d},\varepsilon,k,r}]^{Q_r(0)\times Q_r(0)}$, it is uniformly continuous.

From Lemma 6.5.4 we know that for $A \in \Omega$

$$\begin{aligned}&\left|\left(G_{\Lambda\setminus A}(x,y)\right)_{x,y\in Q_r(0)} - \left(G_{Q_R(0)\setminus A}(x,y)\right)_{x,y\in Q_r(0)}\right|_\infty \\ &\quad\le C_{\mathsf{d}} \exp\left(-\frac{R-r}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right)\frac{1+\mathbb{1}_{\mathsf{d}=4}|\log\varepsilon|^{5/4}}{\varepsilon^{1/2}}\end{aligned}$$

and the right-hand side tends to 0 as $R \to \infty$. Moreover, we have for $A \in \tilde{\Omega}_{0,k}$ that

$$\left(G_{\Lambda\setminus A}(x,y)\right)_{x,y\in Q_r(0)} \in [-X_{\mathsf{d},\varepsilon,k,r}, X_{\mathsf{d},\varepsilon,k,r}]^{Q_r(0)\times Q_r(0)}.$$

Thus, the uniform continuity of $\mathbb{E}(f)$ implies that there is a function $\omega_{\mathsf{d},\varepsilon,f,k,r}(R)$ (independent of Λ) with $\lim_{R\to\infty}\omega_{\mathsf{d},\varepsilon,f,k,r}(R) = 0$ such that for all $A \in \Omega \cap \tilde{\Omega}_{0,k}$

$$\left|\mathbb{E}_{\Lambda\setminus A}(f) - \mathbb{E}_{Q_R(0)\setminus A}(f)\right| \le \omega_{\mathsf{d},\varepsilon,f,k,r}(R).$$

This implies for the first summand in (6.5.22) that

$$\begin{aligned}&\left|\sum_{\substack{A\subset\Lambda\\ A\in\Omega\cap\tilde{\Omega}_{0,k}}} \mathbb{E}_{\Lambda\setminus A}(f)\zeta^\varepsilon_\Lambda(A) - \sum_{A\subset\Lambda}\mathbb{E}_{Q_R(0)\setminus A}(f)\zeta^\varepsilon_\Lambda(A)\right| \\ &\le \left|\sum_{\substack{A\subset\Lambda\\ A\in\Omega\cap\tilde{\Omega}_{0,k}}} \mathbb{E}_{\Lambda\setminus A}(f)\zeta^\varepsilon_\Lambda(A) - \sum_{\substack{A\subset\Lambda\\ A\in\Omega\cap\tilde{\Omega}_{0,k}}} \mathbb{E}_{Q_R(0)\setminus A}(f)\zeta^\varepsilon_\Lambda(A)\right| \\ &\qquad + \left|\sum_{\substack{A\subset\Lambda\\ A\notin\Omega\cap\tilde{\Omega}_{0,k}}} \mathbb{P}_{Q_R(0)\setminus A}(f)\zeta^\varepsilon_\Lambda(A)\right| \\ &\le \sum_{\substack{A\subset\Lambda\\ A\in\Omega\cap\tilde{\Omega}_{0,k}}} \omega_{\mathsf{d},\varepsilon,f,k,r}(R)\zeta^\varepsilon_\Lambda(A) + \|f\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})}\left(\zeta^\varepsilon_\Lambda(\Omega^c) + \zeta^\varepsilon_\Lambda(\tilde{\Omega}^c_{0,k})\right) \\ &\le \omega_{\mathsf{d},\varepsilon,f,k,r}(R) + \left(C_{\mathsf{d}} r^{\mathsf{d}} \exp\left(-\frac{R-r}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right) + \frac{1}{2^k}\right)\|f\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})}\end{aligned} \tag{6.5.24}$$

where we have estimated the error term the same way as in (6.5.23).

We also know that

$$\begin{aligned}\sum_{A\subset\Lambda}\mathbb{E}_{Q_R(0)\setminus A}(f)\zeta^\varepsilon_\Lambda(A) &= \sum_{A'\subset Q_R(0)}\mathbb{E}_{Q_R(0)\setminus A}(f)\sum_{A''\subset\Lambda\setminus Q_R(0)}\zeta^\varepsilon_\Lambda(A'\cup A'') \\ &= \sum_{A'\subset Q_R(0)}\mathbb{E}_{Q_R(0)\setminus A'}(f)\zeta^\varepsilon_\Lambda(\mathcal{A}\cap Q_R(0) = A').\end{aligned} \tag{6.5.25}$$

Putting (6.5.22), (6.5.23), (6.5.24), (6.5.25) together, we find

$$\begin{aligned}&\left|\mathbb{E}^{\varepsilon}_{\Lambda}(f) - \sum_{A'\subset Q_R(0)} \mathbb{E}_{Q_R(0)\setminus A'}(f)\zeta^{\varepsilon}_{\Lambda}(\mathcal{A}\cap Q_R(0) = A')\right| \\ &\quad \leq \omega_{\mathsf{d},\varepsilon,f,k,r}(R) + 2\left(C_{\mathsf{d}} r^{\mathsf{d}} \exp\left(-\frac{R-r}{C_{\mathsf{d}}\lambda_{\mathrm{mac}}}\right) + \frac{1}{2^k}\right)\|f\|_{L^\infty(\mathbb{Z}^{\mathsf{d}})}\,.\end{aligned} \tag{6.5.26}$$

We now want to take the limits $\Lambda \nearrow \mathbb{Z}^{\mathsf{d}}$, $R \to \infty$, $k \to \infty$ in that order. For that purpose, note that the weak convergence of $\zeta^{\varepsilon}_{\Lambda}$ to ζ^{ε} implies that $\lim_{\Lambda\nearrow\mathbb{Z}^{\mathsf{d}}} \zeta^{\varepsilon}_{\Lambda}(\mathcal{A}\cap Q_R(0) = A') = \zeta^{\varepsilon}(\mathcal{A}\cap Q_R(0) = A')$, and so (6.5.26) implies

$$\limsup_{k\to\infty}\limsup_{R\to\infty}\limsup_{\Lambda\nearrow\mathbb{Z}^{\mathsf{d}}}\left|\mathbb{E}^{\varepsilon}_{\Lambda}(f) - \sum_{A'\subset Q_R(0)} \mathbb{E}_{Q_R(0)\setminus A'}(f)\zeta^{\varepsilon}(\mathcal{A}\cap Q_R(0) = A')\right| = 0\,.$$

From this we see that

$$\lim_{\Lambda\nearrow\mathbb{Z}^{\mathsf{d}}} \mathbb{E}^{\varepsilon}_{\Lambda}(f) = \lim_{R\to\infty} \sum_{A'\subset Q_R(0)} \mathbb{E}_{Q_R(0)\setminus A'}(f)\zeta^{\varepsilon}(\mathcal{A}\cap Q_R(0) = A')$$

and that in particular both limits exist. This is what we wanted to show.

The translation invariance follows as in the proof of the first part of the theorem.

□

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