Hyperbolic and dispersive singular stochastic PDEs

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Immanuel Zachhuber
aus
Berlin

Bonn, 4. Juni 2020

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der
Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Massimiliano Gubinelli
2. Gutachter Prof. Herbert Koch

Tag der Promotion: 21.09.2020
Erscheinungsjahr: 2021

Deutsche Zusammenfassung

Diese Dissertation befasst sich mit dem Thema der singulären dispersiven stochastischen partiellen Differentialgleichungen. Genauer gesagt, wird in Kapitel 2, das dem publizierten Paper [49] im Wesentlichen folgt, zuerst der sogannte "Anderson Hamiltonian" auf sowohl dem 2 als auch dem drei dimensionalen Torus definiert. Formal ist dieser gegeben durch

$$A := \Delta + \xi,$$

wobei ξ das räumliche weiße Rauschen ist. Die Hauptschwierigkeit ist die schlechte Regularität von ξ, nämlich unter -1 in 2 Dimensionen und unter $-\frac{3}{2}$ in 3. Um diesen Operator trotzdem als selbstadjungierten und von oben beschränkten Operator auf L^2 definieren zu können, nutzen wir einen *parakontrollierten* Ansatz (eingeführt in [45] und zuerst auf ein solches Problem angewandt in [3]) zusammen mit einer *Renormierung*, in diesem Kontext heißt das wir müssen eine "unendliche Konstante" abziehen, um das Definitionsgebiet des Operators A explizit zu beschreiben als einen Funktionenraum der dicht in L^2 ist aber keine glatten Funktionen enthält. Danach nutzen wir Spektralkalkül und Energiemethoden um für nichtlineare Schrödingergleichungen wie

$$i\partial_t u - Au = -\, u|u|^2 \quad (S)$$

oder Wellengleichungen

$$\partial_t^2 u - Au = -\, u|u|^2 \quad (W)$$

mit geeigneten Anfangsdaten Wohlgestelltheit zu zeigen.
In Kapitel 3(das eng dem Preprint [83] folgt) werden sogenannte Strichartzabschätzungen für den Anderson Hamiltonian bewiesen, nämlich

$$\|e^{-itA}u\|_{L^4_{t;[0,1]}L^4_{\mathbb{T}^2}} \lesssim \|u\|_{\mathcal{H}^{\varepsilon}_{\mathbb{T}^2}}$$
$$\|e^{-itA}u\|_{L^{\frac{10}{3}}_{t:[0,1]}L^{\frac{10}{3}}_{\mathbb{T}^3}} \lesssim \|u\|_{\mathcal{H}^{\frac{1}{2}+\varepsilon}_{\mathbb{T}^3}},$$

wobei das zweidimensionale Resultat so stark wie die klassiche Abschätzung [11] für den Laplaceoperator ist, in drei Dimensionen verliert man im Vergleich eine halbe Ableitung. Die Methodik basiert einerseits auf der parakontrollierten Beschreibung des Operators aus dem vorigen Kapitel und andererseits auf [17], dessen semiklassische Strategie Strichartzabschätzungen zu zeigen hier anwendbar ist. Danach nutzen wir diese Abschätzungen um bessere Resultate für (S) zu beweisen.
In Kapitel 4, dessen Inhalt noch nirgendwo publiziert oder zur Verfügung gestellt wurde, wird der variationelle Ansatz für Wellengleichungen aus [72] angepasst um energiesuperkritische Versionen von (W) zu lösen, das heißt in drei Dimensionen mit Potenzen größer als 5 in der Nichtlinearität. Die Methode liefert globale Existenz von Lösungen (keine Wohlgestelltheit) und basiert auf der Idee, dass die Lösung einer nichtlinearen Wellengleichung formal der Grenzwert von einer Folge von Minimierern von konvexen Raum-Zeit Funktionalen ist.

Acknowledgements

First and foremost I wish to thank my advisor Massimiliano Gubinelli who was both liberal with his support and supportively liberal. In addition, I thank him for being an eclectic cornucopia of knowledge which would frequently be paired with sage advice.

My thanks also go out to Herbert Koch. Firstly because none of this would have to come to pass if he had not assigned me a Master's thesis about Rough Paths which initially piqued my interest in the field and secondly because of the enlightening discussions we had during my PhD.

At this stage I want to express my gratitude to the other members of the group, Francesco, Luigi, Nikolay, Mattia and Lucio (a.k.a. "the twins") which has luckily been growing steadily after I spent the early days of my doctorate quasi-nomadically. Not only did they out up with my antics, including *that* concert and Silvio (special shout-out to my office mate and longstanding partner-in-crime Nikolay), we also shared countless discussions ranging from the ridiculous to the sublime.

Another individual I want to single out is Angelo with whom I traversed the disreputable establishments of Bonn; a fellow connoisseur of cacophonous concerts; a purveyor of fine whiskies and trashy 60s pop culture. The former flowed liberally while consuming the latter along with countless shared interests ranging from Bukowski and Burroughs to Zappa and Zorn.

Further I would like to acknowledge Chiara, who has been more than a little supportive–helping to keep and me caffeinated and (somewhat) sane– particularly in these past trying months.

Local legends I want to pay homage to: Jan, ferocious warrior and salt-of-the-earth man of action rolled into one; Martin, Svengali of Scotch and proprietor of Zone, truly the crème de la crème of dive bars; Andreas for singing in the most infamous local band and resisting the siren song of international fame.

Let me also mention my fellow "applied" mathematicians, especially the card sharks who frequently invited me to join their after lunch Schafkopf/Skat rounds. Moreover, I owe a debt of gratitude to my friends in the "non-applied" Analysis group with whom I share many fond memories like the times we were in Kopp together.

There are countless other people which I would and should thank in addition. In fact, I have written truly marvelous acknowledgments but this page is unfortunately too narrow to contain them all.

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

The topic of *singular hyperbolic and dispersive stochastic PDE* lies, as the name suggests, at the confluence of the topics of singular stochastic PDEs and dispersive PDEs. It is a fairly recent with the papers [47] and [30] perhaps marking the starting point for this particular type of problem, namely that of analysing dispersive PDEs with an additive or multiplicative stochastic forcing term, which is so irregular that it becomes classically ill-posed. In the years since, there have been a multitude of new results about stochastic nonlinear wave(SNLW) [46], [80] [68], stochastic nonlinear Schrödinger(SNLS) [35], [67] to name but a few. The main challenge of these equations is the dual difficulty of having severely irregular noise terms(spatial or space-time white noise is usually the most interesting) on the one hand and the lack of regularising effects coming from the linear part of the equation on the other hand. In some sense, the study of singular dispersive SPDEs is philosophically related to the study of dispersive PDEs with low-regularity inital data pioneered by Bourgain [11] and perhaps, more intimately, the study of dispersive PDEs with randomised(and rough) initial data, see [13] [12], which has seen a considerable growth in recent years following [19] and [20].

The field of singular stochastic PDEs (SSPDEs) has seen a meteoric rise in the past couple of years chiefly due to the emergence of the theories of *Regularity Structures* of Hairer [54] and *Paracontrolled Distributions* due to Gubinelli, Perkowski, and Imkeller [45] (which was later extended by Bailleul and Bernicot [8]). Alternative approaches include the renormalisation group approach of Kupiainen [61] and the one by Otto and Weber [69] which is in some sense a hybrid.

In several ways this development can be traced back to the theory *Rough Paths* due to Lyons [64](see also [36] for a pedagogical introduction), which provides a *pathwise* setting of solving SDEs. The central idea, which also pervades the later theories, is that if one solves an SDE like

$$Y(t) = a + \int_0^t f(Y(s))dX(s), \tag{1.1.1}$$

where a is the initial condition, f is a sufficiently regular function and the driving signal X is a Brownian motion, the Itô solution map

$$\Phi : (a, X) \to Y$$

is *not* continuous in X with respect to the $\mathcal{C}^\alpha$ norm with $\alpha < \frac{1}{2}$. However, if one endows the path X with more information to an object $\mathbb{X} = (X, \mathbb{X}^2)$, called a *Rough Path*, where $\mathbb{X}^2$ contains some "second-order properties" of the path, then it turns out that the solution map

$$\Psi : (a, (X, \mathbb{X}^2)) \to Y$$

is now in fact continuous with respect to the Rough Path topology. Thereafter, one can prove that Brownian motion, even fractional Brownian Motion with Hurst parameter in the correct range, has an almost sure lift to the Rough Path space. This allows to solve an SDE like (1.1.1) in a *pathwise* sense s.t. the solution depends continuously on the driving signal. There are of course some caveats, namely that the lift $X \to \mathbb{X}$ is not unique, which is related to the question of Itô vs Stratonovich integration, see [36]. The field of Rough Paths is still an active research area, in particular since it has been shown to have some real world applications, see e.g. [24] for some applications to machine learning.

An important development to this theory was made by Gubinelli [42] in the form of *Controlled Paths*, which is a reformulation as well a generalisation of Lyons' theory of Rough Paths. The main difference is instead of the SDE (1.1.1), one considers a more general equation like

$$Y(t) = a + \int_0^t Z(s) dX(s), \tag{1.1.2}$$

where X is again the first component of a (fixed) Rough Path $(X, \mathbb{X}^2)$, the difference being that now Z is just some path that should *locally look like* Y in the sense that

$$(Z(t) - Z(s)) - Z'(s)(Y(t) - Y(s)) = o(|t - s|) \text{ for all } s < t, \tag{1.1.3}$$

where Z' is sometimes called the *Gubinelli derivative* and the couple (Z, Z') is called a controlled path w.r.t. Y. Clearly $Z = f(Y), Z' = f'(Y)$ for good enough f satisfies (1.1.3) when Y is not too irregular. Then one solves (1.1.2) for Y in the space of paths controlled by the Rough Path $(X, \mathbb{X}^2)$, usually denoted by $\mathcal{D}_\mathbb{X}$. This is done by rewriting (1.1.2) as a finite difference equation and invoking the *sewing lemma*. A few advantages of this approach are firstly that the space $\mathcal{D}_\mathbb{X}$ of Controlled Paths is actually a Banach space, whereas the space of Rough Paths is not even a linear space; secondly, the fact that Z does not need to be a function of Y allows to treat other interesting cases, for example if there is some non-local dependence; thirdly, this theory was used, employing an infinite dimensional Controlled Path approach, to solve some early examples of singular SPDEs like Burgers [52], KPZ [53] which were instrumental in the further development. See [36] for an accessible introduction to the theory.

The paradigm of having, on the one hand, the underlying Rough Path space which contains the driving signal $\mathbb{X}$ with some higher-order information and, on the other hand, the controlled path space $\mathcal{D}_\mathbb{X}$ as "dual" objects is central to both the theory of Regularity Structures and Paracontrolled Distributions. In Regularity Structures the Rough Path is replaced by a *model* and the Controlled Paths are replaced by *modelled distributions*. In this theory all the "real" objects, i.e. the driving noise and the solution of the equation are replaced by local descriptions thereof, e.g. a function $f : \mathbb{R}^d \to \mathbb{R}$ is replaced by an object $F : \mathbb{R}^d \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$ s.t.

$$F_x(\delta_x^\varepsilon) \to f(x) \quad \text{with } x \in \mathbb{R}^d, (\delta_x^\varepsilon) \subset \mathcal{S}(\mathbb{R}^d) \text{ s.t. } \delta_x^\varepsilon \to \delta_x$$

as well as a continuity condition of F in the first coordinate. Of course the choice of such an F is highly non-unique, but it is usually dictated by the singular operations one wants to perform. Subsequently one has to define the analogues of the necessary operations on such objects, i.e. multiplication, convolution etc. One then obtains the solution via fixed point in the space of local descriptions which has to be defined up to high enough precision. Lastly, in order to re-obtain the "real" object from its local description one has the *Reconstruction Theorem* which does precisely that and can be seen as a generalisation of the sewing lemma of the Controlled Path theory. See [54] or [55] for details.

The theory of Paracontrolled Distributions [45], on the other hand, is based on *paraproducts* as introduced by Bony in [10] which have found myriad applications in nonlinear PDEs, see e.g. [5]. The observation is that when considering a rough ODE like (1.1.1) or similarly a PDE like

$$\partial_t u - \Delta u = u \cdot \xi, \tag{1.1.4}$$

where ξ has regularity worse than -1; The obstruction to solving it is the ill-definedness of the product which appears. This is because of the well-known result about the product between a function and a distribution being extendable continuously to the situation where the sum of the regularities is strictly positive. This should be thought of as being analogous to the condition for the Young integral to exist namely the maps

$$\mathcal{C}^\alpha \times \mathcal{C}^\beta \ni (u,v) \to u \cdot v \quad \text{and} \quad \mathcal{C}^\gamma \times \mathcal{C}^\delta \ni (X,Y) \to \int_0^\cdot X(s)dY(s) \tag{1.1.5}$$

extend continuously to the cases where $\alpha + \beta > 0$ and $\gamma + \delta > 1$ respectively. See Chapter 1.2 for the definition of the Hölder-Besov spaces we use here. The approach of Paracontrolled Distributions employs the tool of paraproducts to isolate the worst part of the product so as to treat it appropriately.
It is well known that any distribution can be decomposed via Littlewood-Paley decomposition into an infinite sum of smooth functions which have almost disjoint Fourier support

$$u = \sum_{i \geqslant -1} \Delta_i u \quad \Delta_{-1} \approx \mathcal{F}^{-1}\mathbb{I}_{B(0,1)}\mathcal{F}, \quad \Delta_i \approx \mathcal{F}^{-1}\mathbb{I}_{B(0,2^{i+1})\setminus B(0,2^i)}\mathcal{F},$$

with smooth cut-off functions instead of indicator functions to be precise. Then one formally splits the product in the following way

$$u \cdot v = \sum_{i,j \geqslant -1} \Delta_i u \Delta_j v = \sum_{i \gtrsim j} \Delta_i u \Delta_j v + \sum_{i \sim j} \Delta_i u \Delta_j v + \sum_{j \gtrsim i} \Delta_i u \Delta_j v =: v \prec u + v \circ u + u \prec v,$$

where $v \prec u$ is called *paraproduct* and is dominated by the high frequencies of u, whereas $u \circ v$ is called *resonant product* and contains the interaction between the same frequencies of u and v. The point is that these objects satisfy the following bounds

$$\begin{aligned} \|u \prec v\|_{\mathcal{C}^{\alpha+0\wedge\beta}} &\lesssim \|u\|_{\mathcal{C}^\beta}\|v\|_{\mathcal{C}^\alpha} \text{ for } \alpha,\beta \in \mathbb{R} \\ \|u \circ v\|_{\mathcal{C}^{\alpha+\beta}} &\lesssim \|u\|_{\mathcal{C}^\beta}\|v\|_{\mathcal{C}^\alpha} \text{ for } \alpha,\beta \in \mathbb{R} : \alpha+\beta > 0, \end{aligned}$$

which immediately gives us the aforementioned result about the product in (1.1.5) and tells us that the paraproduct is **always** defined and contains the most irregular part of the product. The resonant product, on the other hand, only makes sense when the sum of the regularities is strictly positive but then it is regular.

Thus, if one decomposes the product in that way for, say, (1.1.4) one gets

$$\partial_t u - \Delta u = u \prec \xi + u \circ \xi + u \succ \xi,$$

which has two fundamental problems: firstly the paraproduct $u \prec \xi$ has very bad regularity (in fact just worse than -1), which means that u is at best $1-\varepsilon$ for $\varepsilon > 0$. The second problem is that if u indeed has regularity $1-\varepsilon$ then the resonant product $u \circ \xi$ is not well-defined. The idea is then to consider u in such a space that simultaneously mitigates the effect of the irregularity of ξ and allows the resonant product with ξ to be defined as a continuous operation. The ansatz one considers is

$$u = u \prec (\partial_t - \Delta)^{-1}\xi + u^\sharp, \tag{1.1.6}$$

where $u^\sharp$ is a smoother remainder term. This is motivated by the approximate identity

$$(\partial_t - \Delta)(u \prec v) \approx u \prec (\partial_t - \Delta)v,$$

which is strictly speaking only true if we modify the paraproduct in time, see [45]. This says that if u satisfies (1.1.6) then the remainder $u^\sharp$ satisfies the equation

$$\partial_t u^\sharp - \Delta u^\sharp \approx (u \prec (\partial_t - \Delta)^{-1}\xi) \circ \xi + u^\sharp \circ \xi + u \succ \xi,$$

which implies that $u^\sharp$ has better regularity–as it should– given that we are able to make sense of the term

$$(u \prec (\partial_t - \Delta)^{-1}\xi) \circ \xi.$$

This term is then finally treated by using the commutator lemma, see [45] which says

$$(f \prec g) \circ h \approx f(g \circ h) \text{ for } f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta, h \in \mathcal{C}^\gamma \alpha \in (0,1), \alpha+\beta+\gamma > 0, \beta+\gamma < 0,$$

which says we are able to close the equation if we can make sense of $((\partial_t - \Delta)^{-1}\xi) \circ \xi$. Since this is not a continuous function of ξ in the range we are interested in, this means we have to include this object in our data. The couple $(\xi, ((\partial_t - \Delta)^{-1}\xi) \circ \xi)$ is precisely the analogue of the Rough Path and the model in the Controlled Paths and Regularity Structures theories respectively.

The ansatz (1.1.6) is called *paracontrolled ansatz*, we say that u is paracontrolled by $(\partial_t - \Delta)^{-1}\xi$, which is to say that at high frequencies u behaves like $(\partial_t - \Delta)^{-1}\xi$. More generally, one considers

$$u = u' \prec X + u^\sharp,$$

where u' plays the role of the Gubinelli derivative, see (1.1.3). More abstractly, u lives in a space which is parametrised by the couple $(u', u^\sharp)$ which is analogous to the space of controlled paths and modelled distributions and one can see (1.1.6) as a "change of unknown" which is continuous w.r.t. the driving noise and the "new variable" is $u^\sharp$ which is a posteriori the correct one to solve (1.1.4) since its equation is classically solvable if one is given the additional information about the noise.

The last issue which appears when one tries to solve (1.1.4) in the relevant case of ξ being the spatial white noise on the 2- or 3-dimensional torus is that the object $((\partial_t - \Delta)^{-1}\xi) \circ \xi$ actually does not exist a.s. but instead only if one "subtracts an infinite constant". This phenomenon is known as *renormalisation* and in this situation means that for a sequence of smooth functions $\xi_\varepsilon \to \xi$ there exist diverging constants $c_\varepsilon \to \infty$ s.t.

$$((\partial_t - \Delta)^{-1}\xi_\varepsilon) \circ \xi_\varepsilon - c_\varepsilon \to \Xi_2 \text{ in } \mathcal{C}^{-\delta} \text{ for } \delta > 0,$$

for a limit Ξ_2 which is independent of the approximating sequence. This leads us to modifying (1.1.4) to read

$$\partial_t u - \Delta u = u \cdot \xi - \infty u, \tag{1.1.7}$$

which of course has no intrinsic meaning. This is further evidence for the fact that (1.1.7) is the "wrong" way to write the SPDE and the change of variables to $u^\sharp$ leads to the "correct" formulation of (1.1.7), which is in addition continuous w.r.t. the enhanced noise by construction. More precisely, on the level of regularised noise one considers the shifted equation

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = u_\varepsilon \cdot \xi_\varepsilon - c_\varepsilon u_\varepsilon,$$

for which we make the "change of unknown"

$$u^\sharp_\varepsilon = u_\varepsilon - u_\varepsilon \prec (\partial_t - \Delta)^{-1}\xi_\varepsilon$$

which leads for an equation for $u^\sharp_\varepsilon$ of the form

$$\partial_t u^\sharp_\varepsilon - \Delta u^\sharp_\varepsilon \approx u_\varepsilon((\partial_t - \Delta)^{-1}\xi_\varepsilon \circ \xi_\varepsilon - c_\varepsilon) + u^\sharp_\varepsilon \circ \xi_\varepsilon + u_\varepsilon \succ \xi_\varepsilon, \tag{1.1.8}$$

where upon inspection one realises that the right hand side is continuous in $\mathcal{C}^{-\delta}$ as $\varepsilon \to 0$. The last point is that one needs to "invert" the paracontrolled expansion so as to express u in terms of $u^\sharp$. This is discussed in some detail in Chapter 2 and for simplicity we say that $u = \Gamma u^\sharp$ and $u_\varepsilon = \Gamma_\varepsilon u^\sharp_\varepsilon$ for invertible transformations $\Gamma, \Gamma_\varepsilon$ for which $\Gamma_\varepsilon \to \Gamma$ in an appropriate sense. Thus we can pass to the limit in (1.1.8) and get that $u^\sharp$ satisfies

$$\partial_t u^\sharp - \Delta u^\sharp \approx \Gamma u^\sharp \Xi_2 + u^\sharp \circ \xi + \Gamma u^\sharp \succ \xi, \tag{1.1.9}$$

which is classically solvable by fixed point assuming that Γ has good properties. Now, the logical way to make sense of a solution u to (1.1.7) is to say that $u = \Gamma u^\sharp$, where $u^\sharp$ is the solution to (1.1.9). This is made rigorous in [49] or Chapter 2.

A wide-ranging generalisation to this "first-order" Paracontrolled Calculus was developed by Bailleul and Bernicot in [8]. Their theory allows for higher order expansions, meaning that it is possible to treat a larger class of noises and nonlinearities, as well as being applicable on general manifolds. This theory is not required for the topics treated in this thesis, although one could employ it for an alternative construction for the domain of the 3 dimensional Anderson Hamiltonian from Chapter 3.3.2.
In the years since their introduction, these theories have come to fruition, now being applicable to very large classes of locally subcritical – meaning that locally the solution is a perturbation of the solution of a linear problem – semi-linear SSPDEs. Prominent examples include the dynamical Φ^4_3 [54], [22], [65], [44], KPZ [48], Sine Gordon [57] and others including [74], [9], [43] etc.

Moreover, there have been wide-ranging extensions including quasilinear SSPDEs [37], [40], [7], SSPDEs on Manifolds [6], with boundary conditions [39] etc. The drawback of all these results is that they can not be applied to SSPDEs which are not parabolic (or elliptic); There have, however, been some notable developments in this direction with the papers of Gubinelli, Koch, and Oh [47] [46] , on the 2- and 3-dimensional stochastic wave equation with additive space-time white noise being perhaps the most prominent. In particular their work on the 3-dimensional stochastic wave equation with quadratic nonlinearity – which introduces the concept of *paracontrolled operators* and requires some subtle computations with oscillatory integrals – indicates that some fundamentally new ideas are needed to deal with these types of equations. In fact, solving the 3-dimensional stochastic wave equation with cubic nonlinearity (which is akin to Φ^4_3) is an intriguing open problem.

The field of dispersive PDEs has a long and illustrious history the recounting of which is beyond the scope of this thesis; See e.g. [76] and [23] for good textbooks on the topic. Let us instead focus on the major differences with respect to parabolic/elliptic PDEs and how to nonetheless obtain results, as this is the crux of the thesis. The primary difference is, of course, that one does not have smoothing properties of the linear propagator (or less smoothing in the case of the wave equation). Even the fact that one can solve the linear Schrödinger equation

$$\begin{aligned} i\partial_t u - \Delta u &= 0 \\ u(0) &= u_0 \in L^2 \end{aligned}$$

is due to the fact that we can define the *unitary group* $e^{-it\Delta}$ which is a bounded operator on L^2 that is strongly continuous in $t \in \mathbb{R}$, giving a solution via $u(t) = e^{-it\Delta}u_0 \in C_t L^2$. As an important generalisation of this, one gets the exact same type result, if one has instead

$$\begin{aligned} i\partial_t u - Au &= 0 \\ u(0) &= u_0 \in L^2, \end{aligned} \tag{1.1.10}$$

for any A which is self-adjoint on L^2 as a result of Stone's theorem, see [71, Theorem VIII.7]. This will be of importance in Chapter 2, since we are able to define the operator

$$\Delta + \xi,$$

where ξ is a very irregular potential, as a self-adjoint operator. This then directly allows us to solve the linear equation (1.1.10) in L^2, i.e. in a space which is independent of how irregular ξ is (within reason).

Another crucial concept is that of *conserved quantities.* The linear Schrödinger equation conserves e.g. the quantities

$$m(t) := \int |u(t,x)|^2 dx \quad \text{and} \quad E(t) := \int |\nabla u(t,x)|^2 dx,$$

i.e. the *mass* and the *energy*, meaning that they are constant in time. These are particularly useful as they provide uniform in time bounds on the $\mathcal{H}^1$ norm of the solution in terms of the $\mathcal{H}^1$ norm of the initial data. Moreover, there are analogous quantities which are conserved for more complicated PDEs, as long as their nonlinearities are of the correct form. A prominent example which will be of interest to us is the so called cubic nonlinear Schrödinger equation(NLS)

$$\begin{aligned} i\partial_t u - \Delta u &= -u|u|^2 \\ u(0) &= u_0, \end{aligned} \tag{1.1.11}$$

where we have chosen to only consider the *defocussing* nonlinearity(the focussing case being the one where the nonlinearity has the opposite sign). This PDE also has a conserved mass and an energy, given by

$$m(t) := \int |u(t,x)|^2 dx \quad \text{and} \quad E(t) := \frac{1}{2}\int |\nabla u(t,x)|^2 + \frac{1}{4}|u(t,x)|^4 dx,$$

respectively. Broadly speaking, when one has conserved quantities one can extend a local in time solution to a global in time one. Another immediate generalisation is that if one considers the linear PDE (1.1.10) for a general self-adjoint, negative definite operator A one gets the conserved mass and an energy given by

$$m(t) := \int |u(t,x)|^2 dx \quad \text{and} \quad E(t) := \frac{1}{2}(u(t,\cdot), (-A)u(t,\cdot)),$$

and for the corresponding cubic-$A-$NLS

$$\begin{aligned} i\partial_t u - Au &= -u|u|^2 \\ u(0) &= u_0, \end{aligned} \tag{1.1.12}$$

one gets (formally at least)

$$m(t) := \int |u(t,x)|^2 dx \quad \text{and} \quad E(t) := \frac{1}{2}(u(t,\cdot), (-A)u(t,\cdot)) + \frac{1}{4}|u(t,x)|^4 dx$$

which are also conserved in time. Thus one gets a uniform in time bound for the "square root"-norm of A i.e.

$$\|v\|_{\mathcal{D}(\sqrt{-A})} := \|v\|_{L^2} + \sqrt{(v, -Av)},$$

which is of course just the $\mathcal{H}^1$ norm in the case of the Laplacian. Let us mention here that in many cases one does not have an exact conserved quantity, but only an *almost conserved quantity*, meaning broadly that its time derivative is not equal to zero but rather "lower order", leading to a bound using Gronwall-type arguments. This is then often still enough to deduce global well-posedness. The construction of such quantities is the objective of the celebrated *I-method*, see [26], [25] etc.

The aforementioned points do not significantly depend on the domain of definition for the function (self-adjointness of Δ notwithstanding) and hold in very general settings with few caveats. However, one is often interested in more subtle questions of solvability, e.g. well-posedness in low-regularity spaces (local and global), behaviour at criticality, ill-posedness/blow-up etc. One property of the (linear) Schrödinger equation on the Euclidean space $\mathbb{R}^d$ is *dispersion*, which is usually motivated by saying that there are cancellations between "wave packets" of different frequencies. Due to the infinite speed of propagation this is a priori something which is particular to the Euclidean space, in other domains/manifolds there can be non-trivial self-interactions and/or boundary effects. Rather than working with this somewhat nebulous concept, we are instead interested in so-called *Strichartz estimates,* which give a quantitative result due to dispersion. Roughly speaking this type of result says that solutions of the linear eqaution satisfy additional integrability in space if one gives up integrability in time. If we consider, for simplicity, the linear Schrödinger equation on $\mathbb{R}^2$, one gets the bound

$$\|e^{-it\Delta}u_0\|_{L^4_{t,\mathbb{R}}L^4_{\mathbb{R}^2}} \lesssim \|u_0\|_{L^2},$$

in other words, one gains integrability in space while giving up integrability in time, seen by comparing it to the bound

$$\|e^{-it\Delta}u_0\|_{L^\infty_{t,\mathbb{R}}L^2_{\mathbb{R}^2}} \lesssim \|u_0\|_{L^2}.$$

A related result is the *inhomogeneous* Strichartz estimate

$$\left\|\int_{s<t} e^{-i(t-s)\Delta}F(s)ds\right\|_{L^4_{t,\mathbb{R}}L^4_{\mathbb{R}^2}} \lesssim \|F\|_{L^{\frac{4}{3}}_{\mathbb{R}}L^{\frac{4}{3}}_{\mathbb{R}^2}},$$

which is useful when trying to solve e.g. the cubic NLS. Of course analogous results are true for general exponents in general dimensions, see [76]. The chief reason why these bounds are (comparatively) easy to obtain on $\mathbb{R}^d$ is the presence of a *dispersive estimate*

$$\|e^{-it\Delta}u_0\|_{L^\infty_{\mathbb{R}^d}} \lesssim |t|^{-\frac{d}{2}}\|u_0\|_{L^1_{\mathbb{R}^d}},$$

which is simply not true in most other situations. Therefore extensions of these types of results to other situations, such as tori, see [11], [14], [60], or manifolds [17] are more involved. We indisciminately call a bound a "Strichartz estimate", if it is of the form

$$\|e^{-it\Delta}u_0\|_{L^p_{t\in I}L^q} \lesssim \|u_0\|_{\mathcal{H}^s}, \tag{1.1.13}$$

for some exponents $2 \leqslant p, q \leqslant \infty$ and $s < d\left(\frac{1}{2} - \frac{1}{q}\right)$, i.e. we allow a loss of derivates so long as the result is strictly better than what one obtains from the Sobolev embedding. Also, we allow for the bound to hold only on a finite time interval $I \subset \mathbb{R}$, rather than for all times. In Chapter 3 we will furthermore investigate a bound like (1.1.13) for operators other than the Laplacian. Moreover we explain in more detail how such bounds lead to local well-posedness in low-regularity regimes. If one wants to further decrease the regularity one is considering, even more subtle tools such as multilinear estimates and Fourier restriction spaces are required see e.g. [11] or [82].

Let us also mention at this point that there has been a philosophically somewhat related development concerning low-regularity well-posedness of dispersive PDE with randomised initial data. Its study was initiated by Bourgain in the 90s [12], [13] but received more widespread attention after a series of papers by Burq-Tzvetkov [19, 20] about random data supercritical wave equations, see also the notes [81] and [78] for the analogous result for Schrödinger equations. The idea, broadly, is that while a (possibly supercritical) PDE might be ill-posed in a space, it might still be well-posed for "almost every" element of that space meaning that one constructs a measure with respect to which the set of initial data for which one can solve the PDE has full measure. Roughly speaking, one gains integrability from the fact that the random data, in the form of, say, Gaussian coefficients, have a lot of integrability in the probability space, leading to improved "randomised" Strichartz estimates. This is related to the construction of invariant and quasi-invariant measures, which are also active research areas, see e.g. [79] and [51] and the references therein.

1.1.1 Summary of results

We now give a concise overview of the main results of the thesis. In some cases we will sacrifice rigour in favour of readability.

Chapter 2

The aim of this chapter, which very closely follows [49], are twofold: The construction of the *continuum Anderson Hamiltonian* on the 2- and 3-dimensional torus via *Paracontrolled Distributions* and proving well-posedness results for semilinear evolution equations(Schrödinger and wave) whose linear part is given by the Anderson Hamiltonian.

The continuum Anderson Hamiltonian on the torus $\mathbb{T}^d$ is the operator formally given by

$$A = \Delta + \xi, \tag{1.1.14}$$

where ξ is *spatial white noise*, which is the Gaussian random field with covariance

$$\mathbb{E}(\xi(f)\xi(g)) = (f,g)_{L^2} \text{ for } f,g \in C^\infty(\mathbb{T}^3),$$

which has regularity $\xi \in \mathcal{C}^{-\frac{d}{2}-}$(we use this notation to mean $\in \mathcal{C}^{-\frac{d}{2}-\varepsilon}$ for any $\varepsilon > 0$). See Chapter 1.2 for the definition of the *Besov-Hölder* spaces we employ here.

One sees that A can be defined as an operator from $\mathcal{H}^k \to \mathcal{H}^{-\frac{d}{2}-}$ for some large enough k, keeping in mind that the product between an $\mathcal{H}^r$ and a $\mathcal{C}^\alpha$ function can be defined unconditionally as long as $\alpha + r > 0$. This is, however, not so useful and we aim here to define it as a self-adjoint operator on L^2. This was first achieved by Allez and Chouk in [3] on $\mathbb{T}^2$, after suitably *renormalising* A, which in this case amounts to "subtracting an infinite constant". They are able to construct an explicit domain for this operator using the theory of Paracontrolled Distributions, introduced in [45]. We look for $u \in L^2$ s.t.

$$\Delta u + \xi \cdot u \in L^2,$$

and one observes that both terms can not *simultaneously* be in L^2, but rather that there has to be a cancellation between them. In order to quantify the idea that the worst contribution of the Laplacian should cancel the worst contribution one is led to an ansatz for u like

$$u = u_\xi + \mathcal{H}^2,$$

where u_ξ is chosen in such a way that Δu_ξ should cancel the worst part of $u \cdot \xi$. This is done by using a *paracontrolled ansatz* for u and u_ξ can be chosen(upto higher correction terms) as the paraproduct

$$u_\xi = u \prec (-\Delta)^{-1}\xi + \mathcal{H}^2,$$

which means that Δu_ξ cancels $u \prec \xi$, which is the most irregular contribution of the product $u \cdot \xi$. In order to actually have $Au \in L^2$, one needs to make a slightly more subtle ansatz. Moreover, one needs to control a second-order object related to ξ which makes the renormalisation necessary (the resonant product $\xi \circ (1-\Delta)^{-1}\xi$ can only be made sense of after formally subtracting a constant). In Chapter 3.3.1 we recall the construction of Allez-Chouk in two dimensions and prove a couple of novel results about the domain and the form-domain(domain of the square root) of A as well as some related functional inequalities, self-adjointness, and norm resolvent convergence of smooth approximations of A to A.

In Chapter 3.3.2 we make a similar construction on $\mathbb{T}^3$. This was the first time this operator was constructed; It was independently studied by Labbé using Regularity Structures in [62]. The results we obtain are quite analogous to the ones we obtain in the two dimensional case, despite some increased technicality. Since the noise in this case is in $\mathcal{C}^{-\frac{3}{2}-}$(whereas in two dimensions it was $\mathcal{C}^{-1-}$) it turns out that a simple paracontrolled ansatz as in two dimensions is insufficient. The remedy is to introduce an exponential transform inspired by [30] and [56], which removes the worst terms and creates some more complicated but more regular terms. The relevant computation associated to this transform is

$$\Delta(e^W v) + (e^W v)\cdot\xi = e^W(\Delta v + 2\nabla W\cdot\nabla v + |\nabla W|^2 v + (\Delta W)v + v\xi),$$

and one chooses W in order to cancel the most irregular terms in the bracket, i.e. ξ and the worst part of $|\nabla W|^2$. After choosing W in such a way, we perform a paracontrolled analysis on the level of v, which will now be controlled by some higher order expressions of ξ, some of which need to be renormalised.

Despite requiring this two-step construction, we are still able to prove virtually all the results we did in the two dimensional setting, i.e. an explicit description/parametrisation of the domain and the form domain, functional inequalitities, self-adjointness, and norm resolvent convergence.

Using these results, in Chapter 2.3, we turn to well-posedness questions of multiplicative stochastic PDEs like

$$\begin{aligned} i\partial_t u - \Delta u &= u\cdot\xi - u|u|^2 \text{on } \mathbb{T}^d \\ u(0) &= u_0, \end{aligned} \tag{1.1.15}$$

and

$$\begin{aligned} \partial_t^2 u - \Delta u &= u\cdot\xi - u|u|^2 \text{ on } \mathbb{T}^d \\ (u,\partial_t u)(0) &= (u_0,u_1), \end{aligned} \tag{1.1.16}$$

which we recast as

$$\begin{aligned} i\partial_t u - Hu &= -u|u|^2 \text{on } \mathbb{T}^d \\ u(0) &= u_0, \end{aligned} \tag{1.1.17}$$

and

$$\begin{aligned} \partial_t^2 u - Hu &= -u|u|^2 \text{on } \mathbb{T}^d \\ (u,\partial_t u)(0) &= (u_0,u_1), \end{aligned} \tag{1.1.18}$$

respectively, where H denotes the Anderson Hamiltonian shifted by a constant making it uniformly negative. The natural spaces in which to study (1.1.17) and (1.1.18) are $\mathcal{D}(H)$ (the replacement of $\mathcal{H}^2$, so called *strong solutions*) and $\mathcal{D}\left(\sqrt{-H}\right)$(the replacement of $\mathcal{H}^1$, so called *energy solutions*). The functional inequalities

$$\begin{aligned}\|u\|_{L^p} &\lesssim \|u\|_{\mathcal{D}(\sqrt{-H})} \text{ for all } p \in [2,\infty),\\ \|u\|_{L^\infty} &\lesssim \|u\|_{\mathcal{D}(H)}\end{aligned}$$

in two dimensions, together with a *Brezis-Gallouet*-type inequality(see Lemma 2.2.31) allow us to prove global well-posedness(GWP) of (1.1.17) in the space $\mathcal{D}(H)$, i.e. a solution

$$u \in C([0,T];\mathcal{D}(H)) \cap C^1([0,T];L^2) \quad T > 0,$$

for $u_0 \in \mathcal{D}(H)$. Using this, we can prove global *existence* (not well-posedness) of *energy solutions*, i.e.

$$u \in C\left([0,T];\mathcal{D}\left(\sqrt{-H}\right)\right) T > 0$$

for $u_0 \in \mathcal{D}\left(\sqrt{-H}\right)$, the name coming from the conserved (positive) energy

$$E(u) = -\frac{1}{2}(u, Hu) + \frac{1}{4}\int |u|^4 dx.$$

Similarly, we get GWP for (1.1.18) for both $(u, \partial_t u) \in C_t\mathcal{D}(H) \times C_t\mathcal{D}\left(\sqrt{-H}\right)$ and $(u, \partial_t u) \in C_t\mathcal{D}\left(\sqrt{-H}\right) \times C_t L^2$. In this case the conserved energy is given by

$$E(u) = \frac{1}{2}\int |\partial_t u|^2 dx - \frac{1}{2}(u, Hu) + \frac{1}{4}\int |u|^4 dx.$$

On $\mathbb{T}^3$, we get the same results for (1.1.18), however with a smaller range of powers in the nonlinearity. For (1.1.17) we get *local* well-posedness in the space $\mathcal{D}(H)$.

Chapter 3

In this chapter, which is based on [83], we pursue further the study of the equation (1.1.17), this time with some slightly more involved techniques compared to those used the previous chapter, where we essentially used only the self-adjointness/negativity of the operator H, together with the L^p bounds and the conservation of energy. Here we try to establish *Strichartz estimates* for the Anderson Hamiltonian; the bounds we are able to achieve are

$$\|e^{-itH^\sharp} v\|_{L^4_{[0,1]}L^4_{\mathbb{T}^2}} \lesssim \|v\|_{\mathcal{H}^\varepsilon_{\mathbb{T}^2}} \text{for } \varepsilon > 0, \tag{1.1.19}$$

and

$$\|e^{-itH^\sharp} v\|_{L^{\frac{10}{3}}_{[0,1]}L^{\frac{10}{3}}_{\mathbb{T}^3}} \lesssim \|v\|_{\mathcal{H}^{\frac{1}{2}+\varepsilon}_{\mathbb{T}^3}} \text{for } \varepsilon > 0, \tag{1.1.20}$$

respectively where $H^\sharp$ denotes a suitable transformation of H related to the paracontrolled expansion. Essentially the idea is that while H acts on paracontrolled functions(which is a complicated space), while $H^\sharp$ acts on the (smooth) remainder of the paracontrolled functions(which is a simple space). This is possible using the map

$$\Gamma : \text{“smooth remainder”} \to \text{“paracontrolled function with smooth remainder”}$$

introduced in [49], see Chapter 2.2, which is invertible and parametrises the domain of H. Thus we are able to use $H^\sharp$, which turns out to be the Laplacian to highest order, instead of H which corresponds to a "change of unknown" . This change of variables together with the "semiclassical" approach of Burq, Gerard and Tzvetkov [17] to proving Strichartz estimates–based on the idea that for an initial datum localised in frequency, upto a short time depending on that frequency one is morally on the whole space– allow us to prove Strichartz estimates in a perturbative way for $H^\sharp$ using that $H^\sharp$ being close to Δ also gives that $e^{-itH^\sharp}$ is close to $e^{-it\Delta}$ in some sense.

Note that our result on $\mathbb{T}^2$ is the same as the one for the Laplacian from [11], the one on $\mathbb{T}^3$ however loses half a derivative due to the worse regularity of the noise in this setting.

Furthermore, we employ (1.1.19) to show low-regularity local well-posedness of (1.1.17) in $\mathcal{H}^s$ for $s \in \left(\frac{1}{2}, 1\right)$ as well as global well-posedness in the energy space, solving a problem which had remained open in [49] where only global existence was proved.

Chapter 4

This chapter is concerned with solving the PDE

$$\begin{aligned} \partial_t^2 u - Hu &= -u|u|^{p-2} \text{on } \mathbb{T}^3 \\ (u, \partial_t u)(0) &= (u_0, u_1), \end{aligned} \tag{1.1.21}$$

for general exponents $p > 2$, including the so-called *energy supercritical* ones. This nomenclature refers to the fact that in the energy

$$E(u) = \frac{1}{2}\int |\partial_t u|^2 dx - \frac{1}{2}(u, Hu) + \frac{1}{4}\int |u|^p dx,$$

the "potential energy" $\frac{1}{4}\int |u|^p dx$ is no longer controlled by the "kinetic energy" $\frac{1}{2}\int |\partial_t u|^2 dx - \frac{1}{2}(u, Hu)$ via Sobolev embedding for $p > 6$. One does not expect well-posedness in this case, we get global in time existence of solutions in the energy space(we have to assume *both* the initial data u_0 and the initial velocity u_0 to live in the energy space and have L^p integrability, a somewhat unnatural assumption) using the variational approach to wave equations due to Serra and Tilli [72, 73].

We use the exponential transformation introduced in Chapter 3.3.2 to transform (1.1.21) from a PDE on the abstract "energy space" w.r.t. H to a PDE in $\mathcal{H}^1$. In this case the formulation is considerably simpler than the "full" transformed operator $H^\sharp$ we consider in Chapter 3 as we do not need to perform the paracontrolled expansion. Due to this relatively simple transformation, we are almost able to apply the results from [72, 73] directly to this setting; We adapt their method which requires fairly modest modifications.

The variational approach to wave equations due to Serra and Tilli originates from a conjecture of de Giorgi [28], which was that the minimisers of the space-time functional

$$F_\varepsilon(u) := \int_0^\infty e^{-\frac{t}{\varepsilon}} \int \frac{1}{2}|\partial_t^2 u|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{p}|u|^p dx dt,$$

which exist and are unique because of its convexity, should converge (in some sense) to a solution to the nonlinear wave equation

$$\begin{aligned}\partial_t^2 u - \Delta u &= -u|u|^{p-2} \\ (u, \partial_t u)(0) &= (u_0, u_1).\end{aligned}$$

The resolution of this problem was achieved in [72] and considerably generalised in [73] and [77]. As the analysis happens entirely in the time variable, making a modification in only the space variable(as we consider only spatial noise here) barely interferes with the method.

1.2 Preliminaries

We collect some relevant background material. In some cases there will be some overlap with the material in the appendices of the papers.

1.2.1 Paracontrolled calculus/Littlewood-Paley theory

We introduce the concept of Bony's paraproducts and how they behave with respect to Sobolev, Hölder and general Besov spaces; we also collect some results about products of distributions. We work primarily on the $d-$dimensional torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ for $d = 2, 3$, however all the following analysis works equally well in any dimension as well as on the Euclidean space $\mathbb{R}^d$. For any $f \in \mathcal{S}'(\mathbb{T}^d)$, i.e. tempered distributions on $\mathbb{T}^d$, the Fourier transform of f is denoted by $\hat{f} : \mathbb{Z}^d \to \mathbb{C}$ (or $\mathcal{F}f$) and is defined for $k \in \mathbb{Z}^d$ by

$$\hat{f}(k) := \langle f, \exp(2\pi i \langle k, \cdot \rangle) = \int_{\mathbb{T}^d} f(x) \exp(-2\pi i \langle k, x \rangle) dx.$$

Recall that for any $f \in L^2(\mathbb{T}^d)$ and a.e. $x \in \mathbb{T}^d$, we have

$$f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \exp(2\pi i \langle k, x \rangle). \tag{1.2.1}$$

We define the *Sobolev space* $\mathcal{H}^\alpha(\mathbb{T}^d)$ with index $\alpha \in \mathbb{R}$ as

$$\mathcal{H}^\alpha(\mathbb{T}^d) := \{ f \in \mathcal{S}'(\mathbb{T}^d) : \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^\alpha \, |\hat{f}(k)|^2 < +\infty \} \, .$$

Before introducing Besov spaces, we recall the definition of Littlewood-Paley blocks. We denote by χ and ρ two nonnegative smooth and compactly supported radial functions $\mathbb{R}^d \to \mathbb{R}$ such that

1. The support of χ is contained in a ball $\{x \in \mathbb{R}^d : |x| \le R\}$ and the support of ρ is contained in an annulus $\{x \in \mathbb{R}^d : a \le |x| \le b\}$;
2. For all $\xi \in \mathbb{R}^d$, $\chi(\xi) + \sum_{j \ge 0} \rho(2^{-j}\xi) = 1$;
3. For $j \ge 1$, $\chi\rho(2^{-j}\cdot) \equiv 0$ and $\rho(2^{-i}\cdot)\rho(2^{-j}\cdot) \equiv 0$ for $|i - j| \ge 1$.

For the existence of such functions, see e.g. Proposition 2.10 in [5]. The Littlewood-Paley blocks $(\Delta_j)_{j\geq -1}$ acting on $f \in \mathcal{S}'(\mathbb{T}^d)$ are defined by

$$\mathcal{F}(\Delta_{-1} f) = \chi \hat{f} \quad \text{and for} \quad j \geq 0, \quad \mathcal{F}(\Delta_j f) = \rho(2^{-j}.)\hat{f}.$$

Note that, for $f \in \mathcal{S}'(\mathbb{T}^d)$, the Littlewood-Paley blocks $(\Delta_j f)_{j\geq -1}$ define smooth functions, as their Fourier transforms have compact supports. We also set, for $f \in \mathcal{S}'$ and $j \geq 0$,

$$S_j f := \sum_{i=-1}^{j-1} \Delta_i f$$

and note that $S_j f$ converges in the sense of distributions to f as $j \to \infty$.
We can now introduce the *Besov space* with parameters $p, q \in [1, \infty), \alpha \in \mathbb{R}$ whose definition is given by

$$B^{\alpha}_{p,q}(\mathbb{T}^d) := \left\{ u \in \mathcal{S}'(\mathbb{T}^d); \quad \|u\|_{B^{\alpha}_{p,q}} = \left(\sum_{j\geq -1} 2^{jq\alpha} \|\Delta_j u\|^q_{L^p} \right)^{1/q} < +\infty \right\}. \tag{1.2.2}$$

We also define the *Besov-Hölder* spaces

$$\mathcal{C}^{\alpha} := B^{\alpha}_{\infty,\infty}$$

which are naturally equipped with the norm $\|f\|_{\mathcal{C}^{\alpha}} := \|f\|_{B^{\alpha}_{\infty,\infty}} = \sup_{j\geq -1} 2^{j\alpha} \|\Delta_j f\|_{L^{\infty}}$. For $\alpha \in (0, 1)$ these spaces coincide with the classical Hölder spaces. See the books [32] and [5] for more information about these types of spaces.
We can formally decompose the product fg of two distributions f and g as

$$fg = f \prec g + f \circ g + f \succ g$$

where

$$f \prec g := \sum_{j\geq -1} S_{j-1} f \Delta_j g \qquad \text{and} \qquad f \succ g := \sum_{j\geq -1} S_{j-1} g \Delta_j f$$

are usually referred to as the *paraproducts* whereas

$$f \circ g := \sum_{j\geq -1} \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g \tag{1.2.3}$$

is called the *resonant product*.
Moreover we will frequently write $f \preccurlyeq g := f \prec g + f \circ g$ and $f \succcurlyeq g := f \succ g + f \circ g$ as short-hand notations.
The paraproduct terms are always well defined irrespective of regularities. The resonant product is a priori only well defined if the sum of regularities is strictly greater than zero. This is reminiscent of the well known fact that one can not multiply distributions in general. The following result makes those comments precise and gives simple but extremely vital estimates for paraproducts.

Lemma 1.2.1 (cf. Theorem 3.17 [66]). *Let $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ and $p, p_1, p_2, q \in [1, \infty]$ be such that*

$$\alpha_1 \neq 0 \quad \alpha = (\alpha_1 \wedge 0) + \alpha_2 \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then we have the bound

$$\|f \prec g\|_{B^{\alpha}_{p,q}} \lesssim \|f\|_{B^{\alpha_1}_{p_1,\infty}} \|g\|_{B^{\alpha_2}_{p_2,q}}$$

and in the case where $\alpha_1 + \alpha_2 > 0$ we have the bound

$$\|f \circ g\|_{B^{\alpha_1+\alpha_2}_{p,q}} \lesssim \|f\|_{B^{\alpha_1}_{p_1,\infty}} \|g\|_{B^{\alpha_2}_{p_2,q}}.$$

Primarily the cases $p = p_2 = q = 2$ $p_1 = \infty$(Sobolev times Hölder) and $p = p_1 = p_2 = q = \infty$ (Hölder times Hölder) are of interest to us. In particular, one immediately gets conditions for the "full" product to be well-defined as a continuous bilinear operator.
The next result is Bernstein's inequality, which gives quantitative bounds for the differentiability and integrability for functions with compact support in frequency. Note that for functions which are spectrally supported in annuli one has two sided bounds but for functions spectrally localised in balls only one-sided bounds.

Lemma 1.2.2 (Bernstein's inequality, [45]). *Let $\mathcal{A}$ be an annulus and $\mathcal{B}$ be a ball in $\mathbb{R}^d$. For any $k \in \mathbb{N}, \lambda > 0$,and $1 \leq p \leq q \leq \infty$ we have*

1. *if $u \in L^p(\mathbb{R}^d)$ is such that* $\operatorname{supp}(\mathcal{F}u) \subset \lambda\mathcal{B}$ *then*

$$\max_{\mu \in \mathbb{N}^d : |\mu| = k} \|\partial^\mu u\|_{L^q} \lesssim_k \lambda^{k + d\left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^p}$$

2. *if $u \in L^p(\mathbb{R}^d)$is such that* $\operatorname{supp}(\mathcal{F}u) \subset \lambda\mathcal{A}$ *then*

$$\lambda^k \|u\|_{L^p} \lesssim_k \max_{\mu \in \mathbb{N}^d : |\mu| = k} \|\partial^\mu u\|_{L^p}.$$

Another useful result is the following embedding result between Besov spaces, which holds on either $\mathbb{T}^d$ or $\mathbb{R}^d$.

Lemma 1.2.3 (Besov embedding, [45]). *Let $\alpha < \beta \in \mathbb{R}$ and $p \geqslant r \in [1, \infty]$ be such that*

$$\beta = \alpha + d\left(\frac{1}{r} - \frac{1}{p}\right),$$

then we have the following bound for $q \in [1, \infty]$

$$\|f\|_{B^{\alpha}_{p,q}} \lesssim \|f\|_{B^{\beta}_{r,q}}.$$

We also cite the following result, which can be seen as a "first-order Taylor expansion", saying that–up to a smoother remainder– the nonlinear composition of a smooth function F with a function f of limited regularity is given by the paraproduct $F'(f) \prec f$.

Proposition 1.2.4 (Paralinearisation, [48])**.** *Let* $\alpha \in (0,1)$ *and* $F \in C^2$*. Then there exists a locally bounded map* $R_F : \mathcal{C}^\alpha \to \mathcal{C}^{2\alpha}$ *such that*

$$F(f) = F'(f) \prec f + R_F(f) \ \text{ for all } \ f \in \mathcal{C}^\alpha.$$

The next result, which is quite vital to the theory of Paracontrolled Distributions, is a sort of "commutator" between the paraproduct and the resonant product.

Proposition 1.2.5 (Commutator lemma, [45], [3])**.** *Given* $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}$ *such that* $\beta + \gamma < 0$ *and* $\alpha + \beta + \gamma > 0$, *there exists a trilinear operator* C *with the following bound*

$$\|C(f,g,h)\|_{\mathcal{H}^{\alpha+\beta+\gamma}} \lesssim \|f\|_{\mathcal{H}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{C}^\gamma}$$

in either the case

a) $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$ *and* $h \in \mathcal{C}^\gamma$ *or*

b) $f \in \mathcal{H}^\alpha$, $g \in \mathcal{C}^\beta$ *and* $h \in \mathcal{C}^\gamma$.

The restriction of C *to smooth functions satisfies*

$$C(f,g,h) = (f \prec g) \circ h - f(g \circ h).$$

Chapter 2

Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions

2.1 Introduction

The aim of this chapter is to study the following random Cauchy problems

$$i\partial_t u = Hu - u|u|^2, \quad u(0) = u_0 \tag{2.1.1}$$
$$\partial_t^2 u = Hu - u^3, \quad (u, \partial_t u)|_{t=0} = (u_0, u_1) \tag{2.1.2}$$

on the d-dimensional torus $\mathbb{T}^d$ with $d = 2, 3$. Here H is formally the Anderson Hamiltonian $H = \Delta + \xi$, where ξ is a space white noise and Δ the Laplacian with periodic boundary conditions.

The presence of white noise makes this kind of problem not well-posed in classical function spaces. Indeed, it is well known that white noise is almost surely only a distribution of regularity $-d/2 - \varepsilon$ in Hölder-Besov spaces, where $\varepsilon > 0$. One difficulty that arises from this is the fact that the above equations have to be properly renormalized by formally subtracting an infinite constant in order to obtain well defined limits.

In the parabolic setting there is a, by now, well developed theory of such *singular SPDEs*, thanks to Hairer's invention of the theory of Regularity Structures [54] and the parallel development of the paracontrolled approach [45] by Gubinelli, Imkeller and Perkowski. The first results for non parabolic evolution equations have been obtained in [30], where the authors solve the linear and the cubic nonlinear (with a range of powers) Schrödinger equations with multiplicative noise on $\mathbb{T}^2$ by first applying a transform inspired by [56] and then using mass and energy conservation along with certain interpolation arguments. The wave equations in $d = 2$ with polynomial non-linearities and additive space-time white noise have been considered in [47]. The main difficulty is that the absence of parabolic regularisation makes the control of the non-linear terms involving the singular noise contributions non-trivial.

Here we exploit the insights of [3] in order to identify an appropriately renormalized version of H as a self-adjoint operator on $L^2(\mathbb{T}^d)$ and use the related spectral decomposition to give a meaning to the above equations as abstract evolution equations in Hilbert space. Our first contribution is then the study of the Anderson Hamiltonian on $\mathbb{T}^3$ and the derivation of some additional results when $d = 2$, for example the characterisation of the "form domain" of the operator (the domain of the operator $\sqrt{-H}$) and some related functional inequalities which are needed in the abstract treatment of the evolution equations.

For the sake of completeness, and also to illustrate the proof strategy in the $d = 3$ case, we pursue a complete treatment of the $d = 2$ case showing the self-adjointness of the Hamiltonian and the convergence of suitable regularised operators in norm resolvent sense. Norm resolvent convergence is used in the second part to "prepare"suitable initial conditions adapted to prove convergence of approximations. We mention also the proof of a version of the classical Brezis-Gallouet inequality [15] for the Anderson Hamiltonian in $d = 2$. For $d = 3$ we prove that the Anderson Hamiltonian satisfies an inequality which is analogous to the classical Agmon's inequality, see Lemma 2.2.55. These functional inequalities are instrumental in the second part of this work in order to control the non-linear terms of the evolution equations.

An interesting byproduct of our approach is an estimate which expresses the fact that the paraproduct is "almost" adjoint to the resonant product whose definitions we recall in the Appendix. This implies in particular that the energy norm with respect to the Anderson Hamiltonian can be estimated from below in a precise way and allows us to characterise (see Proposition 2.2.23) both the domain and the form domain of H by using certain Sobolev norms.

Section 2.3 is concerned with the solution of the above equations with different regularities of the initial conditions and with the proof of convergence of solutions of approximate equations where the noise has been regularised to the singular limit. While the general methodology is the same adopted in [30], namely the use of conservation laws and functional inequalities to control the non-linear term, one of the main contributions of this work is to clarify the role of the spectral theory of the Anderson Hamiltonian and of relative function spaces in the apriori control of the solutions and in the analysis of the non-linear terms. This simplifies and unifies the analysis of the $d = 2$ and $d = 3$ cases.

Thereafter, having all the necessary Sobolev and L^p-estimates at our disposal along with an analogue of the Brezis-Gallouet inequality and proper approximation tools, in Section 2.3 we move on to the study of the nonlinear Schrödinger and wave equations for the Anderson Hamiltonian (properly shifted for positivity) in dimensions 2 and 3. One important point is that, after having performed the analysis of the Anderson Hamiltonian using Paracontrolled Distributions(which involves dealing with the stochastic terms), we are in a position to address the PDE problems by using classical techniques, which makes the approach somewhat more transparent.

To recap, we study the well-posedness of the PDEs (2.1.1) and (2.1.2) (with a range of powers for the nonlinearity) with operator domain and finite energy data.

We also work out the convergence of the solutions of regularised equations, obtained by suitable approximations of the initial data and the Gaussian white noise, to the solutions of the above

PDEs:

$$i\partial_t u_\varepsilon = H_\varepsilon u_\varepsilon - u_\varepsilon |u_\varepsilon|^2, u_\varepsilon(0) = u_0^\varepsilon. \tag{2.1.3}$$

$$\partial_t^2 u_\varepsilon = H_\varepsilon u_\varepsilon - u_\varepsilon^3 \text{ on } \mathbb{T}^d, \quad (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (u_0^\varepsilon, u_1^\varepsilon). \tag{2.1.4}$$

In Theorem 2.3.5, we establish the well-posedness of (2.1.1) with operator domain data in $d = 2$. This is achieved, in part, using our version of the Brezis–Gallouet inequality for the Anderson Hamiltonian. In Theorem 2.3.9 we show that the solutions to the regularised equations, namely to (2.1.3), converge to that of equation (2.1.1). Observe that, in this context, establishing this convergence is important as the domain of the Anderson Hamiltonian is contained in $\mathcal{H}^{1-}$ whereas the domain of the approximations lie in $\mathcal{H}^2$. So there is a drop in smoothness that needs to be addressed carefully. Extensions of some of these results to $d = 3$ and the focusing case are possible as we prove an analogue of Agmon's inequality in Lemma 2.2.55 to replace the Brezis-Gallouet inequality, see Remark 2.3.10.

Since we characterise the energy domain for the Anderson Hamiltonian in Lemma 2.2.23, we can also make sense of *energy solutions* for the NLS (2.1.1). In fact, in Theorem 2.3.11, we show the existence of such solutions. Observe that in this case we were not able to show uniqueness, in fact one needs *Strichartz estimates* to get this, see Chapter 3. Furthermore, as in the domain case, we show in Corollary 2.3.15 the convergence of the regularised solutions.

Being able to characterise the energy domain for the Anderson Hamiltonian both in dimensions 2 and 3 enables us to also treat nonlinear stochastic wave equations in either dimension. In Section 2.3.3, we prove some results regarding the well-posedness of (2.1.2) in 2 and 3 dimensions. In Theorem 2.3.17, we obtain the well-posedness with initial data/velocity in the domain/energy domain. Similarly to the Schrödinger case we also show convergence of regularised solutions in Theorem 2.3.19. We then conclude by stating Theorem 2.3.20, which details the well-posedness for initial data/velocity in the energy domain and L^2 for (2.1.2) and whose proof follows from our earlier considerations in the same section. By our version of Agmon's inequality and similar methods, certain extensions to the different power nonlinearities are possible, see Remark 2.3.21 for a discussion on this.

Although we solve the PDEs with an Anderson Hamiltonian which is properly shifted to result in a positive operator, this does not cause any weaker results. As known, this shift simply causes a phase shift (i.e. multiplication by e^{iCt} for some constant C) in the NLS case, which one can simply rotate back to the solution of the original equation. In the wave case one can simply add a linear term to the equation to undo it.

In the sequel, we use $\mathcal{H}$ for Sobolev spaces, L for L^p-spaces and $\mathcal{C}$ for the Besov-Hölder spaces. As we work either on $\mathbb{T}^2$ or $\mathbb{T}^3$ and it is very clear in what setting we consider throughout the paper, we drop the domain parameter i.e. for $\mathcal{H}^2(\mathbb{T}^3)$ we simply write $\mathcal{H}^2$; We denote the Gaussian white noise by ξ and enhanced noise by Ξ (see Definition 2.2.3 and Theorem 2.2.33).

We reserve the letter A for the Anderson Hamiltonian and we use the letter H to denote the operator shifted by a specific constant K_Ξ, namely $H := A - K_\Xi$. We denote by C_Ξ the constants depending on certain norms (which will be clear from the context) of the (enhanced) noise. This constant may change value from line to line. We use the notation $\mathcal{X}$ for the enhanced noise space both in $d = 2$ and $d = 3$. We will use the phrase "form domain"(or equivalently energy domain) to refer to the domain of the operator $\sqrt{-H}$ throughout.

After the completion of the present work we became aware of recent work of C. Labbé [62] where he constructs the Anderson Hamiltonian in $d \leq 3$ with Dirichlet boundary conditions using regularity structures and produces some results about the law of its eigenvalues.

2.2 The Anderson Hamiltonian in two and three dimensions

We collect some concepts and definitions that we will use throughout this section. Firstly we recall the definition of Gaussian white noise on $\mathbb{T}^d$.

To get an intuitive description, let $\hat{\xi}(k)$ be i.i.d. centred complex Gaussian random variables with $\hat{\xi}(k) = \bar{\hat{\xi}}(-k)$ and covariance

$$\mathbb{E}(\hat{\xi}(k)\bar{\hat{\xi}}(l)) = \delta_{k,l}.$$

Formally the Gaussian white noise on the torus can be thought as the following random series

$$\xi(x) = \sum_{k\in\Lambda} \hat{\xi}(k) e^{2\pi i k\cdot x},$$

where in this section we will respectively take Λ to be $\mathbb{Z}^2$ and $\mathbb{Z}^3\backslash\{0\}$. That is, in the 3d case we simply take out the zero mode for ease of computations.

We also define the regularised spatial white noise as

$$\xi_\varepsilon(x) = \sum_{k\in\Lambda} m(\varepsilon k) e^{2\pi i k\cdot x}\hat{\xi}(k), \tag{2.2.1}$$

where m is a smooth radial function on $\mathbb{R}\backslash\{0\}$ with compact support such that

$$\lim_{x\to 0} m(x) = 1.$$

We also recall the Anderson Hamiltonian, which is formally the following operator

$$A = \Delta + \xi \tag{2.2.2}$$

where ξ is the Gaussian white noise. As we have articulated in the introduction, this operator can not be naïvely defined in $L^2(\mathbb{T}^{2,3})$ because of the low Hölder regularity of ξ. The Besov-Hölder regularity of Gaussian white noise on $\mathbb{T}^d$ is $-\frac{d}{2} - \delta$, that is $\xi \in \mathcal{C}^{-\frac{d}{2}-\delta}$ almost surely, for any positive $\delta > 0$ [45].

Therefore, we will consider a renormalisation of this operator in the context of paracontrolled distributions which is formally

$$A = \Delta + \xi - \infty \tag{2.2.3}$$

and to which we will give meaning as a suitable limit $\varepsilon \to 0$ of the regularised Hamiltonians

$$A_\varepsilon = \Delta + \xi_\varepsilon - c_\varepsilon, \tag{2.2.4}$$

for suitably diverging constants c_ε.

Accordingly, in this section, we define the Anderson Hamiltonian and introduce suitable regularisations in the setting of paracontrolled distributions in two and three dimensional torus, respectively in the following subsections. Namely, we construct a suitable (dense) domain for the operator and then show closedness, symmetry, self-adjointness and norm resolvent convergence (of the regularised Hamiltonians). At the end of both 2d and 3d cases, we prove certain functional inequalities which we will use in the PDE part of the paper, namely in Section 2.3.

2.2.1 The two dimensional case

In this part, we work on the 2d torus. We follow the same line of thought as in [3] with important modifications. In [3] the authors worked in the 2d case but our modifications will enable us to use similar proofs in Section 2.2.2, namely for the 3d case, and also obtain certain functional inequalities such as the Brezis-Gallouet inequality for the Anderson Hamiltonian. In this section, for paraproducts we use the notations "$\prec$" and "$\succ$" and for the resonant product we use "$\circ$"; please see the appendix for precise definitions of the function spaces and concepts from harmonic analysis that will be used throughout this section.

Enhanced noise, the domain and the Γ-map

In order to introduce the paracontrolled ansatz, which will enable us to define the domain of the operator, we need the following definition.

Definition 2.2.1. *For $\alpha \in \mathbb{R}$, we define $\mathcal{E}^\alpha := \mathcal{C}^\alpha \times \mathcal{C}^{2\alpha+2}$ and $\mathcal{X}^\alpha$ as the closure of the set $\{(\eta, \eta \circ (1-\Delta)^{-1}\eta + c) : \eta \in \mathcal{C}^\infty(\mathbb{T}^2), c \in \mathbb{R}\}$ w.r.t. the $\mathcal{E}^\alpha$ topology, where $\mathcal{C}^\alpha = B^\alpha_{\infty\infty}$ denotes the Besov-Hölder space.*

We refer to $\mathcal{X}^\alpha$ as the space of "enhanced noise". In some sense one needs to lift the noise into a larger space which encodes some higher-order properties. It is desirable that this space contains both smooth approximations to the singular noise as well as the noise itself, whose lift should be independent of the approximation. This is the content of the following result, which was proved in [3, Theorem 5.1].

Theorem 2.2.2. *For any $\alpha < -1$ we have*

$$\Xi^\varepsilon := (\xi_\varepsilon, \xi_\varepsilon \circ (1-\Delta)^{-1}\xi_\varepsilon - c_\varepsilon) \to \Xi = (\Xi_1, \Xi_2) \in \mathcal{X}^\alpha, \tag{2.2.5}$$

where the convergence holds as $\varepsilon \to 0$ in $L^p(\Omega; \mathcal{E}^\alpha)$ for all $p > 1$ and almost surely in $\mathcal{E}^\alpha$. Moreover, the limit is independent of the mollifier and $\Xi_1 = \xi$.

By this result, one can see that

$$\|\xi\|_{\mathcal{C}^\alpha}, \quad \|\Xi_2\|_{\mathcal{C}^{2\alpha+2}}, \quad \|(1-\Delta)^{-1}\xi\|_{\mathcal{C}^{\alpha+2}} < \infty \text{ a.s.}$$

by Schauder estimates.

Before we introduce the domain of the operator, we give some motivation for the enhanced noise and the ansatz, which will appear in the following definition. Neglecting the invertibility issues(of the Laplacian) let us formally write the Anderson Hamiltonian (applied to a domain element) as

$$\Delta u + \xi u - \infty u.$$

We want to define this expression in L^2, where "∞u" will be absorbed to the enhanced noise, as we will demonstrate below. If we write the product $u\xi$ in its paraproduct components we have

$$u\xi = u \prec \xi + u \succ \xi + u \circ \xi.$$

Recall that the paraproducts are always well defined and have the regularity of their "high" parts. Namely, in terms of regularity, $f \prec g$ behaves like g where f only acts as a "modulation" to g in large scales, as reflected in the paraproduct estimates (see Proposition 2.3.22). But as can already be seen, the term $u \prec \xi$ is problematic: it has the same (low) regularity of the white noise, hence so does ξu.

To get some intuition for the (expected) regularity of u for a domain element, we consider u to solve the following resolvent equation for some $f \in L^2$

$$\Delta u + \xi u - \infty u = f.$$

We rewrite it as

$$u = (-\Delta)^{-1}(-f + \xi u - \infty u)$$

which suggests that $u \in \mathcal{H}^{-1-\delta+2} = \mathcal{H}^{1-\delta}$, where $-1-\delta$ comes from the Besov-Hölder regularity of the white noise.

A reasonable first ansatz for such a u is

$$u = u \prec (-\Delta)^{-1}\xi + u^\sharp$$

for some $u^\sharp \in \mathcal{H}^2$, since the paraproduct term removes the worst contribution of the product $u\xi$. Then we have (using the notation $\approx$ to mean equal up to regular terms for clarity)

$$\begin{aligned}
&\Delta u + \xi u - \infty u \\
&= \Delta(u \prec (-\Delta)^{-1}\xi + u^\sharp) + u \prec \xi + u \succ \xi + u \circ \xi - \infty u \\
&\approx \Delta u^\sharp - u \prec \xi + u \prec \xi + u \succ \xi + u \circ \xi - \infty u \\
&= \Delta u^\sharp - u \succ \xi + \xi \circ (u \prec (-\Delta)^{-1}\xi + u^\sharp) - \infty u \\
&\approx \Delta u^\sharp + \xi \circ u^\sharp + (\xi \circ (-\Delta)^{-1}\xi - \infty)u + C(u, (-\Delta)^{-1}\xi, \xi)
\end{aligned}$$

where $C(u, (-\Delta)^{-1}\xi, \xi) := \xi \circ (u \prec (-\Delta)^{-1}\xi) - (\xi \circ (-\Delta)^{-1}\xi)u$ is the commutator from Proposition 2.3.23. As seen, the first term in the ansatz basically lead to the cancellation of the most irregular term $u \prec \xi$. In addition, the singular term $\xi \circ (u \prec (-\Delta)^{-1}\xi)$ is dealt with by using the commutator C, which has better regularity than the terms separately. We also nicely see the appearance of the second component of the enhanced noise $(\xi \circ (-\Delta)^{-1}\xi - \infty)$ in the last step, which was given in Definition 2.2.1 and Theorem 2.2.2.

So far, although we got rid of the bad term and defined the term $u \circ \xi$, this basic ansatz alone does not define the operator in L^2, as an inspection of regularities will reveal according to

Proposition 2.3.22. But the rest of the reasoning is similar: one identifies the singular (worse than L^2) terms and includes these terms composed with the Green's operator in the ansatz (as we denoted by $B_\Xi(u)$ below in (2.2.6)) which induces further cancellations, similar to the case we presented above. This way the operator can finally be defined in L^2. In conclusion, the ansatz and the commutator lead to the cancellation of the most singular term and the isolation of the noise terms which then can be renormalised and defined in the "enhanced noise" space.

We can now recall the following definition which describes the domain of the Anderson Hamiltonian, first introduced in [3].

Definition 2.2.3. *Assume $-\frac{4}{3} < \alpha < -1$ and $-\frac{\alpha}{2} < \gamma \leq \alpha + 2$. Then we define the space of functions paracontrolled by the enhanced noise Ξ as follows*

$$\mathcal{D}^\gamma_\Xi := \{u \in \mathcal{H}^\gamma \text{ s.t. } u = u \prec X + B_\Xi(u) + u^\sharp, \text{ for } u^\sharp \in \mathcal{H}^2\} \tag{2.2.6}$$

where $X = (1-\Delta)^{-1}\xi \in \mathcal{C}^{\alpha+2}$ and

$$B_\Xi(u) := (1-\Delta)^{-1}(\Delta u \prec X + 2\nabla u \prec \nabla X + \xi \prec u + u \prec \Xi_2) \in \mathcal{H}^{2\gamma}.$$

This space is equipped with the scalar product given by, $u, w \in \mathcal{D}^\gamma_\Xi$,

$$\langle u, w\rangle_{\mathcal{D}^\gamma_\Xi} := \langle u, w\rangle_{\mathcal{H}^\gamma} + \langle u^\sharp, w^\sharp\rangle_{\mathcal{H}^2}.$$

Several remarks are in order.

Remark 2.2.4. For the rest of the paper, we set

$$\mathcal{D}(A) := \mathcal{D}^\gamma_\Xi.$$

This suggestive notation will be justified in Proposition 2.2.23, which yields the equality of Banach Spaces $(\mathcal{D}(A), \|\cdot\|_{\mathcal{D}(A)}) = (\mathcal{D}^\gamma_\Xi, \|\cdot\|_{\mathcal{D}^\gamma_\Xi})$ where $\|\cdot\|_{\mathcal{D}(A)}$ denotes the standard domain (i.e. graph) norm.

We make the following modification of the above ansatz (2.2.6) to fit our purposes. Assume u is of the form

$$u = \Delta_{>N}(u \prec X + B_\Xi(u)) + u^\sharp, \tag{2.2.7}$$

for $2/3 < \gamma < 1$ and $\Delta_{>N}$ denotes a frequency cut-off at 2^N, more precisely,

$$\Delta_{>N} f := \mathcal{F}^{-1}\chi_{|\cdot|>2^N}\mathcal{F}f,$$

with $N \in \mathbb{N}$ which will be chosen depending on the (enhanced) noise Ξ. Where, as above, we define

$$B_\Xi(u) := (1-\Delta)^{-1}(\Delta u \prec X + 2\nabla u \prec \nabla X + \xi \prec u + u \prec \Xi_2). \tag{2.2.8}$$

Note that by Schauder estimates we have the following bound for B,

$$\|B_\Xi(u)\|_{\mathcal{H}^s} \lesssim_s C_\Xi \|u\|_{\mathcal{H}^{s-\gamma}}, \qquad s \in [0, 2\gamma].$$

Recall that C_Ξ denotes a constant that depends explicitly on the norm of the realization of the enhanced noise Ξ and may change from line to line.

Remark 2.2.5. This modification changes the decomposition by a smooth function so it does not change the space. Strictly speaking, one obtains a different norm depending on N, which is equivalent to the $\mathcal{D}(A)$ norm above. In fact, assume that for a function f and some $N \geq 1$ we have

$$f = f \prec X + B_\Xi(f) + f_1^\sharp$$

and

$$f = \Delta_{>N}(f \prec X + B_\Xi(f)) + f_2^\sharp.$$

Then we readily have the estimate

$$\begin{aligned}\|f_1^\sharp\|_{\mathcal{H}^2} &= \|f - f \prec X + B_\Xi(f)\|_{\mathcal{H}^2} \\ &= \|f - \Delta_{>N}(f \prec X + B_\Xi(f)) - \Delta_{\leq N}(f \prec X + B_\Xi(f))\|_{\mathcal{H}^2} \\ &\leq \|f_2^\sharp\|_{\mathcal{H}^2} + C(N,\Xi)\|f\|_{\mathcal{H}^\gamma}\end{aligned}$$

and analogously $\|f_2^\sharp\|_{\mathcal{H}^2} \leq \|f_1^\sharp\|_{\mathcal{H}^2} + C(N,\Xi)\|f\|_{\mathcal{H}^\gamma}$. This proves the norm equivalence.

With this modification of the ansatz, we can write u as a function of $u^\sharp$. In order to do so, we define the following linear map Γ

$$\Gamma f = \Delta_{>N}(\Gamma f \prec X + B_\Xi(\Gamma f)) + f,$$

so that $u = \Gamma u^\sharp$. For N large enough, depending on the realization of Ξ, we can show that this map exists and has useful bounds.
In fact, observe that the map that sends u to $u^\sharp$ is of the form $Id -$ "small" in $\mathcal{H}^s$ for $s < \alpha + 2$ by choosing N appropriately. Then it has an inverse on $\mathcal{H}^s$, which we call Γ, which is in particular injective on $\mathcal{H}^2$ and $\mathcal{H}^1$ and thus it makes sense to define the space $\Gamma\mathcal{H}^2$ which is by construction equal to $\mathcal{D}(A)$.

Remark 2.2.6. In the following, we will utilize this map Γ to show density of the domain, symmetry and norm resolvent convergence. The key point is the map Γ can also be defined in the 3d case and be used there in a similar manner, which we will do in the 3d section.

By these considerations we can bound certain Sobolev norms of u by those of $u^\sharp$, which is the content of the following result.

Proposition 2.2.7. *We can choose N large enough depending only on C_Ξ and s so that*

$$\|\Gamma f\|_{L^\infty} \leq 2\|f\|_{L^\infty}, \tag{2.2.9}$$

$$\|\Gamma f\|_{\mathcal{H}^s} \leq D_\Xi\|f\|_{\mathcal{H}^s}. \tag{2.2.10}$$

for some constant D_Ξ for $s \in [0,\gamma]$ and $D_\Xi = 3$ for $s \in [0,\gamma)$.

Proof. Let us start with proving the L^∞ bound. Choose $\delta > 0$ and let $g = \Gamma f$, we have

$$\begin{aligned}\|B_\Xi(g)\|_{\mathcal{C}^{\gamma-\delta}} &\leq \|\Delta g \prec X\|_{\mathcal{C}^{\gamma-\delta-2}} + 2\|\nabla g \prec \nabla X\|_{\mathcal{C}^{\gamma-\delta-2}} \\ &+ \|\xi \prec g\|_{\mathcal{C}^{\gamma-\delta-2}} + \|g \prec \Xi_2\|_{\mathcal{C}^{\gamma-\delta-2}} \\ &\leq 3\|g\|_{\mathcal{C}^{-\delta}}\|X\|_{\mathcal{C}^\gamma} + \|\xi\|_{\mathcal{C}^{\gamma-2}}\|g\|_{\mathcal{C}^{-\delta}} \\ &+ \|g\|_{\mathcal{C}^{-\delta}}\|\Xi_2\|_{\mathcal{C}^{\gamma-2}} \lesssim C_\Xi\|g\|_{\mathcal{C}^{-\delta}} \lesssim C_\Xi\|g\|_{L^\infty}\end{aligned}$$

by paraproduct estimates and the fact that $\|g\|_{\mathcal{C}^{-\delta}} \lesssim \|g\|_{L^\infty}$ for any small $\delta > 0$. Now we can write,

$$\begin{aligned}\|g\|_{L^\infty} &\leq \|\Delta_{>N}(\Gamma f \prec X + B_\Xi(\Gamma f))\|_{L^\infty} + \|f\|_{L^\infty} \leq 2^{(\delta-\gamma)N}\|g \prec X + B_\Xi(g)\|_{\mathcal{C}^{\gamma-\delta}} + \|f\|_{L^\infty} \\ &\leq 2^{(\delta-\gamma)N}(\|X\|_{\mathcal{C}^{\gamma-\delta}} + C_\delta C_\Xi)\|g\|_{L^\infty} + \|f\|_{L^\infty} \leq C_\delta C_\Xi 2^{(\delta-\gamma)N}\|g\|_{L^\infty} + \|f\|_{L^\infty}\end{aligned}$$

and choose N large enough so that $2C_\delta C_\Xi 2^{(\delta-\gamma)N} \leq 1$ which implies $\|g\|_{L^\infty} \leq 2\|f\|_{L^\infty}$.

For the $\mathcal{H}^s$ bound we can proceed more simply by noting that

$$\|B_\Xi(g)\|_{\mathcal{H}^\gamma} \leq (3\|X\|_{\mathcal{C}^\gamma} + \|\xi\|_{\mathcal{C}^{\gamma-2}} + \|\Xi_2\|_{\mathcal{C}^{\gamma-2}})\|g\|_{L^2}$$

and if $s \leq \gamma$ we have

$$\|g\|_{\mathcal{H}^s} \leq CC_\Xi 2^{(s-\gamma)N}\|g\|_{L^2} + \|f\|_{\mathcal{H}^s}.$$

If $s = 0$ we can choose N large enough so that $\|g\|_{L^2} \leq 2\|f\|_{L^2}$ and as a consequence we have also

$$\|g\|_{\mathcal{H}^s} \leq 2CC_\Xi 2^{(s-\gamma)N}\|f\|_{L^2} + \|f\|_{\mathcal{H}^s}$$

for all $s \leq \gamma$. If $s < \gamma$ we can have N large enough (depending on s) so that $\|g\|_{\mathcal{H}^s} \leq 3\|f\|_{\mathcal{H}^s}$. □

Remark 2.2.8. Note that $\mathcal{D}_\Xi^\gamma$ is actually independent of γ, since for $\gamma, \gamma' \in (2/3, 1)$ we can compute

$$\|u\|_{\mathcal{H}^\gamma} \lesssim \|u^\sharp\|_{\mathcal{H}^2} \lesssim \|u\|_{\mathcal{D}_\Xi^{\gamma'}},$$

and vice versa, so the $\mathcal{D}_\Xi^\gamma$ and $\mathcal{D}_\Xi^{\gamma'}$ norms are equivalent and we will from now on drop the γ and write simply $\mathcal{D}_\Xi$. That is, we have $\mathcal{D}(A) = \mathcal{D}_\Xi$.

As a first step we prove that the domain of A, now defined to be $\mathcal{D}(A)$, is dense in L^2. Before that we note the following remark and then a lemma.

Remark 2.2.9. In the sequel, we put

$$X_\varepsilon = (1 - \Delta)^{-1}\xi_\varepsilon.$$

and similar to the operator Γ in Lemma 2.2.7 we define Γ_ε as follows

$$\Gamma_\varepsilon u := \Delta_{>N}(\Gamma_\varepsilon u \prec X_\varepsilon + B_{\Xi_\varepsilon}(\Gamma_\varepsilon u)) + u^\sharp,$$

where $\Xi_\varepsilon \to \Xi$ in $\mathcal{X}^\alpha$. Note that we may choose N to be independent of ε.

For the above introduced Γ_ε we prove the following lemma, which will be useful in the sequel.

Lemma 2.2.10. *We have that $\|Id - \Gamma\Gamma_\varepsilon^{-1}\|_{\mathcal{H}^\gamma \to \mathcal{H}^\gamma} \to 0$.*

Proof. For $f \in \mathcal{H}^\gamma$, we can write, by using Proposition 2.2.7

$$\begin{aligned}\|f - \Gamma\Gamma_\varepsilon^{-1}(f)\|_{\mathcal{H}^\gamma} &= \|\Gamma(f - f \prec X + B_\Xi(f)) - \Gamma(f - f \prec X_\varepsilon + B_{\Xi^\varepsilon}(f))\|_{\mathcal{H}^\gamma} \\ &= \|\Gamma(f \prec (X_\varepsilon - X) + B_{(\Xi^\varepsilon - \Xi)}(f))\|_{\mathcal{H}^\gamma} \\ &\leq D_\Xi\|f\|_{\mathcal{H}^\gamma}\|\Xi^\varepsilon - \Xi\|_{\mathcal{X}^\alpha}\end{aligned}$$

which shows that $\Gamma\Gamma_\varepsilon^{-1}$ converges to $Id = \Gamma\Gamma^{-1}$ in operator norm. We even get a Lipschitz dependence on the noise. □

Corollary 2.2.11. *The space $\mathcal{D}(A)$, as defined in Definition 2.2.3, is dense in $\mathcal{H}^\gamma$, therefore dense in L^2.*

Proof. For $\delta > 0$ and $f \in \mathcal{H}^\gamma$, we have $f_\delta \in \mathcal{H}^2$ s.t. $\|f - f_\delta\|_{\mathcal{H}^\gamma} < \delta$ and by Lemma 2.2.10, we can find an $\varepsilon = \varepsilon(\delta)$ s.t.

$$\|f_\delta - \Gamma\Gamma_\varepsilon^{-1} f_\delta\|_{\mathcal{H}^\gamma} < \delta$$

Since $\Gamma_\varepsilon^{-1} f_\delta \in \mathcal{H}^2$ and thus $\Gamma\Gamma_\varepsilon^{-1} f_\delta \in \mathcal{D}(A)$ we are done. □

We are now in a position to define the operator A in L^2 on its domain $\mathcal{D}(A)$.

Definition 2.2.12. *We define the operator $A : \mathcal{D}(A) \to L^2$ as*

$$Au := \Delta u^\sharp + u^\sharp \circ \xi + G(u), \tag{2.2.11}$$

where we have defined

$$\begin{aligned} G(u) :=\ & \Delta_{\le N}(u \prec \xi + u \succ \xi + u \prec \Xi_2) \\ & + \Delta_{>N}(-B_\Xi(u) - u \prec X + u \succcurlyeq \Xi_2 + C(u, X, \xi) - (\Delta_{\le N}(u \prec X)) \circ \xi + B_\Xi(u) \circ \xi). \end{aligned}$$

Remark 2.2.13. By using the regularities in Definition 2.2.3, one can easily check, through Proposition 2.3.22, that Au is in fact in L^2. Then, in Proposition 2.2.16 and Theorem 2.2.26 we obtain this operator as a norm resolvent limit of A_ε which motivates the informal identity

$$A = \Delta + \xi - \infty.$$

In the following result, we show that the $\mathcal{H}^2$-norm of $u^\sharp$ can be bounded above by the (standard) domain norm of A.

Proposition 2.2.14. *There exists a constant $C_\Xi > 0$ depending on the enhanced noise such that*

$$\|u^\sharp\|_{\mathcal{H}^2} \le 2\|Au\|_{L^2} + C_\Xi \|u\|_{L^2}. \tag{2.2.12}$$

Proof. First, we note that $\Delta u^\sharp \in L^2$ by assumption. For the resonant term we compute

$$\begin{aligned} \|u^\sharp \circ \xi\|_{L^2} &\le \|(\Delta_{\le M} u^\sharp) \circ \xi\|_{L^2} + \|(\Delta_{>M} u^\sharp) \circ \xi\|_{L^2} \\ &\le C_\Xi 2^{2M} \|u^\sharp\|_{L^2} + \|\Delta_{>M} u^\sharp\|_{\mathcal{H}^{1+\delta}} \|\xi\|_{\mathcal{C}^{-1-2\delta}} \end{aligned} \tag{2.2.13}$$

for δ sufficiently small, giving, for any $M \ge 0$,

$$\|u^\sharp \circ \xi\|_{L^2} \lesssim_\Xi (2^{2M} \|u^\sharp\|_{L^2} + 2^{(\delta-1)M} \|u^\sharp\|_{\mathcal{H}^2}),$$

where we have used Bernstein's inequality (Lemma 2.3.24) and Theorem 2.2.2 for the noise. Using again Bernstein's inequality for the low-frequency terms and the paraproduct estimates for the high-frequency terms, we obtain the bound

$$\|G(u)\|_{L^2} \le C_\Xi \|u\|_{\mathcal{H}^\gamma},$$

for $\gamma < 1$, where the constant can be chosen as

$$C_\Xi = C 2^{2N} (\|\xi\|_{\mathcal{C}^\alpha} + \|\Xi_2\|_{\mathcal{C}^{2\alpha+2}})$$

with $\alpha < -1$ as before.

By using these, for the $\mathcal{H}^2$ bound, we compute

$$\|\Delta u^\sharp\|_{L^2} \le \|Au\|_{L^2} + \|u^\sharp \circ \xi\|_{L^2} + \|G(u)\|_{L^2}.$$

Now, as above we have

$$\|u^\sharp \circ \xi\|_{L^2} \lesssim_\Xi (2^{2M}\|u^\sharp\|_{L^2} + 2^{\gamma M}\|u^\sharp\|_{\mathcal{H}^2})$$

and

$$\|G(u)\|_{L^2} \lesssim_\Xi \|u\|_{\mathcal{H}^\gamma} \lesssim_\Xi \|u^\sharp\|_{\mathcal{H}^\gamma} \lesssim_\Xi \|\Delta_{>M} u^\sharp\|_{\mathcal{H}^\gamma} + \|\Delta_{\le M} u^\sharp\|_{\mathcal{H}^\gamma} \tag{2.2.14}$$

and using again Bernstein's inequality for the low-frequency part we get

$$\|G(u)\|_{L^2} \lesssim C_\Xi (2^{2M}\|u\|_{L^2} + 2^{-\gamma M}\|u^\sharp\|_{\mathcal{H}^2})$$

where we have used the straightforward bound $\|u^\sharp\|_{L^2} \le C_\Xi \|u\|_{L^2}$. Finally, choosing M large enough (depending on Ξ), we obtain

$$\|u^\sharp\|_{\mathcal{H}^2} \le 2\|Au\|_{L^2} + C_\Xi \|u\|_{L^2}.$$

Hence the result. □

Density, symmetry, self-adjointness and convergence

In the following, we show that A is a closed and symmetric operator on $\mathcal{D}(A)$. We first establish closedness.

Proposition 2.2.15. *We have that A is a closed operator on its dense domain $\mathcal{D}(A)$.*

Proof. Assume $(u_n) \subset \mathcal{D}(A)$ is a sequence s.t.

$$\begin{aligned} u_n \to u &\quad \text{in} \quad L^2 \\ &\quad \text{and} \\ Au_n \to g &\quad \text{in} \quad L^2 \end{aligned}$$

for some $g \in L^2$. Then $u_n^\sharp := \Gamma^{-1} u_n$ forms a Cauchy sequence in $\mathcal{H}^2$ and thus converges to a limit that we call $w^\sharp$. Moreover $\Gamma w^\sharp = u$, so $u \in \mathcal{D}(A)$. Thus

$$\begin{aligned} \|Au - g\|_{L^2} &\le \|Au - Au_n\|_{L^2} + \|Au_n - g\|_{L^2} \\ &\le \|w^\sharp - u_n^\sharp\|_{\mathcal{H}^2} + C_\Xi \|u - u_n\|_{L^2} + \|Au_n - g\|_{L^2} \end{aligned}$$

where the second step comes from the proof of Proposition 2.2.14, see also Proposition 2.2.23. Since both terms on the right-hand side tend to zero as $n \to \infty$ we get $Au = g$, namely that A is closed.

□

Now, we are also ready to show the norm convergence of the approximating operators.

Proposition 2.2.16. *Let $u^\sharp \in \mathcal{H}^2$, $u = \Gamma u^\sharp$ and $u_\varepsilon = \Gamma_\varepsilon u^\sharp$. We have that*

$$\|Au - A_\varepsilon u_\varepsilon\|_{L^2} \lesssim_\Xi \|\Xi_\varepsilon - \Xi\|_{\mathcal{X}^\alpha} \|u^\sharp\|_{\mathcal{H}^2}. \tag{2.2.15}$$

Consequently, this implies that

$$\|A\Gamma - A_\varepsilon \Gamma_\varepsilon\|_{\mathcal{H}^2 \to L^2} \to 0. \tag{2.2.16}$$

That is to say, $A_\varepsilon \Gamma_\varepsilon \to A\Gamma$ in norm.

Proof. By using the formula (2.2.11), we observe that all terms in $Au - A_\varepsilon u_\varepsilon$ are bilinear. For the upper bound, by addition and subtraction of cross terms, one obtains terms of the form

$$\|\Xi_\varepsilon - \Xi\|_{\mathcal{X}^\alpha} \|u^\sharp\|_{\mathcal{H}^2} + \|u_\varepsilon - u\|_{\mathcal{H}^\gamma} \|\Xi\|_{\mathcal{X}^\alpha}. \tag{2.2.17}$$

Now, recall that $u = \Gamma u^\sharp, u_\varepsilon = \Gamma_\varepsilon u^\sharp$. Then we obtain terms of the form

$$\|\Xi_\varepsilon - \Xi\|_{\mathcal{X}^\alpha} \|u^\sharp\|_{\mathcal{H}^2} + \|\Gamma_\varepsilon - \Gamma\|_{\mathcal{H}^2 \to \mathcal{H}^\gamma} \|u^\sharp\|_{\mathcal{H}^2} \|\Xi\|_{\mathcal{X}^\alpha}. \tag{2.2.18}$$

By using Lemma 2.2.10 and the estimate in its proof, the result (2.2.15) is now immediate.

□

After this we immediately obtain the symmetry of the operator.

Corollary 2.2.17. *Let $u, v \in \mathcal{D}(A)$ and $u_\varepsilon, v_\varepsilon \in \mathcal{H}^2$ be as in Proposition 2.2.16. Then we obtain*

$$\langle u_\varepsilon, A_\varepsilon v_\varepsilon \rangle = \langle A_\varepsilon u_\varepsilon, v_\varepsilon \rangle \to \langle Au, v \rangle = \langle u, Av \rangle. \tag{2.2.19}$$

Consequently, we have that A is a symmetric operator on its dense domain $\mathcal{D}(A)$.

Proof. This directly follows from Proposition 2.2.16. Using the symmetry of A_ε implies the symmetry of A through the equalities

$$\langle u, Av \rangle = \lim_{\varepsilon \to 0} \langle u_\varepsilon, A_\varepsilon v_\varepsilon \rangle = \lim_{\varepsilon \to 0} \langle A_\varepsilon u_\varepsilon, v_\varepsilon \rangle = \langle Au, v \rangle.$$

Hence, the result. □

The next result shows that the quadratic form given by $-A$ is, through addition of a constant, bounded from below by the $\mathcal{H}^1$ norm of $u^\sharp$. We will later use this estimate to bound certain norms by a (conserved) energy when dealing with the NLS and the nonlinear wave equations.

Proposition 2.2.18. *There exists a constant $C_\Xi > 0$ such that*

$$\frac{1}{2} \langle \nabla u^\sharp, \nabla u^\sharp \rangle \le -\langle u, Au \rangle + C_\Xi \|u\|_{L^2}^2. \tag{2.2.20}$$

Proof. Expanding the Ansatz and integrating by parts we get

$$\begin{aligned}\langle u, Au\rangle =&\langle u, \Delta u^\sharp\rangle + \langle u, u^\sharp \circ \xi\rangle + \langle u, G(u)\rangle \\ =&\langle \Delta_{>N}(u \prec X), \Delta u^\sharp\rangle + \langle u^\sharp, \Delta u^\sharp\rangle + \langle u, u^\sharp \circ \xi\rangle + \langle u, G(u)\rangle + \langle \Delta_{>N} B_\Xi(u), \Delta u^\sharp\rangle \\ =& -\langle \Delta_{>N}(u \prec \xi), u^\sharp\rangle + \langle \Delta_{>N}(u \prec X), u^\sharp\rangle - \langle \nabla u^\sharp, \nabla u^\sharp\rangle + \langle u, u^\sharp \circ \xi\rangle \\ &+ \langle u, G(u)\rangle + \langle \Delta_{>N}(\Delta u \prec X), u^\sharp\rangle + 2\langle \Delta_{>N}(\nabla u \prec \nabla X), u^\sharp\rangle + \langle \Delta_{>N} B_\Xi(u), \Delta u^\sharp\rangle \\ =&D(u, \xi, \Delta_{>N} u^\sharp) - \langle \nabla u^\sharp, \nabla u^\sharp\rangle + \langle u, (\Delta_{\le N} u^\sharp) \circ \xi\rangle + \langle \Delta_{>N}(u \prec X), u^\sharp\rangle \\ &+ \langle u, G(u)\rangle + \langle \Delta_{>N}(\Delta u \prec X), u^\sharp\rangle + 2\langle \Delta_{>N}(\nabla u \prec \nabla X), u^\sharp\rangle + \langle \Delta_{>N} B_\Xi(u), \Delta u^\sharp\rangle\end{aligned}$$

where

$$D(u, \xi, \Delta_{>N} u^\sharp) := \langle u, (\Delta_{>N} u^\sharp) \circ \xi\rangle - \langle u \prec \xi, \Delta_{>N} u^\sharp\rangle.$$

Now fix a sufficiently small $\delta > 0$, then we bound

$$|\langle u, (\Delta_{\le N} u^\sharp) \circ \xi\rangle| \lesssim 2^{2(1+2\delta)N} \|\xi\|_{\mathcal{C}^{-1-\delta}} \|u^\sharp\|_{L^2} \|u\|_{L^2} \lesssim C_\Xi 2^{2(1+2\delta)N} \|u\|_{L^2}^2$$

since from the ansatz we readily have $\|u^\sharp\|_{L^2} \le C_\Xi \|u\|_{L^2}$. Moreover

$$\begin{aligned}|\langle u, G(u)\rangle| &\lesssim_\Xi \|u\|_{L^2}^2 + \|u^\sharp\|_{\mathcal{H}^{1-\delta}}^2 \\ |\langle \Delta u \prec X, u^\sharp\rangle| + |\langle \nabla u \prec \nabla X, u^\sharp\rangle| &\lesssim \|u\|_{\mathcal{H}^{1-\delta}} \|X\|_{\mathcal{C}^{1-\delta}} \|u^\sharp\|_{\mathcal{H}^{2\delta}} \le C_\Xi \|u^\sharp\|_{\mathcal{H}^{1-\delta}}^2 \\ |\langle \Delta_{>N} B_\Xi(u), \Delta u^\sharp\rangle| = |\langle \Delta_{>N} \Delta B_\Xi(u), u^\sharp\rangle| &\le \|B_\Xi(u)\|_{\mathcal{H}^{2-2\delta}} \|u^\sharp\|_{\mathcal{H}^{2\delta}} \le C_\Xi \|u^\sharp\|_{\mathcal{H}^{1-\delta}}^2\end{aligned}$$

and similarly we bound the term $\langle \Delta_{>N}(u \prec X), u^\sharp\rangle$. By the proof of Proposition 2.3.26, we have also

$$|D(u, \xi, \Delta_{>N} u^\sharp)| \lesssim \|\xi\|_{\mathcal{C}^{-1-\delta}} \|u\|_{\mathcal{H}^{(1+\delta)/2}} \|\Delta_{>N} u^\sharp\|_{\mathcal{H}^{(1+\delta)/2}} \lesssim C_\Xi \|u^\sharp\|_{\mathcal{H}^{1-\delta}}^2$$

Using that

$$\|u^\sharp\|_{\mathcal{H}^{1-\delta}}^2 \lesssim \|\Delta_{>M} u^\sharp\|_{\mathcal{H}^{1-\delta}}^2 + \|\Delta_{\le M} u^\sharp\|_{\mathcal{H}^{1-\delta}}^2 \lesssim 2^{2M(1-\delta)} \|u\|_{L^2} + 2^{-2\delta M} \|u^\sharp\|_{\mathcal{H}^1}^2$$

and choosing M large enough we can obtain that

$$\frac{1}{2}\langle \nabla u^\sharp, \nabla u^\sharp\rangle \le -\langle u, Au\rangle + C_\Xi \|u\|_{L^2}^2.$$

□

Remark 2.2.19. One can check that the preceding analysis is valid as well for the approximate Hamiltonians A_ε, given by (2.2.4), simply by replacing the noise Ξ by its regularisation Ξ_ε. Moreover, since all the constants we obtain are polynomials in the $\mathcal{X}^\alpha$ norm of the noise, one sees that they can be chosen to hold uniformly in ε, since $\|\Xi_\varepsilon\|_{\mathcal{X}^\alpha} \le \|\Xi\|_{\mathcal{X}^\alpha}$. In particular, the result in Proposition 2.2.18 is true for A_ε and Ξ_ε for the same constant C_Ξ.

Now we are in a position to define the form domain of the operator. We first shift the operators A and A_ε by a constant to obtain a positive operator.

Proposition 2.2.20. *There exists a constant K_Ξ which is independent of ε s.t.*

$$(K_\Xi - A)^{-1} : L^2 \to \mathcal{D}(H) \tag{2.2.21}$$

$$(K_\Xi - A_\varepsilon)^{-1} : L^2 \to \mathcal{H}^2 \tag{2.2.22}$$

are bounded.

Proof. We will prove the statement for A using a generalization of Lax-Milgram, see [4]. The proof for A_ε follows the same lines with the same constant K_Ξ, in virtue of Remark 2.2.19.

Fix the constant $K_\Xi > C_\Xi > 0$ (C_Ξ as in (2.2.20)) such that

$$\|u\|_{L^2}^2 < \langle -(A - K_\Xi)u, u\rangle \ \forall u \in \mathcal{D}(A),$$

which is possible by Proposition 2.2.18.

Define the bilinear map

$$\begin{aligned} B : \mathcal{D}(A) \times L^2 \quad &\to \quad \mathbb{R} \\ B(u,v) \quad &:= \quad \langle -(A - K_\Xi)u, v\rangle, \end{aligned}$$

then B is continuous, namely

$$|B(u,v)| \lesssim \|u\|_{\mathcal{D}(A)}\|v\|_{L^2}, \ \forall u \in \mathcal{D}(A), \ v \in L^2,$$

and it is weakly coercive i.e.

$$\|u\|_{\mathcal{D}(A)} = \| - (A - K_\Xi)u\|_{L^2} = \sup_{\|v\|_{L^2}=1} \langle -(A - K_\Xi)u, v\rangle \ \text{for any } u \in \mathcal{D}(A).$$

The last property to check is that for any $0 \neq v \in L^2$,

$$\sup_{\|u\|_{\mathcal{D}(A)}=1} |B(u,v)| > 0.$$

Assume for the sake of contradiction that there is a $0 \neq v \in L^2$ s.t.

$$|B(u,v)| = 0, \quad \forall u \in \mathcal{D}(A),$$

This means that

$$\langle u, v\rangle_{\mathcal{D}(A),\mathcal{D}(A)^*} = 0 \ \text{ for all } \ u \in \mathcal{D}(A),$$

i.e. $v = 0$ in $\mathcal{D}(A)^*$. But since $\mathcal{D}(A)$ is dense in L^2, this implies $v = 0$ in L^2 which is a contradiction. Then the Babuska-Lax-Milgram Theorem says that for any $f \in (L^2)^* = L^2$ there exists a unique $u_f \in \mathcal{D}(A)$ with

$$B(u_f, v) = \langle f, v\rangle \ \text{ for all } \ v \in L^2$$

with the bound $\|u_f\|_{\mathcal{D}(A)} \lesssim \|f\|_{L^2}$. In other words

$$(-A + K_\Xi)^{-1} : L^2 \to \mathcal{D}(A)$$

is bounded. ∎

Definition 2.2.21. *We define the following shifted operators*

$$H_\varepsilon := A_\varepsilon - K_\Xi$$
$$H := A - K_\Xi.$$

We are now in a position to use the above estimates to give a characterisation of the domain and the form domain of H in terms of standard Sobolev norms of $u^\sharp$. Firstly, we define the form domain.

Definition 2.2.22. *The form domain of H, that we denote as $\mathcal{D}(\sqrt{-H})$, is defined as the closure of the domain under the following norm*

$$\|u\|_{\mathcal{D}(\sqrt{-H})} := \sqrt{\langle u, -Hu\rangle}.$$

Proposition 2.2.23.

1. *$\Gamma u^\sharp \in \mathcal{D}(H) \Leftrightarrow u^\sharp \in \mathcal{H}^2$, where Γ is the map from Proposition 2.2.7. More precisely, on $\mathcal{D}(H)$ we have the following norm equivalence*

$$\|u^\sharp\|_{\mathcal{H}^2} \lesssim \|H\Gamma u^\sharp\|_{L^2} \lesssim \|u^\sharp\|_{\mathcal{H}^2}. \tag{2.2.23}$$

2. *$\Gamma u^\sharp \in \mathcal{D}(\sqrt{-H}) \Leftrightarrow u^\sharp \in \mathcal{H}^1$, where the form domain of $-H$ is given by the closure of $\mathcal{D}(H)$ under the norm*

$$\|\Gamma u^\sharp\|_{\mathcal{D}(\sqrt{-H})} := \sqrt{\langle \Gamma u^\sharp, -H\Gamma u^\sharp\rangle}. \tag{2.2.24}$$

We will see in the following that the operator $-H$ is self-adjoint and positive, so this is in fact a norm. Then the precise statement is that on $\mathcal{D}(H)$ the following norm equivalence holds

$$\|u^\sharp\|_{\mathcal{H}^1} \lesssim \|\Gamma u^\sharp\|_{\mathcal{D}(\sqrt{-H})} \lesssim \|u^\sharp\|_{\mathcal{H}^1},$$

and hence the closures with respect to the two norms coincide.

Proof.
1. The first inequality in (2.2.23) follows directly from (2.2.12) and the second by first expanding using (2.2.11) and then estimating as in the proof of Theorem 2.2.14.
2. In (2.2.24), the first inequality follows directly from the Proposition 2.2.18. For the second term, one plugs in the definition (2.2.11) and then the only non-trivial term is $\langle u^\sharp \circ \xi, u^\sharp\rangle$. For this term, we also have
$$|\langle u^\sharp \circ \xi, u^\sharp\rangle| \le C_\Xi \|u^\sharp\|^2_{\mathcal{H}^1}$$
by similar arguments as in the proof of Proposition 2.2.18.

□

In order to show self-adjointness we would like to use the following result.

Proposition 2.2.24. *[70, X.1] A closed symmetric operator on a Hilbert space $\mathcal{H}$ is self-adjoint if it has at least one real number in its resolvent set.*

Now, we can show self-adjointness.

Lemma 2.2.25. *The operators $H : \mathcal{D}(H) \to L^2$ and $H_\varepsilon : \mathcal{H}^2 \to L^2$ from Definition 2.2.21 are self-adjoint.*

Proof. This follows from Proposition 2.2.24. Observe that Proposition 2.2.20 implies K_Ξ is in the resolvent of A and A_ε. The result follows. □

Now, in Theorem 2.2.26, we prove the norm resolvent convergence of the operators H_ε to H. This result was obtained in [3, Lemma 4.15] but we give a simplified proof in our framework which can also be applied in the 3d case mutatis mutandis.

Theorem 2.2.26. *We have*

$$\|H^{-1} - H_\varepsilon^{-1}\|_{L^2 \to \mathcal{H}^\gamma} \lesssim \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}$$

In other words H_ε converges to H in the norm resolvent sense.

Proof. Recall that $\Gamma : \mathcal{H}^2 \to \mathcal{D}(H)$ and $\Gamma_\varepsilon : \mathcal{H}^2 \to \mathcal{H}^2$ in which case we have $\Gamma^{-1} : \mathcal{D}(H) \to H^2$ and $\Gamma_\varepsilon^{-1} : \mathcal{H}^2 \to \mathcal{H}^2$. Recall that in Proposition 2.2.16 we obtained

$$\|H_\varepsilon \Gamma_\varepsilon - H\Gamma\|_{\mathcal{H}^2 \to L^2} \lesssim \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}$$

This implies the bound on the resolvents

$$\|\Gamma_\varepsilon^{-1} H_\varepsilon^{-1} - \Gamma^{-1} H^{-1}\|_{L^2 \to \mathcal{H}^2} \lesssim \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}$$

To conclude, by using Proposition 2.2.7 we can write the estimate

$$\begin{aligned}
\|H^{-1} - H_\varepsilon^{-1}\|_{L^2 \to \mathcal{H}^\gamma} &\lesssim \|\Gamma^{-1} H^{-1} - \Gamma^{-1} H_\varepsilon^{-1}\|_{L^2 \to \mathcal{H}^\gamma} \\
&\lesssim \|\Gamma^{-1} H^{-1} - \Gamma_\varepsilon^{-1} H_\varepsilon^{-1}\|_{L^2 \to \mathcal{H}^\gamma} + \|(\Gamma_\varepsilon^{-1} - \Gamma^{-1}) H_\varepsilon^{-1}\|_{L^2 \to \mathcal{H}^\gamma} \\
&\lesssim \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}
\end{aligned}$$

Hence, the result. □

We recall a fundamental result about the functional calculus of self-adjoint operators applied to our situation.

Corollary 2.2.27 (cfr. [71], VIII.20)**.** *For any bounded continuous function $f : [0, \infty) \to \mathbb{C}$ we get*

$$f(H_\varepsilon)g \to f(H)g \text{ in } L^2$$

for any $g \in L^2$ i.e. strong operator convergence.

Functional inequalities

In this section, we obtain certain inequalities for the Anderson Hamiltonian which will be crucial when we study related PDEs.

The first one is an L^p-embedding result.

Lemma 2.2.28 (L^p estimates)**.** *For $u \in \mathcal{D}(\sqrt{-H})$ and $p \in [1, \infty)$ we have*

$$\|u\|_{L^p} \lesssim_\Xi \|u\|_{\mathcal{D}(\sqrt{-H})}. \tag{2.2.25}$$

Moreover, for $v \in \mathcal{D}(\sqrt{-H_\varepsilon}) = \mathcal{H}^1$, we have

$$\|v\|_{L^p} \lesssim_\Xi \|v\|_{\mathcal{D}(\sqrt{-H_\varepsilon})}, \tag{2.2.26}$$

the point being that the constant may be chosen uniformly in ε.

Proof. For $p < \infty$ and $\delta(p) > 0$ small enough we have by Sobolev embedding and Propositions 2.2.7 and 2.2.23

$$\|u\|_{L^p} \lesssim \|u\|_{\mathcal{H}^{1-\delta}} \lesssim \|u^\sharp\|_{\mathcal{H}^{1-\delta}} \lesssim \|u^\sharp\|_{\mathcal{H}^1} \lesssim_\Xi \|\sqrt{-H}u\|_{L^2} \lesssim_\Xi \|u\|_{\mathcal{D}(\sqrt{-H})}.$$

and by Remark 2.2.19, the same computation works for the second inequality with constants independent of ε. □

In light of Proposition 2.2.23, the following result is an analogue of the embedding $\mathcal{H}^2 \subset L^\infty$ in 2d.

Lemma 2.2.29. *For $u \in \mathcal{D}(H)$ we have*

$$\|u\|_{L^\infty} \lesssim_\Xi \|Hu\|_{L^2}.$$

Moreover, for any $\alpha < 1$ one has

$$\|u\|_{\mathcal{C}^\alpha} \lesssim_\Xi \|Hu\|_{L^2}.$$

Proof. By using the Sobolev to Hölder embedding $\mathcal{H}^2 \subset \mathcal{C}^\alpha$ and Propositions 2.2.7 and 2.2.23 we have the following chain of inequalities:

$$\|u\|_{\mathcal{C}^\alpha} \lesssim_\Xi \|u^\sharp\|_{\mathcal{C}^\alpha} \lesssim_\Xi \|u^\sharp\|_{\mathcal{H}^2} \lesssim_\Xi \|Hu\|_{L^2},$$

using that the L^∞ bound is simply the case $\alpha = 0$.
Hence, the result. □

In addition to the above result, we can also prove an inequality that, in some sense, interpolates the L^∞-norm between the energy norm and the logarithm of the domain norm. Namely, we prove a version of Brezis-Gallouet inequality for the Anderson Hamiltonian. We first recall below the original version of the inequality.

Theorem 2.2.30. *[15] Let Ω be a domain in $\mathbb{R}^2$ with smooth boundary. Then, for $v \in \mathcal{H}^2(\Omega)$ we have*

$$\|v\|_{L^\infty} \leq C\left(1 + \sqrt{1 + \log(1 + \|v\|_{\mathcal{H}^2})}\right).$$

for every v that satisfies $\|v\|_{\mathcal{H}^1(\Omega)} \leq 1$.

Our version for the Anderson Hamiltonian is as follows.

Theorem 2.2.31. *For $v \in \mathcal{D}(H)$ we have*

$$\|v\|_{L^\infty} \lesssim_\Xi \|v\|_{\mathcal{D}(\sqrt{-H})}\left(1+\sqrt{1+\log(1+\frac{\|v\|_{\mathcal{D}(H)}}{\|v\|_{\mathcal{D}(\sqrt{-H})}})}\right).$$

As a corollary, we obtain, for $v \in \mathcal{D}(H_\varepsilon) = \mathcal{H}^2$,

$$\|v\|_{L^\infty} \lesssim_\Xi \|\sqrt{-H_\varepsilon}v\|_{L^2}\left(1+\sqrt{1+\log(1+\frac{\|H_\varepsilon v\|_{L^2}}{\|\sqrt{-H_\varepsilon}v\|_{L^2}})}\right),$$

where the constant depends on the limiting noise Ξ and can be chosen independently of ε.

Proof. After fixing $v \in \mathcal{H}^2$ with $\|v\|_{\mathcal{H}^1} \le 1$ we start by observing that for any $M > 0$,which will be fixed later, we have

$$\|v\|_{L^\infty} \le \|\Delta_{\le M} v\|_{L^\infty} + \|\Delta_{>M} v\|_{L^\infty}.$$

By Bernstein's inequalities, Lemma 2.3.24 (in $d = 2$), we can bound

$$\begin{aligned}\|\Delta_{\le M} v\|_{L^\infty} &\le \sum_{i=0}^{M-1} \|\Delta_i v\|_{L^\infty} + \|\Delta_{-1} v\|_{L^\infty} \lesssim \sum_{i=0}^{M-1} 2^i \|\Delta_i v\|_{L^2} + 1\\ &\lesssim \sum_{i=0}^{M-1} \|\Delta_i v\|_{\mathcal{H}^1} + 1 \lesssim \left(\sum_{i=0}^{M-1} 1\right)^{\frac{1}{2}} \|v\|_{\mathcal{H}^1} + 1 \lesssim M^{\frac{1}{2}} + 1\end{aligned}$$

On the other hand, one can use the embedding $\mathcal{H}^{\frac{3}{2}} \hookrightarrow L^\infty$ we have

$$\|\Delta_{>M} v\|_{L^\infty} \lesssim \|\Delta_{>M} v\|_{\mathcal{H}^{\frac{3}{2}}} \lesssim 2^{-\frac{M}{2}} \|v\|_{\mathcal{H}^2}$$

so

$$\|v\|_{L^\infty} \lesssim 1 + M^{\frac{1}{2}} + 2^{-\frac{M}{2}} \|v\|_{\mathcal{H}^2}.$$

Now we choose M s.t.

$$2^{-\frac{M}{2}} = \frac{\sqrt{1+\log(1+\|v\|_{\mathcal{H}^2})}}{1+\|v\|_{\mathcal{H}^2}}$$

since the fraction is clearly less than one this is in fact possible. This leads us to a bound like

$$M \lesssim \log(1+\|v\|_{\mathcal{H}^2})$$

and thus we can conclude

$$\|v\|_{L^\infty} \lesssim 1 + \sqrt{1+\log(1+\|v\|_{\mathcal{H}^2})}.$$

By using this and Propositions 2.2.18, and 2.2.7, we obtain

$$\begin{aligned}\|v\|_{L^\infty} \lesssim_\Xi \|v^\sharp\|_{L^\infty} &\lesssim_\Xi \|v^\sharp\|_{\mathcal{H}^1}\left(1+\sqrt{1+\log(1+\frac{\|v^\sharp\|_{\mathcal{H}^2}}{\|v^\sharp\|_{\mathcal{H}^1}})}\right)\\ &\lesssim_\Xi \|v\|_{\mathcal{D}(\sqrt{-H})}\left(1+\sqrt{1+\log(1+\frac{\|v\|_{\mathcal{D}(H)}}{\|v\|_{\mathcal{D}(\sqrt{-H})}})}\right)\end{aligned}$$

By Remark 2.2.19, the same estimates as for $\mathcal{D}(H)$ are also true for $\mathcal{D}(H_\varepsilon)$, in particular the estimates in Proposition 2.2.18 hold with constants independent of ε. $\square$

2.2.2 The three-dimensional case

In this section we study the Anderson Hamiltonian in 3d. As in the 2d case we will perform a paracontrolled analysis of the Anderson Hamiltonian, however this case is more technical since the noise term has the lower Hölder regularity of $\mathcal{C}^{-3/2-}$. This means a paracontrolled ansatz as in the 2-d case is insufficient. We follow a two step procedure for defining the operator. As a first step, similarly to [30], we perform an exponential transformation depending on the noise and as a second step we make an ansatz for the transformed operator using Paracontrolled Distributions.

Enhanced noise in 3d

Recall that in the 2d case we needed to define the space of enhanced noise (see Def. 2.2.1), namely $\mathcal{X}^\alpha$, for the renormalisation. In this section we define the analogue of this space in the 3d case.

The following results prove that $X = (-\Delta)^{-1}\xi$ can be lifted to an element Ξ in the space $\mathcal{X}^\alpha$ of enhanced distributions such that all the stochastic terms we will need for the ansatz in the next section exist with correct regularities. In sequel, we construct the enhanced white noise space in 3d and prove related approximation results. In particular, we show that the lifts Ξ_ε (of the regularised noise ξ_ε) converge to an element– which we denote by Ξ– in $\mathcal{X}^\alpha$.

Definition 2.2.32. *For $0 < \alpha < \frac{1}{2}$, we define the space $\mathcal{X}^\alpha$ to be the closure of the set*

$$\left\{\left(\phi, \phi_a^{\mathbf{V}}, \phi^{\mathbf{Y}}, \phi^{\mathbf{\Psi}}, \phi_b^{\mathbf{W}}, \nabla\phi \circ \nabla\phi^{\mathbf{\Psi}}\right) : (a,b) \in \mathbb{R}^2, \phi \in \mathcal{C}^2(\mathbb{T}^3)\right\}$$

with respect to the $\mathcal{C}^\alpha(\mathbb{T}^3) \times \mathcal{C}^{2\alpha}(\mathbb{T}^3) \times \mathcal{C}^{\alpha+1}(\mathbb{T}^3) \times \mathcal{C}^{\alpha+1}(\mathbb{T}^3) \times \mathcal{C}^{4\alpha}(\mathbb{T}^3) \times \mathcal{C}^{2\alpha-1}(\mathbb{T}^3)$ norm. Here, we defined

$$\begin{aligned}
\phi_a^{\mathbf{V}} &:= (1-\Delta)^{-1}(|\nabla\phi|^2 - a) \\
\phi^{\mathbf{Y}} &:= 2(1-\Delta)^{-1}(\nabla\phi \cdot \nabla\phi_a^{\mathbf{V}}) \\
\phi^{\mathbf{\Psi}} &:= (1-\Delta)^{-1}(\nabla\phi \cdot \nabla\phi^{\mathbf{Y}}) \\
\phi_b^{\mathbf{W}} &:= (1-\Delta)^{-1}(|\nabla\phi_a^{\mathbf{V}}|^2 - b).
\end{aligned}$$

Theorem 2.2.33. *For ξ_ε given by (2.2.1) we define*

$$\begin{aligned}
X_\varepsilon &= (-\Delta)^{-1}\xi_\varepsilon \\
X_\varepsilon^{\mathbf{V}} &= (1-\Delta)^{-1}(|\nabla X_\varepsilon|^2 - c_\varepsilon^1) \\
X_\varepsilon^{\mathbf{Y}} &= 2(1-\Delta)^{-1}(\nabla X_\varepsilon \cdot \nabla X_\varepsilon^{\mathbf{V}}) \\
X_\varepsilon^{\mathbf{\Psi}} &= (1-\Delta)^{-1}(\nabla X_\varepsilon \cdot \nabla X_\varepsilon^{\mathbf{Y}}) \\
X_\varepsilon^{\mathbf{W}} &= (1-\Delta)^{-1}(|\nabla X_\varepsilon^{\mathbf{V}}|^2 - c_\varepsilon^2),
\end{aligned}$$

where the c_ε are diverging constants which can be chosen as

$$c_\varepsilon^1 = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{|m(\varepsilon k)|^2}{|k|^2} \sim \frac{1}{\varepsilon}$$
$$c_\varepsilon^2 = \sum_{k_1,k_2\neq 0} |m(\varepsilon k_1)|^2 |m(\varepsilon k_2)|^2 \frac{|k_1\cdot k_2|}{|k_1-k_2|^2|k_1|^4|k_2|^2} \sim \left(\log\frac{1}{\varepsilon}\right)^2.$$

Then the sequence $\Xi^\varepsilon \in \mathcal{X}^\alpha$, given by

$$\Xi^\varepsilon := (X_\varepsilon, X_\varepsilon^{\vee}, X_\varepsilon^{\curlyvee}, X_\varepsilon^{\check{\curlyvee}}, X_\varepsilon^{\vee\vee}, \nabla X_\varepsilon \circ \nabla X_\varepsilon^{\check{\curlyvee}})$$

converges a.s. to a unique limit $\Xi \in \mathcal{X}^\alpha$, given by

$$\Xi := (X, X^{\vee}, X^{\curlyvee}, X^{\check{\curlyvee}}, X^{\vee\vee}, \nabla X \circ \nabla X^{\check{\curlyvee}}), \tag{2.2.27}$$

where

$$\begin{aligned} X &= (-\Delta)^{-1}\xi \\ X^{\vee} &= (1-\Delta)^{-1}(: |\nabla X| :^2) \\ X^{\curlyvee} &= 2(1-\Delta)^{-1}(\nabla X \cdot \nabla X^{\vee}) \\ X^{\check{\curlyvee}} &= (1-\Delta)^{-1}(\nabla X \cdot \nabla X^{\curlyvee}) \\ X^{\vee\vee} &= (1-\Delta)^{-1}(: |\nabla X^{\vee}| :^2). \end{aligned}$$

Proof. We omit the proof, which goes in a similar way to Theorem 7.11 in [21] (see also Chapter 9 of [48]). Note that their estimates are for the parabolic case, but by using the resolvent identity

$$\int_0^\infty e^{-t} e^{t\Delta} \mathrm{d}t = (1-\Delta)^{-1},$$

one can easily adapt their computations to our setting, essentially by multiplying by e^{-t} and integrating over t. This, in particular, implies that the diverging constants are the same. Note that the last term in our enhanced noise (2.2.27) is slightly different from the one in [21]. However one can easily show that the most singular part of $\nabla X \circ \nabla X^{\check{\curlyvee}}$ is given by $\nabla X \circ \nabla(1-\Delta)^{-1}\nabla X$, which is the term from [21]. In fact, we have

$$\begin{aligned} \nabla X \circ \nabla X^{\check{\curlyvee}} =& \nabla X \circ \nabla (1-\Delta)^{-1} \left(\nabla X \prec \nabla X^{\curlyvee} + \nabla X^{\curlyvee} \prec \nabla X + \nabla X \circ \nabla X^{\curlyvee}\right) \\ =& \nabla X \circ (1-\Delta)^{-1} \left(\nabla \left(\nabla X \prec \nabla X^{\curlyvee} + \nabla X \circ \nabla X^{\curlyvee}\right) + \nabla^2 X^{\curlyvee} \prec \nabla X\right) \\ &+ \nabla X \circ (1-\Delta)^{-1} \left(\nabla X^{\curlyvee} \prec \nabla^2 X\right), \end{aligned}$$

where first expression makes sense assuming the correct regularity for the other stochastic terms. For the second term, we apply the commutator Lemma 2.3.28 (or more precisely its Hölder version) and Proposition 2.3.23. We compute

$$\begin{aligned}\nabla X\circ(1-\Delta)^{-1}\left(\nabla X^{\Psi}\prec\nabla^2 X\right)&=\nabla X\circ\left(\nabla X^{\Psi}\prec(1-\Delta)^{-1}\nabla^2 X+R\left(\nabla X^{\Psi},\nabla^2 X\right)\right)\\&=\nabla X^{\Psi}(\nabla X\circ\nabla(1-\Delta)^{-1}\nabla X)+C\left(\nabla X^{\Psi},(1-\Delta)^{-1}\nabla^2 X,\nabla X\right)\\&+\nabla X\circ R\left(\nabla X^{\Psi},\nabla^2 X\right).\end{aligned}$$

This proves that $\nabla X\circ\nabla(1-\Delta)^{-1}\nabla X\in\mathcal{C}^{2\alpha-1}$ which in turn implies that $\nabla X\circ\nabla X^{\Psi}\in\mathcal{C}^{2\alpha-1}$. Thus our result follows from Theorem 7.11 in [21].
See also Theorems 9.1 and 9.3 in [48] where a similar renormalization was performed with 1d space-time white noise which has the same regularity as 3d spatial white noise. □

Lemma 2.2.34. *Let* $\alpha, X, X^{\vee}, X^{\Psi}, X_\varepsilon, X_\varepsilon^{\vee}, X_\varepsilon^{\Psi}$ *be as above, then* $e^X\in\mathcal{C}^\alpha, e^{X^{\vee}}\in\mathcal{C}^{2\alpha}, e^{X^{\Psi}}\in\mathcal{C}^{\alpha+1}$ *and*

$$\begin{aligned}e^{X_\varepsilon}&\to e^X \text{ in } \mathcal{C}^\alpha\\e^{X_\varepsilon^{\vee}}&\to e^{X^{\vee}} \text{ in } \mathcal{C}^{2\alpha}\\e^{X_\varepsilon^{\Psi}}&\to e^{X^{\Psi}} \text{ in } \mathcal{C}^{\alpha+1}.\end{aligned}$$

Proof. We prove the result for X, the others are proved in the same way. Since $\alpha>0$, we use the equivalent classical Hölder norms on $\mathcal{C}^\alpha$. One easily sees that the spaces $\mathcal{C}^\alpha$ are Banach Algebras, so $e^X=\sum_{n\geq 0}\frac{1}{n!}X^n\in\mathcal{C}^\alpha$ and since $X_\varepsilon\to X$ in $\mathcal{C}^\alpha$, we can estimate

$$\begin{aligned}\|e^X-e^{X_\varepsilon}\|_{\mathcal{C}^\alpha}&\leq\|e^X\|_{\mathcal{C}^\alpha}\|1-e^{X_\varepsilon-X}\|_{\mathcal{C}^\alpha}=\|e^X\|_{\mathcal{C}^\alpha}\left\|\sum_{n\geq 1}\frac{1}{n!}(X_\varepsilon-X)^n\right\|_{\mathcal{C}^\alpha}\\&\leq\|e^X\|_{\mathcal{C}^\alpha}(e^{\|X_\varepsilon-X\|_{\mathcal{C}^\alpha}}-1),\end{aligned}$$

and conclude that $e^{X_\varepsilon}\to e^X$ in $\mathcal{C}^\alpha$. □

Lemma 2.2.35. *For* α, X, X^{Ψ} *as above,* $W:=X+X^{\vee}+X^{\Psi}$ *and*

$$Z\;=\;(1-\Delta)^{-1}\left(\left|\nabla X^{\Psi}\right|^2+2\nabla X^{\vee}\cdot\nabla X^{\Psi}-X^{\vee}-X^{\Psi}\right)+X^{\vee\vee}+2X^{\Psi}.$$

we have

$$\nabla e^X\cdot\nabla e^{X^{\Psi}}\in\mathcal{C}^{\alpha-1},$$

which implies that $e^{2W}(1-\Delta)Z\in\mathcal{C}^{\alpha-1}$.

Proof. We use paralinearisation, see Lemma 2.3.25, to rewrite

$$\begin{aligned}e^X&=e^X\prec X+g^\sharp\\e^{X^{\Psi}}&=e^{X^{\Psi}}\prec X^{\Psi}+f^\sharp,\end{aligned}$$

where $g^\sharp \in \mathcal{C}^{2\alpha}$ and $f^\sharp \in \mathcal{C}^{2\alpha+1}$. Thus,

$$\nabla e^X = (\nabla e^X) \prec X + e^X \prec \nabla X + \nabla g^\sharp$$

and

$$\nabla e^{X^{\curlyvee}} = \left(\nabla e^{X^{\curlyvee}}\right) \prec X^{\curlyvee} + e^{X^{\curlyvee}} \prec \nabla X^{\curlyvee} + \nabla f^\sharp.$$

Note that the only problematic term in the product is

$$(e^X \prec \nabla X)(e^{X^{\curlyvee}} \prec \nabla X^{\curlyvee}). \tag{2.2.28}$$

More precisely, we only have to make sense of the resonant product in (2.2.28) since the paraproducts are always defined. We compute

$$\begin{aligned}(e^X \prec \nabla X) \circ \left(e^{X^{\curlyvee}} \prec \nabla X^{\curlyvee}\right) =& e^{X^{\curlyvee}}\left(\nabla X^{\curlyvee} \circ (e^X \prec \nabla X)\right) + C\left(e^{X^{\curlyvee}}, \nabla X^{\curlyvee}, (e^X \prec \nabla X)\right)\\ =& e^{X+X^{\curlyvee}}\left(\nabla X^{\curlyvee} \circ \nabla X\right) + e^{X^{\curlyvee}} C\left(e^X, \nabla X, \nabla X^{\curlyvee}\right)\\ &+ C\left(e^{X^{\curlyvee}}, \nabla X^{\curlyvee}, (e^X \prec \nabla X)\right).\end{aligned}$$

Now, since $\nabla X^{\curlyvee} \circ \nabla X$ is assumed to be in $\mathcal{C}^{2\alpha-1}$, the above resonant product is also in $\mathcal{C}^{2\alpha-1}$. This finishes the proof that $\nabla e^X \cdot \nabla e^{X^{\curlyvee}} \in \mathcal{C}^{\alpha-1}$. Moreover, by reinserting the definitions we obtain

$$\begin{aligned}&e^{2W}(1-\Delta)Z\\ &= e^{2X+2X^{\vee}+2X^{\curlyvee}}\left(:\left|\nabla X^{\vee}\right|^2: + \left|\nabla X^{\curlyvee}\right|^2 + 2\nabla X \cdot \nabla X^{\curlyvee} + 2\nabla X^{\vee} \cdot \nabla X^{\curlyvee} - X^{\vee} - X^{\curlyvee}\right)\\ &= e^{2X+2X^{\vee}+2X^{\curlyvee}}\left(:\left|\nabla X^{\vee}\right|^2: + \left|\nabla X^{\curlyvee}\right|^2 + 2\nabla X^{\vee} \cdot \nabla X^{\curlyvee} - X^{\vee} - X^{\curlyvee}\right)\\ &+ \frac{1}{2} e^{2X^{\vee}} \nabla\left(e^{2X}\right) \nabla\left(e^{2X^{\curlyvee}}\right)\end{aligned}$$

by using the previous computations and the fact that all the terms in the first bracket have regularity at least $2\alpha - 1$. We can finally conclude that

$$\|e^{2W}(1-\Delta)Z\|_{\mathcal{C}^{\alpha-1}} \lesssim \|e^{2W}\|_{\mathcal{C}^\alpha}\|\Xi\|_{\mathcal{X}^\alpha}.$$

□

The domain, the Γ-map and the definition of the 3-d Hamiltonian

In this section, building on our work in Section 2.2.2, we perform the renormalisation of the Anderson Hamiltonian in 3d. Recall the following quantities we introduced and justified in Section 2.2.2.

$$X = (-\Delta)^{-1}\xi(x) \in \mathcal{C}^{1/2-}$$

$$X^{\vee} = (1-\Delta)^{-1} : |\nabla X|^2 : \in \mathcal{C}^{1-} \qquad X^{\curlyvee} = 2(1-\Delta)^{-1}\left(\nabla X \cdot \nabla X^{\vee}\right) \in \mathcal{C}^{3/2-}$$

$$X^{\Psi} = (1-\Delta)^{-1}\left(\nabla X \cdot \nabla X^{\curlyvee}\right) \in \mathcal{C}^{3/2-} \qquad X^{\vee\vee} = (1-\Delta)^{-1} : \left|\nabla X^{\vee}\right|^2 : \in \mathcal{C}^{2-}.$$

In the following, we first motivate the ansatz via formal calculations and then conclude rigorously in Definition 2.2.37.

Initially we make the following ansatz for the domain of the Hamiltonian

$$u = e^{X+X^{\vee}+X^{\curlyvee}} u^\flat,$$

where the form of $u^\flat$ will be specified later. We begin by computing

$$\begin{aligned}\Delta u + u\xi =& e^{X+X^{\vee}+X^{\curlyvee}}\Big(\Delta\left(X+X^{\vee}+X^{\curlyvee}\right)u^\flat + \left|\nabla\left(X+X^{\vee}+X^{\curlyvee}\right)\right|^2 u^\flat \\ &+ \Delta u^\flat + 2\nabla\left(X+X^{\vee}+X^{\curlyvee}\right)\nabla u^\flat + u^\flat\xi\Big) \\ =& e^{X+X^{\vee}+X^{\curlyvee}}\Big(\Delta u^\flat + \Big(|\nabla X|^2 - :|\nabla X|^2: + \left|\nabla X^{\vee}\right|^2 + \left|\nabla X^{\curlyvee}\right|^2 \\ &+ 2\nabla X\cdot\nabla X^{\curlyvee} + 2\nabla X^{\vee}\cdot\nabla X^{\curlyvee} - X^{\vee} - X^{\curlyvee}\Big)u^\flat + 2\nabla\left(X+X^{\vee}+X^{\curlyvee}\right)\cdot\nabla u^\flat\Big).\end{aligned}$$

Note that the regularity of $X^{\vee}$ is too low for the term $\left|\nabla X^{\vee}\right|^2$ to be defined so we have to replace it by by its Wick ordered version, also note the appearing difference $|\nabla X|^2 - :|\nabla X|^2:$. Here one sees the two divergences that arise, since we formally have

$$:|\nabla X|^2: \;=\; |\nabla X|^2 - \infty, \qquad :\left|\nabla X^{\vee}\right|^2 := \left|\nabla X^{\vee}\right|^2 - \infty.$$

However, this notation is somewhat misleading since the rate of divergence is different in both cases, recall the constants c_1^ε and c_2^ε from Theorem 2.2.33. This again suggests that, as in 2d, the renormalised Hamiltonian can be formally written in the suggestive form

$$A = \Delta + \xi - \infty.$$

We set

$$Au = A(e^W u^\flat) = e^W(\Delta u^\flat + 2(1-\Delta)\widetilde{W}\cdot\nabla u^\flat + (1-\Delta)Zu^\flat), \tag{2.2.29}$$

for functions $u^\flat$ for which this expression makes sense, for brevity we have set

$$\begin{aligned} W &= X + X^{\vee} + X^{\curlyvee} \\ \widetilde{W} &= (1-\Delta)^{-1}\nabla W \\ Z &= (1-\Delta)^{-1}\left(\left|\nabla X^{\curlyvee}\right|^2 + 2\nabla X^{\vee}\cdot\nabla X^{\curlyvee} - X^{\vee} - X^{\curlyvee}\right) + X^{\vee\vee} + 2X^{\curlyvee\!\curlyvee}.\end{aligned}$$

As we have seen in Section 2.2.2, these stochastic terms have the following regularities

$$X, W \in \mathcal{C}^{1/2-}, X^{\vee} \in \mathcal{C}^{1-}, X^{\curlyvee}, X^{\curlyvee\!\curlyvee}, \widetilde{W}, Z \in \mathcal{C}^{3/2-} \text{ and } X^{\vee\vee} \in \mathcal{C}^{2-}.$$

This suggests to make a paracontrolled ansatz for $u^\flat$ in terms of Z and $\widetilde{W}$ since the products appearing are classically ill-defined. In fact, we make the following ansatz

$$u^\flat = u^\flat \prec Z + \nabla u^\flat \prec \widetilde{W} + B_\Xi(u^\flat) + u^\sharp, \tag{2.2.30}$$

with $u^\sharp \in \mathcal{H}^2$ and for a correction term that we denote by $B_\Xi(u^\flat)$. Into the correction term we will absorb the terms which have regularity not worse than $\mathcal{H}^{2-}$. Similarly to the 2d case, we will introduce a frequency cut-off that will allow us to write $u^\flat$ as a function of $u^\sharp$ but, in order not to overburden the notation, we will omit this for the time being.

For the remainder of this section, we define

$$L := (1-\Delta) \quad \text{and} \quad L^{-1} = (1-\Delta)^{-1}.$$

Note that the ansatz (2.2.30) directly implies $u^\flat \in \mathcal{H}^{3/2-}$ by the paraproduct estimates in Lemma 2.3.22.

We want to determine the form of the corrector term $B_\Xi(u^\flat)$ in (2.2.30). We first compute

$$\begin{aligned}\Delta u^\flat =&\Delta u^\flat \prec Z + 2\nabla u^\flat \prec \nabla Z + u^\flat \prec \Delta Z + \nabla \Delta u^\flat \prec \widetilde{W} + 2\nabla^2 u^\flat \prec \nabla \widetilde{W}\\ &+ \nabla u^\flat \prec \Delta \widetilde{W} + \Delta B_\Xi + \Delta u^\sharp\\ =&\Delta u^\flat \prec Z + 2\nabla u^\flat \prec \nabla Z - u^\flat \prec (LZ - Z) + \nabla \Delta u^\flat \prec \widetilde{W} + 2\nabla^2 u^\flat \prec \nabla \widetilde{W}\\ &- \nabla u^\flat \prec (L\widetilde{W} - \widetilde{W}) - LB_\Xi(u^\flat) - B_\Xi(u^\flat) + \Delta u^\sharp.\end{aligned}$$

By using the paraproduct decomposition, we obtain

$$\Delta u^\flat + 2L\widetilde{W} \cdot \nabla u^\flat + LZu^\flat = \Delta u^\sharp + \widetilde{G}(u^\flat) + 2L\widetilde{W} \circ \nabla u^\flat + LZ \circ u^\flat, \tag{2.2.31}$$

where we have defined

$$\begin{aligned}\widetilde{G}(u^\flat) :=&\Delta u^\flat \prec Z + 2\nabla u^\flat \prec \nabla Z + u^\flat \prec Z + \nabla \Delta u^\flat \prec \widetilde{W} + 2\nabla^2 u^\flat \prec \nabla \widetilde{W} + \nabla u^\flat \prec \widetilde{W}\\ &- LB_\Xi(u^\flat) - B_\Xi(u^\flat) + 2L\widetilde{W} \prec \nabla u^\flat + LZ \prec u^\flat.\end{aligned}$$

These are the "non-problematic" terms that can also be absorbed into B_Ξ. We still have to take care of the resonant product $L\widetilde{W} \circ \nabla u^\flat$, which is not a priori defined and the other resonant product which is actually defined as is, but we shall see at a later time that it is necessary to decompose it further. To be precise, we insert the ansatz and use Proposition 2.3.23

$$\begin{aligned}L\widetilde{W} \circ \nabla u^\flat =&L\widetilde{W} \circ (\nabla u^\flat \prec Z + u^\flat \prec \nabla Z + \nabla^2 u^\flat \prec \widetilde{W} + \nabla u^\flat \prec \nabla \widetilde{W} + \nabla B_\Xi(u^\flat) + \nabla u^\sharp)\\ =&\nabla u^\flat (L\widetilde{W} \circ Z) + C(\nabla u^\flat, Z, L\widetilde{W}) + u^\flat (L\widetilde{W} \circ \nabla Z) + C(u^\flat, \nabla Z, L\widetilde{W})\\ &+ L\widetilde{W} \circ (\nabla^2 u^\flat \prec \widetilde{W}) + \nabla u^\flat (L\widetilde{W} \circ \nabla \widetilde{W})\\ &+ C(\nabla u^\flat, \nabla \widetilde{W}, L\widetilde{W}) + L\widetilde{W} \circ (\nabla B_\Xi(u^\flat) + \nabla u^\sharp).\end{aligned}$$

In section 2.2.2 we have seen that the following stochastic terms can be defined and have regularity

$$\begin{aligned}L\widetilde{W} \circ Z &\in \mathcal{C}^{1-}\\ L\widetilde{W} \circ \nabla Z &\in \mathcal{C}^{0-}\\ L\widetilde{W} \circ \nabla \widetilde{W} &\in \mathcal{C}^{0-}.\end{aligned}$$

We furthermore expand the products appearing above as

$$
\begin{aligned}
&L\widetilde{W} \circ \nabla u^\flat \\
&= \nabla u^\flat \prec (L\widetilde{W} \circ Z) + \nabla u^\flat \succcurlyeq (L\widetilde{W} \circ Z) + C(\nabla u^\flat, Z, L\widetilde{W}) + u^\flat \prec (L\widetilde{W} \circ \nabla Z) \\
&+ u^\flat \succcurlyeq (L\widetilde{W} \circ \nabla Z) + C(u^\flat, \nabla Z, L\widetilde{W}) + L\widetilde{W} \circ (\nabla^2 u^\flat \prec \widetilde{W}) + \nabla u^\flat \prec (L\widetilde{W} \circ \nabla\widetilde{W}) \\
&+ \nabla u^\flat \succcurlyeq (L\widetilde{W} \circ \nabla\widetilde{W}) + C(\nabla u^\flat, \nabla\widetilde{W}, L\widetilde{W}) + L\widetilde{W} \circ (\nabla B_\Xi(u^\flat) + \nabla u^\sharp).
\end{aligned}
$$

For the other resonant product in (2.2.31), we do the same and get

$$
\begin{aligned}
LZ \circ u^\flat =& u^\flat \prec (LZ \circ Z) + u^\flat \succcurlyeq (LZ \circ Z) + C(u^\flat, Z, LZ) + \nabla u^\flat \prec (LZ \circ \widetilde{W}) \\
&+ \nabla u^\flat \succcurlyeq (LZ \circ \widetilde{W}) + C(\nabla u^\flat, \widetilde{W}, LZ) + LZ \circ (B_\Xi(u^\flat) + u^\sharp).
\end{aligned}
$$

Now we are in a position to give the precise definition of the correction term; we put

$$
\begin{aligned}
B_\Xi(u^\flat) := L^{-1} \Big[&\Delta u^\flat \prec Z + 2\nabla u^\flat \prec \nabla Z + u^\flat \prec Z + \nabla \Delta u^\flat \prec \widetilde{W} + 2\nabla^2 u^\flat \prec \nabla\widetilde{W} \\
&- \nabla u^\flat \prec \widetilde{W} + 2L\widetilde{W} \prec \nabla u^\flat + LZ \prec u^\flat \\
&+ 2\nabla u^\flat \prec (L\widetilde{W} \circ Z) + 2\nabla u^\flat \succ (L\widetilde{W} \circ Z) \\
&+ 2u^\flat \prec (L\widetilde{W} \circ \nabla Z) + 2u^\flat \succ (L\widetilde{W} \circ \nabla Z) + 2\nabla u^\flat \prec (L\widetilde{W} \circ \nabla\widetilde{W}) \\
&+ 2\nabla u^\flat \succ (L\widetilde{W} \circ \nabla\widetilde{W}) + u^\flat \prec (LZ \circ Z) + u^\flat \succ (LZ \circ Z) \\
&+ \nabla u^\flat \prec (LZ \circ \widetilde{W}) + \nabla u^\flat \succ (LZ \circ \widetilde{W})\Big] .
\end{aligned}
\tag{2.2.32}
$$

Using again the paraproduct estimates from Lemma 2.3.22, one sees that the terms in the brackets are at least of regularity $\mathcal{H}^{0-}$, which implies $B_\Xi(u^\flat) \in \mathcal{H}^{2-}$. We make this precise in the following result.

Lemma 2.2.36. *Let B_Ξ be defined as above, then we have the following bounds for $\sigma < 2$ and $\varepsilon > 0$*

1. $\|B_\Xi(v)\|_{\mathcal{H}^\sigma} \leq C_\Xi \|v\|_{\mathcal{H}^{\sigma-1/2+\varepsilon}}$
2. $\|B_\Xi(v)\|_{\mathcal{C}^\sigma} \leq C_\Xi \|v\|_{\mathcal{C}^{\sigma-1/2}}$,

where for the the constant we can choose $C_\Xi = C\|\Xi\|_{\mathcal{X}^{\sigma-3/2}}$, see Definition 2.2.32 for the precise definition of the norm and $C > 0$ is an independent constant.

Proof. This follows from the paraproduct estimates, Lemma 2.3.22, for the first case. The second case works precisely in the same way using the paraproduct estimates for Besov-Hölder spaces and Schauder estimates, see e.g. [45]. □

Finally we collect everything in the following rigorous definition which describes the domain of the Anderson Hamiltonian.

Definition 2.2.37. *Let $W, \widetilde{W}, Z$ be as above. Then, for $1 < \gamma < 3/2$, we define the space*

$$
\mathcal{W}^\gamma_\Xi := e^W \mathcal{U}^\gamma_\Xi := e^W \{u^\flat \in \mathcal{H}^\gamma \text{ s.t. } u^\flat = u^\flat \prec Z + \nabla u^\flat \prec \widetilde{W} + B_\Xi(u^\flat) + u^\sharp, \text{ for } u^\sharp \in \mathcal{H}^2\},
$$

where $B_\Xi(u^\flat)$ is as in (2.2.32). We furthermore equip the space with the scalar product given by, for $u, w \in \mathcal{W}_\Xi^\gamma$,

$$\langle u, w\rangle_{\mathcal{W}_\Xi^\gamma} := \langle u^\flat, w^\flat\rangle_{\mathcal{H}^\gamma} + \langle u^\sharp, w^\sharp\rangle_{\mathcal{H}^2}.$$

Given $u = e^W u^\flat \in \mathcal{W}_\Xi^\gamma$ we define the renormalised Anderson Hamiltonian acting on u in the following way

$$Au = e^W(\Delta u^\sharp + LZ \circ u^\sharp + 2L\widetilde{W} \circ \nabla u^\sharp + G(u^\flat)), \tag{2.2.33}$$

where

$$\begin{aligned} G(u^\flat) :=& B_\Xi(u^\flat) + 2\nabla u^\flat \circ (L\widetilde{W} \circ Z) + 2C(\nabla u^\flat, Z, L\widetilde{W}) + u^\flat \circ (L\widetilde{W} \circ \nabla Z) \\ &+ C(u^\flat, \nabla Z, L\widetilde{W}) + 2L\widetilde{W} \circ (\nabla^2 u^\flat \prec \widetilde{W}) \\ &+ 2\nabla u^\flat \circ (L\widetilde{W} \circ \nabla\widetilde{W}) + 2C(\nabla u^\flat, \nabla\widetilde{W}, L\widetilde{W}) + 2L\widetilde{W} \circ \nabla B_\Xi(u^\flat) \end{aligned}$$

and C denotes the commutator from Proposition 2.3.23. Note that this definition is equivalent to (2.2.29) by construction.

After this definition, some remarks are in order.

Remark 2.2.38. In view of (2.2.29), for regularised white noise ξ_ε, we set

$$\begin{aligned} A_\varepsilon u &:= e^{W_\varepsilon}(\Delta u^\flat + 2(1-\Delta)\widetilde{W}_\varepsilon \cdot \nabla u^\flat + (1-\Delta)Z_\varepsilon u^\flat) && (2.2.34) \\ &= \Delta u + \xi_\varepsilon u - (c_\varepsilon^1 + c_\varepsilon^2)u, && (2.2.35) \end{aligned}$$

where we have defined

$$\begin{aligned} W_\varepsilon =& X_\varepsilon + X_\varepsilon^{\vee} + X_\varepsilon^{\vee\!\!\vee} \\ X_\varepsilon =& (-\Delta)^{-1}\xi_\varepsilon \\ X_\varepsilon^{\vee} =& (1-\Delta)^{-1}(|\nabla X_\varepsilon|^2 - c_\varepsilon^1) \\ X_\varepsilon^{\vee\!\!\vee} =& 2(1-\Delta)^{-1}\left(\nabla X_\varepsilon \cdot \nabla X_\varepsilon^{\vee}\right) \\ \widetilde{W}_\varepsilon =& (1-\Delta)^{-1}\nabla W_\varepsilon \\ Z_\varepsilon =& (1-\Delta)^{-1}\left(|\nabla X_\varepsilon^{\vee}|^2 - c_\varepsilon^2 + |\nabla X_\varepsilon^{\vee\!\!\vee}|^2 + 2\nabla X_\varepsilon \cdot \nabla X_\varepsilon^{\vee\!\!\vee} + 2\nabla X_\varepsilon^{\vee} \cdot \nabla X_\varepsilon^{\vee\!\!\vee} - X_\varepsilon^{\vee} - X_\varepsilon^{\vee\!\!\vee}\right) \\ &\text{and} \\ u^\flat :=& e^{-W_\varepsilon}u. \end{aligned}$$

Recall that the renormalisation constants, from Theorem 2.2.33, are

$$c_\varepsilon^1 = O(\varepsilon^{-1}) \text{ and } c_\varepsilon^2 = O(\log\varepsilon).$$

Observe that this now makes the constant c_ε in (2.2.4) precise as $c_\varepsilon = c_\varepsilon^1 + c_\varepsilon^2$.

Remark 2.2.39. As in the 2d case, the space $\mathcal{W}_\Xi^\gamma$ is actually independent of γ and we will denote it simply by $\mathcal{W}_\Xi$. Moreover, one can introduce a Fourier cut-off $\Delta_{>N}$ at level 2^N and write

$$u^\flat = \Delta_{>N}(u^\flat \prec Z + \nabla u^\flat \prec \widetilde{W} + B_\Xi(u^\flat)) + u^\sharp. \tag{2.2.36}$$

This again does not change the space, see Remark 2.2.5 for the analogous argument in 2d.

We set up some suggestive notation in the following remark, namely that $\mathcal{W}_\Xi$ will turn out to be the domain of A.

Remark 2.2.40. Similarly to Remark 2.2.4, we introduce the notation

$$\mathcal{D}(A) := \mathcal{W}_\Xi$$

which will be justified later.

We can furthermore introduce the 3d-version of the Γ-map; we use the same notation since there is never any danger of mistaking the two. We define the linear map Γ as

$$\Gamma f = \Delta_{>N}(\Gamma f \prec Z + \nabla(\Gamma f) \prec \tilde{W} + B_\Xi(\Gamma f)) + f, \tag{2.2.37}$$

i.e. the inverse of the modified paracontrolled ansatz. This allows us to write $u^\flat = \Gamma u^\sharp$. Similarly to the 2d case, for N large enough depending on the $\mathcal{X}^\alpha$ norm of Ξ, we can show this map exists and has useful bounds and obtain the following generalisation of Proposition 2.2.7 to 3d.

Proposition 2.2.41. *We can choose N large enough depending only on Ξ and s so that*

$$\|\Gamma f\|_{L^\infty} \le 2\|f\|_{L^\infty}, \tag{2.2.38}$$

$$\|\Gamma f\|_{\mathcal{H}^s} \le 2\|f\|_{\mathcal{H}^s}, \tag{2.2.39}$$

for $s \in \left[0, \frac{3}{2}\right)$.

Proof. With slight modifications, the proof is basically the same as in the 2d case, namely Propostion 2.2.7. For (2.2.38), choose again a small $\delta > 0$, then

$$\begin{aligned}
\|\Gamma f\|_{L^\infty} &\le \|f\|_{L^\infty} + \|\Delta_{>N}(\Gamma f \prec Z + \nabla(\Gamma f) \prec \widetilde{W} + B_\Xi(\Gamma f))\|_{\mathcal{C}^\delta} \\
&\le \|f\|_{L^\infty} + 2^{-\delta N}\|\Gamma f \prec Z + \nabla(\Gamma f) \prec \widetilde{W} + B_\Xi(\Gamma f)\|_{\mathcal{C}^{2\delta}} \\
&\text{and} \\
\|\Gamma f \prec Z\|_{\mathcal{C}^{2\delta}} &\lesssim \|\Gamma f\|_{\mathcal{C}^{-\delta}}\|Z\|_{\mathcal{C}^{3\delta}} \lesssim C_\Xi \|\Gamma f\|_{L^\infty} \\
\|\nabla \Gamma f \prec \widetilde{W}\|_{\mathcal{C}^{2\delta}} &\lesssim \|\nabla(\Gamma f)\|_{\mathcal{C}^{-1-\delta}}\|\widetilde{W}\|_{\mathcal{C}^{1+3\delta}} \lesssim C_\Xi \|\Gamma f\|_{L^\infty} \\
\|B_\Xi(\Gamma f)\|_{\mathcal{C}^{2\delta}} &\lesssim C_\Xi \|\Gamma f\|_{\mathcal{C}^{2\delta-1/2}} \lesssim C_\Xi \|\Gamma f\|_{L^\infty},
\end{aligned}$$

which allows us to conclude by choosing N large enough depending on the norm of the enhanced noise Ξ. The proof of the Sobolev case is similar. □

Remark 2.2.42. Analogously to Remark 2.2.9, we define Γ_ε, using the approximations in Theorem 3.6.5.

By using the map Γ we can obtain an analysis very similar to the 2d case. To begin with, we state the following result about the convergence in norm of the Γ_ε to Γ and conclude this section.

Lemma 2.2.43. *Let γ be as in Definition 2.2.37. We have that $\|id - \Gamma\Gamma_\varepsilon^{-1}\|_{\mathcal{H}^\gamma \to \mathcal{H}^\gamma} \to 0$.*

Proof. The proof is very similar to that of Lemma 2.2.10. □

Density, symmetry and self-adjointness

Firstly, we prove the density of the domain of A, as stated in Definition 2.2.37.

Proposition 2.2.44. *Let $\beta < 1/2$, then the space $\mathcal{W}_\Xi$, as introduced in Definition 2.2.37 is dense in $\mathcal{H}^\beta$ and thus dense in L^2.*

Proof. For an element $g \in \mathcal{H}^\beta$ we first approximate it by

$$e^W e^{-W_\varepsilon} g,$$

next we approximate $e^{-W_\varepsilon} g$ by an $\mathcal{H}^2$ function f_δ s.t. $\|e^{-W_\varepsilon} g - f_\delta\|_{\mathcal{H}^\beta} < \delta$. Lastly we approximate f_δ by

$$\Gamma\Gamma_\varepsilon^{-1} f_\delta,$$

which is close to f_δ in $\mathcal{H}^\beta$. Hence, for arbitrary $g \in \mathcal{H}^\beta$ we can construct the element $e^W \Gamma\Gamma_\varepsilon^{-1} f_\delta$ which is close to g in $\mathcal{H}^\beta$.

□

The following is an analogue of Theorem 2.2.14 for the 3d Hamiltonian.

Theorem 2.2.45. *The renormalised Anderson Hamiltonian $A : \mathcal{D}(A) \to L^2$ is a bounded operator and we get the following $\mathcal{H}^2$ bound for $u^\sharp$*

$$\|u^\sharp\|_{\mathcal{H}^2} \lesssim \|e^{-W} Au\|_{L^2} + C_\Xi \|u^\flat\|_{L^2}. \tag{2.2.40}$$

Similarly we get the bound

$$\|Au\|_{L^2} \lesssim C_\Xi(\|u^\sharp\|_{\mathcal{H}^2} + \|u\|_{L^2}).$$

Proof. By the definition of A we have

$$e^{-W} Au = \Delta u^\sharp + LZ \circ u^\sharp + 2L\widetilde{W} \circ \nabla u^\sharp + G(u^\flat),$$

then we estimate

$$\begin{aligned} \|LZ \circ u^\sharp\|_{L^2} &\lesssim \|Z\|_{\mathcal{C}^{3/2-\delta}} \|u^\sharp\|_{\mathcal{H}^{1/2+2\delta}} \le C_{\varepsilon,\delta} C_\Xi \|u^\flat\|_{L^2} + \varepsilon \|u^\sharp\|_{\mathcal{H}^2} \\ &\text{and} \\ \|L\widetilde{W} \circ \nabla u^\sharp\|_{L^2} &\lesssim \|\widetilde{W}\|_{\mathcal{C}^{3/2-\delta}} \|u^\sharp\|_{\mathcal{H}^{3/2+2\delta}} \le C_{\varepsilon,\delta} C_\Xi \|u^\flat\|_{L^2} + \varepsilon \|u^\sharp\|_{\mathcal{H}^2} \end{aligned}$$

for any $\varepsilon > 0$ using Young's inequality, Sobolev interpolation, and the straightforward bound $\|u^\sharp\|_{L^2} \le C_\Xi \|u^\flat\|_{L^2}$. Moreover we bound $G(u^\flat)$ via

$$\|G(u^\flat)\|_{L^2} \quad \le \quad C_\Xi \|u^\flat\|_{\mathcal{H}^{1+\delta}} \le C_{\varepsilon,\delta} C_\Xi \|u^\flat\|_{L^2} + \varepsilon \|u^\sharp\|_{\mathcal{H}^2},$$

where the first estimate follows from the paraproduct estimates, Proposition 2.3.22, and the commutator bounds (Proposition 2.3.23). This allows us to conclude

$$\|Au\|_{L^2} = \|e^W e^{-W} Au\|_{L^2} \le \|e^W\|_{L^\infty} \|e^{-W} Au\|_{L^2} \le C_\Xi(\|u^\sharp\|_{\mathcal{H}^2} + \|u^\flat\|_{L^2}), \tag{2.2.41}$$

and, in a similar manner,

$$\begin{aligned}\|u^\sharp\|_{\mathcal{H}^2} &\le \|e^{-W}Au\|_{L^2} + \|LZ \circ u^\sharp + 2L\widetilde{W} \circ \nabla u^\sharp\|_{L^2}\\ &\le \|e^{-W}Au\|_{L^2} + C_\Xi \|u^\flat\|_{L^2} + \frac{1}{2}\|u^\sharp\|_{\mathcal{H}^2},\end{aligned}$$

using the above bounds. □

Proposition 2.2.46. *We have that A is a closed operator over its dense domain $\mathcal{D}(A)$.*

Proof. For $u_n \in \mathcal{D}(A)$, suppose that

$$\begin{aligned} u_n &\to u\\ Au_n &\to g.\end{aligned}$$

Then, by (2.2.40), we have that $u_n^\sharp$ is a Cauchy sequence and $\|w - u_n^\sharp\|_{\mathcal{H}^2} \to 0$ for some w. We observe that then $u = e^W \Gamma w$, that is $u \in \mathcal{D}(A)$. After that, writing the same estimate in the end of the proof of Proposition 2.2.15 concludes the proof, this time utilising (2.2.41) instead. □

For the domain what we know is $\mathcal{D}(A) \subset e^W \mathcal{H}^\gamma$. But in the sequel we will need a precise approximation by smooth elements in $\mathcal{H}^2$.

Proposition 2.2.47. *For every $u \in \mathcal{D}(A)$ there exists $u_\varepsilon \in \mathcal{H}^2$ such that*

$$\|u^\flat - u_\varepsilon^\flat\|_{\mathcal{H}^\gamma} + \|u^\sharp - u_\varepsilon^\sharp\|_{\mathcal{H}^2} \to 0$$

as $\varepsilon \to 0$. For $u, v \in \mathcal{D}(A)$, with approximations $u_\varepsilon, v_\varepsilon$ as above, we obtain

$$\langle A_\varepsilon u_\varepsilon, v_\varepsilon\rangle \to \langle Au, v\rangle.$$

Consequently, A is a closed symmetric operator.

Proof. The proof is similar to that of 2d case, this time using Proposition 2.2.41. In this case, for $u^\sharp = u_\varepsilon^\sharp \in \mathcal{H}^2$, we take $u_\varepsilon^\flat = \Gamma_\varepsilon u^\sharp$ and $u_\varepsilon = e^{W_\varepsilon}\Gamma_\varepsilon u^\sharp$ for the approximations. We omit the details. □

Before we introduce the resolvent and the form domain we need the following result.

Proposition 2.2.48. *Let W be as above, then there exists a constant $C_\Xi > 0$ such that*

$$\|\nabla u^\flat\|_{L^2}^2 \lesssim \|e^{-2W}\|_{L^\infty}(-\langle u, Au\rangle + C_\Xi \|u\|_{L^2}),$$

where $u = e^W u^\flat \in \mathcal{D}(A)$.

Proof. Using (2.2.29), we write

$$\begin{aligned}\langle u, Au\rangle &= \langle e^{2W}u^\flat, \Delta u^\flat + 2\nabla u^\flat \nabla W + LZu^\flat\rangle\\ &= -\langle e^{2W}\nabla u^\flat, \nabla u^\flat\rangle + \langle e^{2W}u^\flat, LZu^\flat\rangle,\end{aligned}$$

where the gradient term disappeared because we integrated by parts. Thus

$$\begin{aligned}
\|\nabla u^\flat\|_{L^2}^2 &\leq \|e^{-2W}\|_{L^\infty}\|e^W\nabla u^\flat\|_{L^2}^2 \\
&= \|e^{-2W}\|_{L^\infty}(\langle e^{2W}u^\flat, LZu^\flat\rangle - \langle u, Au\rangle) \\
&\leq \|e^{-2W}\|_{L^\infty}(\|u^\flat\|_{\mathcal{H}^{1/2+\varepsilon}}\|e^{2W}LZu^\flat\|_{\mathcal{H}^{-1/2-\varepsilon}} - \langle u, Au\rangle) \\
&\leq \|e^{-2W}\|_{L^\infty}(\|u^\flat\|_{\mathcal{H}^{1/2+\varepsilon}}^2\|e^{2W}LZ\|_{\mathcal{C}^{-1/2-\varepsilon}} - \langle u, Au\rangle) \\
&\leq \|e^{-2W}\|_{L^\infty}(C_\Xi\|e^{2W}\|_{\mathcal{C}^{1/2-\varepsilon}}\|u^\flat\|_{\mathcal{H}^{1/2+\varepsilon}}^2 - \langle u, Au\rangle),
\end{aligned}$$

where we have used Lemma 2.2.35. Using again Sobolev interpolation and Young's inequality we can conclude by choosing $\varepsilon > 0$ small enough and pick a proper constant $C_\Xi > 0$ for the conclusion. □

After this, we are ready to conclude the self-adjointness of the operator.

Theorem 2.2.49. *The operator A with domain $\mathcal{D}(A)$ is self-adjoint.*

Proof. Choosing $C_\Xi > 0$ (using Proposition 2.2.48) large enough, we again want to prove that

$$(C_\Xi - A)^{-1} : \mathcal{D}(A) \to L^2 \text{ is bounded}.$$

This can be done in precisely the same way as the 2d case, similar to the proof of Proposition 2.2.20, by applying again the Babuska-Lax-Milgram theorem to the the bilinear map

$$\begin{aligned}
B : \mathcal{D}(A) \times L^2 &\to \mathbb{R} \\
B(u,v) &:= \langle (C_\Xi - A)u, v\rangle.
\end{aligned}$$

Afterwards, one concludes self-adjointness by using Proposition 2.2.24. □

Observe that the Proposition 2.2.48 implies the positivity of the form for $C_\Xi - A$. Accordingly, we introduce the shifted operators.

Definition 2.2.50. *For a constant $K_\Xi > C_\Xi$, where C_Ξ is as in the proof of Theorem 2.2.49, we define the following shifted operators*

$$\begin{aligned}
H_\varepsilon &:= A_\varepsilon - K_\Xi \\
H &:= A - K_\Xi
\end{aligned}$$

where in the future the constant K_Ξ may be updated to be larger, if needed.

Now we define the form domain.

Definition 2.2.51. *From Proposition 2.2.41 recall that $u = e^W \Gamma u^\sharp$. We define the form domain of H, denoted by $\mathcal{D}(\sqrt{-H})$, as the closure of the domain under the following norm*

$$\|u\|_{\mathcal{D}(\sqrt{-H})} := \sqrt{\langle u, -Hu\rangle}.$$

We furthermore have the following classification for the domain and the form domain of H; this is the 3d version of Proposition 2.2.23.

Proposition 2.2.52. *We have the following characterisation for the domain and the form domain:*

1. $\Gamma u^\sharp \in e^{-W}\mathcal{D}(H) \Leftrightarrow u^\sharp \in \mathcal{H}^2$. *More precisely, on* $\mathcal{D}(H) = e^W \mathcal{U}_\Xi$ *we have the following norm equivalence*
$$\|u^\sharp\|_{\mathcal{H}^2} \lesssim_\Xi \|H\Gamma u^\sharp\|_{L^2} \lesssim_\Xi \|u^\sharp\|_{\mathcal{H}^2}.$$
2. $u \in \mathcal{D}(\sqrt{-H}) \Leftrightarrow e^{-W}u \in \mathcal{H}^1$. *Then the precise statement is that on* $\mathcal{D}(H)$ *the following norm equivalence holds*
$$\|e^{-W}u\|_{\mathcal{H}^1} \lesssim_\Xi \|u\|_{\sqrt{-H}} \lesssim_\Xi \|e^{-W}u\|_{\mathcal{H}^1},$$
and hence the closures with respect to the two norms coincide.

Proof. This follows from Theorem 2.2.45 and Proposition 2.2.48 similarly to the proof of Proposition 2.2.23. □

Norm resolvent convergence

In this section, we address the resolvent convergence results for the regularised operators as introduced in Remark 2.2.38 and Definition 2.2.50. We first address the norm convergence of approximating Hamiltonians composed with the Γ-maps.

Proposition 2.2.53. *Let* $u^\sharp \in \mathcal{H}^2$, $u = e^W\Gamma u^\sharp$, $u^\flat_\varepsilon = \Gamma_\varepsilon u^\sharp$ *and* $u_\varepsilon = e^{W_\varepsilon}u^\flat_\varepsilon$. *We have that*

$$\|Hu - H_\varepsilon u_\varepsilon\|_{L^2} \lesssim_\Xi \|\Xi_\varepsilon - \Xi\|_{\mathcal{X}^\alpha}\|u^\sharp\|_{\mathcal{H}^2}. \tag{2.2.42}$$

Consequently, this implies that

$$\|He^W\Gamma - H_\varepsilon e^{W_\varepsilon}\Gamma_\varepsilon\|_{\mathcal{H}^2\to L^2} \to 0. \tag{2.2.43}$$

That is to say, $H_\varepsilon e^{W_\varepsilon}\Gamma_\varepsilon \to He^W\Gamma$ *in norm.*

Proof. The proof is similar to that of Proposition 2.2.15. This time one uses the formula (2.2.33) and then proceeds in the same way by using Lemma 2.2.43 instead. Hence, the result. □

In the following results, using the techniques we have used in the 2d part, we address the notions of strong resolvent and norm resolvent convergence.

Theorem 2.2.54. *Let* β *be as defined in Proposition 2.2.44. Then, we have*

$$\|H^{-1} - H_\varepsilon^{-1}\|_{L^2\to\mathcal{H}^\beta} \lesssim_\Xi \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}$$

for $\varepsilon > 0$. *In particular* H_ε *converges to* H *in the norm resolvent sense.*

Proof. This proof is similar to that of Theorem 2.2.26. We only mention the points where it differs.

By Proposition 2.2.53 we have that

$$\|H_\varepsilon e^{W_\varepsilon}\Gamma_\varepsilon u^\sharp - He^W\Gamma u^\sharp\|_{\mathcal{H}^2\to L^2} \lesssim_\Xi \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}\|u^\sharp\|_{\mathcal{H}^2}$$

This implies
$$\|\Gamma_\varepsilon^{-1} e^{-W_\varepsilon} H_\varepsilon^{-1} - \Gamma^{-1} e^{-W} H^{-1}\|_{L^2 \to \mathcal{H}^2} \lesssim_\Xi \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

By using the same tricks as in the proof of Theorem 2.2.26, this time using Proposition 2.2.41 and Lemma 2.2.43, one obtains
$$\|e^{-W_\varepsilon} H_\varepsilon^{-1} - e^{-W} H^{-1}\|_{L^2 \to H^\beta} \lesssim_\Xi \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.$$

We can write the estimate
$$\begin{aligned}\|e^{(W-W_\varepsilon)} H_\varepsilon^{-1} - H^{-1}\|_{L^2 \to H^\beta} &= \|e^W (e^{-W_\varepsilon} H_\varepsilon^{-1} - e^{-W} H^{-1})\|_{L^2 \to H^\beta} \\ &\le \|e^W\|_{H^\beta \to H^\beta} \|e^{-W_\varepsilon} H_\varepsilon^{-1} - e^{-W} H^{-1}\|_{L^2 \to H^\beta}.\end{aligned}$$

which allows us to conclude. □

Lastly, we give a version of Agmon's inequality which can be seen as a 3d analogue of Theorem 2.2.31.

Lemma 2.2.55. *For $u \in \mathcal{D}(H)$ and $\mathcal{D}(H_\varepsilon)$ respectively, we have the following L^∞ bounds*
$$\|u\|_{L^\infty} \lesssim_\Xi \|Hu\|_{L^2}^{1/2} \|\sqrt{-H} u\|_{L^2}^{1/2}$$

$$\|u\|_{L^\infty} \lesssim_\Xi \|H_\varepsilon u\|_{L^2}^{1/2} \|\sqrt{-H_\varepsilon} u\|_{L^2}^{1/2}.$$

Let us mention again, that the point of the latter bound is that the bound is independent of ε.

Proof. The classical version of Agmon's inequality [1] gives the bound
$$\|v\|_{L^\infty} \lesssim \|v\|_{\mathcal{H}^1}^{1/2} \|v\|_{\mathcal{H}^2}^{1/2}.$$

Now we compute
$$\begin{aligned}\|u\|_{L^\infty} &\le \|e^W\|_{L^\infty} \|\Gamma u^\sharp\|_{L^\infty} \lesssim_\Xi \|u^\sharp\|_{L^\infty} \lesssim_\Xi \|u^\sharp\|_{\mathcal{H}^1}^{1/2} \|u^\sharp\|_{\mathcal{H}^2}^{1/2} \\ &\lesssim_\Xi \|Hu\|_{L^2}^{1/2} \|\sqrt{-H} u\|_{L^2}^{1/2},\end{aligned}$$

where we have used Propositions 2.2.41 and 2.2.52 in addition to Agmon's inequality and the straightforward bound $\|u^\sharp\|_{\mathcal{H}^1} \lesssim_\Xi \|u^\flat\|_{\mathcal{H}^1}$. The second inequality follows by the same argument noting that the constant is independent of ε. □

2.3 Semilinear evolution equations

To recall, in the previous section we have introduced the operators H and H_ε (Definitions 2.2.21 and 2.2.50 respectively) along with their domains $\mathcal{D}(H), \mathcal{D}(H_\varepsilon) = \mathcal{H}^2$ (Remarks 2.2.4 and 2.2.40 respectively) and energy domains $\mathcal{D}(\sqrt{-H}), \mathcal{D}(\sqrt{H_\varepsilon}) = \mathcal{H}^1$ (Definitions 2.2.22 and 2.2.51 respectively). We have also studied their resolvents and the norm resolvent convergence of regularised operators (Theorems 2.2.26 and 2.2.54 respectively). Furthermore we have obtained some functional inequalities which will be useful in the present section.

In this part we utilise this preceding analysis in the study of some semilinear PDEs, more precisely nonlinear Schrödinger and wave-type equations with the linear part given by the 2-d and 3-d Anderson Hamiltonian. As a preliminary, we derive and record some simple results for the corresponding linear equations as well as for PDEs with sufficiently nice nonlinearities.

2.3.1 Linear equations and bounded nonlinearities

In this section, before proceeding with the nonlinear PDEs, we first give some straightforward results about the linear evolution and PDEs with bounded/Lipschitz nonlinearities. We also obtain convergence of the solutions to the regularised equations in an appropriate sense.

Abstract Cauchy theory for the linear and bounded nonlinear equations

We want to apply Theorem 3.3.1 from Cazenave [23]. This proves global well-posedness of

$$\begin{aligned} i\partial_t u &= Qu + g(u) \\ u(0) &= u_0 \in \mathcal{D}(Q) \end{aligned}$$

in the strong sense, meaning $u \in C(\mathbb{R}; \mathcal{D}(Q)) \cap C^1(\mathbb{R}; X)$, for a sufficiently nice nonlinearity g and Q self-adjoint on some Hilbert space X.

Theorem 2.3.1. *Consider the abstract Cauchy problem*

$$\begin{cases} i\partial_t u = Qu + g(u) \\ u(0) = u_0 \end{cases} \tag{2.3.1}$$

where Q is a self-adjoint operator on a Hilbert space X. Then we have the following two results for Schrödinger and wave equations respectively.

1. *Assume $(Qu, u) \leq 0$ for $u \in \mathcal{D}(Q)$ and $g : X \to X$ is Lipschitz on bounded sets as well as $(g(x), ix)_X = 0$ for all $x \in X$ and $g = G'$ where $G \in C^1(\mathcal{D}(\sqrt{-Q}))$. Concretely, if we fix $Q = H$, $X = L^2(\mathbb{T}^d)$ $d = 2, 3$ $u_0 \in \mathcal{D}(H)$and $g(u) := K_\Xi u + u\varphi'(|u|^2)$ where $\varphi \in C_b^2$ we get a unique global strong solution of (2.3.1)*

   $$u \in C([0, \infty); \mathcal{D}(H)) \cap C^1([0, \infty); L^2).$$

   We can also relax this slightly if we ask for $u_0 \in \mathcal{D}(\sqrt{-H})$. We get a unique global energy solution

   $$u \in C([0, \infty); \mathcal{D}(\sqrt{-H})) \cap C^1([0, \infty); \mathcal{D}^*(\sqrt{-H})).$$

   In both cases conservation of mass and energy holds for all times.

2. *For the wave equation, with $d = 2, 3$, we set*

   $$Q = i \begin{pmatrix} 0 & \mathbb{I} \\ H & 0 \end{pmatrix}, \quad \mathcal{D}(Q) = \mathcal{D}(H) \oplus \mathcal{D}(\sqrt{-H})$$

   $$X = (L^2(\mathbb{T}^d))^2, \quad g(u) = \begin{pmatrix} 0 \\ -K_\Xi u \end{pmatrix}.$$

Then the abstract linear wave equation

$$
\begin{aligned}
i\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} u \\ \partial_t u \end{pmatrix} &= i\begin{pmatrix} 0 & \mathbb{I} \\ -H & 0 \end{pmatrix}\begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ -K_\Xi u \end{pmatrix} \\
\begin{pmatrix} u \\ \partial_t u \end{pmatrix}_{t=0} &= \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}
\end{aligned}
$$

has a unique global strong solution $(u, \partial_t u) \in C([0,\infty); \mathcal{D}(H)) \times C^1([0,\infty); L^2)$ *i.e.* $u \in C([0,\infty); \mathcal{D}(H)) \cap C^1([0,\infty); \mathcal{D}(\sqrt{-H})) \cap C^2([0,\infty); L^2)$ *and energy conservation holds.*

Proof. 1. The properties of the Hamiltonian have already been verified, it remains to check that the nonlinearity g satisfies all the conditions. We claim that

$$
g(v) = G'(v) \text{ with } G(v) = \int \frac{K_\Xi}{2}|v|^2 + \frac{1}{2}\int \varphi(|v|^2) \in C^1(\mathcal{D}(\sqrt{-H}); \mathbb{R}).
$$

Moreover $g : L^2 \to L^2$ is locally Lipschitz and $\langle g(u), iu\rangle = 0$ for $u \in L^2$.

By construction we have

$$
\langle g(u), iu\rangle = \mathrm{Re}\, i \int K_\Xi |u|^2 + |u|^2 \varphi'(|u|^2) = 0.
$$

Next we show the differentiability of G. Let $u, v \in \mathcal{D}(\sqrt{-H})$, then

$$
\begin{aligned}
& G(u) - G(v) - G'(v)(u-v) \\
&= \int \frac{K_\Xi}{2}|u|^2 + \frac{1}{2}\int \varphi(|u|^2) - \int \frac{K_\Xi}{2}|v|^2 - \frac{1}{2}\int \varphi(|v|^2) - (g(v), u-v) \\
&= \int \frac{K_\Xi}{2}|u-v|^2 + \frac{1}{2}\int f(u) - f(v) - f'(v)(\overline{u-v}) \\
|\ldots| &\leq (K_\Xi + \|\varphi\|_{C_b^2})\|u-v\|_{L^2}^2 \\
&\leq (K_\Xi + \|\varphi\|_{C_b^2})\|u-v\|_{\mathcal{D}(\sqrt{-H})}^2,
\end{aligned}
$$

with $f(u) := \varphi(|u|^2)$. This proves the differentiability. Lastly we prove the L^2 local Lipschitz property of g. Fix $v \in L^2$ and $u \in B_M(v)$, for some $M > 0$. Then

$$
\begin{aligned}
\|g(u) - g(v)\|_{L^2} &\leq K_\Xi \|u-v\|_{L^2} + \|u\varphi'(|u|^2) - v\varphi'(|v|^2)\|_{L^2} \\
&\leq K_\Xi \|u-v\|_{L^2} + \|\varphi'\|_\infty \|u-v\|_{L^2} + \|v\|_{L^2}\|\varphi'(|u|^2) - \varphi'(|v|^2)\|_{L^2} \\
&\leq K_\Xi \|u-v\|_{L^2} + \|\varphi'\|_\infty \|u-v\|_{L^2} + \|v\|_{L^2}\|\varphi''\|_\infty \|u-v\|_{L^2}
\end{aligned}
$$

hence g is locally Lipschitz as a map from L^2 to L^2.

2. See [70, Chapter X.13].

□

The linear multiplicative Schrödinger equation

In this part, we discuss the solution to the linear Schrödinger equation

$$i\partial_t u = Hu \quad \text{on } \mathbb{T}^d, \tag{2.3.2}$$

with initial data in the domain of H. A simple but important observation is that the Schrödinger equation conserves the L^2 norm. Also observe that $\partial_t u$ formally satisfies

$$i\partial_t\partial_t u = H(\partial_t u),$$

so it solves the same equation and in particular we have that $\|\partial_t u(t)\|_{L^2}$ is conserved and that

$$\|Hu\|_{L^2} = \|\partial_t u(t)\|_{L^2} = \|\partial_t u(0)\|_{L^2} = \|Hu(0)\|_{L^2},$$

which we will assume to be finite. This gives us quite a natural condition that the initial data should satisfy. Therefore we will assume $u_0 \in \mathcal{D}(H)$ which implies $\|Hu_0\|_{L^2} < \infty$ by Theorem 2.2.14 . To make this precise, we write

$$\begin{aligned} u(t) &= e^{-itH}u_0 \\ \frac{d}{dt}u(t) &= -ie^{-itH}Hu_0 \\ \left\|\frac{d}{dt}u(t)\right\|_{L^2} &= \|Hu_0\|_{L^2} = \|Hu(t)\|_{L^2}. \end{aligned}$$

So $\|\partial_t u(t)\|_{L^2}$ is conserved for for the solution u as above. For the regularised equation, the unique solution is given by

$$u_\varepsilon(t) = e^{-itH_\varepsilon}u_0^\varepsilon \in \mathcal{H}^2,$$

where $u_0^\varepsilon \in \mathcal{H}^2$ is the regularised initial datum. If we choose the regularisation

$$u_0^\varepsilon := H_\varepsilon^{-1}Hu_0 \in \mathcal{H}^2,$$

then we have $H_\varepsilon u_0^\varepsilon = Hu_0$ and we readily get $u_0^\varepsilon \to u_0$ in L^2 by norm resolvent convergence, namely Theorem 2.2.26. By [71, Theorem VIII.21], $e^{-itH_\varepsilon} \to e^{-itH}$ strongly for any time t, which implies

$$\begin{aligned} e^{-itH_\varepsilon}u_0^\varepsilon &\to e^{-itH}u_0 \\ &\text{and} \\ e^{-itH_\varepsilon}H_\varepsilon u_0^\varepsilon &\to e^{-itH}Hu_0 \end{aligned}$$

in L^2 for any $t \in \mathbb{R}$.

We summarize these results in the following theorem

Theorem 2.3.2. *Let $T > 0$, $u_0 \in \mathcal{D}(H)$. Then there exists a unique solution $u \in C([0,T];\mathcal{D}(H)) \cap C^1([0,T];L^2)$ to the equation*

$$\begin{cases} i\partial_t u = Hu \\ u(0,\cdot) = u_0 \end{cases} \quad \text{on } [0,T] \times \mathbb{T}^d.$$

Moreover, this agrees with the L^2-limit of the solutions $u_\varepsilon \in C([0,T];\mathcal{H}^2)\cap C^1([0,T];L^2)$ to

$$\begin{cases} i\partial_t u_\varepsilon = H_\varepsilon u_\varepsilon \\ u(0,\cdot) = u_0^\varepsilon \end{cases} \quad \text{on } [0,T]\times\mathbb{T}^d,$$

with the regularised data given as

$$u_0^\varepsilon := H_\varepsilon^{-1} H u_0 \in \mathcal{H}^2.$$

One also obtains the convergence of $\partial_t u_\varepsilon$ and $H_\varepsilon u_\varepsilon$ to $\partial_t u$ and $H_\varepsilon u$ in L^2.

Remark 2.3.3. One could also get global well-posedness for the equation with initial data in $\mathcal{D}(\sqrt{-H})$ or in L^2. Moreover, one could treat a bounded nonlinearity as above.

The linear multiplicative wave equation

Similarly to the Schrödinger case, we now consider the linear wave equation

$$\partial_t^2 u = Hu$$

with initial data $(u_0,u_1)\in\mathcal{D}(H)\times\mathcal{D}(\sqrt{-H})$. For the regularised equation

$$\partial_t^2 u_\varepsilon = H_\varepsilon u_\varepsilon$$
$$(u_\varepsilon,\partial_t u_\varepsilon)|_{t=0} = (u_0^\varepsilon,u_1^\varepsilon)\in\mathcal{H}^2\times\mathcal{H}^1$$

the solution is given by

$$\begin{pmatrix} u_\varepsilon \\ \partial_t u_\varepsilon \end{pmatrix} = e^{itQ_\varepsilon}\begin{pmatrix} u_0^\varepsilon \\ u_1^\varepsilon \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{-H_\varepsilon})u_0^\varepsilon \\ \frac{\sin(t\sqrt{-H_\varepsilon})}{\sqrt{-H_\varepsilon}}u_1^\varepsilon \end{pmatrix}, \qquad \text{where} \qquad Q_\varepsilon = i\begin{pmatrix} 0 & \mathbb{I} \\ -H_\varepsilon & 0 \end{pmatrix}.$$

and the $\sin,\cos$ objects are defined via functional calculus. We again choose the same approximation for u_0 as in the Schrödinger case

$$u_0^\varepsilon := (-H_\varepsilon)^{-1}(-H)u_0 \in \mathcal{H}^2$$
$$u_1^\varepsilon := (\sqrt{-H_\varepsilon})^{-1}\sqrt{-H}u_1 \in \mathcal{H}^1$$

for initial data $(u_0,u_1)\in\mathcal{D}(H)\times\mathcal{D}(\sqrt{-H})$. Then we again have

$$\begin{aligned} u_0^\varepsilon &\to u_0 \text{ in } L^2 \\ H_\varepsilon u_0^\varepsilon &\to Hu_0 \text{ in } L^2. \end{aligned}$$

For the initial velocity, we also have

$$u_1^\varepsilon \to u_1 \text{ in } L^2.$$

Then for any time t we get as in the Schrödinger case

$$\begin{pmatrix} u_\varepsilon(t) \\ \partial_t u_\varepsilon(t) \end{pmatrix} = e^{itQ_\varepsilon}\begin{pmatrix} u_0^\varepsilon \\ u_1^\varepsilon \end{pmatrix} \to e^{itQ}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{-H})u_0 \\ \frac{\sin(t\sqrt{-H})}{\sqrt{-H}}u_1 \end{pmatrix} \text{ in } L^2$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{pmatrix} u_\varepsilon(t) \\ \partial_t u_\varepsilon(t) \end{pmatrix} = ite^{itQ_\varepsilon}Q_\varepsilon\begin{pmatrix} u_0^\varepsilon \\ u_1^\varepsilon \end{pmatrix} \to ite^{itQ}Q\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \text{ in } L^2.$$

Moreover, we have that the convergence of the energies, namely

$$\begin{aligned} E_\varepsilon(t) &:= \left\langle \begin{pmatrix} u_\varepsilon(t) \\ \partial_t u_\varepsilon(t) \end{pmatrix}, \begin{pmatrix} -H_\varepsilon & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} u_\varepsilon(t) \\ \partial_t u_\varepsilon(t) \end{pmatrix} \right\rangle \\ &\to \left\langle \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix}, \begin{pmatrix} -H & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} \right\rangle = E(t) \end{aligned}$$

for any time t and thus in particular the energy conservation passes to the limit. We record these observations in the following theorem.

Theorem 2.3.4. *Let $T > 0$ and $(u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H})$. Then there exists a unique solution $(u, \partial_t u) \in C([0,T]; \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H})) \cap C^1([0,T]; \mathcal{D}(\sqrt{-H}) \times L^2)$ to the equation*

$$\begin{aligned} \partial_t^2 u &= Hu \text{ in } (0,T) \times \mathbb{T}^d \\ (u, \partial_t u)|_{t=0} &= (u_0, u_1) \end{aligned}$$

moreover it is equal to the $L^\infty((0,T); L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d))$ limit of the approximate solutions $(u_\varepsilon, \partial_t u_\varepsilon)$ to

$$\begin{aligned} \partial_t^2 u_\varepsilon &= H_\varepsilon u_\varepsilon \text{ in } (0,T) \times \mathbb{T}^d \\ (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} &= (u_0^\varepsilon, u_1^\varepsilon), \end{aligned}$$

and moreover we have the following convergence at any fixed time t

$$\begin{aligned} u_\varepsilon(t) &\to u(t) \text{ in } L^2 \\ H_\varepsilon u_\varepsilon(t) &\to Hu(t) \text{ in } L^2 \\ \sqrt{-H_\varepsilon}\partial_t u_\varepsilon(t) &\to \sqrt{-H}\partial_t u(t) \text{ in } L^2 \\ \partial_t^2 u_\varepsilon(t) &\to \partial_t^2 u(t) \text{ in } L^2 \end{aligned}$$

with $(u_0^\varepsilon, u_1^\varepsilon)$ as above. Also, the energies converge and are conserved in time.

Proof. The computations above prove that the L^2 limit of the solutions we obtain is equal to the solution of the abstract Cauchy problem in Theorem 2.3.1 for all times. Hence, the two are equal. □

2.3.2 Nonlinear Schrödinger equations in two dimensions

In this section we are interested in solving the following defocussing cubic Schrödinger-type equation

$$i\partial_t u = Hu - u|u|^2, \tag{2.3.3}$$

with domain and energy space data.

Recall that for the operator H we have

$$\langle u, -Hu \rangle \geq 0 \text{ for all } u \in \mathcal{D}(H).$$

We consider the mild formulation of (2.3.3)

$$u(t) = e^{-itH}u_0 + i\int_0^t e^{i(s-t)H}u(s)|u(s)|^2\mathrm{d}s \tag{2.3.4}$$

Furthermore, we introduce the energy for u as

$$E(u)(t) := -\frac{1}{2}\langle u(t), Hu(t)\rangle + \frac{1}{4}\int |u(t)|^4. \tag{2.3.5}$$

Using the equation one sees that the energy is formally conserved in time.

Solutions with initial condition in $\mathcal{D}(H)$

In this section we assume $u_0 \in \mathcal{D}(H)$. This is similar in spirit to the global strong well-posedness of the classical cubic NLS with initial data in $\mathcal{H}^2$, which was solved in [15]. We obtain global in time strong solutions in our setting, which is the best one can hope for in view of the classical result. We regularise the initial data in the following way

$$u_0^\varepsilon = H_\varepsilon^{-1}Hu_0 \in \mathcal{D}(H_\varepsilon)$$

so that by the norm resolvent convergence of H_ε to H (see Theorem 2.2.26) we have

$$\lim_{\varepsilon\to 0} u_0^\varepsilon = u_0 \in L^2$$
$$H_\varepsilon u_0^\varepsilon = Hu_0 \in L^2.$$

Note that $\mathcal{D}(H_\varepsilon) = \mathcal{H}^2$ so there exists global solutions $u_\varepsilon \in C([0,T], \mathcal{D}(H_\varepsilon)) \cap C^1([0,T], L^2)$. While this follows as in [15], it is also an immediate consequence of the following result which says that for the operators H_ε and H we obtain global in time strong solutions of the associated cubic NLS on $\mathbb{T}^2$.

Theorem 2.3.5. *For an arbitrary time $T > 0$, there exist unique solutions $u_\varepsilon \in C([0,T];\mathcal{H}^2) \cap C^1([0,T];L^2)$ and $u \in C([0,T];\mathcal{D}(H)) \cap C^1([0,T];L^2)$ to*

$$u_\varepsilon(t) = e^{-itH_\varepsilon}u_0^\varepsilon + i\int_0^t e^{-isH_\varepsilon}u_\varepsilon|u_\varepsilon|^2(t-s)ds, \tag{2.3.6}$$

and

$$u(t) = e^{-itH}u_0 + i\int_0^t e^{-isH}u|u|^2(t-s)ds \tag{2.3.7}$$

respectively, with initial data $u_0^\varepsilon \in \mathcal{H}^2$ and $u_0 \in \mathcal{D}(H)$.

Before we prove the theorem, we need the following technical lemmas which will be used throughout the proof. The first one is a logarithmic Gronwall lemma.

Lemma 2.3.6. *Let $C_2, \log C_1 \geq 1$ and $\theta(t) \geq 1$ satisfy*

$$\theta(t) \leq C_1 + C_2\int_0^t \theta(s)\log(1+\theta(s))\mathrm{d}s = h(t).$$

Then we have

$$h(t) \leq \exp(\log h(0)e^{C_2 t}) - 1.$$

Proof. We have that h is a subsolution of the equation

$$\partial_t h(t) = C_2\theta(t)\log(1+\theta(t)) \leq C_2(h(t)+1)\log(h(t)+1).$$

So taking $\rho(t)$ to be a solution of $\partial_t\rho(t) = C_2(\rho(t)+1)\log(\rho(t)+1), \quad \rho(0) = h(0)$, we have $\rho(t) \geq h(t)$. Indeed $\rho(0) = h(0)$ and whenever we have $\rho(t) = h(t)$ then

$$\partial_t(\rho(t) - h(t)) \geq C_2(\rho(t)+1)\log(\rho(t)+1) - C_2(h(t)+1)\log(h(t)+1) = 0.$$

Observe moreover that

$$\partial_t \log(\rho(t)+1) = C_2 \log(\rho(t)+1) \Rightarrow \log(\rho(t)+1) = (\log h(0))e^{C_2 t}.$$

□

Lemma 2.3.7. *For $v \in C([0,T];\mathcal{H}^2) \cap C^1([0,T];L^2)$, $f(v)(t) = |v(t)|^2 v(t)$ is C^1 as a map from $[0,T]$ to L^2. The same is true for $v \in C([0,T];\mathcal{D}(H)) \cap C^1([0,T];L^2)$.*

Proof. We write

$$\frac{|v(t+h)|^2 v(t+h) - |v(t)|^2 v(t)}{h}$$

and add and subtract the term $v^2(t+h)\bar{v}(t)$ which yields

$$\frac{\bar{v}(t+h)(v(t+h))v(t+h) - v^2(t+h)\bar{v}(t) + v^2(t+h)\bar{v}(t) - \bar{v}(t)v(t)v(t)}{h}.$$

This can be rearranged as

$$v^2(t+h)\frac{\bar{v}(t+h) - \bar{v}(t)}{h} + (\bar{v}(t)(v(t+h)+v(t)))\,\frac{v(t+h) - v(t))}{h}$$

where all the terms converge individually in L^2 as $h \to 0$. Indeed, one can easily check that the multiplication map $(f,g) \to f \cdot g$ defines a continuous map $\mathcal{H}^2 \times L^2 \to L^2$. This follows from the embedding $\mathcal{H}^2 \hookrightarrow L^\infty$ in 2d. Since, by Lemma 2.2.29, we also have the embedding $\mathcal{D}(H) \hookrightarrow L^\infty$, the same holds in this case. □

Proof of Theorem 2.3.5. This is a fixed point argument which is essentially the same in both cases so we only treat one of them. For fixed $u_0 \in \mathcal{D}(H)$, we define the operator

$$\Phi(u)(t) := e^{-itH}u_0 + i\int_0^t e^{-isH}u|u|^2(t-s)ds$$

and claim that is in fact a contraction on $X = C([0,T_E];\mathcal{D}(H)) \cap C^1([0,T_E];L^2)$, where the time $T_E > 0$ depends on the initial data and the energy, which is conserved. This will allow us to obtain a global in time solution.

We bound, for $\|u\|_X \leq M$ with M chosen below, using Theorem 2.2.31

$$\begin{aligned}\|\partial_t\Phi(u)\|_{L^2}(t) \leq & \|Hu_0\|_{L^2} + \int_0^t \|\partial_t(u|u|^2)(s)\|_{L^2}\,\mathrm{ds} + C\|u_0\|_{L^6}^3 \\ \leq & \|Hu_0\|_{L^2} + \int_0^t \|\partial_t u(s)\|_{L^2}\|u(s)\|_{L^\infty}^2\,\mathrm{ds} + C\|u_0\|_{L^6}^3 \\ \leq & \|Hu_0\|_{L^2} + \int_0^t CME(u)(s)(1+\log(M+1))\,\mathrm{ds} + C\|u_0\|_{L^6}^3 \leq \frac{M}{2}\end{aligned}$$

for $t \leq T_E$ small enough such that

$$\int_0^{T_E} CME(u)(s)(1+\log(M+1))\,\mathrm{ds} \leq \frac{M}{2} - (\|Hu_0\|_{L^2} + C\|u_0\|_{L^6}^3)$$

and M such that $\frac{M}{2} - (\|Hu_0\|_{L^2} + C\|u_0\|_{L^6}^3) > 0$.

Analogously, we compute

$$\begin{aligned}\|H\Phi(u)\|_{L^2}(t) \leq & \|Hu_0\|_{L^2} + \|\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(e^{isH})|u|^2 u(t-s)\mathrm{d}s\|_{L^2} \\ \leq & \|Hu_0\|_{L^2} + \int_0^t CME(u)(s)(1+\log(M+1))\,\mathrm{ds} + C\|u_0\|_{L^6}^3 \leq \frac{M}{2},\end{aligned}$$

since we can integrate by parts in the integral. Furthermore, Stone's theorem [71, Theorem VIII.7] implies the time regularity of $\Phi(u)$. For the contraction property, we need to estimate

$$\partial_t\Phi(u)(t) - \partial_t\Phi(v)(t) = \int_0^t e^{-isH}\partial_t(u|u|^2 - v|v|^2)(t-s)\,\mathrm{ds}\,.$$

We obtain, by using Theorem 2.2.31 and Lemma 2.2.29,

$$\begin{aligned}&\|\partial_t\Phi(u)(t) - \partial_t\Phi(v)(t)\|_{L^2} \leq \\ \leq & 3\int_0^t \|u(s)\|_{L^\infty}^2\|\partial_t u - \partial_t v\|_{L^\infty L^2} + \|\partial_t v\|_{L^\infty L^2}\|Hu - Hv\|_{L^\infty L^2}(\|u(s)\|_{L^\infty} + \|v(s)\|_{L^\infty})\,\mathrm{ds} \\ \leq & 3\|u-v\|_X \int_0^t E(u)(s)(1+\log(1+M)) + M\sqrt{(1+\log(\mathrm{M}+1))}(E^{1/2}(u)(s) + E^{1/2}(v)(s))\,\mathrm{ds} \\ < & \|u-v\|_X\end{aligned}$$

for $t \leq T_E$ by possibly making T_E smaller depending on $E(u)$. This gives us short time wellposedness, but since the time span depends only on the energy and the initial data, this can be iterated to yield a global strong solution. In fact, the only thing that is left to show is that $\|Hu(T_E)\|_{L^2}$ can be bounded by $\|Hu_0\|_{L^2}$, i.e. a priori bounds. This allows us to choose a global M and then also a fixed time span T_E which immediately implies a global solution. For the solution that exists up to time T_E, we have the estimate

$$\begin{aligned}\|Hu(T_E)\|_{L^2} &\lesssim \|Hu_0\|_{L^2} + \|u_0\|_{L^6}^3 + \int_0^{T_E} \|\partial_t u(|u|^2)(s)\|_{L^2} ds \\ &\lesssim \|Hu_0\|_{L^2} + \|u_0\|_{L^6}^3 + \int_0^{T_E} \|\partial_t u(s)\|_{L^2} \|u(s)\|_{L^\infty}^2 ds \\ &\lesssim \|Hu_0\|_{L^2} + \|u_0\|_{L^6}^3 + \int_0^{T_E} (\|Hu(s)\|_{L^2} \\ &+ E^{3/2}(u_0))E(u_0)(1+\log(1+\|Hu(s)\|_{L^2}))ds,\end{aligned}$$

where we have used again Theorem 2.2.31 and the fact that one can estimate $\|\partial_t u\|_{L^2}$ by $\|Hu\|_{L^2}$ using the equation. Now, we can conclude by using Lemma 2.3.6. This gives us a bound, by possibly taking larger constants, of the form

$$\|Hu(T_E)\|_{L^2} \lesssim C_\Xi E^{3/2}(u_0) + \exp(e^{cE(u_0)T} \log[C_\Xi E^{3/2}(u_0) + \|Hu_0\|_{L^2}) - 1, \tag{2.3.8}$$

where T is the maximum time of existence. Hence M, and therefore T_E, can be chosen globally which means that we can solve the cubic NLS on the whole interval $[0, T]$ by iterating. The proof for the regularised Hamiltonian follows the same lines with the crucial note that the inequality constant in Theorem 2.2.31 does not blow up, namely the constant does not depend on ε but only on Ξ. □

Remark 2.3.8. One sees from the proof that the same remains true for NLS with lower power nonlinearity, i.e.

$$i\partial_t u = Hu - u|u|^{p-1},$$

with $p \in (1, 3)$. The result will also remain true in the focussing case under some suitable smallness conditions on u_0.

We will moreover prove that the approximate solutions u_ε, which are strong solutions of

$$\begin{aligned} i\partial_t u_\varepsilon &= H_\varepsilon u_\varepsilon - u_\varepsilon |u_\varepsilon|^2, \\ u_\varepsilon(0) &= u_0^\varepsilon \in \mathcal{D}(H_\varepsilon) = \mathcal{H}^2. \end{aligned} \tag{2.3.9}$$

converge to the solution u of the limiting problem. We prove the following result.

Theorem 2.3.9. *Let $u_0 \in \mathcal{D}(H)$ and $T > 0$ be an arbitrary time. Solutions to the regularised equations with initial data $u_0^\varepsilon := (-H_\varepsilon)^{-1}(-H)u_0 \in \mathcal{H}^2$, (the unique global strong solutions u_ε of* (2.3.9)*) converges to the unique global strong solutions $u \in C([0, T]; \mathcal{D}(H)) \cap C^1([0, T]; L^2)$ of*

$$\begin{aligned} i\partial_t u &= Hu - |u|^2 u, \\ u(0) &= u_0, \end{aligned}$$

which is obtained in Theorem 2.3.5. In fact, we get the following convergence results

$$\begin{aligned} u_\varepsilon(t) &\to u(t) \text{ in } L^2 \\ H_\varepsilon u_\varepsilon(t) &\to Hu(t) \text{ in } L^2 \\ \partial_t u_\varepsilon(t) &\to \partial_t u(t) \text{ in } L^2 \end{aligned}$$

for all $t \in [0, T]$.

Proof. We know that the u_ε satisfy the mild formulation

$$u_\varepsilon(t) = e^{-itH_\varepsilon}u_0^\varepsilon + i\int_0^t e^{-i(t-s)H_\varepsilon}u_\varepsilon(s)|u_\varepsilon(s)|^2\mathrm{d}s$$

and u satisfies

$$u(t) = e^{-itH}u_0 + i\int_0^t e^{-i(t-s)H}u(s)|u(s)|^2\mathrm{d}s.$$

We compute

$$\begin{aligned}
&Hu(t) - H_\varepsilon u_\varepsilon(t) = \\
=&(e^{-itH} - e^{-itH_\varepsilon})Hu_0 + \int_0^t e^{-i(t-s)H}\partial_s(u|u|^2(s))\mathrm{d}s - \int_0^t e^{-i(t-s)H_\varepsilon}\partial_s(u_\varepsilon|u_\varepsilon|^2(s))\mathrm{d}s \\
&+ u|u|^2(t) - u_\varepsilon|u_\varepsilon|^2(t) \\
=&(e^{-itH} - e^{-itH_\varepsilon})Hu_0 + \int_0^t (e^{-i(t-s)H} - e^{-i(t-s)H_\varepsilon})\partial_s(u|u|^2(s))\mathrm{d}s \\
&- \int_0^t e^{-i(t-s)H_\varepsilon}(\partial_s(u_\varepsilon|u_\varepsilon|^2(s)) - \partial_s(u|u|^2(s)))\mathrm{d}s \\
&+ \int_0^t \partial_s(u|u|^2(s)) - \partial_s(u_\varepsilon|u_\varepsilon|^2(s))\mathrm{d}s + u_0|u_0|^2 - u_0^\varepsilon|u_0^\varepsilon|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|Hu(t) - H_\varepsilon u_\varepsilon(t)\|_{L^2} \lesssim &\|(e^{-itH} - e^{-itH_\varepsilon})Hu_0\|_{L^2} + \|u_0|u_0|^2 - u_0^\varepsilon|u_0^\varepsilon|^2\|_{L^2} \\
&+ \int_0^t \|(e^{-i(t-s)H} - e^{-i(t-s)H_\varepsilon})\partial_s(u|u|^2(s))\|_{L^2}\mathrm{d}s \\
&+ \int_0^t \|\partial_s(u|u|^2(s)) - \partial_s(u_\varepsilon|u_\varepsilon|^2(s))\|_{L^2}\mathrm{d}s,
\end{aligned}$$

where the first three terms converge to zero by norm resolvent convergence and Theorem VIII.21 in [71]. For the last term we can bound similarly to the proof of Theorem 2.3.5,

$$\begin{aligned}
&\int_0^t \|\partial_s(u|u|^2(s)) - \partial_s(u_\varepsilon|u_\varepsilon|^2(s))\|_{L^2}\mathrm{d}s \\
\lesssim &\int_0^t \|\partial_s u(s) - \partial_s u_\varepsilon(s)\|_{L^2}(\|u\|^2_{L^\infty L^\infty} + \|u_\varepsilon\|^2_{L^\infty L^\infty}) \\
&+ \|u(s) - u_\varepsilon(s)\|_{L^\infty}\|\partial_t u\|_{L^\infty L^2}(\|u\|_{L^\infty L^\infty} + \|u_\varepsilon\|_{L^\infty L^\infty})\mathrm{d}s \\
\lesssim &C(T, u_0, \Xi)\int_0^t \|\partial_s u(s) - \partial_s u_\varepsilon(s)\|_{L^2} + \|Hu(s) - H_\varepsilon u_\varepsilon(s)\|_{L^2} \\
&+ \|u(s) - u_\varepsilon(s)\|_{L^2}\mathrm{d}s + C(T, u_0, \Xi)\|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha},
\end{aligned}$$

where we have used the a priori bounds obtained in the proof of Theorem 2.3.5 and the bound

$$\begin{aligned}
\|u(s) - u_\varepsilon(s)\|_{L^\infty} &\lesssim_\Xi \|u^\sharp(s) - u_\varepsilon^\sharp(s)\|_{\mathcal{H}^2} \\
&\lesssim_\Xi \|Hu(s) - H_\varepsilon u_\varepsilon(s)\|_{L^2} + \|u(s) - u_\varepsilon(s)\|_{L^2} + C(T, u_0)\|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}
\end{aligned}$$

which can be proved in the same way as Theorem 2.2.14, using Proposition 2.2.7 and the embedding $\mathcal{H}^2 \hookrightarrow L^\infty$.
Similarly we can bound (by $O(\varepsilon)$ we denote terms that converge to zero as $\varepsilon \to 0$)

$$\begin{aligned}
&\|\partial_t u(t) - \partial_t u_\varepsilon(t)\|_{L^2} \\
\lesssim &\|(e^{-itH} - e^{-itH_\varepsilon})Hu_0\|_{L^2} + \|e^{-itH}u_0|u_0|^2 - e^{-itH_\varepsilon}u_0^\varepsilon|u_0^\varepsilon|^2\|_{L^2} \\
&+ \left\|\int_0^t e^{-isH}\partial_t(u|u|^2(t-s))\mathrm{d}s - \int_0^t e^{-isH_\varepsilon}\partial_t(u_\varepsilon|u_\varepsilon|^2(t-s))\mathrm{d}s\right\|_{L^2} \\
\lesssim &O(\varepsilon) + C(T, u_0, \Xi)\int_0^t \|\partial_s u(s) - \partial_s u_\varepsilon(s)\|_{L^2} \\
&+ \|Hu(s) - H_\varepsilon u_\varepsilon(s)\|_{L^2} + \|u(s) - u_\varepsilon(s)\|_{L^2}\mathrm{d}s
\end{aligned}$$

and we have

$$\begin{aligned}
&\|u(t) - u_\varepsilon(t)\|_{L^2} \\
&\lesssim \|e^{-itH}u_0 - e^{-itH_\varepsilon}u_0^\varepsilon\|_{L^2} + \left\|\int_0^t e^{-i(t-s)H}(u|u|^2(s))\mathrm{d}s - \int_0^t e^{-i(t-s)H_\varepsilon}(u_\varepsilon|u_\varepsilon|^2(s))\mathrm{d}s\right\|_{L^2} \\
&\lesssim O(\varepsilon) + C(T, u_0, \Xi)\int_0^t \|u(s) - u_\varepsilon(s)\|_{L^2}\mathrm{d}s.
\end{aligned}$$

Thus, for $\phi_\varepsilon(t) := \|u(t) - u_\varepsilon(t)\|_{L^2} + \|\partial_t u(t) - \partial_t u_\varepsilon(t)\|_{L^2} + \|Hu(t) - H_\varepsilon u_\varepsilon(t)\|_{L^2}$ we have

$$\phi_\varepsilon(t) \lesssim O(\varepsilon) + \int_0^t \phi_\varepsilon(s)\mathrm{d}s$$

and by Gronwall we can conclude that $\phi_\varepsilon(t) \to 0$ for all t as $\varepsilon \to 0$.

This finishes the proof. □

Remark 2.3.10. Observe that the above also works in three dimensions. The only difference being that one uses Lemma 2.2.55 instead of Theorem 2.2.31. But note that this gives only local in time strong solutions and as in Theorem 2.3.9 we also obtain the convergence of solutions to the approximated PDEs. This is due to the fact that, unlike the 2d case, one uses a polynomial type Gronwall [31], as opposed to a logarithmic Gronwall, which leads to an estimate that blows up in finite time. In fact, this can be formulated as a blow up alternative (with respect to the L^∞-norm) similarly to the classical case of $\mathcal{H}^2$-solutions [23].

Energy solutions

In this section we solve (2.3.3) in the energy space. For the global well-posedness of (standard) cubic NLS on the 2d torus see [11]. Note that the result we get here is somewhat weaker, since we obtain only existence and partial regularity in time. This is as good as what one would get in the classical case without the use of Strichartz estimates, see [17] and references therein for further information. This issue is addressed in Chapter 3 where Strichartz estimates are proved and applied to well-posedness in different low-regularity regimes which include energy solutions in 2 dimensions.

In this section we denote the dual of $\mathcal{D}(\sqrt{-H})$ by $\mathcal{D}(\sqrt{-H})^*$; We naturally have $\mathcal{D}(\sqrt{-H}) \subseteq L^2 \subseteq \mathcal{D}(\sqrt{-H})^*$.

Theorem 2.3.11. *For $u_0 \in \mathcal{D}(\sqrt{-H})$, the equation* (2.3.3) *has a solution u such that $u \in \mathcal{C}^{1/2}([0,T];L^2) \cap C([0,T];\mathcal{D}(\sqrt{-H}))$.*

For the initial datum u_0, we construct the following approximation

$$u_0^\varepsilon := (1+\varepsilon\sqrt{-H})^{-1}u_0 \in \mathcal{D}(H).$$

Note that by continuous functional calculus the operator $(\sqrt{-H})^{-1} : L^2 \to \mathcal{D}(\sqrt{-H})$ is bounded and we have $u_0^\varepsilon \to u_0$ in $\mathcal{D}(\sqrt{-H})$.

Lemma 2.3.12. *For u_0, u_0^ε as above, we have the following convergence of energies.*

$$E(u_0^\varepsilon) := -\frac{1}{2}\langle u_0^\varepsilon, Hu_0^\varepsilon\rangle + \frac{1}{4}\int |u_0^\varepsilon|^4 \to -\frac{1}{2}\langle u_0, Hu_0\rangle + \frac{1}{4}\int |u_0|^4 = E(u_0)$$

Proof. By the above observation the first terms converge. For the L^4 terms, we can conclude using Lemma 2.2.28 and the $\mathcal{D}(\sqrt{-H})$ convergence.

□

Consider the nonlinear Schrödinger equation

$$\begin{aligned} i\partial_t u_\varepsilon &= Hu_\varepsilon - u_\varepsilon|u_\varepsilon|^2 \\ u_\varepsilon(0) &= u_0^\varepsilon \in \mathcal{D}(H). \end{aligned} \tag{2.3.10}$$

As we have seen in section 2.3.2, there exists a unique solution u_ε to this equation in $C([0,T];\mathcal{D}(H)) \cap C^1([0,T];L^2)$ which conserves the energy

$$E(u_\varepsilon(0)) = E(u_\varepsilon(t)) = -\frac{1}{2}\langle u_\varepsilon(t), Hu_\varepsilon(t)\rangle + \frac{1}{4}\|u_\varepsilon(t)\|_{L^4}^4.$$

Lemma 2.3.13 (A priori bounds)**.** *For solutions u_ε to* (2.3.10)*, we have the following uniform bounds.*

$$\begin{aligned} \|u_\varepsilon\|_{L^\infty L^2} &\lesssim \|u_0\|_{L^2} \\ \|\sqrt{-H}u_\varepsilon\|_{L^\infty L^2} &\lesssim E^{1/2}(u_0) \\ \|(\sqrt{-H})^{-1}\partial_t u_\varepsilon\|_{L^\infty L^2} &\lesssim_\Xi E^{1/2}(u_0) + E^{3/2}(u_0) \end{aligned}$$

Proof. Since we have conservation of mass and energy, the first and second follow directly, using also Lemma 2.3.12 and the positivity of the energy. For the third bound, we use the equation and the fact that

$$\|u_\varepsilon\|_{L^6}^3 \lesssim_\Xi E^{3/2}(u_0),$$

which follows from Lemma 2.2.28.

□

Lemma 2.3.14 (Compactness)**.** *Given u_ε as above, we can extract a subsequence u_{ε_n} and obtain a limit $u \in L^\infty([0,T];\mathcal{D}(\sqrt{-H}))$ s.t.*

$$u_{\varepsilon_n}(t) \to u(t) \text{ in } L^2 \tag{2.3.11}$$

$$\sqrt{-H}u_{\varepsilon_n}(t) \to \sqrt{-H}u(t) \text{ in } L^2 \tag{2.3.12}$$

for all times $t \in [0,T]$.

Proof. By weak compactness in the Hilbert space $\mathcal{D}(\sqrt{-H})$ we obtain a subsequence u_{ε_n} and a limit u s.t.

$$\begin{aligned} &u_{\varepsilon_n}(t) \to u(t) \text{ in } L^2, \\ &u_{\varepsilon_n}(t) \rightharpoonup u(t) \text{ in } \mathcal{D}(\sqrt{-H}), \end{aligned}$$

for a dense set of times and using the third a priori bound from Lemma 2.3.13 we can extend this to all times $t \in [0,T]$. In particular get the L^∞ bound in time. Lastly, we can use the convergence of energies to deduce the convergence of the $\mathcal{D}(\sqrt{-H})$ norms of u_ε and thus conclude that in fact strong convergence holds.

□

Now we can conclude this section by proving Theorem 2.3.11.

Proof of Theorem 2.3.11. We prove that the limit we obtain in the previous lemma solves the mild formulation of (2.3.3). We have by construction that the u_ε solves

$$u_\varepsilon(t) = e^{-itH}u_0^\varepsilon + i\int_0^t e^{i(s-t)H}u_\varepsilon(s)|u_\varepsilon(s)|^2 \mathrm{d}s$$

for all $t \in [0,T]$. Now we can prove that this converges in L^2 as $\varepsilon \to 0$ for all times. The first term converges precisely as in the linear case from section 2.3.1. For the nonlinear term the convergence follows from the fact that $u_\varepsilon(t) \to u(t)$ strongly in L^6 for all times. This is due to the fact that the embedding $\mathcal{D}(\sqrt{-H}) \hookrightarrow \mathcal{H}^{1-\delta}$ is continuous and the embedding $\mathcal{H}^{1-\delta} \hookrightarrow L^6$ is compact (in fact this is true for any L^p with $p < \infty$).

For continuity in $\mathcal{D}(\sqrt{-H})$, we simply observe

$$\begin{aligned} \|\sqrt{-H}u(t) - \sqrt{-H}u(s)\|_{L^2} \leq{}& \|\sqrt{-H}u(t) - \sqrt{-H}u_{\varepsilon_n}(t)\|_{L^2} + \|\sqrt{-H}u_{\varepsilon_n}(t) - \sqrt{-H}u_{\varepsilon_n}(s)\|_{L^2} \\ &+ \|\sqrt{-H}u_{\varepsilon_n}(s) - \sqrt{-H}u(s)\|_{L^2}. \end{aligned}$$

By using Lemma 2.3.14, for a given $\delta > 0$ we can choose N large such that

$$\sup_\tau \|\sqrt{-H}u(\tau) - \sqrt{-H}u_{\varepsilon_N}(\tau)\|_{L^2} < \delta/3$$

for this chosen N we can choose $\kappa > 0$ such that; $|t-s| < \kappa$ implies

$$\|\sqrt{-H}u_{\varepsilon_N}(t) - \sqrt{-H}u_{\varepsilon_N}(s)\|_{L^2} < \delta/3.$$

That is, we have found a $\kappa > 0$ for arbitrary $\delta > 0$. Hence, the continuity.

Next, we prove the time regularity. By using Lemma 2.3.13 we can write

$$\begin{aligned}\|u(t)-u(s)\|_{L^2}^2 &\leq \|\sqrt{-H}(u(t)-u(s))\|_{L^2}\|(\sqrt{-H})^{-1}(u(t)-u(s))\|_{L^2}\\ &\lesssim \left\|\int_s^t(\sqrt{-H})^{-1}\partial_t u(\tau)d\tau\right\|_{L^2}\lesssim |t-s|.\end{aligned}$$

So we can conclude that $u \in \mathcal{C}^{1/2}([0,T],L^2)\cap C([0,T];\mathcal{D}(\sqrt{-H}))$. □

In the following corollary, we show that a solution can be obtained by solving the approximating PDEs.

Corollary 2.3.15. *Consider the following PDE*

$$i\partial_t u_\varepsilon = H_\varepsilon u_\varepsilon - u_\varepsilon|u_\varepsilon|^2$$

with initial data $u_0^\varepsilon = H_\varepsilon^{-1}H(1-\varepsilon\sqrt{-H})^{-1}u_0$, *where* $u_0 \in \mathcal{D}(\sqrt{-H})$ *and* $0<\varepsilon<1$. *There exists a subsequence* ε_n *such that* $u_{\varepsilon_n} \to u$ *and* $\sqrt{-H_{\varepsilon_n}}u_{\varepsilon_n} \to \sqrt{-H}u$ *in* L^2. *In addition,* u *solves* (2.3.3).

Proof. Consider the initial data $u_0^{\varepsilon,\delta} = H_\varepsilon^{-1}H(1-\delta\sqrt{-H})^{-1}u_0$. Then, by Theorem 2.3.9, taking $\epsilon \to 0$ we obtain $u_\delta \in \mathcal{D}(H)$ which solves the equation

$$i\partial_t u_\delta = Hu_\delta - u_\delta|u_\delta|^2$$

with initial data $u_0^\delta = (1-\delta\sqrt{-H})^{-1}u_0 \in \mathcal{D}(H)$. For this solution, we also have $\sqrt{-H_{\varepsilon_n}}u_{\varepsilon_n,\delta} \to \sqrt{-H}u_\delta$ in L^2 and in particular $u_{\varepsilon_n,\delta} \to u_\delta$. Now, as in Theorem 2.3.11, we take $\delta \to 0$ and obtain an energy solution to (2.3.3). Taking a diagonal sequence yields the stated result. □

In the following remarks, we compare those results with the ones in domain case.

Remark 2.3.16. Note that the solution we obtain is not necessarily unique, as opposed to the solution with initial data in $\mathcal{D}(H)$, however see Chapter 3.

2.3.3 Two and three dimensional cubic wave equations

In this section we consider the cubic wave equations

$$\begin{aligned}\partial_t^2 u &= Hu - u^3 \text{ on } \mathbb{T}^d\\ (u,\partial_t u)|_{t=0} &= (u_0,u_1),\end{aligned} \tag{2.3.13}$$

in two and three dimensions simultaneously.

We are interested in the case

$$(u_0,u_1)\in\mathcal{D}(H)\times\mathcal{D}(\sqrt{-H}).$$

However as we shall see, in a similar way we can also treat the case

$$(u_0,u_1)\in\mathcal{D}(\sqrt{-H})\times L^2.$$

We refer to [76] and [33] for classical results about well-posedness of semilinear wave equations. We obtain global strong well-posedness for a range of exponents including the standard case $p = 3$, which we will consider in detail for simplicity. Also in 3d, the range of exponents which are covered by our methods is as good as what one can achieve in the classical case with similar methods.

We fix an approximating sequence $(u_0^\varepsilon, u_1^\varepsilon) \in \mathcal{H}^2 \times \mathcal{H}^1$ such that

$$\begin{aligned} H_\varepsilon u_0^\varepsilon &\to H u_0 \ \text{ in } \ L^2, \\ (u_1^\varepsilon, H_\varepsilon u_1^\varepsilon) &\to (u_1, H u_1). \end{aligned}$$

To be precise, we may choose

$$\begin{aligned} u_0^\varepsilon &:= H_\varepsilon^{-1} H u_0 \\ u_1^\varepsilon &:= (\sqrt{-H_\varepsilon})^{-1} \sqrt{-H} u_1. \end{aligned}$$

We will– as in the NLS case– prove that the solution to (2.3.13) is given by the limit of the solutions of regularised equations (for $d = 2, 3$)

$$\begin{aligned} \partial_t^2 u_\varepsilon =& H_\varepsilon u_\varepsilon - u_\varepsilon^3 \ \text{ on } \ \mathbb{T}^d \\ (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} =& (u_0^\varepsilon, u_1^\varepsilon), \end{aligned} \tag{2.3.14}$$

in an appropriate sense.

We begin by proving global strong well-posedness of (2.3.13) and (2.3.14) by a fixed point argument as in Section 2.3.2.

Theorem 2.3.17. *For $(u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H})$ and $(u_0^\varepsilon, u_1^\varepsilon) \in \mathcal{H}^2 \times \mathcal{H}^1$, there exist unique global in time solutions $u \in C([0,T]; \mathcal{D}(H)) \cap C^1([0,T]; \mathcal{D}(\sqrt{-H})) \cap C^2([0,T]; L^2)$ and $u_\varepsilon \in C([0,T]; \mathcal{H}^2) \cap C^1([0,T]; \mathcal{H}^1) \cap C^2([0,T]; L^2)$ satisfying*

$$u(t) = \cos\left(t\sqrt{-H}\right) u_0 + \frac{\sin\left(t\sqrt{-H}\right)}{\sqrt{-H}} u_1 + \int_0^t \frac{\sin\left((t-s)\sqrt{-H}\right)}{\sqrt{-H}} u^3(s)\,\mathrm{ds}$$

and

$$u_\varepsilon(t) = \cos\left(t\sqrt{-H_\varepsilon}\right) u_0^\varepsilon + \frac{\sin\left(t\sqrt{-H_\varepsilon}\right)}{\sqrt{-H_\varepsilon}} u_1^\varepsilon + \int_0^t \frac{\sin\left((t-s)\sqrt{-H_\varepsilon}\right)}{\sqrt{-H_\varepsilon}} u_\varepsilon^3(s)\,\mathrm{ds}$$

respectively.

Before we come to the proof, we prove some auxiliary lemmas. We define the conserved energies for (2.3.13) and (2.3.14) respectively as

$$E(u) := \frac{1}{2}\langle \partial_t u, \partial_t u\rangle - \frac{1}{2}\langle u, Hu\rangle + \frac{1}{4}\int |u|^4,$$

and

$$E(u_\varepsilon) := \frac{1}{2}\langle \partial_t u_\varepsilon, \partial_t u_\varepsilon\rangle - \frac{1}{2}\langle u_\varepsilon, H_\varepsilon u_\varepsilon\rangle + \frac{1}{4}\int |u_\varepsilon|^4.$$

Also, we introduce the *almost conserved* energies for the time derivatives

$$\tilde{E}(\partial_t u) = \frac{1}{2}\langle \partial_t^2 u, \partial_t^2 u\rangle - \frac{1}{2}\langle \partial_t u, H\partial_t u\rangle + \frac{3}{2}\int |u|^2|\partial_t u|^2,$$

and

$$\tilde{E}(\partial_t u_\varepsilon) = \frac{1}{2}\langle \partial_t^2 u_\varepsilon, \partial_t^2 u_\varepsilon\rangle - \frac{1}{2}\langle \partial_t u_\varepsilon, H_\varepsilon\partial_t u_\varepsilon\rangle + \frac{3}{2}\int |u_\varepsilon|^2|\partial_t u_\varepsilon|^2.$$

We clarify what we mean by almost conserved in the following lemma.

Lemma 2.3.18. *Let $u \in C([0,T];\mathcal{D}(H)) \cap C^1([0,T];\mathcal{D}(\sqrt{-H})) \cap C^2([0,T];L^2)$ and $u_\varepsilon \in C([0,T];\mathcal{H}^2) \cap C^1([0,T];\mathcal{H}^1) \cap C^2([0,T];L^2)$ be solutions of* (2.3.13) *and* (2.3.14) *respectively. Then the energies $\tilde{E}(\partial_t u)$ and $\tilde{E}(\partial_t u_\varepsilon)$ satisfy the following bounds*

$$\begin{aligned}\tilde{E}(\partial_t u)(t) &\lesssim \exp(tC\tilde{E}(u_1))E(u_0),\\ \tilde{E}(\partial_t u_\varepsilon)(t) &\lesssim \exp(tC\tilde{E}(u_1^\varepsilon))E(u_0^\varepsilon),\end{aligned}$$

for some universal constant $C > 0$.

Proof. We give the proof only for the regularised case. The other case can be done analogously by replacing $\mathcal{H}^2$ by $\mathcal{D}(H)$ and $\mathcal{H}^1$ by $\sqrt{-H}$.
First note that $\partial_t u_\varepsilon$ solves the equation

$$\partial_t^2\partial_t u_\varepsilon \quad = \quad H_\varepsilon\partial_t u_\varepsilon - 3\partial_t u_\varepsilon u_\varepsilon^2 \ \text{ in } \ C([0,T];\mathcal{H}^{-1}).$$

Then one can formally compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\tilde{E}(\partial_t u_\varepsilon)(t) &= \langle \partial_t^2 u, \partial_t^3 u_\varepsilon - H_\varepsilon\partial_t u_\varepsilon + 3\partial_t u_\varepsilon u_\varepsilon^2\rangle + 3\int u_\varepsilon \partial_t u_\varepsilon |\partial_t u_\varepsilon|^2\\ &= 3\int u_\varepsilon\partial_t u_\varepsilon|\partial_t u_\varepsilon|^2.\end{aligned}$$

and conclude by Gronwall. However, since $\tilde{E}(\partial_t u_\varepsilon)$ is not C^1 in time, this is not justified. But one can argue that this computation is true in the integrated version. We claim that we get the following weak differentiability, for any $\phi \in C_c([0,\infty))$

$$\int_{\mathbb{R}} \phi'(t)\tilde{E}(\partial_t u_\varepsilon)(t)\,\mathrm{dt} = -3\int_{\mathbb{R}}\phi(t)\int u_\varepsilon\partial_t u_\varepsilon|\partial_t u_\varepsilon|^2(t)\,\mathrm{dt} + \tilde{E}(\partial_t u_\varepsilon(0))\phi(0). \tag{2.3.15}$$

Moreover, this also holds in the integrated form

$$\tilde{E}(\partial_t u_\varepsilon)(t) = \tilde{E}(\partial_t u_\varepsilon)(0) + 3\int_0^t u_\varepsilon\partial_t u_\varepsilon|\partial_t u_\varepsilon|^2(s)\,\mathrm{ds}, \tag{2.3.16}$$

for any $t \in [0,T]$. We prove this by a spectral approximation. For, consider $(e_n)_{n\in\mathbb{Z}^3} \in \mathcal{H}^2$ an orthonormal eigenbasis of H_ε with eigenvalues $\{\lambda_n\}$ and set

$$u_\varepsilon^N(t,x) := \sum_{|n|\leq N}(u_\varepsilon(t,\cdot), e_n)e_n(x).$$

Then one has

$$\partial_t^k u_\varepsilon^N \to \partial_t^k u_\varepsilon \text{ in } C([0,T];\mathcal{H}^{2-k})$$

for $0 \le k \le 3$, which in turn implies that

$$E(u_\varepsilon^N) \to E(u_\varepsilon) \text{ and } \tilde{E}(\partial_t u_\varepsilon^N) \to \tilde{E}(\partial_t u_\varepsilon).$$

One also directly deduces

$$\partial_t^3 u_\varepsilon^N = H_\varepsilon u_\varepsilon^N - 3\sum_{|n|\le N}(\partial_t u_\varepsilon u_\varepsilon^2(t), e_n)e_n(x).$$

Thus, we have

$$\begin{aligned}\int_{\mathbb{R}} \phi'(t)\tilde{E}(\partial_t u_\varepsilon^N)(t)\,\mathrm{dt} &= -\int_{\mathbb{R}} \phi(t)\frac{\mathrm{d}}{\mathrm{dt}}\tilde{E}(\partial_t u_\varepsilon^N)(t)\,\mathrm{dt} + \tilde{E}(\partial_t u_\varepsilon(0))\phi(0)\\ &= -\int_{\mathbb{R}} \phi(t)\left((\partial_t^2 u_\varepsilon^N, \partial_t^3 u_\varepsilon^N)(t) - (\partial_t^2 u_\varepsilon^N, H_\varepsilon \partial_t u_\varepsilon^N)(t)\right.\\ &\quad \left.+3\left(\partial_t^2 u_\varepsilon^N, \partial_t u_\varepsilon^N (u_\varepsilon^N)^2\right)(t) + 3(\partial_t u_\varepsilon^N, (\partial_t u_\varepsilon^N)u_\varepsilon^N)(t)\right)\mathrm{dt} \qquad (2.3.17)\\ &\quad + \tilde{E}(\partial_t u_\varepsilon(0))\phi(0)\\ &= -\int_{\mathbb{R}} \phi(t)[3(\partial_t^2 u_\varepsilon^N, \partial_t u_\varepsilon^N (u_\varepsilon^N)^2 - \sum_{|n|\le N}(\partial_t u_\varepsilon u_\varepsilon^2, e_n)e_n(x))(t)\\ &\quad + 3(\partial_t u_\varepsilon^N, (\partial_t u_\varepsilon^N)^2 u_\varepsilon^N)(t)]\,\mathrm{dt} + \tilde{E}(\partial_t u_\varepsilon(0))\phi(0). \qquad (2.3.18)\end{aligned}$$

Now we have

$$\begin{aligned}\partial_t u_\varepsilon^N (u_\varepsilon^N)^2 \quad &\to \quad \partial_t u_\varepsilon (u_\varepsilon)^2 \text{ in } L^2\\ &\text{and}\\ \sum_{|n|\le N}(\partial_t u_\varepsilon u_\varepsilon^2, e_n)e_n(x) \quad &\to \quad \partial_t u_\varepsilon (u_\varepsilon)^2 \text{ in } L^2.\end{aligned}$$

Therefore, we see that for $N \to \infty$ (2.3.18) converges to (2.3.15). To prove (2.3.16), it suffices to take a sequence ϕ_n in (2.3.15) that converges to the characteristic function $\chi_{[0,t]}$ monotonically. We can thus compute

$$\begin{aligned}\tilde{E}(\partial_t u_\varepsilon)(t) \le& \tilde{E}(u_1^\varepsilon) + 3\int_0^t \int |u_\varepsilon(s)||\partial_t u_\varepsilon(s)||\partial_t u_\varepsilon|^2(s)\,\mathrm{ds}\\ \le& \tilde{E}(u_1^\varepsilon) + 3\int_0^t \|\partial_t u_\varepsilon\|_{L^2}\|u_\varepsilon\|_{L^6}\|\partial_t u_\varepsilon\|_{L^6}^2(s)\,\mathrm{ds}\\ \le& \tilde{E}(u_1^\varepsilon) + 3CE(u_0^\varepsilon)\int_0^t \tilde{E}(\partial_t u_\varepsilon)(s)\,\mathrm{ds},\end{aligned}$$

where we have used the bounds

$$\|\partial_t u_\varepsilon\|_{L^2}^2 \leq E(u_\varepsilon), \qquad \|u_\varepsilon\|_{L^6}^2 \lesssim_\Xi E(u_\varepsilon), \qquad \|\partial_t u_\varepsilon\|_{L^6}^2 \lesssim_\Xi \tilde{E}(\partial_t u_\varepsilon).$$

From this, we conclude by using Gronwall. □

Proof of Theorem 2.3.17. This is similar to the NLS case (Section 2.3.2), except that the time T_E is going to depend on the conserved energy $E(u)$ and the almost conserved energy $\tilde{E}(\partial_t u)$. We again give the proof only for the $\mathcal{D}(H)$ case, as the $\mathcal{H}^2$ case can be proved in a similar way. We claim that for $(u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}\left(\sqrt{-H}\right)$ there exists a unique fixed point of

$$\Phi(u)(t) = \cos\left(t\sqrt{-H}\right) u_0 + \frac{\sin\left(t\sqrt{-H}\right)}{\sqrt{-H}} u_1 + \int_0^t \frac{\sin\left((t-s)\sqrt{-H}\right)}{\sqrt{-H}} u^3(s)\,\mathrm{ds}$$

in $X = C([0,T]; \mathcal{D}(H)) \cap C^1\left([0,T]; \mathcal{D}\left(\sqrt{-H}\right)\right) \cap C^2([0,T]; L^2)$.
For the contraction property, we compute the following, for $\|u\|_X \leq M$ with $M > 0$ fixed later,

$$\begin{aligned}
&\|H\Phi(u)(t) - H\Phi(v)(t)\|_{L^2} = \\
=&\| \int_0^t \sqrt{-H} \sin((t-s)\sqrt{-H})(u^3(s) - v^3(s))\,\mathrm{ds}\, \|_{L^2} \\
=&\| \int_0^t \partial_s(\cos((t-s)\sqrt{-H}))(u^3(s) - v^3(s))\,\mathrm{ds}\, \|_{L^2} \\
=&\| \int_0^t \cos((t-s)\sqrt{-H})\partial_s(u^3(s) - v^3(s))\,\mathrm{ds} + v^3(t) - u^3(t)\|_{L^2} \\
\leq& 2\int_0^t \|\partial_t(u^3 - v^3)(s)\|_{L^2} \\
\leq& 6\int_0^t \|\partial_t u - \partial_t v\|_{L^\infty_{[0,T]}L^6}\|u\|_{L^6}^2(s) + \|\partial_t v\|_{L^\infty_{[0,T]}L^4}\|u - v\|_{L^\infty_{[0,T]}L^\infty}(\|u\|_{L^4}(s) + \|v\|_{L^4}(s))\,\mathrm{ds} \\
\leq& C\|u - v\|_X \int_0^t (E(u)(s) + ME^{1/2}(u)(s) + ME^{1/2}(v)(s))\,\mathrm{ds} \\
<& \frac{1}{3}\|u - v\|_X
\end{aligned}$$

for small enough time depending on the energy and M. Here we have used the bounds $\|\partial_t u\|_{L^6} \lesssim \left\|\sqrt{-H}\partial_t u\right\|_{L^2}$ and $\|u\|_{L^4} \lesssim E^{1/2}(u)$. For the other terms, we similarly compute

$$\begin{aligned}
\left\|\sqrt{-H}\partial_t\Phi(u)(t) - \sqrt{-H}\partial_t\Phi(v)(t)\right\|_{L^2} \leq& 2\int_0^t \|\partial_t(u^3 - v^3)(s)\|_{L^2} \\
<& \frac{1}{3}\|u - v\|_X
\end{aligned}$$

and

$$\|\partial_t^2\Phi(u)(t) - \partial_t^2\Phi(v)(t)\|_{L^2} < \frac{1}{3}\|u - v\|_X.$$

Lastly, we argue that Φ maps a ball to itself. Let $\|u\|_X \leq M$ for M specified below, then we have

$$\begin{aligned}\|H\Phi(u)(t)\|_{L^2} &\lesssim \|Hu_0\|_{L^2} + \|\sqrt{-H}u_1\|_{L^2} + \int_0^t \|\partial_t u\|_{L^6}(s)\|u\|^2_{L^6}(s)\,\mathrm{ds}\\ &\lesssim \|Hu_0\|_{L^2} + \|\sqrt{-H}u_1\|_{L^2} + \int_0^t \tilde{E}^{1/2}(\partial_t u)(s)E(u)(s)\,\mathrm{ds}\\ &\leq \frac{M}{3}\end{aligned}$$

for large M depending on the data and $t \leq T_E$, small depending on $E(u)$ and $\tilde{E}(\partial_t u)$.
Analogously, we also have

$$\|\partial_t^2\Phi(u)\|_{L^\infty_{[0,T]}L^2} \leq \frac{M}{3} \quad \text{and} \quad \left\|\sqrt{-H}\partial_t\Phi(u)\right\|_{L^\infty_{[0,T]}L^2} \leq \frac{M}{3}.$$

Moreover, the time regularity is again a consequence of Stone's Theorem. Thus there exists a unique strong solution up to the time T_E that depends on (almost) conserved quantities and we can conclude that this yields a strong solution up to any time. More precisely, we get a priori estimates that allow us to choose a globally valid $M > 0$ and then iterate the solution map to obtain a solution up to any given time $T > 0$.
Assuming we have a solution on the interval $[0, T_E]$, then we can estimate similarly to above as,

$$\begin{aligned}\|Hu(T_E)\|_{L^2} &\lesssim \|Hu_0\|_{L^2} + \|\sqrt{-H}u_1\|_{L^2} + \int_0^{T_E} \tilde{E}^{1/2}(\partial_t u)(s)E(u)(s)\,\mathrm{ds}\\ &\lesssim \|Hu_0\|_{L^2} + \|\sqrt{-H}u_1\|_{L^2} + T\exp(CT\tilde{E}(u_1))E^{3/2}(u_0)\end{aligned}$$

and also similarly for $\|\sqrt{-H}\partial_t u(T_E)\|_{L^2}$. Thus we can choose M globally and solve on the interval $[T_E, 2T_E]$ and so on.

□

From the above considerations, we obtain a priori bounds for the quantities

$$\|u_\varepsilon\|_{L^\infty_{[0,T]}L^2}, \|H_\varepsilon u_\varepsilon\|_{L^\infty_{[0,T]}L^2} \text{ and } \sup_{t\in[0,T]}(\partial_t u_\varepsilon, H_\varepsilon \partial_t u_\varepsilon),$$

independently of ε. By the same arguments, as in the previous sections, we can also prove convergence of the approximate solutions.

Theorem 2.3.19. *Assume we are in the above setting, i.e. we have unique global strong solutions to* (2.3.13) *and* (2.3.14) *and the initial data are given by* $(u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H})$ *and*

$$\begin{aligned}u_0^\varepsilon &:= H_\varepsilon^{-1}Hu_0\\ u_1^\varepsilon &:= (\sqrt{-H_\varepsilon})^{-1}\sqrt{-H}u_1.\end{aligned}$$

Then the solutions u_ε *converge to* u *in the following way*

$$\begin{aligned}u_\varepsilon(t) &\to u(t) \text{ in } L^2\\ H_\varepsilon u_\varepsilon(t) &\to Hu(t) \text{ in } L^2\\ \sqrt{-H_\varepsilon}\partial_t u_\varepsilon(t) &\to \sqrt{-H}\partial_t u(t) \text{ in } L^2\\ \partial_t^2 u_\varepsilon(t) &\to \partial_t^2 u(t) \text{ in } L^2\end{aligned}$$

for all $t \in [0,T]$.

Proof. The proof is similar to that of Theorem 2.3.9. Since we have strong convergence for the initial data, together with the fact that $\frac{\sin(t\sqrt{-H_\varepsilon})}{\sqrt{-H_\varepsilon}} \to \frac{\sin(t\sqrt{-H})}{\sqrt{-H}}$ strongly, we can compute for any fixed time $t \in [0,T]$ using the mild formulation for u and u_ε

$$
\begin{aligned}
&\|Hu(t) - H_\varepsilon u_\varepsilon(t)\|_{L^2} \\
\lesssim& O(\varepsilon) + \int_0^t \|\partial_s(u^3)(s) - \partial_s(u_\varepsilon^3)(s)\|_{L^2}\mathrm{d}s \\
\lesssim& O(\varepsilon) + \int_0^t \|\partial_s u(s) - \partial_s u_\varepsilon(s)\|_{L^6}\|u\|^2_{L^\infty_{[0,T]}L^6} \\
&+ \|u(s) - u_\varepsilon(s)\|_{L^6}(\|u\|_{L^\infty_{[0,T]}L^6} + \|u_\varepsilon\|_{L^\infty_{[0,T]}L^6})\|\partial_t u_\varepsilon\|_{L^\infty L^6}\mathrm{d}s \\
\lesssim& O(\varepsilon) + C(T,u_0,u_1)\int_0^t \|\sqrt{-H}\partial_s u(s) - \sqrt{-H_\varepsilon}\partial_s u_\varepsilon(s)\|_{L^2} + \|Hu(s) - H_\varepsilon u_\varepsilon(s)\|_{L^2}\mathrm{d}s.
\end{aligned}
$$

Here, we have used the a priori bounds obtained in Theorem 2.3.17 and the estimate

$$
\begin{aligned}
\|\partial_t u(s) - \partial_t u_\varepsilon(s)\|_{L^6} &\lesssim_\Xi \|\partial_t u^\sharp(s) - \partial_t u_\varepsilon^\sharp(s)\|_{\mathcal{H}^1} \\
&\lesssim_\Xi \|\sqrt{-H}\partial_t u(s) - \sqrt{-H_\varepsilon}\partial_t u_\varepsilon(s)\|_{L^2} + C(u_0,u_1,T)\|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha},
\end{aligned}
$$

where the first estimate follows by Sobolev embedding and Proposition 2.2.7 and 2.2.41. The second one can be proved analogously to Proposition 2.2.18 and 2.2.48 for 2 and 3d respectively. In a similar manner, we have the bound

$$
\|u(s) - u_\varepsilon(s)\|_{L^6} \lesssim_\Xi \|Hu(s) - H_\varepsilon u_\varepsilon(s)\|_{L^2} + C(u_0,u_1,T)\|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}.
$$

Analogously we can also write

$$
\begin{aligned}
\|\sqrt{-H}\partial_t u(t) - \sqrt{-H_\varepsilon}\partial_t u_\varepsilon(t)\|_{L^2} \lesssim& O(\varepsilon) + C(T,u_0,u_1)\int_0^t \|\sqrt{-H}\partial_s u(s) - \sqrt{-H_\varepsilon}\partial_s u_\varepsilon(s)\|_{L^2} \\
&+ \|Hu(s) - H_\varepsilon u_\varepsilon(s)\|_{L^2}\mathrm{d}s, \\
\|\partial_t^2 u(t) - \partial_t^2 u_\varepsilon(t)\|_{L^2} \lesssim& O(\varepsilon) + C(T,u_0,u_1)\int_0^t \|\sqrt{-H}\partial_s u(s) - \sqrt{-H_\varepsilon}\partial_s u_\varepsilon(s)\|_{L^2} \\
&+ \|Hu(s) - H_\varepsilon u_\varepsilon(s)\|_{L^2}\mathrm{d}s \\
\|u(t) - u_\varepsilon(t)\|_{L^2} \lesssim& O(\varepsilon) + C(T,u_0,u_1)\int_0^t \|Hu(s) - H_\varepsilon u_\varepsilon(s)\|_{L^2}\mathrm{d}s.
\end{aligned}
$$

Thus, by defining

$$
\begin{aligned}
\phi_\varepsilon(t) :=& \|Hu(t) - H_\varepsilon u_\varepsilon(t)\|_{L^2} + \|\sqrt{-H}\partial_t u(t) - \sqrt{-H_\varepsilon}\partial_t u_\varepsilon(t)\|_{L^2} \\
&+ \|\partial_t^2 u(t) - \partial_t^2 u_\varepsilon(t)\|_{L^2} + \|u(t) - u_\varepsilon(t)\|_{L^2},
\end{aligned}
$$

we can rewrite the above estimates as

$$
\phi_\varepsilon(t) \le O(\varepsilon) + C(T,u_0,u_1)\int_0^t \phi_\varepsilon(s)\mathrm{d}s
$$

and conclude by Gronwall that $\phi_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ for all $t \in [0,T]$.
Hence, the result. □

Lastly, we state the analogous result for the energy space, i.e. with data $(u_0, u_1) \in \mathcal{D}(\sqrt{-H}) \times L^2$. In a nutshell, one can repeat the above arguments. For global well-posedness, one can use a fixed point argument in the space $C([0,T]; \mathcal{D}(\sqrt{-H})) \cap C^1([0,T]; L^2) \cap C^2([0,T]; \mathcal{D}(\sqrt{-H})^*)$ together with energy conservation and convergence can also be proved as above. We omit the proofs.

Theorem 2.3.20. *Let $(u_0, u_1) \in \mathcal{D}(\sqrt{-H}) \times L^2$ and $T > 0$, then* (2.3.13) *has a unique solution $u \in C([0,T]; \mathcal{D}(\sqrt{-H})) \cap C^1([0,T]; L^2) \cap C^2([0,T]; \mathcal{D}(\sqrt{-H})^*)$. Moreover,* (2.3.14) *has a unique solution $u_\varepsilon \in C([0,T]; \mathcal{H}^1) \cap C^1([0,T]; L^2) \cap C^2([0,T]; \mathcal{H}^{-1})$ with initial data $(u_0^\varepsilon, u_1) \in \mathcal{H}^1 \times L^2$, where $u_0^\varepsilon := (-H_\varepsilon)^{-1/2}(-H)^{1/2} u_0$ and the following convergence holds*

$$\begin{aligned} u_\varepsilon(t) &\to u(t) \text{ in } L^2 \\ \sqrt{-H_\varepsilon} u_\varepsilon(t) &\to \sqrt{-H} u(t) \text{ in } L^2 \\ \partial_t u_\varepsilon(t) &\to \partial_t u(t) \text{ in } L^2 \end{aligned}$$

for all $t \in [0,T]$.

Remark 2.3.21. The same result is also true in 2d for any power $p \in (1, \infty)$ both for the domain and energy space case. In 3d, our proof for global wellposedness also works for powers up to 5 in the domain case using an analogue of Agmon's inequality, which we included for completeness as Lemma 2.2.55. See also Chapter 4 for the a treatment of *energy supercritical* powers.

Paracontrolled distributions and function spaces

We recall the definitions of Bony paraproducts, Besov and Sobolev spaces and collect some results about products of distributions. We work on the d-dimensional torus $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ for $d = 2, 3$. For any f in the space $\mathcal{S}'(\mathbb{T}^d)$ of tempered distributions on $\mathbb{T}^d$, the Fourier transform of f will be denoted by $\hat{f} : \mathbb{Z}^d \to \mathbb{C}$ (or sometimes $\mathcal{F}f$) and is defined for $k \in \mathbb{Z}^d$ by

$$\hat{f}(k) := \langle f, \exp(2\pi i \langle k, \cdot \rangle) = \int_{\mathbb{T}^d} f(x) \exp(-2\pi i \langle k, x \rangle) dx.$$

Recall that for any $f \in L^2(\mathbb{T}^d)$ and a.e. $x \in \mathbb{T}^d$, we have

$$f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \exp(2\pi i \langle k, x \rangle). \tag{2.3.19}$$

The Sobolev space $\mathcal{H}^\alpha(\mathbb{T}^d)$ with index $\alpha \in \mathbb{R}$ is defined as

$$\mathcal{H}^\alpha(\mathbb{T}^d) := \{ f \in \mathcal{S}'(\mathbb{T}^d; \mathbb{R}) : \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^\alpha \, |\hat{f}(k)|^2 < +\infty \} .$$

Before introducing Besov spaces, we recall the definition of Littlewood-Paley blocks. We denote by χ and ρ two nonnegative smooth and compactly supported radial functions $\mathbb{R}^d \to \mathbb{R}$ such that

1. The support of χ is contained in a ball $\{x \in \mathbb{R}^d : |x| \leq R\}$ and the support of ρ is contained in an annulus $\{x \in \mathbb{R}^d : a \leq |x| \leq b\}$;
2. For all $\xi \in \mathbb{R}^d$, $\chi(\xi) + \sum_{j\geq 0} \rho(2^{-j}\xi) = 1$;
3. For $j \geq 1$, $\chi\rho(2^{-j}\cdot) \equiv 0$ and $\rho(2^{-i}\cdot)\rho(2^{-j}\cdot) \equiv 0$ for $|i-j| \geq 1$.

The Littlewood-Paley blocks $(\Delta_j)_{j\geq -1}$ acting on $f \in \mathcal{S}'(\mathbb{T}^d)$ are defined by

$$\mathcal{F}(\Delta_{-1}f) = \chi\hat{f} \quad \text{and for} \quad j \geq 0, \quad \mathcal{F}(\Delta_j f) = \rho(2^{-j}.)\hat{f}.$$

Note that, for $f \in \mathcal{S}'(\mathbb{T}^d)$, the Littlewood-Paley blocks $(\Delta_j f)_{j\geq -1}$ define smooth functions, as their Fourier transforms have compact supports. We also set, for $f \in \mathcal{S}'$ and $j \geq 0$,

$$S_j f := \sum_{i=-1}^{j-1} \Delta_i f$$

and note that $S_j f$ converges in the sense of distributions to f as $j \to \infty$.
The Besov space with parameters $p, q \in [1,\infty), \alpha \in \mathbb{R}$ can now be defined as

$$B^{\alpha}_{p,q}(\mathbb{T}^d) := \left\{ u \in \mathcal{S}'(\mathbb{T}^d); \quad \|u\|_{B^{\alpha}_{p,q}} = \left(\sum_{j\geq -1} 2^{jq\alpha} \|\Delta_j u\|^q_{L^p} \right)^{1/q} < +\infty \right\}. \tag{2.3.20}$$

We also define the Besov-Hölder spaces

$$\mathcal{C}^{\alpha} := B^{\alpha}_{\infty,\infty}$$

which are naturally equipped with the norm $\|f\|_{\mathcal{C}^{\alpha}} := \|f\|_{B^{\alpha}_{\infty,\infty}} = \sup_{j\geq -1} 2^{j\alpha}\|\Delta_j f\|_{L^\infty}$. For $\alpha \in (0,1)$ these spaces coincide with the classical Hölder spaces.
We can formally decompose the product fg of two distributions f and g as

$$fg = f \prec g + f \circ g + f \succ g$$

where

$$f \prec g := \sum_{j\geq -1} \sum_{i=-1}^{j-2} \Delta_i f \Delta_j g \qquad \text{and} \qquad f \succ g := \sum_{j\geq -1} \sum_{i=-1}^{j-2} \Delta_i g \Delta_j f$$

are usually referred to as the *paraproducts* whereas

$$f \circ g := \sum_{j\geq -1} \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g \tag{2.3.21}$$

is called the *resonant product*.
Moreover, we define the notations $f \preccurlyeq g := f \prec g + f \circ g$ and $f \succcurlyeq g := f \succ g + f \circ g$.
The paraproduct terms are always well defined irrespective of regularities. The resonant product is a priori only well defined if the sum of regularities is strictly greater than zero. This is reminiscent of the well known fact that one can not multiply distributions in general. The following result makes those comments precise and gives simple but extremely vital estimates for paraproducts.

Proposition 2.3.22 (Bony estimates, [3])**.** *Let $\alpha, \beta \in \mathbb{R}$. We have the following bounds:*

1. *If $f \in L^2$ and $g \in \mathcal{C}^\beta$, then*

   $\|f \prec g\|_{\mathcal{H}^{\beta-\delta}} \leq C_{\delta,\beta} \|f\|_{L^2} \|g\|_{\mathcal{C}^\beta}$ *for all* $\delta > 0$.
2. *if $f \in \mathcal{H}^\alpha$ and $g \in L^\infty$ then*

   $\|f \succ g\|_{\mathcal{H}^\alpha} \leq C_{\alpha,\beta} \|f\|_{\mathcal{H}^\alpha} \|g\|_{\mathcal{C}^\beta}$.
3. *If $\alpha < 0$, $f \in \mathcal{H}^\alpha$ and $g \in \mathcal{C}^\beta$, then*

   $\|f \prec g\|_{\mathcal{H}^{\alpha+\beta}} \leq C_{\alpha,\beta} \|f\|_{\mathcal{H}^\alpha} \|g\|_{\mathcal{C}^\beta}$.
4. *If $g \in \mathcal{C}^\beta$ and $f \in \mathcal{H}^\alpha$ for $\beta < 0$ then*

   $\|f \succ g\|_{\mathcal{H}^{\alpha+\beta}} \leq C_{\alpha,\beta} \|f\|_{\mathcal{H}^\alpha} \|g\|_{\mathcal{C}^\beta}$
5. *If $\alpha + \beta > 0$ and $f \in \mathcal{H}^\alpha$ and $g \in \mathcal{C}^\beta$, then*

   $||f \circ g\|_{\mathcal{H}^{\alpha+\beta}} \leq C_{\alpha,\beta} \|f\|_{\mathcal{H}^\alpha} \|g\|_{\mathcal{C}^\beta}$.

where $C_{\alpha,\beta}$ is a finite positive constant.

Proposition 2.3.23. *Given $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$, there exists a trilinear operator C with the following bound*

$$\|C(f,g,h)\|_{\mathcal{H}^{\alpha+\beta+\gamma}} \lesssim \|f\|_{\mathcal{H}^\alpha} \|g\|_{\mathcal{C}^\beta} \|h\|_{\mathcal{C}^\gamma}$$

for all $f \in \mathcal{H}^\alpha$, $g \in \mathcal{C}^\beta$ and $h \in \mathcal{C}^\gamma$.

The restriction of C to the smooth functions satisfies

$$C(f,g,h) = (f \prec g) \circ h - f(g \circ h).$$

Proof. This is a restatement of the commutator lemma in [3], and the proof follows along the same lines. □

Lemma 2.3.24 (Bernstein's inequality, [45])**.** *Let $\mathcal{A}$ be an annulus and $\mathcal{B}$ be a ball. For any $k \in \mathbb{N}, \lambda > 0$, and $1 \leq p \leq q \leq \infty$ we have*

1. *if $u \in L^p(\mathbb{R}^d)$ is such that* $\mathrm{supp}(\mathcal{F}u) \subset \lambda\mathcal{B}$ *then*

$$\max_{\mu \in \mathbb{N}^d : |\mu| = k} \|\partial^\mu u\|_{L^q} \lesssim_k \lambda^{k + d\left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^p}$$

2. *if $u \in L^p(\mathbb{R}^d)$is such that* $\mathrm{supp}(\mathcal{F}u) \subset \lambda\mathcal{A}$ *then*

$$\lambda^k \|u\|_{L^p} \lesssim_k \max_{\mu \in \mathbb{N}^d : |\mu| = k} \|\partial^\mu u\|_{L^p}.$$

Proposition 2.3.25 (Paralinearisation, [48])**.** *Let $\alpha \in (0,1)$ and $F \in C^2$. Then there exists a locally bounded map $R_F : \mathcal{C}^\alpha \to \mathcal{C}^{2\alpha}$ such that*

$$F(f) = F'(f) \prec f + R_F(f) \ \text{ for all } \ f \in \mathcal{C}^\alpha.$$

Lemma 2.3.26. *Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha+\beta<0$, $\alpha+\beta+\gamma \geq 0$, and $f \in \mathcal{H}^\alpha, g \in \mathcal{C}^\beta, h \in \mathcal{H}^\gamma$, then there exists a map $D(f,g,h)$ with the following bound*

$$|D(f,g,h)| \lesssim \|g\|_{\mathcal{C}^\beta} \|f\|_{\mathcal{H}^\alpha} \|h\|_{\mathcal{H}^\gamma}. \tag{2.3.22}$$

Moreover the restriction of $D(f,g,h)$ to smooth functions f,g,h is as follows:

$$D(f,g,h) = \langle f, h \circ g \rangle - \langle f \prec g, h \rangle.$$

Proof. We define

$$D(f,g,h) := \left(\sum_{i \geq k-1, |j-k| \leq L} - \sum_{i \sim k, 1<|j-k| \leq L} \right) \langle \Delta_i f, \Delta_j h \Delta_k g \rangle.$$

So we get, for some $\delta > 0$,

$$\begin{aligned}
|D(f,g,h)| &\lesssim \sum_{i \gtrsim k, j \sim k} |\langle \Delta_i f, \Delta_j h \Delta_k g \rangle| \\
&\leq \sum_{i \gtrsim k, j \sim k} \|\Delta_i f\|_{L^2} \|\Delta_j h\|_{L^2} \|\Delta_k g\|_{L^\infty} \\
&\leq \|g\|_{\mathcal{C}^{-1-\delta}} \sum_{i \gtrsim k, j \sim k} 2^{k(1+\delta)} \|\Delta_i f\|_{L^2} \|\Delta_j h\|_{L^2} \\
&\leq \|g\|_{\mathcal{C}^{-1-\delta}} \|f\|_{\mathcal{H}^{(1+\delta)/2}} \|h\|_{\mathcal{H}^{(1+\delta)/2}}
\end{aligned}$$

and this argument can be adapted to show (2.3.22) by simply observing
$1 \leq 2^{k(\beta+\alpha+\gamma)} = 2^{k\beta} 2^{k(\alpha+\gamma)}$, since $\beta+\alpha+\gamma \geq 0$. Moreover, for smooth functions f,g,h; we can compute

$$\begin{aligned}
\langle f, h \circ g \rangle - \langle f \prec g, h \rangle &= \sum_{i, |j-k| \leq 1} \langle \Delta_i f, \Delta_j h \Delta_k g \rangle - \sum_{i<k-1, j} \langle \Delta_i f \Delta_k g, \Delta_j h \rangle \\
&= \left(\sum_{i, |j-k| \leq 1} - \sum_{i<k-1, |j-k| \leq L} \right) \langle \Delta_i f \Delta_k g, \Delta_j h \rangle \\
&= \left(\sum_{i, |j-k| \leq L} - \sum_{i<k-1, |j-k| \leq L} - \sum_{i, 1<|j-k| \leq L} \right) \langle \Delta_i f, \Delta_j h \Delta_k g \rangle \\
&= \left(\sum_{i \geq k-1, |j-k| \leq L} - \sum_{i, 1<|j-k| \leq L} \right) \langle \Delta_i f, \Delta_j h \Delta_k g \rangle \\
&= \left(\sum_{i \geq k-1, |j-k| \leq L} - \sum_{i \sim k, 1<|j-k| \leq L} \right) \langle \Delta_i f, \Delta_j h \Delta_k g \rangle = D(f,g,h).
\end{aligned}$$

Hence the result. □

Remark 2.3.27. Proposition 2.3.26 says that the paraproduct is *almost* the adjoint of the resonant product, meaning up to a more regular remainder term as is often the case in paradifferential calculus.

Lemma 2.3.28. *Let* $f \in \mathcal{H}^\alpha, g \in \mathcal{C}^\beta$, *with* $\alpha \in (0,1), \beta \in \mathbb{R}$, *there exists a bilinear map* $R(f,g)$ *that satisfies the following bound*

$$\|R(f,g)\|_{\mathcal{H}^{\alpha+\beta+2}} \lesssim \|f\|_{\mathcal{H}^\alpha}\|g\|_{\mathcal{C}^\beta},$$

and restricts to smooth functions as

$$R(f,g) = (1-\Delta)^{-1}(f \prec g) - f \prec (1-\Delta)^{-1}g.$$

Proof. The proof is basically a straightforward modification of the proof of [3, Proposition A.2], which has almost the same statement. □

Chapter 3

Strichartz estimates and low-regularity solutions to multiplicative stochastic NLS

3.1 Introduction

This chapter is devoted to proving Strichartz estimates and low-regularity well-posedness of defocussing cubic NLS(nonlinear Schrödinger equations) with very rough potentials ξ, so

$$i\partial_t u - \Delta u = u \cdot \xi - u|u|^2 \text{on } \mathbb{T}^d \tag{3.1.1}$$

$$u(0) = u_0, \tag{3.1.2}$$

where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is the $d-$dimensional torus. Many results still hold true in the whole space, see the discussion in section 3.6, but our chief interest is the case where ξ is *spatial white noise*, which is a distribution whose regularity is only $\mathcal{C}^{-\frac{d}{2}-\varepsilon}$ for $\varepsilon > 0$, see Definition 3.6.1 for the precise definition of *white noise* and the appendix for a reminder of the definition of the *Hölder-Besov spaces* $\mathcal{C}^\alpha$.

In the case of the white noise potential there turns out to be a peculiarity in the form of *renormalisation*, which means that in order to make sense of (3.1.2) one is required to shift by an infinite correction term, formally "$\infty \cdot u$". This is reminiscent of the theory of singular SPDEs which has seen a rapid growth in recent years following the introduction of the theory of *Regularity Structures* by Hairer [54] and the theory of *Paracontrolled Distributions* by Gubinelli, Perkowski, and Imkeller [45].

The approach we follow in this paper is to put the potential ξ into the definition of the operator, i.e. we try to define the operator

$$\Delta + \xi$$

as a self-adjoint and semi-bounded operator on $L^2(\mathbb{T}^d)$. This was first done by Allez and Chouk in [3], where the operator together with its domain were constructed in 2d with the white noise potential–hereafter called the *Anderson Hamiltonian*–using Paracontrolled Distributions. A similar approach was used in [49] to construct the operator and its domain in 3d with an eye also on solving PDEs like (3.1.2). In Section 3.3 we recall the main ideas of [49] since the results are integral to the current work. The domain of the Anderson Hamiltonian was also constructed by Labbé using Regularity Structures [62].

The equation (3.1.2) with white noise potential in 2d was solved, but not shown to be well-posed, by Debussche and Weber [30] and on the whole space with a smaller power in the nonlinearity by Debussche and Martin in [29]. In [49] global well-posedness(GWP) was proved in the domain of the Anderson Hamiltonian in 2d, whereas in 3d one gets a blow-up alternative when starting in the domain analogously to the case of classical $\mathcal{H}^2$ solutions in [23]. Furthermore, in [49] global existence in the energy space in 2d was shown, but not well-posedness. Achieving GWP for energy solutions to (3.1.2) is one of the results of this paper.

The (nonlinear) Schrödinger equation (3.1.2) with a potential has certain physical interpretations, see [38] and the references therein. In this paper we are able to treat a large class of *subcritical* potentials and all results are continuous w.r.t. the potential in the "correct" topology, see Section 3.6 for a discussion; Since the chief application is the white noise potential this aspect is not always emphasised. Some potentials of interest are actually *critical* in the sense of scaling, like the Dirac Delta in 2d (see the monograph [2]) or the potential $|\cdot|^{-2}$ treated in [18]. Our method does not apply in these cases, but in the aforementioned examples the analysis depends in a crucial way on the structure of the potential. We stress here that our method relies **only** on the (Besov) regularity of the potential and possibly some related objects and does **not** depend on its sign, radial symmetry, homogeneity etc.

Stochastic NLS of the form (3.1.2) but with different noises(e.g. white in time coloured in space) have also been considered, see [27], [16], [34] to name but a few. Other stochastic dispersive PDEs which have been studied in recent years include stochastic NLS with additive space-time noise [67], [35] and nonlinear stochastic wave equations with additive space-time noise in [47] and [46]. Let us also mention [41], where the theory of *Rough Paths*–the precursor to both Regularity Structures and Paracontrolled Distributions–is used to solve the *deterministic* low-regularity KdV equation and which showcases nicely how tools from singular SPDEs can be applied to non-stochastic PDE problems.

We state the main (shortened) results of the paper relating to the multiplicative stochastic NLS. H"="$\Delta + \xi - \infty$ is the Anderson Hamiltonian whose exact definition and properties are recalled in Section 3.3.

Theorem 3.1.1. *[2d Anderson Strichartz Estimates]Let* $r \geqslant 4$, *then we have for any* $\delta > 0$

$$\|e^{-itH}u\|_{L^r_{t;[0,1]}L^r_{\mathbb{T}^2}} \lesssim \|u\|_{\mathcal{H}^{1-\frac{4}{r}+\delta}_{\mathbb{T}^2}} \tag{3.1.3}$$

Theorem 3.1.2. *[2d low regularity local well-posedness] The PDE*

$$\begin{array}{rcl}(i\partial_t - H)u & = & -u|u|^2 on\ \mathbb{T}^2\\ u(0) & = & u_0\end{array}$$

is locally well-posed(LWP) in $\mathcal{H}^s$ *for* $s \in \left(\frac{1}{2}, 1\right)$.

Theorem 3.1.3. *[Corollary [2d GWP for energy solutions] The PDE*

$$\begin{array}{rcl}(i\partial_t - H)u & = & -u|u|^2 \, on \; \mathbb{T}^2 \\ u(0) & = & u_0\end{array}$$

is globally well-posed(GWP) in the energy space, $\mathcal{D}\left(\sqrt{-H}\right)$, whose definition is recalled in Section 3.3.1.

Theorem 3.1.4. *[3d Anderson Strichartz Estimates] Let $d = 3$ and $r \geqslant \frac{10}{3}$, then for any $\delta > 0$ we have*

$$\|e^{-itH^\sharp} u^\sharp\|_{L^r_{t;[0,1]} L^r_{\mathbb{T}^3}} \lesssim \|u^\sharp\|_{\mathcal{H}^{2-\frac{5}{r}+\delta}_{\mathbb{T}^3}},$$

where $H^\sharp$ is the transformation of the operator H introduced in Section 3.3.2.

The paper is organised as follows: In Section 3.2 we recall the well-known Strichartz estimates on the whole space and how their counterparts on the torus differ. Section 3.3 is meant to recapitulate the construction of the Anderson Hamiltonian and its domain following [49]. In Section 3.4 we prove the Strichartz estimates for the Anderson Hamiltonian in 2- and 3-dimensions, i.e. Theorems 3.1.1 and 3.1.4. Then in Section 3.5 we utilise these bounds to solve the multiplicative stochastic NLS. Lastly we outline in Section 3.6 how the results can straightforwardly be adapted to other potentials.

Notations and conventions

The spaces we work in are L^p-spaces, for $p \in [1, \infty]$, meaning the usual p-integrable Lebesgue functions; $\mathcal{H}^\alpha, W^{\alpha,p}$ spaces, with $\alpha \in \mathbb{R}, p \in [1, \infty]$ the usual Sobolev potential spaces with $\mathcal{H}^\alpha = W^{\alpha,2}$; and $B^s_{r,q}$, the Besov spaces, whose definition is recalled in the appendix and which cover $\mathcal{H}^\alpha$ and $\mathcal{C}^\alpha$–so called Hölder-Besov spaces–as special cases.

Also we write

$$\|f\|_{X_\Omega} := \|f\|_{X(\Omega)} \text{and } \|f(t)\|_{X_{t;\Omega}} := \|f\|_{X(\Omega)},$$

where X is a function space and Ω is the domain, the relevant cases being $\Omega \in \{\mathbb{R}^d, \mathbb{T}^d, [0, T]\}$ for $T > 0$ and $d = 1, 2, 3$.

We write, as is quite common,

$$a \lesssim b$$

to mean $a \leqslant Cb$ for a constant $C > 0$ independent of a, b and their arguments. Also we write

$$a \sim b \Leftrightarrow a \lesssim b \text{ and } b \lesssim a.$$

For the sake of brevity we also allow **every** constant to depend exponentially on the relevant noise norm $\|\Xi\|$, see Section 3.6 for the exact definition of the norms; This can be written schematically as

$$\lesssim \Leftrightarrow \lesssim_\Xi,$$

this comes with the tacit understanding that everything is continuous with respect to this norm, see Section 3.6 for a discussion on this. Another convention is that if we write something like

$$\|F(u)\|_X \lesssim \|u\|_{\mathcal{H}^{\alpha+\varepsilon}} \text{ for } \varepsilon > 0,$$

we of course mean

$$\|F(u)\|_X \leqslant C_\varepsilon \|u\|_{\mathcal{H}^{\alpha+\varepsilon}} \text{ with } C_\varepsilon \to \infty \text{ as } \varepsilon \to 0.$$

3.2 Classical Strichartz estimates on the torus

We start by recalling the well-known Strichartz estimates for Schrödinger equations on $\mathbb{R}^d$.

Theorem 3.2.1. *[[76];Theorem 2.3]Let $d \geqslant 1$ and (p,q) be a Strichartz pair, i.e.*

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} \text{and } (d,p,q) \neq (2,2,\infty),$$

we also take (r', s') to be a dual Strichartz pair, which means that they are Hölder duals of a Strichartz pair (r,s), explicitly

$$\frac{2}{r'} + \frac{d}{s'} = \frac{d+4}{2},$$

then the following are true

i. $\|e^{it\Delta}u\|_{L^p_{t;\mathbb{R}} L^q_{\mathbb{R}^d}} \lesssim \|u\|_{L^2_{\mathbb{R}^d}}$ *"homogeneous Strichartz estimate"*

ii. $\left\|\int_{\mathbb{R}} e^{-it\Delta} F(t)dt\right\|_{L^2_{\mathbb{R}^d}} \lesssim \|F\|_{L^{r'}_{\mathbb{R}} L^{s'}_{\mathbb{R}^d}}$ *"dual homogeneous Strichartz estimate"*

iii. $\left\|\int_{t'<t} e^{i(t-t')\Delta} F(t')dt'\right\|_{L^p_{t;\mathbb{R}} L^q_{\mathbb{R}^d}} \lesssim \|F\|_{L^{r'}_{\mathbb{R}} L^{s'}_{\mathbb{R}^d}}$ *"inhomogeneous Strichartz estimates".*

Next we cite some classical Strichartz estimates on the torus and how they differ from those on the whole space. Moreover we sketch how they allow to solve NLS in spaces below $\mathcal{H}^{\frac{d}{2}+\varepsilon}$,which is an algebra.

The first results we state are the Strichartz estimates proved by Burq-Gerard-Tzvetkov in [17]. They hold on general manifolds, i.e. not only the torus. The results are not optimal for the torus but we nonetheless cite them because the methods we use are strongly inspired by this paper.

Theorem 3.2.2. *[Strichartz estimates on compact manifolds, [17] Theorem 1] Let*

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}.$$

We have on the finite time interval $[0,1]$

$$\|e^{-it\Delta}u\|_{L^p_{t;[0,1]} L^q} \lesssim \|u\|_{\mathcal{H}^{\frac{1}{p}}}$$

and

$$\left\|\int_0^t e^{-i(t-s)\Delta} f(s)ds\right\|_{L^p_{t;[0,1]} L^q} \lesssim \int_0^1 \|f(s)\|_{\mathcal{H}^{\frac{1}{p}}} ds.$$

Note that, as opposed to the whole space, we have a loss of $\frac{1}{p}$ derivatives. The next result is the "state of the art" for Strichartz estimates on the torus due to Bourgain and Demeter in [14] which were refined in [60] by Killip and Visan whose version we cite because it is more amenable to our situation. With this result we are able to reduce the loss of derivative to be arbitrarily small. The result is stated for functions which are localised in frequency but the corresponding Sobolev bound is immediate.

Theorem 3.2.3. *[Improved Strichartz, Theorem 1.2 [60], [14]]Let $d \geqslant 1$ and $p \geqslant \frac{2(d+2)}{d}$, then, for any $\varepsilon > 0$ we have*

$$\|e^{-it\Delta} P_{\leqslant N} f\|_{L^p_{t;[0,1]} L^p_{\mathbb{T}^d}} \lesssim N^{\frac{d}{2} - \frac{d+2}{p} + \varepsilon} \|f\|_{L^2_{\mathbb{T}^d}}.$$

For $d = 2$ this means $p \geqslant 4$ and for $d = 3$ $p \geqslant \frac{10}{3}$.

For future reference we state the above result for short times that may depend on the localised frequency. We write for a function h defined on the torus the short-hand notation

$$h_N := P_{\leqslant N} h, \text{ where } P_{\leqslant N} := \mathcal{F}^{-1} \mathbb{I}_{\lesssim N} \mathcal{F}.$$

Proposition 3.2.4. *[Strichartz for short times]*

Assume, as above, that f is defined on the interval $[0,1]$. Let $N > 0$ an integer and $I_N = [t_0, t_1]$ a subinterval of $[0,1]$ of length $\sim \frac{1}{N}$, $p \geqslant \frac{2(d+2)}{d}$ and $\varepsilon > 0$, then

$$\|e^{-it\Delta} u_N\|_{L^p_{t;I_N} L^p_{\mathbb{T}^d}} \lesssim N^{\frac{d}{2} - \frac{d+2}{p} - \frac{1}{p} + \varepsilon} \|u_N\|_{L^2_{\mathbb{T}^d}}$$

and

$$\begin{aligned} \left\| \int_{t_0}^t e^{-i(t-s)\Delta} f_N(s) ds \right\|_{L^p_{t;I_N} L^p_{\mathbb{T}^d}} &\lesssim N^{\frac{d}{2} - \frac{d+2}{p} - \frac{1}{p} + \varepsilon} \int_{I_N} \|f_N(s)\|_{L^2_{\mathbb{T}^d}} ds \\ &\lesssim N^{\frac{d}{2} - \frac{d+2}{p} - \frac{1}{p} - 1 + \varepsilon} \|f_N\|_{L^\infty_{I_N} L^2_{\mathbb{T}^d}}. \end{aligned}$$

Proof. We prove the first inequality, the second one follows from the first in the usual way. For definiteness we set

$$I_i := \left[\frac{i}{N}, \frac{i+1}{N}\right] \text{ and } u_N := \sum_{|n| \leqslant N} \lambda_n e^{2\pi i n \cdot x}.$$

The first thing we show is, setting $e_n(x) := e^{2\pi i n \cdot x}$,

$$\|e^{-it\Delta} \lambda_n e_n\|_{L^p_{t;I_j} L^p_{\mathbb{T}^d}} = \|e^{-it\Delta} \lambda_n e_n\|_{L^p_{t;I_k} L^p_{\mathbb{T}^d}} \text{ for all } 0 \leqslant j,k \leqslant N-1 \text{ and } n \in \mathbb{Z}^2, \tag{3.2.1}$$

since then we have for any j and n

$$\begin{aligned} N \|e^{-it\Delta} \lambda_n e_n\|^p_{L^p_{t;I_j} L^p_{\mathbb{T}^d}} &= \sum_{k=1}^N \|e^{-it\Delta} \lambda_n e_n\|^p_{L^p_{t;I_k} L^p_{\mathbb{T}^d}} \\ &= \|e^{-it\Delta} \lambda_n e_n\|^p_{L^p_{t;[0,1]} L^p_{\mathbb{T}^d}} \\ &\lesssim N^{\frac{dp}{2} - d - 2 + p\varepsilon} \|\lambda_n e_n\|^p_{L^2_{\mathbb{T}^d}}, \end{aligned}$$

having used the Strichartz estimate from Theorem 3.2.3 in the last step.
We proceed to prove (3.2.1) for $0 = j < k$ w.l.o.g. Then for $t \in I_0 = [0, \frac{1}{N}]$ we have

$$e^{-it\Delta} \lambda_n e_n = \lambda_n e^{i(2\pi n)^2 t + 2\pi i n \cdot x} =: g_n(t,x)$$

and for $t \in I_k = \left[\frac{k}{N}, \frac{k+1}{N}\right]$ we have

$$\begin{aligned} e^{-it\Delta}\lambda_n e_n &= e^{-i(t-\frac{k}{N})\Delta} e^{-i\frac{k}{N}\Delta}\lambda_n e_n \\ &= e^{-i(t-\frac{k}{N})\Delta}\lambda_n e^{2\pi i n\cdot(x+(2\pi)^2\frac{k}{N}n)} \\ &= g_n(t-\frac{k}{N}, x+(2\pi)^2\frac{k}{N}n), \end{aligned}$$

so we have

$$\|g_n(t-\frac{k}{N}, \cdot + (2\pi)^2\frac{k}{N}n)\|_{L^p_{\mathbb{T}^d}} = \|g_n(t-\frac{k}{N}, \cdot)\|_{L^p_{\mathbb{T}^d}}$$

by a change of variables and periodicity. Further, we have

$$\begin{aligned} \|g_n(t-\frac{k}{N}, \cdot + (2\pi)^2\frac{k}{N}n)\|^p_{L^p_{t;I_k}L^p_{\mathbb{T}^d}} &= \|g_n(t-\frac{k}{N})\|^p_{L^p_{t;I_k}L^p_{\mathbb{T}^d}} \\ &= \int_{\frac{k}{N}}^{\frac{k+1}{N}} \|g_n(t-\frac{k}{N})\|^p_{L^p_{\mathbb{T}^d}} dt \\ &= \int_0^{\frac{1}{N}} \|g_n(s)\|^p_{L^p_{\mathbb{T}^d}} ds \\ &= \|e^{-it\Delta}u_N\|^p_{L^p_{t;[0,\frac{1}{N}]}L^p_{\mathbb{T}^d}}, \end{aligned}$$

which means we have showed the bound

$$\|e^{-it\Delta}\lambda_n e_n\|_{L^p_{t;I_j}L^p_{\mathbb{T}^d}} \lesssim N^{-\frac{1}{p}} N^{\frac{d}{2}-\frac{d+2}{p}+\varepsilon}\|\lambda_n e_n\|_{L^2_{\mathbb{T}^d}}.$$

We proceed to sum over the frequencies $\leqslant N$. To do so we first recall the square function estimate

$$\|g\|_{L^p} \sim \|\|\Delta_j g\|_{l^2_j}\|_{L^p},$$

see e.g. [32] and then we start by bounding

$$\begin{aligned} \|\sum_{|n|\leqslant N} e^{-it\Delta}\lambda_n e_n\|^2_{L^p_{\mathbb{T}^d}} &\sim \|\sum_{|n|\leqslant N} |e^{-it\Delta}\lambda_n e_n|^2\|_{L^{\frac{p}{2}}_{\mathbb{T}^d}} \\ &\lesssim \sum_{|n|\leqslant N} \|e^{-it\Delta}\lambda_n e_n\|^2_{L^p_{\mathbb{T}^d}} \end{aligned}$$

having used the square function bound and the triangle inequality. Next we take the $L^{\frac{p}{2}}$ norm in time over the interval I_j

$$\begin{aligned}
\| \sum_{|n|\leqslant N} e^{-it\Delta}\lambda_n e_n\|^2_{L^p_{t;I_j}L^p_{\mathbb{T}^d}} &= \|\|\| \sum_{|n|\leqslant N} e^{-it\Delta}\lambda_n e_n\|^2_{L^p_{\mathbb{T}^d}}\|_{L^{\frac{p}{2}}_{I_j}} \\
&\lesssim \| \sum_{|n|\leqslant N} \|e^{-it\Delta}\lambda_n e_n\|^2_{L^p_{\mathbb{T}^d}}\|_{L^{\frac{p}{2}}_{I_j}} \\
&\lesssim \sum_{|n|\leqslant N} \|e^{-it\Delta}\lambda_n e_n\|^2_{L^p_{t;I_j}L^p_{\mathbb{T}^d}} \\
&\lesssim N^{-\frac{2}{p}} N^{d-\frac{2(d+2)}{p}+2\varepsilon} \sum_{|n|\leqslant N} \|\lambda_n e_n\|^2_{L^2_{\mathbb{T}^d}} \\
&= N^{-\frac{2}{p}} N^{d-\frac{2(d+2)}{p}+2\varepsilon} \| \sum_{|n|\leqslant N} \lambda_n e_n\|^2_{L^2_{\mathbb{T}^d}}
\end{aligned}$$

in other words

$$\|e^{-it\Delta}u_N\|_{L^p_{t;I_N}L^p_{\mathbb{T}^d}} \lesssim N^{-\frac{1}{p}} N^{\frac{d}{2}-\frac{(d+2)}{p}+\varepsilon}\|u_N\|_{L^2_{\mathbb{T}^d}}$$

as claimed. □

Furthermore, we give a quick sketch about why these kinds of estimates are needed for NLS on the torus.

Take for simplicity the cubic NLS on the two-dimensional torus.

$$\begin{aligned}
i\partial_t u - \Delta u &= -u|u|^2 \text{ on } \mathbb{T}^2 \\
u(0) &= u_0.
\end{aligned}$$

The Duhamel formula reads

$$u(t) = e^{-it\Delta}u_0 + i\int_0^t e^{-i(t-s)\Delta}|u|^2 u(s)ds. \tag{3.2.2}$$

Since for $u_0 \in \mathcal{H}^\sigma$ with $\sigma \in \mathbb{R}$ we have

$$e^{-it\Delta}u_0 \in C_t\mathcal{H}^\sigma$$

it is natural to try to solve (3.2.2) in a space like $C_t\mathcal{H}^\sigma$ for, say, $\sigma \geqslant 0$. Now, since the unitary group $e^{-it\Delta}$ has no smoothing properties, the "best" possible way to bound the nonlinear expression in (3.2.2) is

$$\left\|\int_0^t e^{-i(t-s)\Delta}|u|^2 u(s)ds\right\|_{C_{t;[0,T]}\mathcal{H}^\sigma_{\mathbb{T}^2}} \lesssim \int_0^T \||u|^2 u(s)\|_{\mathcal{H}^\sigma_{\mathbb{T}^2}} ds \tag{3.2.3}$$

$$\lesssim \int_0^T \|u(s)\|^2_{L^\infty_{\mathbb{T}^2}}\|u(s)\|_{\mathcal{H}^\sigma_{\mathbb{T}^2}} ds, \tag{3.2.4}$$

where the second inequality follows from the "tame" estimate (see Lemma 3.6.14). In the case that $\sigma > \frac{d}{2}(= 1 \text{ in 2d})$ the L^∞ norm is controlled by the $\mathcal{H}^\sigma$ norm so it is easy to close the fixed point argument. But even in the case of $\mathcal{H}^1$, which is natural as it is the "energy space", one is not able to close the contraction argument without additional input.

The key observation to make – and to see where the Strichartz estimates enter – is that in (3.2.4) we can apply Hölder's inequality in time to obtain

$$\int_0^T \|u(s)\|_{L^\infty_{\mathbb{T}^2}}^2 \|u(s)\|_{\mathcal{H}^\sigma_{\mathbb{T}^2}} ds \lesssim \|u\|^2_{L^p_{[0,T]}L^\infty_{\mathbb{T}^2}} \|u\|_{L^q_{[0,T]}\mathcal{H}^\sigma_{\mathbb{T}^2}} \text{ for } \frac{2}{p}+\frac{1}{q}=1$$

and note that we need not control the $L^\infty_t L^\infty_x$ norm of the solution but it suffices to control the $L^p_t L^\infty_x$ norm for some suitable $2 \leqslant p < \infty$.

So, if we were able to control the $L^p_t L^\infty_x$ norm of the right-hand side of (3.2.2) by the $C\mathcal{H}^\sigma$ norm of u we would be able to get a local-in-time contraction. Of course, bounding the L^∞ norm directly is hopeless but recall the Sobolev embedding in $d-$dimensions

$$W^{\frac{d}{q}+\varepsilon,q} \hookrightarrow L^\infty \text{ for any } q \in (1,\infty) \text{ and } \varepsilon > 0.$$

This is the stage where the Strichartz estimates come in, since, for example by Theorem 3.2.2, one gets the bound

$$\|e^{-it\Delta}u_0\|_{L^p_{t;[0,T]}L^\infty_{\mathbb{T}^2}} \lesssim \|e^{-it\Delta}u_0\|_{L^p_{t;[0,T]}W^{\frac{2}{q}+\varepsilon,q}_{\mathbb{T}^2}} \lesssim_T \|u_0\|_{\mathcal{H}^{\frac{1}{p}+\frac{2}{q}+\varepsilon}_{\mathbb{T}^2}}$$

for the linear evolution where (p,q) is a Strichartz pair. Note that by choosing ε small we get

$$\frac{1}{p}+\frac{2}{q}+\varepsilon = \left(\frac{2}{p}+\frac{2}{q}\right) - \frac{1}{p} + \varepsilon = 1 - \frac{1}{p} + \varepsilon < 1$$

so this is strictly better than what we would get from estimating the L^∞ norm by the $\mathcal{H}^{1+\varepsilon}$ norm.

For the nonlinear term we get similarly (assuming for simplicity $T \leq 1$)

$$\begin{aligned}\left\|\int_0^t e^{-i(t-s)\Delta}|u|^2u(s)ds\right\|_{L^p_{t;[0,T]}L^\infty_{\mathbb{T}^2}} &\lesssim \left\|\int_0^t e^{-i(t-s)\Delta}|u|^2u(s)ds\right\|_{L^p_{t;[0,T]}W^{\frac{2}{q}+\varepsilon,q}_{\mathbb{T}^2}} \\ &\lesssim \int_0^T \||u|^2u(s)\|ds_{\mathcal{H}^{\frac{1}{p}+\frac{2}{q}+\varepsilon}_{\mathbb{T}^2}} \\ &\lesssim T\|u\|^3_{L^\infty_{[0,T]}\mathcal{H}^{\tilde{\sigma}}_{\mathbb{T}^2}}\end{aligned}$$

where $\frac{1}{p}+\frac{2}{q}+\varepsilon < \tilde{\sigma} < 1$ can be computed explicitly using the fractional Leibniz rule, see Lemma 3.6.13.

Clearly these bounds can be sharpened in different ways but the important thing is that Strichartz estimates lead to local-in-time well-posedess for some range of $\sigma \leqslant 1$.

3.3 The Anderson Hamiltonian in 2 and 3 dimensions

The main aim of this work is to establish Strichartz estimates for the Anderson Hamiltonian which is formally given on the 2-/3-dimensional torus by

$$H\text{“=”}\Delta + \xi - \infty,$$

where $\xi = \xi(x)$ is spatial white noise, see Definition 3.6.1. This operator was initially studied by Allez-Chouk in [3] and later by Gubinelli,Ugurcan and the author in [49] using the theory of *Paracontrolled Distributions* which was introduced in [45]. The operator was also studied by Labbé in [62] using the theory of *Regularity Structures* introduced in [54]. Naïvely one might think that it is simply a suitably well-behaved perturbation of the Laplacian in which case Theorem 6 in [17] would more or less directly apply. However, it was shown that the domain of H in both $2d$ and $3d$ can be explicitly determined and one even has

$$\mathcal{D}(H) \cap \mathcal{H}^2 = \{0\},$$

so it is tricky to directly compare the operators H and Δ.

3.3.1 The 2d Anderson Hamiltonian

We briefly recall some of the main ideas from [49] in the 2d setting. An observation made in [3] was that a function u is in the domain of H if

$$u - (u \prec (1-\Delta)^{-1}\xi + B_\Xi(u)) \in \mathcal{H}^2, \tag{3.3.1}$$

see the appendix for the definition and properties of paraproducts. By the paraproduct estimates(Lemma 3.6.8) and the regularity of the noise, the term $u \prec (1-\Delta)^{-1}\xi$ is no better than $\mathcal{H}^{1-\varepsilon}$. The "lower order" correction term B_Ξ is also worse than $\mathcal{H}^2$ (in fact it is $\mathcal{H}^{2-\varepsilon}$). This for example rules out that u is regular, rather it fixes its regularity at $\mathcal{H}^{1-\varepsilon}$; See Definition 3.6.2 for the exact definition of the *enhanced noise* Ξ.

One of the chief innovations in [49] as opposed to [3] was to observe that the statement (3.3.1) is equivalent to

$$u - P_{>N}(u \prec (1-\Delta)^{-1}\xi + B_\Xi(u)) \in \mathcal{H}^2,$$

where $P_{>N} = \mathcal{F}^{-1}\mathbb{I}_{>N}\mathcal{F}$, for any $N > 0$ and subsequently choosing N large enough depending on the $\mathcal{X}^\alpha$ norm of Ξ (see Definition 3.6.2) one can show that the map

$$\begin{array}{rcl} \Phi(u) & := & u - P_{>N(\Xi)}(u \prec (1-\Delta)^{-1}\xi + B_\Xi(u)) \\ \mathcal{D}(H) & \mapsto & \mathcal{H}^2 \end{array}$$

which sends a paracontrolled function to its remainder admits an inverse which we call Γ; We also rename Φ as Γ^{-1}. In the following we use the short-hand notation

$$u = \Gamma u^\sharp = P_{>N(\Xi)}(\Gamma u^\sharp \prec (1-\Delta)^{-1}\xi + B_\Xi(\Gamma u^\sharp)) + u^\sharp, \tag{3.3.2}$$

where the term B_Ξ is explicitly given by

$$B_\Xi(u) := (1-\Delta)^{-1}(\Delta u \prec X + 2\nabla u \prec \nabla X + \xi \prec u - u \prec \Xi_2),$$

where

$$X = (1-\Delta)^{-1}\xi \text{ and } \Xi_2 \text{ is the second component of } \Xi.$$

Consistently with our convention, we write N instead of $N(\Xi)$ in the sequel. Moreover, in the new coordinates, $u^\sharp$, the operator H is given by

$$\begin{aligned} H\Gamma u^\sharp =& \Delta u^\sharp + u^\sharp \circ \xi + P_{\leqslant N}(\Gamma u^\sharp \prec \xi + \Gamma u^\sharp \succ \xi) \\ &+ P_{>N}(-B_\Xi(\Gamma u^\sharp) - \Gamma u^\sharp \prec X + \Gamma u^\sharp \succeq \Xi_2 + C(\Gamma u^\sharp, X, \xi) + B_\Xi(\Gamma u^\sharp) \circ \xi). \end{aligned} \tag{3.3.3}$$

It is also natural to consider the operator H conjugated by Γ, i.e.

$$H^\sharp := \Gamma^{-1} H \Gamma, \tag{3.3.4}$$

which can be expressed as

$$\begin{aligned} H^\sharp =& H\Gamma u^\sharp - P_{>N}(H\Gamma u^\sharp \prec X + B_\Xi(H\Gamma u^\sharp)) \\ =& \Delta u^\sharp + u^\sharp \circ \xi + P_{\leqslant N}(\Gamma u^\sharp \prec \xi + \Gamma u^\sharp \succ \xi) - P_{>N}(H\Gamma u^\sharp \prec X + B_\Xi(H\Gamma u^\sharp)) \\ & + P_{>N}(-B_\Xi(\Gamma u^\sharp) - \Gamma u^\sharp \prec X + \Gamma u^\sharp \succeq \Xi_2 + C(\Gamma u^\sharp, X, \xi) + B_\Xi(\Gamma u^\sharp) \circ \xi) \end{aligned} \tag{3.3.5}$$

We remark that while H was shown to be self-adjoint on L^2, the conjugated operator $H^\sharp$ is not and in particular the map Γ is not unitary.

We now quote some results from [49]; We tacitly assume $-H$ to be positive as opposed to just being semi-bounded, this can be achieved by adding a finite Ξ-dependent constant.

Theorem 3.3.1. *[Proposition 2.27, Lemma 2.33, Lemma 2.34 [49]] We have, writing again $u = \Gamma u^\sharp$,*

i. $\|Hu\|_{L^2_{\mathbb{T}^2}} \sim \|u^\sharp\|_{\mathcal{H}^2_{\mathbb{T}^2}}$

ii. $\left\|\sqrt{-H}u\right\|_{L^2_{\mathbb{T}^2}} = (-(u, Hu)_{L^2_{\mathbb{T}^2}})^{\frac{1}{2}} \sim \|u^\sharp\|_{\mathcal{H}^1_{\mathbb{T}^2}}$

iii. $\mathcal{D}(H) \hookrightarrow L^\infty$ *and* $\mathcal{D}\left(\sqrt{-H}\right) \hookrightarrow L^p$ *for any* $p < \infty$*.*

The following proposition quantifies the idea that the transformed operator $H^\sharp$ is a lower-order perturbation of the Laplacian.

Proposition 3.3.2. *Take $u^\sharp \in \mathcal{H}^2$, then the following holds for any $s, \varepsilon > 0$ s.t. $1 + s + \varepsilon \le 2$*

$$\|(H^\sharp - \Delta)u^\sharp\|_{\mathcal{H}^s_{\mathbb{T}^2}} \lesssim \|u^\sharp\|_{\mathcal{H}^{1+s+\varepsilon}_{\mathbb{T}^2}}.$$

Proof. This essentially follows by noting that in terms of regularity the worst terms to bound in (3.3.5) are $u^\sharp \circ \xi$ and $H\Gamma u^\sharp \prec X$ which are bounded by(see Lemma 3.6.8)

$$\|u^\sharp \circ \xi\|_{\mathcal{H}^s_{\mathbb{T}^2}} \lesssim \|u^\sharp\|_{\mathcal{H}^{1+s+\varepsilon}_{\mathbb{T}^2}} \|\xi\|_{\mathcal{C}^{-1-\varepsilon}_{\mathbb{T}^2}}$$

and

$$\|H\Gamma u^\sharp \prec X\|_{\mathcal{H}^s_{\mathbb{T}^2}} \lesssim \|\Delta u^\sharp\|_{\mathcal{H}^{-1+s+\varepsilon}_{\mathbb{T}^2}} \|X\|_{\mathcal{C}^{1-\varepsilon}_{\mathbb{T}^2}}$$

respectively.

The other terms are bounded similarly by $\mathcal{H}^s$ norms of $u^\sharp$ with $s < 1$ multiplied by Hölder norms of objects related to ξ which appear in the $\mathcal{X}^\alpha$-norm, see Definition 3.6.2. □

We collect all relevant results about the map Γ.

Lemma 3.3.3. *The map Γ is bounded with a bounded inverse in the following situations(recall that we think of α as $-1-\delta$ for small δ.)*

i. $\Gamma : \mathcal{H}^s \to \mathcal{H}^s$ *for any* $s \in [0, \alpha + 2)$;

ii. $\Gamma : L^p \to L^p$ *for any* $p \in [2, \infty]$;

iii. $\Gamma : W^{s,p} \to W^{s,p}$ *for any* $s \in [0, \alpha + 2)$ *and* $p \in [2, \infty)$;

iv. $\Gamma : \mathcal{H}^1 \to \mathcal{D}\left(\sqrt{-H}\right)$;

v. $\Gamma : \mathcal{H}^2 \to \mathcal{D}(H)$.

Proof. Everything but (ii) and (iii) was proved in Section 2.1.1 of [49]. The cases $p = 2, \infty$ of (ii) were also already proved. For a different p we note that the result follows by interpolation. Lastly, (iii) can be proved by using the embeddings

$$B^s_{p,2} \hookrightarrow W^{s,p} \hookrightarrow B^s_{p,p},$$

see e.g. [32] together with the Besov embedding Lemma 3.6.11 and the paraproduct estimates for Besov spaces, Lemma 3.6.8. In fact, we may bound

$$\begin{aligned}
\|\Gamma f\|_{W^{s,p}_{\mathbb{T}^2}} &\leq \|f\|_{W^{s,p}_{\mathbb{T}^2}} + \|(\Gamma - 1)f\|_{W^{s,p}_{\mathbb{T}^2}} \\
&\lesssim \|f\|_{W^{s,p}_{\mathbb{T}^2}} + \|(\Gamma - 1)f\|_{B^s_{p,2}(\mathbb{T}^2)} \\
&\lesssim \|f\|_{W^{s,p}_{\mathbb{T}^2}} + \|f\|_{B^{s-\varepsilon}_{p,2}(\mathbb{T}^2)} \\
&\lesssim \|f\|_{W^{s,p}_{\mathbb{T}^2}} + \|f\|_{B^s_{p,p}(\mathbb{T}^2)} \\
&\lesssim \|f\|_{W^{s,p}_{\mathbb{T}^2}}.
\end{aligned}$$

□

Lastly we prove a statement about the “sharpened” group, which is the transformation of the unitary group e^{-itH}

$$e^{-itH^\sharp} := \Gamma^{-1} e^{-itH} \Gamma.$$

It is clear that one has the bounds

$$\begin{aligned}
\|e^{-itH} u\|_{L^2_{\mathbb{T}^2}} &\lesssim \|u\|_{L^2_{\mathbb{T}^2}} \\
\|e^{-itH} u\|_{\mathcal{D}(H)} &\lesssim \|u\|_{\mathcal{D}(H)}.
\end{aligned}$$

We briefly show the analogous results for the transformed group.

Lemma 3.3.4. *For* $s \in [0, 2]$ *we get the following at any time* $t \in \mathbb{R}$

$$\|e^{-itH^\sharp} u^\sharp\|_{\mathcal{H}^s_{\mathbb{T}^2}} \lesssim \|u^\sharp\|_{\mathcal{H}^s_{\mathbb{T}^2}}.$$

Proof. The case $s = 0$ follows since both Γ and Γ^{-1} are bounded on L^2. The case $s = 2$ can be proved using Lemma 3.3.3. In fact

$$\begin{aligned}
\|e^{-itH^\sharp}u^\sharp\|_{\mathcal{H}^2_{\mathbb{T}^2}} &= \|\Gamma^{-1}e^{-itH}\Gamma u^\sharp\|_{\mathcal{H}^2_{\mathbb{T}^2}} \\
&\lesssim \|He^{-itH}\Gamma u^\sharp\|_{L^2_{\mathbb{T}^2}} \\
&\sim \|e^{-itH}H\Gamma u^\sharp\|_{L^2_{\mathbb{T}^2}} \\
&\lesssim \|H\Gamma u^\sharp\|_{L^2_{\mathbb{T}^2}} \\
&\lesssim \|u^\sharp\|_{\mathcal{H}^2_{\mathbb{T}^2}}
\end{aligned}$$

The case $s \in (0,2)$ is proved by interpolation. □

3.3.2 The 3d Anderson Hamiltonian

We recall the main results about the Anderson Hamiltonian in 3d, which are analogous to– yet slightly more technical than– the 2d case. For definiteness we fix an enhanced white noise $\Xi \in \mathcal{H}^\alpha$ for $\alpha = \frac{1}{2} - \varepsilon$ for small $\varepsilon > 0$; see Definition 3.6.4 and Theorem 3.6.5 where the definition of the "noise space" is recalled and the fact that almost every realisation of white noise has a lift in it.

The main difference with respect to the 2d case is that due to the higher irregularity of the noise in 3d (in fact $\mathcal{C}^{-\frac{3}{2}-\varepsilon}$ for any $\varepsilon > 0$), a simple paracontrolled ansatz does not suffice. Instead, we first make an exponential transformation as in [56] to remove the most irregular terms. This leads to some lower-order terms involving gradients and we subsequently perform a paracontrolled ansatz. As opposed to the 2d case, where there were only two noise terms, in this case there are more. We give a formal argument, which was made rigorous in [49].

Since this section gives only a formal justification, we use notations like $\mathcal{C}^{1-}$ to mean $\mathcal{C}^\sigma$ for any $\sigma < 1$ etc. We start with $\xi \in \mathcal{C}^{-\frac{3}{2}-}$ and assume that the following objects exist:

$$X = (-\Delta)^{-1}\xi \in \mathcal{C}^{\frac{1}{2}-} \qquad X^{\vee} = (1-\Delta)^{-1} :|\nabla X|^2: \in \mathcal{C}^{1-}$$

$$X^{\vee_{\vee}} = 2(1-\Delta)^{-1}\left(\nabla X \cdot \nabla X^{\vee}\right) \in \mathcal{C}^{\frac{3}{2}-} \qquad X^{\vee_{\vee_{\vee}}} = (1-\Delta)^{-1}\left(\nabla X \cdot \nabla X^{\vee_{\vee}}\right) \in \mathcal{C}^{\frac{3}{2}-}$$

and

$$X^{\vee\vee} = (1-\Delta)^{-1} :\left|\nabla X^{\vee}\right|^2: \in \mathcal{C}^{2-},$$

where the "Wick squares" should be thought of as

$$\begin{aligned}
:|\nabla X|^2: &= |\nabla X|^2 - \infty \\
:\left|\nabla X^{\vee}\right|^2: &= \left|\nabla X^{\vee}\right|^2 - \infty,
\end{aligned} \tag{3.3.6}$$

see [59] for background information about these objects.

We make the following auxiliary ansatz for the domain of the Hamiltonian which removes the three most singular terms

$$u = e^{X + X^{\vee} + X^{\vee_{\vee}}} u^\flat;$$

where the form of $u^\flat$ will be specified later. We begin by computing

$$\begin{aligned}\Delta u + u\xi =& e^{X+X^{\vee}+X^{\Psi}}\Big(\Delta\left(X+X^{\vee}+X^{\Psi}\right)u^\flat + \left|\nabla\left(X+X^{\vee}+X^{\Psi}\right)\right|^2 u^\flat + \\ & + \Delta u^\flat + 2\nabla\left(X+X^{\vee}+X^{\Psi}\right)\nabla u^\flat + u^\flat\xi\Big) \\ =& e^{X+X^{\vee}+X^{\Psi}}\Big(\Delta u^\flat + \Big(|\nabla X|^2 - :|\nabla X|^2: + \left|\nabla X^{\vee}\right|^2 + \left|\nabla X^{\Psi}\right|^2 \\ & + 2\nabla X\cdot\nabla X^{\Psi} + 2\nabla X^{\vee}\cdot\nabla X^{\Psi} - X^{\vee} - X^{\Psi}\Big)u^\flat + 2\nabla\left(X+X^{\vee}+X^{\Psi}\right)\cdot\nabla u^\flat\Big),\end{aligned}$$

note that the regularity of $X^{\vee}$ is too low for the term $\left|\nabla X^{\vee}\right|^2$ to be defined so we have to replace it by by its Wick ordered version, also note the appearing difference $|\nabla X|^2 - :|\nabla X|^2:$. Here one sees the two divergences that arise in the definition of H, since their difference is exactly the infinite correction term in (3.3.6).

So we can summarise the above by saying that for

$$W := X + X^{\vee} + X^{\Psi},$$

we "almost" can define the operator $\Delta + \xi - \infty$ on functions of the form $e^W u^\flat$. In fact

$$Hu := H(e^W u^\flat) := e^W(\Delta u^\flat + 2(1-\Delta)\tilde{W}\cdot\nabla u^\flat + (1-\Delta)Zu^\flat), \tag{3.3.7}$$

makes sense for $u^\flat$ in, say, $\mathcal{H}^2$, where we have defined

$$\begin{aligned}\tilde{W} &= (1-\Delta)^{-1}\nabla W \\ Z &= (1-\Delta)^{-1}\left(\left|\nabla X^{\Psi}\right|^2 + 2\nabla X^{\vee}\cdot\nabla X^{\Psi} - X^{\vee} - X^{\Psi}\right) + X^{\vee\vee} + 2X^{\Psi'}\end{aligned}$$

and the regularities are

$$X, W \in \mathcal{C}^{\frac{1}{2}-}, X^{\vee} \in \mathcal{C}^{1-}, X^{\Psi}, X^{\Psi'}, \tilde{W}, Z \in \mathcal{C}^{\frac{3}{2}-}, \text{and } X^{\vee\vee} \in \mathcal{C}^{2-}.$$

The problem with (3.3.7) is of course that the right hand side will not be in L^2, since both the noise terms have negative regularity, so for any $u^\flat \in \mathcal{H}^2$ we have

$$\begin{aligned}\Delta u^\flat &\in L^2 \\ 2(1-\Delta)\tilde{W}\cdot\nabla u^\flat, (1-\Delta)Zu^\flat &\in \mathcal{H}^{-\frac{1}{2}-}.\end{aligned}$$

The remedy is once again a paracontrolled ansatz, this time of the form

$$u^\flat = u^\flat \prec Z + \nabla u^\flat \prec \tilde{W} + B_\Xi(u^\flat) + u^\sharp,$$

with $u^\sharp \in \mathcal{H}^2$ and the correction term $B_\Xi(u^\flat) \in \mathcal{H}^{2-}$ defined below.

We cite the rigorous definition of the operator and its domain; we use the short-hand notation

$$L := (1-\Delta) \text{ and } L^{-1} := (1-\Delta)^{-1}.$$

Definition 3.3.5. *Let $W, \tilde{W}, Z$ be as above. Then, for $0 < \gamma < \frac{3}{2}$, we define the space*

$$\mathcal{W}_\Xi^\gamma := e^W \mathcal{U}_\Xi^\gamma := e^W \left\{ u^\flat \in \mathcal{H}^\gamma : u^\flat = u^\flat \prec Z + \nabla u^\flat \prec \tilde{W} + B_\Xi(u^\flat) + u^\sharp \text{ with } u^\sharp \in \mathcal{H}^2 \right\},$$

where $B_\Xi(u^\flat)$is given as

$$\begin{aligned} B_\Xi(u^\flat) := L^{-1}\big(&\Delta u^\flat \prec Z + 2\nabla u^\flat \prec \nabla Z + u^\flat \prec Z + \nabla \Delta u^\flat \prec \tilde{W} + 2\nabla^2 u^\flat \prec \nabla \tilde{W} - \nabla u^\flat \prec \tilde{W} + \\ &+ 2L\tilde{W} \prec \nabla u^\flat + LZ \prec u^\flat + 2\nabla u^\flat \prec (L\tilde{W} \circ Z) + 2\nabla u^\flat \succ (L\tilde{W} \circ Z) + \\ &+ 2u^\flat \prec (L\tilde{W} \circ \nabla Z) + 2u^\flat \succ (L\tilde{W} \circ \nabla Z) + 2\nabla u^\flat \prec (L\tilde{W} \circ \nabla \tilde{W}) + \\ &+ 2\nabla u^\flat \succ (L\tilde{W} \circ \nabla \tilde{W}) + u^\flat \prec (LZ \circ Z) + u^\flat \succ (LZ \circ Z) + \\ &+ \nabla u^\flat \prec (LZ \circ \tilde{W}) + \nabla u^\flat \succ (LZ \circ \tilde{W})\big). \end{aligned}$$

Given $u \in \mathcal{W}_\Xi^\gamma$ we define the renormalised Anderson Hamiltonian acting on u in the following way

$$Hu := e^W(\Delta u^\sharp + LZ \circ u^\sharp + L\tilde{W} \circ \nabla u^\sharp + G(u^\flat)), \tag{3.3.8}$$

where

$$\begin{aligned} G(u^\flat) := &B_\Xi(u^\flat) + 2\nabla u^\flat \circ (L\tilde{W} \circ Z) + 2C(\nabla u^\flat, Z, L\tilde{W}) + u^\flat \circ (L\tilde{W} \circ \nabla Z) + \\ &+ C(u^\flat, \nabla Z, L\tilde{W}) + 2L\tilde{W} \circ (\nabla^2 u^\flat \prec \tilde{W}) + 2\nabla u^\flat \circ (L\tilde{W} \circ \nabla \tilde{W}) + \\ &+ 2C(\nabla u^\flat, \nabla \tilde{W}, L\tilde{W}) + 2L\tilde{W} \circ \nabla B_\Xi(u^\flat) \end{aligned}$$

and C denotes the commutator from Proposition 3.6.12. Note that this definition is equivalent to (3.3.7) by construction, we have merely defined $u^\flat$ in the proper way.

Remark 3.3.6. *As was seen in [49], the space $\mathcal{W}_\Xi^\gamma = e^W \mathcal{U}_\Xi^\gamma$ does not really depend on γ in the sense that if one equips the space $\mathcal{U}_\Xi^\gamma$ with the norm*

$$\|(u^\flat, u^\sharp)\|_{\mathcal{U}_\Xi^\gamma} := \|u^\flat\|_{\mathcal{H}^\gamma_{\mathbb{T}^3}} + \|u^\sharp\|_{\mathcal{H}^2_{\mathbb{T}^3}},$$

then these norms are equivalent for different values of γ. This is because the paracontrolled relation enforces a certain regularity, i.e. if

$$v = v \prec Y + v^\sharp,$$

for $v^\sharp \in \mathcal{H}^2$ and $Y \in \mathcal{C}^\alpha$, $\alpha > 0$, then $v \in \mathcal{H}^\alpha$ but not better.

As in 2d, we introduce a Fourier cut-off and obtain the map Γ given by

$$\Gamma f = \Delta_{>N}(\Gamma f \prec Z + \nabla(\Gamma f) \prec \tilde{W} + B_\Xi(\Gamma f)) + f, \tag{3.3.9}$$

choosing N large enough depending on the norm of Ξ. This again allows us to write $u = e^W \Gamma u^\sharp$. It is straightforward to adapt the above definition of H to involve the Fourier cut-off but we do not spell this out as the only thing we care about is the fact that

$$He^W \Gamma u^\sharp = e^W(\Delta u^\sharp + L\tilde{W} \circ \nabla u^\sharp + l.o.t.)$$

We collect the results about Γ, this is analogous to Lemma 3.3.3.

Lemma 3.3.7. *We can choose N large enough depending only on Ξ and s so that we have*

$$\|\Gamma f\|_{L^\infty_{\mathbb{T}^3}} \lesssim \|f\|_{L^\infty_{\mathbb{T}^3}} \tag{3.3.10}$$

$$\|\Gamma f\|_{\mathcal{H}^s_{\mathbb{T}^3}} \lesssim \|f\|_{\mathcal{H}^s_{\mathbb{T}^3}} \tag{3.3.11}$$

for $s \in \left[0, \frac{3}{2}\right)$.

$$\|\Gamma f\|_{W^{s,p}_{\mathbb{T}^3}} \lesssim \|f\|_{W^{s,p}_{\mathbb{T}^3}} \tag{3.3.12}$$

for $s \in \left[0, \frac{3}{2}\right)$ and $2 \leqslant p < \infty$.

In all cases Γ is invertible with a bounded inverse.

Proof. (3.3.10) and (3.3.11) were proved in Proposition 2.46 in [49], (3.3.12) can be proved as in the 2d case, see Lemma 3.3.3. □

We cite the main result about the domain; We again shift the operator by a constant depending on the norm of Ξ to make it non-positive, see Definition 2.55 in [49].

Lemma 3.3.8. *For the operator H the following holds*

i. $\Gamma u^\sharp \in e^{-W}\mathcal{D}(H) \Leftrightarrow u^\sharp \in \mathcal{H}^2$, *more precisely on $\mathcal{D}(H) = \mathcal{W}^\gamma_\Xi = e^W \mathcal{U}^\gamma_\Xi$ we have the following norm equivalence*

$$\|u^\sharp\|_{\mathcal{H}^2_{\mathbb{T}^3}} \sim \|H\Gamma u^\sharp\|_{L^2_{\mathbb{T}^3}};$$

ii. $u \in \mathcal{D}\left(\sqrt{-H}\right) \Leftrightarrow e^{-W}u \in \mathcal{H}^1$,

where the form domain of H is given by the closure of $\mathcal{D}(H)$ under the norm

$$\|u\|_{\mathcal{D}(\sqrt{-H})} := \sqrt{-(u, Hu)_{L^2_{\mathbb{T}^3}}}.$$

Then the precise statement is that on $\mathcal{D}(H)$ the following norm equivalence holds

$$\|e^{-W}u\|_{\mathcal{H}^1_{\mathbb{T}^3}} \sim \|u\|_{\mathcal{D}(\sqrt{-H})}$$

and hence the closures with respect to the two norms coincide.

Now the only part we truly need is that the transformed operator, the analogue of (3.3.4), is

$$H^\sharp := \Gamma^{-1} e^{-W} H e^W \Gamma$$

and that it is "close" to the Laplacian in the sense that

$$H^\sharp u^\sharp = \Delta u^\sharp + L\tilde{W} \circ \nabla u^\sharp + l.o.t.$$

In fact, we have the following result.

Proposition 3.3.9. *For $u^\sharp \in \mathcal{H}^2$, we have the following bounds for $s, \varepsilon > 0$ s.t. $\frac{3}{2} + s + \varepsilon \leq 2$*

i. $\|(H^\sharp - \Delta)u^\sharp\|_{\mathcal{H}^s_{\mathbb{T}^3}} \lesssim \|u^\sharp\|_{\mathcal{H}^{\frac{3}{2}+s+\varepsilon}_{\mathbb{T}^3}}$

ii. For $s \in [0,2]$ we get the following at any time $t \in \mathbb{R}$

$$\|\Gamma^{-1} e^{-W} e^{-itH} e^{W} \Gamma u^\sharp\|_{\mathcal{H}^s_{\mathbb{T}^3}} \lesssim \|u^\sharp\|_{\mathcal{H}^s_{\mathbb{T}^3}}$$

Proof. (i) essentially follows by noting that in terms of regularity the worst terms to bound are $L\tilde{W} \circ \nabla u^\sharp$ and $\nabla(e^{-W} H e^{W} \Gamma u^\sharp) \prec \tilde{W}$ which are bounded like

$$\|L\tilde{W} \circ \nabla u^\sharp\|_{\mathcal{H}^s_{\mathbb{T}^3}} \lesssim \|\nabla u^\sharp\|_{\mathcal{H}^{\frac{1}{2}+s+\varepsilon}_{\mathbb{T}^3}} \|L\tilde{W}\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}_{\mathbb{T}^3}} \lesssim \|u^\sharp\|_{\mathcal{H}^{\frac{3}{2}+s+\varepsilon}_{\mathbb{T}^3}}$$

and

$$\|\nabla(e^{-W} H e^{W} \Gamma u^\sharp) \prec \tilde{W}\|_{\mathcal{H}^s_{\mathbb{T}^3}} \lesssim \|\nabla \Delta u^\sharp\|_{\mathcal{H}^{-\frac{3}{2}+s+\varepsilon}_{\mathbb{T}^3}} \|\tilde{W}\|_{\mathcal{C}^{\frac{3}{2}-\varepsilon}_{\mathbb{T}^3}} \lesssim \|u^\sharp\|_{\mathcal{H}^{\frac{3}{2}+s+\varepsilon}_{\mathbb{T}^3}},$$

respectively.
(ii) For $s = 0$ it follows directly by the properties of Γ and e^W. For $s = 2$ we can, using (i) and Lemma 3.3.8 , compute

$$\begin{aligned}\|\Gamma^{-1} e^{-W} e^{-itH} e^{W} \Gamma u^\sharp\|_{\mathcal{H}^2_{\mathbb{T}^3}} &\lesssim \|\Gamma^{-1} e^{-W} H e^{-itH} e^{W} \Gamma u^\sharp\|_{L^2_{\mathbb{T}^3}} \\ &\lesssim \|H e^{W} \Gamma u^\sharp\|_{L^2_{\mathbb{T}^3}} \\ &\lesssim \|u^\sharp\|_{\mathcal{H}^2_{\mathbb{T}^3}}.\end{aligned}$$

The case $s \in (0,2)$ follows again by interpolation as in the 2d case. □

3.4 Strichartz estimates for the Anderson Hamiltonian

3.4.1 The 2d case

In this section we prove Theorems 3.1.1, 3.1.2, and 3.1.3.

Proposition 3.4.1. *We have the following identity for a regular function, say $u^\sharp \in \mathcal{H}^2$, and at any time $t \in \mathbb{R}$:*

$$(e^{-itH^\sharp} - e^{-it\Delta}) u^\sharp = i \int_0^t e^{-i(t-s)\Delta}((H^\sharp - \Delta)(e^{-isH^\sharp} u^\sharp)) ds, \tag{3.4.1}$$

moreover, fixing some $t_0 \in \mathbb{R}$, we get a related result

$$(e^{-i(t-t_0)H^\sharp} - e^{-i(t-t_0)\Delta}) u^\sharp = i \int_{t_0}^t e^{-i(t-s)\Delta}((H^\sharp - \Delta)(e^{-i(s-t_0)H^\sharp} u^\sharp)) ds, \tag{3.4.2}$$

where we recall, from Proposition 3.3.2, that there is a cancellation between $H^\sharp$ and the Laplacian. Moreover, on the interval $[0,T]$ with $T \leqslant 1$, we have for any small $\delta > 0$ the bounds

$$\|(e^{-itH^\sharp} - e^{-it\Delta}) u^\sharp\|_{L^\infty_{t;[0,T]} \mathcal{H}^\sigma_{\mathbb{T}^2}} \lesssim T \|u^\sharp\|_{\mathcal{H}^{\sigma+1+\delta}_{\mathbb{T}^2}} \tag{3.4.3}$$

and

$$\|(e^{-i(t-t_0)H^\sharp} - e^{-i(t-t_0)\Delta}) u^\sharp\|_{L^\infty_{t;[t_0,t_1]} \mathcal{H}^\sigma_{\mathbb{T}^2}} \lesssim |t_1 - t_0| \|u^\sharp\|_{\mathcal{H}^{\sigma+1+\delta}_{\mathbb{T}^2}} \tag{3.4.4}$$

for $\sigma \in [0, 1-\delta)$.

Also, for $r \geqslant 4$ *we have*

$$\|(e^{-itH^\sharp} - e^{-it\Delta})u^\sharp\|_{L^r_{t;[0,T]}W^{\sigma,r}_{\mathbb{T}^2}} \lesssim \int_0^T \|e^{-i(s-t_0)H^\sharp}u^\sharp\|_{\mathcal{H}^{\sigma+2-\frac{4}{r}+\delta}_{\mathbb{T}^2}} \tag{3.4.5}$$

$$\lesssim T\|u^\sharp\|_{\mathcal{H}^{\sigma+2-\frac{4}{r}+\delta}_{\mathbb{T}^2}} \tag{3.4.6}$$

and

$$\|(e^{-i(t-t_0)H^\sharp} - e^{-i(t-t_0)\Delta})u^\sharp\|_{L^r_{t;[t_0,t_1]}W^{\sigma,r}_{\mathbb{T}^2}} \lesssim \int_{t_0}^{t_1} \|e^{-i(s-t_0)H^\sharp}u^\sharp\|_{\mathcal{H}^{\sigma+2-\frac{4}{r}+\delta}_{\mathbb{T}^2}} \tag{3.4.7}$$

$$\lesssim |t_1 - t_0|\|u^\sharp\|_{\mathcal{H}^{\sigma+2-\frac{4}{r}+\delta}_{\mathbb{T}^2}} \tag{3.4.8}$$

for $\sigma \geq 0$ *s.t.* $\sigma + 2 - \frac{4}{r} + \delta \leq 2$.

Proof. To prove (3.4.1), note that the l.h.s. solves a PDE. Set $v_1^\sharp(t) = e^{-it\Delta}u^\sharp$, $v_2^\sharp(t) = e^{-itH^\sharp}u^\sharp$ and $v^\sharp = v_1^\sharp - v_2^\sharp$. Then

$$\begin{aligned}
(i\partial_t - \Delta)v_1^\sharp &= 0\\
v_1^\sharp(0) &= u^\sharp\\
(i\partial_t - \Delta)v_2^\sharp &= (H^\sharp - \Delta)v_2^\sharp\\
v_2^\sharp(0) &= u^\sharp\\
(i\partial_t - \Delta)v^\sharp &= -(H^\sharp - \Delta)v_2^\sharp\\
v^\sharp(0) &= 0
\end{aligned}$$

From this we deduce that the mild formulation for $v^\sharp$ reads

$$v^\sharp(t) = i\int_0^t e^{-i(t-s)\Delta}((H^\sharp - \Delta)(v_2^\sharp))(s)ds$$

which is (3.4.1). To prove (3.4.2), we proceed as above, with the difference that we replace t by $t - t_0$ and do a change of variables in the integral.
The bound (3.4.3) is clear using Lemma 3.3.4 and (3.4.4) is analogous. For the bound (3.4.6), we apply first (3.4.1) then we use the inhomogeneous Strichartz estimate from Theorem 3.2.3 to the right hand side and then Proposition 3.3.2 to bound the term inside the integral. Subsequently, (3.4.6) follows by applying Lemma 3.3.4 and noting that the integrand does not depend on s any more.
The bound (3.4.8) follows in the same way by using (3.4.2) instead of(3.4.1). □

Now we are able to combine the above results to get the first new result.

Theorem 3.4.2. *[2-D Anderson Strichartz] Let* $r \geqslant 4$, $\sigma \geq 0, \delta > 0$ *s.t.* $\sigma + 1 - \frac{4}{r} + \delta < 1$.*Then we have on a finite time interval* $[0,T]$, $T \leqslant 1$ *the following bound*

$$\|e^{-itH^\sharp}u^\sharp\|_{L^r_{t;[0,T]}W^{\sigma,r}_{\mathbb{T}^2}} \lesssim \|u^\sharp\|_{\mathcal{H}^{\sigma+1-\frac{4}{r}+\delta}_{\mathbb{T}^2}} \tag{3.4.9}$$

and

$$\left\|\int_0^t e^{-i(t-s)H^\sharp} f^\sharp(s)ds\right\|_{L^r_{t;[0,T]}W^{\sigma,r}_{\mathbb{T}^2}} \lesssim \int_0^T \|f^\sharp(s)\|_{\mathcal{H}^{\sigma+1-\frac{4}{r}+\delta}_{\mathbb{T}^2}} ds \tag{3.4.10}$$

Proof. We start by proving (3.4.9) with $\sigma = 0$. By Proposition 3.4.1 and the Strichartz estimates in Theorem 3.2.3 we can write, setting $u^\sharp_N = P_{\leqslant N} u^\sharp$, $I := [t_0, t_1]$ a subinterval of length $\sim \frac{1}{N}$ and $\delta > 0$

$$\begin{aligned} P_{\leqslant N} e^{-itH^\sharp} u^\sharp_N &= P_{\leqslant N} e^{-i(t-t_0)H^\sharp} e^{-it_0 H^\sharp} u^\sharp_N \\ &= e^{-i(t-t_0)\Delta} P_{\leqslant N} e^{-it_0 H^\sharp} u^\sharp_N + P_{\leqslant N}(e^{-i(t-t_0)H^\sharp} - e^{-i(t-t_0)\Delta}) e^{-it_0 H^\sharp} u^\sharp_N \\ &= e^{-i(t-t_0)\Delta} P_{\leqslant N} e^{-it_0 H^\sharp} u^\sharp_N + i \int_{t_0}^t P_{\leqslant N} e^{-i(t-s)\Delta} (H^\sharp - \Delta)(e^{-i(s-t_0)H^\sharp} u^\sharp_N) ds. \end{aligned}$$

First we decompose the time interval into slices $\cup_j I_j = [0,T]$ with $|I_j| \sim \frac{1}{N}$

$$\begin{aligned} \|P_{\leq N} e^{-itH^\sharp} u^\sharp_N\|^r_{L^r_{t;[0,T]} L^r_{\mathbb{T}^2}} &= \sum_{I_j = [t^j_0, t^j_1]} \|P_{\leq N} e^{-itH^\sharp} u^\sharp_N\|^r_{L^r_{t;I_j} L^r_{\mathbb{T}^2}} \\ &\lesssim \sum_{I_j = [t^j_0, t^j_1]} \|e^{-it\Delta} P_{\leq N} e^{-it^j_0 H^\sharp} u^\sharp_N\|^r_{L^r_{t;I_j} L^r_{\mathbb{T}^2}} + \|P_{\leq N}(e^{-i(t-t^j_0)H^\sharp} - e^{-i(t-t^j_0)\Delta}) e^{-it^j_0 H^\sharp} u^\sharp_N\|^r_{L^r_{t;I_j} L^r_{\mathbb{T}^2}} \\ &\lesssim \sum_{I_j = [t^j_0, t^j_1]} N^{-1} \|e^{-it^j_0 H^\sharp} u^\sharp_N\|^r_{\mathcal{H}^{1-\frac{4}{r}+\delta}_{\mathbb{T}^2}} + (\star) \\ &\lesssim \sum_{I_j = [t^j_0, t^j_1]} N^{-1} \|u^\sharp_N\|^r_{\mathcal{H}^{1-\frac{4}{r}+\delta}_{\mathbb{T}^2}} + (\star) \\ &\lesssim \|u^\sharp_N\|^r_{\mathcal{H}^{1-\frac{4}{r}+\delta}_{\mathbb{T}^2}} + \sum_{I_j = [t^j_0, t^j_1]} (\star). \end{aligned}$$

Here we have used (3.4.2) in each subinterval and applied the triangle inequality from the first to the second line. In the next step we have used the short-time bound from Proposition 3.2.4 and lastly Lemma 3.3.4 and the fact that there are $\sim N$ summands allow us to conclude.
Next we treat the perturbative part.

$$\begin{aligned}\|P_{\leq N}(e^{-i(t-t_0)H^\sharp} - e^{-i(t-t_0)\Delta})e^{-it_0H^\sharp}u_N^\sharp\|^r_{L^r_{t;I_j}L^r_{\mathbb{T}^2}} &= \left\|\int_{t_0^j}^t e^{-i(t-s)\Delta}P_{\leq N}(H^\sharp - \Delta)(e^{-i(s-t_0^j)H^\sharp}u_N^\sharp)ds\right\|^r_{L^r_{t;I}L^r_{\mathbb{T}^2}} \\ &\lesssim \left(\int_{I_j} N^{-\frac{1}{r}}N^{1-\frac{4}{r}+\delta}\|P_{\leq N}(H^\sharp - \Delta)(e^{-i(s-t_0^j)H^\sharp}u_N^\sharp)\|_{L^2_{\mathbb{T}^2}}ds\right)^r \\ &\lesssim N^{-1+r-4+r\delta}\left(\int_{I_j}\|e^{-i(s-t_0^j)H^\sharp}u_N^\sharp\|_{\mathcal{H}^{1+\delta}_{\mathbb{T}^2}}\right)^r \\ &\lesssim N^{-1+r-4+r\delta}\left(\int_{I_j}\|u_N^\sharp\|_{\mathcal{H}^{1+\delta}_{\mathbb{T}^2}}ds\right)^r \\ &\lesssim N^{-1}\|u_N^\sharp\|^r_{\mathcal{H}^{1-\frac{4}{r}+2\delta}_{\mathbb{T}^2}},\end{aligned}$$

having used the second bound in 3.2.4 to get to the second line and thereafter Proposition 3.3.2, Lemma 3.3.4 and Bernstein's inequality, Lemma 3.6.10.
Thus we can conclude

$$\|P_{\leq N}e^{-itH^\sharp}u_N^\sharp\|_{L^r_{t;[0,T]}L^r_{\mathbb{T}^2}} \lesssim \|u^\sharp\|^r_{\mathcal{H}^{1-\frac{4}{r}+2\delta}_{\mathbb{T}^2}}$$

for any $\delta > 0$, which directly implies the result. The case $\sigma > 0$ is analogous.
The second Strichartz estimate (3.4.10) follows from the first in the usual way. □

Remark 3.4.3. *Our estimates are as good as those for the Laplacian, up to a loss of δ derivatives.*

3.4.2 The 3d case

It turns out that, due to the most part to the lower regularity of the noise, we lose half a derivative in the Strichartz estimates of the Anderson Hamiltonian compared to those of the Laplacian. This means that we actually–as opposed to the 2d case–need the full power of the improved Strichartz estimates, from Theorem 3.2.3. Recall that in three dimensions the transformation is a bit more complicated, namely

$$u = e^W u^\flat = e^W\Gamma u^\sharp,$$

where $W \in \mathcal{C}^{\frac{1}{2}-\varepsilon}$ is a stochastic term and Γ is analogous to the map from the 2d case. The important thing is that

$$H^\sharp u^\sharp := \Gamma^{-1}e^{-W}He^W\Gamma u^\sharp = \Delta u^\sharp + 2\nabla u^\sharp \circ L\tilde{W} + l.o.t.$$

where $L\tilde{W} \in \mathcal{C}^{-\frac{1}{2}-\varepsilon}$. If we repeat the proof of Theorem 3.4.2 we obtain.

Theorem 3.4.4. *Let $d = 3$ and $p \geqslant \frac{10}{3}$, and $T \leqslant 1$ then for any $\varepsilon > 0, \sigma \in [0, \frac{5}{p} - \varepsilon]$, we have*

$$\|e^{-itH^\sharp}u^\sharp\|_{L^p_{[0,T]}W^{\sigma,p}_{\mathbb{T}^3}} \lesssim \|u^\sharp\|_{\mathcal{H}^{\sigma+2-\frac{5}{p}+\varepsilon}_{\mathbb{T}^3}}$$

Proof. Again we consider $\sigma = 0$ and we consider a frequency localised $u_N^\sharp := P_{\leqslant N} u^\sharp$ and subintervals $I_j = [t_0^j, t_1^j]$ of length $\sim \frac{1}{N}$. We again have the representation for $t \in I_j$

$$P_{\leq N} e^{-itH^\sharp} u_N^\sharp = e^{-i(t-t_0^j)\Delta} P_{\leq N} e^{-it_0 H^\sharp} u_N^\sharp + i \int_{t_0^j}^t e^{-i(t-s)\Delta} P_{\leq N} (H^\sharp - \Delta)(e^{-i(s-t_0^j)H^\sharp} u_N^\sharp) ds$$

which leads us, using Proposition 3.3.9 (instead of Proposition 3.3.2 and Lemma 3.3.4 in the 2d case) and Proposition 3.2.4, to the bound

$$\begin{aligned}
\|P_{\leq N} e^{-itH^\sharp} u_N^\sharp\|^p_{L^p_{t;[0,T]} L^p_{\mathbb{T}^3}} &= \sum_{I_j = [t_0^j, t_1^j]} \|P_{\leq N} e^{-itH^\sharp} u_N^\sharp\|^p_{L^p_{t;I_j} L^p_{\mathbb{T}^3}} \\
&\lesssim \sum_{I_j = [t_0^j, t_1^j]} \|P_{\leq N} e^{-it\Delta} e^{-it_0^j H^\sharp} u_N^\sharp\|^p_{L^p_{t;I_j} L^p_{\mathbb{T}^3}} + \|P_{\leq N} (e^{-i(t-t_0^j)H^\sharp} - e^{-i(t-t_0^j)\Delta}) e^{-it_0^j H^\sharp} u_N^\sharp\|^p_{L^p_{t;I_j} L^p_{\mathbb{T}^3}} \\
&\lesssim \sum_{I_j = [t_0^j, t_1^j]} N^{-1} \|P_{\leq N} e^{-it_0^j H^\sharp} u_N^\sharp\|^p_{\mathcal{H}^{\frac{3}{2} - \frac{5}{p} + \delta}_{\mathbb{T}^3}} + N^{-1} \left(\int_{I_j} \|(H^\sharp - \Delta)(e^{-i(s-t_0^j)H^\sharp} u_N^\sharp)\|_{\mathcal{H}^{\frac{3}{2} - \frac{5}{p} + \delta}_{\mathbb{T}^3}} \right)^p \\
&\lesssim \sum_{I_j = [t_0^j, t_1^j]} N^{-1} \|u_N^\sharp\|^p_{\mathcal{H}^{\frac{3}{2} - \frac{5}{p} + \delta}_{\mathbb{T}^3}} + N^{-1} \|u_N^\sharp\|^p_{\mathcal{H}^{2 - \frac{5}{p} + \delta}_{\mathbb{T}^3}} \\
&\lesssim \|u_N^\sharp\|^p_{\mathcal{H}^{2 - \frac{5}{p} + \delta}_{\mathbb{T}^3}},
\end{aligned}$$

which is completely analogous to the 2 dimensional case; The case $\sigma > 0$ then follows as above by applying Lemma 3.3.4. ☐

3.5 Solving stochastic NLS

We turn our attention to “low-regularity” solutions to the stochastic NLS

$$\begin{aligned} (i\partial_t - H)u &= -u|u|^2 \text{ on } \mathbb{T}^2 \\ u(0) &= u_0, \end{aligned} \tag{3.5.1}$$

which is formally

$$(i\partial_t - \Delta)u = u \cdot \xi + \infty u - u|u|^2.$$

In [49] this PDE was studied in the “high regularity” regime, meaning $u_0 \in \mathcal{D}(H)$ or $\mathcal{D}\left(\sqrt{-H}\right)$. Now we employ the Strichartz estimates to solve it in spaces with less regularity. In particular now we solve it in a space that does *not* depend on the realisation of the noise ξ.

Theorem 3.5.1. *[LWP below energy space] Take $s \in \left(\frac{1}{2}, 1\right)$. Then (3.5.1) is LWP in $\mathcal{H}^s$.*

Proof. The mild formulation is

$$u(t) = e^{-itH} u_0 + i \int_0^t e^{-i(t-\tau)H} u|u|^2(\tau) d\tau,$$

by applying Γ^{-1} to both sides and renaming $\Gamma^{-1}u_{(0)} = u^\sharp_{(0)}$ this becomes

$$u^\sharp(t) = e^{-itH^\sharp}u^\sharp_0 + i\int_0^t e^{-i(t-\tau)H^\sharp}\Gamma^{-1}((\Gamma u^\sharp)|\Gamma u^\sharp|^2)(\tau)d\tau.$$

We want to show that this equation has a solution for a short time by setting up a fixed point argument in the space

$$C_{[0,T]}\mathcal{H}^s \cap L^4_{[0,T]}W^{\sigma,4}$$

where σ is chosen s.t.

$$W^{\sigma,4} \hookrightarrow L^\infty$$

i.e. $\sigma > \frac{1}{2}$, we fix $\sigma = \frac{1}{2} + \delta$ for definiteness. Also we fix $s = \sigma + \delta = \frac{1}{2} + 2\delta$.
Thus we may bound, using our new Strichartz estimates (also the “tame” estimates– see Lemma 3.6.14– which say that $H^s \cap L^\infty$is an algebra),

$$\begin{aligned}
\|u^\sharp\|_{L^4_{[0,T]}W^{\sigma,4}_{\mathbb{T}^2}} &\lesssim \|u^\sharp_0\|_{\mathcal{H}^s_{\mathbb{T}^2}} + \int_0^T \|\Gamma^{-1}(\Gamma u^\sharp|\Gamma u^\sharp|^2)(\tau)\|_{\mathcal{H}^{\sigma+\delta}_{\mathbb{T}^2}}d\tau \\
&\lesssim \|u^\sharp_0\|_{\mathcal{H}^s_{\mathbb{T}^2}} + \int_0^T \|\Gamma u^\sharp|\Gamma u^\sharp|^2(\tau)\|_{\mathcal{H}^s_{\mathbb{T}^2}}d\tau \\
&\lesssim \|u^\sharp_0\|_{\mathcal{H}^s_{\mathbb{T}^2}} + \int_0^T \|\Gamma u^\sharp(\tau)\|^2_{L^\infty_{\mathbb{T}^2}}\|\Gamma u^\sharp(\tau)\|_{\mathcal{H}^s_{\mathbb{T}^2}}d\tau \\
&\lesssim \|u^\sharp_0\|_{\mathcal{H}^s_{\mathbb{T}^2}} + \|u^\sharp\|^2_{L^4_{[0,T]}L^\infty_{\mathbb{T}^2}}\|u^\sharp\|_{L^2_{[0,T]}\mathcal{H}^s_{\mathbb{T}^2}} \\
&\lesssim \|u^\sharp_0\|_{\mathcal{H}^s_{\mathbb{T}^2}} + T^{\frac{1}{2}}\|u^\sharp\|^2_{L^4_{[0,T]}W^{\sigma,4}_{\mathbb{T}^2}}\|u^\sharp\|_{L^\infty_{[0,T]}\mathcal{H}^s_{\mathbb{T}^2}}.
\end{aligned}$$

For the other term we bound

$$\begin{aligned}
\|u^\sharp\|_{L^\infty_{[0,T]}\mathcal{H}^s_{\mathbb{T}^2}} &\lesssim \|u^\sharp_0\|_{\mathcal{H}^s_{\mathbb{T}^2}} + \int_0^T \|\Gamma^{-1}(\Gamma u^\sharp|\Gamma u^\sharp|^2)(\tau)\|_{\mathcal{H}^s_{\mathbb{T}^2}}d\tau \\
&\lesssim \|u^\sharp_0\|_{\mathcal{H}^s_{\mathbb{T}^2}} + T^{\frac{1}{2}}\|u^\sharp\|^2_{L^p_{[0,T]}W^{\sigma,p}_{\mathbb{T}^2}}\|u^\sharp\|_{L^\infty_{[0,T]}\mathcal{H}^s_{\mathbb{T}^2}}.
\end{aligned}$$

From here we can get a contraction for small times in the usual way.

Thus we solve the sharpened equation and by applying Γ we get a solution to the original equation. □

Remark 3.5.2. *This result is analogous to Proposition 3.1 in [17] and we get the same range of regularities despite the presence of the noise. Note however that the classical cubic NLS is well-posed on the torus in two dimensions for any $s > 0$, see [11], but in this case one needs some multilinear bounds and Fourier restriction spaces. It is not clear at this moment if such a result can be proved in the current setting.*

Remark 3.5.3. *The fact that $s < 1$ as opposed to $s \geqslant 1$ makes the bound for the term $\Gamma^{-1}(u|u|^2)$ easier since the paraproducts and other correction terms are actually more regular than u and $u^\sharp$.*

Theorem 3.5.4. *[GWP in the energy space] The PDE*

$$\begin{aligned}(i\partial_t - H)u &= -u|u|^2 \\ u(0) &= u_0 \in \mathcal{D}\left(\sqrt{-H}\right)\end{aligned}$$

is GWP.

Proof. In Section 3.2.2 of [49], global in time existence was proved by approximation. Moreover, we can use the fact that $\mathcal{D}\left(\sqrt{-H}\right) \hookrightarrow \mathcal{H}^{1-\delta}$ and applying the previous result we can conclude (local in time) uniqueness. Lastly, to get the continuous dependence on the inital data, we can interpolate between $\mathcal{H}^{1-\delta}$ and $\mathcal{D}(H) = \Gamma\mathcal{H}^2$. □

We conclude this section by noting that it seems plausible to get a result for the Anderson NLS in the three dimensional case, we have decided to postpone this for future work. The difficulty in that case, as opposed to the two dimensional case, is that–on the one hand– one has a loss of *more* than $\frac{1}{2}$ derivatives in the Strichartz estimate and–on the other hand– as one sees in the Duhamel formula

$$u^\sharp(t) = e^{-itH^\sharp}u_0^\sharp + i\int_0^t e^{-i(t-s)H^\sharp}\Gamma^{-1}(e^{2W}|\Gamma u^\sharp|^2\Gamma u^\sharp)(s)ds, \tag{3.5.2}$$

the nonlinearity contains the term e^{2W}, which has strictly *less* than $\frac{1}{2}$ derivative.

3.6 Results for general rough potentials and the whole space

While we have thus far focussed on the case of white noise potential in 2- and 3-d, our results are applicable to a much larger class of rough potentials. For the sake of completeness we quickly recall the definition of spatial white noise on the torus here.

Definition 3.6.1. *[Spatial white noise on $\mathbb{T}^d$, Definition 2.1 in [49]] The Gaussian white noise ξ is a family of centered Gaussian random variables $\{\xi(\varphi), \varphi \in L^2(\mathbb{T}^d)\}$, whose covariance is given by $\mathbb{E}(\xi(\varphi)\xi(\psi)) = (\varphi,\psi)_{L^2(\mathbb{T}^d)}$.*

More explicitly, set $(\hat{\xi}(k))_{k\in\mathbb{Z}^d}$ to be i.i.d centred complex Gaussian random variables such that $\overline{\hat{\xi}(k)} = \hat{\xi}(-k)$ and with covariance $\mathbb{E}(\hat{\xi}(k)\overline{\hat{\xi}(l)}) = \delta(k-l)$. Then the Gaussian white noise on $\mathbb{T}^d$ is given as the following random series

$$\xi(x) = \sum_{k\in\mathbb{Z}^d}\hat{\xi}(k)e^{2\pi ik\cdot x},$$

of course with the understanding that, despite the notation, ξ can not be evaluated point-wise. Note that in the 3d case we actually remove the zero-mode as it simplifies some computations; so the definition we use in Section 3.3.2 is actually

$$\xi(x) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}}\hat{\xi}(k)e^{2\pi ik\cdot x}.$$

We recall the two dimensional "noise space" with respect to which everything is continuous.

Definition 3.6.2. *[2d noise space] For $\alpha \in \mathbb{R}$,we define the spaces*

$$\mathcal{E}^\alpha := \mathcal{C}^\alpha \times \mathcal{C}^{2\alpha+2} \textit{and}$$

$$\mathcal{G}^\alpha \textit{ as the closure of } \{(\xi, \xi \circ (1-\Delta)^{-1}\xi + c) : \xi \in C^\infty(\mathbb{T}^2),\ c \in \mathbb{R}\} \textit{ w.r.t. the } \mathcal{E}^\alpha \textit{topology,}$$

where $\mathcal{C}^\alpha = B^\alpha_{\infty\infty}$ denotes the usual Besov-Hölder space.

The next result tells us that the 2d spatial white noise does in fact a.s. have a lift in $\mathcal{G}^\alpha$.

Theorem 3.6.3. *[[3], Theorem 5.1] For any $\alpha < -1$, the spatial white noise ξ lies in $\mathcal{C}^\alpha$ and the following convergence holds for the enhanced white noise.*

$$\Xi^\varepsilon := (\xi_\varepsilon, \xi_\varepsilon \circ (1-\Delta)^{-1}\xi_\varepsilon + c_\varepsilon) \to \Xi \in \mathcal{G}^\alpha,$$

where the convergence holds as $\varepsilon \to 0$ in $L^p(\Omega; \mathcal{E}^\alpha)$ for all $p > 1$and almost surely in $\mathcal{E}^\alpha$ and the limit is independent of the mollifier. However, the renormalisation constant, which can be chosen as

$$c_\varepsilon := \sum_{k\in\mathbb{Z}^2} \frac{|\theta(\varepsilon|k|)|^2}{1+|k|^2} \sim \log\left(\frac{1}{\varepsilon}\right),$$

depends on the choice of mollifier. Note that our regularised spatial white noise is given by

$$\xi_\varepsilon(x) = \sum_{k\in\mathbb{Z}^2} \theta(\varepsilon|k|) e^{2\pi i k\cdot x}\hat{\xi}(k)$$

where $\hat{\xi}(k)$are i.i.d. complex Gaussians with $\bar{\hat{\xi}}(k) = \hat{\xi}(-k)$ and with covariance

$$\mathbb{E}[\hat{\xi}(k)\bar{\hat{\xi}}(l)] = \delta(k-l)$$

and θ is a smooth function on $\mathbb{R}\backslash\{0\}$ with compact support such that $\lim_{x\to 0}\theta(x) = 1$.

Next we recall the definition of the "noise space" in three dimensions.

Definition 3.6.4. *[3d noise space] Let $0 < \alpha < \frac{1}{2}$, then we define the space $\mathcal{T}^\alpha$ to be the closure of the set*

$$\left\{\left(\phi, \phi^{\vee}_a, \phi^{\curlyvee}, \phi^{\Ydown}, \phi^{\vee\vee}_b, \nabla\phi \circ \nabla\phi^{\Ydown}\right) : (a,b) \in \mathbb{R}^2, \phi \in C^2(\mathbb{T}^3)\right\}$$

with respect to the $\mathcal{C}^\alpha(\mathbb{T}^3) \times \mathcal{C}^{2\alpha}(\mathbb{T}^3) \times \mathcal{C}^{\alpha+1}(\mathbb{T}^3) \times \mathcal{C}^{\alpha+1}(\mathbb{T}^3) \times \mathcal{C}^{4\alpha}(\mathbb{T}^3) \times \mathcal{C}^{2\alpha-1}(\mathbb{T}^3)$ norm. Here we defined

$$\begin{aligned}
\phi^{\vee}_a &:= (1-\Delta)^{-1}(|\nabla\phi|^2 - a) \\
\phi^{\curlyvee} &:= 2(1-\Delta)^{-1}\left(\nabla\phi \cdot \nabla\phi^{\vee}_a\right) \\
\phi^{\Ydown} &:= (1-\Delta)^{-1}\left(\nabla\phi \cdot \nabla\phi^{\curlyvee}\right) \\
\phi^{\vee\vee}_b &:= (1-\Delta)^{-1}\left(\left|\nabla\phi^{\vee}_a\right|^2 - b\right).
\end{aligned}$$

And also the corresponding result for the lift of the 3d spatial white noise.

Theorem 3.6.5. *[[49], Theorem 2.38] For ξ_ε given by*

$$\xi_\varepsilon(x) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} m(\varepsilon k)\hat{\xi}(k)e^{2\pi i k\cdot x}, \tag{3.6.1}$$

where m is a radial, compactly supported function with $m(0)=1$ and we define

$$\begin{aligned} X_\varepsilon &= (-\Delta)^{-1}\xi_\varepsilon \\ X_\varepsilon^{\vee} &= (1-\Delta)^{-1}(|\nabla X_\varepsilon|^2 - c_\varepsilon^1) \\ X_\varepsilon^{\curlyvee} &= 2(1-\Delta)^{-1}\left(\nabla X_\varepsilon \cdot \nabla X_\varepsilon^{\vee}\right) \\ X_\varepsilon^{\Upsilon} &= (1-\Delta)^{-1}\left(\nabla X_\varepsilon \nabla X_\varepsilon^{\curlyvee}\right) \\ X_\varepsilon^{\vee\vee} &= (1-\Delta)^{-1}\left(\left|\nabla X_\varepsilon^{\vee}\right|^2 - c_\varepsilon^2\right), \end{aligned}$$

where the c_ε are diverging constants which can be chosen as

$$c_\varepsilon^1 = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{|m(\varepsilon k)|^2}{|k|^2} \sim \frac{1}{\varepsilon} \text{ and } c_\varepsilon^2 = \sum_{k_1,k_2\neq 0} |m(\varepsilon k_1)|^2|m(\varepsilon k_2)|^2 \frac{|k_1\cdot k_2|}{|k_1-k_2|^2|k_1|^4|k_2|^2} \sim \left(\log\frac{1}{\varepsilon}\right)^2.$$

Then the sequence $\Xi^\varepsilon \in \mathcal{T}^\alpha$ given by

$$\Xi^\varepsilon := \left(X_\varepsilon, X_\varepsilon^{\vee}, X_\varepsilon^{\curlyvee}, X_\varepsilon^{\Upsilon}, X_\varepsilon^{\vee\vee}, \nabla X_\varepsilon \circ \nabla X_\varepsilon^{\Upsilon}\right)$$

converges a.s. to a unique limit $\Xi \in \mathcal{T}^\alpha$ which is given by

$$\Xi := \left(X, X^{\vee}, X^{\curlyvee}, X^{\Upsilon}, X^{\vee\vee}, \nabla X \circ \nabla X^{\Upsilon}\right),$$

where

$$\begin{aligned} X &= (-\Delta)^{-1}\xi \\ X^{\vee} &= (1-\Delta)^{-1}(:|\nabla X|:^2) \\ X^{\curlyvee} &= 2(1-\Delta)^{-1}\left(\nabla X \cdot \nabla X^{\vee}\right) \\ X^{\Upsilon} &= (1-\Delta)^{-1}\left(\nabla X \cdot \nabla X^{\curlyvee}\right) \\ X^{\vee\vee} &= (1-\Delta)^{-1}\left(:\left|\nabla X^{\vee}\right|:^2\right). \end{aligned}$$

For potentials better than $\mathcal{C}^{-1}$ everything works unconditionally, meaning that such potentials have unique canonical lifts inside our "noise spaces", whereas if one wants rougher potentials it is necessary to ensure the existence of certain nonlinear expressions of them. We state the "general" version of the theorems so as to highlight the fact that we do not use any property of white noise other than that it has a lift in the correct "noise space". We omit the proofs since they can be carried over verbatim. We further remark on the fact that paraproducts and all related operations are equally well defined on the whole space and Definitions 3.6.2 and 3.6.4 clearly still make sense if one replaces $\mathbb{T}^d$ by $\mathbb{R}^d$.

For brevity we write

$$\mathbb{Y}^d \in \{\mathbb{T}^d, \mathbb{R}^d\} \text{ with } d \in \{2,3\},$$

in the remainder of the section.

Theorem 3.6.6. *[Results for general potentials]*

i. For $d = 2$, and $-\frac{4}{3} < \alpha < -1$ the maps

$$\Xi \mapsto (-H_\Xi)^{-1} \text{and } \Xi \mapsto \Gamma_\Xi$$

are locally Lipschitz as maps from $\mathcal{G}^\alpha \to \mathcal{L}(L^2(\mathbb{Y}^2); \mathcal{H}^{\alpha+2-\varepsilon}(\mathbb{Y}^2))$ for any $\varepsilon > 0$, where Γ_Ξ is defined as in (3.3.2) and H_Ξ is defined as in (3.3.3). Moreover, we get the following homogeneous Strichartz estimate for $T \leqslant 1$

$$\|\Gamma_\Xi^{-1} e^{-itH_\Xi} \Gamma_\Xi v\|_{L^4_{t;[0,T]} L^4_{\mathbb{Y}^2}} \lesssim \|v\|_{\mathcal{H}^{-1-\alpha+\delta}_{\mathbb{Y}^2}},$$

for any $\delta > 0$. In addition, this bound is locally Lipschitz in Ξ w.r.t. $\mathcal{G}^\alpha$ and we get LWP of the PDE

$$\begin{aligned} i\partial_t u &= H_\Xi u - u|u|^2 \text{ on } \mathbb{Y}^2 \\ u(0) &= u_0 \end{aligned}$$

on H^s with $s \in \left(-\alpha - \frac{1}{2}, 1\right)$.

ii. For $d = 3$ and $0 < \alpha < \frac{1}{2}$ the maps

$$\Xi \mapsto (-H_\Xi)^{-1} \text{and } \Xi \mapsto \Gamma_\Xi$$

are locally Lipschitz as maps from $\mathcal{T}^\alpha \to \mathcal{L}(L^2(\mathbb{Y}^3); \mathcal{H}^{\alpha-\varepsilon}(\mathbb{Y}^3))$ for any $\varepsilon > 0$,where H_Ξ is defined as in Definition 3.3.5 and Γ_Ξ as in (3.3.9). Moreover, we get the following homogeneous Strichartz estimate for $T \leqslant 1$

$$\|\Gamma_\Xi^{-1} e^{-(\Xi_1+\Xi_2+\Xi_3)} e^{-itH_\Xi} e^{(\Xi_1+\Xi_2+\Xi_3)} \Gamma_\Xi v\|_{L^{\frac{10}{3}}_{t;[0,T]} L^{\frac{10}{3}}_{\mathbb{Y}^3}} \lesssim \|v\|_{\mathcal{H}^{1-\alpha+\varepsilon}_{\mathbb{Y}^3}},$$

for any $\varepsilon > 0$.

Remark 3.6.7. *The result on the whole space does not apply to the white noise potential, since it is only in a weighted Besov space, see [44]. However, one can split the white noise into an irregular but small part and a regular but large part, as was done in [44]*

$$\xi = \xi_< + \xi_\geqslant, \xi_< \in L^\infty_{\langle \cdot \rangle^\sigma}, \xi_\geqslant \in \mathcal{C}^{-\frac{d}{2}-}$$

for some $\sigma > 0$ and then the results will apply to the potential $\xi_\geqslant$, whose lift can be constructed analogously.

Paracontrolled Distributions etc.

We collect some elementary results about paraproducts, see [45], [3], [5] for more details. For the most part we work on the $d-$dimensional torus

$$\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d \text{ for } d = 2, 3.$$

However, to emphasise the fact that most things work equally well on the whole space, see Section 3.6, we write $\mathbb{Y}^d \in \{\mathbb{T}^d, \mathbb{R}^d\}$. The Sobolev space $\mathcal{H}^\alpha(\mathbb{Y}^d)$ with index $\alpha \in \mathbb{R}$ is defined as

$$\mathcal{H}^\alpha(\mathbb{Y}^d) := \left\{u \in \mathcal{S}'(\mathbb{Y}^d) : \left\|(1-\Delta)^{\frac{\alpha}{2}} u\right\|_{L^2} < \infty\right\}.$$

Next, we recall the definition of Littlewood-Paley blocks. We denote by χ and ρ two non-negative smooth and compactly supported radial functions $\mathbb{R}^d \to \mathbb{C}$ such that

i. The support of χ is contained in a ball and the support of ρ is contained in an annulus $\{x \in \mathbb{R}^d : a \leqslant |x| \leqslant b\}$;

ii. For all $\xi \in \mathbb{R}^d$, $\chi(\xi) + \sum_{j\geqslant 0} \rho(2^{-j}\xi) = 1$;

iii. For $j \geqslant 1$, $\chi(\cdot)\rho(2^{-j}\cdot) = 0$ and $\rho(2^{-j}\cdot)\rho(2^{-i}\cdot) = 0$ for $|i-j| > 1$.

The Littlewood-Paley blocks $(\Delta_j)_{j\geqslant -1}$ associated to $f \in \mathcal{S}'(\mathbb{Y}^d)$ are defined by

$$\Delta_{-1} f := \mathcal{F}^{-1}\chi\mathcal{F}f \text{ and } \Delta_j f := \mathcal{F}^{-1}\rho(2^{-j}\cdot)\mathcal{F}f \text{ for } j \geqslant 0.$$

We also set, for $f \in \mathcal{S}'(\mathbb{Y}^d)$ and $j \geqslant 0$

$$S_j f := \sum_{i=-1}^{j-1} \Delta_i f.$$

Then the Besov space with parameters $p, q \in [1, \infty], \alpha \in \mathbb{R}$ can now be defined as

$$B^\alpha_{p,q}(\mathbb{Y}^d) := \{u \in \mathcal{S}'(\mathbb{Y}^d) : \|u\|_{B^\alpha_{p,q}} < \infty\},$$

where the norm is defined as

$$\|u\|_{B^\alpha_{p,q}} := \left(\sum_{k\geqslant -1} (2^{\alpha k}\|\Delta_j u\|_{L^p})^q\right)^{\frac{1}{q}},$$

with the obvious modification for $q = \infty$. We also define the *Besov-Hölder* spaces

$$\mathcal{C}^\alpha := B^\alpha_{\infty\infty}.$$

Using this notation, we can formally decompose the product $f \cdot g$ of two distributions f and g as

$$f \cdot g = f \prec g + f \circ g + f \succ g,$$

where

$$f \prec g := \sum_{j\geqslant -1} S_{j-1} f \Delta_j g \quad \text{and} \quad f \succ g := \sum_{j\geqslant -1} \Delta_j f S_{j-1} g$$

are referred to as the *paraproducts*, whereas

$$f \circ g := \sum_{j \geqslant -1} \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

is called the *resonant product*. An important point is that the paraproduct terms are always well defined whatever the regularity of f and g. The resonant product, on the other hand, is a priori only well defined if the sum of their regularities is positive. We collect some results.

Lemma 3.6.8. *[cf.Theorem 3.17 [66]]Let $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ and $p, p_1, p_2, q \in [1, \infty]$ be such that*

$$\alpha_1 \neq 0 \quad \alpha = (\alpha_1 \wedge 0) + \alpha_2 \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Then we have the bound

$$\|f \prec g\|_{B^{\alpha}_{p,q}} \lesssim \|f\|_{B^{\alpha_1}_{p_1,\infty}} \|g\|_{B^{\alpha_2}_{p_2,q}}$$

and in the case where $\alpha_1 + \alpha_2 > 0$ we have the bound

$$\|f \circ g\|_{B^{\alpha_1+\alpha_2}_{p,q}} \lesssim \|f\|_{B^{\alpha_1}_{p_1,\infty}} \|g\|_{B^{\alpha_2}_{p_2,q}}.$$

Remark 3.6.9. *For the majority of the paper we care only about the case where $p = p_2 = q = 2$ and $p_1 = \infty$.*

We frequently make liberal use of Bernstein's inequality, so for the sake of completeness we state it here.

Lemma 3.6.10. *[Bernstein's inequality] Let $\mathcal{A}$ be an annulus and $\mathcal{B}$ be a ball. For any $k \in \mathbb{N}, \lambda > 0,$and $1 \leqslant p \leqslant q \leqslant \infty$ we have*

1. *if $u \in L^p(\mathbb{R}^d)$is such that $\operatorname{supp}(\mathcal{F}u) \subset \lambda\mathcal{B}$ then*

$$\max_{\mu \in \mathbb{N}^d : |\mu| = k} \|\partial^{\mu} u\|_{L^q} \lesssim_k \lambda^{k + d\left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^p}$$

2. *if $u \in L^p(\mathbb{R}^d)$is such that $\operatorname{supp}(\mathcal{F}u) \subset \lambda\mathcal{A}$ then*

$$\lambda^k \|u\|_{L^p} \lesssim_k \max_{\mu \in \mathbb{N}^d : |\mu| = k} \|\partial^{\mu} u\|_{L^p}.$$

Lemma 3.6.11. *[Besov embedding] Let $\alpha < \beta \in \mathbb{R}$ and $p \geqslant r \in [1, \infty]$ be such that*

$$\beta = \alpha + d\left(\frac{1}{r} - \frac{1}{p}\right),$$

then we have the following bound for $q \in [1, \infty]$

$$\|f\|_{B^{\alpha}_{p,q}(\mathbb{Y}^d)} \lesssim \|f\|_{B^{\beta}_{r,q}(\mathbb{Y}^d)}.$$

Proposition 3.6.12. *[Commutator Lemma, Proposition 4.3 in [3]]*
Given $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}$ such that $\beta + \gamma < 0$ and $\alpha + \beta + \gamma > 0$, the following trilinear operator C defined for any smooth functions f, g, h by

$$C(f,g,h) := (f \prec g) \circ h - f(g \circ h)$$

can be extended continuously to the product space $\mathcal{H}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$. Moreover, we have the following bound

$$||C(f,g,h)||_{\mathcal{H}^{\alpha+\beta+\gamma-\delta}} \lesssim ||f||_{\mathcal{H}^\alpha} ||g||_{\mathcal{C}^\beta} ||h||_{\mathcal{C}^\gamma}$$

for all $f \in \mathcal{H}^\alpha$, $g \in \mathcal{C}^\beta$ and $h \in \mathcal{C}^\gamma$, and every $\delta > 0$.

Lemma 3.6.13. *[Fractional Leibniz, [50]] Let $1 < p < \infty$ and p_1, p_2, p_1', p_2' such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p}.$$

Then for any $s, \alpha \geqslant 0$ there exists a constant s.t.

$$\|\langle\nabla\rangle^s (fg)\|_{L^p} \leqslant C \|\langle\nabla\rangle^{s+\alpha} f\|_{L^{p_2}} \|\nabla^{-\alpha} g\|_{L^{p_1}} + C \|\langle\nabla\rangle^{-\alpha} f\|_{L^{p_2'}} \|\nabla^{s+\alpha} g\|_{L^{p_1'}}.$$

Lemma 3.6.14. *[Tame estimate,Corollary 2.86 in [5]] For any $s > 0$ and $(p,q) \in [1,\infty]^2$, the space $B^s_{p,r} \cap L^\infty$ is an algebra and the bound*

$$\|u \cdot v\|_{B^s_{p,q}} \lesssim \|u\|_{B^s_{p,q}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{B^s_{p,q}}$$

holds.

Chapter 4

Variational approach to stochastic wave equations

4.1 Introduction

In Chapter 2 we have seen how to solve the stochastic PDE

$$\begin{aligned} \partial_t^2 u - Hu &= -u|u|^{p-2} \text{ on } \mathbb{R}_+ \times \mathbb{T}^d \\ (u, \partial_t u)|_{t=0} &= (u_0, u_1), \end{aligned} \tag{4.1.1}$$

in different situations, where $H \approx \Delta + \xi$ is the *Anderson Hamiltonian*, ξ being *spatial white noise*, which was introduced and discussed in detail in Chapter 2, see also [3] where H was first introduced in two dimensions.

In fact, for dimensions $d = 2, 3$, we focussed on proving well-posedness for (4.1.1) on the one hand in the *strong* regime, by which we mean initial data $(u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}\left(\sqrt{-H}\right)$ and solutions $(u, \partial_t u) \in C_t\mathcal{D}(H) \times C_t\mathcal{D}\left(\sqrt{-H}\right)$, and on the other hand in the *energy* regime, by which we mean initial data $(u_0, u_1) \in \mathcal{D}\left(\sqrt{-H}\right) \times L^2$ and solutions $(u, \partial_t u) \in C_t\mathcal{D}\left(\sqrt{-H}\right) \times C_t L^2$. We primarily analysed the cubic case, i.e. $p = 4$, but the same analysis works for any *energy subcritical* power.

The motivation for this terminology comes from the energy which is conserved by (4.1.1) given by

$$E(u) := \frac{1}{2}\int_{\mathbb{T}^d} |\partial_t u|^2 dx - \frac{1}{2}(u, Hu) + \frac{1}{p}\int_{\mathbb{T}^d} |u|^p dx.$$

In Chapter 2.2 we have seen that the same Sobolev embedding is true for $\mathcal{D}\left(\sqrt{-H}\right)$ as for $\mathcal{H}^1$, i.e.

$$\mathcal{D}\left(\sqrt{-H}\right) \hookrightarrow L^q \text{ for } q \in \left[2, \frac{2d}{d-2}\right] \text{ or } q \in [2, \infty) \text{ for } d = 2.$$

Energy supercritical refers to powers s.t. the potential energy, i.e. $\frac{1}{p}\int_{\mathbb{T}^d} |u|^p dx$, is not controlled by the kinetic energy i.e. $\frac{1}{2}\int_{\mathbb{T}^d} |\partial_t u|^2 dx - \frac{1}{2}(u, Hu)$. In other words if $p > \frac{2d}{d-2}$. Accordingly, energy subcritical refers to the case $p < \frac{2d}{d-2}$ and energy critical is the case of equality.

Now we turn our attention to the problem of solving energy supercritical problems in $d = 3$ in the energy regime. In two dimensions all powers are energy subcritical, so we will only work in three dimensions. We refer to [58] for some well- /ill-posedness results concerning energy super-critical wave equations.
We briefly recall the definition of the energy domain of H in three dimensions which is considerably more simple than that of the domain which is rather cumbersome. We are somewhat formal, see Chapter 2.2.2 for the rigorous treatment. We have $\xi \in \mathcal{C}^{-\frac{3}{2}-}$ as well as the following objects:

$$X = (-\Delta)^{-1}\xi \in \mathcal{C}^{\frac{1}{2}-} \qquad X^{\vee} = (1-\Delta)^{-1} : |\nabla X|^2 :\in \mathcal{C}^{1-} \qquad X^{\Psi} = 2(1-\Delta)^{-1}\left(\nabla X \cdot \nabla X^{\vee}\right) \in \mathcal{C}^{\frac{3}{2}-}$$

$$\text{and} \quad X^{\Upsilon} = (1-\Delta)^{-1}\left(\nabla X \cdot \nabla X^{\Psi}\right) \in \mathcal{C}^{\frac{3}{2}-} \quad X^{\vee\vee} = (1-\Delta)^{-1} : \left|\nabla X^{\vee}\right|^2 :\in \mathcal{C}^{2-}.$$

Furthermore we set

$$\begin{aligned}
W &:= X + X^{\vee} + X^{\Psi} \in \mathcal{C}^{\frac{1}{2}-} \\
\tilde{W} &:= (1-\Delta)^{-1}\nabla W \in \mathcal{C}^{\frac{3}{2}-} \\
Z &:= \left|\nabla X^{\Psi}\right|^2 + 2\nabla X \cdot \nabla X^{\Upsilon} + 2\nabla X^{\vee} \cdot \nabla X^{\Psi} - X^{\vee} - X^{\Psi} + (1-\Delta)(X^{\vee\vee} + 2X^{\Upsilon}) \in \mathcal{C}^{-\frac{1}{2}-} \\
Z' &:= (1-\Delta)^{-1}Z + K_{\Xi},
\end{aligned}$$

where we have added constant K_{Ξ} which may depend on the norm of the enhanced noise Ξ. This is done as in Chapter 2.2 so as to make the operator $-H$ uniformly positive. We can then define the action of H on a function of the form $e^W v$ with $v \in \mathcal{H}^2$ as

$$H(e^W v) := e^W(\Delta v + 2\nabla W \cdot \nabla v + Z'v). \tag{4.1.2}$$

Moreover, we can write down the associated quadratic form which has a particularly nice form (compared to the description of the domain)

$$-(e^W v, H(e^W v)) = \int_{\mathbb{T}^3} e^{2W}|\nabla v|^2 dx - (v^2, e^{2W}Z'), \tag{4.1.3}$$

which makes sense for $v \in \mathcal{H}^1$, see Proposition 2.2.48.

If we look for solutions to (4.1.1) of the form $u = e^W v$, the new equation for v reads

$$\begin{aligned}
e^W\partial_t^2 v - H(e^W v) &= -e^{(p-1)W}v^{(p-1)} \\
(v, \partial_t v)|_{t=0} &= (e^{-W}u_0, e^{-W}u_1) =: (v_0, v_1).
\end{aligned} \tag{4.1.4}$$

The actual form of the equation we will use henceforth is

$$\begin{aligned}
e^{2W}\partial_t^2 v - e^W H(e^W v) &= -e^{pW}v^{(p-1)} \\
(v, \partial_t v)|_{t=0} &= (v_0, v_1),
\end{aligned} \tag{4.1.5}$$

which is of course simply the previous one multiplied by e^W. This formulation has a few advantages, namely the operator $e^W H e^W$ is self-adjoint, while $He^W$is not, further it has the weak formulation

$$\int_0^\infty (\partial_t e^W v, \partial_t e^W \varphi) + (He^W v, e^W\varphi) - (e^{(p-1)W}v^{(p-1)}, e^W\varphi)dt = 0 \quad \text{for } \varphi \in C_c^\infty(\mathbb{R}_+; \mathbb{T}^3), \tag{4.1.6}$$

which is equivalent to the weak formulation for (4.1.1) if we undo the transformation.

We will now sketch the variational approach to solving nonlinear wave equations due to Serra-Tilli after a conjecture of de Giorgi. To that purpose, we take for simplicity the usual defocussing nonlinear wave equation

$$\partial_t^2 w - \Delta w = - w|w|^{p-2} \text{ on } \mathbb{R}\times\mathbb{R}^d$$
$$(w,\partial_t w)|_{t=0} = (w_0,w_1) \in (\mathcal{H}^1 \cap L^p)^2.$$

Then the observation made by de Giorgi was that, at least formally, the solution w is the limit of the sequence w_ε of minimisers of space-time functionals

$$F_\varepsilon(w) := \int_0^\infty e^{-\frac{t}{\varepsilon}} \int \left(\frac{\varepsilon^2}{2}|\partial_t^2 w|^2 + \frac{1}{2}|\nabla w|^2 + \frac{1}{p}|w|^p\right) dxdt. \tag{4.1.7}$$

In fact, one readily checks that the minimisers w_ε of F_ε (which exist and are unique by the direct method of the Calculus of Variations) solve the Euler-Lagrange equation

$$\varepsilon^2\partial_t^4 w_\varepsilon - 2\varepsilon\partial_t^3 w_\varepsilon + \partial_t^2 w_\varepsilon - \Delta w_\varepsilon + w_\varepsilon^{p-1} = 0. \tag{4.1.8}$$

Moreover, the way to ensure initial data $(w_\varepsilon, \partial_t w_\varepsilon)|_{t=0} = (w_0, w_1)$, is that one puts these as "boundary conditions" on the set of functions with respect to which one is minimising (this condition is closed and convex), see Remark 2.1 in [72], where also the somewhat unnatural assumptions on the initial velocity is discussed.

The approach of Serra-Tilli then allows us to pass to the limit in the PDE (4.1.8) after taking subsequences, using compactness which one gains after obtaining uniform in ε bounds from the minimising functional F_ε. The method relies heavily on deriving a quantity called the *approximate energy,* which is a time-averaged version of the conserved energy of the usual wave equation and is close to it for small ε. This approximate energy turns out to be decreasing in time, which is quite crucial to the argument.
In fact they get a result of the following form.

Theorem 4.1.1. *[Theorem 1.1 from [72]] For $p \geqslant 2$ and $\varepsilon > 0$ let v_ε denote the unique minimiser of the strictly convex functional F_ε defined as in (4.1.7) under the boundary conditions*

$$(w,\partial_t w)|_{t=0} = (w_0,w_1) \in (\mathcal{H}^1 \cap L^p) \times (\mathcal{H}^1 \cap L^p).$$

Then the following statements hold

a) ***Estimates:*** *There exists a constant $C = C(w_0, w_1, p, d)$ s.t. for every $\varepsilon \in (0,1)$*

$$\int_0^T \int (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^p) dxdt \leqslant CT \quad \forall T \geqslant \varepsilon,$$

$$\int |\partial_t v_\varepsilon(t,x)|^2 dx \leqslant C, \quad \int |v_\varepsilon(t,x)|^2 dx \leqslant C(1+t^2) \quad \forall t \geqslant 0$$

and for every function $h \in \mathcal{H}^1 \cap L^p$

$$\left|\int \partial_t^2 v_\varepsilon(t,x)h(x)dx\right| \leqslant C(\|h\|_{L^p} + \|\nabla h\|_{L^2}) \quad \textit{for a.e. } t>0.$$

b) ***Convergence:*** *Every sequence v_{ε_i} (with $\varepsilon_i \downarrow 0$) admits a subsequence that converges strongly in $L^q((0,T)\times A)$ for any $T>0$ any bounded open set $A\subset\mathbb{R}^d$ (with $q\in[2,p)$ if $p>2$ and $q=2$ if $p=2$) almost everywhere on $\mathbb{R}_+\times\mathbb{R}^d$ and weakly in $\mathcal{H}^1((0,T)\times\mathbb{R}^d)$ for any $T>0$ to a function w such that*

$$\begin{aligned} w &\in L^\infty(\mathbb{R}_+;L^p), \qquad &\nabla w \in L^\infty(\mathbb{R}_+;L^2) \\ \partial_t w &\in L^\infty(\mathbb{R}_+;L^2), \qquad &w\in L^\infty((0,T);\mathcal{H}^1) \quad \forall T>0, \end{aligned}$$

which solves the PDE

$$\begin{aligned} \partial_t^2 w - \Delta w &= -|w|^{p-2}w \\ (w,\partial_t w)|_{t=0} &= (w_0,w_1) \end{aligned}$$

weakly.

c) ***Energy inequality:*** *Letting*

$$\mathcal{E}(t) := \int \frac{1}{2}|w'(t,x)|^2 + \frac{1}{2}|\nabla w(t,x)|^2 + \frac{1}{p}|w(t,x)|^p dx$$

we get that the solution w satisfies the energy inequality

$$\mathcal{E}(t) \leqslant \mathcal{E}(0) = \int \frac{1}{2}|w_1(x)|^2 + \frac{1}{2}|\nabla w_0(x)|^2 + \frac{1}{p}|w_0(x)|^p dx \text{ for a.e.} t>0.$$

Remark 4.1.2. *We make a few comments about this result and its scope. The first thing to note is that one only gets existence of global energy solutions; One has to assume more regularity for the initial velocity (i.e. $\mathcal{H}^1$) than one usually would; Although stated on the whole Euclidean space, the result can be extended to the periodic setting or the setting of a bounded set with boundary conditions.*
Their method works as well for more general hyperbolic PDEs like

$$\partial_t^2 w = -\nabla\mathcal{W}(w) \tag{4.1.9}$$

for some (fairly general) functional $\mathcal{W}$ either on the whole space or the torus, moreover one may add dissipative terms as well, see [73]. Note that this vast generalisation is possible due to the fact that the entire analysis happens in the time-direction, while the space-direction is never touched.

In Section 4.2 we recall their approach since our PDE does not fall into their framework, but we need to modify it slightly. See also [77] for an extension of the theory to include forcing terms in the equation which are (locally) in $L_t^2 L_x^2$.

Remark 4.1.3. *As was remarked on in [72], one can obtain the existence of solutions to wave equations with general power nonlinearities by other means, see e.g. [75], [63]. More precisely, Theorem 4.1.1 can be alternatively proved using finite dimensional approximation, apriori estimates and compactness. Similarly one could obtain a result like Theorem 4.2.1 by first proving global existence for the equation with regularised noise, i.e. by replacing the Anderson Hamiltonian by a regularised version of the form (see Chapter 2)*

$$H_\varepsilon = \Delta + \xi_\varepsilon - c_\varepsilon$$

and regularising the initial data similarly to Theorem 2.3.20, i.e. solving the (classical) PDE

$$\partial_t^2 u_\varepsilon - \Delta u_\varepsilon - \xi_\varepsilon u_\varepsilon + c_\varepsilon u_\varepsilon = -u_\varepsilon |u_\varepsilon|^{p-2}$$
$$(u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (u_0^\varepsilon, u_1^\varepsilon)$$

by compactness methods. Subsequently one can take a suitable sequence of regular initial data such that the energies converge and extract converging subsequences s.t.

$$u_\varepsilon(t) \to u(t) \text{ in } L^2$$
$$\sqrt{-H_\varepsilon} u_\varepsilon \hookrightarrow \sqrt{-H} u$$
$$u_\varepsilon(t) \hookrightarrow u(t) \text{ in } L^p$$

for some limit function u. We do not spell out all the details here, but simply mention that the approach we present here is not the only or, for that matter, shortest way, but rather a "proof of concept"of applying the variational method of Serra and Tilli to the singular stochastic setting. In future works we plan to pursue this avenue of research further and hopefully apply the method to more complicated equations.

4.2 The variational approach to Anderson wave equations

In this section we adopt the same convention for constants as in Chapter 3, namely that every constant may depend on the enhanced noise Ξ, i.e.

$$\lesssim \Leftrightarrow \lesssim_\Xi$$

The most straightforward approach would be to simply try to apply the general theory from [73] to (4.1.9) with the functional

$$\mathcal{W}(w) := -\frac{1}{2}(w, Hw) + \frac{1}{p}\int_{\mathbb{T}^3} |w|^p dx \tag{4.2.1}$$

and accordingly

$$F_\varepsilon(w) := \int_0^\infty e^{-\frac{t}{\varepsilon}} \int_{\mathbb{T}^3} \frac{\varepsilon^2}{2} |\partial_t^2 w|^2 dx dt + \int_0^\infty e^{-\frac{t}{\varepsilon}} \mathcal{W}(w(t)) dt.$$

This almost works, the problem being that the assumption that the domain of $\mathcal{W}$ (which is given by $\mathcal{D}\left(\sqrt{-H}\right)$) contains smooth functions is *not* satisfied. This is in principle not an insurmountable problem, however it does cause some awkwardness when deriving the Euler-Lagrange equations. So, instead of pursuing this path, we will instead perform the "change of variables" $w = e^W v$ in (4.2), where W is defined in (4.1.2). This leads us to a new functional

$$G_\varepsilon(v) := F_\varepsilon(e^W v) := \int_0^\infty e^{-\frac{t}{\varepsilon}} \int_{\mathbb{T}^3} \left(\frac{\varepsilon^2}{2} e^{2W} |\partial_t^2 v|^2 + \frac{1}{2} e^{2W} |\nabla v|^2 + \frac{1}{p} e^{pW} |v|^p \right) - \frac{1}{2}(v^2, e^{2W} Z'). \tag{4.2.2}$$

Now we have a functional (namely $\mathcal{W}(e^W v)$) whose domain is $\mathcal{H}^1$ and which is coercive(even strictly convex if we recall the computation from Chapter 2.2.2 and the fact that we have shifted

Z' by a constant depending on the noise) but the leading term, i.e. the one with $|\partial_t^2 v|^2$ contains the function e^{2W} so this is also not a direct corollary of the Serra-Tilli result from [73]. We shall see that since e^{2W} is an L^∞ function which is time-independent, uniformly positive and bounded trivially via

$$0 < e^{-2\|W\|_\infty} \leqslant e^{2W} \leqslant e^{2\|W\|_\infty} < \infty,$$

this should not cause major problems. We will recall the method of Serra and Tilli and make the appropriate modifications needed wherever necessary.

The theorem we wish to prove in the end is as follows.

Theorem 4.2.1. *For $p \geqslant 2$ and $\varepsilon > 0$ let v_ε denote the unique minimiser of the strictly convex functional G_ε defined in (4.2.2) under the boundary conditions*

$$(v, \partial_t v)|_{t=0} = (e^{-W} u_0, e^{-W} u_1) =: (v_0, v_1) \in (\mathcal{H}^1 \cap L^p(\mathbb{T}^3))^2.$$

Then the following statements hold

a) ***Estimates:*** *There exists a constant $C = C(v_0, v_1, p, \Xi)$ s.t. for every $\varepsilon \in (0,1)$*

$$\int_0^T \int_{\mathbb{T}^3} (|\nabla v_\varepsilon|^2 + |v_\varepsilon|^p) dx dt \leqslant CT \quad \forall T \geqslant \varepsilon, \tag{4.2.3}$$

$$\int_{\mathbb{T}^3} |\partial_t v_\varepsilon(t,x)|^2 dx \leqslant C, \quad \int_{\mathbb{T}^3} |v_\varepsilon(t,x)|^2 dx \leqslant C(1+t^2) \quad \forall t \geqslant 0 \tag{4.2.4}$$

and for every function $h \in \mathcal{H}^1 \cap L^p(\mathbb{T}^3)$

$$\left| \int_{\mathbb{T}^3} e^{2W(x)} \partial_t^2 v_\varepsilon(t,x) h(x) dx \right| \leqslant C(\|h\|_{L^p} + \|\nabla h\|_{L^2}) \quad \text{for a.e. } t > 0. \tag{4.2.5}$$

b) ***Convergence:*** *Every sequence v_{ε_i} (with $\varepsilon_i \downarrow 0$) admits a subsequence that converges strongly in $L^q((0,T) \times \mathbb{T}^3)$ for any $T > 0$ (with $q \in [2,p)$ if $p > 2$ and $q = 2$ if $p = 2$) almost everywhere on $\mathbb{R}_+ \times \mathbb{T}^3$ and weakly in $\mathcal{H}^1((0,T) \times \mathbb{T}^3)$ for any $T > 0$ to a function v such that*

$$v \in L^\infty(\mathbb{R}_+; L^p), \qquad \nabla v \in L^\infty(\mathbb{R}_+; L^2) \tag{4.2.6}$$

$$\partial_t v \in L^\infty(\mathbb{R}_+; L^2), \qquad v \in L^\infty((0,T); \mathcal{H}^1) \quad \forall T > 0, \tag{4.2.7}$$

which solves the PDE

$$\begin{aligned} e^{2W} \partial_t^2 v - e^W H(e^W v) &= -e^{pW} v|v|^{(p-2)} \\ (v, \partial_t v)|_{t=0} &= (e^{-W} u_0, e^{-W} u_1) =: (v_0, v_1). \end{aligned} \tag{4.2.8}$$

c) ***Energy inequality:*** *Letting*

$$\mathcal{E}(t) := \int_{\mathbb{T}^3} \frac{1}{2} e^{2W(x)} |\partial_t v(t,x)|^2 + \frac{1}{p} e^{pW(x)} |v(t,x)|^p dx - \frac{1}{2}(e^W v, H(e^W v)) \tag{4.2.9}$$

we get that the solution w satisfies the energy inequality which holds for a.e. $t > 0$

$$\mathcal{E}(t) \leqslant \mathcal{E}(0) = \int_{\mathbb{T}^3} \frac{1}{2} e^{2W(x)} |v_1(t,x)|^2 + \frac{1}{p} e^{pW(x)} |v_0(x)|^p dx - \frac{1}{2}(e^W v_0, H(e^W v_0)). \tag{4.2.10}$$

Following the general approach of [73], we write (by a slight abuse of notation)

$$\mathcal{W}(v) := -\frac{1}{2}(e^W v, He^W v) + \frac{1}{p}\int_{\mathbb{T}^3} e^{pW}|v|^p dx \tag{4.2.11}$$

$$= \int_{\mathbb{T}^3} \frac{1}{2}e^{2W}|\nabla v|^2 + \frac{1}{p}e^{pW}|v|^p - \frac{1}{2}(v^2, e^{2W}Z'). \tag{4.2.12}$$

We readily check that we get the following kind of bound

$$\|\nabla\mathcal{W}(v)\|_{(\mathcal{H}^1\cap L^p)^*} \lesssim 1 + (\mathcal{W}(v))^{\frac{1}{2}} \text{ for any } v \in \mathcal{H}^1 \cap L^p \tag{4.2.13}$$

and consequently also a linear bound

$$\|\nabla\mathcal{W}(v)\|_{(\mathcal{H}^1\cap L^p)^*} \lesssim 1 + (\mathcal{W}(v)) \tag{4.2.14}$$

and “Lipschitz continuity along rays” in the sense that one has

$$\sup_{[a,a+b]} \mathcal{W} \lesssim 1 + \mathcal{W}(a) + \|b\|^2_{\mathcal{H}^1\cap L^p} \text{ for any } a, b \in \mathcal{H}^1 \cap L^p$$

and

$$\sup_{[a,a+b]} ||\nabla\mathcal{W}||_{(\mathcal{H}^1\cap L^p)^*} \lesssim 1 + \mathcal{W}(a) + \|b\|^2_{\mathcal{H}^1\cap L^p} \text{for any } a, b \in \mathcal{H}^1 \cap L^p$$

where $[a, a+b]$ denotes all functions f s.t. there exists $\lambda \in [0,1]$ with $f = a + \lambda b$. This then gives us by the mean value theorem for $\delta > 0$

$$\begin{aligned} \left|\frac{\mathcal{W}(a+\delta b) - \mathcal{W}(a)}{\delta}\right| &\lesssim \|b\|_{\mathcal{H}^1\cap L^p} \sup_{[a,a+\delta b]} ||\nabla\mathcal{W}||_{(\mathcal{H}^1\cap L^p)^*} \\ &\lesssim \|b\|_{\mathcal{H}^1\cap L^p}(1 + \mathcal{W}(a) + \delta^2\|b\|^2_{\mathcal{H}^1\cap L^p}) \end{aligned}$$

in particular this implies

$$\mathcal{W}(a+\delta b) \leqslant \mathcal{W}(a) + C_\Xi \delta\|b\|_{\mathcal{H}^1\cap L^p}(1 + \mathcal{W}(a) + \delta^2\|b\|^2_{\mathcal{H}^1\cap L^p}). \tag{4.2.15}$$

Next, we introduce the simpler time-rescaled functional J_ε given by

$$J_\varepsilon(u) := \int_0^\infty e^{-t}\int_{\mathbb{T}^3} \frac{1}{2\varepsilon^2}e^{2W(x)}|\partial_t^2 u(t,x)|^2 dx + \mathcal{W}(u(t))dt, \tag{4.2.16}$$

which is equivalent to G_ε in the sense that

$$G_\varepsilon(v) = \varepsilon J_\varepsilon(u) \text{ for } u(t,x) = v(\varepsilon t, x). \tag{4.2.17}$$

The boundary conditions are scaled in the following way

$$u(0) = v_0 \text{ and } \partial_t u(0) = \varepsilon v_1 \tag{4.2.18}$$

and the existence of minimisers to J_ε is clearly equivalent to the existence of minimisers to G_ε. We have the following first bound.

Lemma 4.2.2 (cf Lemma 3.1 [73])**.** *For $\varepsilon \in (0,1)$ and $v_0, v_1 \in \mathcal{H}^1 \cap L^p$ the functional J_ε defined in (4.2.16) has a unique minimiser $u_\varepsilon \in \mathcal{H}^2_{\mathrm{loc}}(\mathbb{R}_+; L^2)$ under the boundary conditions (4.2.18) which satisfies the bound*

$$J_\varepsilon(u_\varepsilon) \leqslant \mathcal{W}(v_0) + C\varepsilon \lesssim 1 \tag{4.2.19}$$

Proof. This is identical to the proof of Lemma 3.1 in [73]. One observes that the function $\psi(t,x) := v_0(x) + \varepsilon t v_1(x)$ is an admissable competitor in the minimisation class of J_ε for which $\partial_t^2 \psi \equiv 0$ and in order to bound $\mathcal{W}(\psi)$ one uses (4.2.15). We omit the details. □

Next we introduce some further notation

$$\begin{aligned}
\mathcal{W}_\varepsilon(t) &:= \mathcal{W}_\varepsilon(u_\varepsilon(t,\cdot)) \\
D_\varepsilon(t) &:= \frac{1}{2\varepsilon^2}\int_{\mathbb{T}^3} e^{2W}|\partial_t^2 u_\varepsilon|^2 dx \\
L_\varepsilon(t) &:= D_\varepsilon(t) + \mathcal{W}_\varepsilon(t) \\
K_\varepsilon(t) &:= \frac{1}{2\varepsilon^2}\int_{\mathbb{T}^3} e^{2W}|\partial_t u_\varepsilon|^2 dx,
\end{aligned}$$

where one may think of L_ε as being the *Lagrangian* and K_ε the *kinetic energy.*

One gets $K_\varepsilon \in W^{1,1}(0,T)$ with

$$K'_\varepsilon(t) = \frac{1}{\varepsilon^2}\int_{\mathbb{T}^3} e^{2W}\partial_t u_\varepsilon \partial_t^2 u_\varepsilon$$

from Lemma 4.2.4 below which is analogous to Lemma 3.4 in [73]. The proof relies on a simple inequality which we cite here for the sake of completeness.

Lemma 4.2.3. *[[72] Lemma 2.3] Let $v \in \mathcal{H}^1_{\mathrm{loc}}(\mathbb{R}_+; L^2)$, then*

$$\int_0^\infty \int e^{-t}|v(t,x)|^2 dx dt \leqslant 2\int |v(0,x)|^2 dx + 4\int_0^\infty \int e^{-t}|\partial_t v(t,x)|^2 dx.$$

Lemma 4.2.4. *The minimisers u_ε satisfy*

$$\int_0^\infty e^{-t} D_\varepsilon(t) dt = \int_0^\infty e^{-t}\frac{1}{2\varepsilon^2}\int_{\mathbb{T}^3} e^{2W}|\partial_t^2 u_\varepsilon|^2 dx \lesssim 1 \tag{4.2.20}$$

$$\int_0^\infty e^{-t} K_\varepsilon(t) dt = \int_0^\infty e^{-t}\frac{1}{2\varepsilon^2}\int_{\mathbb{T}^3} e^{2W}|\partial_t u_\varepsilon|^2 dx \lesssim 1 \tag{4.2.21}$$

Proof. This is again completely analogous to Lemma 3.4 in [73]; The first bound follows from the bound on J_ε, while the second follows from the first together with Lemma 4.2.3. □

4.2.1 The approximate energy

We introduce the following *time averaging operator* for positive measurable functions $f : \mathbb{R}_+ \to [0,\infty]$

$$\mathcal{A}f(s) := \int_s^\infty e^{-(t-s)} f(t) dt \quad s \geqslant 0 \tag{4.2.22}$$

and note that $\mathcal{A}f$ is always defined–albeit possibly infinite– and in the case

$$\mathcal{A}f(0) = \int_0^\infty e^{-t} f(t) dt < \infty$$

it is absolutely continuous with derivative given by

$$(\mathcal{A}f)' = \mathcal{A}f - f. \tag{4.2.23}$$

It turns out to be very convenient to also consider the twice iterated time average operator, i.e. $\mathcal{A}^2$, which is explicitly given by

$$\mathcal{A}^2 f(s) = \int_s^\infty e^{-(t-s)}(t-s) f(t) dt.$$

With this notation in hand we will now introduce a fundamental quantity, namely the *approximate energy* which contains a time-average of $\mathcal{W}$, which can be seen as the "potential energy".

Definition 4.2.5. *Let u_ε be the minimiser of J_ε. Then the approximate energy is the function*

$$E_\varepsilon := K_\varepsilon + \mathcal{A}^2 \mathcal{W}_\varepsilon$$

or more concretely

$$E_\varepsilon(s) = K_\varepsilon(s) + \int_s^\infty e^{-(t-s)}(t-s)\mathcal{W}(u_\varepsilon(t)) dt.$$

Remark 4.2.6. *The name approximate energy is justified by recalling that if one undoes the time rescaling, one gets*

$$E_\varepsilon\left(\frac{s}{\varepsilon}\right) = \int e^{2W} |\partial_t v_\varepsilon(s)|^2 + \int_s^\infty e^{-\varepsilon(t-s)} \varepsilon^2 (t-s) \mathcal{W}(v_\varepsilon(t)) dt,$$

where $v_\varepsilon(s) = u_\varepsilon\left(\frac{s}{\varepsilon}\right)$ is the minimiser of G_ε. Now $e^{-\varepsilon(t-s)}\varepsilon^2(t-s)$ is a probability kernel on the set $t \geqslant s$ concentrating around s for small ε. Thus one can see that $E_\varepsilon\left(\frac{s}{\varepsilon}\right)$ reasonably approximates the "real" energy $\mathcal{E}(s)$. This approximation means that we get around ever having to directly take a time derivative of $\mathcal{W}_\varepsilon$.

Moreover, we observe that we have have

$$\mathcal{A}\mathcal{W}_\varepsilon(0) \leqslant \mathcal{A}L_\varepsilon(0) = J_\varepsilon(u_\varepsilon) \lesssim 1, \tag{4.2.24}$$

so $\mathcal{A}\mathcal{W}_\varepsilon$ is well-defined, however we still have to determine whether $\mathcal{A}^2\mathcal{W}_\varepsilon$ is.

In fact, we will show not only that E_ε is finite but also that it is *decreasing* which is the crucial point that then allows us to conclude. The next result is almost verbatim Proposition 4.4 from [73], however we restate it and its proof nonetheless since this is really the crux of the method.

Proposition 4.2.7. *Let u_ε be a minimiser of J_ε. For every $g \in C^2(\mathbb{R}_+)$ s.t. $g(0) = 0$ and $g(t)$is constant for large t we have*

$$\int_0^\infty e^{-s}(g'(s) - g(s)) L_\varepsilon(s) ds - \int_0^\infty e^{-s}(4D_\varepsilon(s) g'(s) + K_\varepsilon'(s) g''(s)) ds = g'(0) R(u_\varepsilon), \tag{4.2.25}$$

where

$$R(u_\varepsilon) := -\varepsilon \int_0^\infty e^{-s} s \langle \nabla \mathcal{W}(u_\varepsilon(s)), w_1 \rangle ds,$$

which satisfies the bound

$$|R(u_\varepsilon)| \leqslant C\varepsilon. \tag{4.2.26}$$

Proof. We introduce for δ with $|\delta| \ll 1$ the function

$$\varphi(t) := t - \delta g(t)$$

for g as above. This is a C^2 diffeomorphism of $\mathbb{R}_+$, we also denote by ψ its inverse

$$\psi(s) = \varphi^{-1}(s).$$

Consider the test function

$$U(t) := u_\varepsilon(\varphi(t)) + t\delta\varepsilon g'(0) w_1$$

which satisfies the boundary conditions

$$U(0) = w_0 \text{ and } U'(0) = \varepsilon w_1.$$

Moreover, we have

$$\begin{aligned} U'(t) &= u_\varepsilon'(\varphi(t))\varphi'(t) + \delta\varepsilon g'(0) w_1 \\ U''(t) &= u_\varepsilon''(\varphi(t))|\varphi'(t)|^2 + u_\varepsilon'(\varphi(t))\varphi''(t) \end{aligned}$$

and hence

$$J_\varepsilon(U) = \int_0^\infty e^{-t} \left(\frac{1}{2\varepsilon^2} \left\| e^W u_\varepsilon''(\varphi(t))|\varphi'(t)|^2 + e^W u_\varepsilon'\left(\varphi\left(t\right)\right)\varphi''(t) \right\|_{L^2}^2 + \mathcal{W}(u_\varepsilon(\varphi(t)) + t\delta\varepsilon g'(0) w_1) \right) dt.$$

Note that for $\delta = 0$ $J_\varepsilon(U)|_{\delta=0} = J_\varepsilon(u_\varepsilon)$.

We change variables in the integral above via $t = \psi(s)$ i.e. $s = \varphi(t)$ and get

$$\begin{aligned} J_\varepsilon(U) = \int_0^\infty \psi'(s) e^{-\psi(s)} \Big(& \frac{1}{2\varepsilon^2} \left\| e^{2W} u_\varepsilon''(s)|\varphi'(\psi(s))|^2 + e^{2W} u_\varepsilon'(s)\varphi''(\psi(s)) \right\|_{L^2}^2 \\ & + \mathcal{W}(u_\varepsilon(s) + \psi(s)\delta\varepsilon g'(0) w_1) \Big) ds. \end{aligned} \tag{4.2.27}$$

As in the paper we note $e^{-\psi(s)} \leqslant e^{\delta\|g\|_\infty} e^{-s}$ to argue the finiteness of the expression.

We use the Lipschitz bound for $\mathcal{W}$ in (4.2.15) to get

$$\mathcal{W}(u_\varepsilon(s) + \psi(s)\delta\varepsilon g'(0) w_1) \leqslant C(1 + \mathcal{W}(u_\varepsilon(s)) + \psi(s)^2)$$

This implies the finiteness of the above integral expression and thus it is an admissable "competitor" for the minimisation of J_ε. The minimality of u_ε thus implies

$$\frac{d}{d\delta} J_\varepsilon(U)|_{\delta=0} = 0.$$

In order to compute this, we first note

$$\frac{d}{d\delta}(\psi'(s)e^{-\psi(s)})|_{\delta=0} = g(s)e^{-s} - g'(s)e^{-s}$$

and

$$\frac{d}{d\delta}|\varphi'(\psi(s))|^2|_{\delta=0} = -2g'(s), \quad \frac{d}{d\delta}\varphi''(\psi(s))|_{\delta=0} = -g''(s).$$

We denote by Θ the function inside the large bracket in (4.2.27) and note

$$\Theta(s)|_{\delta=0} = \frac{1}{2\varepsilon^2}\|e^W u_\varepsilon''(s)\|_{L^2}^2 + \mathcal{W}_\varepsilon(s) = L_\varepsilon(s)$$

and moreover

$$\begin{aligned}
\frac{d}{d\delta}\Theta(s)|_{\delta=0} &= -\frac{1}{\varepsilon^2}\langle u_\varepsilon''(s)e^W, 2e^W u_\varepsilon''(s)g'(s) + e^W u_\varepsilon'(s)g''(s)\rangle + \varepsilon g'(0)s\langle\nabla\mathcal{W}(u_\varepsilon(s)), w_1\rangle \\
&= -4D_\varepsilon(s)g'(s) - K_\varepsilon'(s)g''(s) + \varepsilon g'(0)s\langle\nabla\mathcal{W}(u_\varepsilon(s)), w_1\rangle
\end{aligned}$$

combining these two identities we obtain

$$\begin{aligned}
0 &= \frac{d}{d\delta}(\psi'(s)e^{-\psi(s)}\Theta(s))|_{\delta=0} \\
&= e^{-s}(g'(s) - g(s))L_\varepsilon(s) - e^{-s}(4D_\varepsilon(s)g'(s) + K_\varepsilon'(s)g''(s)) + e^{-s}(\varepsilon g'(0)s\langle\nabla\mathcal{W}(u_\varepsilon(s)), w_1\rangle)
\end{aligned}$$

which yields (4.2.25) after integrating in s. Lastly we need to prove the bounds for the remainder. It remains to show the bound (4.2.26); we compute

$$\begin{aligned}
\left|\int_0^\infty e^{-s}s\langle\nabla\mathcal{W}(u_\varepsilon(s)), w_1\rangle ds\right| &\lesssim \|w_1\|_{\mathcal{H}^1\cap L^p}\int_0^\infty e^{-s}s\|\nabla\mathcal{W}(u_\varepsilon(s))\|_{(\mathcal{H}^1\cap L^p)^*} \\
&\lesssim \int_0^\infty e^{-s}s\left(1 + (\mathcal{W}(u_\varepsilon(s)))^{\frac{1}{2}}\right) \\
&\lesssim 1 + \int_0^\infty e^{-s}s^2 ds + \int_0^\infty e^{-s}\mathcal{W}(u_\varepsilon(s))ds \\
&\lesssim 1 + J_\varepsilon(u_\varepsilon) \\
&\lesssim 1
\end{aligned}$$

where we have used (4.2.14) and (4.2.19). We can thus conclude $|R(u_\varepsilon)| \lesssim \varepsilon$.

This finishes the proof. □

By monotone approximation the same result is also true for $g \in C^{1,1}$(not necessarily bounded). This is proved in Corollary 4.5 in [73] whose proof we omit. By inserting the special case $g(t) = t$ this then yields the identity

$$\mathcal{A}^2 L_\varepsilon(0) + 4\mathcal{A}D_\varepsilon(0) = \mathcal{A}L_\varepsilon(0) - R(u_\varepsilon). \tag{4.2.28}$$

Similarly, for almost every $T > 0$, one obtains(see Corollary 4.7 in [73])

$$\mathcal{A}^2 L_\varepsilon(T) - \mathcal{A}L_\varepsilon(T) + K_\varepsilon'(T) = -4\mathcal{A}D_\varepsilon(T) \tag{4.2.29}$$

whose proof we also omit since it is a relatively straightforward approximation argument.

Finally, one can conclude that the approximate energy E_ε is decreasing in time. Recalling the definition

$$E_\varepsilon := K_\varepsilon + \mathcal{A}^2\mathcal{W}_\varepsilon,$$

we note that, using (4.2.28) and (4.2.24), we get

$$\begin{aligned}
E_\varepsilon(0) &= \frac{1}{2}\int e^{2W}|v_1|^2 + \int_0^\infty e^{-s}s\mathcal{W}(u_\varepsilon(s))ds \\
&\leqslant C\|v_1\|_{L^2} + \mathcal{A}^2 L_\varepsilon(0) \\
&= C\|v_1\|_{L^2} - 4\mathcal{A}D_\varepsilon(0) + \mathcal{A}L_\varepsilon(0) - R(u_\varepsilon) \\
&\lesssim \|v_1\|_{L^2} + \mathcal{A}L_\varepsilon(0) - R(u_\varepsilon) \\
&\lesssim \|v_1\|_{L^2} + \mathcal{A}L_\varepsilon(0) + C\varepsilon \\
&\lesssim \|v_1\|_{L^2} + J_\varepsilon(u_\varepsilon) + C\varepsilon \\
&\lesssim \|v_1\|_{L^2} + \mathcal{W}(v_0) + C\varepsilon.
\end{aligned}$$

This finally justifies why E_ε is well-defined. Furthermore, the next result gives us that it is even decreasing in time.

Theorem 4.2.8. *The following is true for all $T \geqslant 0$*

$$E_\varepsilon'(T) \leqslant 0$$

and further

$$E_\varepsilon(T) \lesssim \|v_1\|_{L^2} + \mathcal{W}(v_0) + \varepsilon. \tag{4.2.30}$$

Proof. The second bound follows from the first together with the preceding computation. To prove the first, we compute

$$E_\varepsilon' = K_\varepsilon' - \mathcal{A}\mathcal{W}_\varepsilon + \mathcal{A}^2\mathcal{W}_\varepsilon$$

using (4.2.23). Next, recalling that $\mathcal{W}_\varepsilon = L_\varepsilon - D_\varepsilon$ and using (4.2.29) we deduce

$$E_\varepsilon'(T) = -3\mathcal{A}D_\varepsilon(T) - \mathcal{A}^2 D_\varepsilon(T) \leqslant 0.$$

□

4.2.2 Proving the apriori estimates

Now we proceed to show the bounds (4.2.3),(4.2.4),(4.2.5) from Theorem 4.2.1.

Firstly we rescale in time, using the relation (4.2.17) between v_ε and u_ε to obtain

$$\frac{1}{2}\int_{\mathbb{T}^3}|\partial_t v_\varepsilon(t,x)|^2dx \lesssim \frac{1}{2}\int_{\mathbb{T}^3} e^{2W(x)}|\partial_t v_\varepsilon(t,x)|^2dx = K_\varepsilon\left(\frac{t}{\varepsilon}\right) \lesssim 1,$$

where the last bound follows from the approximate energy bound from Theorem 4.2.8. This shows the first part of (4.2.4), the second bound in (4.2.4) follows readily from the first using fact that in particular $w_0 \in L^2$.

To get the bound (4.2.3), we first recall that

$$\int_0^T \int |\nabla v_\varepsilon(t,x)|^2 + |v_\varepsilon(t,x)|^p dxdt = \int_0^{\frac{T}{\varepsilon}} \varepsilon \int |\nabla u_\varepsilon(t,x)|^2 + |u_\varepsilon(t,x)|^p dxdt$$
$$\lesssim \int_0^{\frac{T}{\varepsilon}} \varepsilon \mathcal{W}(u_\varepsilon(t))dt \tag{4.2.31}$$

then we distinguish two cases: $\varepsilon \leqslant T \leqslant 2\varepsilon$ and $T > 2\varepsilon$. In the first case we bound

$$e^{-2}\int_0^{\frac{T}{\varepsilon}} \mathcal{W}(u_\varepsilon(t))dt \leqslant \int_0^2 e^{-t}\mathcal{W}(u_\varepsilon(t))dt \leqslant \int_0^2 e^{-t}L_\varepsilon(t)dt \leqslant J_\varepsilon(u_\varepsilon) \lesssim 1.$$

For the second case we split the integral

$$\int_0^{\frac{T}{\varepsilon}} = \int_0^2 + \sum_{i=1}^{\lfloor \frac{T}{\varepsilon}\rfloor - 2} \int_{i+1}^{i+2} + \int_{\lfloor \frac{T}{\varepsilon}\rfloor}^{\frac{T}{\varepsilon}},$$

where we of course bound the first integral as in the first case. For the remaining terms we proceed as follows

$$e^{-2}\int_{i+1}^{i+2} \mathcal{W}_\varepsilon(t)dt \leqslant \int_{i+1}^{i+2} (t-i)e^{-(t-i)}\mathcal{W}_\varepsilon(t)dt \leqslant \mathcal{A}^2\mathcal{W}_\varepsilon(i) \leqslant E_\varepsilon(i) \lesssim 1 \tag{4.2.32}$$

and in the same way for the last summand. Finally we can conclude (4.2.3) from (4.2.31), since there are $\sim \frac{T}{\varepsilon}$ summands.

We furthermore note that (4.2.32) combined with (4.2.13) gives us the following bound

$$\int_t^{t+1} \|\nabla \mathcal{W}(u_\varepsilon(s))\|^2_{(\mathcal{H}^1\cap L^p)^*} ds \lesssim 1 \quad \forall t \geqslant 0, \tag{4.2.33}$$

which we will employ in proving the next lemma, which is essentially Lemma 5.1 in [73].

Lemma 4.2.9 (Euler-Lagrange equations)**.** *Suppose that* $\eta(t,x) = \varphi(t)h(x)$ *with* $\varphi \in C^{1,1}(\mathbb{R}_+)$, $\varphi(0) = \varphi'(0) = 0$ *and* $h \in \mathcal{H}^1 \cap L^p$. *Then*

$$\int_0^\infty e^{-t}\left(\frac{1}{\varepsilon^2}(e^{2W}\partial_t^2 u_\varepsilon(t), \partial_t^2\eta(t)) + (\nabla\mathcal{W}(u_\varepsilon(t)), \eta(t))\right)dt = 0. \tag{4.2.34}$$

The same conclusion holds true for test functions $\eta \in C_c^\infty$.

Proof. The Euler-Lagrange equation (4.2.34) is formally obtained by asking $f'(0) = 0$ for the function $f(\delta) := J_\varepsilon(u_\varepsilon + \delta\eta)$. So one needs to justify pulling the derivative into the integral. In fact, using (4.2.33) one can conclude that

$$\frac{d}{d\delta}\mathcal{W}(u_\varepsilon(t) + \delta\eta(t))|_{\delta=0} = (\nabla\mathcal{W}(u_\varepsilon(t)), \eta(t)) = \varphi(t)(\nabla\mathcal{W}(u_\varepsilon(t)), h)$$

multiplied by e^{-t} is integrable in time. The case of general η follows by density, see [73] for details. □

Now we come to the proof of (4.2.5). In fact this bound comes from the following representation formula for $\partial_t^2 u_\varepsilon$ (actually for $e^{2W}\partial_t^2 u_\varepsilon$), namely

$$\frac{1}{\varepsilon^2}(e^{2W}\partial_t^2 u_\varepsilon(T),h) = -\mathcal{A}^2(\nabla\mathcal{W}(u_\varepsilon(\cdot)),h)(T) \text{ for a.e. } T>0, \forall h\in\mathcal{H}^1\cap L^p. \tag{4.2.35}$$

This can be proved by applying (4.2.34) with $\eta(t,x)=g_\delta(t)h(x)$ with g_δ s.t. $\partial_t^2 g_\delta(t)=\delta^{-1}\chi_{(T,T+\delta)}(t)$ and $g_\delta\to(t-T)^+$. In fact, inserting this into (4.2.34) and multiplying both sides by e^T gives

$$\frac{e^T}{\varepsilon^2\delta}\int_T^{T+\delta} e^{-t}(\partial_t^2 u_\varepsilon(t),h)dt = -\int_T^\infty e^{-(t-T)}g_\delta(t)(\nabla\mathcal{W}(u_\varepsilon(t)),h)dt.$$

We can thus conclude (4.2.35) for a.e. $T>0$ after taking $\delta\downarrow 0$.

Finally, we can use (4.2.35) to prove (4.2.5). Using the bound

$$|(\nabla\mathcal{W}(u_\varepsilon(t)),h)| \leqslant \|\nabla\mathcal{W}(u_\varepsilon(t))\|_{(\mathcal{H}^1\cap L^p)^*}\|h\|_{\mathcal{H}^1\cap L^p} \lesssim \|h\|_{\mathcal{H}^1\cap L^p}(1+\mathcal{W}(u_\varepsilon(t)))$$

as well as recalling $\mathcal{A}^2\mathcal{W}_\varepsilon \leqslant E_\varepsilon \lesssim 1$ we have

$$|\mathcal{A}^2(\nabla\mathcal{W}(u_\varepsilon(\cdot)),h)(T)| \leqslant \mathcal{A}^2|(\nabla\mathcal{W}(u_\varepsilon(\cdot)),h)|(T) \lesssim \|h\|_{\mathcal{H}^1\cap L^p},$$

which allows us to conclude (4.2.5).

4.2.3 Passing to the limit and energy inequality

In this section we conclude the proof of Theorem 4.2.1 by passing to the limit and proving (4.2.6),(4.2.7) and (4.2.8) as well as the energy inequality (4.2.10).

In the following we will extract subsequences via compactness but, as is commonly done, we will not relabel them, in fact we just write w_ε for all the subsequences for the sake of brevity.

Firstly we use (4.2.3) and (4.2.4) to deduce the uniform bounds for any $T\geq 0$

$$\|v_\varepsilon\|^2_{\mathcal{H}^1((0,T);L^2)\cap L^2((0,T);\mathcal{H}^1)} = \int_0^T \|\partial_t v_\varepsilon(t)\|^2_{L^2} + \|\nabla v_\varepsilon(t)\|^2_{L^2} + \|v_\varepsilon(t)\|^2_{L^2}dt \leqslant C(T),$$

and

$$\|v_\varepsilon\|^p_{L^p((0,T);\mathbb{T}^3)} = \int_0^T ||v_\varepsilon(t)||^p_{L^p}dt \leqslant C(T),$$

i.e. equiboundedness in $\mathcal{H}^1_{\text{loc}}(\mathbb{R}_+;L^2(\mathbb{T}^3))\cap L^2_{\text{loc}}(\mathbb{R}_+;\mathcal{H}^1(\mathbb{T}^3))\cap L^p_{\text{loc}}(\mathbb{R}_+;L^p(\mathbb{T}^3))$. This allows us to extract a subsequence and a limit s.t. for any $T>0$

$$v_\varepsilon \rightharpoonup v \quad \text{in} \quad \mathcal{H}^1((0,T);L^2(\mathbb{T}^3))\cap L^2((0,T);\mathcal{H}^1(\mathbb{T}^3)) \tag{4.2.36}$$

$$v_\varepsilon \rightharpoonup v \quad \text{in} \quad L^p((0,T);\mathbb{T}^3) \tag{4.2.37}$$

$$v_\varepsilon \to v \quad \text{in} \quad L^q((0,T);\mathbb{T}^3) \text{ for any } q\in[2,p). \tag{4.2.38}$$

Note that in the last one we have strong convergence since by the compact embedding $\mathcal{H}^1\hookrightarrow L^2$ we get strong convergence in $L^2((0,T);\mathbb{T}^3)$ and interpolating this with the L^p bound, we can deduce strong convergence in $L^q((0,T);\mathbb{T}^3)$.

Now we note that from the uniform bounds in (4.2.4) and (4.2.5) we get for the limit

$$\partial_t v \in L^\infty(\mathbb{R}_+; L^2) \text{ and } e^{2W}\partial_t^2 v \in L^\infty(\mathbb{R}_+; (\mathcal{H}^1 \cap L^p)^*). \tag{4.2.39}$$

In order to confirm that the limit satisfies the correct initial conditions, we firstly note that the condition $v(0) = v_\varepsilon(0) = v_0$ follows from (4.2.36). Moreover, the bounds (4.2.4) and (4.2.5) together with the embedding $L^2 \hookrightarrow (\mathcal{H}^1 \cap L^p)^*$ allow us to get a uniform bound

$$\|e^{2W}\partial_t v_\varepsilon\|_{W^{1,\infty}(\mathbb{R}_+;(\mathcal{H}^1\cap L^p)^*)} \lesssim 1.$$

This allows us to conclude that

$$e^{2W}\partial_t v(0) = e^{2W}\partial_t v_\varepsilon(0) = e^{2W} v_1$$

as elements in $(\mathcal{H}^1 \cap L^p)^*$.

Next, we show that our limit v actually solves the correct equation, i.e. (4.2.8). Note that the v_ε satisfy the Euler-Lagrange equations (4.2.34) which are –slightly rewritten–

$$-\int_0^\infty \int e^{2W}\partial_t v_\varepsilon \partial_t\left(e^{-\frac{t}{\varepsilon}}\partial_t^2\eta\right) dxdt + \int_0^\infty \int e^{-\frac{t}{\varepsilon}} e^{2W}(\nabla v_\varepsilon \cdot \nabla\eta - v_\varepsilon \eta Z' + |v_\varepsilon|^{p-2} v_\varepsilon \eta) = 0.$$

Now we choose $\eta(t,x) = e^{-\frac{t}{\varepsilon}}\varphi(t,x)$ for any $\varphi \in C_c^\infty$ which yields

$$-\int_0^\infty \int e^{2W}\partial_t v_\varepsilon \partial_t(\varepsilon^2\partial_t^2\varphi + \varepsilon\partial_t\varphi + \varphi)dxdt + \int_0^\infty \int e^{2W}(\nabla v_\varepsilon \cdot \nabla\varphi - v_\varepsilon\varphi Z' + |v_\varepsilon|^{p-2}v_\varepsilon\varphi) = 0.$$

Now we are able to pass to the limit in this formulation, using the fact that

$$\begin{aligned} \partial_t v_\varepsilon &\rightharpoonup \partial_t v \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^2) \\ \nabla v_\varepsilon &\rightharpoonup \nabla v \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^2) \\ v_\varepsilon &\to v \quad \text{in} \quad L^2_{\text{loc}}\left(\mathbb{R}_+; \mathcal{H}^{\frac{3}{4}}\right) \\ v_\varepsilon &\to v \quad \text{in} \quad L^{p-1}_{\text{loc}}(\mathbb{R}_+; L^{p-1}). \end{aligned}$$

The limiting equation is thus

$$-\int_0^\infty \int e^{2W}\partial_t v \partial_t\varphi dxdt + \int_0^\infty \int e^{2W}(\nabla v \cdot \nabla\varphi - v\varphi Z') + e^{pW}|v|^{p-2}v\varphi = 0.$$

which is the weak formulation of (4.2.8) i.e. (4.1.6).

Lastly we prove the energy inequality (4.2.10) as in [73]. This follows from the following lemma.

Lemma 4.2.10 (Lemma 6.1 from [73])**.** *Let $l(t), m(t)$ be nonnegative functions in L^1_{loc} s.t.*

$$\mathcal{A}^2 l(t) \leqslant m(t) \quad \textit{for a.e. } t > 0,$$

then for $\delta \in (0,1), a > 0$ we have

$$\left(\int_0^{\delta a} se^{-s}ds\right)\int_{T+\delta a}^{T+a} l(t)dt \leqslant \int_T^{T+a} m(t)dt \quad \forall T \geqslant 0.$$

We want to apply Lemma 4.2.10 with $l(t) = \mathcal{W}_\varepsilon(t)$ and

$$m(t) = -K_\varepsilon(t) + C(\|v_1\|_{L^2} + \mathcal{W}(v_0) + \varepsilon),$$

where C is the constant from (4.2.30). This is allowed in light of (4.2.30) and by recalling the definition

$$E_\varepsilon = K_\varepsilon + \mathcal{A}^2 \mathcal{W}_\varepsilon.$$

Thus we have

$$Y(\delta a)\int_{T+\delta a}^{T+a} \mathcal{W}_\varepsilon(t)dt \leqslant -\int_T^{T+a} K_\varepsilon(t)dt + aC\mathcal{E}(0) + aC\varepsilon,$$

for all $T \geqslant 0, \delta \in (0,1)$ and $a > 0$, where we have defined $Y(s) := \int_0^s te^{-t}dt$. Now we rescale, i.e. make the re-substitution $u(t,x) = v(\varepsilon t, x)$, which leads to

$$Y\left(\frac{\delta a}{\varepsilon}\right)\int_{T+\delta a}^{T+a} \mathcal{W}(v_\varepsilon(t))dt + \int_T^{T+a} \frac{1}{2}\|e^W \partial_t v_\varepsilon\|_{L^2}^2 dt \leqslant aC\mathcal{E}(0) + aC\varepsilon$$

after having also replaced T by $\frac{T}{\varepsilon}$ and a by $\frac{a}{\varepsilon}$. Now we can take $\varepsilon \to 0$ in this bound, using that $Y\left(\frac{\delta a}{\varepsilon}\right) \to 1$ and lower-semicontinuity/Fatou to obtain

$$\int_{T+\delta a}^{T+a} \mathcal{W}(v(t))dt + \int_T^{T+a} \frac{1}{2}\|e^W \partial_t v\|_{L^2}^2 dt \leqslant aC\mathcal{E}(0),$$

next we take $\delta \to 0$, divide both sides by a and take $a \to 0$ to finally obtain

$$\mathcal{W}(v(T)) + \frac{1}{2}\|e^W \partial_t v(T)\|_{L^2}^2 \leqslant \mathcal{E}(0),$$

which is precisely (4.2.10).

This concludes the proof of Theorem 4.2.1.

We have thus constructed solutions to the energy supercritical stochastic wave equation (4.1.1). With the same method we are able to prove that for solutions u_δ to the Wave equation with a smooth approximation to the Anderson Hamiltonian, as in (2.2.35)

$$\begin{aligned} \partial_t^2 u_\delta - (\Delta + \xi_\delta - c_\delta) &= -u_\delta |u_\delta|^p - 2 \text{ on } \mathbb{R}_+ \times \mathbb{T}^3 \\ (u_\delta, \partial_t u_\delta)|_{t=0} &= (u_0^\delta, u_1^\delta) = (e^{W_\delta} v_0, e^{W_\delta} v_1), \end{aligned} \tag{4.2.40}$$

exist globally for $(v_0, v_1) \in (\mathcal{H}^1 \cap L^p)^2$. Moreover, there exists a subsequence which converges to a solution to (4.1.1) with initial data

$$((u_0, u_1) = (e^W v_0, e^W v_1)).$$

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