External Spanier-Whitehead duality for $\mathcal{C}$-spectra and applications to $\mathcal{C}$-homology theories

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Malte Felix Lackmann

aus
Bordesholm

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Summary

This thesis studies $\mathcal{C}$-homology theories, i. e. homology theories for diagram spaces over an index category $\mathcal{C}$, from a topological and from an algebraic point of view.

To begin with, we construct a Spanier-Whitehead type external duality functor relating finite $\mathcal{C}$-spectra to finite $\mathcal{C}^{\mathrm{op}}$-spectra and prove that every $\mathcal{C}$-homology theory can be represented by taking the homotopy groups of a balanced smash product with a fixed $\mathcal{C}^{\mathrm{op}}$-spectrum.

More specifically, our duality comes from an abstract framework of adjoint 1-morphisms in closed bicategories, applied to a certain bicategory $\mathrm{DerMod}_{\mathrm{Sp}^O}$ of spectral categories, derived bimodules and morphisms between these which we construct in a first step, using advanced methods from enriched homotopy theory. We also deal with the question how this closed bicategory depends on the model of spectra used to define it, proving compatibility results for comparison along monoidal Quillen adjunctions.

As an application, we use the homology representation result mentioned above to construct Chern characters for certain rational $\mathcal{C}$-homology theories. This leads to the algebraic question to characterise which category algebras are hereditary, which we achieve under certain mild combinatorial conditions. The algebras which occur here are rings with approximate unit, and we adapt certain methods for unital rings to this setup.

A prominent example to which our methods may be applied is the orbit category of a discrete finite or infinite group G, whose homotopy theory of diagram spaces is equivalent to the homotopy of G-spaces. We characterise very concretely when these have hereditary category algebras and discuss interesting examples where G is the fundamental group of a graph of groups.

Contents

0. Introduction

This thesis deals with the homotopy theory of pointed $\mathcal{C}$-spaces, or pointed diagram spaces over $\mathcal{C}$, which are functors

$$X : \mathcal{C} \longrightarrow \mathrm{Top}_* ,$$

where $\mathcal{C}$ is a small category.

Example 0.1. Let $\mathcal{C}$ consist of two objects c and d, with $\mathcal{C}(c,c) = \{\mathrm{id}_c\}$, $\mathcal{C}(d,d) = \{\mathrm{id}_d\}$ and $\mathcal{C}(c,d)$ consisting of a single morphism, whereas $\mathcal{C}(d,c) = \emptyset$. Then a pointed $\mathcal{C}$-space can equivalently be described as a triple (X, Y, f), where X and Y are pointed spaces and

$$f : X \longrightarrow Y$$

is a pointed continuous map. We see here where the name "diagram space" comes from: A pointed $\mathcal{C}$-space is a diagram of pointed spaces of a given, in this case quite simple, form. A morphism of $\mathcal{C}$-spaces from $f : X \to Y$ to $f' : X' \to Y'$ is given by a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y' \end{array}$$

and composition is given by juxtaposition of squares.

Example 0.2. A prominent class of examples of index categories $\mathcal{C}$, studied for reasons explained below, is given by the so-called *orbit categories*: Let G be a discrete group, then $\mathrm{Or}(G)$ is the category whose objects are all transitive G-sets G/H and whose morphisms are all G-equivariant maps between them. Certain full subcategories of the orbit category, where the subgroup H is restricted to a so-called *family* $\mathcal{F}$, see Definition 5.27, are also studied in the literature and in this thesis.

The famous Elmendorf Theorem [Elm83] identifies the homotopy theory of G-spaces for a discrete group G with the homotopy theory of diagram spaces over the orbit category $\mathrm{Or}(G)$. This may seem surprising at the first glance: we single out a small number of G-spaces, namely only the transitive ones, and the corresponding subcategory already governs the homotopy theory of *all* G-spaces. This theorem is one of the most fundamental reason why the homotopy theory of diagram spaces attracted mathematical attention.

Similarly to classical homotopy theory, major tools to study $\mathcal{C}$-spaces are $\mathcal{C}$-homology theories, which are collections of functors

$$h_n^{\mathcal{C}} : \mathrm{Fun}(\mathcal{C}, \mathrm{Top}_*) \to \mathrm{Ab}$$

satisfying the usual Eilenberg-Steenrod axioms, cf. Section 4.1. A well-known way to construct a $\mathcal{C}$-homology theory is by setting

$$h_n^{\mathcal{C}}(X; E) \coloneqq \pi_n(E \wedge_{\mathcal{C}} X) \tag{0.1}$$

where

$$E\colon \mathcal{C} \longrightarrow \mathrm{Sp}^O$$

is a (cofibrant) orthogonal $\mathcal{C}^{\mathrm{op}}$-spectrum. Here, $\wedge_{\mathcal{C}}$ denotes the balanced smash product over $\mathcal{C}$, a well-known categorical construction which we will recall in Section 1.2. The construction (0.1) can be traced back to the very beginning of the theory of spectra in the case that $\mathcal{C}$ is the trivial category and was first formulated by Davis and Lück [DL98] in this general form. It since has proved useful in many contexts, primarily in work on the Farrell-Jones conjecture [LR05, BLR08, BL12, Weg15, KLR16, Rüp16, Wu16, KUWW18, BB19].
This thesis puts $\mathcal{C}$-homology theories into the spotlight and recognises them as worthy of study in their own right. The first aspect we investigate is the question whether *every* $\mathcal{C}$-homology theory arises in the way described above. This is answered in the positive by our first theorem, the homology representation theorem, proved as Theorem 4.7:

Theorem A . *Suppose that $\mathcal{C}$ is countable. Let $h_*^{\mathcal{C}}$ be any $\mathcal{C}$-homology theory. Then there is a $\mathcal{C}^{\mathrm{op}}$-spectrum E and a natural isomorphism*

$$h_*^{\mathcal{C}}(-) \cong h_*^{\mathcal{C}}(-; E)\,. \tag{0.2}$$

Moreover, every morphism of homology theories

$$h_*^{\mathcal{C}}(-; E) \longrightarrow h_*^{\mathcal{C}}(-; E')$$

is induced by a morphism $E \longrightarrow E'$ in the derived category of $\mathcal{C}^{\mathrm{op}}$-spectra.

In practical situations, for many $\mathcal{C}$-homology theories a concrete representing diagram spectrum can be written down explicitly, or the theory arose from an application of the Davis–Lück construction in the first place, so that the above theorem is not needed. Examples include Borel homology [Lac16, Sec. 3.4], Bredon homology (see Lemma 5.17) and the G-homology theories defined from the K- and L-theory spectra over the orbit category [DL98, Sec. 2]. However, there are exceptions, the most prominent one being equivariant bordism:

Example 0.3. Let G be a countable discrete group. A *G-manifold* is a smooth manifold M, with or without boundary, together with a smooth G-action. Such a G-manifold is called *proper* if the stabilisers of all points are finite, and *cocompact* if $G\backslash M$ is compact. For any pointed G-space X, we define the *n-th G-bordism group*

$$\mathcal{N}_n^G(X) := \{(M, \partial M) \xrightarrow{f} (X, *) \mid M \text{ proper cocompact } G\text{-manifold}\}/\sim\,,$$

where we divide out proper cocompact G-bordisms over $(X, *)$. This defines a homology theory on proper G-spaces, i. e. a $\mathcal{C}$-homology theory for the opposite orbit category $\mathcal{C} = \mathrm{Or}(G, \mathcal{FIN})^{\mathrm{op}}$ of G with respect to the family of finite subgroups [Lac16, Sec. 5]. Since the latter category is countable, our Theorem A yields an $\mathrm{Or}(G, \mathcal{FIN})$-spectrum E such that

$$\mathcal{N}_n^G(X) \cong \pi_n(X \wedge_{\mathrm{Or}(G,\mathcal{FIN})} E) .$$

To the author's knowledge, this is a new result, and no concrete construction of such a diagram spectrum is contained in the literature. Our Theorem A doesn't yield any information on how to construct such a spectrum neither.

Proof strategy of Theorem A

It is a well-known theme in algebraic topology that cohomology is much easier to access for representability arguments than homology. This is technically due to the Yoneda Lemma. In the basic case $\mathcal{C} = *$, the cohomological analogue of Theorem A is the famous Brown representability theorem [Bro62]. Neeman [Nee01] has vastly generalised this argument to a triangulated category setup that is sufficient to treat the case of $\mathcal{C}$-cohomology theories. Specific references for the case of $\mathcal{C}$-spaces are [Bár14, Lac16].

The classical strategy for deducing the homological Theorem A from the cohomological consists of the following two steps:

1. Use Spanier-Whitehead duality to switch between cohomology and homology.
2. Then use Adams' version of Brown's representability theorem to deal with the arising difficulty that the duality functor is only defined on finite spectra.

The latter point can easily be carried out in our setup, since Adams' result was also generalised by Neeman [Nee97] in a form suitable for our applications.

Remark 0.4. Neeman's results, and all similar results contained in the literature, have countability hypotheses on the triangulated category, and it is at this point where the countability hypothesis of Theorem A comes in. We do not know whether Theorem A holds true for uncountable categories. However, most of the categories of practical interest are countable. Indeed, orbit categories of finite groups supply a large class of categories where the result is of interest and the category $\mathcal{C}$ is even finite.

To adapt the first point of the strategy described above, i. e. to find a suitable generalisation of Spanier-Whitehead duality, is more difficult. The main innovation here is that the correct notion of duality is *not* incorporated by a functor

$$D\colon \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)^{\mathrm{op}} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O) ,$$

but by a functor

$$D\colon \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)^{\mathrm{op}} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}^O) .$$

This is the reason why we call it the *external (Spanier-Whitehead) duality functor* in this thesis. Note that in the technical sense, the term "duality" is not justified: It refers to the canonical isomorphism

$$DDX \cong X \tag{0.3}$$

for dualisable X. However, the two D's here are not, as in the classical case, the same functor, but only formally given by the same construction, applied to $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$, respectively.
These two aspects originate from the fact that instead of classical duality theory, which takes place in a monoidal category, the correct framework for us is duality theory in a closed bicategory. This was first developed in [MS06, Ch. 16]. We give a slightly simplified exposition in Chapter 3. With the correct setup at hand, the following statement, which is our Corollary 3.16, may be proved mostly analogously to the classical case:

Theorem B . *Every finite $\mathcal{C}$-CW-spectrum is dualisable.*

More precisely, we apply bicategorical duality theory to a closed bicategory of spectrally enriched categories, derived bimodules and morphisms between these, constructed in Theorem 1.16. Chapter 1 recalls the basic enriched homotopy theory that we need for this setup. Whereas we restricted to discrete categories $\mathcal{C}$ in this introduction, we treat more general topological or even spectral categories $\mathcal{C}$ in the rest of the thesis. Theorem A holds true for all spectral categories satisfying a certain cofibrancy condition (C), and we also explain ways how to deal with categories not satisfying (C).
This is a good place to mention that the literature contains a plethora of well-known models for the stable homotopy category other than orthogonal spectra. Chapter 2 makes the point that we could equally well have taken one of the other models to set up our theory and finally construct homology theories: we list mild conditions that a category of spectra[1] has to satisfy in order that our setup from Chapter 1 can be built on this category as well, and we argue why two monoidally Quillen equivalent models yield the same theory by writing down comparison maps between the corresponding balanced smash products. Some technical problems occur in this program, and we install different solutions on several layers of generality, as explained in the introduction to the chapter.

Relation to genuine G-homotopy theory

Let us lose a few words about the relation between our results and the classical *genuine G-homotopy theory* which is prominent in the literature. This exists for a finite group G and is a very sophisticated and rich theory. However, it relies on the abundance of the orthogonal representation theory of G. In particular, the construction of Spanier-Whitehead duality takes advantage of the fact that every finite G-CW-complex embeds into a representation sphere. This totally breaks down for infinite G, as the following example shows.

Example 0.5. A result of Grothendieck [Gro70, Corollaire 2.1] says that if G is a finitely generated group without proper normal subgroups of finite index, then every finite-dimensional representation of G over any field is trivial. In characteristic 0,

[1] or more generally an arbitrary monoidal model category, which could as well model a different homotopy theory

this is due to Mal'tsev [Mal40, Thm. 7]. An example of such a group, constructed by Higman [Hig51], is given by the following presentation:

$$G = \langle a, b, c, d \mid a^{-1}ba = b^2,\ b^{-1}cb = c^2,\ c^{-1}dc = d^2,\ d^{-1}ad = a^2 \rangle .$$

The overall goal of this thesis can be summarised as remedying this defect: Applying our constructions to $\mathrm{Or}(G)$ yields a suitable duality functor on finite G-spaces, where G is any discrete group, e. g. the Higman group we just mentioned. This route is thus completely independent of the representation theory of G.
Recently, the paper [DHL+19] developed a different approach to *proper* equivariant homotopy theory for infinite discrete groups, generalising the genuine theory. We want to stress that for (finite or infinite) groups, our results are neither generalisations nor special cases of the results for the genuine theory. We refer the reader to Remark 1.5 for a more detailed discussion.

The rational case

After having proved the representation theorem, we shift our focus by investigating *rational* $\mathcal{C}$-homology theories. It is well-known from the classical case $\mathcal{C} = *$ that studying these theories can be translated into a purely algebraic theory, and the same turns out to be the case for general (countable) $\mathcal{C}$.
We will always focus on the question whether a *Chern character* exists for a given rational $\mathcal{C}$-homology theory $h_*^{\mathcal{C}}$. This means that $h_*^{\mathcal{C}}$ can be written as a sum of so-called *Bredon homology theories* which have a simple algebraic construction and can be seen as the counterparts of singular homology in the equivariant setup, see Section 5.4. In the classical case $\mathcal{C} = *$, every rational homology theory has a Chern character, or equivalently, is isomorphic to a direct sum of shifted singular homology theories with $\mathbb{Q}$ coefficients, by a folklore result which is based on the computation of the rational stable homotopy groups of the sphere in Serre's thesis [Ser51].
An application of Theorem 4.7 already brings us halfway to the existence of Chern characters. The idea here is that rational spectra are well-known to be Quillen equivalent to unbounded rational chain complexes by the Dold-Kan correspondence, so that the balanced smash products of spectra we dealt with before translate into balanced tensor products of chain complexes, which in turn are isomorphic to tensor products over the so-called *category algebra* $\mathbb{Q}\mathcal{C}$ of $\mathcal{C}$. We thus get the following result, proved as Corollary 5.7:

Theorem C . *If E is a chain complex of right $\mathbb{Q}\mathcal{C}$-modules, then*

$$h_*^{\mathcal{C}}(X; E) = \mathrm{H}_*(E \otimes_{\mathbb{Q}\mathcal{C}} \overline{X})$$

defines a rational reduced $\mathcal{C}$-homology theory. Here $\overline{X}$ denotes a certain resolution of the normalised singular chain complex of X.
Conversely, if $h_*^{\mathcal{C}}$ is a rational $\mathcal{C}$-homology theory, then there are a chain complex E and a natural isomorphism of homology theories as above. Any morphism of rational homology theories is represented by a morphism in the derived category of chain complexes.

Implementing the proof idea sketched above reveals some technical difficulties: Firstly, one has to realise the Dold-Kan correspondence in a suitably monoidal way so that it is compatible with balanced smash products. This was done in the paper [Shi07], which we review in Section 5.1. For us, the Dold-Kan correspondence is a certain zig-zag of weak monoidal Quillen equivalences, and we can now profit from our comparison results proved in Chapter 2, in particular in Section 2.2.1, to prove that balanced smash products are preserved. One caveat is that $\mathcal{C}$ has to be discrete from this point on.
Secondly, if the category $\mathcal{C}$ has infinitely many objects, then the ring $\mathbb{Q}\mathcal{C}$ fails to have a unit. However, there is a good substitute at hand: It always has a so-called *approximate unit*. This technical replacement for a unit, see Definition 5.4, makes it possible to transfer virtually all results which are well-known for unital rings, though one usually has to take a little more care. We discuss this in Section 5.3.
Having arrived at this point, the existence of a Chern character boils down to a purely algebraic question. In the situation of Theorem C, a Chern character exists for $h_*^{\mathcal{C}} = h_*^{\mathcal{C}}(-;E)$ if and only if E is *decomposable*, i. e. is isomorphic in the derived category to a complex with zero differentials, cf. Lemma 5.21. We provide two approaches to proving decomposability.
The first approach uses a spectral sequence argument to show that if all homology modules $\mathrm{H}_s(E)$ are flat as right $\mathbb{Q}\mathcal{C}$-modules, then E decomposes. This is formulated as Proposition 5.23. An important special case, explained in Section 5.5, is when $\mathrm{H}_s(E)$ has an extension to a Mackey functor, as happens in many examples. This approach follows ideas of [Lüc02], see Remarks 5.24 and 5.35.

Hereditary category algebras

The second approach, discussed in Chapter 6, has no hypothesis on the derived chain complex E, but on the category $\mathcal{C}$: It asks when *all* rational $\mathcal{C}$-homology theories possess a Chern character. This leads to the task of characterising hereditary category algebras. This task, which is interesting in its own right, has been accomplished (for EI categories satisfying a mild combinatorial assumption) in joint work with Liping Li [LL20], based on earlier work of Li in the finite case [Li11]. The following is proved as a corollary of Theorem 6.7:

Theorem D . *Let $\mathcal{C}$ be a countable discrete EI category. Suppose that*

- *$\mathcal{C}$ has the finite factorisation property FFP and the unique factorisation property UFP,*
- *all group algebras $\mathbb{Q}G_c$ are hereditary,*
- *the opposite category $\mathcal{C}^{\mathrm{op}}$ satisfies conditions (A_d) and (B_d) for $k = \mathbb{Q}$.*

Then $\mathbb{Q}\mathcal{C}$ is hereditary and every *rational $\mathcal{C}$-homology theory has a Chern character.*

Remark 0.6. This is actually an 'if and only if' statement (if the FFP is assumed), as proved in [LL20]. We focus on the sufficiency implication in this thesis. Using the 'only if' application, we get the result that there certainly exist rational $\mathcal{C}$-homology theories which *do not* possess a Chern character, for example over the category

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

since this doesn't have the UFP.

Let us comment shortly on the list of hypotheses appearing here. The finite factorisation property FFP is a combinatorial assumption satisfied in most practical examples, cf. Definition 6.4. The unique factorisation property already appears in the finite case [Li11] as the main pattern governing hereditarity. It refers to the factorisability properties of morphisms, and categories with the UFP can be seen as analogues of the unique factorisation domains in commutative algebra. The conditions (A_d) and (B_d) refer to a governing pattern which only appears if $\mathcal{C}$ is infinite, namely the structure of $\mathcal{C}(c, d)$ as a (G_d, G_c)-bimodule for arbitrary objects $c \neq d$. All these conditions are explained in detail in Section 6.1.
To prove Theorem D, we first translate it to a purely algebraic statement about the global dimension of certain tensor algebras, see Theorem 6.26, which we prove by mimicking a strategy from the paper [CQ95].
Applying Theorem D to orbit categories yields the following result, proved as Theorem 6.12:

Theorem E . *Let G be a discrete group and $\mathcal{F}$ a family of finite subgroups. Suppose that*

- *G is either countable locally finite or the fundamental group of a connected graph of finite groups,*
- *all members of $\mathcal{F}$ are cyclic of prime power order, and their Weyl groups are finite (except possibly for the Weyl group of $\{1\}$).*

Then every rational $\mathrm{Or}(G, \mathcal{F})$-homology theory possesses a Chern character.

Relation to the preprints [Lac19] and [LL20]

Many of the results from Part I and Chapter 5 can already be found in my arxiv preprint [Lac19]. There are, however, some technical improvements concerning the setup of the theory and the comparison methods. Referring to the five questions formulated in the introduction of the paper, this thesis succeeds in answering Questions 2 and 5. Chapter 6 is based on the paper [LL20] of the author together with Liping Li. I included mainly those of our results which are relevant for the question of the existence of a Chern character, mentioning the more general results as a side remark from time to time.

Organisation of the thesis

The thesis is divided into two parts: Part I, comprising Chapters 2 through 5, deals with $\mathcal{C}$-homology theories in general and is thus the topological part. In Part II, we treat rational $\mathcal{C}$-homology theories and quickly arrive at a purely algebraic situation. Chapter 2, together with Section 5.1, serve as a bridge between the two parts, translating the topological constructions into algebraic ones.
Here is a more detailed overview:

- Chapter 1 recalls some notions from enriched homotopy theory and constructs the closed bicategory of spectrally enriched categories.
- Chapter 2 discusses what happens if orthogonal spectra are replaced by another model category of spectra as the target category of our diagram spectra.
- Chapter 3.1 develops external duality theory in closed bicategories and applies this to $\mathcal{C}$-spectra, proving Theorem B.
- Chapter 4 proves Theorem A via the route sketched above.
- Chapter 5 introduces Chern characters and proves Theorem C. It also recalls the background on rings with approximate unit which is needed for this and the next chapter.
- Chapter 6 treats hereditarity criteria for category algebras, leading to Theorems D and E.

The thesis has three appendices:

- Appendix A describes an alternative way to develop the setup underlying our theory, based on the paper [Shu06].
- Appendix B proves a finiteness result for the compact-open orbit categories of certain p-adic Lie groups, opening the way to applications of Theorem A.
- Appendix C discusses combinatorial and geometric conditions, in terms of group actions on trees, under which the hypotheses of Theorem E are satisfied.

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Part I.

Topology

1. The closed bicategory $\mathrm{DerMod}_{\mathrm{Sp}^O}$

The correct setup for developping external duality theory, as is done in Chapter 3, is given by the notion of a closed bicategory. This will then be applied to deduce results about $\mathcal{C}$-spectra. In this first chapter, we will introduce the actors, i. e. recall the basics about model structures on the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)$ of $\mathcal{C}$-spectra in Section 1.1, introduce the notion of a closed bicategory in Section 1.2 and show how $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)$ can be endowed with this structure, cf. Proposition 1.12, and then show that we can preserve this structure when passing to the homotopy category in Section 1.3, especially Theorem 1.16. Our closed bicategories will, in the underived, resp. derived case, consist of small spectrally enriched categories (with cofibrant mapping objects), (derived) bimodules over these and morphisms (in the homotopy category) of bimodules.

1.1. Recapitulations about the homotopy category of $\mathcal{C}$-spectra

Let Sp^O denote the category of orthogonal spectra with the stable model structure, as discussed in [MMSS01], and let $\mathcal{C}$ be a small category enriched in Sp^O. Let $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)$ denote the category of enriched functors from $\mathcal{C}$ to Sp^O and enriched natural transformations [Bor94b, Def. 6.2.4]. Prominent objects of this category are the representable functors

$$\underline{c} = \mathcal{C}(c, ?)$$

for $c \in \mathrm{Ob}(\mathcal{C})$, or more generally $X \wedge \underline{c}$ for some spectrum X, where the smash product is meant objectwise.

We want to endow $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)$ with a model structure in which the fibrations and weak equivalences are given by the objectwise fibrations and weak equivalences. This determines the model structure, if it exists, uniquely, justifying that we call it 'the' projective model structure.

For usual Set-enriched categories $\mathcal{C}$, the existence of the projective model structure is folklore since Sp^O is a cofibrantly generated model category [Hir03, Thm. 11.6.1, Prop. 11.6.3]. For spectrally enriched $\mathcal{C}$, the situation is more subtle. The first assertion of the following theorem is due to Shipley and Schwede [SS03a, Thm. 6.1(i)[1]]. The topological case was first written down in [Pia91, Thm. 5.4]. The second assertion

[1] This theorem is stated with the assumption that every spectrum is small with respect to the whole category of spectra, which is not true for orthogonal spectra, but the conclusion of the Theorem still holds, as remarked on p. 330.

then follows formally and can be found in [Shu06, Thm. 24.4] or [GM20, Thm. 4.32]. All these theorems use the fact that Sp^O satisfies the monoid axiom [SS00, Def. 3.3], as is proved in [MMSS01, Thm. 12.1(iii)].

Theorem 1.1 . *(i) The projective model structure exists for any spectral category* $\mathcal{C}$*. A class of generating cofibrations is given by morphisms of the form* $X \wedge \underline{c} \to Y \wedge \underline{c}$*, where* $X \to Y$ *runs through a class of generating cofibrations of* Sp^O *and* c *through the objects of* $\mathcal{C}$*; a class of generating trivial cofibrations is described similarly.*
(ii) A cofibration in the projective model structure is objectwise a cofibration if $\mathcal{C}$ *satisfies the following condition:*

(C) The mapping spectra $\mathcal{C}(c,d)$ *are cofibrant for all* $c, d \in \mathcal{C}$*.*

Because of this theorem, if not explicitly mentioned otherwise,

we assume from now on that our category $\mathcal{C}$ *satisfies (C).*

We denote
$$\mathscr{SHC}_{\mathcal{C}} = \mathrm{Ho}(\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O))$$
and use square brackets to indicate that we are talking about morphisms in the homotopy category:

$$[X,Y]_{\mathcal{C}} := \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Fun}(\mathcal{C},\mathrm{Sp}^O))}(X,Y) = \mathrm{Hom}_{\mathscr{SHC}_{\mathcal{C}}}(X,Y)\,.$$

Remark 1.2. The paper [SS03b] shows that (if spectra are simplicial symmetric spectra) the model categories $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^{\Sigma}_{sSet})$ are exactly the simplicial, cofibrantly generated, proper, stable model categories with a set of compact generators, up to zig-zags of Quillen equivalences.

$\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)$ is a stable model category, so the homotopy category admits a preferred *triangulated structure*, even in the strong sense of [Hov99, Sec. 7]. We refer to the fact that

$$X \xrightarrow{f} Y \longrightarrow Z \to \Sigma X \tag{1.1}$$

is a distinguished triangle sloppily as $Z = C(f)$. Note that this notion makes sense already in the pointed model category of pointed $\mathcal{C}$-spaces [Hov99, Sec. 6]. If f is a cofibration between cofibrant objects, then $C(f) = Y/X$.
It is a well-known fact about triangulated categories that a distinguished triangle (1.1) induces a long exact sequence

$$\ldots \longrightarrow [\Sigma Y, B]_{\mathcal{C}} \longrightarrow [\Sigma X, B]_{\mathcal{C}} \longrightarrow [Cf, B]_{\mathcal{C}} \longrightarrow [Y,B]_{\mathcal{C}} \longrightarrow [X,B]_{\mathcal{C}} \longrightarrow \ldots \tag{1.2}$$

and similarly for $[B, -]_{\mathcal{C}}$.
A triangulated subcategory of $\mathscr{SHC}_{\mathcal{C}}$ is a full subcategory closed under Σ and Σ^{-1} with the property that if it contains a morphism $f\colon X \to Y$, then also its cone Cf. Recall from [MMSS01] that Sp^O is inhabited by various spheres $F_k\mathrm{S}^n$ with $F_0\mathrm{S}^0 = \mathbb{S}$ and $F_k(X \wedge Y) = (F_kX) \wedge Y$. In the homotopy category, $F_k\mathrm{S}^n$ becomes a k-fold

desuspension of S^n. The canonical maps $F_k(S^n_+) \to F_k(D^{n+1}_+)$, with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, define a class of generating cofibrations in Sp^O. A class of generating cofibrations in $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)$ is thus given by $F_k(S^n_+) \wedge \underline{c} \to F_k(D^{n+1}_+) \wedge \underline{c}$ for $k \in \mathbb{Z}$, $n \in \mathbb{N}$ and $c \in \mathrm{Ob}(\mathcal{C})$. We will call an object of $\mathscr{SHC}_\mathcal{C}$ a *finite $\mathcal{C}$-CW-spectrum* if it can be obtained from the trivial functor $*$ by a finite number of gluing steps using these generating cofibrations. The $\mathcal{C}$-Spanier-Whitehead category $\mathscr{SW}_\mathcal{C}$ is the full subcategory of $\mathscr{SHC}_\mathcal{C}$ on the finite $\mathcal{C}$-CW-spectra.
The name is justified by the following lemma:

Lemma 1.3. *(a) $\mathscr{SW}_\mathcal{C}$ is the full subcategory of $\mathscr{SHC}_\mathcal{C}$ on objects of the form $\Sigma^N\Sigma^\infty A$ for some integer N and some finite pointed $\mathcal{C}$-CW-complex A.*
(b) If A is a finite $\mathcal{C}$-CW-complex and B is an arbitrary $\mathcal{C}$-CW-complex, then

$$\mathrm{Hom}_{\mathscr{SW}_\mathcal{C}}(\Sigma^N\Sigma^\infty A, \Sigma^M\Sigma^\infty B) \cong \mathrm{colim}_k \left\{\Sigma^{N+k}A, \Sigma^{M+k}B\right\}_\mathcal{C},$$

where the curly brackets on the right denote (unstable) homotopy classes of maps of $\mathcal{C}$-spaces.
(c) $\mathscr{SW}_\mathcal{C}$ is the smallest triangulated subcategory of $\mathscr{SHC}_\mathcal{C}$ containing the objects $\underline{c}$ for all $c \in \mathrm{Ob}(\mathcal{C})$.

Note that statement (b) serves as an alternative definition of $\mathscr{SW}_\mathcal{C}$, not using $\mathscr{SHC}_\mathcal{C}$.

Proof. Part (a) is an easy induction. In part (c), the fact that $\mathscr{SW}_\mathcal{C}$ is triangulated is clear as well. For the minimality, note that this would be clear inductively if we had defined finite $\mathcal{C}$-CW-spectra using attaching maps $F_k(S^n) \wedge \underline{c} \to F_k(D^{n+1}) \wedge \underline{c}$ since $D^{n+1} = C(S^n)$. Unfortunately, D^{n+1}_+ is *not* the cone of S^n_+. However, in the homotopy category, we may suspend as often as we want since this is an isomorphism. After one suspension, the basepoint problem vanishes: the inclusion $S^n_+ \to D^{n+1}_+$ becomes the inclusion of the boundary B of an $(n+2)$-disk D with two boundary points identified (to the basepoint). The cone of the quotient map $S^{n+1} \to B$ can be identified with D.
For part (b), note that it suffices to prove this statement for ΣA and ΣB. Fix B. As in the proof of (c), we only need to show that it holds true for all $\underline{c}$ and if it is true for A and A', and if $f\colon A \to A'$ is a morphism, then it is true for Cf. For corepresentable functors $\underline{c}$, the statement boils down to the well-known corresponding statement for $\mathscr{SHC}$. Use Theorem 1.16 (c) and Lemma 1.23 below to deal with the left-hand side. For the cone argument, first prove that the right-hand side functor (for fixed B) turns cofibre sequences into long exact sequences, similarly to (1.2). There is a natural map from the right-hand side to the left-hand side which is compatible with these two cone long exact sequences, and thus the claim follows via induction and the five lemma. □

Remark 1.4. If $\mathcal{C} = \mathrm{Or}(G)$ is the orbit category of a group G, then Marc Stephan [Ste16] has shown that Elmendorf's Theorem holds in orthogonal spectra, i. e. there is a model structure on naive orthogonal G-spectra (G-objects in the category of orthogonal spectra) and a Quillen equivalence between this model category and $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)$. However, this may fail in other categories of spectra with the properties discussed in Section 2.1 below. For instance, it definitely fails in $\mathrm{Ch}_{\mathbb{Q}}$. The reason is that Stephan's paper has a cellularity condition that is satisfied by Sp^O, but not by $\mathrm{Ch}_{\mathbb{Q}}$. We will not use the spectral Elmendorf Theorem in this thesis.

Remark 1.5. As promised in the introduction, we want to compare our approach to the classical one of classical genuine G-equivariant homotopy theory. Surveys on this topic are [May96], [Sch18, Ch. 3] and [HHR16, Sec. 2,3, App. A,B]. In this context, G is a finite (or compact Lie) group, and usually not $\mathbb{Z}$-graded, but so-called $RO(G)$-graded (co-)homology theories are considered and this leads to a stable category in which not only S^1, but all representation spheres S^V are invertible with respect to the smash product, where V runs through all finite subrepresentations of a so-called universe $\mathscr{U}$. Using Remark 1.4 above, one sees that (for Sp^O as the category of spectra) we invert subrepresentations of the trivial universe $\mathbb{R}^\infty$, an approach sometimes called naive equivariant stable homotopy theory in the genuine context.
This framework in all its generality breaks down when G becomes an infinite group. Recently, the authors of [DHL$^+$19] developed a generalisation for infinite (or non-compact Lie) groups G with respect to the family of finite (or compact) subgroups. In their setup, smashing with all Thom spaces S^ξ, with ξ a G-vector bundle over $\underline{E}G$, is inverted. Thus, this gives a different setup than the one we treat here, and in particular does not relate to the Davis-Lück construction of homology theories occuring in our homology representation theorem 4.7. Also, our theory is more general in that it treats diagram spaces over arbitrary countable categories $\mathcal{C}$.

1.1.1. Topological categories without (C)

Let us shortly dwell on the case that our index category $\mathcal{C}$ doesn't satisfy condition (C). This becomes important in relation with forthcoming work of Bartels-Lück on the algebraic K-theory of Hecke algebras, where certain topological categories appear whose mapping spaces are totally disconnected.
As will become clear soon, the derived balanced smash product $- \wedge^L_{\mathcal{C}} -$ is not even defined, so even the *formulation* of our homology representation theorem, as well as the application of most other results and methods in this thesis, is out of reach. However, we can approximate $\mathcal{C}$ by a suitable category $\mathcal{C}'$ satisfying (C) and yielding the same homotopy theory of diagram spaces, starting from the following construction. A continuous functor $\nu\colon \mathcal{C} \to \mathcal{D}$ induces a functor

$$\nu^*\colon \mathrm{Fun}(\mathcal{D}, \mathrm{Sp}^O) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)$$

given by precomposition with ν, which has a left adjoint $\nu_!$ given by left Kan extension.

Definition 1.6. A continuous functor $\nu\colon \mathcal{C} \to \mathcal{D}$ is called *weakly fully faithful* if $\nu\colon \mathcal{C}(c,d) \to \mathcal{D}(\nu(c), \nu(d))$ is a weak equivalence for all $c, d \in \mathrm{Ob}(\mathcal{C})$. It is called a *weak equivalence* if it is weakly fully faithful, and essentially surjective in the usual sense.

Proposition 1.7 [GM20, Prop. 2.4]. *Let $\mathcal{C}$ be a small category enriched in* Sp^O*, not necessarily satisfying (C). The adjunction $(\nu_!, \nu^*)$ is a Quillen adjunction, which is a Quillen equivalence if and only if ν is a weak equivalence.*

Given a spectral category $\mathcal{C}$, one may thus try to find a category $\mathcal{C}'$ satisfying (C) together with a weak equivalence $\nu\colon \mathcal{C}' \to \mathcal{C}$, so that ν induces an equivalence of categories

$$\mathscr{SHC}_{\mathcal{C}'} \cong \mathscr{SHC}_{\mathcal{C}}$$

and we can apply the methods developed in this thesis to $\mathcal{C}'$ to study $\mathcal{C}$-homology theories.

In case that the index category comes from a topological category (by adding basepoints to and then applying the infinite suspension functor to the mapping spaces), this is always possible. The reason is that Top has a strong monoidal cofibrant replacement functor

$$Q = |\mathrm{Sing}(-)|\,,$$

cf. [GJ09, Prop. 2.4, Thm. 11.4]. Moreover, applying the suspension functor Σ^∞ preserves weak equivalences in this case by [MMSS01, Thm. 6.9 (i)] since we suspend well-pointed spaces by definition (we have added a basepoint).

Remark 1.8. It is not known to the author whether there is a similar construction for pointed topological categories, let alone general spectral categories.

Remark 1.9. As a side remark, we want to warn the reader that the suspension spectra doesn't in general preserve weak equivalences [Kar20].

Corollary 1.10. *Let $\mathcal{C}$ be an arbitrary topological category. Then there is a topological category $\mathcal{C}'$ together with a continuous functor $\nu\colon \mathcal{C}' \to \mathcal{C}$ such that*

- $\Sigma^\infty \mathcal{C}'_+$ *satisfies (C) and*
- *the induced functor $\Sigma^\infty \mathcal{C}'_+ \to \Sigma^\infty \mathcal{C}_+$ is a weak equivalence.*

The topological category $\mathcal{C}'$ has the same objects as $\mathcal{C}$, and

$$\mathcal{C}'(c,d) = |\mathrm{Sing}(\mathcal{C}(c,d))|\,.$$

Summarising, the homotopy theory that we study in this thesis doesn't see the point-set topology on the mapping spaces of a topological category. This can be remedied by considering the Čech model structure [Dug99]. This model structure plays no role in this thesis, however.

1.2. $\wedge_{\mathcal{C}}$ and $\mathrm{map}_{\mathcal{C}}$

From now on, the letters $\mathcal{A}, \mathcal{B}$ and $\mathcal{D}$ also refer to spectrally enriched small categories satisfying (C). The spectrally enriched category $\mathcal{A} \wedge \mathcal{B}^{\mathrm{op}}$ has objects $\mathrm{Ob}(\mathcal{A}) \times \mathrm{Ob}(\mathcal{B})$ and

$$(\mathcal{A} \wedge \mathcal{B}^{\mathrm{op}})((a,b),(a',b')) = \mathcal{A}(a,a') \wedge \mathcal{B}(b',b)\,.$$

An $(\mathcal{A},\mathcal{B})$-bimodule is a continuous functor $\mathcal{A} \wedge \mathcal{B}^{\mathrm{op}} \to \mathrm{Sp}^O$. We denote the category of $(\mathcal{A},\mathcal{B})$-bimodules and $(\mathcal{A},\mathcal{B})$-linear morphisms (i. e. natural transformations of enriched functors) by

$$\mathrm{Mod}(\mathcal{A},\mathcal{B}) = \mathrm{Fun}(\mathcal{A} \wedge \mathcal{B}^{\mathrm{op}}, \mathrm{Sp}^O)\,,$$

with Hom sets denoted by $\mathrm{Hom}_{(\mathcal{A},\mathcal{B})}(-,-)$ and homotopy sets (i. e. Hom sets in the homotopy category of $(\mathcal{A} \wedge \mathcal{B}^{\mathrm{op}})$-spectra) denoted by $[-,-]_{(\mathcal{A},\mathcal{B})}$. $(\mathcal{C},*)$- and $(*,\mathcal{C})$-bimodules are just called left, respectively right $\mathcal{C}$-modules.
If X is a right and Y a left $\mathcal{C}$-module, then $X \wedge_{\mathcal{C}} Y$ is the spectrum

$$\mathrm{coequ}\left(\bigvee_{(c,d)\in \mathrm{Ob}(\mathcal{C})^2} Y(c) \wedge \mathcal{C}(c,d) \wedge X(d) \rightrightarrows \bigvee_{c \in \mathrm{Ob}(\mathcal{C})} Y(c) \wedge X(c)\right).$$

Here, the upper arrow is defined on any (c,d)-summand via the morphism corresponding to

$$X^*\colon \mathcal{C}(c,d) \to \mathrm{map}(X(d),X(c))$$

under the adjunction between $- \wedge X(d)$ and $\mathrm{map}(X(d),-)$. The lower arrow is defined similarly, using Y instead of X.

Remark 1.11. One could understand the above coequaliser in two ways: Either as a colimit in the usual sense in the category Sp^O, or as a Sp^O-colimit, meaning that the mapping spectrum out of it is isomorphic, as a spectrum, to a suitable limit of mapping spectra. Since we are working with Sp^O itself as target category here, there is no distinction between these two notions, compare the discussion in [Shu06, § 11].

More generally, the balanced smash product $X \wedge_{\mathcal{B}} Y$ of an $(\mathcal{A},\mathcal{B})$-bimodule X and a $(\mathcal{B},\mathcal{C})$-bimodule Y is the $(\mathcal{A},\mathcal{C})$-bimodule defined by

$$X \wedge_{\mathcal{B}} Y(a,c) = X(a,?) \wedge_{\mathcal{B}} Y(?,c)\,.$$

Similarly, the mapping spectrum $\mathrm{map}_{\mathcal{C}^{\mathrm{op}}}(U,X)$ between two right $\mathcal{C}$-modules U and X is defined as

$$\mathrm{equ}\left(\prod_{c\in \mathrm{Ob}(\mathcal{C})} \mathrm{map}(U(c),X(c)) \rightrightarrows \prod_{(c,d)\in \mathrm{Ob}(\mathcal{C})^2} \mathrm{map}\left(\mathcal{C}(c,d), \mathrm{map}(U(d),X(c))\right)\right).$$

More generally, for an $(\mathcal{A},\mathcal{B})$-bimodule X and a $(\mathcal{C},\mathcal{B})$-bimodule U, we have an $(\mathcal{A},\mathcal{C})$-bimodule $\mathrm{map}_{\mathcal{B}^{\mathrm{op}}}(U,X)$. We can similarly define the mapping spectrum between two left $\mathcal{C}$-modules, or between an $(\mathcal{A}, \mathcal{B})$-bimodule and an $(\mathcal{A}, \mathcal{C})$-bimodule. We also introduce the $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{A}$ defined by

$$(a, a') \mapsto \mathcal{A}(a', a)\,,$$

this not being a tautology, but referring to the mapping spectra of the category $\mathcal{A}$. The constructions just introduced can not only be defined in Sp^O, but in any cosmos $\mathcal{V}$. They are linked in various ways that can be subsumed using the notion of a closed bicategory. Recall that a *bicategory* $\mathscr{A}$ consists of a class of objects $\mathrm{Ob}(\mathscr{A})$, and a small category of 1-morphisms $\mathscr{A}(A, B)$ between any two objects A and B, together with composition functors that are associative and have units up to coherent isomorphisms [Bor94a, Def. 7.7.1]. The morphisms between the 1-morphisms are called 2-morphisms. A bicategory is called *closed* [MS06, Def. 16.3.1] if for every 1-morphism $f\colon A \to B$ and every object C, the precomposition with f, $f^*\colon \mathscr{A}(B, C) \longrightarrow \mathscr{A}(A, C)$, as well as the postcomposition with f, $f_*\colon \mathscr{A}(C, A) \longrightarrow \mathscr{A}(C, B)$, have a right adjoint. Since adjoints are unique up to unique isomorphism if they exist, this is a property of a bicategory, not an additional structure on it.

Proposition 1.12. *Let $\mathcal{V}$ be a cosmos. Then there is a closed bicategory $\mathrm{Mod}_{\mathcal{V}}$ in which the objects are given by small $\mathcal{V}$-enriched categories; 1-morphisms from $\mathcal{A}$ to $\mathcal{B}$ are $(\mathcal{A}, \mathcal{B})$-bimodules, with composition given by balanced product and $\mathrm{id}_{\mathcal{A}}$ given by the $(\mathcal{A}, \mathcal{A})$-bimodule $\mathcal{A}$; the 2-morphisms are given by morphisms of bimodules; if X is an $(\mathcal{A}, \mathcal{B})$-bimodule, then the right adjoints of pre- and postcomposition with X are given by $\mathrm{map}_{\mathcal{A}}(X, -)$ and $\mathrm{map}_{\mathcal{B}^{\mathrm{op}}}(X, -)$.*

Proof. The bicategory structure was first discussed in [Bén73] for $\mathcal{V} = \mathrm{Set}$; [HV92, Prop. 2.6] is a classical reference for $\mathcal{V} = \mathrm{Top}$, though it omits bicategorical language. A general reference is [Shu13, Sec. 3, esp. Lemmas 3.25, 3.27]. □

Remark 1.13. In the literature, there are three different names for what we call bimodules here, all of which seem to be common in some circles; the other two are distributors and profunctors. Consequently, the bicategory introduced above is sometimes also called $\mathrm{Dist}_{\mathcal{V}}$ or $\mathrm{Prof}_{\mathcal{V}}$.

Remark 1.14. The category $\mathrm{Mod}(\mathcal{A}, \mathcal{B})$ can again be jazzed up to a spectrally enriched category: If we view two $(\mathcal{A}, \mathcal{B})$-bimodules X and Y as left $(\mathcal{A} \wedge \mathcal{B}^{\mathrm{op}})$-modules (or right $(\mathcal{A}^{\mathrm{op}} \wedge \mathcal{B})$-modules), we can define a mapping spectrum $\mathrm{map}_{(\mathcal{A},\mathcal{B})}(X, Y)$ with underlying set $\mathrm{Hom}_{(\mathcal{A},\mathcal{B})}(X, Y)$. Thus, $\mathrm{Mod}_{\mathrm{Sp}^O}$ is a spectrally enriched closed bicategory in the obvious sense. We don't give further details since we won't use this enrichment.

Example 1.15. In addition to $\mathcal{V} = \mathrm{Set}$ and $\mathcal{V} = \mathrm{Sp}^O$, another interesting example of a cosmos is $\mathcal{V} = \mathrm{Ab}$. An Ab-enriched category is usually called a preadditive category, and a preadditive category with one element is the same as a ring, with a bimodule in the sense discussed here corresponding to a bimodule in the usual sense (whence the name). Thus, we get as a full sub-bicategory of $\mathrm{Mod}_{\mathrm{Ab}}$ the bicategory of rings, (R, S)-bimodules and (R, S)-linear homomorphisms between them, which is sometimes called the Morita category. More generally, one can take $\mathcal{V} = R{-}\mathrm{Mod}$ for some commutative ring R. One may also take $\mathcal{V} = \mathrm{Ch}_R$. A Ch_R-category is the same as an R-linear dg-category. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are R-linear categories (concentrated in degree 0), then an $(\mathcal{A}, \mathcal{B})$-bimodule is the same as a chain complex of $(\mathcal{A}, \mathcal{B})$-bimodules over $R{-}\mathrm{Mod}$. Thus we get as a full sub-bicategory of $\mathrm{Mod}(\mathrm{Ch}_R)$ the bicategory of R-linear categories and chain complexes of $(\mathcal{A}, \mathcal{B})$-bimodules. We will study this in detail for $R = \mathbb{Q}$ in Chapter 5.

1.3. Deriving $\wedge_{\mathcal{C}}$ and $\mathrm{map}_{\mathcal{C}}$

We will now derive the whole setup in the sense that we pass to the homotopy category of every bimodule category $\mathrm{Mod}(\mathcal{A}, \mathcal{B})$, and define a derived version of the balanced smash product which allows us to view the collection of all derived bimodule categories as a bicategory, as well as derived versions of the mapping spectra which exhibit this bicategory as closed. Technically, we achieve this by using the notion of a Quillen adjunction of two variables [Hov99, Sec. 4.1]. The paper [Shu06] presents a slightly different approach based on the two-sided bar construction. This has stronger assumptions, but avoids certain technical difficulties arising in this section and the next chapter. We review this approach in Appendix A.
Throughout the rest of this section, let X be an $(\mathcal{A}, \mathcal{B})$-bimodule, Y a $(\mathcal{B}, \mathcal{C})$-bimodule, Z a $(\mathcal{C}, \mathcal{D})$-bimodule, U an $(\mathcal{A}, \mathcal{D})$-bimodule, and V an $(\mathcal{A}, \mathcal{C})$-bimodule. (This convention will always be clear from the context.)

Theorem 1.16. *The following data defines a closed bicategory* $\mathrm{DerMod}_{\mathrm{Sp}^O}$*: objects are small* Sp^O*-enriched categories satisfying (C);* 1*-morphisms from* $\mathcal{A}$ *to* $\mathcal{B}$ *are* $(\mathcal{A}, \mathcal{B})$*-bimodules;* 2*-morphisms are given by*

$$(\mathrm{DerMod}(\mathcal{A}, \mathcal{B}))(X, Y) = [X, Y]_{(\mathcal{A},\mathcal{B})}\,.$$

The identity 1*-morphism of an object* $\mathcal{A}$ *is the* $(\mathcal{A}, \mathcal{A})$*-bimodule* $\mathcal{A}$ *and the identity* 2*-morphism of a* 1*-morphism* X *is* id_X*. The composition of* 1*-morphisms and their adjoints are given by the functors*

$$- \wedge^L_{\mathcal{B}} - \colon \quad \mathrm{DerMod}(\mathcal{A}, \mathcal{B}) \times \mathrm{DerMod}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{DerMod}(\mathcal{A}, \mathcal{C})\,,$$

$$R\mathrm{map}_{\mathcal{C}^{\mathrm{op}}} \colon \quad \mathrm{DerMod}(\mathcal{B}, \mathcal{C}))^{\mathrm{op}} \times \mathrm{DerMod}(\mathcal{A}, \mathcal{C}) \longrightarrow \mathrm{DerMod}(\mathcal{A}, \mathcal{B})\,,$$

and

$$R\mathrm{map}_{\mathcal{A}} \colon \quad \mathrm{DerMod}(\mathcal{A}, \mathcal{B})^{\mathrm{op}} \times \mathrm{DerMod}(\mathcal{A}, \mathcal{C}) \longrightarrow \mathrm{DerMod}(\mathcal{B}, \mathcal{C})\,,$$

which are the total derived functors of $- \wedge_{\mathcal{B}} -$*,* $\mathrm{map}_{\mathcal{C}^{\mathrm{op}}}$ *and* $\mathrm{map}_{\mathcal{A}}$*. Explicitly, for* Q *a functorial cofibrant replacement and* R *a functorial fibrant replacement, we have*

$$X \wedge^L_{\mathcal{B}} Y \cong QX \wedge_{\mathcal{B}} QY\,, \quad R\mathrm{map}_{\mathcal{C}^{\mathrm{op}}}(Y, V) \cong \mathrm{map}_{\mathcal{C}^{\mathrm{op}}}(QY, RV)$$

and

$$R\mathrm{map}_{\mathcal{A}}(X, U) \cong \mathrm{map}_{\mathcal{C}^{\mathrm{op}}}(QX, RU)\,.$$

In particular, the closed bicategory structure induces the following natural isomorphisms:

(a) $\mathcal{A} \wedge^L_{\mathcal{A}} X \cong X \cong X \wedge^L_{\mathcal{B}} \mathcal{B}$ *in* $\mathrm{DerMod}(\mathcal{A},\mathcal{B})$,

(b) $(X \wedge^L_{\mathcal{B}} Y) \wedge^L_{\mathcal{C}} Z \cong X \wedge^L_{\mathcal{B}} (Y \wedge^L_{\mathcal{C}} Z)$ *in* $\mathrm{DerMod}(\mathcal{A},\mathcal{D})$,

(c) $[X \wedge^L_{\mathcal{B}} Y, V]_{(\mathcal{A},\mathcal{C})} \cong [X, R\mathrm{map}_{\mathcal{C}^{\mathrm{op}}}(Y,V)]_{(\mathcal{A},\mathcal{B})} \cong [Y, \mathrm{map}_{\mathcal{A}}(X,V)]_{(\mathcal{B},\mathcal{C})}$,

(d) $R\mathrm{map}_{\mathcal{A}}(\mathcal{A},X) \cong X \cong R\mathrm{map}_{\mathcal{B}^{\mathrm{op}}}(\mathcal{B},X)$ *in* $\mathrm{DerMod}(\mathcal{A},\mathcal{B})$,

(e) $R\mathrm{map}_{\mathcal{A}}(X \wedge^L_{\mathcal{B}} Y, U) \cong R\mathrm{map}_{\mathcal{B}}(Y, R\mathrm{map}_{\mathcal{A}}(X,U))$ *in* $\mathrm{DerMod}(\mathcal{C},\mathcal{D})$,

(f) $R\mathrm{map}_{\mathcal{D}^{\mathrm{op}}}(Z, R\mathrm{map}_{\mathcal{A}}(X,U)) \cong R\mathrm{map}_{\mathcal{A}}(X, R\mathrm{map}_{\mathcal{D}^{\mathrm{op}}}(Z,U))$ *in* $\mathrm{DerMod}(\mathcal{B},\mathcal{C})$.

Remark 1.17. This theorem allows one to construct a $\mathrm{Ho}(\mathrm{Sp}^O)$-enrichment of $\mathscr{SHC}_{\mathcal{C}}$, interpreting $\mathcal{C}$-spectra as $(*,\mathcal{C})$-spectra and using $R\mathrm{map}_{\mathcal{C}}(-,-)$.

Proof. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ denote small spectrally enriched categories satisfying (C). The closedness of the bicategory $\mathrm{Mod}_{\mathrm{Sp}^O}$ gives natural isomorphisms

$$\varphi_l \colon \mathrm{Hom}_{(\mathcal{A},\mathcal{C})}(X \wedge_{\mathcal{B}} Y, V) \xrightarrow{\cong} \mathrm{Hom}_{(\mathcal{B},\mathcal{C})}(Y, \mathrm{map}_{\mathcal{A}}(X,V))$$

and

$$\varphi_r \colon \mathrm{Hom}_{(\mathcal{A},\mathcal{C})}(X \wedge_{\mathcal{B}} Y, V) \xrightarrow{\cong} \mathrm{Hom}_{(\mathcal{A},\mathcal{B})}(X, \mathrm{map}_{\mathcal{C}^{\mathrm{op}}}(Y,V))\,.$$

The categories $\mathrm{Mod}(\mathcal{A},\mathcal{B})$, $\mathrm{Mod}(\mathcal{B},\mathcal{C})$ and $\mathrm{Mod}(\mathcal{A},\mathcal{C})$ with the quintuple consisting of $\wedge_{\mathcal{B}}$, $\mathrm{Hom}_r = \mathrm{map}_{\mathcal{C}^{\mathrm{op}}}$, $\mathrm{Hom}_l = \mathrm{map}_{\mathcal{A}}$ and the two isomorphisms φ_r and φ_l form an adjunction of two variables in the sense of [Hov99, Def. 4.1.12]. We want to apply [Hov99, Cor. 4.2.5] to show that $\wedge_{\mathcal{B}}$ is a Quillen bifunctor.
For this we have to check that the pushout product of two generating cofibrations is a cofibration, and that it is a trivial cofibration if one of the factors is a generating trivial cofibration. For the definition of the pushout product $\Box$, see [Hov99, Def. 4.2.1]. We check the first statement, the other two being similar. We may choose the generating cofibrations of the form $f \wedge \underline{(a,b)}$ and $g \wedge \underline{(b',c)}$, where f and g belong to a class of generating cofibrations of Sp^O. Up to isomorphism of morphisms, we have the identity

$$(f \wedge \underline{(a,b)}) \Box_{\mathcal{B}} (g \wedge \underline{(b',c)}) \cong (f \Box g) \wedge \mathcal{B}(b,b') \wedge \underline{(a,c)}\,. \tag{1.3}$$

By the pushout-product axiom for Sp^O, $f \Box g$ is a cofibration. Now, $\mathcal{B}(b,b')$ is cofibrant by (C) and thus $(f \Box g) \wedge \mathcal{B}(b,b')$ is a cofibration, since it is a smash product of a cofibration with a cofibrant object. Here we use the pushout-product axiom for Sp^O again. Thus, the right hand side of (1.3) has the left lifting property with respect to all trivial fibrations and is thus a cofibration.
Proposition 4.3.1 of [Hov99] then applies to show that we have total derived functors as in the statement of the theorem and that the quintuple

$$(\wedge^L_{\mathcal{B}}, R\mathrm{map}_{\mathcal{C}^{\mathrm{op}}}, R\mathrm{map}_{\mathcal{A}}, R\varphi_r, R\varphi_l)$$

defines an adjunction of two variables. This gives the isomorphism (c). Isomorphism (b) follows from the explicit description of $\wedge^L_{\mathcal{B}}$ together with the fact that the balanced smash product of two cofibrant bimodules is cofibrant, which follows from the Quillen bifunctor property.
To show that $\mathrm{DerMod}_{\mathrm{Sp}^{\mathcal{O}}}$ is actually a bicategory, we are left to deal with two points: Firstly, that there is an associativity isomorphism satisfying a coherence square. This follows directly from the corresponding fact for $\mathrm{Mod}_{\mathrm{Sp}^{\mathcal{O}}}$, as in the proof of [Hov99, Prop. 4.3.1 or Prop. 4.3.2]. Secondly, that we have an identity 1-morphism at every object. Surprisingly, this is the more difficult part, since the identity $\mathcal{A}$ might be non-cofibrant. However, we may use Corollary 1.21 below to see that

$$\mathcal{A} \wedge^L_{\mathcal{A}} X \cong \mathcal{A} \wedge_{\mathcal{A}} X \cong X$$

since $\mathcal{A}$ is obviously right flat in the sense of Definition 1.20. The coherence conditions for this unitality isomorphism are readily checked.
The fact that the derived mapping functors are right adjoints of the derived smash products is part of the adjunction of two variables statement. Summarising, we have now proved that $\mathrm{DerMod}_{\mathrm{Sp}^{\mathcal{O}}}$ is a closed bicategory, amounting to isomorphisms (a) to (c).
Now the punchline is that (d) to (f) are valid in any closed bicategory: (d) follows from (a) – if pre- and postcomposition with $\mathcal{A}$ is isomorphic to the identity, then the same has to be true for their adjoints. Similarly, (e) and (f) follow from (b). □

Proposition 1.18. *If X is a cofibrant $(\mathcal{A}, \mathcal{B})$-spectrum, then $X \wedge_{\mathcal{B}} -$ preserves weak equivalences.*

Remark 1.19. To the author's knowledge, it is not clear whether this holds for arbitrary X, even for $\mathcal{A} = \mathcal{B} = *$.

Proof. Let $Y \to Y'$ be any weak equivalence of $(\mathcal{B}, \mathcal{C})$-spectra which will be fixed throughout the proof. We first treat the case where $X = F_k A \wedge \underline{(a, b)}$, with A a pointed CW-complex. Smashing with $F_k A \wedge \underline{(a, b)}$, we get the map

$$F_k A \wedge \mathcal{A}(a, -) \wedge Y(b, -) \longrightarrow F_k A \wedge \mathcal{A}(a, -) \wedge Y(b, -)\,.$$

Now, $\mathcal{A}(a, -)$ is objectwise a cofibrant spectrum by (C), and so is $F_k A$. But smashing with a cofibrant spectrum preserves weak equivalences by [MMSS01, Prop. 12.3].
Now we want to reduce to the general case. By general theory of cofibrantly generated model categories, a cofibrant object is a retract of a cell complex. Since weak equivalences are closed under retracts, we may assume that X is a (transfinite) cell complex, i. e. a transfinite composition [Hir03, Def. 10.2.2] of pushouts along generating cofibrations. Suppose that the transfinite composition is indexed by some ordinal β and denote the intermediate 'skeleta' by X_α, $\alpha < \beta$, where $X_{\alpha+1}$ can be obtained from X_α by a cobase change along a coproduct of generating cofibrations. In particular, $X_\alpha \hookrightarrow X_{\alpha+1}$ is a cofibration in the projective model structure, but this property is not preserved when smashing (over $\mathcal{C}$) with an arbitrary spectrum. This is

why we have to use the more subtle notion of h-cofibration. This is a concept which is not available in an arbitrary model category, but in many topological examples, in particular in Sp^O. Our use of h-cofibrations is restricted to this proof.
We define a map of $\mathcal{C}$-spectra $A \to B$ to be an h-cofibration if $B \wedge I_+$ retracts onto $A \wedge I_+ \cup_A B \wedge \{0\}_+$, cf. [MMSS01, p. 457]. Since the generating cofibrations are h-cofibrations, the same is true for the inclusions $X_\alpha \hookrightarrow X_{\alpha+1}$. Moreover, h-cofibrations are preserved under balanced smash products by definition.
Now we are in shape to prove the proposition for general X by transfinite induction on β. Suppose that $X_\alpha \wedge_{\mathcal{B}} -$ preserves weak equivalences. We have a pushout diagram

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ X_\alpha & \hookrightarrow & X_{\alpha+1} \end{array}$$

with a generating cofibration $A \hookrightarrow B$. This will stay be a pushout diagram after applying $- \wedge_{\mathcal{B}} Y$ and $- \wedge_{\mathcal{B}} Y'$, since these functors have right adjoints. We get a comparison diagram

$$\begin{array}{ccccc} X_\alpha \wedge_{\mathcal{B}} Y & \longleftarrow & A \wedge_{\mathcal{B}} Y & \hookrightarrow & B \wedge_{\mathcal{B}} Y \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ X_\alpha \wedge_{\mathcal{B}} Y' & \longleftarrow & A \wedge_{\mathcal{B}} Y' & \hookrightarrow & B \wedge_{\mathcal{B}} Y' \end{array}$$

where the hooked arrows denote h-cofibrations. The left vertical arrow is a weak equivalence by induction hypothesis, and the two right vertical arrows are by the beginning of this proof, for $A = \mathrm{S}^n_+$ and $A = \mathrm{D}^n_+$. It follows that the map on the pushout is a weak equivalence by [MMSS01, Thm. 8.12(iv)]. For limit ordinals β, we know that X_β is the colimit of X_α, $\alpha < \beta$. This is preserved when smashing with Y and Y'. In orthogonal spectra, a stable equivalence is the same as a π_*-isomorphism [MMSS01, Prop. 8.7]. Computing the stable homotopy groups commutes with colimits along h-cofibrations, since these are levelwise closed inclusions. □

Definition 1.20. An $(\mathcal{A}, \mathcal{B})$-spectrum F is *right flat* if the functor

$$F \wedge_{\mathcal{B}} -\colon \mathrm{Mod}(\mathcal{B}) \longrightarrow \mathrm{Mod}(\mathcal{A})$$

preserves weak equivalences. $f\colon F \to X$ is called a *right flat replacement* of X if F is right flat and f is a weak equivalence.

Since weak equivalences are defined objectwise, for a right flat $(\mathcal{A}, \mathcal{B})$-spectrum F and arbitrary $\mathcal{C}$, the functor

$$F \wedge_{\mathcal{B}} -\colon \mathrm{Mod}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Mod}(\mathcal{A}, \mathcal{C})$$

will preserve weak equivalences as well.

Corollary 1.21. *Let $f\colon F \to X$ be a right flat replacement of X. Then there is a natural isomorphism*

$$X \wedge^L_{\mathcal{B}} Y \cong F \wedge_{\mathcal{B}} Y$$

for any $(\mathcal{B}, \mathcal{C})$-bimodule Y.

Proof. There are weak equivalences

$$X \wedge^L_{\mathcal{B}} Y = QX \wedge_{\mathcal{B}} QY \xrightarrow{\sim} X \wedge_{\mathcal{B}} QY \xleftarrow{\sim} F \wedge_{\mathcal{B}} QY \xrightarrow{\sim} F \wedge_{\mathcal{B}} Y\,,$$

where the first and second weak equivalence follow from Proposition 1.18. □

Left flat replacements are defined similarly and the statement of the corollary carries over mutatis mutandis.

Remark 1.22. Proposition 1.18 and Corollary 1.21 have been proved to show the isomorphism $\mathcal{A} \wedge^L_{\mathcal{A}} X \cong X$. The proofs are technically much more advanced than the rest of the proofs in this chapter and in particular harder to generalise to other model categories of spectra than orthogonal spectra, cf. Chapter 2. In the understanding of the author, this is inevitable for Proposition 1.18 since the corresponding statement for $\mathcal{C} = *$ is a subtle point in all treatments he could find. A different setup in which the fact that $\mathcal{A} \wedge^L_{\mathcal{A}} X \cong X$ can be proved without Proposition 1.18 is presented in Appendix A.

The one-object Yoneda lemma carries over to the derived setting without trouble since $\underline{c}$ is a cofibrant $\mathcal{C}^{\mathrm{op}}$-spectrum. We state it here for later use.

Lemma 1.23. *For a $\mathcal{C}$-spectrum X and $c \in \mathrm{Ob}(\mathcal{C})$, there are natural isomorphisms in $\mathscr{SHC}$*

$$\underline{c} \wedge_{\mathcal{C}} X \cong X(c)$$

and

$$R\mathrm{map}_{\mathcal{C}}(\underline{c}, X) \cong X(c)\,.$$ □

2. Changing the category of spectra

Throughout the thesis hitherto, we investigated $\mathcal{C}$-spectra in the sense of functors from $\mathcal{C}$ to the category Sp^O of orthogonal spectra. However, the literature also uses several other model categories of spectra, which are either Quillen equivalent to orthogonal spectra (respecting the smash product in one sense or the other, as discussed below), or describe a slightly different version of spectra, e. g. connective spectra or rational spectra. In addition, there are of course monoidal model categories describing completely different homotopy theories, but our results may still be helpful in these categories. The purpose of this chapter is to bring all these other models in, in the following two ways:

- Firstly, we state conditions under which much of the framework built up so far can be built up with another monoidal model category instead of orthogonal spectra.

- Secondly, suppose we have built up the framework for two different model categories $\mathcal{S}$ and $\mathcal{T}$, and we have a Quillen adjunction between the two. Then we want to compare our constructions, performed in $\mathcal{S}$, with the same constructions, performed in $\mathcal{T}$.

The first item will be carried out in Section 2.1. We will write down a list of assumptions on the monoidal model category and then deduce a substantial part of Chapter 1. At this point, we introduce three different layers of generality.
In the first layer, the minimalist approach, we generalise enough to write down derived smash products and mapping spectra, and prove the various adjunctions between them, cf. Proposition 2.5. What we will *not* prove is the derived Yoneda Lemma

$$\mathcal{A} \wedge^L_{\mathcal{A}} X \cong X \cong X \wedge^L_{\mathcal{B}} \mathcal{B}$$

since the way we proved it used rather specific properties of orthogonal spectra, cf. the proof of Proposition 1.18. The minimalist approach can, by definition, be applied to so-called *nice enriching categories*. In addition, we introduce *very nice enriching categories*, in which the derived Yoneda Lemma holds, and *very very nice enriching categories*, in which the derived Yoneda Lemma holds and can be derived in the same way as for orthogonal spectra.

Remark 2.1. A completely different approach to the first item, which has slightly stronger assumptions, but in which the derived Yoneda Lemma doesn't need specific treatment, is presented in Appendix A.

The second item is dealt with in Section 2.2. The Quillen adjunction between $\mathcal{S}$ and $\mathcal{T}$ has to be compatible with the smash product. The literature knows (at least) two different ways in which a Quillen adjunction can be compatible with monoidal structures on its source and target: strong and weak monoidal Quillen adjunctions. Their definitions will be recalled below. In many cases, it is possible to compare two categories of spectra by a strong monoidal Quillen adjunctions, and then the comparison result is trivial (for discrete categories). For instance, all pairs of model categories of spectra discussed in [MMSS01] are linked by strong monoidal Quillen equivalences. However, we also discuss the comparison along the less restrictive notion of a weak monoidal Quillen adjunction, which is not trivial any longer, cf. Subsection 2.2.1.
The agenda of the first item may be carried out for spectrally enriched categories $\mathcal{C}$ satisfying (C). For the second item, we can pass to enriched categories for strong monoidal Quillen adjunctions only. This is discussed in Subsection 2.2.2, which is based on the paper [GM20].
The reason that we get into this discussion in detail is twofold: Firstly, it is intrinsically satisfying to know that our results are independent of the choice of a model category of spectra. Secondly, and more concretely, our comparison results will become crucial in Section 5.1, where they are used in the rational case to pass from rational spectra to rational chain complexes. It is here where it becomes important that we can also deal with weak monoidal Quillen equivalences.

Remark 2.2. We want to comment the way we intend to apply the comparison results of this chapter. Suppose $\mathcal{S}$ is a model category of spectra which is Quillen equivalent to orthogonal spectra, and we are interested in Theorem A from the Introduction for $\mathcal{S}$. Then we will use the result for Sp^O, to be proved below, and then compare the balanced smash product occuring (secretly) on the right hand side of (0.2) to the corresponding balanced smash product in $\mathcal{S}$, using the machinery we are just about to develop, for instance the isomorphism (2.2).
Another strategy would be to develop bicategorical duality theory over $\mathcal{S}$ and then *prove* Theorem A separately for $\mathcal{S}$. Although this is also a totally valid approach, it is *not* the one we will use here – mainly because of the technical problem mentioned above that we potentially cannot prove the derived Yoneda Lemma for $\mathcal{S}$ and thus do not have a clean bicategory at hand.

2.1. Nice enriching categories

We start by distilling properties of Sp^O we used to set up the framework of Chapter 1. Let $(\mathcal{S}, \wedge, \mathbb{S})$ denote a model category which also has a monoidal structure. As already explained above, we rather pursue a minimalist approach here, comprising the following goals: Set up homotopy categories, as in Section 1.1 up to and including the discussion of the triangulated structure; define balanced smash products and mapping spectra, Section 1.2; derive these as in Theorem 1.16 to get $\wedge^L_{\mathcal{C}}$ and $R\mathrm{map}_{\mathcal{C}}$ and isomorphisms (b) through (f).

To prove these statements, we used the following list of properties of Sp^O:

- The smash product and mapping spectra furnish Sp^O with the structure of a cosmos, i. e. a closed symmetric monoidal category with all small limits and colimits.
- It has a cofibrantly generated model structure. There is a class of generating cofibrations and generating trivial cofibrations whose sources are cofibrant, and small with respect to all relative cell complexes in any diagram category $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^O)$.
- The unit of the smash product is cofibrant.
- The pushout-product axiom [SS00, Def. 3.1] holds.
- The monoid axiom [SS00, Def. 3.3] holds.

Definition 2.3. We call a monoidal model category $(\mathcal{S}, \wedge, \mathbb{S})$ satisfying the above list of properties a *nice enriching category.*

We don't claim that a nice enriching category is stable [Hov99, Ch. 7] or models the same homotopy theory as Sp^O. This has the advantage that we can also apply the following 'meta theorem' to different situations which play a supporting role in this thesis.

Example 2.4. The categories Top and Top_* of (pointed) compactly generated weak Hausdorff spaces with the model structure from [Hov99, Sec. 2.4] are nice enriching categories. The monoid axiom is shown on p. 7 of [Hov98].

Proposition 2.5. *If $(\mathcal{S}, \wedge, \mathbb{S})$ is a nice enriching category, then the statements of Theorems 1.1 and 1.16 hold for $\mathcal{S}$ in the place of* Sp^O*, except that* $\mathrm{DerMod}(\mathcal{S})$ *may fail to have identities, thus is not a bicategory, and that isomorphism (a) may not hold.* □

Remark 2.6. The fact that Sp^O is a cosmos (with respect to the smash product) was crucially needed to construct balanced smash products and mapping spectra, and the compatibility with the model structure to derive these, cf. Sections 1.2 and 1.3. The cofibrant generation is needed to construct model structures on $\mathcal{C}$-spectra. The smallness condition goes into Theorem 1.1, see [GM20, Rem. 4.34]. The same remark explains why it follows from standard arguments in all cases we consider. The cofibrancy of the sources is used in the proof of Proposition 2.13 below. The facts that Sp^O is a cosmos, the unit is cofibrant and the pushout-product axiom holds imply that it is a monoidal model category in the sense of [Hov99, Def. 4.2.6]. The latter notion is slightly weaker than the three mentioned facts and would technically also suffice for our purposes. The monoid axiom is needed for Theorem 1.1.

If $(\mathcal{S}, \wedge, \mathbb{S})$ satisfies the above list of properties and in addition the derived Yoneda Lemma

$$\mathcal{A} \wedge^L_{\mathcal{A}} X \cong X \cong X \wedge^L_{\mathcal{A}} X \tag{2.1}$$

holds, then we say that $(\mathcal{S}, \wedge, \mathbb{S})$ is a *very nice enriching category.* In this case, the full Theorem 1.16 holds and there is a bicategory $\mathrm{DerMod}_{\mathcal{S}}$ of $\mathcal{S}$-categories satisfying (C), bimodules over $\mathcal{S}$ and $\mathcal{S}$-bilinear maps.

Remark 2.7. In Appendix A, we review some results from the paper [Shu06] which shows essentially that simplicial monoidal model categories are very nice. Actually, we can only show that they are almost very nice due to some technical problems. This almost means that they are very nice, see Definition A.3 for a more serious explanation. All monoidal model categories mentioned in this section are simplicial.

One way to show that $(\mathcal{S}, \wedge, \mathbb{S})$ is a very nice enriching category is to show that Proposition 1.18 holds over $\mathcal{S}$: Any cofibrant bimodule is left and right flat. In this case, we call $(\mathcal{S}, \wedge, \mathbb{S})$ a *very very nice enriching category.*
For instance, this is satisfied if any cofibrant object $S \in \mathcal{S}$ is flat, and there is a meaningful notion of h-cofibrations, as in many topological models for spectra, which gets the proof started as above. But it also holds in other cases.

Example 2.8. We prove in Lemma 5.9 below (see also Remark 5.6) that $\mathrm{Ch}_{\mathbb{Q}}$ is a very very nice enriching category.

Example 2.9. We don't know whether Top_* is very nice. We will show in Appendix A that Top_* is almost very nice. However, Top_* is not very very nice.
We find a counterexample to Proposition 1.18 already in the category itself: Smashing with the cofibrant object S^1 does not preserve weak equivalences in this monoidal model category. In fact, the map

$$\mathbb{N}_0 \to \{0\} \cup \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}, \quad 0 \mapsto 0, \quad n \mapsto \frac{1}{n},$$

where source and target are endowed with the subspace topology from $\mathbb{R}$, is a weak equivalence. The smash product of the right hand side with S^1 can easily be identified with the Hawaiian earrings which are not weakly equivalent to a wedge of circles since their fundamental group is too large [deS92].

The literature in stable homotopy theory contains many different model categories of spectra. The paper [MMSS01] further treats the model categories of $\mathscr{W}$-spaces and sequential spectra. The treatment of $\mathscr{W}$-spaces and orthogonal spectra is completely analogous, so that all results will be true for $\mathscr{W}$-spaces, with the same references in [MMSS01] applying. In particular, $\mathscr{W}$-spaces are a very very nice enriching category. All model categorical aspects apply to sequential spectra as well, but this is not a closed symmetric monoidal category and will be treated separately in Section 2.3.
We will also use the category $\mathrm{Sp}^{\Sigma}_{sSet}$ of simplicial symmetric spectra with the stable model structure from [HSS00].

Lemma 2.10. *The model category* $\mathrm{Sp}^{\Sigma}_{sSet}$ *is a nice enriching category.*

Proof. See [HSS00, Thm. 2.2.10, Thm. 3.4.4, Cor. 5.3.8, Cor. 5.5.2]. Note that the authors of [HSS00] call a monoidal model category what we defined as a model category satisfying the pushout-product axiom. The fact that the unit is cofibrant is remarked on p. 53 of [HSS00]. □

We will now discuss some model categories of rational spectra originally introduced in [Shi07]. These will be the main actors of Section 5.1. The four monoidal model categories are:

- the category $H\mathbb{Q} - \mathcal{M}od$ of modules over the monoid $H\mathbb{Q}$ in $\mathrm{Sp}^{\Sigma}_{sSet}$ with model structure as explained in [SS00, Thm. 4.1(1)];
- the model category of unbounded rational chain complexes [Hov99, Sec. 2.3];
- the category $\mathrm{Sp}^{\Sigma}(s\mathrm{Vect}_{\mathbb{Q}})$ of symmetric spectra over simplicial $\mathbb{Q}$-vector spaces [Hov01b];
- the category $\mathrm{Sp}^{\Sigma}(\mathrm{ch}^{+}_{\mathbb{Q}})$ of symmetric spectra over non-negatively graded rational chain complexes [Hov01b].

The latter two model structures are constructed following the general construction [Hov99] of a model category of symmetric spectra over a given (nice) monoidal model category. It is applied to the categories of simplicial objects in $\mathbb{Q}$-vector spaces with the model structure from [Qui67, Ch. II.4] and to $\mathrm{ch}^{+}_{\mathbb{Q}}$ with the projective model structure [DS95, Sec. 7].

Lemma 2.11. *The four model categories mentioned above are nice enriching categories.*

Proof. The Standing Assumptions 2.4 of [Shi07], proved for our four model categories in Section 3, comprise all our assumptions except the cofibrancy of the sources of the generating (trivial) cofibrations. For $H\mathbb{Q} - \mathcal{M}od$, this can be seen as follows: Generating cofibrations for $H\mathbb{Q}$-modules can be obtained from generating cofibrations in $\mathrm{Sp}^{\Sigma}_{sSet}$ by smashing with $H\mathbb{Q}$ (cf. [SS00, Lemma 2.3]). Since these have cofibrant sources and $H\mathbb{Q}$ is cofibrant, the smash product is cofibrant in $\mathrm{Sp}^{\Sigma}_{sSet}$ and thus also in $H\mathbb{Q} - \mathcal{M}od$ since this has less cofibrations. For $\mathrm{Ch}_{\mathbb{Q}}$, the sources are cofibrant since they are bounded and (trivially) degreewise projective. For the latter two categories, the stable model structures on symmetric spectra have the same cofibrant objects as the projective model structures introduced [Hov01b, Thm. 8.2] and the generating cofibrations of these have cofibrant sources since this is true for $s\mathrm{Vect}_{\mathbb{Q}}$ and $\mathrm{ch}^{+}_{\mathbb{Q}}$. □

2.2. Comparison between different enriching categories

Let $(\mathcal{S}, \wedge, \mathbb{S})$ and $(\mathcal{T}, \otimes, \mathbb{T})$ denote nice enriching categories. Let

$$F\colon (\mathcal{S}, \wedge, \mathbb{S}) \rightleftarrows (\mathcal{T}, \otimes, \mathbb{T})\colon G$$

be a Quillen adjunction, where F is the left adjoint. If $\mathcal{A}$ is an $\mathcal{S}$-category, we want to compare the derived category of $\mathcal{A}$-modules over $\mathcal{S}$ with $\mathcal{B}$-modules over $\mathcal{T}$, where $\mathcal{B}$ is a suitable $\mathcal{T}$-category. This comparison should be compatible with derived balanced smash products and mapping spaces. In case that $\mathcal{A}$ is the free $\mathcal{S}$-category on a discrete category, as in the first subsection, we may take the free $\mathcal{T}$-category on the same category as $\mathcal{B}$. For an arbitrary enriched category $\mathcal{A}$, the first difficulty will be to find the right category $\mathcal{B}$. Note that the natural candidate $F\mathcal{A}$, which has the same objects and morphism spaces are obtained by application of F from those of $\mathcal{A}$, only defines a $\mathcal{T}$-category if F is strong monoidal, see Subsection 2.2.2.

Recall the definition of weak and strong monoidal Quillen adjunctions from [SS03a, Sec. 3.2]: A Quillen adjunction is called strong monoidal if F is strong monoidal and $F(Q\mathbb{S}) \to F(\mathbb{S}) \cong \mathbb{T}$ is a weak equivalence for the unit $\mathbb{S}$. It is called weak monoidal if G is lax monoidal, thus F lax comonoidal, such that the maps

$$\nabla \colon F(x \wedge y) \to F(x) \otimes F(y)$$

are weak equivalences for all cofibrant x and y, and the composite

$$F(Q\mathbb{S}) \to F(\mathbb{S}) \to \mathbb{T}$$

is a weak equivalence as well. In our case, the unit $\mathbb{S}$ is cofibrant, so the unit condition is vacuous for strong monoidal Quillen equivalences and boils down to the fact that $F(\mathbb{S}) \to \mathbb{T}$ is a weak equivalence in the weak monoidal case.

In many cases of interest, a strong monoidal Quillen equivalence exists. However, we will also need the case of weak monoidal Quillen equivalences when studying the Dold-Kan correspondence in Section 5.1.

Example 2.12. If $\mathcal{S} = \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}$ and $\mathcal{T} = \mathrm{Sp}^{O}$, then a Quillen equivalence can be constructed by first comparing with topological symmetric spectra (via degreewise application of geometric realisation and singular set, see [MMSS01, Thm. 19.4]) and then moving on to orthogonal spectra [MMSS01, p. 442]. Both Quillen equivalences are strong monoidal, and so is their composition.

If $\mathcal{C}$ denotes a discrete category, then we get a Quillen equivalence

$$F \colon \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^{\Sigma}_{\mathrm{sSet}}) \rightleftarrows \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^{O}) \colon G\,.$$

Since F respects colimits and is strong monoidal, there is an obvious natural isomorphism

$$F(X \wedge_{\mathcal{C}} Y) \cong F(X) \wedge_{\mathcal{C}} F(Y)$$

for all X and Y. This descends to the homotopy category: If Φ and Γ denote the derived functors of F and G, then

$$\Phi(X \wedge^{L}_{\mathcal{C}} Y) \cong \Phi(X) \wedge^{L}_{\mathcal{C}} \Phi(Y)$$

and consequently

$$\Gamma(X' \wedge^{L}_{\mathcal{C}} Y') \cong \Gamma(X') \wedge^{L}_{\mathcal{C}} \Gamma(Y')\,.$$

Furthermore, one checks easily that the functors preserve suspension spectra in the sense that the suspension symmetric spectrum of a $\mathcal{C}$-simplicial set is isomorphic to the suspension orthogonal spectrum of its objectwise geometric realisation; and the suspension orthogonal spectrum of a $\mathcal{C}$-space is isomorphic (in the derived category) to the suspension symmetric spectrum of its objectwise singular complex.

2.2.1. Weak monoidal Quillen adjunctions and discrete categories

Throughout this subsection, $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ are discrete[1] categories. An $(\mathcal{A}, \mathcal{A}')$-bimodule is just a functor in the usual non-enriched sense from $\mathcal{A} \times (\mathcal{A}')^{\mathrm{op}}$ to $\mathcal{S}$, respectively $\mathcal{T}$. We thus have a Quillen adjunction

$$F_*\colon \mathrm{Fun}(\mathcal{A} \wedge (\mathcal{A}')^{\mathrm{op}}, \mathcal{S}) \rightleftarrows \mathrm{Fun}(\mathcal{A} \otimes (\mathcal{A}')^{\mathrm{op}}, \mathcal{T})\colon G_*$$

which is again a Quillen equivalence if (F, G) is [Hir03, Thm. 11.6.5].
The comonoidal transformation ∇ induces the following natural commutative diagram for $a \in \mathrm{Ob}(\mathcal{A})$ and $a'' \in \mathrm{Ob}(\mathcal{A}'')$:

$$\begin{array}{ccc} \bigvee_{a_1' \to a_2'} F(X(a, a_2') \wedge Y(a_1', a'')) & \rightrightarrows & \bigvee_{a'} F(X(a, a') \wedge Y(a', a'')) \\ \downarrow \nabla & & \downarrow \nabla \\ \bigvee_{a_1' \to a_2'} F(X(a, a_2')) \otimes F(Y(a_1', a'')) & \rightrightarrows & \bigvee_{a'} F(X(a, a')) \otimes F(Y(a', a'')) \end{array}$$

and thus induces a map on the colimits of the rows. Since F commutes with colimits, we get $\nabla\colon F_*(X \wedge_{\mathcal{A}'} Y) \to F_*(X) \otimes_{\mathcal{A}'} F_*(Y)$.

Proposition 2.13. *∇ is a weak equivalence if X and Y are cofibrant.*

Proof. We first treat the case where $X = A \wedge \underline{(a, a')}$ for some cofibrant spectrum A. Then ∇ is isomorphic to

$$\nabla\colon F(A \wedge \underline{a} \wedge Y(a', -)) \to F(A \wedge \underline{a}) \otimes F(Y(a', -))$$

which is a weak equivalence since $A \wedge \underline{a}$ is objectwise cofibrant by discreteness of $\mathcal{A}$ and $\mathcal{C}$, and $Y(a', -)$ is objectwise cofibrant by Theorem 1.1 (ii), which follows already from [Hir03, Prop. 11.6.3] here since $\mathcal{A}'$ and $\mathcal{A}''$ are discrete. Note the natural isomorphism

$$F_*(A \wedge \underline{(a, a')}) \cong F_*\left(\bigvee_{\underline{a'}} A \wedge \underline{a}\right) \cong \bigvee_{\underline{a'}} F_*(A \wedge \underline{a}) \cong F_*(A \wedge \underline{a}) \otimes \underline{a'}\,,$$

since F commutes with colimits.

[1] as opposed to: enriched

In the general case, X is a retract of a (transfinite) cell complex. We may thus assume that X is itself a cell complex. Arguing by transfinite induction, we have to show that the property that ∇ is a weak equivalence is preserved under gluing along coproducts of generating cofibrations and under passage to colimits along cofibrations.
For the first point, we use the first step of the proof and the Cube Lemma [Hov99, Lemma 5.2.6]. The two comparison diagrams consist of cofibrant objects and one cofibration since F_* is left Quillen and $\wedge_{\mathcal{A}'}$ and $\otimes_{\mathcal{A}'}$ are Quillen bifunctors, cf. the proof of Theorem 1.16.
For the second point, suppose that we have a chain of cofibrations of some shape κ. This is a cofibrant diagram in the projective model structure on the functor category of κ-sequences: The lifting property can be proved by transfinite induction. Since the colimit is a left Quillen functor [Hir03, Thm. 11.6.8], it preserves weak equivalences between cofibrant objects. □

For the derived functor

$$\Phi = \Phi_{(\mathcal{A},\mathcal{A}')} = \mathrm{Ho}(F_*)\colon \mathrm{DerMod}_{\mathcal{S}}(\mathcal{A},\mathcal{A}') \to \mathrm{DerMod}_{\mathcal{T}}(\mathcal{A},\mathcal{A}')\,,$$

we get a natural isomorphism in $\mathrm{DerMod}_{\mathcal{T}}(\mathcal{A},\mathcal{A}'')$

$$\Phi(X \wedge^L_{\mathcal{A}'} Y) = F_*(QX \wedge_{\mathcal{A}'} QY) \xrightarrow[\nabla]{\cong} F_*(QX) \otimes_{\mathcal{A}'} F_*(QY) \cong \Phi(X) \otimes^L_{\mathcal{A}'} \Phi(Y)\,. \quad (2.2)$$

This induces by adjunction and Yoneda Lemma an isomorphism of derived $(\mathcal{A},\mathcal{A}')$-bimodules over $\mathcal{T}$

$$\Gamma(R\mathrm{map}_{\mathcal{A}''}(\Phi(Y), V')) \cong R\mathrm{map}_{\mathcal{A}''}(Y, \Gamma(V')) \quad (2.3)$$

for any derived $(\mathcal{A},\mathcal{A}'')$-bimodule V' over $\mathcal{T}$, where $\Gamma = \mathrm{Ho}(G_*)$.
If (F, G) is a Quillen equivalence, then we get further isomorphisms

$$\Gamma(X' \wedge^L_{\mathcal{A}'} Y') \cong \Gamma(X') \otimes^L_{\mathcal{A}'} \Gamma(Y') \quad (2.4)$$

as well as

$$R\mathrm{map}_{\mathcal{A}}(\Phi(X), \Phi(U)) \cong \Phi(R\mathrm{map}_{\mathcal{A}}(X, U)) \quad (2.5)$$

and

$$R\mathrm{map}_{\mathcal{A}}(\Gamma(X'), \Gamma(U')) \cong \Gamma(R\mathrm{map}_{\mathcal{A}}(X', U')) \quad (2.6)$$

and similar isomorphisms for $R\mathrm{map}_{\mathcal{A}'}$.
If $\mathcal{S}$ and $\mathcal{T}$ are very good categories of spectra, we can formulate this in terms of the bicategories $\mathrm{DerMod}_{\mathcal{S}}$ and $\mathrm{DerMod}_{\mathcal{T}}$. This uses the notion of a *pseudofunctor*.
Let $\mathcal{A}$ and $\mathcal{B}$ denote bicategories. Recall that a pseudofunctor between $\mathcal{A}$ and $\mathcal{B}$ consists of the following data:

- an assignment
$$F\colon \mathrm{Ob}(\mathcal{A}) \longrightarrow \mathrm{Ob}(\mathcal{B})\,,$$

- for all $A, A' \in \mathrm{Ob}(\mathcal{A})$, a functor

$$F\colon \mathcal{A}(A, A') \longrightarrow \mathcal{B}(FA, FA'),$$

- for all $A \in \mathrm{Ob}(\mathcal{A})$, a 2-isomorphism

$$\alpha\colon F(\mathrm{id}_A) \to \mathrm{id}_{FA},$$

- for all pairs (g, f) of composable 1-morphisms, a 2-isomorphism

$$m_{gf}\colon F(g \circ f) \longrightarrow F(g) \circ F(f),$$

satisfying certain coherence conditions detailed in [Hov99, Def. 1.4.2]. We call a pseudofunctor F a *biequivalence* if it induces equivalences of categories on all categories on 1-morphisms and 2-morphisms.

Remark 2.14. Biequivalences are sometimes called equivalences of bicategories in the literature. For every biequivalence, there exists a pseudofunctor in the other direction which is its inverse up to a suitably coherent notion of pseudonatural transformation.

Definition 2.15. Let $\mathcal{S}$ be a very nice enriching category. The locally full sub-bicategory[2] of $\mathrm{DerMod}_{\mathcal{S}}$ on all free $\mathcal{S}$-categories on discrete categories is denoted by $\mathrm{DerMod}_{\mathcal{S}}^{\mathrm{disc}}$.

Corollary 2.16. *Suppose that $\mathcal{S}$ and $\mathcal{T}$ are very nice enriching categories. A weak monoidal Quillen adjunction induces a pseudofunctor*

$$\Phi\colon \mathrm{DerMod}_{\mathcal{S}}^{\mathrm{disc}} \longrightarrow \mathrm{DerMod}_{\mathcal{T}}^{\mathrm{disc}}.$$

If (F, G) is a Quillen equivalence, then Φ is a biequivalence.

Proof. The functor Φ is given by the identity on objects, viewed as discrete categories, and by application of Φ on 1- and 2-morphisms. The unit isomorphisms α are induced by the natural weak equivalences

$$F(\mathbb{S} \wedge \mathcal{A}) \cong F(\mathbb{S}) \wedge \mathcal{A} \xrightarrow{\sim} \mathbb{T} \wedge \mathcal{A}.$$

The isomorphisms m_{gf} are given by the composition (2.2). The coherence diagrams are checked to commute already before passing to the derived category. □

[2]By this we mean a bicategory which consists of a subclass of objects, and all 1- and 2-morphisms on these objects.

2.2.2. Strong monoidal Quillen adjunctions and enriched categories

In case that (F, G) is a strong monoidal Quillen equivalence, we can deal with enriched categories as well. We follow [GM20, Sec. 3], and prove that certain Quillen equivalences given there are compatible with balanced smash products. Throughout the following, $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ are $\mathcal{S}$-categories satisfying (C), and $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}''$ are $\mathcal{T}$-categories satisfying (C).
There is a $\mathcal{T}$-category $F\mathcal{A}$ with the same objects as $\mathcal{A}$ and

$$F\mathcal{A}(x,y) = F(\mathcal{A}(x,y))\,,$$

and similarly an $\mathcal{S}$-category[3] $G\mathcal{B}$.

Remark 2.17. Note, however, that $F\mathcal{A}$ in general has a different underlying (Set-enriched) category than $\mathcal{A}$, while $G\mathcal{B}$ and $\mathcal{B}$ have the same underlying category. Also note that if $\mathcal{C}_{\mathcal{A}}$ denotes the free $\mathcal{S}$-enriched category over a discrete category $\mathcal{C}$, as discussed in the last subsection, then $F\mathcal{C}_{\mathcal{S}} \cong \mathcal{C}_{\mathcal{T}}$, but in general $G\mathcal{C}_{\mathcal{T}}$ and $\mathcal{C}_{\mathcal{S}}$ are different.

Lemma 2.18. *(i) If $\mathcal{A}$ satisfies (C), then so does $F\mathcal{A}$.*
(ii) Every $\mathcal{A}$-module X defines an $F\mathcal{A}$-module FX by objectwise application of F. □

Let $\mathcal{A}$ be a small $\mathcal{S}$-category. We want to compare $\mathcal{A}$-modules (over $\mathcal{S}$) with $F\mathcal{A}$-modules (over $\mathcal{T}$). More generally, we state our comparison result for a small $\mathcal{T}$-category $\mathcal{B}$ with the same objects as $\mathcal{A}$, together with a $\mathcal{T}$-functor $\psi\colon F\mathcal{A} \to \mathcal{B}$ which is the identity on objects. An equivalent datum is the §-linear adjoint $\phi = \psi^\flat\colon \mathcal{A} \to G\mathcal{B}$. This setup, which was already studied in [GM20], has the advantage that given $\mathcal{B}$, we can set $\mathcal{A} = G\mathcal{B}$ and let ψ be the $\mathcal{T}$-functor $FG\mathcal{B} \to \mathcal{B}$ given by objectwise applying the counit of the adjunction. We thus study the relation between $\mathcal{B}$-modules and $G\mathcal{B}$-modules in the same breath. Throughout the following, $(\mathcal{A}, \psi, \mathcal{B})$, $(\mathcal{A}', \psi', \mathcal{B}')$ and $(\mathcal{A}'', \psi'', \mathcal{B}'')$ denote three such triples.
We recall the decisive construction from [GM20, (3.11)], with a changed notation and adjusted to bimodules. Our goal is a Quillen adjunction

$$(F_{(\psi,\psi')}, G_{(\psi,\psi')})\colon \mathrm{Mod}_{\mathcal{S}}(\mathcal{A}, \mathcal{A}') \rightleftarrows \mathrm{Mod}_{\mathcal{T}}(\mathcal{B}, \mathcal{B}')\,. \tag{2.7}$$

If Y is a $(\mathcal{B}, \mathcal{B}')$-bimodule over $\mathcal{T}$, then $G_{(\psi,\psi')}Y$ is given by $G(Y(-))$ on objects and on morphisms by the adjoint of

$$\begin{aligned}\mathcal{A}'(a_1', a_2') \wedge G(Y(a_1, a_2')) \wedge \mathcal{A}(a_1, a_2) &\xrightarrow{\phi\wedge\phi'} G(\mathcal{B}(a_1', a_2')) \wedge G(Y(a_1, a_2')) \wedge G(\mathcal{B}(a_1, a_2)) \\ &\longrightarrow G(\mathcal{B}(a_1', a_2') \otimes Y(a_1, a_2') \otimes \mathcal{B}(a_1, a_2)) \\ &\longrightarrow G(Y(a_2, a_1'))\,.\end{aligned}$$

If X is an $(\mathcal{A}, \mathcal{A}')$-bimodule over $\mathcal{S}$, then FX is an $(F\mathcal{A}, F\mathcal{A}')$-bimodule over $\mathcal{T}$. We now consider the $(\mathcal{B}, F\mathcal{A})$-bimodule $\mathscr{L}$ over $\mathcal{T}$ given by

$$\mathcal{B} \otimes F\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{id}\otimes\psi^{\mathrm{op}}} \mathcal{B} \otimes \mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{B}} \mathcal{T}\,.$$

[3] For G, this is in fact already true for a weak monoidal Quillen equivalence.

The $(F\mathcal{A}', \mathcal{B}')$-bimodule $\mathscr{R}'$ is obtained by swapping the order of $\mathcal{B}$ and $F\mathcal{A}$, and inserting primes everywhere. We set $F_{(\psi,\psi')}X = \mathscr{L} \otimes_{F\mathcal{A}} FX \otimes_{F\mathcal{A}'} \mathscr{R}'$. It is easy to check that these two functors are well-defined and adjoint, using Proposition 1.12. Moreover, $(F_{(\psi,\psi')}, G_{(\psi,\psi')})$ is a Quillen adjunction: $G_{(\psi,\psi')}$ preserves fibrations and acyclic fibrations since these are defined objectwise.

Proposition 2.19. *Let X be an $(\mathcal{A}, \mathcal{A}')$-bimodule and Y an $(\mathcal{A}', \mathcal{A}'')$-bimodule. Suppose that the following conditions are satisfied:*

- *The functor $\psi' \colon F\mathcal{A}' \to \mathcal{B}'$ is weakly fully faithful.*
- *$\mathscr{L} \otimes_{F\mathcal{A}} FX$ is a right flat $(\mathcal{B}, F\mathcal{A}')$-bimodule and $FY \otimes_{F\mathcal{A}''} \mathscr{R}''$ is a left flat $(F\mathcal{A}', \mathcal{B}'')$-bimodule in the sense of Definition 1.20.*

Then there is a natural weak equivalence of $(\mathcal{B}, \mathcal{B}'')$-bimodules over $\mathcal{T}$

$$F_{(\psi,\psi'')}(X \wedge_{\mathcal{A}'} Y) \longrightarrow F_{(\psi,\psi')}X \otimes_{\mathcal{B}'} F_{(\psi,\psi'')}Y .$$

Proof. We have

$$F_{(\psi,\psi')}X \otimes_{\mathcal{B}'} F_{(\psi',\psi'')}Y = \mathscr{L} \otimes_{F\mathcal{A}} FX \otimes_{F\mathcal{A}'} \mathscr{R}' \otimes_{\mathcal{B}'} \mathscr{L}' \otimes_{F\mathcal{A}'} FY \otimes_{F\mathcal{A}''} \mathscr{R}'' .$$

By the Yoneda Lemma, the $(F\mathcal{A}', F\mathcal{A}')$-bimodule $\mathscr{R}' \otimes_{\mathcal{B}'} \mathscr{L}'$ is given by $\mathcal{B}'(-,-)$ on objects and on morphisms by

$$F\mathcal{A}'(a_1', a_2') \otimes F\mathcal{A}'(a_1', a_2') \xrightarrow{\psi' \otimes \psi'} \mathcal{B}'(a_1', a_2') \otimes \mathcal{B}'(a_3', a_4') \xrightarrow{\mathcal{B}'} \mathcal{T}(\mathcal{B}'(a_4', a_1'), \mathcal{B}'(a_3', a_2')) .$$

There is a weak equivalence of $(F\mathcal{A}', F\mathcal{A}')$-bimodules

$$F\mathcal{A}' \xrightarrow{\sim} \mathscr{R}' \otimes_{\mathcal{B}'} \mathscr{L}'$$

given by applying ψ' objectwise. Since $\mathscr{L} \otimes_{F\mathcal{A}} FX$ and $FY \otimes_{F\mathcal{A}''} \mathscr{R}''$ are right, resp. left flat, this induces a weak equivalence

$$\begin{aligned} F_{(\psi,\psi'')}(X \wedge_{\mathcal{A}'} Y) &\cong \mathscr{L} \otimes_{F\mathcal{A}} F(X \wedge_{\mathcal{A}'} Y) \otimes_{F\mathcal{A}''} \mathscr{R}'' \\ &\cong \mathscr{L} \otimes_{F\mathcal{A}} FX \otimes_{F\mathcal{A}'} FY \otimes_{F\mathcal{A}''} \mathscr{R}'' \\ &\cong \mathscr{L} \otimes_{F\mathcal{A}} FX \otimes_{F\mathcal{A}'} F\mathcal{A}' \otimes_{F\mathcal{A}'} FY \otimes_{F\mathcal{A}''} \mathscr{R}'' \\ &\xrightarrow{\sim} \mathscr{L} \otimes_{F\mathcal{A}} FX \otimes_{F\mathcal{A}'} \mathscr{R}' \otimes_{\mathcal{B}'} \mathscr{L}' \otimes_{F\mathcal{A}'} FY \otimes_{F\mathcal{A}''} \mathscr{R}'' \\ &= F_{(\psi,\psi')}X \otimes_{\mathcal{B}'} F_{(\psi',\psi'')}Y . \end{aligned}$$

□

Recall that a functor $K \colon \mathcal{N} \to \mathcal{M}$ between model categories is said to *create the weak equivalences* if it has the property that f is a weak equivalence in $\mathcal{N}$ if and only if $K(f)$ is a weak equivalence in $\mathcal{M}$.

Lemma 2.20. *Suppose that*

$$H \colon \mathcal{M} \rightleftarrows \mathcal{N} \colon K$$

is a Quillen equivalence and the right adjoint K creates the weak equivalences. Then the adjunction unit

$$\eta \colon M \longrightarrow K(H(M))$$

is a weak equivalence for any cofibrant $M \in \mathrm{Ob}(\mathcal{M})$.

Proof. In general for a Quillen equivalence, the composition

$$M \xrightarrow{\eta} K(H(M)) \xrightarrow{K(f)} K(R)$$

is a weak equivalence, where $f\colon H(M) \xrightarrow{\sim} R$ is a fibrant replacement [Hov99, Prop. 1.3.13]. Since K creates the weak equivalences, $K(f)$ is a weak equivalence and thus η is, by 2-out-of-3. □

Theorem 2.21 [GM20, Thm. 3.17]. *If (F, G) is a strong monoidal Quillen equivalence, G creates the weak equivalences, and ψ and ψ' are weakly fully faithful, then $(F_{(\psi,\psi')}, G_{(\psi,\psi')})$ is a Quillen equivalence.*

Proof. Apply the theorem cited above to $\mathscr{V} = \mathscr{V}_+ = \mathcal{S}$. We use here our assumption that the unit of the smash product is cofibrant. The case of bimodules works in the same way as for modules.
The cited theorem has the assumption that instead of ψ, the adjoint ϕ is fully faithful. But these conditions are actually equivalent in our situation: Let $A = \mathcal{A}(a_1, a_2)$ and $B = \mathcal{B}(a_1, a_2)$ for some objects a_1, a_2 of $\mathcal{A}$. Then A is cofibrant by (C). We have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta} & GFA \\ & \searrow^{\phi} & \downarrow{\scriptstyle G(\psi)} \\ & & GB \end{array}$$

with η a weak equivalence by Lemma 2.20. Thus, ϕ is a weak equivalence if and only if $G(\psi)$ is if and only if ψ is. □

Corollary 2.22. *Suppose that $\mathcal{T}$ is a very very nice enriching category. Then $F_{(\mathrm{id},\mathrm{id})}$ induces a functor*

$$\Phi\colon \mathrm{DerMod}_{\mathcal{S}}(\mathcal{A}^{(i)}, \mathcal{A}^{(j)}) \longrightarrow \mathrm{DerMod}_{\mathcal{T}}(\mathcal{B}^{(i)}, \mathcal{B}^{(j)})$$

such that there is a natural isomorphism

$$\Phi(X \wedge^L_{\mathcal{A}'} Y) \xrightarrow{\cong} \Phi(X \otimes^L_{\mathcal{B}'} \Phi(Y))\,.$$

Now, suppose additionally that (F, G) is a Quillen equivalence, and G creates the weak equivalences. Then Φ is an equivalence of categories. In this case, the isomorphisms (2.4), (2.5) *and* (2.6) *follow.* □

Proof. We apply Proposition 2.19 to $\psi^{(i)} = \mathrm{id}\colon F\mathcal{A}^{(i)} \to F\mathcal{A}^{(i)}$. If X is cofibrant, then $FX = F_{(\mathrm{id},\mathrm{id})}X$ is cofibrant and thus right flat since $\mathcal{T}$ is very very good. Moreover, $\mathscr{L} = \mathcal{A}$ is right flat. □

Corollary 2.23. *Suppose that $\mathcal{S}$ is a very nice enriching category and $\mathcal{T}$ is a very very nice enriching category. A strong monoidal Quillen adjunction induces a pseudofunctor*

$$\Phi\colon \mathrm{DerMod}_{\mathcal{S}} \longrightarrow \mathrm{DerMod}_{\mathcal{T}}$$

which is given on objects by $\mathcal{A} \mapsto F\mathcal{A}$, and on Hom categories by $\mathrm{Ho}(F_{(\mathrm{id},\mathrm{id})})$. If (F, G) is a Quillen equivalence and G creates weak equivalences, then Φ is a biequivalence. □

2.3. Sequential spectra

We finally treat sequential spectra. Anticipating the next chapter, we argue how we can substitute orthogonal spectra with sequential spectra in Theorem 4.7 even if these don't have a well-behaved smash product.
Sequential spectra do not form a monoidal model category, only a model category tensored and cotensored over spaces. The tensor and cotensor structure can be derived by the same Quillen adjunction argument as in Theorem 1.16. In this case, even Proposition 1.18 may be proved in the same way as above, relying on the same references in [MMSS01] as this paper treats sequential and orthogonal spectra uniformly.
Let $\mathcal{C}$ denote a discrete category. While it is impossible to formulate duality for sequential spectra over $\mathcal{C}$ (using our methods), it *is* possible to write down a homology theory for $\mathcal{C}$-spaces from a sequential $\mathcal{C}^{\mathrm{op}}$-spectrum as in (4.1). For this construction, Theorem 4.7 actually holds true as well. To see this, we compare with a Quillen equivalence to orthogonal $\mathcal{C}$-spectra and only have to show that the balanced smash products are translated into one another.
Let $\mathbb{U}_*(Y)$ denote the underlying $(\mathcal{B}, \mathcal{C})$-sequential spectrum of an $(\mathcal{B}, \mathcal{C})$-orthogonal spectrum Y. Let X be an $(\mathcal{A}, \mathcal{B})$-space. Then there is a tautological isomorphism of sequential spectra

$$\mathbb{U}_*(\Sigma^\infty X \wedge Y) \cong X \wedge \mathbb{U}_* Y$$

inducing the same isomorphism for $\wedge_{\mathcal{B}}$ instead of $\wedge$ since $\mathbb{U}_*$ commutes with colimits. To pass to the derived functor, it suffices to cofibrantly replace X by an argumentation similar to Corollary 1.21. Thus, we get a natural isomorphism

$$\mathrm{Ho}(\mathbb{U}_*)(\Sigma^\infty X \wedge^L_{\mathcal{B}} Y) \cong X \wedge^L_{\mathcal{B}} \mathrm{Ho}(\mathbb{U}_*)(Y)\,.$$

Similarly,

$$\mathrm{Ho}(\mathbb{U}_*)(R\mathrm{map}_{\mathcal{A}}(\Sigma^\infty X, U)) \cong R\mathrm{map}_{\mathcal{A}}(X, \mathrm{Ho}(\mathbb{U}_*)(U))$$

where we use in the derivation process that the right adjoint $\mathbb{U}_*$ preserves fibrant objects.
Everything we said is more generally true for a topological category $\mathcal{C}$ satisfying (C). This can be proved as in Subsection 2.2.2. The category $\mathcal{A}$ doesn't have to be changed since the left adjoint $\mathbb{P}$ of $\mathbb{U}$ ('prolongation') is compatible with the tensor structure over spaces.

3. External Spanier-Whitehead duality

In this chapter, we develop a duality theory in the closed bicategory $\mathrm{DerMod}_{\mathrm{Sp}^O}$ discussed in Theorem 1.16. We give this the name "external duality" – which is slightly oxymorontic, as explained in the Introduction –, since the dual of an object lives in another category than the object itself: finite $\mathcal{C}$-spectra are paired with finite $\mathcal{C}^{\mathrm{op}}$-spectra. The constructions of this chapter will allow us to go back and forth between homology theories on finite $\mathcal{C}$-spectra and cohomology theories on finite $\mathcal{C}^{\mathrm{op}}$-spectra in the proof of Theorem 4.7.
We will begin by formulating the problem, i. e. by defining the notion of a dual pair. This is carried out in Section 3.1, which uses only the bicategory structure of $\mathrm{DerMod}_{\mathrm{Sp}^O}$. There are several equivalent formulations of this notion, the equivalence of which is proved in Proposition 3.2. The discussion in the end of Section 3.1 uses the symmetry of the bicategory $\mathrm{DerMod}_{\mathrm{Sp}^O}$. Finally, the closedness allows us to write down, in Section 3.2, an *ansatz* for the solution of the above problem: We construct a functor D for which it is plausible that (X, DX) is a dual pair. Finally, we prove in the usual way, using an inductive argument, that finite spectra are dualisable.

Remark 3.1. The yoga of the first section could be carried out in any bicategory, or a symmetric bicategory for the results after Remark 3.6. The ansatz for the functor D can be written down in any closed bicategory, so that dualisable objects are well-defined. The proof that finite $\mathcal{C}$-spectra are dualisable can be absorbed into the more general context of compatibly triangulated closed symmetric bicategories. In our exposition, we focussed on the closed bicategory category $\mathrm{DerMod}_{\mathrm{Sp}^O}$, which we will need later, for simplicity. A more general treatment of all three steps can be found in [MS06].

3.1. Bicategorical duality theory

The discussion in this section is essentially equivalent to [MS06, Ch. 16], slightly simplified for our purposes. Also compare [LMSM86, Ch. III]. We change our standing notation from the last chapter: In this chapter, X will always denote an $(\mathcal{A}, \mathcal{B})$-bimodule, Y a $(\mathcal{B}, \mathcal{A})$-bimodule, Z a $(\mathcal{C}, \mathcal{A})$-bimodule, U a $(\mathcal{B}, \mathcal{C})$-bimodule, V an $(\mathcal{A}, \mathcal{C})$-bimodule and W a $(\mathcal{C}, \mathcal{B})$-bimodule. All morphisms between bimodules are morphisms in the homotopy category – in other words, we are working in the bicategory $\mathrm{DerMod}_{\mathrm{Sp}^O}$.
Given a morphism

$$\varepsilon \colon X \wedge^L_{\mathcal{B}} Y \xrightarrow{(\mathcal{A},\mathcal{A})} \mathcal{A}\,,$$

we may define

$$\varepsilon^1_*\colon [W, Z \wedge^L_{\mathcal{A}} X]_{(\mathcal{C},\mathcal{B})} \to [W \wedge^L_{\mathcal{B}} Y, Z]_{(\mathcal{C},\mathcal{A})}$$

where $\varepsilon^1_*(f)$ is the composition

$$W \wedge^L_{\mathcal{B}} Y \xrightarrow{f\wedge^L_{\mathcal{B}} Y} Z \wedge^L_{\mathcal{A}} X \wedge^L_{\mathcal{B}} Y \xrightarrow{Z\wedge^L_{\mathcal{A}}\varepsilon} Z \wedge^L_{\mathcal{A}} \mathcal{A} \cong Z\,.$$

Similarly, we may define

$$\varepsilon^2_*\colon [U, Y \wedge^L_{\mathcal{A}} V]_{(\mathcal{B},\mathcal{C})} \to [X \wedge^L_{\mathcal{B}} U, V]_{(\mathcal{A},\mathcal{C})}\,.$$

On the other hand, a morphism

$$\eta\colon \mathcal{B} \xrightarrow{(\mathcal{B},\mathcal{B})} Y \wedge^L_{\mathcal{A}} X$$

yields

$$\eta^1_*\colon [W \wedge^L_{\mathcal{B}} Y, Z]_{(\mathcal{C},\mathcal{A})} \to [W, Z \wedge^L_{\mathcal{A}} X]_{(\mathcal{C},\mathcal{B})}$$

and

$$\eta^2_*\colon [X \wedge^L_{\mathcal{B}} U, V]_{(\mathcal{A},\mathcal{C})} \to [U, Y \wedge^L_{\mathcal{A}} V]_{(\mathcal{B},\mathcal{C})}\,.$$

In the following, the letters ε and η are reserved for morphisms with source and target as above. The next proposition is the main point of our discussion of duality since it shows that the notion of a dual pair can equivalently formulated in terms of ε and η, or only one of them – the other one can be recovered uniquely. It is essentially [LMSM86, Thm. III.1.6] or [MS06, Prop. 16.4.6].

Proposition 3.2. *The following data determine one another:*

(I) morphisms ε and η such that the composition

$$X \cong X \wedge^L_{\mathcal{B}} \mathcal{B} \xrightarrow{X\wedge^L_{\mathcal{B}}\eta} X \wedge^L_{\mathcal{B}} Y \wedge^L_{\mathcal{A}} X \xrightarrow{\varepsilon\wedge^L_{\mathcal{A}} X} \mathcal{A} \wedge^L_{\mathcal{A}} X \cong X$$

equals id_X *and the composition*

$$Y \cong \mathcal{B} \wedge^L_{\mathcal{B}} Y \xrightarrow{\eta\wedge^L_{\mathcal{B}} Y} Y \wedge^L_{\mathcal{A}} X \wedge^L_{\mathcal{B}} Y \xrightarrow{Y\wedge^L_{\mathcal{A}}\varepsilon} Y \wedge^L_{\mathcal{A}} \mathcal{A} \cong Y$$

equals id_Y*;*

(II) a morphism ε such that ε^1_* is a bijection for all W and Z;

(III) a morphism ε such that ε^2_* is a bijection for all U and V;

(IV) a morphism η such that η^1_* is a bijection for all W and Z;

(V) a morphism η such that η^2_* is a bijection for all W and Z.

Proof. If ε and η as in (I) are given, then a direct check reveals that ε^1_* and η^1_* are inverse bijections, as are ε^2_* and η^2_*. Thus we recover (II) through (V). We now show how to recover (I) from (II), with the proceeding starting from another point being analogous.
Suppose that ε^1_* is always a bijection. With $\mathcal{C} = \mathcal{B}$, $W = \mathcal{B}$ and $Z = Y$, we get an isomorphism

$$\varepsilon^1_*\colon [\mathcal{B}, Y \wedge^L_{\mathcal{A}} X]_{(\mathcal{B},\mathcal{B})} \to [\mathcal{B} \wedge^L_{\mathcal{B}} Y, Y]_{(\mathcal{B},\mathcal{A})} .$$

Choosing η as the preimage of the canonical isomorphism $\mathcal{B} \wedge^L_{\mathcal{B}} Y \cong Y$, we get the second of the two compositions in (I) to equal id_Y. Note that we have no other choice for η if we want (I) to hold. Moving on, note that $\varepsilon^1_*\eta^1_*$ is the identity for all W and Z. Since ε^1_* is a bijection, this exhibits η^1_* as a bijection as well and implies that the other composition $\eta^1_*\varepsilon^1_*$ also equals the identity. Now, the first composition in (I), viewed as a morphism $X \to \mathcal{A} \wedge^L_{\mathcal{A}} X$ (i. e., forget the last canonical isomorphism φ_X), equals $\eta^1_*(\varepsilon)$, so its image under η^1_* equals ε. But the same is true for φ_X^{-1}, so the two are equal.
It is obvious that the presented constructions are inverse to each other – one way, we forgot about η, and going back, we had a unique choice for η. □

Remark 3.3. Condition (I) says that (X, Y) is an adjoint pair in the sense of adjointness between 1-morphisms in bicategories [Bor94a, Def. 7.7.2].

Definition 3.4. $(X, Y; \varepsilon, \eta)$ – equivalently $(X, Y; \varepsilon)$ or $(X, Y; \eta)$ – is called a *dual pair* of bimodules if the equivalent conditions of Proposition 3.2 hold.

Note that we can omit one of ε and η from the quadruple $(X, Y; \varepsilon, \eta)$, but not both: for instance, ε is not uniquely determined by X and Y, since we might change it by an automorphism of its source or target.

Remark 3.5. The discussion above is not symmetric in $\mathcal{A}$ and $\mathcal{B}$. We could equally well have formulated a second kind of duality where we interchanged the role of the source and target of a 1-morphism, as well as the order of the composition (i. e. balanced smash product) everywhere. This would have given a *different* notion of duality with *different* dual pairs.

The bicategory $\mathrm{DerMod}_{\mathrm{Sp}^O}$ has a special kind of symmetry available: By definition, there is a canonical isomorphism of categories between $(\mathcal{A}, \mathcal{B})$-bimodules and $(\mathcal{B}^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$-bimodules which we denote by

$$X \mapsto X^{\mathrm{op}} .$$

This assignment is involutive, and we have canonical isomorphisms

$$\gamma\colon \left(X \wedge^L_{\mathcal{B}} Y\right)^{\mathrm{op}} \xrightarrow{\cong} Y^{\mathrm{op}} \wedge^L_{\mathcal{B}^{\mathrm{op}}} X^{\mathrm{op}}$$

of $(\mathcal{A}, \mathcal{C})$-bimodules, and

$$\delta\colon (\mathrm{id}_{\mathcal{A}})^{\mathrm{op}} \xrightarrow{\cong} \mathrm{id}_{\mathcal{A}^{\mathrm{op}}}$$

of $(\mathcal{A}^{\mathrm{op}}, \mathcal{A}^{\mathrm{op}})$-bimodules.

Remark 3.6. In the language of [MS06, Sec. 16.2], this refers to the fact that $\mathrm{DerMod}_{\mathrm{Sp}^{\mathcal{O}}}$ is a symmetric bicategory, with involution $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{op}}$.

In this notation, Remark 3.5 says that the fact that $(X, Y; \varepsilon, \eta)$ is a dual pair is *not* equivalent to the fact that $(X^{\mathrm{op}}, Y^{\mathrm{op}}; \varepsilon', \eta')$ is a dual pair for some ε' and η'. However, there is the following tautological observation which we will use later:

Proposition 3.7. *$(X, Y; \varepsilon, \eta)$ is a dual pair if and only if the pair $(Y^{\mathrm{op}}, X^{\mathrm{op}}; \delta\varepsilon^{\mathrm{op}}\gamma^{-1}, \gamma\eta^{\mathrm{op}}\delta^{-1})$ is.*

Proof. Trivial for condition (I) of Proposition 3.2. □

Proposition 3.8. *If $(X, Y; \varepsilon, \eta)$ and $(U, W; \zeta, \theta)$ are dual pairs, then so is $(X \wedge_{\mathcal{B}} U, W \wedge_{\mathcal{B}} Y; \nu, \xi)$ where ν is the composition*

$$X \wedge^L_{\mathcal{B}} U \wedge^L_{\mathcal{C}} W \wedge^L_{\mathcal{B}} Y \xrightarrow{X \wedge^L_{\mathcal{B}} \zeta \wedge^L_{\mathcal{B}} Y} X \wedge^L_{\mathcal{B}} \mathcal{B} \wedge^L_{\mathcal{B}} Y \cong X \wedge^L_{\mathcal{B}} Y \xrightarrow{\varepsilon} \mathcal{A}$$

and ξ is defined similarly.

Proof. The proof is trivial for condition (I), cf. [MS06, Thm. 16.5.1]. □

3.2. Proof of Theorem B

We now make use of the closedness of $\mathrm{DerMod}_{\mathrm{Sp}^{\mathcal{O}}}$ – i. e., the existence of derived mapping bimodules. The following two propositions are essentially Propositions 16.4.13 and 16.4.12 of [MS06].

Proposition 3.9. *If $(X, Y; \varepsilon)$ is a dual pair, then we have the following natural isomorphisms:*

$$Z \wedge^L_{\mathcal{A}} X \xrightarrow[\cong]{(\mathcal{C},\mathcal{B})} R\mathrm{map}_{\mathcal{A}^{\mathrm{op}}}(Y, Z) \tag{3.1}$$

$$Y \wedge^L_{\mathcal{A}} V \xrightarrow[\cong]{(\mathcal{B},\mathcal{C})} R\mathrm{map}_{\mathcal{A}}(X, V)\,, \tag{3.2}$$

and

$$Y \cong R\mathrm{map}_{\mathcal{A}}(X, \mathcal{A})\,. \tag{3.3}$$

Proof. For the first two isomorphisms, use condition (II) and Theorem 1.16 (c) – and the (usual form of the) Yoneda lemma. Setting $\mathcal{C} = \mathcal{A}$ and $V = \mathcal{A}$ in (3.2) yields (3.3). □

Considering Equation (3.3) above, we will now reverse the logic, define Y as $R\mathrm{map}_{\mathcal{A}}(X, \mathcal{A})$ and investigate when this yields a dual pair.

Definition 3.10. For an $(\mathcal{A}, \mathcal{B})$-spectrum X, define the *dual of* X to be the $(\mathcal{B}, \mathcal{A})$-spectrum

$$DX = D_{(\mathcal{A},\mathcal{B})}X = R\mathrm{map}_{\mathcal{A}}(X, \mathcal{A})\,.$$

Remark 3.11. The notation $D_{(\mathcal{A},\mathcal{B})}$ above should draw the reader's attention to the fact that the dual of an $(\mathcal{A},\mathcal{B})$-spectrum depends on the pair $(\mathcal{A},\mathcal{B})$, and not only on the indexing category $\mathcal{A} \wedge \mathcal{B}^{\mathrm{op}}$. However, we will only write D from now on.

Remark 3.12. If we are sloppy for the moment and ignore the derivation process, we may think of D as given by the formula

$$DX(c) = \mathrm{map}_{\mathcal{C}}(X(-), \mathcal{C}(c,-)) .$$

We have the evaluation map

$$\varepsilon_X : X \wedge^L_{\mathcal{B}} DX \cong R\mathrm{map}_{\mathcal{A}}(\mathcal{A}, X) \wedge^L_{\mathcal{B}} R\mathrm{map}_{\mathcal{A}}(X,\mathcal{A}) \xrightarrow{(\mathcal{A},\mathcal{A})} \mathcal{A} .$$

Definition 3.13. X is called *dualisable* if $(X, DX; \varepsilon_X)$ is a dual pair, i. e. if the map $(\varepsilon_X)^*_1$ from Proposition 3.2 is a bijection for all W and Z.

ε_X has the following naturality property: For every morphism $f : X \to X'$ in $\mathrm{DerMod}(\mathcal{A},\mathcal{B})$, the diagram

$$\begin{array}{ccc} X \wedge^L_{\mathcal{B}} DX' & \xrightarrow{f \wedge^L_{\mathcal{B}} \mathrm{id}} & X' \wedge^L_{\mathcal{B}} DX' \\ \Big\downarrow{\scriptstyle \mathrm{id} \wedge^L_{\mathcal{B}} Df} & & \Big\downarrow{\scriptstyle \varepsilon_{X'}} \\ X \wedge^L_{\mathcal{B}} DX & \xrightarrow{\varepsilon_X} & \mathcal{A} \end{array}$$

commutes. It follows that for all W and Z (which we consider fixed from now on),

$$(\varepsilon_X)^1_* : [W, Z \wedge^L_{\mathcal{A}} X]_{(\mathcal{C},\mathcal{B})} \to [W \wedge^L_{\mathcal{B}} DX, Z]_{(\mathcal{C},\mathcal{A})}$$

is a natural transformation.
Recall that an exact functor between triangulated categories is a functor which commutes with the shift functor and sends distinguished triangles to distinguished triangles. If $\mathscr{S}$ is a triangulated category, then $\mathscr{S}^{\mathrm{op}}$ becomes a triangulated category with shift functor the opposite of Σ^{-1}, abusively denoted by Σ^{-1} again, where a triangle

$$X \to Y \to Z \to \Sigma^{-1} X$$

is distinguished if and only if

$$\Sigma^{-1} X \to Z \to Y \to X$$

is distinguished in $\mathscr{S}$.

Lemma 3.14. *(a)* $D : (\mathrm{DerMod}(\mathcal{A},\mathcal{B}))^{\mathrm{op}} \to \mathrm{DerMod}(\mathcal{B},\mathcal{A})$ *is an exact functor.*
(b) X *is dualisable if and only if* ΣX *is.*
(c) If $X \to X' \to X'' \to \Sigma X$ *is a distinguished triangle and* X *and* X' *are dualisable, then so is* X''.

Proof. (a): By Theorem 1.16 (e),

$$D(\Sigma X) = R\mathrm{map}_{\mathcal{A}}(\mathbb{S} \wedge^L X, \mathcal{A}) \cong R\mathrm{map}(\mathbb{S}, DX) \cong \Sigma^{-1} DX\,.$$

To show that D preserves cofiber sequences, we may assume that our cofiber sequence is of the form

$$X \xrightarrow{f} Y \to Cf \to \Sigma X$$

with X and Y cofibrant and f a cofibration. By using the explicit cofibrant models and the properties of (underived) mapping spectra, cf. Section 1.2, the image of the sequence under D is identified with the sequence

$$\Omega DX \to \mathrm{hofib}(Df) \to DY \xrightarrow{Df} DX$$

which is a fiber sequence in the sense of [Hov99, Def. 6.2.6]. But fiber and cofiber sequences coincide in a stable model category by [Hov99, Thm. 7.1.11].
(b): There is a commutative diagram

$$\begin{array}{ccc} [W, Z \wedge^L_{\mathcal{A}} \Sigma X]_{(\mathcal{C},\mathcal{B})} & \xrightarrow{(\varepsilon_{\Sigma X})^1_*} & [W \wedge^L_{\mathcal{B}} D(\Sigma X), Z]_{\mathcal{C},\mathcal{A}} \\ \downarrow \cong & & \downarrow \cong \\ [W, \Sigma Z \wedge^L_{\mathcal{A}} X]_{(\mathcal{C},\mathcal{B})} & \xrightarrow{(\varepsilon_X)^1_*} & [W \wedge^L_{\mathcal{B}} DX, \Sigma Z]_{\mathcal{C},\mathcal{A}} \end{array}$$

where the vertical arrows are the isomorphisms from Theorem 1.16 (b) and (c), and the right one uses in addition the isomorphisms $\Omega E \cong \Sigma^{-1}\mathbb{S} \wedge E$ and $R\mathrm{map}(\Sigma^{-1}\mathbb{S}, F) \cong \Sigma F$ in $\mathscr{SHC}$.
(c): Fix W and Z. Note that $Z \wedge^L_{\mathcal{A}} -$ and $W \wedge^L_{\mathcal{B}} -$ preserve distinguished triangles since they are left adjoints. By equation (1.2) on p. 4, the rows of the following ladder are exact:

$$\begin{array}{ccccccccc} [W, Z \wedge^L_{\mathcal{A}} X]_{(\mathcal{C},\mathcal{B})} & \longrightarrow & [W, Z \wedge^L_{\mathcal{A}} X']_{(\mathcal{C},\mathcal{B})} & \longrightarrow & [W, Z \wedge^L_{\mathcal{A}} X'']_{(\mathcal{C},\mathcal{B})} & \longrightarrow & [W, Z \wedge^L_{\mathcal{A}} \Sigma X]_{(\mathcal{C},\mathcal{B})} & \longrightarrow & [W, Z \wedge^L_{\mathcal{A}} \Sigma X']_{(\mathcal{C},\mathcal{B})} \\ \downarrow (\varepsilon_X)^1_* & & \downarrow (\varepsilon_{X'})^1_* & & \downarrow (\varepsilon_{X''})^1_* & & \downarrow (\varepsilon_{\Sigma X})^1_* & & \downarrow (\varepsilon_{\Sigma X'})^1_* \\ [W \wedge^L_{\mathcal{B}} DX, Z]_{(\mathcal{C},\mathcal{A})} & \to & [W \wedge^L_{\mathcal{B}} DX', Z]_{(\mathcal{C},\mathcal{A})} & \to & [W \wedge^L_{\mathcal{B}} DX'', Z]_{(\mathcal{C},\mathcal{A})} & \to & [W \wedge^L_{\mathcal{B}} \Sigma DX, Z]_{(\mathcal{C},\mathcal{A})} & \to & [W \wedge^L_{\mathcal{B}} \Sigma DX', Z]_{(\mathcal{C},\mathcal{A})}\,. \end{array}$$

The statement is now deduced via the five-lemma. □

From now on, assume that

(FM) The mapping spectra of $\mathcal{B}$ are finite CW-spectra.

In our applications, $\mathcal{B}$ will always be the trivial category $*$ with mapping spectrum $\mathbb{S}$.

Lemma 3.15. *If condition (FM) holds, then every $(\mathcal{A},\mathcal{B})$-spectrum of the form $\underline{(a,b)}$ is dualisable.*

Proof. For clarity, denote by $\underline{a}$ (as usual) the covariant functor corepresented by a, and by $\underline{\underline{a}}$ the contravariant functor represented by a during this proof. We first treat the case that $\mathcal{B}$ is trivial. Note that $D\underline{a} \cong \underline{\underline{a}}$ by Lemma 1.23 and

$$\varepsilon\colon \underline{a} \wedge^L \underline{\underline{a}} \cong \underline{a} \wedge \underline{\underline{a}} \to \mathcal{A}$$

is just the composition in $\mathcal{A}$. It follows that ε^1_* is given by

$$[W, Z \wedge^L_{\mathcal{A}} \underline{a}]_{(\mathcal{C},*)} \to [W \wedge^L \underline{\underline{a}}, Z \wedge^L_{\mathcal{A}} \underline{a} \wedge^L \underline{\underline{a}}]_{(\mathcal{C},\mathcal{A})} \xrightarrow{\text{compose}} [W \wedge^L \underline{\underline{a}}, Z]_{(\mathcal{C},\mathcal{A})} .$$

Lemma 1.23 exhibits the source and the target as $[W, Z(?, a)]_{(\mathcal{C},*)}$. Here, $Z(a, ?)$ makes sense for a derived module Z because of the definition of weak equivalence. A direct check on elements (assuming that W is cofibrant and Z is fibrant) shows that the above composition is an isomorphism.
In the general case, we have

$$\underline{(a, b)} = \underline{a} \wedge \underline{\underline{b}} \cong \underline{a} \wedge^L \underline{\underline{b}} .$$

Denote by $D\underline{\underline{b}}$ the functor $R\mathrm{map}(\underline{\underline{b}}, \mathbb{S})$. This is the dual of $\underline{\underline{b}}$ viewed as a $(*, \mathcal{B})$-bimodule. This $(*, \mathcal{B})$-bimodule is dualisable by condition (FM). By Proposition 3.8 and the first part of the proof, $\underline{(a, b)}$ is dualisable with dual

$$D\underline{(a, b)} \cong D\underline{\underline{b}} \wedge^L \underline{\underline{a}} . \qquad \square$$

The following corollary summarises the last two sections and comprises Theorem B from the Introduction.

Corollary 3.16. *Suppose that condition (FM) holds. Then every finite $(\mathcal{A}, \mathcal{B})$-CW-spectrum is dualisable. Consequently, for every finite $(\mathcal{A}, \mathcal{B})$-spectrum X, any $(\mathcal{A}, \mathcal{C})$-spectrum V and any $(\mathcal{C}, \mathcal{A})$-spectrum Z, there are natural isomorphisms*

$$Z \wedge^L_{\mathcal{A}} X \cong R\mathrm{map}_{\mathcal{A}^{\mathrm{op}}}(DX, Z)$$

and

$$DX \wedge^L_{\mathcal{A}} V \cong R\mathrm{map}_{\mathcal{A}}(X, V) ;$$

in particular, there is a natural isomorphism

$$D_{(\mathcal{A}^{\mathrm{op}},\mathcal{B}^{\mathrm{op}})}(D_{(\mathcal{A},\mathcal{B})}X)^{\mathrm{op}} \cong X \tag{3.4}$$

for finite X.

Remark 3.17. It follows from the proof of Lemma 3.15 that if $\mathcal{B} = *$, then the dual of a finite $(\mathcal{A}, *)$-spectrum is a finite $(*, \mathcal{A})$-spectrum. This is false for general $\mathcal{B}$.

Remark 3.18. In practice, we will refer to (3.4) sloppily as $DDX \cong X$. The 'op' in (3.4) refers to the fact that we have to consider DX as an $(\mathcal{A}^{\mathrm{op}}, \mathcal{B}^{\mathrm{op}})$-spectrum, instead of as a $(\mathcal{B}, \mathcal{A})$-spectrum, which implies that the duality functor is taken with respect to the (contravariant) $\mathcal{A}$-variance again.

Proof of Corollary 3.16. The full subcategory of dualisable objects contains all corepresentable functors $\underline{(a,b)}$ by Lemma 3.15 and is a triangulated subcategory by Lemma 3.14 (b) and (c). Thus, it contains all finite $(\mathcal{A},\mathcal{B})$-spectra by Lemma 1.3 (c). The two isomorphisms follow from Proposition 3.9. The isomorphism $X \cong DDX$ follows from the first one by setting $\mathcal{C} = \mathcal{A}$ and $Z = \mathcal{A}$ (or from Proposition 3.7). □

In particular, D constitutes an equivalence of triangulated categories

$$\mathscr{SW}_{\mathcal{C}} \to \mathscr{SW}_{\mathcal{C}^{\mathrm{op}}}^{\mathrm{op}}$$

for an arbitrary spectrally enriched category $\mathcal{C}$ satisfying (C).

Example 3.19. Let $\mathcal{C}$ be the orbit category of finite subgroups of the integers. It has one object and automorphism group $\mathbb{Z}$. We will view $\mathcal{C}$ as a spectrally enriched category by adjoining a basepoint and smashing with $\mathbb{S}$. Let X be the $\mathbb{Z}$-space $\mathbb{R}$ with the usual translation action. This is a free, thus proper action, so it defines a $\mathcal{C}^{\mathrm{op}}$-space $X^?$ that we abusively also denote by X. We want to describe the dual of X_+ which is a $\mathcal{C}$-spectrum. Suspending once, we get a cofibre sequence

$$\Sigma\underline{\underline{x}} \xrightarrow{F} \Sigma\underline{\underline{x}} \longrightarrow \Sigma X_+ \,.$$

Here, x denotes the unique object of $\mathcal{C}$ and the map F can be described as follows: In the S^1 coordinate, it collapses the antipodal point of the base point to the base point. Then it maps the first half of the circle to the circle in the target with the same $\underline{\underline{x}}$ coordinate n, and the second half of the circle to the $(n+1)$-st circle. Dualising and rotating, we thus get a cofibre sequence

$$\Sigma^{-1}\underline{x} \xrightarrow{DF} \Sigma^{-1}\underline{x} \longrightarrow D(X_+) \,.$$

4. Homological representation theorems

Having established external Spanier-Whitehead duality, we can now prove our homology representation theorem, Theorem 4.7, via the route sketched in the Introduction. Section 4.1 first recollects some well-known information about $\mathcal{C}$-homology theories, before Section 4.2 uses results of Neeman, as well as the results of Chapter 3, to prove the main result. It has the hypothesis that $\mathscr{SW}_{\mathcal{C}^{\mathrm{op}}}$ is a countable category. For discrete $\mathcal{C}$, this turns out to be equivalent to the countability of $\mathcal{C}$ itself (up to equivalence of categories), as proved in Section 4.3.
From now on, $\mathcal{C}$ is a topological index category satisfying (C).

4.1. $\mathcal{C}$-homology theories

Let Λ be a ring, which is set by default to $\Lambda = \mathbb{Z}$ if not mentioned explicitely otherwise. Recall that a *$\mathcal{C}$-homology theory with values in Λ-modules* consists of a sequence of functors

$$h_n^{\mathcal{C}} \colon \mathrm{Fun}(\mathcal{C}, \mathrm{Top}_*) \to \mathrm{Mod}_\Lambda$$

for $n \in \mathbb{Z}$, together with natural Λ-linear isomorphisms $\sigma_n \colon h_n^{\mathcal{C}}(\Sigma X) \cong h_{n-1}^{\mathcal{C}}(X)$ such that:

- If $A \xrightarrow{f} X$ is a map of pointed $\mathcal{C}$-spaces, then the sequence

  $$h_n^{\mathcal{C}}(A) \to h_n^{\mathcal{C}}(X) \to h_n^{\mathcal{C}}(Cf)$$

  is exact.

- For a collection (X_i) of pointed $\mathcal{C}$-spaces, the canonical homomorphism

  $$\bigoplus_{i \in I} h_n^{\mathcal{C}}(X_i) \to h_n^{\mathcal{C}}\left(\bigvee_{i \in I} X_i\right)$$

  is an isomorphism.

- If $f \colon X \to Y$ is a weak equivalence of $\mathcal{C}$-spaces, then $h_n^{\mathcal{C}}(f)$ is an isomorphism for all n.

$\mathcal{C}$-cohomology theories $(h^n)_n$ are defined similarly, only that they are contravariant functors and the wedge axiom has a product instead of a sum.
If the functors $h_n^{\mathcal{C}}$ are only defined on finite $\mathcal{C}$-CW-complexes, then we call $h_*^{\mathcal{C}}$ a *homology theory on finite $\mathcal{C}$-CW-complexes*. For homology theories, this is the same

datum since the homology of a $\mathcal{C}$-CW-complex is the colimit of the homologies of its finite subcomplexes, by a telescope argument well-known from the classical setting. This is, however, *not* true for cohomology theories. In both cases however, the wedge axiom is void since it follows from the cone axiom for finite wedge sums.

Remark 4.1. There are variations in this definition which give equivalent notions of homology theories. For example, the homology theory may only be defined on pointed $\mathcal{C}$-CW-complexes, with the weak equivalence axiom left out (being void on $\mathcal{C}$-CW-complexes). Such a theory can be extended to all pointed $\mathcal{C}$-spaces via a functorial CW-approximation. Also, one might define unreduced homology theories which are functors from pairs of (unpointed) $\mathcal{C}$-spaces to abelian groups, satisfying the usual Eilenberg-Steenrod axioms. The notions of reduced and unreduced $\mathcal{C}$-homology theories are proved to be equivalent in the classical way, see [Lac16] for discrete $\mathcal{C}$. All combinations of these two variations occur in the literature.

Recall the notion of a (co-)homological functor on a triangulated category from [Nee01, Def. 1.1.7, Rem. 1.1.9].

Lemma 4.2. *A (co-)homology theory on finite pointed $\mathcal{C}$-CW-complexes with values in Λ-modules is the same datum as a (co-)homological functor with target* Mod_Λ *on the triangulated category $\mathscr{SW}_{\mathcal{C}}$.*

Proof. We use the description of $\mathscr{SW}_{\mathcal{C}}$ given in Lemma 1.3. If H is a homological functor, then defining

$$h_n^{\mathcal{C}}(X) = H(\Sigma^{-n}\Sigma^\infty X)$$

together with the obvious suspension isomorphisms yields a homology theory on finite $\mathcal{C}$-CW-complexes. Conversely, if $h_*^{\mathcal{C}}$ is such a theory, then Lemma 1.3 shows that

$$H(\Sigma^N\Sigma^\infty X) = h_{-N}^{\mathcal{C}}(X)$$

defines a functor on $\mathscr{SW}_{\mathcal{C}}$. The short exact cofibre sequence can be turned into a long exact sequence by the usual rotation method, showing that H is a homological functor. It is obvious that these two constructions are inverse to each other. □

The following construction is classical [DL98, Lemma 4.2]:

Lemma 4.3. *Let $E\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}^O$ be a functor. Then*

$$h_n^{\mathcal{C}}(X;E) = \pi_n(E \wedge_{\mathcal{C}}^L \Sigma^\infty X) \tag{4.1}$$

defines a $\mathcal{C}$-homology theory.

Remark 4.4. Strictly speaking, in the right-hand side of the above equation, $\pi_n(-)$ should be $[\Sigma^n\mathbb{S}, -]_{\mathscr{SHC}}$. This coincides with the well-known colimit definition for orthogonal spectra, but not for (all) symmetric spectra, cf. [HSS00, p. 61].

4.2. The homology representation theorem

Our main result, Theorem 4.7, which is Theorem A from the Introduction, can be seen as a converse to Lemma 4.3. It shows that every homology theory can be obtained by this construction, in case $\mathscr{SW}_{\mathcal{C}^{\mathrm{op}}}$ is countable in the following sense.

Definition 4.5. A category is called *countable* if it has countably many objects and morphisms.

Remark 4.6. All our results also apply to categories which are equivalent to countable categories. We decided to require that they *are* countable to keep the exposition simple.

Theorem 4.7. *Suppose that $\mathscr{SW}_{\mathcal{C}^{\mathrm{op}}}$ is countable. Let $h_*^{\mathcal{C}}$ be any $\mathcal{C}$-homology theory. Then there is a $\mathcal{C}^{\mathrm{op}}$-spectrum E and a natural isomorphism*

$$h_*^{\mathcal{C}}(-) \cong h_*^{\mathcal{C}}(-;E)\,.$$

Moreover, every morphism of homology theories

$$h_*^{\mathcal{C}}(-;E) \longrightarrow h_*^{\mathcal{C}}(-;E')$$

is induced by a morphism $E \longrightarrow E'$ in the derived category $\mathscr{SHC}_{\mathcal{C}^{\mathrm{op}}}$.

Remark 4.8. We will prove in Section 4.3 below that if $\mathcal{C}$ is discrete, then the countability of $\mathscr{SW}_{\mathcal{C}^{\mathrm{op}}}$ is equivalent to the countability of $\mathcal{C}$.

Remark 4.9. We want to point out that there is an almost invisible difference between Theorem 4.7 and Theorem A from the Introduction. Indeed, we are dealing with an arbitrary topological category $\mathcal{C}$ satisfying (C) here, whereas we restricted to discrete $\mathcal{C}$ for simplicity in the Introduction.

The morphism in the last statement of Theorem 4.7 is in general not unique, already in the case $\mathcal{C} = *$, due to the existence of phantoms. The proof of the theorem is based on the following two theorems from [Nee97]:

Theorem 4.10 [Nee97, Thm. 5.1]. *Let $\mathscr{S}$ be a countable triangulated category. Then the objects of projective dimension ≤ 1 in $\mathrm{Fun}(\mathscr{S}^{\mathrm{op}}, \mathrm{Ab})$ are exactly the homological functors $\mathscr{S}^{\mathrm{op}} \to \mathrm{Ab}$.*

We cite a second theorem from the same paper. The version in which we state it here seems to be slightly stronger, but the same proofs apply in our case.
In detail: Let $\mathscr{T}$ be a triangulated category with arbitrary small coproducts, and denote by $\mathscr{S}$ a triangulated subcategory which

- is essentially small,
- generates $\mathscr{T}$ [Nee97, Def. 2.5],
- consists of compact objects [Nee97, Def. 2.2].

Neeman insists on $\mathscr{S}$ being the category $\mathscr{T}^c$ of *all* compact objects (and he requires this subcategory to have the other two properties), but this is not really needed.

Theorem 4.11 [Nee97, Prop. 4.11]. *If every homological functor $H\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Ab}$ has projective dimension ≤ 1 as an object of $\mathrm{Fun}(\mathscr{S}^{\mathrm{op}}, \mathrm{Ab})$, then the pair $(\mathscr{T}, \mathscr{S})$ satisfies Brown representability in the sense that the following two assertions hold:*

1. *Every homological functor $H\colon \mathscr{S}^{\mathrm{op}} \to \mathrm{Ab}$ is naturally isomorphic to a restriction*
$$H(-) \cong \mathscr{T}(-, X) \restriction_{\mathscr{S}}$$
for some object X of $\mathscr{T}$.
2. *Given any natural transformation of functors on $\mathscr{S}^{\mathrm{op}}$*
$$\mathscr{T}(-, X) \restriction_{\mathscr{S}} \to \mathscr{T}(-, Y) \restriction_{\mathscr{S}},$$
there is a morphism $f\colon X \to Y$ in $\mathscr{T}$ inducing the natural transformation. The map f is in general not unique.

We apply the two theorems to $\mathscr{T} = \mathscr{SHC}_{\mathcal{C}^{\mathrm{op}}}$ and $\mathscr{S} = \mathscr{SW}_{\mathcal{C}^{\mathrm{op}}}$. The generation and compactness hypotheses are trivial.

Proof. Let H be the homological functor on $\mathscr{SW}_{\mathcal{C}}$ corresponding to (the restriction of) $h_*^{\mathcal{C}}$ by Lemma 4.2. Since D is exact by Lemma 3.14 (a), we can define a homological functor G on $\mathscr{SW}_{\mathcal{C}^{\mathrm{op}}}^{\mathrm{op}}$ by
$$G(Y) = H(DY).$$
By Theorems 4.10 and 4.11, there is a fibrant and cofibrant $\mathcal{C}^{\mathrm{op}}$-spectrum E that represents G. We thus have natural isomorphisms
$$\begin{aligned} h_n^{\mathcal{C}}(X) &\cong H(\Sigma^{-n}\Sigma^\infty X) \cong G(D(\Sigma^{-n}\Sigma^\infty X)) \cong [D(\Sigma^{-n}\Sigma^\infty X), E]_{\mathcal{C}^{\mathrm{op}}} \\ &\overset{(\eta_X)_*^1}{\cong} [\Sigma^n \mathbb{S}, E \wedge_{\mathcal{C}}^L \Sigma^\infty X] \cong \pi_n(E \wedge_{\mathcal{C}} \Sigma^\infty X). \end{aligned}$$
An arbitrary $\mathcal{C}$-CW-complex X is the colimit of its finite subcomplexes, and both homology theories commute with these colimits, so the isomorphism can be pulled over. Finally, an arbitrary $\mathcal{C}$-space can be approximated by a $\mathcal{C}$-CW-complex.
The representation of morphisms of homology theories follows analogously from part (2) of Theorem 4.11. □

4.2.1. $\mathcal{C}$-cohomology theories

A $\mathcal{C}^{\mathrm{op}}$-spectrum E defines a cohomology theory via
$$h_{\mathcal{C}}^*(Y; E) = [\Sigma^{-n}\Sigma^\infty Y, E]_{\mathscr{SHC}_{\mathcal{C}^{\mathrm{op}}}} \cong \pi_{-n}(R\mathrm{map}_{\mathcal{C}^{\mathrm{op}}}(\Sigma^\infty Y, E)).$$
If Y is a $\mathcal{C}$-CW-complex and E is fibrant, the R can be omitted. The fact that every $\mathcal{C}$-cohomology theory has this form, i. e. the generalisation of the classical Brown Representability Theorem, may be obtained by mimicking its original proof [Bár14, Lac16], or by citing a theorem of Neeman again [Nee01, Thm. 8.3.3]. Note that the cohomological case is in any way considerably easier than the homological case. It doesn't need the countability assumption.

4.2.2. Morphisms of $\mathcal{C}$-cohomology theories

These are always represented by morphisms in $\mathscr{SHC}_{\mathcal{C}^{\mathrm{op}}}$: First, replace the representing spectra E, E' by fibrant and cofibrant spectra, and restrict to cofibrant X. Then the n-th degree cohomology theory is just given by $[X, E_n]_{\mathcal{C}}$, thus we get various maps $E_n \to E'_n$ such that the obvious compatibility diagrams commute up to homotopy. Now, rewrite these diagrams using the structure maps $\Sigma E_n \to E_{n+1}$ and use that these have the homotopy extension property since E is cofibrant [MMSS01, Lemma 11.4] to strictify the diagrams inductively. (This is the argument for sequential spectra; use the arguments presented in Section 2.3 to pass to orthogonal spectra.)

4.3. Countability considerations

In practice, it may seem hard to check whether $\mathscr{SW}_{\mathcal{C}^{\mathrm{op}}}$ is countable for a given category $\mathcal{C}$. We want to argue that the contrary is the case, by proving that if $\mathcal{C}$ is discrete, this is equivalent of the countability of the category $\mathcal{C}$ itself:

Proposition 4.12. *Let $\mathcal{C}$ be a discrete category. Then $\mathscr{SW}_{\mathcal{C}}$ is equivalent to a countable category if and only if $\mathcal{C}$ is.*

Example 4.13. If G is a countable group and $\mathcal{F}$ is a family of subgroups which is countable up to conjugation in G, then $\mathrm{Or}(G, \mathcal{F})$ is countable. For instance, $\mathcal{F}$ can be the family of finite subgroups.

Example 4.14. If $G = \mathrm{SL}_n(\mathbb{Q}_p)$ and $\mathcal{COMOP}$ denotes the family of compact open subgroups, then the orbit category $\mathrm{Or}(G, \mathcal{COMOP})$ is countable. This is proved in Appendix B. What is more, the orbit category is even locally finite, i. e. has finite Hom sets. More generally, G can be any semisimple algebraic group over a locally compact nonarchimedean field.

Lemma 4.15. *Let X be a countable pointed CW-complex.*

(a) For every n, $\pi_n(X)$ is countable.

(b) Fix a map $\partial \mathrm{D}^n \to X$. Then the set $[(\mathrm{D}^n, \partial \mathrm{D}^n), X]$ of homotopy classes of maps $\mathrm{D}^n \to X$ rel $\partial \mathrm{D}^n$ is countable.

Proof. Part (a) is contained in Theorem 6.1 of [LW69]. Part (b) can be proved similarly. □

Proof of Proposition 4.12. It is obviously necessary that $\mathcal{C}$ is equivalent to a countable category: For any object c, the 0-th singular homology of $X(c) \cong R\mathrm{map}_{\mathcal{C}}(\underline{c}, X)$ is a well-defined functor H_c on $\mathscr{SW}_{\mathcal{C}}$. The composition with the Yoneda embedding,

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Fun}_{\mathrm{fin.CW}}(\mathcal{C}, \mathrm{Sp}^O) \longrightarrow \mathscr{SW}_{\mathcal{C}} \xrightarrow{(H_c)_c} \mathrm{Fun}(\mathcal{C}, \mathrm{Ab})$$

is the Yoneda embedding which is fully faithful. It follows that the composition of the first two functors sends non-isomorphic objects to non-isomorphic objects and is faithful.

For the sufficiency, it is obviously enough to show that the category of finite pointed $\mathcal{C}$-CW-complexes, with homotopy classes of maps, is countable, compare Lemma 1.3. Note that for a countable $\mathcal{C}$-CW-complex X, all $X(c)$ are themselves countable CW-complexes, because of the condition that $\mathcal{C}$ has countable morphism sets.
First, we show that there are only countably many homotopy types of objects X, via induction on the number of cells of X. There are only countably many 0-dimensional CW-complexes since $\mathrm{Ob}(\mathcal{C})$ is countable. Now, we suppose that X is given and we want to show that there are only countably many possibilities to attach one further cell. This amounts to choosing an object c (countably many choices) and a based homotopy class of an attaching map $\mathrm{S}^n_+ \wedge \underline{c} \to X$. But these are in bijection with free homotopy classes $\mathrm{S}^n \to X(c)$ which is a quotient of $\pi_n(X(c))$ and thus countable by Lemma 4.15 (a).
The countability of the morphism sets follows similarly from Lemma 4.15 (b). □

Part II.

Algebra

5. Rational $\mathcal{C}$-homology theories and Chern characters

Let E be a $\mathcal{C}^{\mathrm{op}}$-spectrum such that the corresponding $\mathcal{C}$-homology theory $h_*^{\mathcal{C}}(-;E)$ is rational, i. e. takes values in $\mathbb{Q}$-vector spaces. By plugging in corepresentable functors $\underline{c}$, it follows that all spectra $E(c)$ have rational homotopy groups for all c. Thus the natural map

$$E \to H\mathbb{Q} \wedge E$$

is a weak equivalence of $\mathcal{C}$-spectra. Note that the right-hand side is not only a functor from $\mathcal{C}$ to spectra, but to $H\mathbb{Q}$-modules. The *stable Dold-Kan correspondence*, discussed in Section 5.1, links these to chain complexes.

We have thus arrived in a purely algebraic setting. More precisely, we study modules over a certain category algebra $\mathbb{Q}\mathcal{C}$, cf. Section 5.2. One of the tools that is available here, and was not available in the case of spectra, is the Künneth spectral sequence for a tensor product of chain complexes. We use this to prove the existence of a Chern character in the case of flat coefficients, cf. Corollary 5.23. The flatness hypothesis is true in the homological case if the coefficients extend to Mackey functors, as discussed in Section 5.5.

Remark 5.1. In this chapter, we work with simplicial symmetric spectra instead of orthogonal spectra for technical reasons. The two approaches are completely equivalent, see Example 2.12.

5.1. The stable Dold-Kan correspondence

We work with the paper [Shi07] which realises the stable Dold-Kan correspondence as a zig-zag of weak monoidal Quillen equivalences (left adjoints on top)

$$H\mathbb{Q}\text{–}\mathcal{M}od \underset{U}{\overset{Z}{\rightleftarrows}} \mathrm{Sp}^{\Sigma}(s\mathrm{Vect}_{\mathbb{Q}}) \underset{\phi^* N}{\overset{L}{\leftrightarrows}} \mathrm{Sp}^{\Sigma}(\mathrm{ch}_{\mathbb{Q}}^{+}) \underset{R}{\overset{D}{\rightleftarrows}} \mathrm{Ch}_{\mathbb{Q}}\,. \tag{5.1}$$

The paper constructs these functors over a general ring R (and concentrates on $R = \mathbb{Z}$ in some parts of the exposition), but we will only need the special case $R = \mathbb{Q}$. The four model categories used here were introduced in Section 2.1. For the definition of the various functors, we refer to [Shi07]. The definitions of some of them will be recalled in the proof of Proposition 5.2. They have the special property that all right adjoints preserve all weak equivalences. This passes to the functor categories and has the consequence that no fibrant replacements are necessary when the derived functor is computed.

For any (Set-enriched) category $\mathcal{C}$, we get Quillen equivalences

$$\mathrm{Fun}(\mathcal{C}, H\mathbb{Q}-\mathcal{M}od) \underset{U_*}{\overset{Z_*}{\rightleftarrows}} \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^\Sigma(s\mathrm{Vect}_\mathbb{Q})) \underset{(\phi^*N)_*}{\overset{L_*}{\leftrightarrows}} \mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^\Sigma(\mathrm{ch}_\mathbb{Q}^+)) \underset{R_*}{\overset{D_*}{\rightleftarrows}} \mathrm{Fun}(\mathcal{C}, \mathrm{Ch}_\mathbb{Q}) \,.$$

By the discussion in Subsection 2.2.1, we thus get an equivalence of categories

$$\mathrm{Ho}(\mathrm{Fun}(\mathcal{C}, H\mathbb{Q}-\mathcal{M}od)) \underset{\Gamma}{\overset{\Phi}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Fun}(\mathcal{C}, \mathrm{Ch}_\mathbb{Q}))$$

where Φ and Γ respect derived balanced smash products and mapping spectra, and are given by

$$\Phi = D_*Q(\phi^*N)_*Z_*Q \quad \text{and} \quad \Gamma = U_*L_*QR_* \,.$$

For a based simplicial set A, let $\widetilde{\mathbb{Q}}A$ denote the simplicial $\mathbb{Q}$-vector space which is the reduced linearisation of A, i. e. it has as a basis in degree n the set of non-basepoint n-simplices $A_n \setminus \{*\}$. Furthermore, let $N\colon s\mathrm{Vect}_\mathbb{Q} \to \mathrm{ch}_\mathbb{Q}^+$ denote the normalised chain complex functor.

Proposition 5.2. *If X is a based simplicial $\mathcal{C}$-set, then there is a natural isomorphism*

$$\Phi(H\mathbb{Q} \wedge \Sigma^\infty X) \cong N\widetilde{\mathbb{Q}}X \,.$$

Proof. We may assume that X is cofibrant in the projective model structure since $N\widetilde{\mathbb{Q}}\colon \mathrm{sSet} \longrightarrow \mathrm{Ch}_\mathbb{Q}$ preserves all weak equivalences [GJ09, Prop. 2.14]. We go through the construction of Φ step by step. The first cofibrant replacement is not needed since $\Sigma^\infty X$ is a cofibrant $\mathcal{C}$-spectrum, and thus $H\mathbb{Q} \wedge \Sigma^\infty X$ is a cofibrant $\mathcal{C}$-$H\mathbb{Q}$-module. The functor Z is given by linearising and then using the canonical morphism $\mu\colon \widetilde{\mathbb{Q}}(H\mathbb{Q}) \to \widetilde{\mathbb{Q}}\mathbb{S}$ to turn the result into a $\widetilde{\mathbb{Q}}\mathbb{S}$-module again.
We thus have

$$\begin{aligned} Z_*Q(H\mathbb{Q} \wedge \Sigma^\infty X) &= \widetilde{\mathbb{Q}}\mathbb{S} \otimes_{\widetilde{\mathbb{Q}}(H\mathbb{Q})} \widetilde{\mathbb{Q}}(H\mathbb{Q} \wedge \Sigma^\infty X) \\ &\cong \widetilde{\mathbb{Q}}\mathbb{S} \otimes_{\widetilde{\mathbb{Q}}(H\mathbb{Q})} \widetilde{\mathbb{Q}}(H\mathbb{Q}) \otimes_{\widetilde{\mathbb{Q}}\mathbb{S}} \widetilde{\mathbb{Q}}(\Sigma^\infty X) \\ &\cong \widetilde{\mathbb{Q}}(\Sigma^\infty X) \,. \end{aligned}$$

Here we used that the functor $\widetilde{\mathbb{Q}}$ is strong monoidal and commutes with colimits. Note that

$$(\widetilde{\mathbb{Q}}(\Sigma^\infty X))_n \cong \widetilde{\mathbb{Q}}\mathrm{S}^n \otimes \widetilde{\mathbb{Q}}X \,,$$

which we refer to as $\widetilde{\mathbb{Q}}(\Sigma^\infty X) = \widetilde{\mathbb{Q}}\mathbb{S} \otimes \widetilde{\mathbb{Q}}X$.
Next, we apply the functor ϕ^*N objectwise. Here N is the normalised chain complex functor, which sends $\widetilde{\mathbb{Q}}(\Sigma^\infty X)$ to an $N(\widetilde{\mathbb{Q}}\mathbb{S})$-module in the category of symmetric sequences of positive chain complexes. This becomes a module over $\mathrm{Sym}(\mathbb{Q}[1])$ (i. e., a symmetric spectrum) via a ring homomorphism

$$\phi\colon \mathrm{Sym}(\mathbb{Q}[1]) \to N(\widetilde{\mathbb{Q}}\mathbb{S})$$

specified on p. 358 of [Shi07]. This ring map is not an isomorphism (it corresponds to a subdivision of a cube into simplices), but a weak equivalence, cf. the proof of Shipley's Proposition 4.4.
Next, we show that $N\widetilde{\mathbb{Q}}X$ is a cofibrant $\mathcal{C}$-chain complex. Since N is an equivalence of categories, it commutes with colimits, and so does $\widetilde{\mathbb{Q}}$. Thus the assertion follows inductively from the fact that $N\widetilde{\mathbb{Q}}(\mathrm{S}^{n-1}_+) \to N\widetilde{\mathbb{Q}}(\mathrm{D}^n_+)$ is a cofibration. The latter is readily checked since cofibrations of chain complexes over the field $\mathbb{Q}$ are just monomorphisms.
A similar inductive argument shows that $\mathrm{Sym}(\mathbb{Q}[1]) \otimes N\widetilde{\mathbb{Q}}X$ is cofibrant in $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp}^\Sigma(\mathrm{ch}^+_\mathbb{Q}))$ and that ϕ induces a weak equivalence

$$\phi \otimes \mathrm{id} \colon \mathrm{Sym}(\mathbb{Q}[1]) \otimes N\widetilde{\mathbb{Q}}X \longrightarrow N\widetilde{\mathbb{Q}}\mathbb{S} \otimes N\widetilde{\mathbb{Q}}X\,.$$

From the right-hand side we go on with the shuffle map of [SS03a, 2.7], applied levelwise:

$$\nabla \colon N\widetilde{\mathbb{Q}}\mathbb{S} \otimes N\widetilde{\mathbb{Q}}X \longrightarrow N(\widetilde{\mathbb{Q}}\mathbb{S} \otimes \widetilde{\mathbb{Q}}X)\,.$$

The shuffle map is always a quasi-isomorphism on the level of chain complexes (even a homotopy equivalence with homotopy inverse the Alexander-Whitney map), thus it induces a weak equivalence on each level. To see that it is a morphism of symmetric spectra, i. e. $\mathrm{Sym}(\mathbb{Q}[1])$-modules, it suffices to show that it is a morphism of $N\widetilde{\mathbb{Q}}\mathbb{S}$-modules. This is an easy diagrammatic check using the fact that N is a lax monoidal transformation [SS03a, p. 256]. Summarising, we have constructed a cofibrant replacement

$$\mathrm{Sym}(\mathbb{Q}[1]) \otimes N\widetilde{\mathbb{Q}}X \xrightarrow[\sim]{\nabla\circ(\phi\otimes\mathrm{id})} \phi^* N(\widetilde{\mathbb{Q}}\mathbb{S} \otimes \widetilde{\mathbb{Q}}X)\,.$$

The last step is to apply the functor D objectwise to the left-hand side. But this is objectwise just the suspension spectrum of $N\widetilde{\mathbb{Q}}X$(?), and D applied to the suspension spectrum of a chain complex yields just the chain complex itself by [Shi07, Lemma 4.6]. (Suspension spectra are denoted by F_0 in Shipley's paper.) □

5.2. Rational $\mathcal{C}$-modules and nondegenerate $\mathbb{Q}\mathcal{C}$-modules

The computation carried through in the last section leads us to considering functors from a fixed category $\mathcal{C}$ to $\mathrm{Ch}_\mathbb{Q}$. Note that such a functor is the same as a chain complex of functors from $\mathcal{C}$ to $\mathbb{Q}$-vector spaces. We now take a closer look at the additive category of these. More generally, we consider functors from $\mathcal{C}$ to Mod_k, where k is an arbitrary commutative ring and $\mathcal{C}$ may be an arbitrary k-linear category. The functor categories in this case consist of k-linear functors.

Definition 5.3. The *category algebra* $k\mathcal{C}$ of $\mathcal{C}$ over k is given by

$$\bigoplus_{c,d\in\mathcal{C}} k\mathrm{Hom}_\mathcal{C}(c,d)\,,$$

with multiplication defined by bilinear extension of the relations

$$g \cdot f = \begin{cases} gf & \text{if } g,\, f \text{ are composable} \\ 0 & \text{else} \end{cases} .$$

If $\mathcal{C}$ happens to be the free k-linear category on a (Set-enriched) category, we have the presentation

$$k\mathcal{C} \cong k\left\langle e_f \text{ for } f\colon c \to d \,\middle|\, e_g e_f = \begin{cases} e_{gf} & \text{if } g,\, f \text{ are composable} \\ 0 & \text{else} \end{cases} \right\rangle ,$$

where the angle brackets indicate that we take the quotient of the free (non-commutative) algebra over the e_f by the said relations. If $\mathcal{C}$ has only finitely many objects, the category algebra $k\mathcal{C}$ has a unit $\sum_{c \in \mathrm{Ob}(\mathcal{C})} \mathrm{id}_c$. For general (i. e. non-object-finite) $\mathcal{C}$, $k\mathcal{C}$ has only an approximate unit in the sense defined below. Recall that a net in a set S is a map $I \to S$ where I is a directed set, i. e. a partially ordered set in which any two elements have a common upper bound.

Definition 5.4. A ring S has an *approximate unit* if there is a net $(e_i)_{i \in I}$ of idempotents in S with the following two properties:

- For every $s \in S$, there is some i such that $e_i s = s = s e_i$.
- For $i \leq j$, we have $e_j e_i = e_i e_j = e_i$.

A left S-module M is called non-degenerate if $SM = M$. Equivalently, if for every $m \in M$ there is some i such that $e_i m = m$. The category of non-degenerate left S-modules and S-linear maps is denoted $\mathcal{NM}od_S$.

For our category algebra $k\mathcal{C}$, the set I consists of all finite sets of objects of $\mathcal{C}$, ordered by inclusion, and the approximate unit sends $F \in I$ to

$$e_F = \sum_{c \in F} \mathrm{id}_c \,.$$

The following result is essentially [Mit72, Thm. 7.1].

Proposition 5.5. *There is an isomorphism of additive categories*

$$\Xi\colon \quad \mathrm{Fun}(\mathcal{C}, \mathrm{Mod}_k) \longrightarrow \mathcal{NM}od_{k\mathcal{C}}$$

and a similar equivalence between contravariant functors and non-degenerate right $k\mathcal{C}$-modules. If this is also denoted by Ξ, then there are natural isomorphisms of k-modules

$$\Xi(X) \otimes_{k\mathcal{C}} \Xi(Y) \cong X \otimes_{\mathcal{C}} Y$$

for a right $\mathcal{C}$-module X and left $\mathcal{C}$-module Y, and

$$\mathrm{Hom}_{k\mathcal{C}}(\Xi(X), \Xi(Z)) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

for two right $\mathcal{C}$-modules X and Z.

Proof. The equivalence is defined as follows: If $X\colon \mathcal{C} \to \mathrm{Mod}_k$ is a functor, define

$$\Xi(X) = \bigoplus_{c \in \mathrm{Ob}(\mathcal{C})} X(c)$$

with the action of $(f : c_0 \to d_0) \in k\mathcal{C}$ on $x = (x_c)_c$ given by

$$(f \cdot x)_d = \begin{cases} X(f)(x_{c_0}) & \text{if } d = d_0 \\ 0 & \text{else.} \end{cases}$$

This yields a non-degenerate $k\mathcal{C}$-module: Every element lies in some submodule $\bigoplus_{c \in F} X(c)$, where F is a finite set of objects, and e_F acts as the identity on this submodule.

An inverse equivalence

$$\Pi\colon \quad \mathcal{NM}od_{k\mathcal{C}} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Mod}_k)$$

is constructed as follows: If M is a non-degenerate $k\mathcal{C}$-module, let

$$(\Pi(M))(c) = \mathrm{id}_c M\,.$$

A morphism $f\colon c \to d$ induces a linear map $\mathrm{id}_c M \to \mathrm{id}_d M$ since $f = \mathrm{id}_d f$.
It is easy to check that $\Pi\Xi$ equals the identity. For the other composition, note that there is a natural map

$$\Xi(\Pi(M)) = \bigoplus_{c \in \mathrm{Ob}(\mathcal{C})} \mathrm{id}_c M \to M$$

induced by the inclusions. This will be an injective $k\mathcal{C}$-linear map in general since the id_c are orthogonal idempotents. If M is non-degenerate, it is surjective. The two asserted natural isomorphisms are straightforward. □

Remark 5.6. The category

$$\mathrm{Ch}(\mathcal{NM}od_{k\mathcal{C}}) \cong \mathrm{Fun}(\mathcal{C}, \mathrm{Ch}_k)$$

can be endowed with a model structure in (at least) two ways. The first one is just the projective model structure as a functor category, coming from the projective model structure on Ch_k. The second one is the projective model structure on chain complexes over $\mathcal{NM}od_{k\mathcal{C}}$. This model structure (for abelian categories different from modules over a unital ring) is defined in [Hov01a, Sec. 3]. It has as input a set $\mathcal{M}$ of monomorphisms, for which we choose all monomorphisms of the form $0 \to k\mathcal{C} \cdot \mathrm{id}_c$ with $c \in \mathrm{Ob}(\mathcal{C})$. One can then check that the hypotheses of [Hov01a, Thm. 3.7] are satisfied and that the two model structures have the same cofibrations and the same weak equivalences, and thus coincide.

The following result was stated in the Introduction as Theorem C.

Corollary 5.7. *If E is a chain complex of right $\mathbb{Q}\mathcal{C}$-modules, then*

$$h_*^{\mathcal{C}}(X;E) = \mathrm{H}_*(E \otimes_{\mathbb{Q}\mathcal{C}} \overline{X}) \tag{5.2}$$

defines a rational reduced $\mathcal{C}$-homology theory. Here

$$\overline{X} = QN\widetilde{\mathbb{Q}}\mathrm{Sing}(X) \cong N\widetilde{\mathbb{Q}}(Q(\mathrm{Sing}(X))) \tag{5.3}$$

denotes a cofibrant replacement of $N\widetilde{\mathbb{Q}}\mathrm{Sing}(X)$ and $\mathrm{Sing}\colon \mathrm{Top}_* \to \mathrm{sSet}_*$ denotes the singular simplicial complex functor.
Conversely, if $\mathcal{C}$ is countable and $h_*^{\mathcal{C}}$ is a rational $\mathcal{C}$-homology theory, then there is a chain complex E, and a natural isomorphism of homology theories as above. Moreover, any morphism of rational homology theories is induced by a morphism in the derived category of $\mathcal{N}\mathcal{M}od_{\mathbb{Q}\mathcal{C}^{\mathrm{op}}}$.

Analogous statements hold for cohomology.

Proof. The isomorphism (5.3) comes from the fact that $N\widetilde{\mathbb{Q}}$ preserves all weak equivalences.
We have to translate the Davis-Lück construction (4.1) from orthogonal spectra to chain complexes. Let X be a $\mathcal{C}$-space and E a rational $\mathcal{C}^{\mathrm{op}}$-spectrum. We apply first the Quillen equivalence from Example 2.12 between orthogonal spectra and simplicial symmetric spectra. The derived balanced smash product $E \wedge_{\mathcal{C}}^{L} X \cong E \wedge_{\mathcal{C}}^{L} \Sigma^\infty X$ then translates into $E' \wedge_{\mathcal{C}}^{L} \Sigma^\infty(\mathrm{Sing}(X))$, where E' is obtained by turning the orthogonal spectrum into a (topological) symmetric spectrum and then applying the singular complex object- and levelwise, as explained in the mentioned example. The spectrum E' is still rational, and thus so is $E' \wedge_{\mathcal{C}}^{L} \Sigma^\infty(\mathrm{Sing}(X))$.
If Φ is the functor introduced in the beginning of Section 5.1, then we can translate further as follows:

$$\begin{aligned}\Phi(E' \wedge_{\mathcal{C}}^{L} \Sigma^\infty(\mathrm{Sing}(X))) &\cong \Phi(E' \wedge_{\mathcal{C}}^{L} H\mathbb{Q} \wedge \Sigma^\infty(\mathrm{Sing}(X))) \\ &\cong \Phi(E') \otimes_{\mathcal{C}}^{L} \Phi(H\mathbb{Q} \wedge \Sigma^\infty(\mathrm{Sing}(X))) \\ &\cong \Phi(E') \otimes_{\mathcal{C}}^{L} N\widetilde{\mathbb{Q}}\mathrm{Sing}(X) \cong Q\Phi(E') \otimes_{\mathbb{Q}\mathcal{C}} QN\widetilde{\mathbb{Q}}\mathrm{Sing}(X)\,.\end{aligned}$$

Here, the second isomorphism used the comparison theory developed in Subsection 2.2.1, in particular Proposition 2.13, the third isomorphism is Proposition 5.2, and the last isomorphism is Proposition 5.5. Any chain complex of right $\mathbb{Q}\mathcal{C}$-modules may be brought into the form $Q\Phi(E')$ by Lemma 5.9 below. We thus have proved the first part of the corollary. The second part follows by an application of Theorem 4.7. □

Remark 5.8. In the second part of Corollary 5.7, if E is cofibrant (as a chain complex over $\mathcal{N}\mathcal{M}od_{\mathbb{Q}\mathcal{C}}$), one might take $N\widetilde{\mathbb{Q}}\mathrm{Sing}(X)$ instead of its cofibrant replacement $\overline{X}$. This is due to the fact that in the aformentioned model category, tensoring with a cofibrant chain complexes preserves weak equivalences, by Lemma 5.9 below, so we get an analogue of Corollary 1.21. However, we will mainly use (5.2) in the form with $\overline{X}$ since this allows us to manipulate E.

Lemma 5.9. *If X is a cofibrant chain complex over $\mathcal{NM}od_{k\mathcal{C}}$, then tensoring with X preserves weak equivalences.*

Proof. Note that a cofibrant chain complex is degreewise projective by the argument from [Hov99, Lemma 2.3.6]. For positive chain complexes, the assertion thus follows from the Künneth spectral sequence [ML63, Thm. 12.1]. In the general case, truncate the chain complexes (the cofibrant one naively, the members of the quasi-isomorphism as in the proof of Corollary 5.23 below) and then pass to the colimit. □

5.3. Basic facts about rings with approximate unit

We recall here some basic facts about rings with approximate unit, and adapt some standard results from module theory to this setting.

Let S be a ring with approximate unit, as in Definition 5.4. The category $\mathcal{NM}od_S$ of non-degenerate left S-modules is an abelian category and thus has a meaningful notion of projective dimension and global dimension [Mit72, Sec. 9]. We refer to this dimension if we talk about the global dimension of a ring with approximate unit. If e_i is idempotent, then Se_i is projective. It follows that the category of non-degenerate S-modules has enough projectives, so that it is hereditary if and only if submodules of projectives are projective.

As in the unital case, projective non-degenerate modules are flat:

Lemma 5.10. *Let S be a ring with approximate unit.*
(a) If M is a non-degenerate left S-module, then there is a natural isomorphism of S-modules

$$S \otimes_S M \cong M \,.$$

(b) A non-degenerate left S-module P which is projective in the category of non-degenerate left S-modules is flat in the sense that $- \otimes_S P$ is an exact from non-degenerate S-modules to abelian groups.

Proof. (a) Define an S-linear map $f\colon S \otimes_S M \longrightarrow M$ by $s \otimes m \mapsto sm$. A map g (of sets) in the other direction is defined as follows: An element $m \in M$ is mapped to $e_i \otimes m$, where $i \in I$ is such that $e_i m = m$. This is well-defined: If j is another such index, choose $k \geq i, j$. Then

$$e_i \otimes m = (e_k e_i) \otimes m = e_k \otimes (e_i m) = e_k \otimes m = e_j \otimes m \,.$$

It is immediate that $f \circ g$ is the identity. For $g \circ f$, we use the fact that S has an approximate unit: Choose e_i with $e_i s = s$, then $e_i sm = sm$ and

$$g(f(s \otimes m)) = e_i \otimes (sm) = s \otimes m \,.$$

(b) A non-degenerate S-module is a quotient of a direct sum of left regular representations S. If it is projective, then it is a direct summand and hence flat by part (a). □

Remark 5.11. Part (b) implies that one can define Tor terms in the usual way via projective resolutions, which are symmetric and yield long exact Tor sequences for every short exact sequence of non-degenerate S-modules.

Let A be an S-algebra with approximate unit, i. e. A is a ring equipped with a ring homomorphism $S \to A$ such that the image of the approximate unit of S constitutes an approximate unit of A. The rest of this section is taken from [LL20]. For unital rings, the results are already discussed in [CQ95, Sec. 2]. See also [HGK07].

Definition 5.12. The (A, A)-*bimodule of differential forms of degree one* is defined as $\Omega^1_S A = A \otimes_S A/S$ with the 'Leibniz' A-bimodule structure

$$a \cdot (b \otimes [c]) \cdot d = ab \otimes [cd] - abc \otimes [d] \,.$$

This is easily checked to be a non-degenerate (A, A)-bimodule.

Definition 5.13. Let S be a ring with approximate unit, A an S-algebra, and M an (A, A)-bimodule. The abelian group of S-*derivations* $\mathrm{Der}_S(A, M)$ consists of all derivations $D\colon A \to M$ with $DS = 0$.

There is a canonical S-derivation $d\colon A \to \Omega^1_S A$ sending a to $e_i \otimes [a]$, where $e_i \in S$ is chosen such that $ae_i = e_i a = a$. This is well-defined: If e_j is another such element, we may assume $i \le j$ and get

$$e_i \otimes [a] = e_j e_i \otimes [a] = e_j \otimes [e_i a] = e_j \otimes [a] \,.$$

It is a derivation by the definition of the Leibniz bimodule structure.

Lemma 5.14. *d is a universal S-derivation, furnishing an isomorphism*

$$\mathrm{Hom}_{(A,A)}(\Omega^1_S A, M) \cong \mathrm{Der}_S(A, M) \,.$$

Proof. We have to show that for an S-derivation $D\colon A \to M$, there is a unique (A, A)-linear map $F\colon \Omega^1_S A \to M$ such that $F \circ d = D$, i. e. $F(e_i \otimes [a]) = Da$ with e_i as above. Uniqueness is clear: Let $a, b \in A$ and choose e_i such that $ae_i = e_i a = a$ and $be_i = e_i b = b$. Then

$$F(a \otimes [b]) = F(a \cdot (e_i \otimes [b])) = a \cdot F(e_i \otimes [b]) = a \cdot Db \,.$$

On the other hand, one easily checks that this furnishes a well-defined (A, A)-bilinear map $F\colon \Omega^1_S A \to M$ which satisfies $F \circ d = D$, so that we have proved existence. □

Proposition 5.15. *Let S be a ring with approximate unit and A an S-algebra with approximate unit. There is a short exact sequence*

$$0 \longrightarrow \Omega^1_S A \xrightarrow{\kappa} A \otimes_S A \xrightarrow{m} A \longrightarrow 0$$

of (A, A)-bimodules, with $m(a_0 \otimes a_1) = a_0 a_1$ and κ defined in the proof below.

Proof. We define an (A,S)-linear section ι of the multiplication map m as follows: For $a \in A$, choose e_i with $e_i a = a e_i = a$ and set $\iota(a) = a \otimes e_i$. This is well-defined: Suppose e_j is another such element. Since I is directed, we may assume that $i \leq j$. Then

$$a \otimes e_i = a \otimes e_i e_j = a e_i \otimes e_j = a \otimes e_j \,.$$

Thus, the kernel of m is identified (as (A,S)-bimodule) with the cokernel of ι via the projection $\mathrm{id}_{A \otimes_S A} - \iota m$. Since ι is given by the canonical isomorphism

$$A \cong S \otimes_S A$$

stemming from the fact that A is a non-degenerate module over S, cf. Lemma 5.10 (a), followed by the canonical morphism $S \to A$, and tensoring is right exact (over an arbitrary non-unital ring), we identify the cokernel of ι with $A \otimes_S A/S = \Omega^1_S A$. It follows that the map

$$\kappa\colon \Omega^1_S A \to A \otimes_S A$$

sending $a \otimes [b]$ to

$$(\mathrm{id} - \iota m)(a \otimes b) = a \otimes b - ab \otimes e_i \,,$$

where e_i is chosen such that $e_i ab = ab = abe_i$, is well-defined and renders the above sequence exact. Finally, one checks that κ is a morphism of (A,A)-bimodules if $\Omega^1_S A$ has the Leibniz bimodule structure. □

5.4. Chern characters

We quickly recall the notion of Bredon homology [DL98, Sec. 3]. Let M be a right $k\mathcal{C}$-module. If X is a pointed $\mathcal{C}$-CW-complex, then applying the cellular complex objectwise yields a left $k\mathcal{C}$-chain complex, and the homology of the tensor product

$$h_n^{\mathcal{C},\mathrm{Br}}(X;M) = \mathrm{H}_n(M \otimes_{k\mathcal{C}} C_*^{\mathrm{cell}}(X;\mathbb{Q}))$$

defines a $\mathcal{C}$-homology theory with values in k-modules – use a CW-approximation to extend it to arbitrary $\mathcal{C}$-spaces.

Definition 5.16. The *coefficient system* of a reduced $\mathcal{C}$-homology theory $h_*^{\mathcal{C}}$ with values in k-modules is the $\mathbb{Z}$-graded right $k\mathcal{C}$-module given by $h_n^{\mathcal{C}} = h_n^{\mathcal{C}}(\mathrm{S}^0 \wedge \underline{c})$.

The Bredon homology with respect to this coefficient system appears in the Atiyah-Hirzebruch spectral sequence

$$h_p^{\mathcal{C},\mathrm{Br}}(X; h_q^{\mathcal{C}}) \quad \Rightarrow \quad h_n^{\mathcal{C}}(X)\,, \tag{5.4}$$

which is proved in the same way as in the case $\mathcal{C} = *$ [Lac16].

Lemma 5.17. *Suppose that the right $k\mathcal{C}$-chain complex E is given by a right $k\mathcal{C}$-module $E_0 = M$ in degree 0, and $E_n = 0$ otherwise. Then there is a natural isomorphism of homology theories*

$$\mathrm{H}_*(E \otimes_{k\mathcal{C}} \overline{X}) \cong h_*^{\mathcal{C},\mathrm{Br}}(X;M)\,.$$

Proof. The coefficient system of the left-hand homology theory is given by E_k in degree k. Since this is 0 in non-zero degrees, the Atiyah-Hirzebruch spectral sequence (5.4) collapses and gives the above isomorphism. □

Remark 5.18. Alternatively to using the Atiyah-Hirzebruch spectral sequence, one could also prove Lemma 5.17 by using a zig-zag of chain complexes between the singular and the cellular chain complex which is natural (in cellular maps) and induces the isomorphism between singular and cellular homology. This then can be upgraded to $\mathcal{C}$-CW-complexes. Such a zig-zag is constructed on p. 121 of [VF04].

We now turn to the question of existence of Chern characters, which means for us that the homology theory splits into a direct sum of shifted Bredon homology theories. By plugging in suspended representable functors $\mathrm{S}^n \wedge \underline{c}$, one sees that there is only one choice for the coefficient systems in every degree, yielding the following definition.

Definition 5.19. Let $h_*^{\mathcal{C}}$ be a $\mathcal{C}$-homology theory with values in k-modules. A *Chern character for* $h_*^{\mathcal{C}}$ is an isomorphism of $\mathcal{C}$-homology theories with values in k-modules

$$h_n^{\mathcal{C}}(X) \cong \bigoplus_{s+t=*} h_s^{\mathcal{C},\mathrm{Br}}(X; h_t^{C}) \,.$$

Remark 5.20. We define Chern characters for $\mathcal{C}$-*co*homology theories in the same way, only with a direct product on the right-hand side. There seems to be no consense in the literature whether to take a product or sum here. However, in most cases cohomological Chern characters are only considered for finite $\mathcal{C}$-CW-complexes and there is no difference.

This notion of Chern character was introduced in [Lüc02], building on [BC98] and [LO01]. The name originates from the (non-equivariant) case of complex K theory, where the Chern character

$$K^0(X) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{H}^{\mathrm{even}}(X; \mathbb{Q})$$

for X a finite CW-complex is constructed using Chern classes.

Lemma 5.21. *Let $\mathcal{C}$ be countable and let M be a right $\mathbb{Q}\mathcal{C}$-chain complex. Then a Chern character exists for $h_*^{\mathcal{C}}(-; M)$ if and only if M is isomorphic to a complex with zero differentials in the derived category of* $\mathrm{Ch}(\mathcal{N}\mathcal{M}od_{\mathbb{Q}\mathcal{C}})$*. The same is true for $\mathcal{C}$-cohomology theories.*

Proof. This follows directly from the second part of Theorem 4.7 (representation of morphisms), together with the Dold-Kan correspondence described in Section 5.1 and Remark 5.6. For cohomology, representation of morphisms is shown in Subsection 4.2.2. □

Remark 5.22. In the case that M is bounded, [Ill02, Sec. 4.5, 4.6] describes how one can find out whether the condition of Lemma 5.21 holds, using a sequence of obstructions living in $\mathrm{Ext}^i_{\mathbb{Q}\mathcal{C}}(H_{p+i-1}(M), H_p(M))$ with $i \geq 2$. The exposition assumes that the ground ring has a unit, but this is not used in the argumentation.

We now discuss one approach to construct Chern characters.

Proposition 5.23. *Let $\mathcal{C}$ be countable. Suppose that $h_*^{\mathcal{C}}$ is a rational $\mathcal{C}$-homology theory with the property that all coefficient systems $h_t^{\mathcal{C}}$ are flat as right $\mathbb{Q}\mathcal{C}$-modules. Then there exists a Chern character for $h_*^{\mathcal{C}}$, which is natural in the homology theory $h_*^{\mathcal{C}}$.*

Remark 5.24. This result is similar to [Lüc02, Thm. 4.4] where the case of the proper orbit category of a discrete group is treated, with the additional assumption that the homology theory is equivariant, i. e. there are proper homology theories for all discrete groups linked via induction isomorphisms. A technical difference is that the flatness assumption is not over the orbit category itself, but over a certain category $\mathbb{Q}\mathrm{Sub}(G, \mathcal{FIN})$, whereas the homology theories are defined on $\mathrm{Or}(G, \mathcal{FIN})$-spaces as usual. Thus, our theorem does not imply Lück's theorem directly.

Remark 5.25. Taking into account Lemma 5.21, we have proved that whenever a chain complex of non-degenerate $\mathbb{Q}\mathcal{C}$-modules has flat homology, then it is isomorphic to a trivial complex in the derived category. For bounded complexes, we may also see this as follows: Using the result that over a countable ring, flat modules have projective dimension at most 1 [Sim74], we see that all higher Ext^i-groups of the homology modules, $i \geq 2$, appearing in Remark 5.22, vanish.

Proof. Start with the representation as in (5.2). First suppose that E is bounded below, say positive. We claim that $\overline{X}$ is degreewise flat. Copying the argument from the proof of [Hov99, Lemma 2.3.6] shows that $\overline{X}$ is degreewise projective in the category $\mathcal{NM}od_{\mathbb{Q}\mathcal{C}}$. Note that a fibration is still the same as a degreewise surjective map. By Lemma 5.10 (b), $\overline{X}$ is degreewise flat.
Having said this, we get a Künneth spectral sequence [ML63, Thm. 12.1]

$$E^2_{p,q} = \bigoplus_{s+t=q} \mathrm{Tor}^p_{\mathbb{Q}\mathcal{C}}(\mathrm{H}_s(E), \mathrm{H}_t(\overline{X})) \quad \Rightarrow \quad \mathrm{H}_{p+q}(E \otimes_{\mathbb{Q}\mathcal{C}} \overline{X}) \,.$$

Since the coefficients $\mathrm{H}_s(E)$ are flat, all higher Tor terms vanish and the E^2 page is concentrated on the line $p = 0$. It thus degenerates and gives an isomorphism

$$\begin{aligned} h_n^{\mathcal{C}}(X) &\cong \mathrm{H}_n(E \otimes_{\mathbb{Q}\mathcal{C}} \overline{X}) \cong \bigoplus_{s+t=n} \mathrm{H}_s(E) \otimes_{\mathbb{Q}\mathcal{C}} \mathrm{H}_t(\overline{X}) \cong \bigoplus_{s+t=n} \mathrm{H}_t(\mathrm{H}_s(E) \otimes_{\mathbb{Q}\mathcal{C}} \overline{X}) \\ &\cong \bigoplus_{s+t=n} h_t^{\mathcal{C},\mathrm{Br}}(X; \mathrm{H}_s(E)) \,, \end{aligned}$$

where we used flatness of $\mathrm{H}_s(E)$ again, and Lemma 5.17. Naturality of the Künneth spectral sequence shows directly that this isomorphism is natural in X. Naturality in the homology theory additionally needs the fact that every morphism of homology theories is induced by a morphism of chain complexes, after possibly replacing E by a fibrant and cofibrant complex, cf. Theorem 4.7.

For arbitrary E, let $\tau_k E$ denote the truncations

$$(\tau_k E)_n = \begin{cases} E_n\,, & n \geq k \\ \ker(d_k)\,, & n = k \\ 0\,, & n < k \end{cases}\,.$$

There are natural injective chain maps

$$\tau_k E \hookrightarrow E$$

inducing a homology isomorphism in all degrees $\geq k$, whereas the homology of $\tau_k E$ in degrees $< k$ is 0. In particular, $\mathrm{H}_t(\tau_k E)$ is flat for all t.
The maps above exhibit E as the colimit of the sequence

$$\tau_0 E \hookrightarrow \tau_{-1} E \hookrightarrow \tau_{-2} E \hookrightarrow \dots\,.$$

We now run the above argument with the truncations $\tau_k E$. Note that we need not assume these to be cofibrant, thanks to Lemma 5.9. The various isomorphisms

$$\mathrm{H}_n(\tau_k E \otimes_{\mathbb{Q}\mathcal{C}} \overline{X}) \cong \bigoplus_{s+t=n} \mathrm{H}_s(\mathrm{H}_t(\tau_k E) \otimes_{\mathbb{Q}\mathcal{C}} \overline{X})$$

are natural with respect to the inclusions $\tau_k E \hookrightarrow \tau_{k-1} E$ by naturality of the Künneth spectral sequence. Passing to the colimit, the right-hand side obviously gives the desired sum of Bredon homologies. The left-hand side gives $\mathrm{H}_n(E \otimes_{\mathbb{Q}\mathcal{C}} \overline{X})$ since homology commutes with filtered colimits, and so does $- \otimes_{\mathbb{Q}\mathcal{C}} \overline{X}$. □

A cohomological version can be proved in a very similar way, except that we have to restrict to finite X here since the truncation argument doesn't go through.

Proposition 5.26. *Let $\mathcal{C}$ be countable, and let $h^*_{\mathcal{C}}$ be a rational $\mathcal{C}$-cohomology theory with* projective *coefficient systems. Then there is a Chern character for the restriction $h^*_{\mathcal{C}}$ to finite X.*

Proof. The proof is analogous, using a cohomological version of Corollary 5.7 and the cohomological Künneth spectral sequence [Rot79, Thm. 11.34]

$$E_2^{p,q} = \bigoplus_{s+t=q} \mathrm{Ext}^p_{\mathbb{Q}\mathcal{C}}(\mathrm{H}_s(\overline{X}), \mathrm{H}_t(E)) \quad \Rightarrow \quad \mathrm{H}_{p+q}(\mathrm{Hom}_{\mathbb{Q}\mathcal{C}}(\overline{X}, E))\,.$$

Note that to formulate Corollary 5.7 with $\mathrm{Hom}_{\mathbb{Q}\mathcal{C}}$ instead of the derived tensor product, we also need to replace E fibrantly, since we don't have a mapping space version of Corollary 1.21 at hand. However, in $\mathrm{Ch}_{\mathbb{Q}}$ and thus in $\mathrm{Fun}(\mathcal{C}, \mathrm{Ch}_{\mathbb{Q}})$, all objects are fibrant. □

5.5. Mackey functors

In this section, $\mathcal{C}$ is the orbit category of a group G. We will show that the flatness assumption of Corollary 5.23 holds if G is finite and the coefficients can be extended to Mackey functors.
The following definitions and the explicit description of the orbit category below are well-known. The restriction that a family is closed under taking subgroups (in particular, it always contains the trivial subgroup) is sometimes dropped in the literature, but we use it in this section as well as in Section 6.2 below.

Definition 5.27. (a) Let G be a group. A family of subgroups of G is a set of subgroups of G which is non-empty and closed under conjugation and taking subgroups.
(b) The orbit category $\mathrm{Or}(G, \mathcal{F})$ has as objects the transitive G-spaces G/H for $H \in \mathcal{F}$, and as morphisms all G-linear maps.

Lemma 5.28. *Let G be an arbitrary group and $\mathcal{F}$ a family of subgroups. For $H, K \in \mathcal{F}$, there is a bijection*

$$\phi^{H,K} \colon K\backslash \mathrm{Trans}_G(H,K) \cong \mathrm{Hom}_{\mathrm{Or}(G,\mathcal{F})}(G/H, G/K),$$
$$g \mapsto \phi^{H,K}(g)$$

with

$$\mathrm{Trans}_G(H,K) = \{g \in G;\, gHg^{-1} \subseteq K\}$$

and

$$(\phi^{H,K}(g))(xH) = xg^{-1}K$$

for $x \in G$. Furthermore, for $L \in \mathcal{F}$ and $g' \in \mathrm{Trans}_G(K, L)$, we have

$$\phi^{K,L}(g') \circ \phi^{H,K}(g) = \phi^{H,L}(g'g)\,.$$

If no confusion can arise, we will only write ϕ for $\phi^{H,K}$.

Proof. This follows immediately from the fact that the objects of $\mathrm{Or}(G, \mathcal{F})$ are transitive G-spaces. □

From now on, G is finite and $\mathcal{F}$ is the family of all subgroups. Recall that a (rational) Mackey functor assigns to any subgroup H of G a $\mathbb{Q}$-vector space $M(H)$ and to any inclusion $K \subseteq H$ two homomorphisms

$$I_K^H \colon M(K) \longrightarrow M(H) \quad \text{and} \quad R_K^H \colon M(H) \longrightarrow M(K)\,,$$

called induction and restriction, and for any $g \in G$ conjugation homomorphisms

$$c_g \colon M(H) \longrightarrow M(gHg^{-1})\,.$$

These have to satisfy certain relations listed for instance in [TW95].

Let $\Omega_{\mathbb{Q}}(G)$ denote the Mackey category of G; we take [TW95, Prop. 2.2] as its definition. It is a category enriched in $\mathbb{Q}$-vector spaces which is not the free $\mathbb{Q}$-linear category on a category. Its objects the finite G-sets. By design, a Mackey functor is just a $\mathbb{Q}$-linear functor $\Omega_{\mathbb{Q}}(G) \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$. The category algebra of $\Omega_{\mathbb{Q}}(G)$ is called $\mu_{\mathbb{Q}}(G)$, the Mackey algebra.

Lemma 5.29. *There is a canonical functor $I\colon \mathrm{Or}(G) \to \Omega_{\mathbb{Q}}(G)$ defined by $I(G/H) = H$ and*

$$I(\phi(g)) = I^K_{gHg^{-1}} c_g$$

for $g \in \mathrm{Trans}_G(H, K)$.

Remark 5.30. Since I is injective on objects, it induces a ring homomorphism I on the category algebras [Xu06, Prop. 3.2.5].

Proof. Let H, K, L, g and g' be as in Lemma 5.28. Calculate:

$$\begin{aligned} I(\phi(g') \circ \phi(g)) = I(\phi(g'g)) = I^L_{g'gH(g'g)^{-1}} c_{g'g} &= I^L_{g'K(g')^{-1}} I^{g'K(g')^{-1}}_{g'gH(g'g)^{-1}} c_{g'} c_g \\ &= I^L_{g'K(g')^{-1}} c_{g'} I^K_{gHg^{-1}} c_g = I(\phi(g'))I(\phi(g)) \,. \end{aligned}$$

□

Definition 5.31. A left (or right) rational $\mathrm{Or}(G)$-module M is said to *extend to a Mackey functor* if it is of the form $I^*\widetilde{M}$ for a left (or right) $\Omega_{\mathbb{Q}}(G)$-module $\widetilde{M}$.

Proposition 5.32. *$\mu_{\mathbb{Q}}(G)$ is a projective left $\mathbb{Q}\mathrm{Or}(G)$-module.*

Remark 5.33. It is not known to us whether the corresponding statement for the right $\mathbb{Q}\mathrm{Or}(G)$-module $\mu_{\mathbb{Q}}(G)$ holds. Thus, Corollary 5.34 cannot be formulated for G-*co*homology theories.

Proof. A $\mathbb{Q}$-basis of $\mu_{\mathbb{Q}}(G)$ is given on the bottom of p. 1875 of [TW95] (cf. Prop. 3.2, 3.3). It consists of all elements

$$I^K_{gLg^{-1}} c_g R^H_L = I(\phi(g))R^H_L \,,$$

for $L \subseteq H$ and $g \in \mathrm{Trans}_G(L, K)$, up to the following identification:

$$I(\phi(g))R^H_L = I(\phi(g'))R^H_{L'} \Leftrightarrow \exists x \in H \cap (g')^{-1}Kg\colon L' = xLx^{-1} \,. \tag{5.5}$$

Let $\mathscr{P}$ denote a set of representatives of pairs (H, L) with $L \subseteq H$, modulo the relation that for fixed H, L may be conjugated by an element from H: $(H, L) \sim (H, hLh^{-1})$. Then we define an $\mathrm{Or}(G)$-linear homomorphism

$$\begin{aligned} F\colon \bigoplus_{(H,L)\in\mathscr{P}} \mathbb{Q}\mathrm{Hom}_{\mathrm{Or}(G)}(G/L, -) \otimes_{\mathbb{Q}N_G(L)} \mathbb{Q}[N_G(L)/(H \cap N_G(L))] &\longrightarrow \mu_{\mathbb{Q}}(G) \,, \\ \phi(g) \otimes n \quad &\mapsto \quad I(\phi(gn))R^H_L \,. \end{aligned}$$

We will show that F is an isomorphism, which implies the result since the left-hand side is a projective module by the semi-simplicity of all $\mathbb{Q}N_G(L)$.

To see that F is surjective, note that by the result cited above, the right-hand side has a basis of elements $I(\phi(g))R_L^H$ with $L \subseteq H$. We only have to achieve $(H, L) \in \mathscr{P}$. For this, choose $h \in H$ such that $(H, L') \in \mathscr{P}$ with $L' = hLh^{-1}$. For $g' = gh^{-1}$, we have

$$h = (g')^{-1} \cdot 1 \cdot g \in (g')^{-1}Kg \cap H$$

and thus $I(\phi(g))R_L^H = I(\phi(g'))R_{L'}^H = F(\phi(g) \otimes 1)$ by (5.5).
Next, we show that F is injective. Fix H and K and consider only morphisms from H to K. Let $\mathcal{L}$ be a set of representatives of subgroups of H up to conjugation (in H). The left-hand side has a basis consisting of all pairs $(L, \phi(g) \otimes 1)$, where $(H, L) \in \mathscr{P}$ and $g \in K\backslash \mathrm{Trans}_G(L, K)/N_G(L)$. Such an element is mapped to the element $I(\phi(gn))R_L^H$ on the right-hand side, which is part of the Thévenaz-Webb basis. Thus, we only have to show that F is injective when restricted to the basis $\{(L, \phi(g) \otimes 1)\}$. Suppose that

$$F(L, \phi(g) \otimes 1) = F(L', \phi(g') \otimes 1)\,.$$

By (5.5), there exists $x \in H \cap (g')^{-1}Kg$ such that $L' = xLx^{-1}$. In particular, L and L' are conjugate in H, i. e. $L = L'$. Then $x \in N_G(L)$. We have $g'x = kg$ for some $k \in K$ and consequently

$$\phi(g) \otimes 1 = \phi(kg) \otimes 1 = \phi(g'x) \otimes 1 = \phi(g') \otimes x = \phi(g') \otimes 1. \qquad \square$$

Corollary 5.34. *Let G be finite and h_*^G a rational G-homology theory with the property that all coefficient systems $h_t^{\mathcal{C}}$ extend to Mackey functors. Then there is a Chern character for h_*^G.*

Proof. Let $M = I^*\widetilde{M}$. By [TW90, Thm. 9.1], the Mackey algebra (over $\mathbb{Q}$) is semisimple. Thus, $\widetilde{M}$ is a projective $\mu_{\mathbb{Q}}(G)$-module and hence M is a projective, thus flat, $\mathbb{Q}\mathrm{Or}(G)$-module by Proposition 5.32. The existence of the Chern character then follows from Corollary 5.23. $\square$

Remark 5.35. A similar result was shown by Lück [Lüc02, Thm. 5.2]. His result holds for arbitrary discrete G (with $\mathcal{F}$ the family of finite subgroups), but refers to equivariant homology theories, and the Mackey condition is formulated for $\mathbb{Q}\mathrm{Sub}(G, \mathcal{FIN})$-modules, cf. Remark 5.24. Lück's definition of Mackey extension is stronger than our definition given below. Thus his examples, namely rationalised equivariant bordism (Ex. 1.4, 6.4) and the equivariant homology theories associated to rationalised algebraic K-theory and rationalised algebraic L-theory of the group ring, as well as rationalised topological K-theory of the reduced group C^*-algebra (Ex. 1.5, Sec. 8) can also serve as examples for us.

In contrast to Lück's result, the argumentation presented here breaks down for infinite G. While Proposition 5.32 still holds true in this case, it is not true any longer that $\mu_{\mathbb{Q}}(G)$ is semi-simple. We give an example showing that it is not even von Neumann regular. Recall from [Goo91] that a ring is called von Neumann regular if every module is flat, and that this is equivalent to the condition that for every ring element a, there exists a ring element x such that $axa = a$.

Example 5.36. Let $G = \mathrm{D}_\infty = \langle s, t \mid s^2 = t^2 = 1\rangle$ be the infinite dihedral group, and let $\Omega_\mathbb{Q}(G)$ and $\mu_\mathbb{Q}(G)$ be defined exactly as above (for finite groups), with the difference that the subgroups H and K are restricted to the finite subgroups of G. One can show that

$$\mathrm{Hom}_{\Omega_\mathbb{Q}(G)}(\langle s\rangle, \langle t\rangle) = \mathbb{Q}\langle\{I_1^{\langle t\rangle} g R_1^{\langle s\rangle}; g \in \langle t\rangle\backslash G/\langle s\rangle\}\rangle \,.$$

Representatives of the $(\langle t\rangle, \langle s\rangle)$-double cosets are given by $(st)^k$ for $k \in \mathbb{Z}$. Let $x_k = I_1^{\langle t\rangle}(st)^k R_1^{\langle s\rangle}$ and $y_k = I_1^{\langle s\rangle}(st)^k R_1^{\langle t\rangle}$. The y_k form a $\mathbb{Q}$-basis of the homomorphisms from $\langle t\rangle$ to $\langle s\rangle$ similarly.
Let $a = y_0 = I_1^{\langle s\rangle} R_1^{\langle t\rangle} \in \mathrm{Hom}_{\Omega_\mathbb{Q}(G)}(\langle t\rangle, \langle s\rangle)$. Compute

$$\begin{aligned} a x_k a &= I_1^{\langle s\rangle} R_1^{\langle t\rangle} I_1^{\langle t\rangle} (st)^k R_1^{\langle s\rangle} I_1^{\langle s\rangle} R_1^{\langle t\rangle} = I_1^{\langle s\rangle}(1+t)(st)^k(1+s)R_1^{\langle t\rangle} \\ &= I_1^{\langle s\rangle}((st)^k + t(st)^k + (st)^k s + t(st)^k s)R_1^{\langle t\rangle} \\ &= I_1^{\langle s\rangle}((st)^k + st(st)^k + (st)^k st + st(st)^k st)R_1^{\langle t\rangle} \\ &= y_k + 2y_{k+1} + y_{k+2}\,. \end{aligned}$$

It follows easily that the linear equation $axa = a$ has no solution. Thus, $\mu_\mathbb{Q}(\mathrm{D}_\infty)$ is not von Neumann regular.

6. Hereditary category algebras

As announced in the Introduction, this chapter is devoted to the characterisation of categories whose category algebra is hereditary. Most of the results are taken from, or applications of results from, the paper [LL20] of the author together with Liping Li. This paper in turn builds on Li's earlier work [Li11] for the finite case.
Recall that a ring is called left (right) hereditary if any submodule of a projective left (right) module is projective. For a ring with approximate unit, we generalise these notions to mean that any non-degenerate submodule of a non-degenerate projective module is projective. By a standard argument, this is equivalent to the vanishing of $\mathrm{Ext}^2(-,-)$. There is an intimate connection between hereditarity and the existence of Chern characters:

Proposition 6.1. *The following are equivalent for a countable category $\mathcal{C}$:*
(a) $\mathbb{Q}\mathcal{C}$ is right hereditary.
(b) Every *rational $\mathcal{C}$-homology theory possesses a Chern character.*

The same statement holds for left hereditarity and cohomology. In the rest of this chapter, we will present sufficient conditions for the hereditarity of the category algebras of certain EI categories.
We arrived here in a purely algebraic situation. The question of hereditarity of the category of representations of a category $\mathcal{C}$ may be formulated in the following most general framework:

- instead of considering only $\mathbb{Q}$-vectorspaces, we work over a fixed commutative ring k from now on,
- $\mathcal{C}$ may be an arbitrary small k-linear category.

This most general case is discussed in Sections 6.3 and 6.4. Slightly simplified conditions for discrete categories are discussed in Section 6.1, with the translation explained in Section 6.5.

Proof of Proposition 6.1. By Lemma 5.21, assertion (b) is equivalent to the fact that every chain complex of non-degenerate right R-modules is isomorphic to a trivial complex in the derived category.
Suppose $\mathbb{Q}\mathcal{C}$ is hereditary. Since $\mathcal{NM}od_{\mathbb{Q}\mathcal{C}}$ has enough projectives, one easily sees that any (right) chain complex is quasi-isomorphic to a degreewise projective one, and it is well-known [Kra07, Sec. 1.6] that these split over right hereditary abelian categories.
Conversely, assume that $\mathbb{Q}\mathcal{C}$ is not right hereditary. Then there are right $\mathbb{Q}\mathcal{C}$-modules M and N such that $\mathrm{Ext}^2_{\mathbb{Q}\mathcal{C}}(N, M) \neq 0$. A straightforward triangulated category

argument, explained for instance in [Ill02, Sec. 4.5, 4.6], shows how this can be used to construct a chain complex L with only nontrivial homology groups $H_0(L) \cong M$ and $H_1(L) \cong N$ which is not isomorphic to the trivial complex $M[0] \oplus N[-1]$ in the derived category. □

6.1. Formulation of the main result for discrete categories

In this section, let $\mathcal{C}$ be a small discrete EI category. This means that all Endomorphisms in $\mathcal{C}$ are Isomorphisms. We can define a preorder $\leqslant$ on the set of objects by letting $c \leqslant d$ if $\mathcal{C}(c,d)$ is nonempty. This preorder induces a partial order $\leqslant$ on the set of isomorphism classes of objects.
Given a pair of objects c and d, the morphism set $\mathcal{C}(c,d)$ is a (G_d, G_c)-biset, i. e. a left $(G_d \times G_c^{\mathrm{op}})$-set. As such, it is a disjoint union of transitive bisets

$$G_d \times G_c^{\mathrm{op}}/H_i$$

where $H_i \subseteq G_d \times G_c^{\mathrm{op}}$ is a subgroup. We call all subgroups H_i occuring like this *biset stabilisers in* $\mathcal{C}(c,d)$, and will study the following conditions on the biset stabilisers.

(A_d) For all $c < d$, all biset stabilisers in $\mathcal{C}(c,d)$ are locally $k^\times$-finite, in the sense that all their finitely generated subgroups are $k^\times$-finite.

(B_d) For all $c < d$, and any biset stabiliser H_i in $\mathcal{C}(c,d)$, $\mathrm{pr}_1(H_i)$ is $k^\times$-finite. Here pr_1 denotes the projection $G_d \times G_c^{\mathrm{op}} \to G_d$.

Here, a set X is called $k^\times$*-finite* if its cardinality is finite and invertible in k. The first condition is symmetric, but the second one is not.

Example 6.2. Let G be an infinite, locally $k^\times$-finite group, for instance $k = \mathbb{Q}$ and $G = \mathbb{Q}/\mathbb{Z}$, and let $\mathcal{C}$ be a category with two objects c and d such that $G_c = G = G_d$, and $\mathcal{C}(c,d)$ as a biset is generated by an element α such that G_d acts on $\mathcal{C}(c,d)$ freely and every element in G_c fixes every morphism in $\mathcal{C}(c,d)$, and $\mathcal{C}(c,d) = \emptyset$. Then we have $H_i = \{1\} \times G_c^{\mathrm{op}}$ for the only biset stabiliser, and $\mathcal{C}$ satisfies (A_d) and (B_d), whereas the opposite category only satisfies (A_d).

Remark 6.3. By a result of Auslander [Aus55], the left and right global dimensions of a Noetherian ring coincide. The proof can easily be adapted to rings with approximate unit via replacing the Noetherian condition by the *approximately Noetherian* condition that for every i, every left S-submodule of Se_i and every right S-submodule of e_iS is finitely generated as a left, resp. right S-module. In particular, using Theorem 6.7 below, the category algebra of the category $\mathcal{C}$ introduced in Example 6.2 is *not* approximately Noetherian.

We now consider factorisation properties of morphisms in $\mathcal{C}$. Recall that a non-invertible morphism in $\mathcal{C}$ is called *unfactorisable* if it can not be written as the composition of two non-invertible morphisms. In the following, we focus on discrete categories with the property that every non-invertible morphism can be written as a finite composition of unfactorisable morphisms.

Definition 6.4. An EI category $\mathcal{C}$ is said to have the finite factorisation property FFP if every non-invertible morphism is a composition of finitely many unfactorisables.

Example 6.5. For an arbitrary EI category $\mathcal{C}$, not every non-invertible morphism can be written as a finite composition of unfactorisable morphisms, or even worse, unfactorisable morphisms do not exist. For example, the poset $\mathbb{R}$ with the usual ordering can be viewed as a category, and in this category every morphism is either invertible or factorisable. Another example is the poset $\mathbb{N} \cup \{\infty\}$ with the usual ordering. The reader can see that unfactorisable morphisms exist, but the unique morphism from 1 to ∞ cannot be expressed as a composition of unfactorisable morphisms. We shall emphasise that there *do* exist categories with hereditary category algebra, but without the FFP, see [LL20, Example 6].

For an EI category $\mathcal{C}$ with finite factorisation property, the way to decompose a non-invertible morphism into unfactorisable morphisms is in general not unique. We now recall the unique factorisation property UFP due to Liping Li [Li11, Def. 2.7]. Here we take a slightly altered version which is appropriate for arbitrary, not necessarily skeletal, EI categories. Loosely speaking, the UFP means that the factorisation of every non-invertible morphism into unfactorisable morphisms is unique up to insertion of automorphisms of objects. Therefore, categories having the UFP are analogous to unique factorisation domains in commutative algebra.

Definition 6.6. The category $\mathcal{C}$ satisfies the *unique factorisation property (UFP)* if for any two chains

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} x_n = y$$

and

$$x = x'_0 \xrightarrow{\alpha'_1} x'_1 \xrightarrow{\alpha'_2} \dots \xrightarrow{\alpha'_{n'}} x'_{n'} = y$$

of unfactorisable morphisms α_i and α'_i which have the same composition $f\colon x \to y$, we have $n = n'$ and there are isomorphisms $h_i\colon x_i \to x'_i$ for $1 \le i \le n-1$ such that

$$h_1\alpha_1 = \alpha'_1, \quad \alpha'_n h_{n-1} = \alpha_n \quad \text{and} \quad \alpha'_i h_{i-1} = h_i\alpha_i \quad \text{for } 2 \le i \le n-1\,,$$

i. e., the following ladder diagram commutes:

$$\begin{array}{ccccccccccc} x & \xrightarrow{\alpha_1} & x_1 & \xrightarrow{\alpha_2} & x_2 & \xrightarrow{\alpha_3} & \dots & \xrightarrow{\alpha_{n-1}} & x_{n-1} & \xrightarrow{\alpha_n} & y \\ \downarrow{\scriptstyle \mathrm{id}_x} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & & & \downarrow{\scriptstyle h_{n-1}} & & \downarrow{\scriptstyle \mathrm{id}_y} \\ x & \xrightarrow{\alpha'_1} & x'_1 & \xrightarrow{\alpha'_2} & x'_2 & \xrightarrow{\alpha'_3} & \dots & \xrightarrow{\alpha'_{n-1}} & x'_{n-1} & \xrightarrow{\alpha'_n} & y\,. \end{array}$$

The following is the main result of [LL20]. We will prove the sufficiency direction at the end of Section 6.5. It follows essentially from Theorem 6.26.

Theorem 6.7. *Let k be a semisimple commutative ring and $\mathcal{C}$ a discrete EI category with the finite factorisation property FFP. The category of left $\mathcal{C}$-modules is hereditary if and only if the following conditions are satisfied:*

1. *the group ring kG_c is hereditary for every object c in $\mathcal{C}$,*
2. *$\mathcal{C}$ has the unique factorisation property UFP,*
3. *Conditions* (A_d) *and* (B_d) *hold.*

The hereditarity of group rings is characterised by a result of Dicks [Dic79] which can be seen as a precursor of Theorem 6.7:

Proposition 6.8 [Dic79, Thm. 1]. *Let k be an arbitrary ring and G a group. Then kG is hereditary if and only if at least one of the following holds:*

(H1) *k is completely reducible and G is the fundamental group of a connected graph of $k^\times$-finite groups.*

(H2) *k is (left) $\aleph_0$-Noetherian and von Neumann regular, and G is countable and locally $k^\times$-finite.*

(H3) *k is hereditary and G is $k^\times$-finite.*

The conditions on the ring k are explained on the first page of [Dic79]. For the definition of the fundamental group of a graph of groups, consult [Ser80].

Remark 6.9. If k is a field, then all conditions on k hold. Thus, kG is hereditary if and only if G is the fundamental group of a connected graph of $k^\times$-finite groups, or is countable and locally $k^\times$-finite.

Corollary 6.10 [LL20, Cor. D]. *Let k and $\mathcal{C}$ be as in Theorem 6.7. Then both the categories of left and right $\mathcal{C}$-modules are hereditary if and only if the following conditions are satisfied:*

1. *the group ring kG_c is hereditary for every object c in $\mathcal{C}$,*
2. *$\mathcal{C}$ has the UFP,*
3. *all biset stabilisers are $k^\times$-finite.*

Remark 6.11. Under the conditions of Theorem 6.7 and certain additional mild combinatorial assumptions, the paper [LL20] proves further that every projective $k\mathcal{C}$-module is a direct sum of modules which are induced up from a group ring G_c, see Theorem B. This however uses a completely different combinatorial approach to Theorem 6.7 that we don't discuss in this thesis.

Returning to our main story, we get that under the hypotheses of Theorem 6.7 for $k = \mathbb{Q}$, every rational $\mathcal{C}$-homology theory possesses a Chern character. This was stated as Theorem D in the Introduction. It generalises [Lac19, Cor. 6.5.4] from the finite to the infinite case. The same statement is true for cohomology if we require $\mathcal{C}$ itself to satisfy (A_d) and (B_d).

6.2. Application to orbit categories

Let G be a group and $\mathcal{F}$ a family of finite subgroups of G. The following result will be derived by an application of Theorem 6.7 to $\mathrm{Or}(G, \mathcal{F})$ and directly implies Theorem E from the Introduction.

Theorem 6.12. *Let k be a field, G a discrete group and $\mathcal{F}$ a family of finite subgroups. Then $k\mathrm{Or}(G, \mathcal{F})$ is hereditary if and only if*

- *G is either countable locally $k^\times$-finite or the fundamental group of a connected graph of $k^\times$-finite groups,*
- *all members of $\mathcal{F}$ are cyclic of prime power order, and their Weyl groups are finite (except possibly for the Weyl group of $\{1\}$).*

Remark 6.13. Note that if the first item is satisfied, then in both cases all finite subgroups of G are $k^\times$-finite. This refers in particular to the subgroups in $\mathcal{F}$ and their Weyl groups, except $W_G(1)$, if the second item is satisfied.

Remark 6.14. Theorem 6.12 is Theorem E of [LL20]. In [LL20, Sec. 5], similar classification results are proved for transporter categories, Quillen categories (including fusion systems as an important special case) and the subgroup category in the sense of Lück [Lüc02].

We now treat some example situations in which Theorem 6.12 applies.

Example 6.15. Suppose that $\mathcal{F} = \mathcal{FIN}$ is the family of all finite subgroups of G and that $k\mathcal{F}$ is hereditary. If G is locally $k^\times$-finite, then any two elements are contained in a finite, thus cyclic subgroup, and G is abelian. Thus, $N_G(K) = G$ for an arbitrary K and G has to be $k^\times$-finite itself and thus cyclic.
On the other hand, if G is the fundamental group of a connected graph of $k^\times$-finite groups, then one can prove that G satisfies the above items if and only if this graph has trivial edge and loop groups. Contracting a spanning tree, we see that G *is a free product, finite or infinite, of finite groups $\mathbb{Z}/p_i^{k_i}$ where p_i is a prime invertible in k, or $p_i = 1$.*

Example 6.16. The groups $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$ and $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}/2 * \mathbb{Z}/3$ have a hereditary orbit category with respect to $\mathcal{F} = \mathcal{FIN}$.

Remark 6.17. G is the fundamental group of a *finite* graph of finite groups if and only if it is virtually finitely generated free abelian [KPS73]. In this case, we give a geometric characterisation of the Weyl group condition in Appendix C. It can be summarised as follows: If F is a finite subgroup of G, then $W_G(F)$ is infinite if and only if F fixes a ray (equivalently, a line) in the Bass-Serre tree, and there is a combinatorial algorithm how to read this off from the graph of groups.

Example 6.18. The group $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$ has a nontrivial normal subgroup $\mathbb{Z}/2$ and thus the orbit category for any family containing this subgroup is not hereditary. However, the subgroup $\mathbb{Z}/3$ (canonically embedded via the second factor of the amalgam) has finite normaliser by Lemma C.1 and thus $\mathrm{Or}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{F}_3)$ is hereditary where $\mathcal{F}_3$ denotes the family of subgroups which are finite 3-groups.

We now turn to the proof of Theorem 6.12.

Lemma 6.19. $\mathrm{Or}(G,\mathcal{F})$ *is an EI category which has the FFP. For* $F \in \mathcal{F}$*, there is a group isomorphism*

$$\mathrm{Hom}_{\mathrm{Or}(G,\mathcal{F})}(G/K, G/K) \cong K\backslash N_G(K) = W_G(K)\,.$$

Proof. This follows from Lemma 5.28: If F is finite, then $gFg^{-1} \subseteq F$ implies by cardinality reasons that $gFg^{-1} = F$ and thus $g^{-1}Fg = F$. Moreover, since any noninvertible morphism strictly increases the cardinality of the finite isotropy group H, $\mathrm{Or}(G,\mathcal{F})$ has the FFP. □

Lemma 6.20. *Let* $K \in \mathcal{F}$ *with* $K \neq \{1\}$*. If Condition* (B_d) *is satisfied for the* $(W_G(K), G)$*-biset* $\mathrm{Hom}_{\mathrm{Or}(G,\mathcal{F})}(G/\{1\}, G/K)$*, then* $W_G(K)$ *is* $k^\times$*-finite.*

Proof. If we set $L = \{1\}$ in Lemma 5.28, we have $\mathrm{Trans}_G(\{1\}, K) = G$ and get the isomorphism

$$\phi^{\{1\},K}\colon K\backslash G \cong \mathrm{Hom}_{\mathrm{Or}(G,\mathcal{F})}(G/\{1\}, G/K)\,,$$

where $K\backslash G$ is furnished with the $(W_G(K), G)$-biset structure given by left and right multiplication. The biset stabiliser H_1 of $\phi^{\{1\},K}(1K)$ thus equals

$$H_1 = \{([g], g^{-1}); g \in N_G(K)\}\,,$$

so pr_1 is surjective onto $W_G(K)$. □

Proposition 6.21. *Let G be a group and $\mathcal{F}$ a family of finite subgroups. The category* $\mathrm{Or}(G,\mathcal{F})$ *satisfies the UFP if and only if $\mathcal{F}$ consists only of cyclic subgroups of prime power order (where different prime bases may occur in the same family).*

This result was already proved in [Lac19, Prop. 6.5.5], but we simplified the proof considerably. Note the formal similarity to Triantafillou's results in [Tri83].

Proof. The 'only if' part. Suppose that $\mathrm{Or}(G,\mathcal{F})$ has the UFP. Let $F \in \mathcal{F}$. Let H and K be two subgroups of F. Let

$$1 \subseteq H_1 \subseteq H_2 \subseteq \ldots H_i = H \subseteq H_{i+1} \subseteq \ldots \subseteq H_n = F$$

be a chain of subgroups such that $H_l \subseteq H_{l+1}$ is a maximal subgroup, and similarly

$$1 \subseteq K_1 \subseteq K_2 \subseteq \ldots K_j = K \subseteq K_{j+1} \subseteq \ldots \subseteq K_m = F\,.$$

Recall the bijection ϕ from Lemma 5.28. We can factor the morphism $\phi(1) \in \mathrm{Hom}_{\mathrm{Or}(G,\mathcal{F})}(G/1, G/F)$ as a product of unfactorisables in two ways: Firstly, as

$$G/1 \xrightarrow{\phi(1)} G/H_1 \xrightarrow{\phi(1)} G/H_2 \xrightarrow{\phi(1)} \ldots \xrightarrow{\phi(1)} G/H_n = G/F$$

and secondly, as

$$G/1 \xrightarrow{\phi(1)} G/K_1 \xrightarrow{\phi(1)} G/K_2 \xrightarrow{\phi(1)} \ldots \xrightarrow{\phi(1)} G/K_m = G/F\,.$$

We infer from the UFP that $m = n$ and that for all l, G/H_l and G/K_l are isomorphic in $\mathrm{Or}(G, \mathcal{F})$ via some $\phi(h)$, $h \in \mathrm{Trans}_G(H_l, K_l)$ such that $\phi(1) \circ \phi(h) = \phi(1)$, i. e., $h = 1 \in F\backslash\mathrm{Trans}_G(H_l, F)$ and $h \in F$. It follows that for two arbitrary subgroups of F, one is subconjugate to the other in F.
This implies that F has to be a p-group for some p. Indeed, suppose that two different primes p and q divide $|F|$. Then we can choose H of order p and K of order q, and neither of H and K can be subconjugate to the other.
Next, we claim that all abelian quotients of F are cyclic. Indeed, any abelian quotient Q of F inherits the property that for any two subgroups, one is subconjugate to the other. But conjugation is trivial here, so this forces Q to be cyclic: just consider an element of maximal order.
Finally, we claim that F is cyclic. We show this claim by induction over the order of F. Since F is a p-group, it has a non-trivial center C. F/C has only cyclic abelian quotients as well, and it follows that F/C is cyclic. It is an easy exercise to show that if the quotient of the group by its center is cyclic, the group has to be abelian. Thus, F is abelian and hence cyclic.
The 'if' part. Now, suppose that $\mathcal{F}$ only has cyclic members of prime power order. Given a chain

$$G/H_0 \xrightarrow{\phi(g_1)} G/H_1 \xrightarrow{\phi(g_2)} G/H_2 \ldots \xrightarrow{\phi(g_n)} G/H_n$$

of unfactorisable morphisms, we first manipulate it as follows using the equivalence relation explained in Definition 6.6: Substitute $H_1' = g_1^{-1} H_1 g_1$, $g_1' = 1$ and $g_2' = g_2 g_1$, i. e. we consider the factorisation

$$G/H_0 \xrightarrow{\phi(1)} G/H_1' \xrightarrow{\phi(g_2 g_1)} G/H_2 \xrightarrow{\phi(g_3)} G/H_3 \ldots \xrightarrow{\phi(g_n)} G/H_n$$

with the same composition as before. Repeating this step at positions 2 through $n-1$, we arrive at a chain

$$G/H_0 \xrightarrow{\phi(1)} G/H_1' \xrightarrow{\phi(1)} G/H_2' \xrightarrow{\phi(1)} G/H_3' \ldots G/H_{n-1}' \xrightarrow{\phi(g')} G/H_n$$

with composition g' modulo H_n. Since our replacement algorithm followed the definition of the UFP, we only need to compare morphisms in such a normal form. Note that since H_{n-1}' is cyclic of order a power of p, the index $[H_i' : H_{i-1}']$ is always p since the morphisms of the chain are unfactorisable. This is true for any other chain from G/H_0 to G/H_n and consequently, the length of such a chain is always n. Let

$$G/H_0 \xrightarrow{\phi(1)} G/H_1'' \xrightarrow{\phi(1)} G/H_2'' \xrightarrow{\phi(1)} G/H_3'' \ldots G/H_{n-1}'' \xrightarrow{\phi(g'')} G/H_n$$

be another chain with the same composition, i. e. $g'' = fg'$ with $f \in H_n$. This implies that

$$(g')^{-1} H_n g' = (g'')^{-1} H_n g''.$$

Thus, H_{n-1}' and H_{n-1}'' are both maximal subgroups of $(g')^{-1} H_n g'$, and since this is a cyclic group, they coincide: $H_{n-1}' = H_{n-1}''$. Similarly, $H_i' = H_i''$ for all $i \leq n-1$. Since $g'' = fg$, we get that $\phi(g') = \phi(g'')$ and thus the two chains are equal. □

Proof of Theorem 6.12. If the two items are satisfied, all automorphism groups have hereditary group rings by Remark 6.9 and $\mathrm{Or}(G, \mathcal{F})$ has the UFP by Proposition 6.21. Moreover, Conditions (A_d) and (B_d) are trivially satisfied: If $G/L < G/K$, and $L \neq \{1\}$, then K is nontrivial, so H_i is a subgroup of the $k^\times$-finite group $W_G(K) \times W_G(L)$ and thus $k^\times$-finite. If $L = \{1\}$, then the biset stabilisers are all isomorphic to $N_G(K)$: For the biset stabiliser of $\phi^{\{1\},K}(1K)$, this follows from the explicit description given in the proof of Lemma 6.20. The isomorphism is given by pr_2^{-1}. For any other element, it follows by transitivity. The group $N_G(K)$ is $k^\times$-finite since K and $W_G(K)$ are.
Now, suppose that $k\mathrm{Or}(G, \mathcal{F})$ is hereditary. Note that $G = W_G(1)$. By Remark 6.9, if $k\mathrm{Or}(G, \mathcal{F})$ is hereditary, then G is either countable locally $k^\times$-finite or the fundamental group of a connected graph of $k^\times$-finite groups. By Lemma 6.20, all Weyl groups of nontrivial members of $\mathcal{F}$ are finite. Finally, the members of $\mathcal{F}$ are cyclic of prime power order by Lemma 6.21. □

The discussion until here treated left $k\mathrm{Or}(G, \mathcal{F})$-modules. Let us comment shortly on right $k\mathrm{Or}(G, \mathcal{F})$-modules, i. e., left $k\mathrm{Or}(G, \mathcal{F})^{\mathrm{op}}$-modules. These appear when studying $(G, \mathcal{F})$-homology theories, cf. Theorem D. Condition (A_d), the UFP, and the Dicks condition for hereditarity of group rings are insensible when passing from a category to its opposite, but Condition (B_d) a priori is not – the two projections are interchanged. However, in the present case, the condition becomes stronger since the first projection (pr_2 above) is an isomorphism in the case $L = \{1\}$. We thus get directly that $N_G(K)$ is $k^\times$-finite and consequently the Weyl group $W_G(K)$. Summarising, we can prove the following in exactly the same way as Theorem 6.12:

Corollary 6.22. *Let k be a field, G a discrete group and $\mathcal{F}$ a family of finite subgroups. Then $k\mathrm{Or}(G, \mathcal{F})$ is right hereditary if and only if it is left hereditary, i. e., the conditions listed in Theorem 6.12 hold.*

Finally, we treat an example where Theorem 6.12 cannot be applied directly to $\mathrm{Or}(G, \mathcal{F})$, but we can still use the characterisation from Theorem 6.7.

Example 6.23. Let p be a prime and let G denote the locally compact totally disconnected topological group $\mathbb{Z}_p$. Let $\mathcal{F} = \mathcal{COMOP}$ be the family of compact open subgroups[1] of G. The orbit category $\mathrm{Or}(\mathbb{Z}_p, \mathcal{COMOP})$ is an EI category with the FFP since G has a well-defined finite volume function μ. The automorphism group of a compact open subgroup $F = p^k\mathbb{Z}_p$ is $W_G(F) = G/F \cong \mathbb{Z}/p^k$ and thus finite. It is $k^\times$-finite if and only if $p \in k^\times$. In this case, (A_d) and (B_d) become automatic. The UFP is always satisfied with a proof similar to the one of Proposition 6.21 since G has exactly one subgroup of any given volume of the form $p^{-k}\mu(G)$. Summarising, $k\mathrm{Or}(\mathbb{Z}_p, \mathcal{COMOP})$ is hereditary if and only if $p \in k^\times$.
In contrast, for $G = (\mathbb{Q}_p, +)$ and $\mathcal{F} = \mathcal{COMOP}$, Condition ($\mathrm{B}_d$) is never satisfied.

[1] This is not a family in the strict sense of Definition 5.27 (a), since it is not closed under taking subgroups. However, the orbit category is still a well-defined category, and Lemma 5.28 still holds. Note that for $F \in \mathcal{F}$, the quotient G/F is discrete and thus the canonical topology on $\mathrm{Or}(G, \mathcal{F})$ is the discrete topology.

6.3. Formulation of the main result for k-linear categories

In this section, let $\mathcal{C}$ be a *directed* k-linear category; that is, if $\mathcal{C}(c,d) \neq 0$ and $\mathcal{C}(d,c) \neq 0$, then c and d are isomorphic. The directedness of $\mathcal{C}$ implies that we can define a partial order on the set of isomorphism classes in $\mathcal{C}$ by writing $[c] \leq [d]$ if $\mathcal{C}(c,d) \neq 0$. As usual, we write $[c] < [d]$ if $[c] \leq [d]$ and $[c] \neq [d]$.
The endomorphism k-algebra of an object c is denoted R_c. If c and d are objects, then $\mathcal{C}(c,d)$ is an (R_d, R_c)-bimodule, i. e. a left $(R_d \otimes R_c^{\mathrm{op}})$-module, where $\otimes$ without subscript denotes $\otimes_k$. Paralleling the combinatorial conditions (A_d) and (B_d) on biset stabilisers of EI categories, we will consider the following two conditions on these bimodules:

(A) For all $c < d$, $\mathcal{C}(c,d)$ is flat as a right R_c-module.

(B) For all $c < d$ and all left R_c-modules M, the tensor product $\mathcal{C}(c,d) \otimes_{R_c} M$ is projective as a left R_d-module.

Note that Condition (B) implies in particular that $\mathcal{C}(c,d)$ is projective as a left R_d-module (take M to be the left regular representation R_c). On the other hand, if $\mathcal{C}(c,d)$ is projective as an (R_d, R_c)-bimodule, and k is semisimple, then (A) and (B) are satisfied. The category in Example 6.2 yields an example where (A) and (B) are satisfied, but $\mathcal{C}(c,d)$ is not projective as a bimodule. This is implied by the results of Section 6.5, in particular Lemmas 6.31 and 6.32.
We now introduce a special type of k-linear categories, called *free tensor categories*, which are analogues of EI categories satisfying the unique factorisation property [Li11, Def. 2.1, Def. 2.2, Prop. 2.8].

Definition 6.24. A *(directed) k-linear tensor quiver* $(X, \mathcal{U})$ consists of

- a partially ordered set X,
- a k-algebra R_x for every $x \in X$, and
- an (R_y, R_x)-bimodule $U(x,y)$ for all $x < y$.

Suppose $(X, \mathcal{U})$ is a k-linear tensor quiver. For every chain $\gamma = (x_0, \dots, x_\ell)$ in X, where $x_i < x_{i+1}$, define

$$U(\gamma) = U(x_{\ell-1}, x_\ell) \otimes_{R_{x_{\ell-1}}} U(x_{\ell-2}, x_{\ell-1}) \otimes_{R_{x_{\ell-2}}} \dots \otimes_{R_{x_1}} U(x_0, x_1)\,.$$

This is an (R_{x_ℓ}, R_{x_0})-bimodule. In particular, if γ consists of two entries $x < y$, then $U(\gamma) = U(x,y)$; if γ has a single entry x, then $U(\gamma) = R_x$; if γ is empty, then $U(\gamma) = 0$.
For a chain $\gamma = (x_0, \dots, x_\ell)$ as above, denote $\ell(\gamma) = \ell$. For two chains $\gamma = (x_0, \dots, x_\ell)$ and $\delta = (y_1, \dots, y_m)$ with $x_\ell = y_1$, let $\delta\gamma$ denote the concatenation $\delta\gamma = (x_0, \dots, x_\ell, y_2, \dots y_m)$. Then we have a canonical map

$$U(\delta) \otimes U(\gamma) \to U(\delta) \otimes_{R_{x_\ell}} U(\gamma) \cong U(\delta\gamma)\,. \tag{6.1}$$

Definition 6.25. The *free tensor category* associated to a k-linear tensor quiver $(X,\mathcal{U})$ is the k-linear category $\mathcal{T}_k(X,\mathcal{U})$ with object set X and

$$\mathcal{T}_k(X,\mathcal{U})(x,y) = \bigoplus_{\gamma\colon x\to y} U(\gamma)$$

where the sum runs over all chains $\gamma = (x, x_1, \ldots, x_{\ell-1}, y)$ in X. Composition is given by (6.1).

The theorem below is [LL20, Thm. 17]. The next section is devoted to its proof. It is explained in the end of Section 6.5 how it implies the sufficiency implication of Theorem 6.7.

Theorem 6.26. *Let $(X,\mathcal{U})$ be a tensor quiver with hereditary R_x for every $x \in X$, satisfying conditions (A) and (B). Then $\mathcal{T}_k(X,\mathcal{U})$ is hereditary.*

Remark 6.27. A very similar reasoning can provide more general sufficient conditions for the global dimension of $k\mathcal{C}$ to equal the maximum of the global dimensions of the endomorphism algebras R_c, see [LL20, Thm. A].

6.4. Cuntz-Quillen proof of Theorem 6.26

The main goal of this section is to prove Theorem 6.26, relying on techniques from [CQ95]. Throughout, $\mathcal{C} = \mathcal{T}_k(X,\mathcal{U})$ is the free tensor category over a k-linear tensor quiver $(X,\mathcal{U})$ with R_x hereditary for every $x \in X$ which satisfies Conditions (A) and (B). Let R denote the ring $\bigoplus_c R_c$. It is a hereditary ring with approximate unit.

Proposition 5.15 gives us a short exact sequence

$$0 \longrightarrow \Omega^1_R\mathcal{C} \xrightarrow{\kappa} \mathcal{C} \otimes_R \mathcal{C} \xrightarrow{m} \mathcal{C} \longrightarrow 0 \tag{6.2}$$

of $(\mathcal{C},\mathcal{C})$-bimodules.

Lemma 6.28. *Let M be any left $\mathcal{C}$-module. There is an exact sequence of left $\mathcal{C}$-modules*

$$0 \longrightarrow \Omega^1_R\mathcal{C} \otimes_{\mathcal{C}} M \longrightarrow \mathcal{C} \otimes_R M \longrightarrow M \longrightarrow 0\,. \tag{6.3}$$

Proof. Tensor the exact sequence (6.2) from the right with M and note that it stays exact since the last term $\mathcal{C}$ is a flat right $\mathcal{C}$-module and thus $\mathrm{Tor}^1_{\mathcal{C}}(\mathcal{C}, M) = 0$. This uses Remark 5.11. □

Lemma 6.29. *Let N be a left R-module. Then the projective dimension of $\mathcal{C} \otimes_R N$ as a left $\mathcal{C}$-module is at most the projective dimension of N over R (which is at most 1).*

Proof. Condition (A) means that $\mathcal{C}$ is flat over R. Take a projective resolution of N over R, tensor it up to $\mathcal{C}$ and it will stay exact. □

Let $U = \bigoplus_{x<y}(x,y)$ denote the (R,R)-bimodule of unfactorisables.

Proposition 6.30. *There is an isomorphism $\Omega_1^R\mathcal{C} \cong \mathcal{C}\otimes_R U \otimes_R \mathcal{C}$ of $(\mathcal{C},\mathcal{C})$-bimodules.*

Proof. The proof is the same as in [CQ95, Prop. 2.6], using Lemma 5.14. □

Now we have collected all prerequisites to prove our main result.

Proof of Theorem 6.26. Let M be an arbitrary left $\mathcal{C}$-module. Consider the short exact sequence (6.3). The first term

$$\Omega_1^R\mathcal{C} \otimes_\mathcal{C} M \cong \mathcal{C} \otimes_R (U \otimes_R M)$$

is a projective left $\mathcal{C}$-module since $U \otimes_R M$ is a projective left R-module by (B). It follows from [Mit72, Lemma 9.1] that

$$\mathrm{p.dim}_\mathcal{C}(M) \leq \max(\mathrm{p.dim}_\mathcal{C}(\mathcal{C}\otimes_R M), 1)$$

and this implies the claim by Lemma 6.29. □

6.5. Translation between the k-linear and the discrete case

In this section we show how the hereditarity conditions introduced above for discrete and k-linear categories translate into one another and deduce Theorem 6.7 from Theorem 6.26. From now on, we assume that the ring k is semisimple.
Part (a) of the following lemma is well-known.

Lemma 6.31. *Let X be a left G-set.*
(a) kX is a projective left kG-module if and only if all stabilisers occuring in X are $k^\times$-finite.
(b) kX is a flat left kG-module if and only if all stabilisers occuring in X are locally $k^\times$-finite, i. e., all their finitely generated subgroups are $k^\times$-finite.

Proof. (a) Every G-set is a disjoint union of transitive G-sets, and a direct sum is projective if and only if each summand is. We may thus assume that $X = G/H$ for some subgroup H, and have to show that $k(G/H)$ is projective if and only if H is $k^\times$-finite. For this, consider the canonical surjection

$$\pi\colon kG \twoheadrightarrow k(G/H)\,.$$

It has a section s if and only if $k(G/H)$ is projective. If s exists, then $s([1])$ has equal entries in all left H-cosets, hence H is finite. Consequently, $\pi(s([1])$ is divisible by $|H|$ and it follows that H is $k^\times$-finite. Finally, if H is k-finite, then a section s can be defined by

$$s([gH]) = \frac{1}{|H|}\sum_{h\in H} gh\,.$$

(b) As in the proof of (a), we may assume that $X = G/H$. Suppose that H is locally $k^\times$-finite. Then H is a filtered union of $k^\times$-finite subgroups H_i, thus G/H is a filtered colimit of G/H_i and $k(G/H)$ is a filtered colimit of projectives $k(G/H_i)$ and hence flat.
Now, suppose that $k(G/H)$ is flat. For $h \in H$, consider the map of right kG-modules $kG \to kG$ given by left multiplication λ_{h-1} with $h-1$. It induces a non-injective map after tensoring with $k(G/H)$. Since $k(G/H)$ is flat, the original map has to have nontrivial kernel. Let $x = (x_g g)_{g \in G}$ be nonzero in the kernel. Let F be the finite, nonempty set of elements g with $x_g \neq 0$. Since $x = hx$, F is invariant under left multiplication with h, i. e. the subgroup generated by h acts on F. This action is free since G is a group. It follows that h has finite order, which we denote by m. Set $N(h) = 1 + h + h^2 + ... + h^{m-1} \in kG$. It is easily checked that

$$\ker(\lambda_{h-1}) = \mathrm{im}(\lambda_{N(h)})\,.$$

By exactness, this remains true after tensoring with $k(G/H)$. But then $1H$ lies in the kernel, and it follows that there exist finitely many k_i and g_i such that

$$1H = N(h) \sum_{i=1}^{n} k_i g_i H = \sum_{i=1}^{n} \sum_{j=1}^{m} k_i h^j g_i H\,.$$

In the above double sum, we now focus on those summands with $h^j g_i H = 1H$. (All other summands cancel each other out.) These satisfy $h^j g_i \in H$ and thus $g_i \in H$. It follows that also all the other $h^{j'} g_i$ lie in H and we have

$$1H = \sum_{\substack{i=1 \\ g_i \in H}}^{n} m k_i g_i H = \left(m \sum_{\substack{i=1 \\ g_i \in H}}^{n} k_i \right) \cdot 1H\,.$$

Thus, m is invertible in k and $\langle h \rangle$ is $k^\times$-finite.
For several elements $h_1, \dots, h_n$, consider similarly the map

$$kG \longrightarrow \bigoplus_{i=1}^{n} kG$$

where the i-th component is given by left multiplication with $h_i - 1$. Again, we find a nonzero x in the kernel, and this time, all h_i have to stabilise F under right multiplication, i. e. $\langle h_1, \dots, h_n \rangle$ acts on F freely and thus is a finite group. To show that it is $k^\times$-finite, run the same argument as above with the sum of all elements of the subgroup $\langle h_1, \dots, h_n \rangle$ as norm element. □

Lemma 6.32. *If $\mathcal{C}$ is a discrete EI category and k is semisimple, then (B) and* (B_d) *are equivalent.*

Proof. If (B) holds, take M to be the trivial left G_c-module $k = k(G_c/G_c)$. We get that

$$k((G_d \times G_c^{\mathrm{op}})/H_i) \otimes_{kG_c} k(G_c/G_c) \cong k((G_d \times G_c^{\mathrm{op}})/H_i \times_{G_c} G_c/G_c) \cong k(G_d/\mathrm{pr}_1(H_i))$$

is a projective left kG_d-module, so $\mathrm{pr}_1(H_i)$ is $k^\times$-finite by Lemma 6.31 (a).
Conversely, suppose that (B_d) holds. Let M be an arbitrary left kG_c-module and H_i a biset stabiliser for $\mathcal{C}(c,d)$. Let $L = \mathrm{pr}_1(H_i)$. Then H_i is contained in $L \times G_c^{\mathrm{op}}$, so $(G_d \times G_c^{\mathrm{op}})/H_i$ can be written as $G_d \times_L ((L \times G_c^{\mathrm{op}})/H_i)$. It follows that

$$\mathcal{C}(c,d) \otimes_{G_c} M \cong kG_d \otimes_{kL} [k((L \times G_c^{\mathrm{op}})/H_i) \otimes_{G_c} M] \ .$$

The term in square brackets is a left kL-module which is projective since kL is semisimple by Maschke's theorem. Inducing it up to kG_d yields a projective left kG_d-module. □

Proof of the sufficiency direction of Theorem 6.7. The UFP translates into the fact that the k-linearisation of $\mathcal{C}$ is a free tensor category over the tensor quiver generated by the unfactorisable morphisms. Part (b) of Lemma 6.31 (respectively, Lemma 6.32) tells us that Condition ($\mathrm{A_d}$) (resp., Condition ($\mathrm{B_d}$)) for discrete EI categories $\mathcal{C}$ is equivalent to Condition (A) (resp., Condition (B)) for its k-linearisation. The conclusion then follows from Theorem 6.26. □

Part III.

Appendix

A. Simplicial monoidal model categories are almost very nice

In this appendix, we review some aspects of the paper [Shu06] which develops a slightly different approach to enriched homotopy theory than the one we took in Chapters 1 and 2. The paper came to the author's attention after finishing the aforementioned groundwork of this thesis. The reason why it is interesting to us is twofold:

- It could serve as an alternative groundwork for the thesis, which does the same job apart from some technical differences, under slightly stronger assumptions (which are always satisfied in practical applications, though).
- It can complement our approach and bridge some of the technical difficulties appearing in it.

We will focus on the second item. In particular, we will be able to show that a great class of simplicial monoidal model categories are almost very nice enriching categories (see p. 18), even if they are not very very nice, so that the way chosen in the main body of the text is barred. Here we wrote 'almost very nice' instead of 'very nice' since we have to impose a slight strengthening (C+) of (C) on our enriched categories $\mathcal{C}$. This explains the title of this appendix.
Concretely, Shulman's approach is based on the enriched two-sided bar construction. For a right $\mathcal{C}$-module X and a left $\mathcal{C}$-module Y, this is the simplicial object $B(X, \mathcal{C}, Y)$ (in the enriching category $\mathcal{S}$) given by the formula

$$B_n(X, \mathcal{C}, Y) = \coprod_{\alpha\colon [n] \to \mathcal{C}} X(\alpha_n) \wedge \mathcal{C}(\alpha_{n-1}, \alpha_n) \wedge \ldots \wedge \mathcal{C}(\alpha_0, \alpha_1) \wedge Y(\alpha_0)\,,$$

see [Shu06, Def. 12.1]. This two-sided bar construction is not fully homotopical, but the idea is that under certain circumstances, it is homotopical if X and Y are *objectwise cofibrant* (instead of cofibrant in the projective model structure). Since this applies to the $(\mathcal{C}, \mathcal{C})$-bimodule $\mathcal{C}$ if (C) holds, this opens a new route to prove the derived Yoneda Lemma (2.1).
We make the following assumptions to get all this to work:

- $(\mathcal{S}, \wedge, \mathbb{S})$ is a nice enriching category.
- $(\mathcal{S}, \wedge, \mathbb{S})$ is a simplicial monoidal model category, i. e. a simplicial model category [Hov99, Def. 4.2.18] such that the canonical adjunction sSet $\rightleftarrows \mathcal{S}$ given by applying tensors to the unit $\mathbb{S}$ is strong monoidal.

- As usual, our $\mathcal{S}$-enriched category $\mathcal{C}$ satisfies (C). Additionally, we claim that all unit inclusions $\mathbb{S} \to \mathcal{C}(c,c)$ are cofibrations in $\mathcal{S}$. We denote this stronger condition by (C+).

This is not the most general framework in which Shulman's approach works. Actually, the paper [Shu06] is written in a much more general context dealing with homotopical categories and deformable adjunctions instead of model categories and Quillen adjunctions, and introduces several layers of generality in which the results hold. Moreover, the paper treats functor categories $\mathrm{Fun}(\mathcal{C}, \mathscr{M})$ with $\mathscr{M}$ an arbitrary complete and cocomplete $\mathcal{S}$-enriched category. For us, always $\mathscr{M} = \mathcal{S}$.

Example A.1. All monoidal model categories discussed in Section 2.1 are simplicial. In particular, this applies to Top_* which is not very very nice, cf. Example 2.9, and we could not show to be very nice in Section 2.1.

Proposition A.2. *Suppose that $(\mathcal{S}, \wedge, \mathbb{S})$ is a simplicial monoidal model category and that $\mathcal{C}$ satisfies (C+). Then the bar construction is homotopical on objectwise cofibrant $\mathcal{C}$-modules.*

Proof. This is proved in §23 of [Shu06]. More precisely, the assumptions on $\mathcal{C}$ say that $\mathcal{C}$ is q-cofibrant in the terminology of the paper, and our proposition follows from Theorem 23.12. □

Definition A.3. We say that a monoidal model category $(\mathcal{S}, \wedge, \mathbb{S})$ is *almost very nice* if Theorem 1.16 holds when we restrict to $\mathcal{S}$-categories satisfying (C+).

Proposition A.4 [Shu06, Prop. 22.11]. *Suppose that $(\mathcal{S}, \wedge, \mathbb{S})$ satisfies the list of properties on p. 77. Then $(\mathcal{S}, \wedge, \mathbb{S})$ is almost very nice.*

Proof sketch: Under the mentioned assumptions, the objectwise cofibrant functors form a deformation retract of all functors. The reason is that cofibrant objects are objectwise cofibrant. This is discussed in [Shu06, §24] and in our Theorem 1.1. It follows that $\mathcal{C}$ is very good in the sense of Shulman. By [Shu06, Thm. 20.4], the bar construction, applied to cofibrant replacements, is a derived functor of $- \wedge_{\mathcal{C}} -$. The derived Yoneda Lemma (2.1) now follows from the fact that the $(\mathcal{C},\mathcal{C})$-bimodule $\mathcal{C}$ is objectwise cofibrant:

$$\mathcal{C} \wedge^L_{\mathcal{C}} Y \cong B(Q\mathcal{C}, \mathcal{C}, QY) \cong B(\mathcal{C}, \mathcal{C}, QY) \cong QY \cong Y\,.$$

Here the second isomorphism comes from Proposition A.2 and the third isomorphism from [Shu06, Lemma 13.5]. □

B. A geometric proof that the orbit category of $\mathrm{SL}_n(\mathbb{Q}_p)$ is locally finite

Our homology representation theorem, Theorem 4.7, has the hypothesis of countability, i. e. it can only be applied to categories with a countable skeleton and countable Hom sets. When analysing the situation of the compact-open orbit category of a p-adic special linear group $\mathrm{SL}_n(\mathbb{Q}_p)$, it turned out that it is even locally finite, i. e. has *finite* Hom sets. To prove this is the goal of the present appendix. More generally, we prove:

Theorem B.1. *Let k be a locally compact nonarchimedean field, and let G be a semisimple algebraic group over k. Let $\mathcal{COMOP}$ denote the family of compact open subgroups*[1] *of G. Then the orbit category $\mathrm{Or}(G, \mathcal{COMOP})$ is locally finite.*

Our main interest is in the case that k is a finite extension of $\mathbb{Q}_p$ for a prime p. G can be one of the well-known groups SL_n, Spin_n and Sp_n, but there are many more examples [Mil17, Ch. 24]. In contrast, our argumentation does *not* apply to $\mathrm{GL}_n(\mathbb{Q}_p)$ which is not semisimple. The existence of an infinite center makes Theorem B.1 impossible in this case.
Note that the orbits G/K for K compact open are discrete topological spaces, so we don't have to worry about a topology on the orbit category and can just treat it as a discrete category.
Our proof of Theorem B.1 is geometric, using the action of G on its Bruhat-Tits building Δ. This is a polysimplicial[2] complex whose dimension equals the rank of G and on which G acts in a nice way, with compact open stabilisers if G is semisimple. Δ has a very subtle combinatorial structure which is comprised in the building axioms [AB08]. More specifically, it is a Euclidean building, and in particular carries a CAT(0) metric preserved by G, so that Δ is a classifying space for the family of compact open subgroup by a standard argument. As an example, the Bruhat-Tits building of $\mathrm{SL}_2(\mathbb{Q}_p)$ famously is a regular $(p+1)$-valent tree [Ser80, Ch. II §1].

Remark B.2. In this remark, we give the (easy) argument that if G is as above and k is a finite extension of $\mathbb{Q}_p$, then the orbit category of G has a countable skeleton. Any compact subgroup fixes a vertex in the Bruhat-Tits building, hence is contained in a vertex stabiliser. These are all conjugate to a stabiliser of the vertex of some

[1] This is not a family in the strict sense of Definition 5.27 (a), since it is not closed under taking subgroups. However, the orbit category is still a well-defined category, and Lemma 5.28 still holds.

[2] A polysimplicial complex is built out of finite products of simplices in the same way as a simplicial complex is built out of simplices.

fundamental chamber, of which there are only finitely many. Let K be a vertex stabiliser. The compact totally disconnected group K has a countable system K_i of compact open subgroups which form a neighbourhood basis of the identity. Thus, every subgroup of K lies between some K_i and K. But for fixed i, there are only finitely many of these, since they correspond to subgroups of the finite group K/K_i. Countability of the Hom sets could be shown in the same way.

B.1. A Lemma about Buildings

Throughout, let Δ denote a thick and locally compact Euclidean building. We use Δ and its geometric realization $|\Delta|$ interchangeably. The Euclidean metric on Δ is denoted by d^E, whereas the Weyl distance function (on chambers) is denoted by δ. Consult [AB08] for the definition of all these notions.

Suppose that a topological group G acts by type-preserving simplicial isometries on Δ such that the stabiliser of a chamber is compact open and the action is Weyl-transitive. The last condition means that if $\delta(C, D) = \delta(C', D')$, then there is $g \in G$ with $gC = C'$ and $gD = D'$.

I thank Bernhard Reinke for the suggestion that a suitable notion of 2-transitivity could be used to prove Proposition B.4, leading to Lemma B.5. The proof I had before was far more complicated.

Remark B.3. A simplicial action of a topological group on a simplicial complex X has open chamber stabilisers iff it is continuous when X is given the discrete topology iff it is continuous when X is given the CW topology.

Let $x \in \Delta$ and let G_x denote the stabiliser of x. For all $y \in \Delta$, the orbit

$$G_x \cdot y \cong G_x \Big/ (G_x \cap G_y)$$

is finite since G_x is compact and $G_x \cap G_y$ is open. For $r \in \mathbb{R}_{\geq 0}$, let

$$f(r) = \min \left\{ |G_x \cdot y|;\ y \in \Delta \text{ with } d^E(x, y) = r \right\} .$$

Note that f is increasing by the uniqueness of geodesics.

Proposition B.4. *We have*

$$\lim_{r \to \infty} f(r) = \infty .$$

More specifically, there are constants $c > 0$ and $a > 1$ such that $f(r) \geq ca^r$.

Let D denote the diameter of a chamber of Δ, and call a point of Δ *generic* if it lies in the interior of a chamber. The set of generic points is open and dense in Δ.

Lemma B.5. *Let x, y be points in the thick building Δ, and assume that y is a generic point and $d^E(x, y) \geq 2D$. Then there is a generic point z with $d^E(y, z) < 3D$ such $G_x \cap G_y$ is a proper subgroup of $G_x \cap G_z$.*

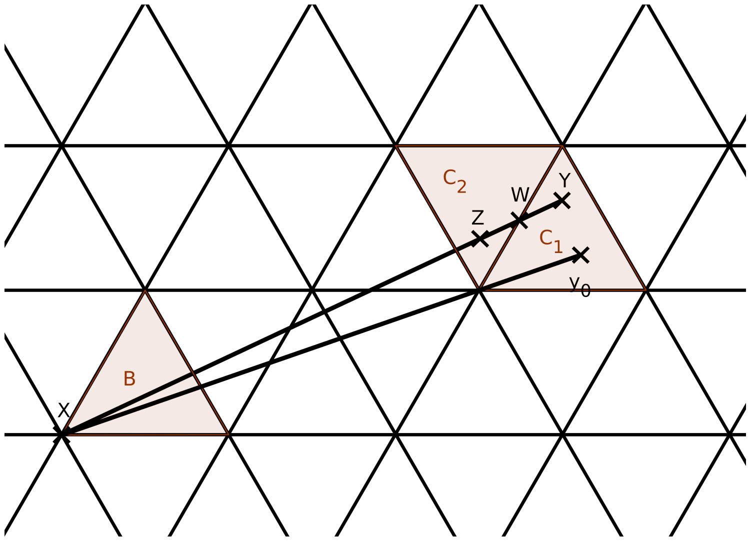

Figure B.1.: The situation of the proof of Lemma B.5. The figure shows the appartment Σ_1 (in the case of type $\widetilde{A}_2$). In the original position of y, denoted y_0, the geodesic to x didn't meet the interior of a close-by panel, so it was moved a little bit.

Proof. The geodesic $[x, y]$ intersects the boundary of a chamber in a point w with $d^E(x, y) \leq D$. Note that we may move y inside the interior of the chamber to which it belongs without changing G_y, since G permutes the chambers. Since the assertion is only about G_y, we do this once and for all (at the cost of adding a distance D) accomplishing that w lies in the interior of a panel (i. e., codimension 1 simplex) σ. Let C_1 be the chamber containing y. The geodesic $[x, y]$ intersects a chamber $C_2 \neq C_1$, which has σ as a face, on the opposite side than C_1 with respect to σ. Let z be a point in the interior of C_2 with $d^E(z, w) < D$, so that $d^E(z, y) < 3D$. Since z lies on the geodesic $[x, y]$ and since geodesics are unique, we have

$$G_x \cap G_y \subseteq G_x \cap G_z$$

and we will now show that this is a proper inclusion.
Let B be a chamber containing x in its closure. By thickness, there is a third chamber $C_3 \neq C_1, C_2$ containing σ. For every chamber C_i, there is an appartment Σ_i containing B and C_i by Axiom (B1). All of them contain x and w, thus the geodesic $[x, w]$ and thus the chamber C_2. Since every Σ_i is a Coxeter complex, the panel σ is contained in exactly two chambers in Σ_i [AB08, p. 5]. It follows that Σ_3 doesn't contain C_1 and consequently $y \notin \Sigma_3$.
By Axiom (B2) and [AB08, Prop. 4.6], there is a type-preserving simplicial isomorphism $\phi\colon \Sigma_1 \to \Sigma_3$ fixing B and C_2 pointwise. Since σ is fixed by ϕ and is contained in exactly two chambers in Σ_3, we have $\phi(C_1) = C_3$. Then $\delta(B, C_1) = \delta(B, C_3)$ since ϕ is type-preserving.

By Weyl-transitivity, there is $g \in G$ fixing B and mapping C_1 to C_3. Since g preserves types, it fixes x and $C_1 \cap C_3 = \sigma$ pointwise. In particular, it fixes w, thus the geodesic $[x, w]$ and thus z. Since $y \notin C_3$, we have $gy \neq y$. Consequently,

$$g \in (G_x \cap G_z) \setminus (G_x \cap G_y) . \qquad \square$$

Proof of Prop. B.4. Let g be defined similarly than f, but only for generic points:

$$g(r) = \min \left\{ |G_x \cdot y|;\ y \text{ generic with } d^E(x, y) = r \right\} .$$

Then for $r \geq 2D$, we have

$$g(r) \geq 2^{\lfloor \frac{r}{3D} \rfloor} .$$

This follows directly from Lemma B.5 since the index $[G_x \cap G_z : G_x \cap G_y]$ has to be at least 2. Thus the asserted inequality holds for g with $a = 2^{\frac{1}{3D}}$.
Now, let y be arbitrary with $d^E(x, y) = r$. By local compactness, y can lie in at most N chambers, where N doesn't depend on y (only on Δ). Choosing a generic point y' in one of these chambers (in distance at most D), $G_{y'}$ is a subgroup of G_y of index at most N. It follows that

$$\begin{aligned} |G_x \cdot y| &\geq \frac{1}{N} |G_x \cdot y'| \geq \frac{1}{N} g(d^E(x, y')) \\ &\geq \frac{c}{N} a^{d^E(x,y')} \geq \frac{c}{N a^D} a^r . \end{aligned} \qquad \square$$

B.2. Application to local finiteness

Let G be a topological group and Δ a simplicial model for the classifying space of the family of compact open subgroups, and assume that Δ has a metric for which it is locally compact and Proposition B.4 holds. In this section, we assume additionally that G is unimodular.

Proposition B.6. *The orbit category of compact open subgroups of G is locally finite.*

Proof. Let G/L and G/L' be objects. If there is any morphism, then L is subconjugate to L' and we may assume $L \subseteq L'$. Let K be a maximal compact subgroup (vertex stabiliser!) containing L'. Recall that

$$\mathrm{Hom}(G/L, G/L') \cong L' \backslash \mathrm{Trans}_G(L, L') .$$

Since $\mathrm{Trans}_G(L, L') \subseteq \mathrm{Trans}_G(L, K)$ and L' has finite index in K, we may assume $L' = K$. Let x be a vertex with stabiliser K. For every $g \in \mathrm{Trans}_G(L, K)$, we have

$$[G_x : G_x \cap G_{gx}] = [K : K \cap gKg^{-1}] \leq [K : K \cap gLg^{-1}] = [K : gLg^{-1}] = [K : L] ,$$

where the last equality used unimodularity. This means that gx has to lie in some ball around x by Proposition B.4. The orbit of x, which consists of vertices and is thus discrete, contains only finitely many points from this ball. Thus, there are only finitely many cosets modulo K in which g can lie. $\square$

I thank Jessica Fintzen for patiently explaining me many of the notions used in the following proof, and finding the correct generality for Theorem B.1.

Proof of Theorem B.1. The paper [Pra20] constructs the Bruhat-Tits building for k and G as in the statement of the theorem. We will now check that all hypotheses of the last two sections are satisfied.
The basic properties of the Bruhat-Tits building are listed in 1.15 and Theorem 3.8, note that $G = G'$ for semisimple G. Euclidean is called affine there. Thickness of Δ is shown in Proposition 3.7. Proposition 3.10 shows that the action is strongly transitive, which implies Weyl transitivity by [AB08, Cor. 6.12]. By [Pra20, 3.3], the action is type-preserving if G is simply connected. Every semisimple group admits a finite covering (surjective morphism with finite kernel) $\widetilde{G}$ which is simply connected and semisimple, and the statement of Theorem B.1 is obviously equivalent for G and $\widetilde{G}$. For local compactness of the building, it is enough to show that every vertex x is contained in finitely many chambers only. There is a certain group scheme $\overline{\mathscr{G}}_x^\circ$, defined over the ring of integers $\mathfrak{o}$ of k, such that the chambers containing x are in bijection with the minimal parabolic subgroups of the reductive group $\overline{\mathscr{G}}_x^\circ(\kappa)$. Here κ is the residue field which is finite, thus there are only finitely many (parabolic) subgroups. For details, consult the proof of [Pra20, Prop. 3.7]. Finally, any reductive k-group is unimodular.
Our geometric argumentation is based on the treatment of buildings in [AB08]. The building constructed in [Pra20] satisfies the definition of building given there, cf. [AB08, Rem. 4.2], except for the technical problem that Prasad allows a building to be a polysimplicial complex and not just a simplicial complex. One can sweep this problem under the rug by arguing that the theory presented in [AB08] and our proof works as well for polysimplicial complexes, or one can argue as follows: The building is a simplicial complex if G is absolutely almost-simple [Pra20, 3.2]. If we now are given an arbitrary semisimple group, we first make it simply connected as above and then apply [Mil17, Thm. 24.3] to write it as a direct product of Weil restrictions of absolutely almost-simple groups (called geometrically almost-simple there). Since a Weil restriction has the same Bruhat-Tits building as the original group, and the statement of our theorem is inherited under direct products, this yields a proof for G. □

C. Normalisers in groups acting on trees

Let $\pi = \pi_1(G, Y, P_0)$ be the fundamental group of a connected *finite* graph of $k^\times$-finite groups (G, Y). In this appendix, we analyse the condition appearing in Theorem 6.12 that a finite subgroup $F \subseteq \pi$ has finite normaliser $N_\pi(F)$. We first treat this question combinatorially in terms of the graph of groups, and then geometrically. The geometry enters by the well-known fact that π acts on the Bass-Serre tree $X = \widetilde{X}(G, Y, T)$ with finite stabilisers and finite quotient. Here T is a spanning tree of the quotient graph Y. We use the notation, constructions and main results of the famous and beautiful book [Ser80].

We now explain how to read off the normaliser of a finite subgroup $F \subseteq \pi$ from the graph of groups. By conjugating (in π) if necessary, we may assume that F fixes a vertex in the chosen lift of T to X.

Let c be a path in Y, given by edges $y_1, \dots, y_n$. We put $P_i = t(y_i) = o(y_{i+1})$. Recall that a *word of type* c is a pair (c, μ) where $\mu = (r_0, \dots, r_n)$ with $r_i \in G_{P_i}$. It is reduced if $n = 0$ and $r_0 \neq 1$, or if $n > 0$ and whenever $y_{i+1} = \overline{y_i}$, we have $r_i \notin G_{y_i}^{y_i}$. Every reduced word with c a circle is nontrivial in π, and every word in π can be written as a reduced word. If we restrict to paths starting and ending in P_0, as we do in $\pi_1(G, Y, T)$, then c is unique and μ is unique up to the equivalence relation generated by

$$(r_0, \dots, r_n) \sim (r_0, \dots, r_i a^{\overline{y_{i+1}}}, (a^{y_{i+1}})^{-1} r_{i+1}, \dots, r_n) \tag{C.1}$$

with $a \in G_{y_{i+1}}$ [Ser80, p. 50].

Let (c, μ) be a reduced word as above. For $0 \leq i \leq n$, let ${}_ic$ denote the starting segment $(y_1, \dots y_i)$ of c, and ${}_i\mu$ the starting segment $(r_0, \dots r_i)$ of μ. We view $({}_ic, {}_i\mu)$ as a reduced word centered at P_0 by going the same path backwards with trivial labels.

Lemma C.1. *Let $F \subseteq G_{P_0}$ be a finite subgroup, and let (c, μ) be a reduced word such that $|c, \mu|$ normalises F. Then*

$$|{}_ic, {}_i\mu|^{-1} F |{}_ic, {}_i\mu| \subseteq G_{y_{i+1}}^{\overline{y_{i+1}}}.$$

Here $G_{y_{i+1}}^{\overline{y_{i+1}}}$ denotes the image of $G_{\overline{y_{i+1}}} = G_{y_{i+1}}$ in $G_{t(\overline{y_{i+1}})} = G_{o(y_{i+1})} = G_{P_i}$ as usual. Intuitively, the lemma says that every element normalising a subgroup F describes a way how to move this subgroup along the graph of groups, starting in P_0 and inserting conjugations at subsequent vertices if necessary to push it into the next edge group. In the end, we arrive at P_0 again, with a subgroup conjugate to F in G_{P_0}.

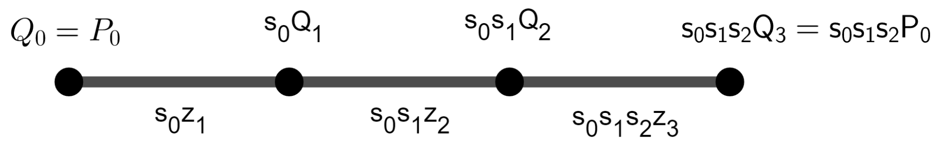

Figure C.1.: The path in X, in the case $m = 3$.

Proof. Let $g = |c, \mu|$. Then F fixes P_0 and gP_0, thus the geodesic between the two in X. We show how to use this geodesic to write g in reduced form satisfying the conjugation assertion of the lemma. By uniqueness of reduced word presentations, up to equivalence as described above, this proves the lemma.

Let $(z_1, \ldots, z_m)$ be the image of the geodesic from P_0 to gP_0 in Y, with $Q_i = t(z_i) = o(z_{i+1})$. We have $Q_0 = Q_m = P_0$. We can write the first edge of the path in X as s_0z_1 with $s_0 \in G_{P_0}$ since it is incident to P_0. The second vertex then equals s_0Q_1. The second edge can be written as $(s_0z_1s_1)z_2$ with $s_1 \in G_{Q_1}$ since it is incident to s_0Q_1. Here, $s_0z_1s_1$ is to be understood as an element of π (we supress inserting the path back to P_0 with trivial labels in this proof) which acts on the edge z_2. Inductively, one gets a description of the path as in Figure C.1.

We have

$$(s_0 \ldots s_{m-1})P_0 = (s_0 \ldots s_{m-1})Q_m = gP_0$$

and thus $g = s_0 \ldots s_{m-1}z_ms_m$ with $s_m \in G_{P_0}$. The z_i and s_i define another reduced word presentation of g. It follows that $m = n$, $z_i = y_i$ and $Q_i = P_i$. The r_i and s_i are linked via the equivalence relation generated by (C.1). But it is easy to check that the statement of the lemma is insensitive to this equivalence relation, and thus we assume $r_i = s_i$ without loss of generality.

Consequently, F fixes the edge $|_ic,_i\mu|y_{i+1}$ and is thus contained in its stabiliser, which yields the claim of the lemma. □

Lemma C.1 can be used to check whether the normaliser $N_\pi(F)$ of F is infinite, for example in conjunction with the equivalence of the first two items of Proposition C.4 below.

Example C.2. Let Y consist of an edge y, with vertex groups A and B and edge group C. We identify C with both its images in A and B. Let $F \subseteq A$.

We can draw the following conclusions from the above lemma:

- If F is not subconjugate to C in A, then $N_\pi(F) = N_A(F)$.
- If $F \subseteq C$ and there exist $a \in N_A(F) \setminus C$ and $b \in N_B(F) \setminus C$, then $N_\pi(F)$ contains the element $1yb\overline{y}a$ of infinite order.

We emphasise that the second condition is *not* necessary for the infinity of the normaliser. For example, let $A = B$ equal the dihedral group

$$D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = (\sigma\tau)^2 \rangle$$

and let

$$C = \langle \tau, \sigma^2\tau \rangle = \{1, \tau, \sigma^2\tau, \sigma^2\} .$$

Let F be the 2-element subgroup generated by τ. Then $N_B(F) = C$, so the second item cannot be satisfied. However, the normaliser of F contains the infinite order element given by the reduced word $1y\sigma\overline{y}\sigma^{-1}$.
What happens here is that $F = \langle \tau \rangle$ is conjugated in the second step into $\langle \sigma^2\tau \rangle$ which still lies in C, and then back into F in the third step. Similarly, there can be situations when a chain of length two doesn't suffice, but length 3 or higher is necessary.

Example C.3. Let Y consist of a loop t, with vertex group A and loop group C. As usual, we identify C with one of its images in A and denote the other by $\iota(C)$. We can draw the following conclusions from the above lemma:

- If F is neither subconjugate to C nor to $\iota(C)$ in A, then $N_\pi(F) = N_A(F)$.
- If $F \subseteq C$ and there exists $a \in \mathrm{Trans}_A(\iota(F), F)$, then $N_\pi(F)$ contains the element $1ta$ of infinite order.

Again, the second condition is not the only way to produce an infinite order element: Let $A = C$ equal the Klein 4-group $\{1, a, b, c\}$, and let $\iota \colon C \to C$ be the order 3 automorphism mapping a to b, b to c and c to a. Then $F = \langle a \rangle$ cannot be normalised by going around the loop once – note that all conjugations in A are trivial –, but the (infinite order) element $t^3 = 1t1t1t1$ normalises F.

A *ray* in X is a geodesic embedding of the metric space $[0, \infty)$ (with the standard metric) into X, and a *line* in X is a geodesic embedding $\mathbb{R} \hookrightarrow X$. The existence of a CAT(0) metric on X equips it with a Gromov boundary. The underlying set can be described as the set of all rays emanating from a fixed point p [BH99, Lemma III.H.3.1]. Since X is a tree, there is no need of an equivalence relation on the rays. Topologically, ∂X is a Cantor space.

Proposition C.4. *Let F be a finite subgroup of π. Then the following are equivalent:*

(i) $N_\pi(F)$ *is infinite,*

(ii) $N_\pi(F)$ *contains an element of infinite order,*

(iii) X^F *is an infinite graph,*

(iv) F *fixes a ray in the tree* X *pointwise,*

(v) F *fixes a line in the tree* X *pointwise,*

(vi) F fixes a point on the Gromov boundary ∂X,

(vii) F fixes two points on the Gromov boundary ∂X.

Proof. (i) ⇒ (ii). $N_\pi(F)$ acts on the tree X and is thus isomorphic to the fundamental group of a certain connected graph of finite groups (G', Y'). This need not be a finite graph of groups, but it inherits from (G, Y) the property that there is a global bound on the orders of the groups G'_P. We show that this suffices for the existence of an element of infinite order if $N_\pi(F)$ is infinite.
Assume that $N_\pi(F)$ is a torsion group. The fundamental group of the graph of groups (G', Y') surjects onto the usual topological fundamental group of Y', which has an element of infinite order unless Y' is a tree. Moreover, any edge group has to equal the vertex groups of one of its two vertices. The reason is that there certainly is an element of infinite order if both inclusions are strict, see Example C.2 with $F = \{1\}$. Note that the fundamental group of the graph of groups on any subgraph of Y' embeds into the whole fundamental group, as can be seen by considering reduced words. Now, let P be a vertex of Y' such that G'_P is of maximal order. Then the two facts mentioned above ensure that the canonical map

$$G'_P \longrightarrow \pi_1(G', Y', P) \cong N_\pi(F)$$

is an isomorphism, thus $N_\pi(F)$ is finite.
(ii) ⇒ (iii). Since F is finite, it fixes a vertex x. Let $g \in N_\pi(F)$ be of infinite order. For all n, we have $g^{-n}Fg^n \subseteq \mathrm{Stab}_\pi(x)$, or, equivalently, $F \subseteq \mathrm{Stab}_\pi(g^n x)$. Since the stabiliser of x is finite, there are infinitely many points of the form $g^n x$.
(iii) ⇒ (i). Since there are finitely many π-orbits, F in particular fixes infinitely many vertices in the same π-orbit. Suppose that this is the G-orbit of x, i. e. there are infinitely many $g \in \pi$ such that

$$F \subseteq \mathrm{Stab}_\pi(gx)$$

or, equivalently,

$$g^{-1}Fg \subseteq \mathrm{Stab}_\pi(x)\,.$$

Thus, the subgroup $M \subseteq \pi$ consisting of all g with the above property is infinite. But M acts on the finite set of π-conjugates of F in $\mathrm{Stab}_\pi(x)$ by conjugation, and the stabiliser of F equals $N_\pi(F)$ which is thus also infinite.
(iii) ⇔ (iv). In the tree X, every vertex is of finite degree since this is true for the quotient $Y = \pi\backslash X$ and all edge stabilisers are finite. The same is thus true for X^F which is connected by uniqueness of geodesics. Finally, a connected graph in which all vertices have finite degree is infinite if and only if it contains a ray [Die17, Prop. 8.2.1].
(iv) ⇔ (vi), (v) ⇔ (vii). In the description of the Gromov boundary as the set of rays emanating from a fixed vertex p recalled directly before this Proposition, take p to be a vertex fixed by F. Then F acts on the set of rays, and the assertions translate into one another.

(ii) ⇒ (v). Let g be of infinite order normalising F and let x be a vertex fixed by F. Then F fixes all $g^n x$ with $n \in \mathbb{Z}$. Let

$$m = \min_{x \in \mathrm{vert} X} d(x, gx) > 0 \,.$$

By the structure theorem for hyperbolic elements [Ser80, Prop. 24], there is a g-invariant line L on which g acts by translation by m. Moreover, if z denotes the point on L closest to x, then the geodesic from x to gx contains z (and gz). Thus, $z \in X^F$. A similar argument applied to $g^n x$ shows that $g^n z$ is contained in X^F. Since X^F is geodesically closed, this implies that it contains the whole line L.
(v) ⇒ (iv). This is a tautology. □

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