On steady Kähler-Ricci solitons

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ABSTRACT

In this thesis we study the existence and uniqueness of steady Kähler-Ricci solitons. We consider two classes of manifolds on which we obtain new examples of steady solitons by using different methods for each class.

In the first part we focus on suitable vector bundles over Kähler manifolds whose Ricci curvature has constant eigenvalues. This condition reduces the soliton equation to an ODE, which we then solve to find new examples. Moreover, we show that these new steady Kähler-Ricci solitons are unique if the Kähler class, the vector field and the asymptotic behavior is fixed.

In the second part we consider certain crepant resolutions $\pi : M \to (\mathbb{C} \times D)/\Gamma$ of orbifolds $(\mathbb{C} \times D)/\Gamma$ for some finite group Γ which acts by rotation on the first factor and preserves a holomorphic volume form on $\mathbb{C} \times D$. To construct new steady Kähler-Ricci solitons on M we use PDE methods for complex Monge-Ampère equations. The solitons obtained this way are asymptotic to a Ricci-flat cylinder.

Contents

INTRODUCTION

In 1982, Hamilton introduced the Ricci flow equation

$$\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Ric}(g(t))$$

for a family of Riemannian metrics $g(t)$ with Ricci curvature $\mathrm{Ric}(g(t))$ ([Ham82]) and outlined a possible approach to proving Thurston's geometrization conjecture in subsequent works. Since then, Ricci flow was used to prove spectacular theorems in geometry. Most notably, various authors worked on Hamilton's program until Perelman completed the proof of Thurston's conjecture. (For an introduction to this topic, see for example [MT07]). In higher dimensions, Böhm and Wilking successfully applied Ricci flow to show that manifolds with positive curvature operator are space forms ([BW08]), and Brendle and Schoen also relied on Ricci flow to prove the differentiable sphere theorem ([BS11]).

On Kähler manifolds, Hamilton's Ricci flow has been the subject of intense study over the past two decades because it has deep connections with the complex geometry of the underlying manifold. In this context there also exists a guiding program which was developed by Song and Tian ([ST07],[ST09],[Tia08]). This so-called Analytic Minimal Model Program proposes how Ricci flow may be used to tackle Mori's Program in birational geometry, which is an essential step towards a possible classification of algebraic varieties (see [BEG13][Chapter 3] and [Tos18] for excellent introductions).

For many applications of Ricci-flow, it is important to understand how the metric $g(t)$ changes if t approaches the maximal existence time $T > 0$. In general, $g(t)$ does not converge to a smooth metric as $t \to T$, but forms singularities whose study is a broad area of current research. Possible models for singularities are provided by Ricci solitons which are self-similar solutions to Ricci flow, i.e. they only evolve by scaling and diffeomorphisms. For instance, Type IIa singularities, which occur if the maximal existence time $T > 0$ of Ricci flow is finite and the curvature blows up faster than $(T-t)^{-1}$, are expected to be related to so-called steady Ricci solitons (compare [CK04][Chapter 2.6], or more recently [CDM20], [CFSZ20], [BCD$^+$21] for results in this direction).

In this work we focus on the special case of so-called *steady Kähler-Ricci solitons* and the goal is to study existence and uniqueness of these objects. Recall that a steady Kähler-Ricci soliton is a triple (M, g, X) consisting of a Kähler manifold (M, g) and a real holomorphic vector field X on M such that

$$\mathrm{Ric}(\omega) = -\frac{1}{2}\mathcal{L}_X\omega, \tag{1}$$

where ω denotes the Kähler form of g, $\mathrm{Ric}(\omega)$ the corresponding Ricci form and $\mathcal{L}_X$ the Lie derivative in direction of X. If X is the gradient field of some function $f : M \to \mathbb{R}$, we say that the soliton (M, g, X) is *gradient*. Interestingly, if $(M, g, \nabla^g f)$ is a gradient steady Kähler-Ricci soliton on a simply-connected manifold M, then (1) is equivalent to

$$\omega^n = i^{n^2} e^{-f}\Omega \wedge \overline{\Omega} \tag{2}$$

for some holomorphic volume form Ω on M, see [Bry08][Theorem 1]. This equation is the starting point for the construction of such solitons since it allows us to reduce (1) to a Monge-Ampère equation for a single scalar function.

However, it is in general not known when a Kähler manifold with trivial canonical bundle admits a steady Kähler-Ricci soliton, and it is also not clear to which extends these solitons are unique. In contrast, the existence and uniqueness of compact Ricci-flat Kähler manifolds is well understood due to Yau's solution to Calabi's conjecture ([Cal54],[Yau78]).

One difficulty in studying non-trivial (i.e. non Ricci-flat) steady solitons is that they can only exist on non-compact manifolds ([Ive93]). While there are general versions of Yau's theorem for non-compact manifolds (most notably [TY90] and [Hei10]), there do not exist comparable results for solutions to (2) in this generality.

Instead, all known examples of steady Kähler-Ricci solitons are constructed by first fixing a specific complex manifold M and then pursuing one of the following two approaches: The first is to consider complex manifolds, on which the scalar Monge-Ampère equation may be further reduced to an ODE, for instance by considering $\mathrm{U}(n)$-invariant metrics on $\mathbb{C}^n$ or on the canonical bundle $K_{\mathbb{CP}^{n-1}}$. (Works in this direction include [Ham88],[Cao96],[DW11] and [Yan12]).

In the second approach, the underlying manifolds are certain resolutions of isolated conical singularities, for example crepant resolutions of orbifolds $\mathbb{C}^n/\Gamma$ for finite subgroups $\Gamma \subset \mathrm{SU}(n)$ that fix the origin. Since there exists an (incomplete) steady soliton on the smooth part of the singular space ($\mathbb{C}^n \setminus \{0\}/\Gamma$ in the example), the idea is to use PDE methods for complex Monge-Ampère equations to construct a (complete) soliton on the resolution (compare [BM17],[CD20]).

Corresponding to these two construction methods, this thesis is divided into two parts. Part I is the preprint [Sch20], in which we unify and extend the existing results obtained by the ODE approach. Moreover, we also study uniqueness of this class of solitons. Part II consists of [Sch21], in which we construct new examples of steady Kähler-Ricci solitons by considering resolutions of orbifolds different from those in [BM17] and [CD20].

In the following two sections, we give a more precise description of our results and briefly discuss their proofs.

1. Overview of Part I.

The ODE approach to solving the Monge-Ampère equation (2) uses an Ansatz which goes back to Calabi ([Cal79]) and was originally introduced to find Ricci-flat metrics on the canonical bundle $K_{\mathbb{CP}^{n-1}}$ over complex projective space. In the case of solitons, it was applied by various authors ([Cao96],[FIK03],[Yan12]) to construct steady Kähler-Ricci solitons on the canonical bundle $\pi : K_M \to (M, g_M)$ over Kähler-Einstein Fano manifolds (M, g_M). If ω_M is the corresponding Kähler form on M, Calabi's Ansatz takes the form

$$\omega_\phi = \pi^*\omega_M + i\partial\bar{\partial}\phi,$$

for some function $\phi = \phi(t)$ only depending on the parameter $t = \log h_M$, where h_M is the Hermitian metric on K_M induced by g_M. The Einstein condition on ω_M then reduces (2) to an ODE in ϕ.

For constructing Kähler-Einstein metrics by Calabi's Ansatz, it was observed by Hwang and Singer ([HS02]) that imposing a weaker condition on g_M suffices to reduce the Einstein equation to an ODE. They merely assumed that the endomorphism

$$g_M^{-1} \cdot \mathrm{Ric}(\omega_M) : T^{1,0}M \to T^{1,0}M$$

on the base manifold (M, g_M) has constant eigenvalues.

We adapt Hwang-Singer's observation to the case of steady Kähler-Ricci solitons and obtain the following

Theorem 1.1 ([Sch20][Theorem 1.1])**.** *Let $\pi : K_M \to (M, g_M)$ be the canonical line bundle over a compact Kähler manifold. Assume that the Ricci form of g_M is positive semi-definite and has constant eigenvalues with respect to g_M. Then K_M admits a 1-parameter family of complete steady Kähler-Ricci solitons in the Kähler class $[\pi^*\omega_M]$.*

This theorem includes all known examples of steady Kähler-Ricci solitons which have previously been constructed on line bundles over Kähler manifolds by ODE methods ([Cao96],[CV96],[PV99],[FIK03], [Yan12]). In addition, Theorem 1.1 also produces new examples, for instance if $M = \mathbb{P}(T^*\mathbb{CP}^n)$ is the projectivization of the cotangent bundle $T^*\mathbb{CP}^n$. Since $\mathbb{P}(T^*\mathbb{CP}^n)$ is a flag variety, there exists a metric g_M for *each* Kähler class on $\mathbb{P}(T^*\mathbb{CP}^n)$, which satisfies the required assumption (see [Sch20][Example 2.5] for details). In comparison, the previous works only found steady solitons with Kähler class that is a multiple of (the pullback of) the first Chern class $\pi^*c_1(\mathbb{P}(T^*\mathbb{CP}^n))$.

Hwang and Singer modified their approach in [HS02][Section 3.2] such that it can also be applied to certain vector bundles of rank ≥ 2. The idea is essentially to work on the corresponding tautological line bundle over the projectivization. However, the conditions one requires on the base manifold are more complicated to state, so we refer the reader to [Sch20][Section 3] for a detailed discussion of the necessary changes. The main result of this section is [Sch20][Theorem 1.2], which constructs steady Kähler-Ricci solitons on vector bundles of higher rank by adapting Hwang-Singer's ideas. As Theorem 1.1, it also unifies and extends the previous results that rely on ODE methods to obtain steady Kähler-Ricci solitons on vector bundles of higher rank ([Li10],[DW11]).

In the second part of the article [Sch20], we study the uniqueness of the solitons constructed in [Sch20][Theorems 1.1 and 1.2]. Before this article, the only known uniqueness result for steady Kähler-Ricci solitons was the following

Proposition 1.2 ([BM17][Proposition 1.2])**.** *Let (M, g, X) be a steady Kähler-Ricci soliton with Kähler form ω. Suppose $\omega + i\partial\bar{\partial}u$ defines another steady Kähler-Ricci soliton on M with the same vector field X, such that at infinity we have*

$$u \to 0, \quad X(u) \to 0 \quad \text{and} \quad |\partial\bar{\partial}u|_g \to 0.$$

Then $u \equiv 0$.

Thus, it is natural to ask if the solitons ω_ϕ constructed in Theorems 1.1 and 1.2 of [Sch20] are unique provided the following three parameters are fixed:

(i) the vector field,
(ii) the de Rahm cohomology class $[\omega_\phi]$ and
(iii) the asymptotic behavior.

To make (iii) precise, suppose ω_ϕ is defined on some vector bundle $E \to M$ and denote the corresponding metric by g_ϕ. We introduce the spaces $C^\infty_\delta(\Lambda^* T^* E)$ that consist of differential forms η on the manifold E such that for all $k \in \mathbb{N}_0$

$$|\nabla^k \eta| = O(d_{g_\phi}^{-\delta-k}).$$

Here, ∇ and $|\cdot|$ are induced by the metric g_ϕ and d_{g_ϕ} is the distance function of g_ϕ to a fixed point. With this notation, we show:

Theorem 1.3 ([Sch20][Theorem 1.3])**.** *Let ω_ϕ be a steady Kähler-Ricci soliton constructed in* [Sch20][Theorem 1.1 or 1.2]*. Assume that ω is a Kähler-Ricci soliton on E with the same vector field as ω_φ such that $[\omega] = [\omega_\varphi] \in H^2(E)$. If moreover $\omega_\phi - \omega \in C^\infty_{-\delta}(\Lambda^2 T^* E)$ for some $\delta > 2$, then $\omega_\phi = \omega$.*

Our approach is to reduce this theorem to Proposition 1.2 which is accomplished by the next

Theorem 1.4 ([Sch20][Theorem 5.4])**.** *Let $\delta > 2$ and $\eta \in C^{\infty}_{-\delta}(\Lambda^* T^* E)$ be a real $(1,1)$ form. If η is d-exact, then $\eta = \sqrt{-1}\partial\bar{\partial}u$ for some $u \in C^2_{2-\delta}(E)$.*

Applying Theorem 1.4 to the difference $\omega_\phi - \omega$ allows us to use Proposition 1.2 since $\delta > 2$. Then Theorem 1.3 follows immediately.

Since the underlying manifold E is non-compact, it is a priori not clear why Theorem 1.4 should be true. Our proof of this theorem uses a strategy that was pursed by Conlon and Hein ([CH13][Section 3]) for conical Kähler manifolds of non-negative Ricci curvature. Why their strategy works in our context can essentially be traced back to two things: Firstly, the spaces $C^{\infty}_{\delta}(\Lambda^* T^* E)$ are well-adapted to the Laplace operator Δ of g_ϕ since we can solve Poisson's equation $\Delta v = h$ for functions v, h with controlled growth (see [Sch20][Proposition 5.6]). And secondly, it turns out that g_ϕ is in fact of non-negative Ricci curvature ([Sch20][Theorem 4.1]). Using these two facts, we conclude Theorem 1.4 in the same way as Conlon and Hein proved [CH13][Theorem 3.1].

After our preprint [Sch20] was uploaded to the arXiv, Conlon and Deruelle also posted their work [CD20], which contains new existence and uniqueness results for steady Kähler-Ricci solitons. There is some overlap between their and our results. For example, in the case of the canonical bundle $K_M \to M$ over a Kähler-Einstein Fano manifold, [CD20][Theorem A] includes both Theorem 1.1 and 1.3 as a special case.

Conlon-Deruelle's existence result takes a different approach than we pursued here. It is based on the continuity method for complex Monge-Ampère equations and their ideas play a crucial role in the second article [Sch21], which is discussed in the next section.

2. Overview of Part II.

Before stating our main result, we briefly explain the work [CD20] because we use the same underlying idea to find new examples of steady Kähler-Ricci solitons. While Conlon and Deruelle study resolutions of general Ricci-flat cones, let us focus on the special case of orbifolds $\mathbb{C}^n/\Gamma$ for finite subgroups $\Gamma \subset \mathrm{SU}(n)$ that only fix the origin. Recall that there is a $\mathrm{U}(n)$-invariant steady Kähler-Ricci soliton on $\mathbb{C}^n$ (due to Cao [Cao96]), and so it descends to a soliton on the (singular) orbifold $\mathbb{C}^n/\Gamma$. By considering resolutions $\pi : M \to \mathbb{C}^n/\Gamma$ of the isolated singularity, one may search for new steady solitons on the complex manifold M, which are asymptotic to Cao's soliton on $\mathbb{C}^n/\Gamma$. This requires that the resolution is

(i) crepant and
(ii) equivariant, i.e. that the $\mathbb{C}^*$-action, given by multiplying a vector in $\mathbb{C}^n$ by $\lambda \in \mathbb{C}^*$, extends equivariently to the resolution M.

Conditon (i) means that M admits a holomorphic volume form Ω and (ii) implies that the radial vector field on $\mathbb{C}^n/\Gamma$ extends to a real holomorphic vector field X on M. This X is the candidate for the soliton field. With these assumptions, Conlon and Deruelle set up a continuity method to solve the Monge-Ampère equation (2).

In [Sch21], our goal is to construct new examples of steady Kähler-Ricci solitons which are geometrically different from those in [CD20]. While Conlon-Deruelle's solitons have rather complicated asymptotics (these are so-called cigar-paraboloids, see [CD20][Section 3]), we aim at finding new examples that are asymptotically cylindrical. So we purse a strategy similar to Conlon and Deruelle, but we consider a different asymptotic model as a starting point.

Recall that Hamilton ([Ham88]) constructed a steady Kähler-Ricci soliton on $\mathbb{C}$, that is asymptotic to the cylinder $\mathbb{R}\times\mathbb{S}^1\cong\mathbb{C}^*$. Then the product $\mathbb{C}\times D$ of Hamilton's soliton and a compact Ricci-flat Kähler manifold D is also a steady Kähler-Ricci soliton, asymptotic to the cylinder $\mathbb{R}\times\mathbb{S}^1\times D$.

To find non-trivial examples, we pass to the quotient $(\mathbb{C}\times D)/\Gamma$ by a suitable finite cyclical group Γ and then consider resolutions $\pi: M\to(\mathbb{C}\times D)/\Gamma$. We essentially require that Γ acts by rotation on the first factor and that D admits a holomorphic volume form Ω_D such that Γ preserves $\Omega := dz\wedge\Omega_D$, so that Ω descends to the quotient. As explained before, we have to assume that the resolution $\pi: M\to(\mathbb{C}\times D)/\Gamma$ is crepant (i.e. Ω extends to M) and satisfies a certain equivariance condition so that the radial vector field on the first factor of $(\mathbb{C}\times D)/\Gamma$ also extends to the resolution.

Under these assumptions our main result is to construct steady Kähler-Ricci solitons on M, which are asymptotic to the Ricci-flat cylinder $\mathbb{R}\times(\mathbb{S}^1\times D)/\Gamma$.

Theorem 2.1 ([Sch21][Theorem 1.2])**.** *Let D^{n-1} be a compact Kähler manifold with nowhere-vanishing holomorphic $(n-1,0)$-form Ω_D. Suppose $\gamma: D\to D$ is a complex automorphism of order $m>1$ such that*

$$\gamma^*\Omega_D = e^{-\frac{2\pi i}{m}}\Omega_D,$$

and consider the orbifold $(\mathbb{C}\times D)/\langle\gamma\rangle$, where γ acts on the product via

$$\gamma(z,w)=\left(e^{\frac{2\pi i}{m}}z,\gamma(w)\right).$$

Let $\pi: M\to(\mathbb{C}\times D)/\langle\gamma\rangle$ be a crepant resolution such that the $\mathbb{C}^*$-action on $(\mathbb{C}\times D)/\langle\gamma\rangle$ given by

$$\lambda*(z,w)=(\lambda z,w),\quad \lambda\in\mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action of $\mathbb{C}^*$ on M.

Let $\overline{M}=M\cup\overline{D}$ be the complex compactification of M by adding the orbifold divisor $\overline{D}:=D/\langle\gamma\rangle$ at infinity. Then for every orbifold Kähler *class $\kappa_{\overline{M}}$ on $\overline{M}$, there exists a steady Kähler-Ricci soliton on M whose Kähler form is contained in the class $\kappa_{\overline{M}}|_M \in H^2(M,\mathbb{R})$.*

The compactification $\overline{M}$ of a resolution $\pi : M \to (\mathbb{C} \times D)/\langle\gamma\rangle$ as in Theorem 2.1 is obtained by replacing the $\mathbb{C}$-factor with the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, i.e. we glue in the divisor $\overline{D} = \{\infty\} \times D/\langle\gamma\rangle$. The reason for passing to $\overline{M}$ is that we obtain a precise characterisation of Kähler classes which admit asymptotically cylindrical Kähler forms. By work of Haskins, Hein and Nordström ([HHN15]), these classes are restrictions of orbifold Kähler classes from $\overline{M}$ to M (also compare [Sch21][Section 4] for details).

Given such a class $\kappa \in H^2(M,\mathbb{R})$, there exists a Kähler form $\omega_0 \in \kappa$, whose corresponding metric is asymptotic to the Ricci-flat cylinder $\mathbb{R} \times (\mathbb{S}^1 \times D)/\langle\gamma\rangle$ (see [HHN15][Section 4.2]). For constructing new solitons, we make the Ansatz $\omega_\varphi = \omega_0 + i\partial\bar{\partial}\varphi$ so that (2) can be rewritten as

$$\omega_\varphi^n = e^{F - \frac{X}{2}(\varphi)}\omega_0^n \tag{3}$$

for a suitable choice of F. Finding a solution φ to (3) by setting up a continuity method requires two key points.

First, we need to define suitable Banach spaces between which the linearisation of (3) is an isomorphism. Here we consider the function spaces $C^\infty_\varepsilon(M)$ that are well-adapted to the cylindrical geometry, see [HHN15][Section 1] for instance. The space $C^\infty_\varepsilon(M)$ contains all functions whose derivatives decay at least like $e^{-\varepsilon t}$, with t denoting the cylindrical parameter. If g_φ denotes the metric associated with ω_φ, then the linearisation of (3) is given by the so-called drift Laplace operator $\Delta_{g_\varphi} + X$. Using the theory of asymptotically translation-invariant operators, we show that $\Delta_{g_\varphi} + X$ is indeed an isomorphism between the chosen function spaces.

The second and most difficult part is to obtain a priori estimates on φ. For these estimates, we essentially adapt Conlon-Deruelle's ideas ([CD20]) to our cylindrical setting. While the uniform estimates in [CD20][Section 7] rely on F being compactly supported, we present a modified proof allowing us to immediately assume that F is merely decaying exponentially, see [Sch21][Section 5].

Combining the previous two parts then finishes the proof of Theorem 2.1.

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Part I.

EXISTENCE AND UNIQUENESS OF S^1-INVARIANT KÄHLER-RICCI SOLITONS*

JOHANNES SCHÄFER

Abstract. We use the momentum construction for S^1-invariant Kähler metrics as developed by Hwang-Singer to construct new examples of steady Kähler-Ricci solitons. We also prove that these solitons are unique in their Kähler class, provided the vector field and the asymptotic behavior are fixed.

1. Introduction

A *steady Kähler-Ricci soliton* is a Kähler manifold (M, g) whose Kähler form ω satisfies

$$\mathrm{Ric}(\omega) = -\mathcal{L}_X \omega \tag{1}$$

for some vector field X which is the real part of a holomorphic vector field. Solutions to (1) are natural generalizations of Ricci-flat metrics and arise as self-similar solutions to Ricci flow.

If the vector field X is non-zero, the manifold must be non-compact [Ive93]. In general, there is no classification for steady Kähler-Ricci solitons available and only few examples are known. Even if a manifold admits a Kähler-Ricci soliton, it is not understood which subset of the Kähler cone contains further examples of Ricci solitons. It is also not clear, how many solitons there are in each Kähler class.

All known examples with $X \neq 0$ are divided into two classes. One class contains explicitly constructed solutions by using ODE methods ([Ham88], [Cao96], [CV96], [PTFV99], [FIK03], [Li10], [DW11],[FW11], [Yan12]), while the other examples are obtained by using PDE gluing methods ([BM17]). The explicit examples are constructed on Euclidean space or on holomorphic vector bundles over Kähler manifolds, while the gluing method produces solitons on certain crepant resolutions of orbifolds $\mathbb{C}^n/G$.

In this article, we use the momentum construction introduced by Hwang-Singer [HS02] to find new examples of steady Kähler-Ricci solitons. More precisely, we prove the following theorem.

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*This chapter is the article:
Johannes Schäfer, *Existence and uniqueness of S^1-invariant Kähler-Ricci solitons*, arXiv:2001.09858v2, to appear in the *Annales de la Faculté des Sciences de Toulouse mathématiques.*

Theorem 1.1. *Let $\pi : K_M \to (M, g_M)$ be the canonical line bundle over a compact Kähler manifold. Assume that the Ricci form of g_M is positive semi-definite and has constant eigenvalues with respect to g_M. Then K_M admits a 1-parameter family of complete steady Kähler-Ricci solitons in the Kähler class $[\pi^*\omega_M]$.*

Theorem 1.1 generalises results obtained in [Cao96], [CV96], [PTFV99], [FIK03], [DW11] and [Yan12]. The main difference is that we do *not* assume (M, g_M) to be a Kähler-Einstein Fano manifold, but only require that $\mathrm{Ric}(\omega_M)$ has constant eigenvalues.

Under the same assumption, Hwang-Singer [HS02] used Calabi's ansatz to construct Kähler-Einstein metrics on line bundles. They observed that the constancy of eigenvalues is sufficient to reduce the Kähler-Einstein equation to a single ODE, which is linear after applying a certain transformation. We prove Theorem 1.1 by adapting their construction to the case of steady Kähler-Ricci solitons.

Theorem 1.1 produces new examples if the base M is a flag variety. More concretely, consider the canonical bundle over $M = \mathbb{P}(T^*\mathbb{CP}^n)$, the projectivization of the cotangent bundle $T^*\mathbb{CP}^n$. Previously, it was only known that compactly supported Kähler classes admit steady solitons ([PTFV99], [DW11], [Yan12]), whereas Theorem 1.1 shows they sweep out the *entire* Kähler cone.

Another interesting feature of Hwang-Singer's construction is that it can also be applied to certain vector bundles of rank ≥ 2. Then we obtain a result analogue to Theorem 1.1.

Theorem 1.2. *Let $\pi : E \to D$ be a holomorphic vector bundle of rank m over a compact Kähler manifold (D, ω_D). Assume that E admits a Hermitian metric h such that the corresponding curvature form γ of the tautological bundle $(\mathcal{O}_{\mathbb{P}(E)}(-1), h)$ is negative semi-definite and has constant eigenvalues with respect to the Kähler metric $\omega_M = p^*\omega_D - \gamma$, where $p : M = \mathbb{P}(E) \to D$ is the natural projection. Additionally, suppose that*

$$\mathrm{Ric}(\omega_M) = -m\gamma. \tag{2}$$

Then E admits a 1-parameter family of complete steady Kähler-Ricci solitons in the class $[\pi^*\omega_D]$.

This can be applied to certain sums of line bundles and again, if the base is a flag variety, it constructs steady solitons in each Kähler class, generalising results in [Li10] and [DW11][Theorem 4.20].

Given a Kähler-Ricci soliton, it is an interesting question whether or not it is unique in its Kähler class. It is natural to fix a vector field for this question because there can be families of solitons as in Theorem 1.1 and 1.2 for instance. In general, this question seems to be largely open.

In the special case of Ricci-flat Kähler metrics, the question of uniqueness is studied under additional assumptions on the asymptotic behaviour of the metric ([Joy00], [Got12], [CH13], [HHN15]). For example, asymptotically conical Ricci-flat metrics are unique in their Kähler class [CH13].

In the different setting of solitons with $X \neq 0$, there are only few results such as [BM17]. Assuming that two steady solitons ω_1, ω_2 with the same vector field are related by $\omega_1 = \omega_2 + \sqrt{-1}\partial\bar{\partial}u$, [BM17][Proposition 1.2] shows that $\omega_1 = \omega_2$ provided u and its derivatives tend to zero at infinity.

In this work, we extend the previous result for the metrics constructed in Theorem 1.1 and 1.2.

Theorem 1.3. *Let $E \to D$ be a holomorphic vector bundle satisfying the assumptions in Theorem 1.1 or 1.2 and denote the steady Kähler-Ricci solitons constructed in Theorem 1.1 or 1.2 by ω_φ. Suppose that ω is a Kähler-Ricci soliton on E with the same vector field as ω_φ such that $[\omega] = [\omega_\varphi] \in H^2(E)$. If moreover $\omega_\varphi - \omega \in C^\infty_{-\delta}(\Lambda^2 T^*E)$ for some $\delta > 2$, then $\omega_\varphi = \omega$.*

We reduce the proof of Theorem 1.3 to [BM17][Proposition 1.2] by proving a $\partial\bar{\partial}$-Lemma with controlled growth. Assuming that $\omega_\varphi - \omega$ is asymptotic to zero, in a suitable sense, we show that there exists a smooth function u such that $\omega_\varphi - \omega = \sqrt{-1}\partial\bar{\partial}u$ and $u \in C^\infty_{-\delta+2}(E)$, i.e. u and all its derivatives tend to zero because $2 - \delta < 0$.

The strategy for finding such a function u is analogue to [CH13][Section 3]. The main point is proving that all harmonic 1-forms of a certain growth behaviour are identically zero which requires non-negative Ricci curvature. We will see that this is indeed true for the metrics ω_φ constructed in Theorem 1.1 and 1.2.

This article is structured as follows. In Section 2, we recall Hwang-Singer's construction of Kähler metrics and prove Theorem 1.1. For proving Theorem 1.2, we have to make some adjustments which are explained in Section 3. The metrics are studied more closely in Section 4. Here we observe in particular that the curvature of these metrics is bounded and that the Ricci curvature is non-negative. Then, in Section 5, we prove Theorem 1.3 by studying the Laplace operator and harmonic 1-forms of the metrics constructed in Theorem 1.1 and 1.2.

After the first version of this paper was uploaded to the arXiv, Conlon and Deruelle [CD20] posted a preprint on the arXiv containing a new existence result for steady Kähler-Ricci solitons. There is some overlap between their main result [CD20][Theorem A] and our Theorems 1.1 and 1.2, compare Remarks 2.6 and 3.4 below.

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2. Calabi's Ansatz for line bundles

Hwang-Singer's construction combines Calabi's ansatz with ideas from symplectic geometry ([HS02]). If $\pi : (L, h) \to (M, \omega_M)$ denotes a Hermitian holomorphic line bundle over a Kähler manifold, then Calabi's idea ([Cal79]) was to search for Kähler metrics of the form

$$\pi^*\omega_M + \sqrt{-1}\partial\bar{\partial} f(t). \tag{3}$$

Here, t denotes the logarithm of the fibre-wise norm function induced by h and f is a convex function of one variable. Instead of describing the metric (3) in terms of the potential f, Hwang-Singer introduced a new variable $\tau = \tau(t)$ and a function $\varphi = \varphi(\tau) : (0, \infty) \to \mathbb{R}_+$ which is related to the Legendre transformation F of f by $\varphi = 1/F''$. In particular, φ determines the metric (3) uniquely.

Assuming that the curvature form of h has constant eigenvalues, we will see in this section that the non-linear Kähler-Ricci soliton equation (1) is equivalent to a single, linear ODE in the function φ, which can be solved explicitly. This leads to a proof of Theorem 1.1. Additionally, we discuss the main examples to which Theorem 1.1 applies.

2.1. **Notation and set-up.** We begin by briefly recalling Calabi's construction of Kähler metrics in the special case of the canonical bundle. We follow the presentation in [HS02][Section 2].

Let (M^n, ω_M) be a Kähler manifold of complex dimension n and equip its canonical line bundle $\pi : K_M \to M$ with the Hermitian metric h induced by ω_M. Let γ be the curvature form of h and assume that $-\gamma \geq 0$, i.e. γ is negative semi-definite. Recall that γ is given by

$$\gamma = -\sqrt{-1}\partial\bar{\partial} \log h(s, \bar{s}) = -\operatorname{Ric}(\omega_M),$$

where $s : U \to K_M$ is a local holomorphic section of K_M and $\operatorname{Ric}(\omega_M)$ denotes the Ricci form of ω_M. We introduce the radial function $r : K_M \to \mathbb{R}_{\geq 0}$ defined by $r(v) = \sqrt{h(v, \bar{v})}$ and outside the zero section, we define a new function $t : K_M \setminus M \to \mathbb{R}$ by $t = 2 \log r$. The pullback $\pi^*\gamma$ is a $\partial\bar{\partial}$-exact form on $K_M \setminus M$ and satisfies

$$\pi^*\gamma = -\sqrt{-1}\partial\bar{\partial} t. \tag{4}$$

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

$$\lim_{t \to -\infty} f'(t) = 0 \quad \text{and} \quad f'' > 0. \tag{5}$$

Then Calabi's Ansatz searches for Kähler metrics ω of the form

$$\omega = \pi^*\omega_M + \sqrt{-1}\partial\bar{\partial} f(t) = \pi^*\omega_M - f'(t)\pi^*\gamma + f''(t)\sqrt{-1}\partial t \wedge \bar{\partial} t. \tag{6}$$

Note that ω is defined on $K_M \setminus M$, the canonical bundle with the zero section removed, and it is positive since we assumed $-\gamma \geq 0$ and (5).

Depending on the behaviour of $f(t)$ as $t \to \pm\infty$, ω can be extended to all of K_M and define a complete metric. When this can happen is explained in the next subsection.

We conclude this subsection by describing the Calabi metric ω in terms of the Legendre transformation of its potential f, which is well-defined since f is convex by (5). We now briefly recall this transformation. Let $I = \operatorname{Im} f' \subset \mathbb{R}_+$ be the image of f' and define the new variable $\tau := f'(t) \in I$. We write $I = (0, \tau_2)$, which means that

$$\lim_{s\to-\infty} \tau(s) = \lim_{t\to-\infty} f'(t) = 0, \quad \lim_{s\to+\infty} \tau(s) = \lim_{t\to+\infty} f'(t) = \tau_2.$$

We point out that in general $\tau_2 \leq +\infty$, but in the case considered in subsequent sections, we have in fact that $\tau_2 = +\infty$. The Legendre transform $F : I \to \mathbb{R}$ is defined by the formula

$$f(t) + F(\tau) = t\tau.$$

One can check that F is also strictly convex, so that we can define a new function $\varphi : I \to \mathbb{R}_+$ by

$$\varphi(\tau) = \frac{1}{F''(\tau)}.$$

Then we obtain the following relations

$$\frac{d\tau}{dt} = f''(t) = \varphi(\tau), \quad f'''(t) = \frac{d\varphi}{dt} = \varphi'(\tau)\varphi(\tau). \tag{7}$$

In particular, (5) translates into

$$\varphi > 0 \quad \text{on } I = (0, \tau_2). \tag{8}$$

We can then express the metric ω obtained from Calabi's construction (6) as

$$\omega = \pi^*\omega_M - \tau\pi^*\gamma + \frac{1}{\varphi(\tau)}\sqrt{-1}\partial\tau \wedge \bar{\partial}\tau \tag{9}$$

by using equations (6) and (7).

The function φ is called the momentum profile of ω. We note that it is possible to reconstruct the Kähler potential f of ω from its momentum profile by

$$f(t) = \int_0^{\tau(t)} \frac{x\,dx}{\varphi(x)}. \tag{10}$$

Hence, the Kähler metric given by Calabi's Ansatz (6) is uniquely determined by its momentum profile. We emphasize this by writing $\omega = \omega_\varphi$.

Completeness of ω_φ. The Kähler metric $\omega = \omega_\varphi$ given by (9) is a priori only defined on $K_M \setminus M$ and is in general not complete. Whether or not ω_φ extends across the zero section to a complete metric is determined by the behaviour of the momentum profile φ toward the endpoints of $I = (0, \tau_2)$. This is well-understood and there is the following well-known proposition, whose proof can be found in [HS02][Section 2] or [FW11][Section 6], for example.

Proposition 2.1. *Let ω_φ be given by (9). Suppose the profile $\varphi : I \to \mathbb{R}$ has a zero of integer order at each endpoint of $I = (0, \tau_2)$. Then ω_φ extends across the zero section if and only if $\varphi(0) = 0$ and $\varphi'(0) = 1$.*

In this case, the resulting metric on K_M is complete if and only if at the upper endpoint τ_2, one of the following conditions (i) and (ii) holds:

(i) *The endpoint τ_2 is finite and φ vanishes at least to second order.*
(ii) *The endpoint τ_2 is infinite and φ grows at most quadratically.*

Remark 2.2. Note that [HS02][Proposition 2.3] is identical with Proposition 2.1, except that Hwang and Singer require $\varphi'(0) = 2$ instead of $\varphi'(0) = 1$. This is due to the fact that our Kähler potential f is *twice* the potential function used by Hwang and Singer; compare (6) with [HS02][(1.1)].

If the metric ω_φ extends to the total space of K_M, we would like to identify its de Rham cohomology class. Since we assumed (8), i.e. $I = (0, \tau_2)$, it follows immediately that $[\omega_\varphi] = [\pi^*\omega_M] \in H^2(K_M)$. We refer to the class $[\pi^*\omega_M]$ as the Kähler class of ω_φ.

More generally, we define a Kähler class on K_M simply to be a class in $H^2(K_M)$ containing positive $(1,1)$ forms and the Kähler cone is the set of all such Kähler classes. Using this definition, the projection map $\pi^* : H^2(M) \to H^2(K_M)$ identifies the Kähler cone of the compact base M with the Kähler cone of K_M. Indeed, given a Kähler form on K_M, its restriction to M clearly is a Kähler form on M. Conversely, given a Kähler form ω_M on M, Calabi's Ansatz always produces a positive $(1,1)$ form in the class $[\pi^*\omega_M]$, for example consider ω_φ with $\varphi(\tau) = \tau$, which extends to K_M by Proposition 2.1.

The Ricci form. In this paragraph, we provide a description of the Ricci-form of ω_φ. The computations can be found, for example, in [HS02][Section 2.1].

Denote the Kähler metric of ω_M by g_M and the curvature form of (K_M, h) by γ. It gives rise to an endomorphism $B : T^{1,0}M \to T^{1,0}M$ of the holomorphic tangent bundle, which is locally defined by $B := g_M^{-1}\gamma = g_M^{\bar{k}i}\gamma_{j\bar{k}}$. As in Theorem 1.1, we assume from now on that the eigenvalues of B are constant over M. This condition is sufficient to reduce the soliton equation (1) to an ODE. These conditions guarantee that the function $Q : I \times M \to \mathbb{R}_+$ defined by

$$Q = \det\left(g_M^{-1}(\omega_M - \tau\gamma)\right) = \det\left(\mathrm{Id} - \tau B\right) \tag{11}$$

only depends on the parameter τ, i.e. is constant over M. Also observe that Q is a positive function because $-\gamma \geq 0$ and $\tau \geq 0$. Q naturally appears in the computation of $\mathrm{Ric}(\omega_\varphi)$. Indeed, the Ricci form is given by

$$\mathrm{Ric}(\omega_\varphi) = \pi^* \mathrm{Ric}(\omega_M) + \frac{(\varphi Q)'}{Q}\pi^*\gamma - \frac{1}{\varphi}\left(\frac{(\varphi Q)'}{Q}\right)' \sqrt{-1}\partial\tau \wedge \bar{\partial}\tau, \tag{12}$$

see [HS02][(2.14)].

2.2. **Reduction to an ODE.** We use the previously derived formula for the Ricci curvature to show that the Kähler-Ricci soliton equation is equivalent to an ODE in the function $\varphi(\tau)$. Our presentation is similar to [FW11][Section 4].

By definition, the soliton vector field X must be the real part of a holomorphic vectorfield, i.e. $\mathcal{L}_X J = 0$. On the line bundle K_M, there is a natural choice for X, which we now describe. K_M admits a holomorphic $\mathbb{C}^*$-action by fibre-wise multiplication and the corresponding holomorphic vector field Z is given by $Z = z_0 \frac{\partial}{\partial z_0}$, where z_0 denotes the fibre coordinate of K_M. In terms of the radial function t defined at the beginning of this section, we can write Z as

$$Z = \mathrm{Re}\, Z + \sqrt{-1}\,\mathrm{Im}\, Z = \frac{\partial}{\partial t} - \sqrt{-1}J\frac{\partial}{\partial t}. \tag{13}$$

So it is natural to set $X := \mu\,\mathrm{Re}\, Z = \mu\frac{\partial}{\partial t}$ for some constant $0 \neq \mu \in \mathbb{R}$. Before deriving the ODE, we need to calculate the following Lie-derivative:

$$\mathcal{L}_X\omega_\varphi = d(\iota_X\omega_\varphi) = \sqrt{-1}\partial\bar{\partial}(\mathcal{L}_X f)(t) = \mu\sqrt{-1}\partial\bar{\partial}f'(t). \tag{14}$$

Here, we used $2\sqrt{-1}\partial\bar{\partial} = dJd$ and $\mathcal{L}_X J = 0$ to obtain the second equality. We shall write out equation (14) in terms of fibre and base direction, as we did for the Ricci-form in (12):

$$-\mathcal{L}_X\omega_\varphi = \mu\varphi(\tau)\pi^*\gamma - \mu\frac{\varphi'}{\varphi}(\tau)\sqrt{-1}\partial\tau \wedge \bar{\partial}\tau. \tag{15}$$

Now we are in position to see by comparing (12) and (15) that the soliton equation (1) for ω_φ is equivalent to the following two equations

$$\mathrm{Ric}(\omega_M) + \frac{(\varphi Q)'}{Q}(\tau)\gamma = \mu\varphi(\tau)\gamma \tag{16}$$

$$\left(\frac{(\varphi Q)'}{Q}\right)'(\tau) = \mu\varphi'(\tau). \tag{17}$$

Since $\mathrm{Ric}(\omega_M) = -\gamma$, we see that differentiating (16) gives (17), so that we proved the following Lemma:

Lemma 2.3. *Suppose that ω_φ is a Kähler metric with momentum profile φ. Then (1) with $X = \mu\frac{\partial}{\partial t}$ is equivalent to the following equation:*

$$\varphi'(\tau) + \left(\frac{Q'}{Q}(\tau) - \mu\right)\varphi(\tau) = 1 \tag{18}$$

For the rest of this paragraph, we study the solution φ to Equation (18). This is a linear ODE of the form $y' + p(x)y = q(x)$, which has an explicit one-parameter family of solutions given by

$$y = \exp\left(-\int p(x)dx\right)\left(\int q(x)\exp\left(\int p(x)dx\right)dx + K\right). \tag{19}$$

Applying (19) to (18), we have

$$\varphi(\tau) = \frac{e^{\mu\tau}}{Q(\tau)}\left(\int_0^\tau e^{-\mu x}Q(x)dx + K\right), \tag{20}$$

where $K \geq 0$ is determined by the initial value $\lim_{\tau\to 0}\varphi(\tau)$. Justified by (ii) of Proposition 2.1, we will assume that $K = 0$.

One can compute the integral (18) explicitly in terms of the coefficients $b_j \geq 0$ of the polynomial $Q(\tau) = \det(\mathrm{Id} - \tau B) = b_k\tau^k + b_{k-1}\tau^{k-1} + \cdots + b_0$. Note that the degree k of Q could be less than n since B is allowed to have zero eigenvalues. In fact, it is straight forward to see that

$$\varphi(\tau) = \nu(0)\frac{e^{\mu\tau}}{Q(\tau)} - \frac{\nu(\tau)}{Q(\tau)}, \tag{21}$$

where ν is given by

$$\nu(\tau) = \sum_{j=0}^{k}\sum_{l=0}^{j} b_j\frac{j!}{l!}\frac{\tau^l}{\mu^{j+1-l}}. \tag{22}$$

We point out that the explicit expression for ν is not relevant, but rather that it has the form

$$\varphi(\tau) = \nu(0)\frac{e^{\mu\tau}}{Q(\tau)} + \frac{(-b_k/\mu)\tau^k + R_{k-1}(\tau)}{Q(\tau)} \tag{23}$$

for a polynomial R_{k-1} of degree $k-1$. Hence, we found an explicit solution for the soliton ODE (18). Also note that φ is defined on $[0, +\infty)$ since $Q(0) > 0$. Moreover, φ is clearly positive on $(0, +\infty)$.

With these observations, we can now finish the proof of Theorem 1.1.

2.3. **Proof of Theorem 1.1.** Let $K_M \to (M, g_M)$ be the canonical bundle whose semi-negative curvature form $\gamma = -\mathrm{Ric}(\omega_M)$ has constant eigenvalues w.r.to g_M. Suppose $\varphi : (0, +\infty) \to \mathbb{R}$ is given by (20) with $K = 0$ and, as before, let ω_φ be defined by (9). Since $\varphi(\tau) > 0$ for all $\tau > 0$, ω_φ defines a Kähler metric and hence is a steady Kähler-Ricci soliton by Lemma 2.3. We note that these metrics can only be complete if $\mu < 0$. This can be proven similarly to [FIK03][Lemma 5.1].

Hence we assume $\mu < 0$. From (23), we have the following asymptotic behaviour for large τ:

$$\varphi(\tau) = -\frac{1}{\mu} + O(1/\tau). \tag{24}$$

Also recall that φ and the potential f are related by

$$\frac{df'}{dt}(t) = \varphi(f'(t)). \tag{25}$$

Using (25) together with (24), we conclude that the corresponding potential $f(t)$ is indeed defined for all $t \in \mathbb{R}$, i.e. ω_φ is defined on $K_M \setminus M$.

It remains to check that ω_φ extends across the zero section and defines a complete metric as $t \to +\infty$. By the first part of Proposition 2.1, ω_φ extends provided $\varphi(0) = 0$ and $\varphi'(0) = 1$. Since we assumed $K = 0$ in (20), we have $\varphi(0) = 0$. Plugging this into (18) gives $\varphi'(0) = 1$, as desired. The completeness as $t \to +\infty$ follows immediately from the asymptotic expansion (24) and (ii) of Proposition 2.1.

2.4. **Examples.** Theorem 1.1 immediately recovers all known examples of steady Kähler-Ricci solitons on the total space of line bundles ([Cao96], [CV96], [PTFV99], [DW11], [Yan12]). In these cases, the base is a product of Kähler-Einstein manifolds and the considered Kähler classes are represented by convex combinations of Kähler-Einstein metrics on each factor.

If the base manifold is a flag variety, Theorem 1.1 produces examples, which have not been mentioned before. In this case, steady solitons sweep out the entire Kähler cone.

Example 2.4 (Products)**.** Let (M_i, ω_i), $i = 1, \ldots, r$ be Kähler-Einstein manifolds with non-negative scalar curvature and denote their canonical bundles by $K_{M_i} \to M_i$. We consider the bundle

$$K_M = p_1^* K_{M_1} \otimes \cdots \otimes p_r^* K_{M_r} \to M := M_1 \times \cdots \times M_r,$$

where $p_i : M \to M_i$ is the projection. Then Theorem 1.1 applies and gives a complete steady soliton in each Kähler class of the form $\sum_{i=1}^r \alpha_i [p_i^* \omega_i] \in H^2(M)$ with $\alpha_i > 0$.

The case $r = 1$ was first considered in [Cao96] and [CV96] for $M = \mathbb{CP}^n$ and in [PTFV99] for a general Kähler-Einstein Fano manifold. For $r > 1$, these solitons are found in [DW11][Theorem 4.20].

Example 2.5 (Flag varieties)**.** Let G be a complex semisimple Lie group, $P \subset G$ a parabolic subgroup and $K \subset G$ a maximal compact subgroup. Then K acts transitively on the flag manifold $M = G/P$. It is well-known that M admits a K-invariant complex structure so that its anti canonical bundle is ample, compare [Bes87][Chapter 8] for example.

The previously mentioned results only produce solitons on the canonical bundle K_M whose Kähler class is a multiple of $[\pi^* c_1(M)]$. In general, however, $H^2(M)$ is not spanned by $[c_1(M)]$.

We claim that *every* Kähler class admits a steady Kähler-Ricci soliton. Indeed, every Kähler class on M admits a K-invariant Kähler form ω_K whose Ricci form $\mathrm{Ric}(\omega_K)$ is also K-invariant. This means that the eigenfunctions of $\mathrm{Ric}(\omega_K)$ w.r.t. ω_K must be K-invariant and hence constant since K acts transitively on M. So Theorem 1.1 can be applied and proves the existence of a steady soliton in the class $[\pi^*\omega_K]$.

Remark 2.6. The new metrics in Example 2.5 can also be obtained from the recent result [CD20][Theorem A], which was posted after the first version of this paper was uploaded to the arXiv.

3. Calabi metrics on vector bundles

Given a vector bundle $E \to D$, Hwang-Singer's idea was to apply their construction to the tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ over $\mathbb{P}(E)$, the projectivization of E ([HS02][Section 3.2]). In this section, we explain the changes which are necessary to prove Theorem 1.2 and provide some examples.

The main difference is that one has to choose a new background metric on $\mathbb{P}(E)$, with respect to which the eigenvalues of the curvature form are computed. Then the discussion of the previous section can be applied and again, the soliton equation (1) reduces to a simple ODE. In this new setting, however, the function Q defined by (11) will have zeros at $\tau = 0$, so there are some details which have to be checked.

3.1. **Constructing a Kähler metric.**

As in [HS02][Section 3.2], we explain how to adapt the machinery from the previous section to the tautological line bundle.

Let $\pi : E \to (D, \omega_D)$ be a holomorphic vector bundle of rank $m \geq 2$ equipped with an Hermitian metric h and assume that the Kähler manifold D has complex dimension d. As in the case of line bundles, we define $r : E \to \mathbb{R}_{\geq 0}$ to be the radial function induced by h and let $t = \log r^2$. Then Calabi's Ansatz has the form

$$\omega = \pi^*\omega_D + \sqrt{-1}\partial\bar{\partial} f(t).$$

By construction, the projectivization of E is naturally a fibre bundle $p : \mathbb{P}(E) \to D$, with fibre isomorphic to $\mathbb{CP}^{m-1}$. Recall that the natural map $\mathcal{O}_{\mathbb{P}(E)}(-1) \subset p^*E \to E$ identifies $\mathcal{O}_{\mathbb{P}(E)}(-1) \setminus \mathbb{P}(E) \cong E \setminus D$. By abuse of notation, we denote the bundle projection of $\mathcal{O}_{\mathbb{P}(E)}(-1)$ also by π, so that we have a commuting diagram

$$\begin{array}{ccc} L := \mathcal{O}_{\mathbb{P}(E)}(-1) & \longrightarrow & E \\ \pi\big\downarrow & & \big\downarrow\pi \\ M := \mathbb{P}(E) & \xrightarrow{p} & D \end{array} \tag{26}$$

In the notation from the previous section, let us denote the complex dimension of M by n, i.e. $n = d+m-1$. Via the natural identification $L \setminus M \cong E \setminus D$, h induces a Hermitian metric on L, which we also simply denote by h. Hence, we can view r as a function on L and, if γ is the curvature form of (L, h), we have as before $\pi^*\gamma = -\sqrt{-1}\partial\bar{\partial}t$ with $t = \log r^2$. We again assume that $-\gamma \geq 0$. Then we are looking for metrics of the form

$$\omega_\varphi = \pi^*\omega_D - f'(t)\pi^*\gamma + f''(t)\sqrt{-1}\partial t \wedge \bar{\partial} t \tag{27}$$

where we require that $f : \mathbb{R} \to \mathbb{R}$ satisfies (5) to obtain a positive form. As before, we set $\tau := f'(t)$ and define $\varphi : (0, \tau_2) \to \mathbb{R}_+$ by (7), so that it also satisfies (8). Hence, ω_φ can also be expressed as in (9).

For the computation of Ricci curvature below, we need to choose a background Kähler metric ω_M on M. Define

$$\omega_M = p^*\omega_D - \gamma, \tag{28}$$

which is clearly positive in base direction of the fibration $p : \mathbb{P}(E) \to D$. To see that ω_M is positive in fibre direction, we note that $-\gamma$ restricts to the Fubini-Study metric on each fibre $\cong \mathbb{CP}^{m-1}$.

The Ricci form. The calculation is in principle the same as in the line bundle case, but the polynomial Q does have zeros. Let $B = g_M^{-1}\gamma$ be the curvature endomorphism of γ, where g_M is the metric with Kähler form given by (28) and assume that the eigenvalues of B are constant over M. Then we define a function Q by

$$Q = \det(g_M^{-1}(p^*\omega_D - \tau\gamma)), \tag{29}$$

which can be viewed as a function $Q : (0, \tau_2) \to \mathbb{R}_{\geq 0}$. Indeed, we can write

$$g_M^{-1}(p^*\omega_D - \tau\gamma) = g_M^{-1}(\omega_M - (\tau - 1)\gamma) = \mathrm{Id} - (\tau - 1)B, \tag{30}$$

so that Q is constant over M, i.e. it only depends on τ. If $\beta_1, \dots, \beta_n$ are the eigenvalues of B, we must have $\beta_{d+1} = \cdots = \beta_n = -1$ by the definition of ω_M and $\beta_1, \dots, \beta_d \leq 0$ by assumption. From (30), we conclude that Q is given by

$$Q(\tau) = \tau^{n-d}\prod_{j=1}^{d}(1 + \beta_j - \tau\beta_j) = \tau^{n-d}\hat{Q}(\tau), \tag{31}$$

for some polynomial $\hat{Q}$. Since $p^*\omega_D$ is positive in base direction, we conclude from (30) that $1 + \beta_j > 0$ for all $j = 1, \dots d$. Hence, $\hat{Q}(0) > 0$ and Q has a zero at $\tau = 0$ of order $n - d = m - 1$.

As in (12), one can find the following expression for the Ricci form:

$$\operatorname{Ric}(\omega_\varphi) = \pi^* \operatorname{Ric}(\omega_M) + \frac{(\varphi Q)'}{Q}\pi^*\gamma - \frac{1}{\varphi}\left(\frac{(\varphi Q)'}{Q}\right)' \partial\tau \wedge \bar{\partial}\tau. \tag{32}$$

3.2. **The ODE.** The natural $\mathbb{C}^*$-action on E by biholomorphisms induces a holomorphic vector field Z. On $L\setminus M$, which is the tautological bundle with the zero section removed, the real part of Z is given by $\operatorname{Re} Z = \partial/\partial t$, so we are looking for Ricci solitons with vector field $X = \mu\partial/\partial t$. Again, we find

$$-\mathcal{L}_X\omega_\varphi = -\mu\sqrt{-1}\partial\bar{\partial}f'(t) = \mu\varphi(\tau)\pi^*\gamma - \mu\frac{\varphi'}{\varphi}(\tau)\sqrt{-1}\partial\tau \wedge \bar{\partial}\tau. \tag{33}$$

Combining (27) with (32) and (33), one can check that the soliton equation (1) is equivalent to

$$\operatorname{Ric}(\omega_M) = c\gamma \tag{34}$$

$$\varphi'(\tau) + \left(\frac{Q'}{Q}(\tau) - \mu\right)\varphi(\tau) = -c \tag{35}$$

for some integration constant $c \in \mathbb{R}$. In fact, we must have $c = -m$ since the first Chern class of $M = \mathbb{P}(E)$ is given by

$$c_1(M) = -mc_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) + p^*c_1(E) + p^*c_1(D). \tag{36}$$

Equation (35) has the same form as (18), but with a different Q. Hence, the solution φ is given by

$$\varphi(\tau) = \frac{e^{\mu\tau}}{Q(\tau)}\left(\int_0^\tau me^{-\mu x}Q(x)dx\right), \tag{37}$$

if we assume the integration constant to be zero.

We end this section by studying the solution φ. Let us write $Q(\tau) = b_{k+n-d}\tau^{k+n-d} + \cdots + b_{n-d}\tau^{n-d}$ with coefficients $b_j \geq 0$ for $j = 1+n-d, \ldots, k+n-d$ and $b_{n-d} = \hat{Q}(0) > 0$. Adapting (21) and (22) to this case, we obtain

$$\varphi(\tau) = \nu(0)\frac{e^{\mu\tau}}{Q(\tau)} - \frac{\nu(\tau)}{Q(\tau)} \tag{38}$$

as well as

$$\nu(\tau) = m\sum_{j=n-d}^{k+n-d}\sum_{l=0}^{j} b_j\frac{j!}{l!}\frac{\tau^l}{\mu^{j+1-l}}. \tag{39}$$

A priori, φ given by (37) is defined on the interval $(0, +\infty)$ and because $Q(0) = 0$ one needs to check that φ and its derivatives have a limit as $\tau \to 0$. To see that this is the case, note that we can rewrite (39) as

$$\nu(0)e^{\mu\tau} - \nu(\tau) = m\sum_{j=n-d}^{k+n-d} b_j\frac{j!}{\mu^{j+1}}\sum_{l=j+1}^{\infty}\frac{(\mu\tau)^l}{l!},$$

i.e. $\tau^{-(n-d)}(\nu(0)e^{\mu\tau} - \nu(\tau))$ tends to zero as $\tau \to 0$. Since Q vanishes of order $n-d$ at $\tau = 0$, we then deduce from (38) that $\lim_{\tau\to 0}\varphi = 0$. Similarly, it follows that all derivatives of φ have a limit as $\tau \to 0$.

3.3. **Proof of Theorem 1.2.** The proof is now analogue to Section 2.3. The only part that might a priori be different is the extension of ω_φ to a complete metric on E. However, one can check that Proposition 2.1 also applies to the vector bundle case, see [HS02][Lemma 3.7].

As before, one can check that φ has the behaviour required by Proposition 2.1. Indeed, one can compute that $\varphi(0) = 0$ and $\varphi'(0) = 1$, as desired. Sending $\tau \to +\infty$, we conclude the following asymptotic expansion from (37) and (38)

$$\varphi(\tau) = -\frac{m}{\mu} + O(1/\tau), \tag{40}$$

and so we obtain a complete metric on the total space E.

3.4. **Examples.** We briefly discuss three different situations to which Theorem 1.2 applies. New examples of steady solitons are given in Example 3.2.

Example 3.1 (Complex plane)**.** We let D be a single point and $E \cong \mathbb{C}^n$ be the trivial bundle over D. Let h be the Euclidean metric on E, so that $\omega_M = -\gamma$ is the Fubini-Study metric on $M = \mathbb{CP}^{m-1}$. This is the situation first studied in [Cao96].

Example 3.2 (Sum of line bundles)**.** Let (D, ω_D) be a Kähler-Einstein Fano manifold of Fano index m. Define $L := K_D^{1/m}$ and consider the m-fold sum of L with itself, i.e. $E = L \otimes \mathbb{C}^m$. Then we have $M = \mathbb{P}(E) = \mathbb{CP}^{m-1} \times D$ and

$$\mathcal{O}_{\mathbb{P}(\mathbb{E})}(-1) = p_1^*\mathcal{O}_{\mathbb{CP}^{m-1}}(-1) \otimes p_2^*L,$$

where p_1, p_2 denote the projections onto the first and second factor of M, respectively. Let ω_{FS} be the Fubini-Study metric on $\mathbb{CP}^{m-1}$, so that $\gamma = -p_1^*\omega_{FS} - 1/mp_2^*\operatorname{Ric}(\omega_D)$ is the curvature form of $\mathcal{O}_{\mathbb{P}(\mathbb{E})}(-1)$, and define $\omega_M = p_2^*\omega_D - \gamma$. Then we clearly have

$$\operatorname{Ric}(\omega_M) = mp_1^*\omega_{FS} + p_2^*\operatorname{Ric}(\omega_D) = -m\gamma, \tag{41}$$

since ω_D is Kähler-Einstein. Moreover, the eigenvalues of $\operatorname{Ric}(\omega_M)$ w.r.t. ω_M are constant, so that Theorem 1.2 can be applied. These examples of steady solitons are obtained in [Li10][Theorem 2.1] and [DW11][Theorem 4.20].

If the base $D = G/P$ is a flag manifold for G a complex semisimple Lie group and $P \subset G$ a parabolic subgroup, one can find steady solitons in *every* Kähler class, similarly as in Example 2.5.

To see this, assume that ω_D represents a given Kähler class (not necessarily the first Chern class of D). We can pick ω_D to be K-invariant, where $K \subset G$ is a maximal compact subgroup. Since $\operatorname{Ric}(\omega_D)$ is also

K-invariant, the form $-\gamma = p_1^*\omega_{FS} + 1/mp_2^* \operatorname{Ric}(\omega_D)$ is invariant under the diagonal action of $SU(m) \times K$ and also positive.

We claim that $\operatorname{Ric}(-\gamma) = -m\gamma$. By [Bes87][Theorem 8.2], we know that there exists a $SU(m) \times K$-invariant Kähler-Einstein metric $\omega_{\mathrm{KE}} \in c_1(\mathbb{CP}^{m-1} \times D)$. Also recall that the Ricci forms of *all* $SU(m) \times K$-invariant Kähler metrics agree, i.e. $\operatorname{Ric}(-\gamma) = \operatorname{Ric}(\omega_{\mathrm{KE}})$. Since $-m\gamma$ and ω_{KE} are in the same Kähler class, we deduce from the uniqueness part of Calabi's conjecture that $-m\gamma = \omega_{\mathrm{KE}} = \operatorname{Ric}(-\gamma)$.

As the form $\omega_M = p_2^*\omega_D - \gamma$ is also invariant under $SU(m) \times K$, we conclude

$$\operatorname{Ric}(\omega_M) = \operatorname{Ric}(-\gamma) = -m\gamma,$$

and hence the assumptions in Theorem 1.2 are satisfied.

Example 3.3 (Cotangent bundle of $\mathbb{CP}^d$)**.** Let $D = \mathbb{CP}^d$ be projective space equipped with the Fubini-Study metric and consider $E = T^*\mathbb{CP}^d$, the cotangent bundle of $\mathbb{CP}^d$. E is naturally a $SU(d+1)$-homogeneous vector bundle, where the fibre action is given by the coadjoint action of $SU(d+1)$ on its Lie algebra. Since $\mathbb{CP}^d$ is a rank 1 symmetric space, the induced action of $SU(d+1)$ on $M = \mathbb{P}(E)$ is transitive. Verifying the assumptions of Theorem 1.2 is now similar to the previous Example. These steady solitons on $T^*\mathbb{CP}^d$ are of cohomogeneity one and are contained in [DW11][Section 5].

Remark 3.4. All the previous examples can also be constructed from [CD20][Theorem A].

4. Properties of ω_φ

In this short section, we study curvature properties of the previously constructed metric ω_φ. We show that ω_φ has bounded curvature and that its Ricci curvature is non-negative. Moreover, we obtain estimates on the growth of the function f and its derivatives.

Recall that $f = f(t)$ is the Kähler potential of ω_φ as defined in (6) and $\varphi = \varphi(\tau)$ is its momentum profile, see (9). If ω_φ is a steady Kähler-Ricci soliton constructed in Theorem 1.1 or Theorem 1.2, then φ satisfies (18) or (35), respectively. This ODE is in turn determined by the polynomial $Q = Q(\tau)$ defined by either (11) or (29). The statements in this section mainly reduce to understanding Q and how it effects the asymptotic behaviour of φ, compare (21) or (38) depending on the rank of the underlying vector bundle.

We begin by considering the Ricci curvature of ω_φ. More precisely, we prove the following theorem, which we need in the subsequent section. It generalises the observation made in [Yan12][Case 7].

Theorem 4.1. *The complete steady Kähler-Ricci solitons constructed in Theorem 1.1 and 1.2 have non-negative Ricci curvature. Moreover,* *if the curvature form $-\gamma$ is positive definite, then the Ricci curvature is positive away from the zero section.*

Proof. First, we consider the solitons constructed on line bundles in Theorem 1.1. Let ω_φ be the Kähler metric given by (9) with φ satisfying (18) and $\varphi(0) = 0$. Recall that the Ricci curvature is given by

$$\mathrm{Ric}(\omega_\varphi) = -\mathcal{L}_X(\omega_\varphi) = \mu\varphi(\tau)\pi^*\gamma - \mu\frac{\varphi'}{\varphi}\sqrt{-1}\partial\tau \wedge \bar{\partial}\tau.$$

Since $\varphi(0) = 0$, $\varphi > 0$ on $(0, \infty)$, and $\mu\gamma \geq 0$, we only need to show that $\varphi' > 0$. To see that this is the case, we define a function

$$H(\tau) := \frac{Q^2}{Q' - \mu Q}e^{-\mu\tau} - \int_0^\tau e^{-\mu x}Q(x)dx. \tag{42}$$

Using the ODE (18), it is straight forward to prove that $\varphi' \geq 0$ iff $H \geq 0$. As $H(0) > 0$ for Q given by (11), we are done if we can show that $H' \geq 0$. From the definition of H, we compute

$$H'(\tau) = e^{-\mu\tau}\frac{Q}{(Q' - \mu Q)^2}\left((Q')^2 - QQ''\right), \tag{43}$$

so that $H' \geq 0$ if and only if $(Q')^2 - QQ'' \geq 0$. The later condition can be checked easily starting from the explicit expression for Q. Indeed, let $\beta_1, \ldots, \beta_n$ be the eigenvalues of the endomorphism $B = g_M^{-1}\gamma : T^{1,0}M \to T^{1,0}M$, and write

$$Q(\tau) = \det\left(\mathrm{Id} - \tau B\right) = \prod_{j=1}^n (1 - \beta_j\tau). \tag{44}$$

Then we have

$$\frac{(Q')^2 - QQ''}{Q^2} = \sum_{j=1}^n \frac{\beta_j^2}{(1 - \beta_j\tau)^2} \geq 0, \tag{45}$$

as required. For the second statement, it suffices to observe that $\varphi'(\tau) > 0$ if and only if $(Q')^2 - QQ'' > 0$, which is certainly true if $\gamma < 0$. This proves Theorem 4.1 for line bundles.

The arguments for the metrics in Theorem 1.2 are analogous. It also reduces to showing that $(Q')^2 - QQ'' \geq 0$, where Q is this time given by (31). □

Note that the non-negativity of Ricci curvature can also be expressed in terms of the potential function f. In particular, we have the following

Corollary 4.2. *Let ω_φ be a steady Kähler-Ricci soliton constructed in Theorem 1.1 or 1.2 and let $f = f(t)$ be defined by (6) or (27), respectively. Then f'' is monotone increasing.*

Proof. Recall from (14) or (33) that we have

$$-\mathcal{L}_X\omega_\varphi = -\mu\sqrt{-1}\partial\bar{\partial}f'(t) = \mu f''(t)\pi^*\gamma - \mu f'''(t)\sqrt{-1}\partial t\wedge\bar{\partial}t,$$

and so $\mathrm{Ric}(\omega_\varphi) = -\mathcal{L}_X\omega_\varphi$ can only be non-negative if $f'''(t)\geq 0$ since $\mu<0$. Thus, Theorem 4.1 implies that f'' is increasing. $\square$

We end this section by pointing out some growth properties of the potential function f.

Lemma 4.3. *Let ω_φ be a steady Kähler-Ricci soliton constructed in Theorem 1.1 or 1.2 and let $f = f(t)$ be related to φ by (10). Then there is a constant $C>0$ such that for all $t\geq C$, we have*

$$C^{-1}\leq f''(t)\leq C \quad \textit{and} \quad C^{-1}t\leq f'(t)\leq Ct. \tag{46}$$

Moreover, for all $j\in\mathbb{N}_0$ and $t\geq C$

$$C^{-1}(1+f'(t))^{-j}\leq |f^{(2+j)}(t)|\leq C(1+f'(t))^{-j}. \tag{47}$$

Proof. First note that the bound on $f''(t)$ in (46) implies the bound on $f'(t)$ after integrating the parameter t, so we only need to find $C>0$ such that

$$C^{-1}\leq f''(t)\leq C \tag{48}$$

for all $t\geq C$. Translating the problem into bounding $\varphi(\tau)$, we recall from (7) that

$$\tau=\tau(t)=f'(t) \quad \text{and} \quad \varphi(\tau(t))=f''(t). \tag{49}$$

Since $f'(t)$ is positive and increasing, we can choose a $C\geq 1$ such that the following estimate

$$\tau(t)=f'(t)\geq C^{-1} \tag{50}$$

holds for all $t\geq C$. Then we recall the asymptotic expansion (40)

$$\varphi(\tau(t)) = -\frac{m}{\mu}+O(1/\tau(t))$$

with $\mu<0$ implying that $\varphi(\tau(t))$ is uniformly bounded from above because of (50). Together with (49), this proves the upper bound for $f''(t)$ in (48). For the lower bound, note that $f''(t)>0$ is increasing and thus is bounded from below by some positive constant if $t\geq C$. Inequality (48) now follows, and so does (46).

Next, consider the case $j>0$, i.e. we estimate $f^{(2+j)}(t)$. Differentiating (49) and using the chain rule, we see that

$$f'''(t)=\varphi'(\tau(t))\frac{d\tau}{dt}(t)=\varphi'(\tau(t))\cdot f''(t).$$

Taking further derivatives of this equation, we conclude that $f^{(2+j)}$ can be written as

$$f^{(2+j)}=\sum_\alpha c_\alpha\cdot\varphi^{(\alpha_1)}\cdot\ldots\cdot\varphi^{(\alpha_i)}\cdot(f'')^j, \tag{51}$$

where the sum is over all multi-indices α with $\alpha_1 + \ldots + \alpha_i = j$ and c_α are constants only depending on the multi-index α. Since $f''(t)$ satisfies (48), it is sufficient to estimate derivatives of φ. In fact, we have for all $\beta \in \mathbb{N}$ that

$$C^{-1}\tau^{-\beta} \leq |\varphi^{(\beta)}(\tau)| \leq C\tau^{-\beta}, \tag{52}$$

because $\varphi(\tau)$ behaves asymptotically like a rational function of the form P/Q with polynomials $P(\tau), Q(\tau)$ having the same degree, see (23). Substituting $\tau(t) = f'(t)$ in (52) and combining the resulting estimate with (51), we finally obtain (47) as desired. □

The important point about Lemma 4.3 is estimate (46), i.e. that $f''(t)$ behaves like a constant and $f'(t)$ growths roughly like the function t in the limit $t \to \infty$. This will be crucial in the next section because we want to understand the asymptotic geometry of ω_φ.

Another interesting consequence of Lemma 4.3 is that the metrics ω_φ have bounded curvature and positive injectivity radius.

Lemma 4.4. *The curvature tensor of the steady solitons constructed in Theorem 1.1 and 1.2 is uniformly bounded and each of these metrics has positive injectivity radius.*

Proof. It is straight forward to see that the first claim reduces to bounding $f''(t), f'''(t)$ and $f^{(4)}(t)$, where φ and f are related by (10), so we focus on the second one.

According to [CGT82][Theorem 4.7], the lower bound on the injectivity radius follows if we can bound the volume of all unit balls uniformly from below. For this, recall that the function t identifies $E\setminus D \cong \mathbb{R}\times S$, where S is the S^1-bundle associated to $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$, see (26). Under this identification, the metric g_φ admits the following decomposition on $\mathbb{R} \times S$

$$g_\varphi = f''(t)\left(dt^2 + (Jdt)^2\right) + f'(t)\pi^*\hat{g} + \pi^* g_D, \tag{53}$$

where J denotes the complex structure on E, and $\hat{g}$, g_D are the (2,0) tensors associated to $-\gamma$, ω_D, respectively. By compactness of D, we only have to consider the set $\{t \gg 1\}$, on which g_φ is uniformly equivalent to the metric

$$g_t := dt^2 + (Jdt)^2 + t\pi^*\hat{g} + \pi^* g_D, \tag{54}$$

compare Lemma 4.3. Let us further denote $g_{S^1} := (Jdt)^2$ and rescale g_t by some fixed constant so that the diameter $\mathrm{diam}(S, g_{S^1})$ of each S^1-fibre satisfies $\mathrm{diam}(S, g_{S^1}) = 1/4$. It then suffices to bound the volume of unit balls w.r.t. g_t on the set $\{t \gg 1\}$ uniformly from below.

Let $x \in E$ with $t(x) \gg 1$ and denote the unit ball of g_t around x by $B_{g_t}(x, 1)$. We introduce families of metrics on M and S by declaring

$$\begin{aligned} g_{M,\tau} &:= \tau\hat{g} + p^* g_D \\ g_{S,\tau} &:= g_{S^1} + \pi^* g_{M,\tau} \end{aligned}$$

for each $\tau \geq 1$, where $p : M \to D$ is the projection as in (26) and $\pi : S \to M$. In particular, the projection π becomes a Riemannian submersion $\pi : (S, g_{S,\tau}) \to (M, g_{M,\tau})$. Using this notation and writing $x = (t(x), y)$, we obtain the following inclusion

$$B := [t(x) - 1/2, t(x)] \times B_{g_{S,t(x)}}(y, 1/2) \subset B_{g_t}(x, 1).$$

This is an immediate consequence of the decomposition (54) together with the fact that for all $p \in B$ we have $t(p) \leq t(x)$ and $g_{S,\tau_0} \leq g_{S,\tau_1}$ for all $\tau_0 \leq \tau_1$. Before estimating the g_t-volume $\mathrm{Vol}_{g_t}(B_{g_t}(x, 1))$ of the unit ball $B_{g_t}(x, 1)$, we observe that

$$g_{S,t(x)-1/2} = g_{S,t(x)} - \frac{1}{2}\pi^*\hat{g} \geq g_{S,t(x)} - \frac{1}{2}(t(x) - 1)\pi^*\hat{g} \geq \frac{1}{2} g_{S,t(x)} \tag{55}$$

provided $t(x) \geq 2$. Using the inclusion $B \subset B_{g_t}(x, 1)$ then implies that

$$\begin{aligned} \mathrm{Vol}_{g_t}(B_{g_t}(x, 1)) &\geq \mathrm{Vol}_{g_t}(B) \\ &\geq \frac{1}{2} \cdot \mathrm{Vol}_{g_{S,t(x)-1/2}}(B_{g_{S,t(x)}}(y, 1/2)) \\ &\geq 2^{-\frac{\dim_{\mathbb{R}} S}{2} - 1} \mathrm{Vol}_{g_{S,t(x)}}(B_{g_{S,t(x)}}(y, 1/2)), \end{aligned}$$

where we applied Fubini's theorem in the second line, and the last inequality follows from (55). Thus, it remains to bound the $g_{S,t(x)}$-volume of $B_{g_{S,t(x)}}(y, 1/2)$ uniformly from below.

We further reduce this volume bound to an integration on M by observing that the projection $\pi : S \to M$ satisfies

$$\pi^{-1}(B_{g_{M,t(x)}}(\pi(y), 1/4)) \subset B_{g_{S,t(x)}}(y, 1/2) \tag{56}$$

Indeed, given a $b \in B_{g_{M,t(x)}}(\pi(y), 1/4)$ and a length-minimizing curve $q : [0, 1] \to M$ from $\pi(y)$ to b, we may lift q to a horizontal curve $\tilde{q}$ in S from $\tilde{q}(0) = y$ to some point $\tilde{q}(1) \in \pi^{-1}(b)$. For any $a \in \pi^{-1}(b)$, the triangle inequality for the distance function $\mathrm{dist}_{g_{S,t(x)}}$ then yields

$$\begin{aligned} \mathrm{dist}_{g_{S,t(x)}}(y, a) &\leq \mathrm{dist}_{g_{S,t(x)}}(y, \tilde{q}(1)) + \mathrm{dist}_{g_{S,t(x)}}(\tilde{q}(1), a) \\ &\leq \mathrm{dist}_{g_{M,t(x)}}(\pi(y), b) + \frac{1}{4} \\ &< \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \end{aligned}$$

where the second inequality holds since we normalised each fibre $\pi^{-1}(b)$ to be of diameter $1/4$ and the third one follows since $\pi : S \to M$ is a Riemannian submersion. Hence, we conclude that $a \in B_{g_{S,t(x)}(y,1/2)}$ as claimed.

Inclusion (56) yields an estimate on the $g_{S,t(x)}$-volume as follows. We write $\omega_{t(x)}$ for the Kähler form of $g_{M,t(x)}$ and χ for the characteristic function of the ball $B_{g_{S,t(x)}}(y,1/2)$, and then observe that

$$\begin{aligned}\int_{B_{g_{S,t(x)}}(y,1/2)}(Jdt)\wedge\pi^*\omega_{t(x)}^{\dim_{\mathbb{C}}M} &= \int_{B_{g_{M,t(x)}}(\pi(y),1/2)}\pi_*(\chi Jdt)\cdot\omega_{t(x)}^{\dim_{\mathbb{C}}M}\\ &\geq \int_{B_{g_{M,t(x)}}(\pi(y),1/4)}\pi_*(Jdt)\cdot\omega_{t(x)}^{\dim_{\mathbb{C}}M}\\ &= \mathrm{Vol}_{g_{S^1}}(S^1)\cdot\int_{B_{g_{M,t(x)}}(\pi(y),1/2)}\omega_{t(x)}^{\dim_{\mathbb{C}}M}.\end{aligned}$$

Here, $\pi_*(\chi Jdt)$ denotes the function on M obtained by integrating χJdt over fibres, i.e. $\pi_*(\chi Jdt)(b)=\int_{\pi^{-1}(b)}\chi Jdt$, so that the first equality follows from Fubini's theorem. In the second line, we used that $\chi\equiv 1$ on the set $\pi^{-1}(B_{g_{M,t(x)}}(\pi(y),1/4))$ by (56) and the final equation holds because the volume of each S^1-fibre is the same by (54). Thus, it remains to find a constant $C>0$, independent of $x=(t(x),y)$, such that

$$\mathrm{Vol}_{g_{M,t(x)}}(B_{g_{M,t(x)}}(\pi(y),1/4))\geq C^{-1}>0. \tag{57}$$

To prove this, let us first assume that $\hat{g}$ is positive definite. Then we can find a constant $C_0>0$, only depending on the eigenvalues of $\hat{g}$ w.r.t. g_M, such that

$$C_0^{-1}\tau g_M\leq g_{M,\tau}\leq C_0\tau g_M,$$

for all $\tau\geq 1$ and with $g_M:=g_{M,1}$. Since we can rescale by a fixed constant, it suffices to bound the τg_M-volume of $B_{\tau g_M}(z,1)$ from below by a constant independent of both $z\in M$ and $\tau\gg 1$. We note that

$$B_{\tau g_M}(z,1)=B_{g_M}(z,\tau^{-\frac{1}{2}})$$

and by compactness, the g_M-volume $\mathrm{Vol}_{g_M}(B_{g_M}(z,\tau^{-\frac{1}{2}}))$ is, up to some uniform constant, bounded from below by $\tau^{-\frac{\dim_{\mathbb{R}}M}{2}}$ for τ sufficiently large. This shows that $\mathrm{Vol}_{\tau g_M}(B_{\tau g_M}(z,1))=\tau^{\frac{\dim_{\mathbb{R}}M}{2}}\mathrm{Vol}_{g_M}(B_{g_M}(z,\tau^{-\frac{1}{2}}))$ is indeed uniformly bounded from below and (57) then follows.

Let us now assume that γ has at least one zero eigenvalue w.r.t. g_M. Since these eigenvalues are assumed to be constant over M, its Kernel $\mathrm{Ker}\,\gamma$ defines a proper subbundle of $T^{1,0}M$. Moreover, γ is closed, so that $\mathrm{Ker}\,\gamma$ is integrable according to Frobenius' theorem. Thus, if n is the complex dimension of M and k the number of positive eigenvalues of γ, we find a chart around each point defined on some neighborhood of the Euclidean unit ball $B(1)\subset\mathbb{C}^n\cong\mathbb{C}^k\times\mathbb{C}^{n-k}$ around the origin such that each slice $\{z_0\}\times\mathbb{C}^{n-k}$ in $B(1)$ is an integral manifold for $\mathrm{Ker}\,\gamma$, i.e.

$$\gamma(v,v)>0\quad\text{and}\quad\gamma(v,w)=0\quad\text{for all } v\in T\mathbb{C}^k,\ w\in T\mathbb{C}^{n-k}.$$

By compactness, we can cover M by finitely many of such Euclidean balls $B_j(1)$ for $j = 1, \ldots, N$ and also find a constant $C_0 > 0$ such that

$$C_0^{-1} g_{\mathbb{C}^n} \leq g_M \leq C_0 g_{\mathbb{C}^n} \quad \text{on each } B_j(1),$$

where $g_{\mathbb{C}^n}$ is the Euclidean metric on $B_j(1)$ and the constant C_0 is independent of the ball $B_j(1)$.

For $\tau \geq 1$, we also consider the following product metric

$$g_{\mathbb{C}^n,\tau} := (1+\tau) g_{\mathbb{C}^k} + g_{\mathbb{C}^{n-k}} \quad \text{on} \quad \mathbb{C}^n \cong \mathbb{C}^k \times \mathbb{C}^{n-k}$$

Then there exists a uniform constant $C > 0$, which only depends on C_0 and the g_M-eigenvalues of γ, such that

$$C^{-1} g_{\mathbb{C}^n,\tau} \leq g_{M,\tau} \leq C g_{\mathbb{C}^n,\tau} \quad \text{on each} \quad B_j(1). \tag{58}$$

Let $\varepsilon > 0$ be the Lebesgue number associated to the cover $\{B_j(1)\}_{j=1,\ldots N}$ of the manifold (M, g_M), i.e. the ball $B_{g_M}(z, \varepsilon)$ is contained in $B_j(1)$ for some j. Note that since $g_M \leq g_{M,\tau}$, we also have $B_{g_{M,\tau}}(z, 1) \subset B_{g_M}(z, 1) \subset B_j(1)$. Additionally, we may assume that $\varepsilon < 1$, so that it suffices to bound the $g_{M,\tau}$-volume of the smaller ball $B_{g_{M,\tau}}(z, \varepsilon)$ from below because the constant $\varepsilon > 0$ is independent of both $z \in M$ and $\tau \geq 1$.

This, in turn, can be reduced to bounding the $g_{\mathbb{C}^n,\tau}$ -volume of $B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon)$. Indeed, this is a direct consequence of (58), which implies that the volume forms of $g_{\mathbb{C}^n,\tau}$ and $g_{M,\tau}$ are uniformly equivalent on $B_{g_{M,\tau}}(z, \varepsilon)$ and also that we have the inclusion

$$B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon) \subset B_{g_{M,\tau}}(z, \varepsilon).$$

For the remaining lower volume bound, observe that the following product of Euclidean balls

$$B_{g_{\mathbb{C}^k}}\left(z, \frac{1}{2} C^{-\frac{1}{2}} \varepsilon (1+\tau)^{-\frac{1}{2}}\right) \times B_{g_{\mathbb{C}^{n-k}}}\left(z, \frac{1}{2} C^{-\frac{1}{2}} \varepsilon\right) \tag{59}$$

is contained in $B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon)$. Applying Fubinis' theorem to the product (59) and using the fact that the volume form of $g_{\mathbb{C}^n,\tau}$ is equal to $(1+\tau)^k$-times the volume form of $g_{\mathbb{C}^n}$ then yields the required lower bound on the $g_{\mathbb{C}^n,\tau}$-volume of $B_{g_{\mathbb{C}^n,\tau}}(z, C^{-\frac{1}{2}}\varepsilon)$, which is independent of both $z \in M$ and $\tau \geq 1$. This finishes the proof.

□

5. Uniqueness in a Kähler class

The purpose of this section is to prove Theorem 1.3. We begin by briefly recalling notation from Sections 2 and 3 and then define the function spaces appearing in Theorem 1.3. We also explain how to reduce the proof to a $\partial\bar{\partial}$-Lemma, which is stated below (Theorem 5.4).

5.1. **A $\partial\bar{\partial}$-Lemma.** Throughout this section, let $\pi : E \to D$ be a rank m holomorphic vector bundle over a compact Kähler manifold (D, ω_D). The complex dimension of E (as a manifold) is denoted by $m+d$, where d is the complex dimension of D. If $m = 1$, we assume that it satisfies the conditions in Theorem 1.1, and if $m \geq 2$, we assume E is given as in Theorem 1.2. Also recall that we defined a radial function $r : E \to \mathbb{R}_{\geq 0}$ by $r(v) = \sqrt{h(v, \bar{v})}$, which vanishes along the zero section of E and we set $t := 2\log r$. Note that we can use the function t to identify E, with its zero section removed, as the product $\mathbb{R} \times S$, where S is the S^1-bundle associated to $\mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathbb{P}(E)$, see Diagram (26). Under this identification, the function t on $E \backslash D$ corresponds to the projection onto the first factor of $\mathbb{R} \times S$.

Let ω_φ be the Kähler Ricci soliton constructed in Theorem 1.1 or 1.2, i.e. ω_φ is defined by (9) with φ satisfying (18) if E is a line bundle or by (27) and (35) if $m \geq 2$. We denote the corresponding Riemannian metric by g_φ.

On the manifold $\mathbb{R} \times S$, we can write the metric g_φ as follows. If J denotes the complex structure on E and g_D and $\hat{g}$ are the (2,0) tensors associated to ω_D and $-\gamma$, respectively, then

$$g_\varphi = f''(t)\left(dt^2 + (Jdt)^2\right) + f'(t)\pi^*\hat{g} + \pi^* g_D, \tag{60}$$

where f can be reconstructed from φ via (10). We would also like to point out that we allowed $-\gamma$ to have zero-eigenvalues, i.e. $\hat{g}$ is only semidefinite. As a consequence, the volume growth of g_φ will be determined by the zero-eigenvalues of $-\gamma$.

Before stating the main theorem of this section, we require a definition of weighted function spaces. As a weight function, we choose $w : E \to \mathbb{R}_+$ to be defined by

$$w(t) := 1 + f'(t). \tag{61}$$

This choice is inspired by the work of Hein [Hei11]. Indeed, the following lemma shows that w has the same properties as the function ρ in [Hei11][Theorem 1.6].

Lemma 5.1. *Fix $x_0 \in E$ and denote the distance function of g_φ by $\rho(x)$. Then there exists a constant $C > 0$ such that*

$$C^{-1}w(t(x)) \leq (1 + \rho(x)) \leq Cw(t(x)) \tag{62}$$

for all $x \in E$ with $w(t(x)) \geq C$. Moreover, w satisfies

$$|\nabla w| + w|\Delta w| \leq C, \tag{63}$$

where $|\cdot|, \nabla$ and Δ are associated with g_φ.

Proof. We identify $E \setminus D \cong \mathbb{R} \times S$ and without loss of generality, we can assume $x_0 = (t_0, y_0) \in \mathbb{R} \times S$. Let $(t, y) \in \mathbb{R} \times S$ with $t_0 \leq t$ and consider a shortest path $q_{t,y} = (q_t, q_y) : [0,1] \to \mathbb{R} \times S$ from (t_0, y_0) to (t, y). Its length $L(q_{t,y})$ is given by

$$L(q_{t,y}) = \int_0^1 \sqrt{g_\varphi\left(\dot{q}_{t,y}(\sigma), \dot{q}_{t,y}(\sigma)\right)}d\sigma.$$

Then (62) reduces to finding a constant $C > 0$ such that

$$C^{-1}w(t) \leq L(q_{t,y}) \leq Cw(t) \tag{64}$$

for all $y \in S$ and all $t \geq C$. In fact, it is sufficient to show inequality (64) with $w(t)$ replaced by t since there is a $C > 0$ such that

$$C^{-1}t \leq w(t) \leq Ct \tag{65}$$

for $t \geq C$, compare (46). Thus, we begin by choosing $C > 0$ such that (65) holds, and we increase $C > 0$ as we go along, if necessary.

For proving the lower bound in (64), we estimate

$$L(q_{t,y}) \geq \int_0^1 \sqrt{f''(q_{t,y})}\dot{q}_t(\sigma)d\sigma \geq \sqrt{f''(t_0)}(t - t_0),$$

as required. Before showing the upper bound, we conclude from (46) that

$$g_\varphi \leq C\left(dt^2 + tg_S\right),$$

where we define $g_S := (Jdt)^2 + \pi^*\hat{g} + \pi^*g_D$ with $\hat{g}$ and g_D as in (60). Also observe that we can now assume q_t to be the linear path in the $\mathbb{R}$-factor, i.e. $q_t(\sigma) = \sigma(t - t_0) + t_0$. Then we obtain

$$\begin{aligned} L(q_{t,y}) &\leq C\int_0^1 \dot{q}_t(\sigma)d\sigma + C\int_0^1 \sqrt{q_t(\sigma)} \cdot \sqrt{g_S(\dot{q}_y(\sigma), \dot{q}_y(\sigma)}d\sigma \\ &\leq Ct + C\operatorname{diam}(S, g_S)\sqrt{t} \qquad (66) \\ &\leq Ct, \end{aligned}$$

for all t sufficiently large and with $\operatorname{diam}(S, g_S)$ denoting the diameter of the compact manifold (S, g_S). Now (5.1) follows immediately.

For the second claim, observe from (27) that we have

$$|\nabla w| = f'',$$

which is uniformly bounded according to Lemma 4.3. For bounding the Laplace operator $\Delta w = \Delta f'$, recall that on a Kähler manifold, the Laplace operator Δ satisfies $\Delta = 2\operatorname{tr}_{\omega_\varphi}\sqrt{-1}\partial\bar{\partial}$, where $\operatorname{tr}_{\omega_\varphi}$ denotes the trace computed w.r.to g_φ. Then we apply (33) to obtain

$$\begin{aligned} \Delta f' &= 2\operatorname{tr}_{\omega_\varphi}\left(\sqrt{-1}\partial\bar{\partial}f'\right) \\ &= \frac{2}{\mu}\operatorname{tr}_{\omega_\varphi}\left(\mathcal{L}_X\omega_\varphi\right) \\ &= 2\left(\varphi\frac{Q'}{Q} + \varphi'\right), \end{aligned}$$

where the last equality holds since there is the following formula

$$\mathrm{tr}_{\omega_\varphi}(-\pi^*\gamma) = \frac{Q'}{Q},$$

see [HS02][(2.22)]. Using the soliton ODE (35), we continue

$$\begin{aligned} w\Delta w &= (1+f')\Delta f' \\ &= 2(1+\tau)\left(\varphi\frac{Q'}{Q} + \varphi'\right) \\ &= 2(1+\tau)\left(m + \mu\varphi\right), \end{aligned}$$

which is also uniformly bounded because of the asymptotic expansion (40). This, together with the uniform bound on $|\nabla w|$, implies (63). □

Lemma 5.1 ensures that our definition of weighted function spaces below coincides with the one used in [Hei11]. These spaces are well-adapted to study the Laplace operator on a wide class of complete manifolds.

Definition 5.2. Let Λ^*T^*E be the exterior algebra of T^*E and consider $\delta \in \mathbb{R}$ and $k \in \mathbb{N}_0$. We define $C^k_\delta(\Lambda^*T^*E)$ to be the space of k-times continuously differentiable sections η of Λ^*T^*E such that the norm

$$||\eta||_{C^k_\delta} := \sum_{j=0}^{k} \sup_E |w^{j-\delta}\nabla^j\eta|$$

is finite, where w is given by (61) and ∇, $|\cdot|$ are associated to g_φ. We also set

$$C^\infty_\delta(\Lambda^*T^*E) := \bigcap_{k\in\mathbb{N}_0} C^k_\delta(\Lambda^*T^*E).$$

In other words, elements in $C^\infty_\delta(\Lambda^*T^*E)$ grow at most like w^δ and their l-th derivatives at most like $w^{\delta-l}$. Having introduced the necessary notation, we can now state the main result of this section.

Theorem 5.3. *Let ω_φ be a steady Kähler-Ricci soliton constructed in Theorem 1.1 or 1.2. Assume that ω is a Kähler-Ricci soliton on E with the same vector field as ω_φ such that $[\omega] = [\omega_\varphi] \in H^2(E)$. If moreover $\omega_\varphi - \omega \in C^\infty_{-\delta}(\Lambda^2T^*E)$ for some $\delta > 2$, then $\omega_\varphi = \omega$.*

The main part of proving Theorem 5.3 will be a $\partial\bar\partial$-Lemma, with controlled growth. In fact, we will prove

Theorem 5.4. *Let $\delta > 2$ and $\eta \in C^\infty_{-\delta}(\Lambda^*T^*E)$ be a real $(1,1)$ form. If η is d-exact, then $\eta = \sqrt{-1}\partial\bar\partial u$ for some $u \in C^\infty_{2-\delta}(E)$.*

Assuming this result, Theorem 5.3 follows immediately.

Proof of Theorem 5.3. By Theorem 5.4, there exists a $u \in C^\infty_{2-\delta}(E)$ such that $\omega_\varphi - \omega = \sqrt{-1}\partial\bar{\partial}u$. Since $2 - \delta < 0$, u and all its derivatives tend to zero at infinity, so we can apply the maximum principle [BM17][Proposition 1.2] and conclude that $\omega_\varphi = \omega$. □

The remainder of this section is devoted to proving Theorem 5.4. We follow the ideas for asymptotically conical metrics given in [CH13][Section 3], which rely on two main ingredients. Firstly, we need to understand solutions to Poisson's equation $\Delta u = h$ and their growth behaviour (Section 5.2). Secondly, we need to show that harmonic (1,0) forms of certain growth behaviour are identically zero (Section 5.3). The proof of Theorem 5.4 will then be finished in Section 5.4.

5.2. **The Laplace Operator.** We start by considering the Laplace operator Δ of the metric g_φ acting on suitably weighted Hölder spaces, which we now define.

Definition 5.5. Let $\mathrm{dist}(x, y)$ be the distance between $x, y \in E$ measured w.r.to g_φ and denote the injectivity radius of g_φ by i_0. (Note that $i_0 > 0$ by Lemma 4.4). For $0 < \alpha < 1$ and $\delta \in \mathbb{R}$, we define a seminorm on the space of all tensor fields T on E by

$$[T]_{C^{0,\alpha}_\delta} := \sup_{\substack{x \neq y \in E \\ \mathrm{dist}(x,y) < \frac{i_0}{2}}} \left(\min(w(x), w(y))^{-\delta} \frac{|T_x - T_y|}{\mathrm{dist}(x, y)^\alpha} \right),$$

where the norm $|\cdot|$ is induced by g_φ and the difference $T_x - T_y$ is defined by using parallel transport along the minimal geodesic from x to y.

The weighted Hölder space $C^{k,\alpha}_\delta(E)$ is then defined to be the subset of all $u \in C^k_\delta(E)$ for which the norm

$$||u||_{C^{k,\alpha}_\delta} := ||u||_{C^k_\delta} + [\nabla^k u]_{C^{0,\alpha}_{\delta-k-\alpha}}$$

is finite.

The Laplace operator Δ acts as

$$\Delta : C^{2,\alpha}_{2+\delta}(E) \to C^{0,\alpha}_\delta(E),$$

for any $\delta \in \mathbb{R}$ and we are interested in the surjectivity of this operator. A partial answer to this question is provided in [Hei11].

Given $h \in C^{0,\alpha}_\delta(E)$ with $\delta < -2$, we can essentially always solve Poisson's equation $\Delta u = h$, but it is not clear how the solution u will behave as $t \to \infty$. This depends on the volume growth of g_φ, which is related to the degree k of the polynomial Q defined in (11) for $m = 1$ or (31) for $m \geq 2$. Alternatively, it is evident from the definition of Q that k is equal to $m + d - 1$ minus the number of zero-eigenvalues of γ. (Recall that $m + d - 1$ is the complex dimension of $\mathbb{P}(E)$.)

More precisely, we will prove the following important proposition about the existence of solutions to $\Delta u = h$.

Proposition 5.6. *Let $\delta > 2$ and suppose $h \in C^{0,\alpha}_{-\delta}(E)$.*

(i) *If $k \leq 1$, assume $\int h\omega_\varphi^{m+d} = 0$ additionally. Then there exists a $u \in C^{2,\alpha}(E)$ such that $\Delta u = f$ and the integral $\int |\nabla u|^2 \omega_\varphi^{m+d}$ is finite.*

(ii) *If $k > 1$ and $2 < \delta < k+1$, then there exists $u \in C^{2,\alpha}(E)$ such that $\Delta u = h$ and $u = O(w^{2-\delta+\varepsilon})$ as well as $|\nabla u| = O(w^{2-\delta+\varepsilon})$ for all $\varepsilon > 0$.*

Before proceeding with its proof, we first of all need to check that we can indeed apply Hein's work [Hei11][Theorem 1.5, 1.6], i.e. we have to verify that the metric (E, g_φ) satisfies Hein's condition SOB(β). For the sake of completeness, we recall [Hei11][Definition 1.1] here.

Definition 5.7 ([Hei11][Definition 1.1])**.** A Riemannian manifold (M, g) is called SOB(β) if there exists a $x_0 \in M$ and a constant $C \geq 1$ satisfying the following:

(i) The set $B(x_0, s_1) \setminus \overline{B}(x_0, s_0)$ is connected for all $s_1 > s_0 \geq C$,

(ii) $\mathrm{Vol}(B(x_0, s)) \leq Cs^\beta$ holds for all $s \geq C$,

(iii) $\mathrm{Vol}(B(x, (1 - C^{-1})\rho(x))) \geq C^{-1}\rho(x)^\beta$ holds for all $x \in M$ with $\rho(x) \geq C$,

(iv) $\mathrm{Ric}_x \geq -C\rho(x)^{-2}$ holds if $\rho(x) \geq C$.

Here $B(x_0, s)$ denotes the geodesic ball around x_0, $\mathrm{Vol}(B(x_0, s))$ its volume and $\rho(x)$ denotes the distance from x to x_0.

As the next lemma shows, the soliton metrics (E, g_φ) constructed in Theorem 1.1 and 1.2 are SOB($k+1$).

Lemma 5.8. *The metric (E, g_φ) is SOB($k+1$), where k is equal to $m+d-1$ minus the number of zero-eigenvalues of the curvature form γ on $\mathbb{P}(E)$.*

Proof. We fix $x_0 \in D \subset E$ to be a point on the zero-section of E. Thanks to Theorem 4.1, Condition (iv) in Definition 5.7 is clearly satisfied, so we focus on (i), (ii), (iii).

Beginning with the volume estimates (ii) and (iii), we consider the volume form of g_φ, which is given by

$$\frac{\omega_\varphi^{m+d}}{(m+d)!} = \frac{\sqrt{-1}}{(m+d)!} f''Q(f')\partial t \wedge \bar{\partial} t \wedge (\pi^*\omega_D - \pi^*\gamma)^{m+d-1}, \tag{67}$$

where the polynomial Q is defined by (11) if $m = 1$ or (29) if $m \geq 2$. Recall from above, that the degree of Q is equal to k as defined in Lemma 5.8. If we then choose $C \geq 1$ such that (46) is satisfied, we obtain for large $t \geq C$:

$$C^{-1}t^k \leq f''(t)Q(f'(t)) \leq Ct^k. \tag{68}$$

Moreover, Lemma 5.1 implies that

$$C^{-1}t(x) \leq \rho(x) \leq Ct(x) \tag{69}$$

if $\rho(x) \geq C$. In the estimates that follow, we increase $C > 0$ if necessary but it still denotes a uniform constant which only depends on the geometry of (E, g_φ) and the choice of base point x_0.

For verifying (ii), let $s \geq C$ and observe that (69) implies

$$B(x_0, s) \subset B(x_0, C) \cup \{y \in E \,|\, 0 \leq t(y) \leq Cs\}\,.$$

Integrating over these sets and using (67), we obtain

$$\begin{aligned}\mathrm{Vol}(B(x_0, s)) &\leq C + C \int_0^{Cs} \int_S f''(t) Q(f'(t)) dt \wedge \bar{\partial} t \wedge (\pi^*(\omega_D - \gamma))^{m+d-1} \\ &\leq C + C \int_0^{Cs} t^k dt \\ &\leq C s^{k+1},\end{aligned}$$

where we used (68) in the second line. This proves (ii) of Definition 5.7 with $\beta = k + 1$.

For showing (iii), the goal is to choose a new $C_0 \geq 1$ such that for all $x \in E$ with $\rho(x) \gg 1$ sufficient large, we have an inclusion of the form

(70)
$$B(x, (1 - C_0^{-1})\rho(x)) \supset \left\{ t(y) \in \left[t(x) + 1, t(x) + C_0^{-1}\rho(x) - \sqrt{\rho(x)} \right] \right\}.$$

Indeed, if (70) holds, we can integrate and use (68) to estimate

$$\begin{aligned}\mathrm{Vol}(B(x, (1 - C_0^{-1})\rho(x))) &\geq C_0^{-1} \int_{t(x)+1}^{t(x)+C_0^{-1}\rho(x)-\sqrt{\rho(x)}} \sigma^k d\sigma \\ &\geq C_0^{-1} \rho(x)^{k+1},\end{aligned}$$

which is (iii) with $\beta = k + 1$ as required. Hence it remains to check inclusion (70). To see that this is true, we again introduce the metric $g_S := (Jdt)^2 + \pi^*\hat{g} + \pi^* g_D$ on the cross-section S as in the proof of Lemma 5.1, so that

$$g_\varphi \leq C \left(dt^2 + t g_S \right) =: g_t. \tag{71}$$

To estimate the distance function of g_t from above, we proceed as in (66). Given $x, y \in E$ with $C \leq t(x)$ and $t(x) \leq t(y)$, we consider a path $q : [0, 1] \to E$ from $q(0) = x$ to $q(1) = y$, which we write as $q = (q_1, q_2)$ under the identification $E \setminus D \cong \mathbb{R} \times S$. Furthermore, we assume that $q_1(\sigma) = \sigma(t(y) - t(x)) + t(x)$ is the linear path from $t(x)$ to $t(y)$, so that we estimate using (46)

$$\begin{aligned}\mathrm{dist}_{g_t}(x, y) &\leq C \int_0^1 \dot{q}_1(\sigma) d\sigma + C \int_0^1 \sqrt{q_1(\sigma)} \sqrt{g_S(\dot{q}_2(\sigma), \dot{q}_2(\sigma))} d\sigma \\ &\leq C(t(y) - t(x)) + C \,\mathrm{diam}(S, g_S) \left(\sqrt{t(y) - t(x)} + \sqrt{t(x)} \right)\end{aligned}$$

Together with (69) and (71), this implies

$$\operatorname{dist}_{g_\varphi}(x,y) \leq C\left(t(y)-t(x)+\sqrt{t(y)-t(x)}\right)+C\sqrt{\rho(x)} \tag{72}$$

from which we can deduce inclusion (70). Indeed, let $C>0$ satisfy (72) and define a new constant $C_0>0$ by $C_0^{-1}=C^{-1}(1-C^{-1})$. If we then assume that $\rho(x)\gg 1$ is large enough so that $C_0^{-1}\rho(x)-\sqrt{\rho(x)}>1$, we estimate for all $y\in E$ with $t(x)+1\leq t(y)\leq t(x)+C_0^{-1}\rho(x)-\sqrt{\rho(x)}$:

$$\begin{aligned}\operatorname{dist}_{g_\varphi}(x,y) &\leq C\left(t(y)-t(x)\right)+C\sqrt{\rho(x)}\\ &\leq CC_0^{-1}\rho(x)-C\sqrt{\rho(x)}+C\sqrt{\rho(x)}\\ &=\left(1-C^{-1}\right)\rho(x).\end{aligned}$$

Here we obtained the first inequality by applying $t(y)-t(x)\geq 1$ to (72) and the second inequality makes use of the upper bound on $t(y)$. This shows inclusion (70) and thus (iii).

It remains to verify Condition (i). By compactness of D, we can choose $C>1$ such that for all $s\geq C$ the ball $B(x_0,s)$ contains a tubular neighborhood of the zero section. Given $x\in E\setminus D$, we denote the complex line thorough x by $L_x\cong\mathbb{C}$. We need to understand the shape of the intersection of L_x with the set $B_{(s_0,s_1)}(x_0):=B(x_0,s_1)\setminus\overline{B}(x_0,s_0)$ for all $s_1>s_0\geq C$.

First, we claim that for each $x\in B(x_0,s)$, the radial path q_{rad} in L_x from x to $0\in L_x$ is entirely contained in the ball $B(x_0,s)$. Note that for this to be true it suffices to show that the function ρ is increasing along q_{rad}. In order to prove this, use the identification $E\setminus D\cong\mathbb{R}\times S$ and write $x=(a_1,b)$. Let $q:[0,1]\to E$ be a shortest geodesic from $q(0)=x_0$ to $q(1)=(a_1,b)$. On $E\setminus D$, we decompose $q=(q_1,q_2)$ and let us assume for the moment that $q_1(\sigma)$ is increasing in $\sigma\in[0,1]$. Given $a_0<a_1$, we then choose a $\sigma_0\in(0,1)$ with $q_1(\sigma_0)=a_0$ and reparameterize the path q by declaring $q_{\sigma_0}(\sigma):=(q_1(\sigma_0\sigma),q_2(\sigma))$, so that q_{σ_0} is a path from x_0 to (a_0,b). It follows from (60) that we have

$$g_\varphi\left(\dot{q}_{\sigma_0}(\sigma),\dot{q}_{\sigma_0}(\sigma)\right)\leq g_\varphi\left(\dot{q}(\sigma),\dot{q}(\sigma)\right)$$

for all $\sigma\in[0,1]$, since f'' and f' are both increasing and we assumed that $q_1(\sigma_0\sigma)\leq q_1(\sigma)$. Then we conclude $L(q_{\sigma_0})\leq L(q)$ and thus $\rho(a_0,b)\leq\rho(a_1,b)$ for all $a_0<a_1$ and $b\in S$, as we claimed.

Hence, the claim holds if we show that q_1 is increasing. Recall that by definition, $q_1=t(q)$ and clearly q_1 increases if and only if $r^2(q)=e^{t(q)}$ does, where $r:E\to\mathbb{R}_{\geq 0}$ is defined at the beginning of Section 5. Since x_0 lies on the zero section of E, we have $r(q(0))=0$ and consequently there is a $\hat{\sigma}\in[0,1)$ such that

$$r(q(\sigma))=0 \;\text{ for all } \sigma\in[0,\hat{\sigma}] \quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}\sigma}r^2(q(\sigma))>0 \quad\text{on}\quad (\hat{\sigma},\hat{\sigma}+\varepsilon)$$

for some small $\varepsilon > 0$. In particular, $\lim_{\sigma\to\hat{\sigma}^+} \dot{q}_1(\sigma) \geq 0$ and we only have to rule out the existence of two points $\sigma_1, \sigma_2 \in [\hat{\sigma}, 1]$ with $\sigma_1 < \sigma_2$ such that

$$q_1(\sigma_1) = q_2(\sigma_2) \quad \text{and } q_1(\sigma_1) < q_1(\sigma) \quad \text{for all } \sigma \in (\sigma_1, \sigma_2). \tag{73}$$

However, if this was the case, then the path q cannot be length-minimizing. Indeed, suppose that there are such numbers σ_1, σ_2 satisfying (73). Then we define a new path $\tilde{q}$ from x_0 to x by

$$\tilde{q}(\sigma) = \begin{cases} q(\sigma) & \text{if } \sigma \in [0,1] \setminus (\sigma_1, \sigma_2) \\ (q_1(\sigma_1), q_2(\sigma)) & \text{if } \sigma \in (\sigma_1, \sigma_2). \end{cases}$$

Using the decomposition (60) and the fact that f'' is increasing, we see that

$$L(\tilde{q}) < L(q),$$

contradicting the minimality of q. It follows that q_1 must be increasing.

Now we can verify Condition (i), so consider any $s_1 > s_0 \geq C$. As shown in the previous paragraph, both $L_x \cap B(x_0, s_j)$ with $j = 1, 2$ are star-shaped regions with center $0 \in L_x$, so the complement $L_x \cap B_{(s_0,s_1)}(x_0)$ is diffeomorphic to a genuine open annulus in $\mathbb{C}$. From this, we deduce that $B_{(s_0,s_1)}(x_0)$ is a fibre bundle over $\mathbb{P}(E)$ with annuli in $\mathbb{C}$ as fibres. In particular, $B_{(s_0,s_1)}(x_0)$ is connected because D is, finishing the proof. □

Before proving Proposition 5.6, we study the spaces $C^{k,\alpha}_\delta(E)$ further. In fact, Lemma 5.1 allows us to obtain the expected embedding theorems and also Schauder estimates for Δ.

Lemma 5.9 (Embeddings)**.** *Let $k, l \in \mathbb{N}$, $0 < \alpha_0, \alpha_1 < 1$ and $\delta_0 \leq \delta_1$. Then there are the following continuous embeddings:*

(i) $C^k_{\delta_0}(E) \subset C^l_{\delta_1}(E)$ *if* $l \leq k$,

(ii) $C^{k,\alpha_0}_{\delta_0}(E) \subset C^{l,\alpha_1}_{\delta_1}(E)$ *if* $l \leq k$ *and* $\alpha_1 \leq \alpha_0$,

(iii) $C^{k+1}_\delta(E) \subset C^{k,1}_\delta(E)$. *In particular,* $C^\infty_\delta(E) = \bigcap_{k\in\mathbb{N}_0} C^{k,\alpha}_\delta(E)$.

The proof of this lemma is analogue to [CSCB79][Lemma 2], so we omit it here.

Now we are in a position to show Proposition 5.6.

Proof of Proposition 5.6. Part (i) is a direct consequence of Theorem 1.5 in [Hei11]. Indeed, (E, g_φ) is SOB$(k + 1)$ by Lemma 5.8, and, because of Lemma 5.1, we have that $|h| = O(\rho^{-\delta})$ with $\delta > 2$, where $\rho(x)$ denotes the distance to some fixed point x_0. Then (i) is precisely [Hei11][Theorem 1.5].

For (ii), we note that the function w satisfies the assumption of Theorem 1.6 in [Hei11], see Lemma 5.1. Consequently, [Hei11][Theorem 1.6] gives a $u \in C^{2,\alpha}(E)$ such that $\Delta u = h$ and $u = O(w^{2-\delta+\varepsilon})$ for all $\varepsilon > 0$.

Then it only remains to verify the decay rate of $|\nabla u|$, which is a consequence of standard Schauder theory. Indeed, since the curvature of g_φ is bounded by Lemma 4.4, we can find $s > 0$ and $Q > 0$ such that for all $x \in E$, there is a chart Φ_x from to the Euclidean ball $B_x(s) \subset \mathbb{R}^{m+d}$ of radius s onto a neighborhood of x so that $\frac{1}{Q} g_{\mathrm{euc}} \leq \Phi_x^* g_\varphi \leq Q g_{\mathrm{euc}}$ and $||\Phi_x^* g_\varphi||_{C^{1,\alpha}(B_x(s))} \leq Q$ ([Pet97][Theorem 4.1]). Here, g_{euc} denotes the flat metric and $|| \cdot ||_{C^{1,\alpha}(B_x(s))}$ the Euclidean Hölder norm. For simplicity, we suppress the chart Φ_x and view $B_x(s)$ as a subset of E. Also note that we can assume that s is strictly smaller than the injectivity radius of (E, g_φ). Applying the Euclidean Schauder estimates ([GT01][Theorem 6.2]) to the balls $B_x(s)$, we obtain that

$$
\begin{aligned}
|\nabla u|_{g_\varphi}(x) &\leq Q |du|_{g_{\mathrm{euc}}}(x) \\
&\leq Q ||u||_{C^{2,\alpha}(B_x(s))} \\
&\leq Q C_0 \left(||h||_{C^{0,\alpha}(B_x(s))} + ||u||_{C^0(B_x(s))} \right)
\end{aligned}
\tag{74}
$$

for some uniform constant $C_0 > 0$ depending only on α, s and Q. Moreover, the weight function w is chosen so that there is a uniform constant $C_1 > 0$ such that for all $x \in E$ with $t(x) \gg 1$ and all $y \in B_x(s)$, we have $\frac{1}{C_1} w(y) \leq w(x) \leq C_1 w(y)$. This follows directly from Lemma 5.1 and the fact that g_φ and g_{euc} are uniformly equivalent on $B_x(s)$. Therefore, we continue to estimate for all $x \in E$ with $t(x) \gg 1$ and all $y \in B_x(s)$:

$$
u(y) \leq C w(y)^{2-\delta+\varepsilon} \leq C C_1^{2-\delta+\varepsilon} w(x)^{2-\delta+\varepsilon},
$$

i.e. $||u||_{C^0(B_x(s))} = O(w(x)^{2-\delta+\varepsilon})$. Similarly, we conclude that

$$
||h||_{C^{0,\alpha}(B_x(s))} = O(w(x)^{-\delta})
$$

because s is chosen strictly smaller than the injectivity radius and $h \in C^{0,\alpha}_{-\delta}(E)$. In combination with (74) we consequently arrive at

$$
|\nabla u|_{g_\varphi} = O(w^{2-\delta+\varepsilon}),
$$

as claimed. □

The issue with (ii) of Proposition 5.6 is that one can in general not conclude $u = O(w^{2-\delta})$ if $u = O(w^{2-\delta+\varepsilon})$ for all $\varepsilon > 0$. For proving Theorem 5.4, however, we would like to conclude that indeed $u = O(w^{2-\delta})$. The following proposition gives a criterion, when this conclusion is true.

Lemma 5.10. *Let $\delta > 0$ and suppose that $\xi \in C^\infty_{-1-\delta}(T^*E)$. If $\xi = du$ for some $u \in C^1(E)$, then there exists a constant function u_c such that $u - u_c \in C^\infty_{-\delta}(E)$. If additionally $u \to 0$ as $t \to \infty$, then $u_c \equiv 0$.*

Proof. This lemma is proven analogously to the corresponding statement for conical metrics [Mar02][Lemma 5.10]. First observe that we only need to find a constant u_c such that $u - u_c \in C^0_{-\delta}(E)$ because $\nabla(u - u_c) = du \in C^\infty_{-1-\delta}(T^*E)$ by assumption.

We work on $E \setminus D \cong \mathbb{R} \times S$ and fix a point $(t_0, y_0) \in \mathbb{R} \times S$. Viewing S as the slice $\{0\} \times S$, we endow S with a metric g_S by restricting g_φ to S. For a different point (t, y), we let $q_{t_0,t}$ be the straight line path from (t_0, y_0) to (t, y_0) and $q_{y_0,y}$ be a path joining the points (t, y_0) and (t, y), so that its projection onto S is a length minimizing geodesic. Then we have by Stoke's theorem

$$u(t,y) - u(t_0,y_0) = \int_{q_{t_0,t}} \xi + \int_{q_{y_0,y}} \xi. \tag{75}$$

As in the proof of [Mar02][Lemma 5.10 (c)], the key is to notice that the integral of ξ along the path $q_{t_0,\infty}$ is finite, where $q_{t_0,\infty}$ is the linear path from (t_0, y_0) to $(+\infty, y_0)$. Indeed, since $\xi \in C^\infty_{-1-\delta}(T^*E)$ and $\delta > 0$, we can estimate

$$\begin{aligned} \left| \int_{q_{t_0,\infty}} \xi \right| &\le \int_{t_0}^{\infty} |\xi(\dot{q}_{t_0,\infty})| \, ds \\ &\le ||\xi||_{C^0_{-1-\delta}} \int_{t_0}^{\infty} f'' w^{-1-\delta} ds \\ &\le ||\xi||_{C^0_{-1-\delta}} \frac{w^{-\delta}(t_0)}{\delta} < +\infty, \end{aligned} \tag{76}$$

Splitting the integral $\int_{q_{t_0,\infty}} \xi$ into two parts, we can rewrite (75) as follows:

$$u(t,y) - u(t_0,y_0) - \int_{q_{t_0,\infty}} \xi = -\int_{q_{t,\infty}} \xi + \int_{q_{y_0,y}} \xi. \tag{77}$$

As in (76), it is easy to see that the right hand side of (77) is bounded by $w^{-\delta}(t)$. In fact, we have

$$\begin{aligned} \left| \int_{q_{y_0,y}} \xi \right| &\le \int_a^b |\xi|_\varphi |\dot{q}_{y_0,y}|_\varphi ds \\ &\le C ||\xi||_{C^0_{-1-\delta}} w^{-1-\delta}(t) \sqrt{f'(t)} \int_a^b |\dot{q}_{y_0,y}|_{g_S} ds \\ &\le C ||\xi||_{C^0_{-1-\delta}} w^{-\delta}(t) \operatorname{diam}(S, g_S), \end{aligned} \tag{78}$$

where $q_{y_0,y}$ is defined on the interval $[a, b]$ and $C > 0$ is some constant independent of t. Combining with (76), we obtain

$$\left| u(t,y) - u(t_0,y_0) - \int_{q_{t_0,\infty}} \xi \right| \le ||\xi||_{C^0_{-1-\delta}} \left(\delta^{-1} + C \operatorname{diam}(S, g_S) \right) w^{-\delta}(t),$$

i.e. $u - u_c \in C^0_{-\delta}(E)$ where we set

$$\begin{aligned} u_c &= u(t_0, y_0) + \int_{q_{t_0,\infty}} \xi \\ &= u(t_0, y_0) + \lim_{t\to\infty} (u(t, y_0) - u(t_0, y_0)) \\ &= \lim_{t\to\infty} u(t, y_0). \end{aligned}$$

Thus, it remains to show that u_c is indeed constant. Let q_{y_0,y_1} be a path in the slice $\{t\} \times S$ connecting two points (t, y_0) and (t, y_1). Then we have

$$u(t, y_1) - u(t, y_0) = \int_{q_{y_0,y_1}} \xi, \tag{79}$$

and by (78), the right hand side of (79) goes to 0 as $t \to \infty$. Hence $\lim_{t\to\infty} u(t, y_0) = \lim_{t\to\infty} u(t, y_1)$ for any $y_0, y_1 \in S$, proving the lemma. □

5.3. **Vanishing of harmonic forms.** We aim at proving a vanishing theorem for harmonic (1,0)-forms on the manifold (E, g_φ). This will be needed for the $\partial\bar{\partial}$-Lemma. We start with a basic observation which is immediate from the standard Bochner formula.

Lemma 5.11. *Any harmonic 1-form β on (E, g_φ) such that $|\beta| \to 0$ as $t \to \infty$ must vanish identically.*

Proof. Since $\mathrm{Ric}(\omega_\varphi)$ is non-negative by Theorem 4.1, the Bochner formula reads

$$\Delta|\beta|^2 \geq 0,$$

and the claim then follows from the Maximum principle. □

It becomes more interesting if we replace the asymptotic condition of β in the previous lemma by requiring that β be square-integrable instead. If β is moreover of type $(1, 0)$, it is also holomorphic and it must be zero by the following Theorem.

Theorem 5.12. *Any L^2-holomorphic (1,0)-form on (E, g_φ) is identically zero.*

Proof. We adapt the idea behind [MW15][Theorem 7]. Let β be a holomorphic (1,0)-form, which is square integrable w.r.to the metric g_φ. Then $\bar{\partial}\beta = \bar{\partial}^*\beta = 0$, and by the Kähler identities $\Delta_d\beta = 0$, i.e. β is harmonic. Since every L^2-harmonic form on a complete manifold is closed and coclosed, we conclude $d\beta = d^*\beta = 0$. Observe that β and $\pi^*j^*\beta$ are in the same de-Rham cohomology class, where $\pi : E \to D$ is the projection and $j : D \to E$ is the inclusion of D as the zero section. Hence $\beta = \pi^*j^*\beta + \partial h$ for some function h. It follows immediately that $\bar{\partial}\partial h = 0$.

For some $\varepsilon > 0$, consider the tube $D_\varepsilon = \{z \in E \mid r(z) \leq \varepsilon\}$ around the zero section. Then by Stoke's theorem, there is the following formula

$$\int_{D_\varepsilon} |\partial h|^2 = -\int_{D_\varepsilon} \langle h, \partial^* \partial h\rangle + \int_{\partial D_\varepsilon} h\iota_\nu(\partial h). \tag{80}$$

Here, $\nu := \frac{X}{|X|}$ denotes the outward pointing unit normal vector to ∂D_ε. As X is a real holomorphic vector field, the function $\iota_X(\partial h)$ is also holomorphic and we claim that it is in L^2. Indeed, using $\iota_X(\pi^* j^* \beta) = 0$, we observe that

$$|\iota_X(\partial h)| = |\iota_X(\beta)| \leq |X| \cdot |\beta|$$

so that $\iota_X(\partial h)$ is square-integrable since X is bounded and β is L^2. Hence, $\iota_X(\partial h)$ is an L^2-holomorphic function and must consequently be zero. Moreover, $2\partial^*\partial h = \Delta h = 0$ because h is pluriharmonic. Thus, ∂h vanishes identically on D_ε by (80). So ∂h must be zero everywhere since it is a holomorphic (1,0)-form.

We conclude that $\beta = \pi^* j^* \beta$. However, a form pulled back from the base can never be in L^2, unless it vanishes identically. Indeed, let α be a 1-form on D which is non-zero at some point p. Keeping the expression (60) in mind, we can always estimate in a neighborhood around p

$$\langle \pi^*\alpha, \pi^*\alpha\rangle \geq C w^{-1} > 0$$

for some $C > 0$ independent of t. It follows that $\int_E |\pi^*\alpha|^2 = +\infty$ since w^{-1} is not integrable. This finishes the proof. □

5.4. **The $\partial\bar{\partial}$-Lemma.** In this paragraph, we prove Theorem 5.4 on the manifold E analogue to [CH13][Theorem 3.11].

The first step is to find a primitive of η, with controlled growth. In fact, one can write down an explicit primitive for η on the product $E \setminus D \cong \mathbb{R} \times S$ and then read off its growth behaviour. This is the idea behind the next proposition.

Proposition 5.13. *Let $\delta > 2$ and $\eta \in C^\infty_{-\delta}(\Lambda^2 T^* E)$ be a d-exact* 2-*form. Then $\eta = d\theta$ for some $\theta \in C^\infty_{-\delta+1}(T^*E)$.*

Proof. As in [CH13][Theorem 3.11], we first reduce the problem to finding a primitive for η on the product $\mathbb{R} \times S$. By assumption, there exists a 1-form ξ such that $\eta = d\xi$. Let $t_1 < t_2$ and define two compact sets K_j with $j = 1, 2$ by

$$K_j = \{z \in E \mid t(z) \leq t_j\},$$

where we view the zero section of E to be the set $\{z \in E \mid t(z) = -\infty\}$. We pick a cut-off function χ so that $\chi \equiv 0$ on K_1 and $\chi \equiv 1$ on the complement of K_2. Then we put $\hat{\xi} := \chi\xi$ and $\hat{\eta} := d\hat{\xi}$. Note that if $\hat{\eta} = d\hat{\theta}$ for some $\hat{\theta}$, then $\theta := \xi - \hat{\xi} + \hat{\theta}$ satisfies

$$d\theta = d\xi - d(\chi\xi) + \hat{\eta} = \eta.$$

Since $\theta = \hat{\theta}$ outside K_2, it suffices to find $\hat{\theta} \in C^\infty_{1-\delta}(\Lambda^* E)$ with $\hat{\eta} = d\hat{\theta}$ and $\hat{\theta} \equiv 0$ on K_1. The following construction of $\hat{\theta}$ can be found in the proof of [Mar02][Proposition 5.8].

For each $t \in \mathbb{R}$, there is an inclusion $i_t : \{t\} \times S \to \mathbb{R} \times S$ given by $i_t(y) = (t, y)$. Write $\hat{\eta} = dt \wedge \hat{\eta}_1 + \hat{\eta}_2$, where $\hat{\eta}_j$ are 1-parameter families of j-forms such that

$$\iota_{\frac{\partial}{\partial t}} \hat{\eta}_j = 0 \quad \text{and} \quad i_t^* \hat{\eta}_j = 0 \quad \text{for all } t \leq t_1. \tag{81}$$

We define a family $\hat{\theta}_t$ with $t \in \mathbb{R}$ of 1 forms on S by

$$\hat{\theta}_t = -\int_t^\infty i_s^*(\hat{\eta}_1) ds. \tag{82}$$

Then we define a 1-form $\hat{\theta}$ on $\mathbb{R} \times S$ by requiring that

$$\iota_{\frac{\partial}{\partial t}} \hat{\theta} = 0 \quad \text{and} \quad i_t^* \hat{\theta} = \hat{\theta}_t \quad \text{for all } t \in \mathbb{R}. \tag{83}$$

We have to prove that $\hat{\theta}$ is well-defined, i.e. that the integral (82) exists. We start by looking at $|\hat{\eta}_1|$. As dt and $\hat{\eta}_1$ are orthogonal to each other, we have that

$$|dt \wedge \hat{\eta}_1| = |dt||\hat{\eta}_1| = \frac{1}{\sqrt{f''(t)}}|\hat{\eta}_1|.$$

Using that $dt \wedge \hat{\eta}_1$ is orthogonal to $\hat{\eta}_2$, we can estimate

$$|\hat{\eta}_1| = \sqrt{f''(t)}|dt \wedge \hat{\eta}_1| \leq \sqrt{f''(t)}|dt \wedge \hat{\eta}_1 + \hat{\eta}_2| = O(w^{-\delta}), \tag{84}$$

since f'' is bounded and $|\hat{\eta}| = O(w^{-\delta})$ by assumption. To compute the integral (82), we work in coordinates. Let $(y_0 = t, y_1, \ldots, y_{2(m+d)-1})$ be real coordinates of $\mathbb{R} \times S$ and write $\hat{\eta}_1 = \sum_{j \geq 1} \hat{\eta}_{1,j} dy_j$. Then (82) becomes

$$\hat{\theta}_t = -\sum_{j \geq 1} \left(\int_t^\infty i_s^* \hat{\eta}_{1,j} ds \right) dy_j. \tag{85}$$

Note that the norms $|dy_j|$ may not have the same asymptotic behaviour for different values of $j = 1, \ldots, m+d-1$. In fact, it follows from (60) that we have

(86)

$$|dy_j| = \begin{cases} O(w^{-\frac{1}{2}}) & \text{if } \pi^* \hat{g}_{jj} > 0, \\ O(1) & \text{if } \pi^* \hat{g}_{jj} = 0, \end{cases} \text{and} \quad \frac{1}{|dy_j|} = \begin{cases} O(w^{\frac{1}{2}}) & \text{if } \pi^* \hat{g}_{jj} > 0, \\ O(1) & \text{if } \pi^* \hat{g}_{jj} = 0. \end{cases}$$

As $|\hat{\eta}| = O(w^{-\delta})$, we conclude that either $|\hat{\eta}_{1,j}| = O(w^{-\delta+\frac{1}{2}})$ or $|\hat{\eta}_{1,j}| = O(w^{-\delta})$ and hence, the integrals in (85) are all finite because we chose $-\delta + 1 < -1$.

We also observe from (81) that $\hat{\theta}_t = \hat{\theta}_s$ for all $s, t \le t_1$, so $\hat{\theta}$ extends to a smooth 1-form on E. Moreover, we can read off (85) that $|\hat{\theta}| = O(w^{-\delta+1})$, i.e. $\hat{\theta} \in C^0_{-\delta+1}(T^*E)$. It is possible to obtain estimates on derivatives of $\hat{\theta}$ and to show that $\hat{\theta} \in C^\infty_{-\delta+1}(T^*E)$. However, this is a long calculation which relies only on two main observations. First, we deduce from Lemma 4.3 that $|\nabla^l dy_j|$ behaves asymptotically like $|dy_j| w^{-l}$ for all $l \ge 0$ and $j = 0, \dots, 2(m+d)-1$. Secondly, we can conclude from $\eta \in C^\infty_{-\delta}(\Lambda^* T^* E)$ that also $|\nabla^l \hat{\eta}_1| = O(w^{-\delta-l})$. Using formula (85), it is then straight forward to verify $|\nabla^l \hat{\theta}| = O(w^{-\delta-l+1})$, as claimed. We leave the details to the reader, but remark that the required estimate is similar to bounding $|\hat{\theta}|$.

It remains to show that $\hat{\eta} = d\hat{\theta}$ by considering its components. In fact, it is an easy calculation ([Mar02][p.80]) to prove that

$$\frac{\partial}{\partial t}\left(i_t^*(\hat{\eta} - d\hat{\theta})\right) = 0,$$

i.e. $i_s^*(\hat{\eta} - d\hat{\theta}) = i_t^*(\hat{\eta} - d\hat{\theta})$ for all $s, t \in \mathbb{R}$. Since $\hat{\eta}, \hat{\theta} \to 0$ as $t \to \infty$, we conclude that $i_t^*(\hat{\eta} - d\hat{\theta}) = 0$ for any $t \in \mathbb{R}$. Moreover, it is shown in [Mar02][p.80] that

$$\iota_{\frac{\partial}{\partial t}} \hat{\eta} = \iota_{\frac{\partial}{\partial t}} d\hat{\theta},$$

and hence $\hat{\eta} = d\hat{\theta}$ as we claimed.

□

Proof of Theorem 5.4. The strategy is the same as for the proof of [CH13][Theorem 3.11]. We start with some basic observations. By Proposition 5.13, there exists a $\theta \in C^\infty_{1-\delta}(\Lambda^* E)$ such that $d\theta = \eta$. Since η is real, θ will also be a real form, i.e. $\theta^{1,0} = \overline{\theta^{0,1}}$ if $\theta = \theta^{1,0} + \theta^{0,1}$ is the decomposition into types. Moreover, η is of type (1,1), so we must have that $\partial\theta^{1,0} = \bar{\partial}\theta^{0,1} = 0$.

If ∂^* denotes the formal dual of ∂ (w.r.to the L^2-metric induced by g_φ), then $\partial^*\theta^{1,0} \in C^\infty_{-\delta}(E)$. We would like to find a solution u to the equation $\Delta u = \partial^*\theta^{1,0}$, whose growth we can control. There are two cases to consider, corresponding to part (i) and (ii) of Proposition 5.6.

First, we consider the case where the degree k of the polynomial Q is greater or equal to 2. By (ii) of Proposition 5.6, there exists $u \in C^{2,\alpha}(E)$ such that $\Delta u = \partial^*\theta^{1,0}$ and $|u| + |\nabla u| = O(w^{2-\delta+\varepsilon})$. It follows that $\partial^*(\partial u - \theta^{1,0}) = \partial(\partial u - \theta^{1,0}) = 0$, and hence the 1-form $\partial u - \theta^{1,0}$ is harmonic by the Kähler identities.

Choosing $\varepsilon > 0$ small enough, we can assume that $2 - \delta + \varepsilon < 0$ and hence we see from $|\nabla u| = O(w^{2-\delta+\varepsilon})$ and $\theta \in C^\infty_{1-\delta}(\Lambda^* E)$ that

$$|\partial u - \theta^{1,0}| \le |du| + |\theta| \to 0,$$

as $t \to \infty$. Then Lemma 5.11 implies $\partial u - \theta^{1,0} = 0$ and consequently,

$$\eta = d\theta = \partial\theta^{0,1} + \bar{\partial}\theta^{1,0} = \partial\bar{\partial}\bar{u} + \bar{\partial}\partial u = -2\sqrt{-1}\partial\bar{\partial}\operatorname{Im} u,$$

where $\operatorname{Im} u$ is the imaginary part of u.

It remains to show that $\operatorname{Im} u \in C^{\infty}_{2-\delta}(E)$ as opposed to only $\operatorname{Im} u \in C^{2,\alpha}(E)$ and $\operatorname{Im} u = O(w^{2-\delta+\varepsilon})$. As we can choose $\varepsilon > 0$ so that $2-\delta+\varepsilon < 0$, this improvement of the decay rate, however, follows immediately from Proposition 5.10 if we can show $d\operatorname{Im} u \in C^{\infty}_{1-\delta}(\Lambda^* E)$. This last condition is clearly satisfied since $\theta^{1,0}, \theta^{0,1} \in C^{\infty}_{1-\delta}(\Lambda^* E)$ and $\theta^{1,0} - \theta^{0,1} = \partial u - \overline{\partial u} = d\operatorname{Re} u + \sqrt{-1} d\operatorname{Im} u$. This settles the first case.

For the second case, assume that $k \leq 1$. We want to use (i) of Proposition 5.6 to solve $\Delta u = \partial^*\theta^{1,0}$. This time, however, we only know that the solution u satisfies $\int |\nabla u|^2 \omega_\varphi^{m+d} < +\infty$, and not necessarily that u decays towards infinity. So the idea is to use the vanishing Theorem 5.12 instead.

Before applying Proposition 5.6 (i), we need to verify that $\int \partial^*\theta^{1,0}\omega_\varphi^{m+d}$ is zero. For any $t_0 \in \mathbb{R}$, define $K_{t_0} = \{z \in E \mid t(z) \leq t_0\}$ and consider the integral

$$\int_{K_{t_0}} \partial^*\theta^{1,0}\omega_\varphi^{m+d} = \int_{K_{t_0}} d * \theta^{1,0} = \int_{\{t_0\}\times S} *\theta^{1,0}, \tag{87}$$

where we used Stoke's for the last equality. If we equip the slice $\{t_0\}\times S$ with the restriction of g_φ and denote the corresponding volume by $\operatorname{Vol}(\{t_0\}\times S)$, then we can estimate

$$\left| \int_{\{t_0\}\times S} *\theta^{1,0} \right| \leq \operatorname{Vol}(\{t_0\}\times S) \sup_{\{t_0\}\times S} |\theta| = O(w^{k+1-\delta}(t_0)),$$

since $|\theta| = O(w^{1-\delta})$ and $\operatorname{Vol}(\{t_0\}\times S) = O(w^k)$. It follows that the right hand side of (87) goes to zero as $t_0 \to +\infty$, as we assumed $k \leq 1$ and $\delta > 2$. Hence $\int \partial^*\theta^{1,0}\omega_\varphi^{m+d} = 0$, as claimed.

So we find a $u \in C^{2,\alpha}(E)$ such that $\Delta u = \partial^*\theta^{1,0}$ and $\int |\nabla u|^2\omega_\varphi^{m+d}$ is finite. In particular, the 1-form $\beta = \theta^{1,0} - \partial u$ is harmonic. Also note that

$$|\theta|\omega_\varphi^{m+d} = O(w^{2-2\delta+k})$$

with $2 - 2\delta + k < -1$, so that θ is in L^2, and thus β is L^2 as well.

It follows that $d\beta = d^*\beta = 0$, and in particular, β is an L^2-holomorphic (1,0)-form. Hence it must be identically zero by Theorem 5.12, i.e. $\theta^{1,0} = \partial u$. The rest of the proof is now analogous to the first case.

□

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Part II.

ASYMPTOTICALLY CYLINDRICAL STEADY KÄHLER-RICCI SOLITONS*

JOHANNES SCHÄFER

Abstract. Let D be a compact Kähler manifold with trivial canonical bundle and Γ be a finite cyclical group of order m acting on $\mathbb{C} \times D$ by biholomorphisms, where the action on the first factor is generated by rotation of angle $2\pi/m$. Furthermore, suppose that Ω_D is a trivialisation of the canonical bundle such that Γ preserves the holomorphic form $dz \wedge \Omega_D$ on $\mathbb{C} \times D$, with z denoting the coordinate on $\mathbb{C}$.

The main result of this article is the construction of new examples of gradient steady Kähler-Ricci solitons on certain crepant resolutions of the orbifolds $(\mathbb{C} \times D)/\Gamma$. These new solitons converge exponentially to a Ricci-flat cylinder $\mathbb{R} \times (\mathbb{S}^1 \times D)/\Gamma$.

1. Introduction

A *steady Ricci soliton* is a Riemannian manifold (M, g) together with a vector field X such that

$$\mathrm{Ric}(g) = \frac{1}{2}\mathcal{L}_X g, \tag{1}$$

where $\mathrm{Ric}(g)$ denotes the Ricci tensor of g and $\mathcal{L}_X$ is the Lie derivative in direction of X. The soliton (M, g, X) is called *gradient* if X is the gradient field of some function on M.

If (M, g) is Kähler and the vector field X real holomorphic, equation (1) is equivalent to

$$\mathrm{Ric}(\omega) = \frac{1}{2}\mathcal{L}_X \omega, \tag{2}$$

where ω is the Kähler form of g and $\mathrm{Ric}(\omega)$ the corresponding Ricci form. A Kähler manifold (M, g) which admits a real holomorphic vector field X satisfying (2) is called a *steady Kähler-Ricci soliton*.

Steady solitons may be viewed as natural generalisations of Einstein manifolds, which correspond to the case $X \equiv 0$. Non-Einstein steady solitons, however, must be non-compact ([Ive93]).

To each steady Ricci soliton (M, g, X) one can associate a self-similar Ricci-flow by rescaling and pulling back g along the flow of X. Thus, steady solitons may be possible candidates for singularity models for

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Ricci-flow. They are also important in the context of so-called Type II singularities, i.e. when a Ricci-flow exists up to the finite time $T > 0$, and the curvature blows up faster than $(T-t)^{-1}$. For recent progress in the study of singularities as well as steady Ricci solitons, we refer the reader to [BCD$^+$21], [CDM20], [Bam20], [CFSZ20], [DZ20], and the references therein.

This article focuses on the case of steady Kähler-Ricci solitons, and our main result is the existence of a new class of such solitons. In contrast to general Ricci-solitons, it suffices to solve a single equation of top-dimensional differential forms in order to construct a gradient Kähler-Ricci soliton. If M is a complex manifold of (complex) dimension n, together with a nowhere-vanishing holomorphic $(n,0)$-form Ω, and a Kähler metric g whose Kähler form ω satisfies

$$\omega^n = e^{-f} i^{n^2} \Omega \wedge \overline{\Omega} \tag{3}$$

for some function $f : M \to \mathbb{R}$, then $(M, g, \nabla^g f)$ defines a gradient steady Kähler-Ricci soliton. In fact, if M is simply-connected, then one can always associate such a form Ω to a gradient steady Kähler-Ricci soliton, compare [Bry08][Theorem 1].

However, given M and a nowhere-vanishing holomorphic n-form Ω on M it is not known if M admits a steady soliton, i.e. there is no general existence theory for steady Kähler-Ricci solitons as is the case for compact Ricci-flat Kähler manifolds due to Yau [Yau78].

All previously known examples of steady Kähler-Ricci solitons may be divided into two classes. The first group consists of solitons constructed by reducing (2) to an ODE, for instance by Hamilton [Ham88], Cao [Cao96], Dancer and Wang [DW11], Yang [Yan12] and the author [Sch20]. Most notably, we mention Hamilton's cigar on $\mathbb{C}$ ([Ham88]) and Cao's soliton on $\mathbb{C}^n$ for $n \geq 2$ ([Cao96]). The cigar is asymptotic to the cylinder $dt^2 + d\theta^2$ on the product $\mathbb{R} \times \mathbb{S}^1 \cong \mathbb{C}^*$, whereas Cao's soliton has a more complicated asymptotic behavior. (It is a so-called cigar-paraboloid whose precise asymptotics are explained in [CD20b][Section 3].)

The second group of examples are constructed by PDE methods ([BM17], [CD20b]). Here, the underlying complex manifolds are equivariant, crepant resolutions of certain orbifolds $\mathbb{C}^n/G$ ([BM17]) and of more general Calabi-Yau cones ([CD20b]). In both cases, the solitons have an asymptotic behavior similar to Cao's soliton.

In this article, we build on ideas developed in [CD20b] and find new examples of steady Kähler-Ricci solitons which are asymptotic to a product $\mathbb{C} \times D$ of Hamilton's cigar and a compact Ricci-flat Kähler manifold D. (Note that this product is also a steady Kähler-Ricci soliton.) These new examples exist on resolutions $\pi : M \to (\mathbb{C} \times D)/\Gamma$ of certain orbifolds $(\mathbb{C} \times D)/\Gamma$. Before introducing the precise conditions on D, Γ and M, consider the following example.

Example 1.1. Let $D = \mathbb{T}$ be the (real) 2-torus and let $\Gamma = \{\pm \mathrm{Id}\}$. Then $(\mathbb{C} \times \mathbb{T})/\Gamma$ has precisely four singular points, each isomorphic to a neighborhood of the origin in $\mathbb{C}^2/\{\pm \mathrm{Id}\}$. Thus, we may blow-up each of these singular points to obtain a resolution $\pi : M \to (\mathbb{C} \times \mathbb{T})/\Gamma$. (Note that previously, certain Calabi-Yau metrics, so-called ALG gravitational instantons, were constructed on this resolution, see [BM11].)

This resolution $\pi : M \to (\mathbb{C} \times \mathbb{T})/\Gamma$ satisfies three essential properties. First, the resolution is crepant, i.e. the holomorphic (2,0)-form Ω on $(\mathbb{C}^* \times \mathbb{T})/\Gamma$, which lifts to the canonical form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$, extends to a nowhere-vanishing form on the entire resolution M.

Second, the $\mathbb{C}^*$-action on $(\mathbb{C}^* \times \mathbb{T})/\Gamma$ given by

$$\lambda * (z, w) = (\lambda z, w), \quad \lambda \in \mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action on M, since the resolution is toric. In particular, the infinitesimal generator $z_1 \frac{\partial}{\partial z_1}$ on $(\mathbb{C}^* \times \mathbb{T})/\Gamma$ extends to a holomorphic vector field Z on M.

And third, M admits a natural complex compactification $\overline{M}$ obtained by adding the divisor $\overline{\mathbb{T}} := \mathbb{T}/\{\pm \mathrm{Id}\}$ 'at infinity', i.e. we compactify $\mathbb{C}$ by the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and let $\overline{M} = M \cup \big(\{\infty\} \times \overline{\mathbb{T}}\big)$. Given a Kähler class $\kappa_{\overline{M}} \in H^2(\overline{M}, \mathbb{R})$ on $\overline{M}$, it is possible to construct a new Kähler form on M in the class $\kappa_{\overline{M}}|_M \in H^2(M, \mathbb{R})$ that is asymptotic to the cylinder

$$|z_1|^{-2} \frac{i}{2} dz_1 \wedge d\bar{z}_1 + \frac{i}{2} dz_2 \wedge d\bar{z}_2. \tag{4}$$

(This construction follows by adapting ideas from the case of asymptotically cylindrical Calabi-Yau manifolds [HHN15].)

Thus, one may ask if there exists a steady Kähler-Ricci soliton on M which is asymptotic to the cylinder (4), whose Kähler form is contained in the class $\kappa_{\overline{M}}|_M$ and whose soliton vector field equals the real part of Z. This is indeed a non-trivial question, because M is *not* a product, but a resolution of the orbifold $(\mathbb{C} \times \mathbb{T})/\Gamma$.

Our main result (Theorem 1.2), however, implies that M *does* admits such solitons. In fact, Theorem 1.2 proves the existence of steady Kähler-Ricci solitons for a more general setup:

Theorem 1.2. *Let D^{n-1} be a compact Kähler manifold with nowhere-vanishing holomorphic $(n-1, 0)$-form Ω_D. Suppose $\gamma : D \to D$ is a complex automorphism of order $m > 1$ such that*

$$\gamma^* \Omega_D = e^{-\frac{2\pi i}{m}} \Omega_D,$$

and consider the orbifold $(\mathbb{C} \times D)/\langle \gamma \rangle$, where γ acts on the product via

$$\gamma(z, w) = \left(e^{\frac{2\pi i}{m}} z, \gamma(w) \right).$$

Let $\pi : M \to (\mathbb{C} \times D)/\langle\gamma\rangle$ be a crepant resolution such that the $\mathbb{C}^*$-action on $(\mathbb{C} \times D)/\langle\gamma\rangle$ given by

$$\lambda * (z, w) = (\lambda z, w), \quad \lambda \in \mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action of $\mathbb{C}^*$ on M.

Let $\overline{M} = M \cup \overline{D}$ be the complex compactification of M by adding the orbifold divisor $\overline{D} := D/\langle\gamma\rangle$ at infinity. Then for every orbifold Kähler class $\kappa_{\overline{M}}$ on $\overline{M}$, there exists a steady Kähler-Ricci soliton on M whose Kähler form is contained in the class $\kappa_{\overline{M}}|_M \in H^2(M, \mathbb{R})$.

As in Example 1.1, M admits a nowhere-vanishing holomorphic $(n, 0)$-form because the resolution is crepant, and the infinitesimal generator of the $\mathbb{C}^*$-action on M provides a candidate for the soliton vector field. Also, the Kähler class is determined by the compactification $\overline{M}$ and the resulting Kähler-Ricci soliton is asymptotic to the cylinder $dt^2 + d\theta^2 + g_D$ on the product $(\mathbb{C}^* \times D)/\langle\gamma\rangle \cong \mathbb{R} \times (\mathbb{S}^1 \times D)/\langle\gamma\rangle$ for some Ricci-flat Kähler metric g_D on D.

The new examples of steady Kähler-Ricci solitons provided by Theorem 1.2 are geometrically different from all previously found examples in complex dimension $n \geq 2$. For instance, their volume grows linearly since they are asymptotically cylindrical, while the examples modelled on Cao's soliton in complex dimension n have volume growth equal to n, compare [Cao96], [BM17] and [CD20b].

Interestingly, our examples also seem to be the only (non-Einstein) steady Kähler-Ricci solitons whose asymptotic model is *Ricci-flat*. This contrasts with the fact that Cao's soliton has *positive* Ricci curvature ([Cao96][Lemma 2.2]).

The strategy for proving Theorem 1.2 is analogue to the proof of [CD20b][Theorem A]. We adapt Conlon and Deruelle's ideas to our setting and reduce (2) to a complex Monge-Ampère equation, whose solution exists by the following result, which is similar to [CD20b][Theorem 7.1]

Theorem 1.3. *Let (M, g, J) be an asymptotically cylindrical Kähler manifold of complex dimension n with Kähler form ω. Suppose that M admits a real holomorphic vector field X such that*

$$X = 2\Phi_* \frac{\partial}{\partial t}$$

outside some compact domain, where Φ denotes the diffeomorphism onto the cylindrical end of (M, g) and t is the radial parameter on this end. Moreover, assume that JX is Killing for g.

If $1 < \varepsilon < 2$ and $F \in C^\infty_\varepsilon(M)$ is JX-invariant, then there exists a unique, JX-invariant $\varphi \in C^\infty_\varepsilon(M)$ such that $\omega + i\partial\bar{\partial}\varphi > 0$ and

$$\left(\omega + i\partial\bar{\partial}\varphi\right)^n = e^{F - \frac{X}{2}(\varphi)}\omega^n.$$

Note that in this theorem, we do allow more general manifolds than those appearing in Theorem 1.2. This is because the proof of Theorem 1.3 essentially only requires that we have a Kähler manifold (M, g, J), asymptotic to a cylinder (in the sense of Definition 2.1 below) and satisfying two further assumptions: Firstly, we need the radial vector field on the cylinder to be extended to a real holomorphic vector field on (M, J) and secondly, JX must be an infinitesimal isometry of g. We will see in Proposition 3.5 below that this ensures $X = \nabla^g f$ for some function f with understood asymptotical behavior.

The spaces $C^\infty_\varepsilon(M)$ in Theorem 1.3 contain all smooth functions on M whose covariant derivatives (with respect to g) decay at least like $e^{-\varepsilon t}$ with t denoting the cylindrical parameter of (M, g) (compare Definitions 2.1 and 2.3). These function spaces are well-adapted to the cylindrical geometry and have previously been used in the construction of asymptotically cylindrical Calabi-Yau manifolds [HHN15].

Following the proof of [CD20b][Theorem 7.1], we also implement a continuity method to conclude Theorem 1.3. To this end, we need to show two things. First, that the linearisation of the Monge-Ampère operator is an isomorphism, which can be deduced from standard results on asymptotically translation invariant differential operators. Second, and most importantly, we have to derive a priori-estimates along the continuity path, where the C^0-estimate is the key part of the proof. To obtain this estimate, we adapt the C^0-estimate of Conlon and Deruelle ([CD20b][Section 7.1]) to our cylindrical setup. These authors first assume that the right-hand side F is *compactly supported* to obtain the C^0-estimate ([CD20b][Theorem 7.1]) and in a second step, they explain how to solve the Monge-Ampère equation for *decaying* F ([CD20b][Theorem 9.2]). We, however, present a modification of their argument, which allows us to achieve the C^0-estimate *directly* for F decaying exponentially in Theorem 1.3.

This article is structured as follows. In Section 2, we recall the notion of asymptotically cylindrical manifolds and the theory of linear asymptotically translation-invariant operators on such manifolds. This is later applied to the linearisation of the Monge-Ampère operator.

The basics of steady Kähler-Ricci solitons are covered in Section 3. We recall the underlying Monge-Ampère equation and also discuss when a soliton is gradient. Most notably, we show at the end of this section that, under the assumptions of Theorem 1.3, X must be a gradient field.

In Section 4, we reduce Theorem 1.2 to Theorem 1.3. We discuss the existence of cylindrical Kähler metrics on manifolds as in Theorem 1.2 in Section 4.1 and also explain which Kähler classes do indeed admit such metrics. Theorem 1.2 is then proven in Section 4.2, before we provide further examples in Section 4.3.

The fifth and final section is entirely devoted to Theorem 1.3. We explain the continuity method and reduce the proof to an a priori-estimate.

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2. Linear analysis on ACyl manifolds

In this section, we review the basic definitions and theorems about asymptotically translation-invariant operators on ACyl manifolds following the presentation in [HHN15][Section 2.1] and [Nor08][Section 2.3]. The goal is to apply the general theory to the special class of operators that arise as the linearisation of the Monge-Ampère operator in Section 5 below.

We begin by recalling the definition of ACyl manifolds. For simplicity, we restrict our attention to the case of only *one* cylindrical end, i.e. a connected cross-section.

Definition 2.1. A complete Riemannian manifold (M, g) is called *asymptotically cylindrical (ACyl) of rate* $\delta > 0$ if there is a bounded open set $U \subset M$, a connected and closed Riemannian manifold (L, g_L) as well as a diffeomorphism $\Phi : [0, \infty) \times L \to M \setminus U$ such that

$$|\nabla^k \left(\Phi^* g - g_{cyl}\right)| = O(e^{-\delta t})$$

for all $k \in \mathbb{N}_0$, where $g_{cyl} := dt^2 + g_L$ is the product metric and both ∇ and $|\cdot|$ are taken with respect to this metric. Here t denotes the projection onto $[0, \infty)$ and we extend the function $t \circ \Phi^{-1}$ smoothly to all of M. This extension is called a *cylindrical coordinate function*, (L, g_L) is called the *cross-section* and Φ the *ACyl map*.

Throughout this section, (M, g) denotes an ACyl manifold of rate $\delta > 0$ as defined above. It will be convenient to suppress Φ and simply view t as smooth a function on M.

Let $E, F \to M$ be tensor bundles over M and denote the corresponding space of smooth sections of E and F by $\Gamma(E)$ and $\Gamma(F)$, respectively. Then we consider a differential operator $P : \Gamma(E) \to \Gamma(F)$ of order l and we would like to understand P on the cylindrical end $M \setminus U \cong [0, \infty) \times L$.

As in [Mar02][Section 4], we cover the compact link L by charts $V_1, \ldots, V_N$ so that both E and F are trivial over each $\mathbb{R}_+ \times V_\alpha$. Given $u \in \Gamma(E)$, we denote the components of u and Pu on $\mathbb{R}_+ \times V_\alpha$ by u_j^α and $(Pu)_i^\alpha$, respectively, where $\alpha = 1, \ldots, N$, $j = 1, \ldots, \operatorname{rank} E$ and $i = 1, \ldots, \operatorname{rank} F$. Moreover, there are smooth functions $P_{ij}^{\alpha\beta}$:

$\mathbb{R}_+ \times V_\alpha \to \mathbb{C}$ such that

$$(Pu)_i^\alpha = \sum_{j=1}^{\operatorname{rank} E} \sum_{0 \le |\beta| \le l} P_{ij}^{\alpha\beta} D^\beta u_j^\alpha \tag{5}$$

where the second sum runs over all multi-indices $\beta = (\beta_0, \dots, \beta_{\dim L})$ of order $|\beta|$ at most l and D^β is defined to be

$$D^\beta := \frac{\partial^{|\beta|}}{\partial t^{\beta_0} x_1^{\beta_1} \cdots \partial x_{\dim L}^{\beta_{\dim L}}}$$

for coordinates $(x_1, \dots, x_{\dim L})$ of V_α.

Given a second operator $Q : \Gamma(E) \to \Gamma(F)$ also of order l, we say that P is *asymptotic* to Q if the coefficients $P_{ij}^{\alpha\beta}, Q_{ij}^{\alpha\beta}$ defined by (5) satisfy

$$\sup_{\{t\} \times V_\alpha} \left| \rho_\alpha D^\gamma \left(P_{ij}^{\alpha\beta} - Q_{ij}^{\alpha\beta} \right) \right| \to 0 \quad \text{as} \quad t \to \infty$$

for all $i = 1, \dots, \operatorname{rank} F$, $j = 1, \dots \operatorname{rank} E$, $\alpha = 1, \dots N$, $|\beta| \le l$ and all multi-indices γ, where $\rho_1, \dots, \rho_N$ is a partition of unity subordinate to the cover $V_1, \dots, V_N$. Note that this definition does neither depend on the choice of covering nor on the partition of unity.

With this notion of asymptotic operators, we may introduce the following definitions, compare [Mar02][Section 4.2.2].

Definition 2.2. Let $P, P_\infty : \Gamma(E) \to \Gamma(F)$ be two differential operators of order l between sections of tensor bundles $E, F \to M$.

(i) P_∞ is called *translation-invariant* if the functions $(P_\infty)_{ij}^{\alpha\beta}$ defined in (5) are invariant under translation in the $\mathbb{R}_+$-factor, for all $i = 1, \dots, \operatorname{rank} F$, $j = 1, \dots, \operatorname{rank} E$, $\alpha = 1, \dots, N$ and all multi-indices β of order at most l.

(ii) P is called *asymptotically translation-invariant* if P is asymptotic to some translation-invariant operator P_∞.

Important examples of asymptotically translation-invariant operators include the Laplacian Δ_g and the operator d^* associated to the ACyl metric g. These are asymptotic to the corresponding operators associated to the cylinder g_{cyl}.

Such operators may in general not be Fredholm between the usual Hölder spaces because M is noncompact. However, this changes if we introduce weight functions.

Definition 2.3. Let (M, g) be an ACyl manifold with cylindrical coordinate t and suppose $E \to M$ is a tensor bundle. The metric on E induced by g is also denoted by g, with corresponding connection ∇.

(i) For $\alpha \in (0,1)$, the Hölder semi-norm $[\,\cdot\,]_{C^{0,\alpha}}$ is defined for any continuous tensor field v over M by

$$[v]_{C^{0,\alpha}} := \sup_{\substack{x \neq y \in M \\ d_g(x,y) < \frac{i(g)}{2}}} \frac{|v_x - v_y|_g}{d_g(x,y)^\alpha},$$

where $v_x - v_y$ is defined by parallel transport along the minimal geodesic from x to y and $i(g) > 0$ denotes the injectivity radius of g.

(ii) For $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and $\varepsilon \in \mathbb{R}$, we define $C^{k,\alpha}_\varepsilon(E)$ to be the space of k-times continuously differentiable sections u of E such that the norm

$$||u||_{C^{k,\alpha}_\varepsilon} := \sum_{j=0}^{k} \sup_M \left| e^{\varepsilon t} \nabla^j u \right|_g + [e^{\varepsilon t} \nabla^k u]_{C^{0,\alpha}}$$

is finite.

(iii) $C^\infty_\varepsilon(E)$ is defined to be the intersection of $C^{k,\alpha}_\varepsilon(E)$ over all $k \in \mathbb{N}_0$.

(iv) If u is a function on M, the corresponding spaces are denoted by $C^{k,\alpha}_\varepsilon(M)$.

In other words, elements in $C^\infty_\varepsilon(E)$, as well as their covariant derivatives, are bounded from above by $e^{-\varepsilon t}$. It is not difficult to see that the definition is independent of the extension of the cylindrical coordinate t. Moreover, there are continuous inclusions

$$C^{k+1}_\varepsilon(E) \subseteq C^{k,\alpha}_\varepsilon(E) \quad \text{and} \quad C^{k,\alpha}_{\varepsilon_1}(E) \subseteq C^{k,\alpha}_{\varepsilon_0}(E),$$

if $\varepsilon_0 \leq \varepsilon_1$.

This notion of weighted Hölder spaces is well-adapted to the study of asymptotically translation-invariant operators. If the operator is moreover elliptic, we have the following weighted Schauder estimates.

Theorem 2.4. *Let (M,g) be ACyl and let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic, asymptotically translation-invariant operator of order l. Suppose $h \in C^{k,\alpha}_\varepsilon(E)$ and that u is a $k+l$-times continuously differentiable solution to $Pu = h$. If $u \in C^0_\varepsilon(E)$, then $u \in C^{k+l,\alpha}_\varepsilon(E)$ and*

$$||u||_{C^{k+l,\alpha}_\varepsilon} \leq C \left(||h||_{C^{k,\alpha}_\varepsilon} + ||u||_{C^0_\varepsilon} \right)$$

for some constant $C > 0$ independent of u.

Proof. This is [MP84][Theorem 3.16]. □

Every translation-invariant operator $P : \Gamma(E) \to \Gamma(F)$ of order l induces a bounded map $P : C^{k+l,\alpha}_\varepsilon(E) \to C^{k,\alpha}_\varepsilon(F)$. If P is moreover elliptic, it depends on the weight $\varepsilon \in \mathbb{R}$ whether or not the induced map $P : C^{k+l,\alpha}_\varepsilon(E) \to C^{k,\alpha}_\varepsilon(F)$ is Fredholm. This naturally leads to the definition of so called critical weights.

Definition 2.5. Let $P : \Gamma(E) \to \Gamma(F)$ be a differential operator asymptotic to a translation-invariant operator $P_\infty : \Gamma(E) \to \Gamma(F)$. $\varepsilon \in \mathbb{R}$ is called a *critical weight* if there exists a non-trivial solution $v = e^{i\lambda t}u : \mathbb{R} \times L \to \mathbb{C}$ to

$$P_\infty(v) = 0 \tag{6}$$

for some $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda = \varepsilon$ and for some smooth section $u = u(t, x)$ of E over $\mathbb{R} \times L$ that is a polynomial in t.

Note that the set of critical weights is discrete in $\mathbb{R}$. In the case of functions, i.e. if E is the trivial line bundle, u in the above definition is simply a polynomial in t with smooth functions on L as coefficients. This is crucial because it allows us to explicitly compute critical weights in examples.

The fundamental result in the theory of asymptotically translation-invariant operators is the following

Theorem 2.6. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic, translation-invariant operator of order l. If ε is not a critical weight, then the map $P : C^{k+l,\alpha}_\varepsilon(E) \to C^{k,\alpha}_\varepsilon(F)$ is Fredholm.*

This result was originally formulated for weighted Sobolev spaces ([LMO85][Theorem 6.2]). However, as explained in [HHN15][Section 2.1], the same proof applies in the Hölder setting as well.

Knowing that the induced map $P : C^{k+l,\alpha}_\varepsilon(E) \to C^{k,\alpha}_\varepsilon(F)$ is Fredholm for all non-critical weights ε, we would now like to have a better understanding of its kernel and image.

Proposition 2.7. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic, translation-invariant operator of order l. If an interval $[\varepsilon_1, \varepsilon_2]$ contains no critical weights, then the kernels of P in $C^{k,\alpha}_{\varepsilon_1}(M)$ and $C^{k,\alpha}_{\varepsilon_2}(M)$ are equal.*

This is proven in [LMO85][Lemma 7.1]. To give a precise characterization of the image of P, we need to introduce the formal adjoint $P^* : \Gamma(F) \to \Gamma(E)$. It is uniquely defined by the condition that

$$\langle Pu, v\rangle_{L^2} = \langle u, P^*v\rangle_{L^2} \tag{7}$$

holds for all smooth, compactly supported sections u, v. Here, the L^2-inner product is defined with respect to the ACyl metric g. Observe that the identity (7) extends to sections u, v in certain Hölder spaces.

Lemma 2.8. *Let $P : \Gamma(E) \to \Gamma(F)$ be an asymptotically translation-invariant operator of order l with formal adjoint $P^* : \Gamma(F) \to \Gamma(E)$. Suppose that $u \in C^{l,\alpha}_{\varepsilon_1}(E)$ and $v \in C^{l,\alpha}_{\varepsilon_2}(F)$ with $\varepsilon_1 + \varepsilon_2 > 0$. Then*

$$\langle Pu, v\rangle_{L^2} = \langle u, P^*v\rangle_{L^2}.$$

The proof is straight forward, and written out in ([Nor08][Lemma 2.3.15]), for example.

Proposition 2.9. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic, asymptotically translation-invariant operator of order l with formal adjoint $P^* : \Gamma(F) \to \Gamma(E)$. If ε is not a critical weight, then the image of $P : C^{k+l,\alpha}_{\varepsilon}(E) \to C^{k,\alpha}_{\varepsilon}(F)$ is precisely the L^2-orthogonal complement to the kernel of $P^* : C^{k+l,\alpha}_{-\varepsilon}(F) \to C^{k,\alpha}_{-\varepsilon}(E)$ in $C^{k,\alpha}_{\varepsilon}(F)$.*

Proof. This can be deduced from Theorem 2.6 and Proposition 2.7, compare [Nor08][Proposition 2.3.16] for details. □

We seek to apply this general theory to a certain subclass of asymptotically translation-invariant operators, which naturally arise as the linearisation of the Monge-Ampère operator in Section 5 below.

Definition 2.10. Let f be a smooth function on an ACyl manifold (M, g). Then the following operator

$$\Delta_f u := \Delta_g u + g(\nabla^g f, \nabla^g u)$$

is called the *drift Laplacian with potential function f.* If additionally $f - 2t \in C^{\infty}_{\delta_0}(M)$ for some $\delta_0 > 0$, we refer to Δ_f as an *ACyl drift Laplace operator.*

Any such operator Δ_f is self-adjoint with respect to the L^2-inner product induced by the measure $e^f \,\mathrm{dV}_g$, i.e.

$$\int_M (\Delta_f u) v \, e^f \,\mathrm{dV}_g = \int_M u(\Delta_f v)\, e^f \,\mathrm{dV}_g$$

for all smooth, compactly supported functions u, v. If Δ_f is moreover an ACyl drift Laplacian, then it is asymptotic to the translation-invariant operator

$$\Delta_{2t} u = \Delta_{g_{cyl}} u + g_{cyl}\left(2\frac{\partial}{\partial t}, \nabla^{g_{cyl}} u\right) = \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} + \Delta_{g_L} u \tag{8}$$

where $g_{cyl} = dt^2 + g_L$ is the product metric. From the general theory, we deduce the next

Theorem 2.11. *Let (M, g) be an ACyl manifold and suppose that Δ_f is an ACyl drift Laplacian with potential function f. Then for any $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$ and $0 < \varepsilon < 2$ the operator*

$$\Delta_f : C^{k+2,\alpha}_{\varepsilon}(M) \to C^{k,\alpha}_{\varepsilon}(M)$$

is an isomorphism.

Proof. Since $\varepsilon > 0$, the injectivity of Δ_f follows immediately from the standard maximum principle, so we only need to show surjectivity. Before using Proposition 2.9, we need to prove the following

Claim. There are no critical weights for Δ_f in the interval $(0, 2)$.

Since Δ_f is asymptotic to the operator given in (8), the definition of critical weights requires us to show that there are no solutions v to the equation

$$\frac{\partial^2 v}{\partial t^2} + 2\frac{\partial v}{\partial t} + \Delta_{g_L} v = 0 \tag{9}$$

of the form

$$v = e^{i\lambda t}\sum_{j=0}^{d} a_j t^j \tag{10}$$

with $\operatorname{Im}\lambda = \varepsilon \in (0,2)$ and functions a_j on L. To see this, we plug (10) into (9) and by considering the coefficient of t^d, we observe that (9) can only be satisfied if

$$\Delta_{g_L} a_d + (-\lambda^2 + 2i\lambda)a_d = 0. \tag{11}$$

This implies that $-\lambda^2+2i\lambda$ must be real and non-negative because Δ_{g_L} is a negative and self-adjoint operator on the closed manifold (L, g_L). Writing $\lambda = \gamma + i\varepsilon$, this translates into

$$2\gamma(1-\varepsilon) = 0, \quad \text{and} \quad -\gamma^2 + \varepsilon(\varepsilon - 2) \geq 0. \tag{12}$$

If $\varepsilon = 1$, the second equation in (12) gives a contradiction, and so $\gamma = 0$. Then, however, the second equation implies $\varepsilon \geq 2$ since $\varepsilon > 0$. Thus, there cannot be a solution to (9) of the form (10) with $\varepsilon \in (0,2)$, proving the claim.

Hence, according to Proposition 2.9, it suffices to show that the formal adjoint Δ_f^* of Δ_f is injective when viewed as a map $\Delta_f^* : C^{k+2,\alpha}_{-\varepsilon}(M) \to C^{k,\alpha}_{-\varepsilon}(M)$ with $0 < \varepsilon < 2$. A simple computation shows that Δ_f^* is given by

$$\Delta_f^* u = \Delta_g u - g(\nabla^g f, \nabla^g u) - u\,\Delta_g f.$$

Assuming that $u \in C^{k+2,\alpha}_{-\varepsilon}(M)$ satisfies $\Delta_f^* u = 0$, we compute

$$\begin{aligned}
\Delta_f(e^{-f}u) &= u\,\Delta_f e^{-f} + e^{-f}\Delta_f u + 2g(\nabla^g e^{-f}, \nabla^g u)\\
&= -ue^{-f}\Delta_f f + e^{-f}\Delta_g u - e^{-f}g(\nabla^g f, \nabla^g u)\\
&= e^{-f}\Delta_f^* u\\
&= 0.
\end{aligned}$$

Since $\varepsilon < 2$, the function $e^{-f}u$ tends to zero as $t \to \infty$, and so the maximum principle implies that $e^{-f}u$ vanishes identically. Thus, the kernel of Δ_f^* is trivial and the theorem follows. $\square$

We end this section by proving a (global) Poincaré-inequality for a certain drift Laplace operator, which is needed later on to obtain L^2-estimates for the Monge-Ampère operator as in [CD20b].

Proposition 2.12. *Let (M, g) be an ACyl manifold. If f is a C^2-function on M satisfying $f - 2t \in C^2_{\delta_0}(M)$ for some $\delta_0 > 0$ then there exists a constant $\lambda > 0$ such that*

$$\lambda \int_M u^2 \frac{e^f}{(f+c)^2} \, \mathrm{dV}_g \leq \int_M |\nabla^g u|^2_g \frac{e^f}{(f+c)^2} \, \mathrm{dV}_g$$

holds for all smooth, compactly supported functions u on M, where $c > 0$ is chosen so that $f + c > 0$.

Proof. First of all note that we can assume that $f + c > 0$ for some $c > 0$ because f is proper since $f - 2t \in C^2_{\delta_0}(M)$. By [CD20b][Lemma 5.1], it is sufficient to find a positive C^2-function v on M and a positive constant λ_0 such that $\Delta_{f-2\log(f+c)} v \leq -\lambda_0 v$ outside some compact subset $K \subset M$.

We claim that this condition holds for the function $v := e^{-\frac{f}{2}}$. Indeed, we first calculate

$$\Delta_{f-2\log(f+c)} e^{-\frac{f}{2}} = -e^{-\frac{f}{2}} \left(\frac{1}{2} \Delta_g f + \left(\frac{1}{4} - \frac{1}{f+c} \right) g(\nabla^g f, \nabla^g f) \right),$$

and, since $f - 2t \in C^2_{\delta_0}(M)$, we then observe that $(f+c)^{-1} \to 0$ in the limit $t \to \infty$, as well as

$$\Delta_g f \to \Delta_{g_{cyl}} t = 0, \quad \text{and} \quad |\nabla^g f|^2_g \to |\nabla^{g_{cyl}} t|^2_{g_{cyl}} = 1 \quad \text{if} \quad t \to \infty.$$

The claim now follows by taking for instance $\lambda_0 = 1/8$ and $K := \{x \in M \mid t(x) \leq m\}$ for $m \gg 1$ large enough. $\square$

3. Preliminaries on Kähler-Ricci solitons

In this section, we recall some basic definitions and facts about steady Kähler-Ricci solitons. In particular, we review when a solitons is gradient. The main result in this direction is Proposition 3.5, which states a criterion for the radial vector field on an ACyl Kähler manifold to be a gradient field.

Definition 3.1. A triple (M, g, X) consisting of a complete Kähler manifold (M, g) and a complete real holomorphic vector field X on M is a *steady Kähler-Ricci soliton* if the corresponding Kähler form ω satisfies

$$\mathrm{Ric}(\omega) = \frac{1}{2} \mathcal{L}_X \omega, \tag{13}$$

where $\mathrm{Ric}(\omega)$ denotes the Ricci form of ω and $\mathcal{L}_X$ is the Lie derivative in direction of X. The vector field X is called the *soliton vector field*.

We say that a steady Kähler-Ricci soliton (M, g, X) is *gradient* if $X = \nabla^g f$ for some smooth real-valued function f on M. In this case, f is called the *soliton potential* and equation (13) becomes

$$\mathrm{Ric}(\omega) = i\partial\bar{\partial} f. \tag{14}$$

For us, the most important example is Hamilton's cigar soliton [Ham88].

Example 3.2 (Cigar soliton)**.** Let $M = \mathbb{C}$ and consider the following metric

$$g_{cig} = \frac{1}{1 + x^2 + y^2}\left(dx^2 + dy^2\right)$$

which is Kähler with Kähler form

$$\omega_{cig} = \frac{1}{1 + |z|^2}\frac{i}{2}dz \wedge d\bar{z},$$

where $z = x + iy$ is the standard coordinate for $\mathbb{C}$. A straight forward computation then shows that $(\mathbb{C}, g_{cig})$ defines a Kähler-Ricci soliton with real holomorphic vector field

$$X = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} = 4\,\mathrm{Re}\left(z\frac{\partial}{\partial z}\right).$$

In fact, $(\mathbb{C}, g_{cig})$ is also an ACyl manifold in the sense of Definition 2.1, with ACyl map $\Phi : \mathbb{R} \times S^1 \to \mathbb{C}^*$ given by

$$\Phi(t, e^{2\pi i\theta}) := e^{t+2\pi i\theta}.$$

Under this change of coordinates, we have

$$\Phi_*\frac{\partial}{\partial t} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{and} \quad \Phi^* g_{cig} = \frac{1}{1 + e^{2t}}(dt^2 + d\theta^2),$$

from which it is easy to see that g_{cig} is indeed asymptotic to $g_{cyl} = dt^2 + d\theta^2$.

In higher dimension, further examples of ACyl Kähler-Ricci solitons can be obtained by taking the product of the cigar soliton with a compact, Ricci-flat Kähler manifold. Such examples, and their finite quotients, are the asymptotic model for the solitons constructed in Theorem 4.6.

Under certain conditions, the soliton equation (13) can be reduced to solving a Monge-Ampère equation, as shown in [CD20b][Proposition 4.5], for example. We adapt their arguments to obtain the next

Lemma 3.3. *Let (M, g, J) be a Kähler manifold of dimension n with Kähler form ω. Let X be a real holomorphic vector field such that $X = \nabla^g f$, for some smooth function $f : M \to \mathbb{R}$, and suppose M admits a nowhere-vanishing, holomorphic $(n, 0)$-form Ω. If there is a smooth function $\varphi : M \to \mathbb{R}$ satisfying*

$$\left(\omega + i\partial\bar{\partial}\varphi\right)^n = e^{-f-\frac{X}{2}(\varphi)} i^{n^2}\,\Omega \wedge \overline{\Omega}, \tag{15}$$

then $\omega + i\partial\bar{\partial}\varphi$ defines a steady Kähler-Ricci soliton with vector field X. Moreover, if φ is JX-invariant, the resulting soliton is gradient.

Proof. We closely follow the computation provided in the proof of [CD20b][Proposition 4.5]. Suppose $\omega_\varphi := \omega + i\partial\bar{\partial}\varphi$ satisfies (15) and compute:

$$\begin{aligned}\mathrm{Ric}(\omega_\varphi) &= -i\partial\bar{\partial}\log\frac{\omega_\varphi^n}{i^{n^2}\Omega\wedge\overline{\Omega}}\\ &= i\partial\bar{\partial}f+\frac{X}{2}(\varphi)\\ &= \frac{1}{2}\mathcal{L}_X\omega+\frac{1}{2}\mathcal{L}_X\left(i\partial\bar{\partial}\varphi\right)=\frac{1}{2}\mathcal{L}_X\omega_\varphi,\end{aligned}$$

where we used in the last line that X is real holomorphic, and

$$\frac{1}{2}\mathcal{L}_X\omega=\frac{1}{2}d\iota_X\omega=\frac{1}{2}dJ\iota_{JX}\omega=-\frac{1}{2}dJdf=i\partial\bar{\partial}f$$

since $X=\nabla^g f$. So, if g_φ is the metric corresponding to ω_φ, the triple (M,g_φ,X) defines a steady Kähler-Ricci soliton.

For the second claim, assume φ to be JX-invariant. It is not difficult to see that $\iota_{JX}\left(2i\partial\bar{\partial}\varphi\right)=-dX(\varphi)$, compare the proof of [CD20b][Lemma 7.3] for instance. Then we conclude

$$\iota_{JX}\left(\omega+i\partial\bar{\partial}\varphi\right)=-d\left(f+\frac{X}{2}(\varphi)\right)$$

i.e. $X=\nabla^{g_\varphi}\left(f+\frac{X}{2}(\varphi)\right)$ as claimed. □

We conclude this section by addressing the question when a given Kähler-Ricci soliton is gradient. It is not difficult to see that if it is gradient, then JX is a Killing vector field for the metric. Under certain conditions, the converse is true as well.

Lemma 3.4. *Let (M,g,X) be a steady Kähler-Ricci soliton and suppose that JX is Killing for g, where J denotes the complex structure of (M,g). If $H^1(M,\mathbb{R})=0$, then the soliton (M,g,X) is gradient.*

Proof. This is a special case of [CD20a][Corollary A.7]. □

In the special case of ACyl manifolds, we can replace the condition $H^1(M,\mathbb{R})=0$ in Lemma 3.4 by an asymptotic condition on the vector field X. In fact, there is the following statement for more general ACyl Kähler manifolds.

Proposition 3.5. *Let (M,g) be an ACyl Kähler manifold of rate $\delta>0$ with complex structure J and ACyl map Φ. Suppose X is a real holomorphic vector field on M such that*

$$X=2\Phi_*\frac{\partial}{\partial t}\tag{16}$$

outside some compact domain. If JX is Killing for g, then there exists a smooth function $f:M\to\mathbb{R}$ with $f-2t\in C^\infty_\delta(M)$ such that $X=\nabla^g f$.

Proof. The idea is to adapt a proof of Frankel for compact manifolds ([Fra59]) to the ACyl setting. This is possible because there is a version of Hodge splitting on such manifolds, see for example [Nor08][Section 2.3.3].

Let ω be the Kähler form of (M, g, J). First, since JX is Killing and X is real holomorphic, we have $\mathcal{L}_{JX} g = \mathcal{L}_{JX} J = 0$ and so $\mathcal{L}_{JX}\omega = 0$. In particular, the 1 form $\iota_{JX}\omega$ is closed. We would like to show that it is in fact exact, for which we need to understand its asymptotic behaviour.

Let $\Phi^{-1} \circ t$ be a cylindrical coordinate function for (M, g), whose smooth extension to all of M is denoted by τ. Then we claim that

$$\iota_{JX}\omega + d\tau \in C^\infty_\delta(\Lambda^1(M)). \tag{17}$$

Indeed, outside of a sufficiently large domain so that (16) is satisfied, we can estimate

$$|d\tau + \iota_{JX}\omega|_g = |\iota_{\Phi_*\partial_t}(\Phi_* g_{cyl}) - \iota_X g|_g \leq |X|_g \cdot |\Phi_* g_{cyl} - g|_g = O(e^{-\delta t})$$

because (M, g) is ACyl of rate $\delta > 0$ and the norm of X is uniformly bounded on M. Here we used that on the product $\mathbb{R} \times L$, the tensors dt and g_{cyl} are related by $\iota_{\partial_t} g_{cyl} = dt$. A similar estimate holds for the first covariant derivative

$$\begin{aligned} |\nabla^g \left(\iota_{\Phi_*\partial t}(\Phi_* g_{cyl}) - \iota_X g\right)|_g &\leq |\nabla^g X|_g \cdot |\Phi_* g_{cyl} - g|_g + |X|_g \cdot |\nabla^g g_{cyl}|_g \\ &= O(e^{-\delta t}) \end{aligned}$$

since $|\nabla^g X|_g = O(1)$ and $|\nabla^g g_{cyl}|_g$ decays exponentially of rate δ. Similarly, we can proceed by induction to obtain bounds on higher derivatives, which implies (17).

By the ACyl version of Hodge splitting ([Nor08][Theorem 2.3.27]), there are 1-forms $h, \alpha, \beta \in C^\infty_\varepsilon(\Lambda^1 M)$ such that

$$\iota_{JX}\omega + d\tau = h + \alpha + \beta, \tag{18}$$

where h is Δ_g-harmonic, α exact and β co-exact. Here, $0 < \varepsilon < \min\{\delta, \lambda\}$, with λ denoting the smallest (positive) critical weight of the Laplace operator Δ_g acting on 1-forms. Moreover, we can write

$$\alpha = d\tilde{f} \quad \text{and} \quad \beta = d^*\gamma$$

for some $\tilde{f} \in C^\infty_\varepsilon(M)$ and $\gamma \in C^\infty_0(\Lambda^2 M)$. (Note that the growth of γ follows from [Nor08][Theorem 2.3.27] since translation-invariant forms on the cylinder $\mathbb{R} \times L$ are bounded with respect to g_{cyl}, and that we can indeed assume $\tilde{f}$ decays at infinity because the only translation-invariant harmonic functions are constants.)

We have to show that both h and β vanish identically. We begin by observing that h is closed. Since h is Δ_g-harmonic and in $C^\infty_\varepsilon(\Lambda^1 M)$, we may, according to Lemma 2.8, integrate by parts to obtain

$$0 = \langle h, \Delta_g h\rangle_{L^2} = \langle dh, dh\rangle_{L^2} + \langle d^* h, d^* h\rangle_{L^2}, \tag{19}$$

i.e. $dh = 0$ and $d^*h = 0$. Hence, we deduce immediately from the decomposition (18) that β is also closed. Integrating by parts then yields

$$\langle \beta, \beta \rangle_{L^2} = \langle \beta, d^*\gamma \rangle_{L^2} = \langle d\beta, \gamma \rangle_{L^2} = 0,$$

so $\beta \equiv 0$ as desired.

Next, we follow the proof of [Fra59][Lemma 2] to show that $h \equiv 0$. By assumption, JX is Killing for g and so

$$\Delta_g \left(\mathcal{L}_{JX} h \right) = \mathcal{L}_{JX} \left(\Delta_g h \right) = 0,$$

but also $\mathcal{L}_{JX} h = d(\iota_{JX} h)$, i.e. $\mathcal{L}_{JX} h$ is a harmonic and exact 1-form in $C^\infty_\varepsilon(\Lambda^1 M)$. Using the orthogonality of Hodge's decomposition, we conclude $\mathcal{L}_{JX} h = 0$

Moreover, the 1-form $Jh(\cdot) := h(J\cdot)$ is also harmonic since the Laplace operator on a Kähler manifold preserves the bi-degree decomposition of the cotangent bundle. Using the same computation as in (19), we conclude that Jh is closed, from which we further deduce that

$$d \left(\iota_{JX}(Jh) \right) = \mathcal{L}_{JX}(Jh) = \mathcal{L}_{JX}(J)h + J\mathcal{L}_{JX} h = 0$$

because JX is real holomorphic, i.e. $\mathcal{L}_{JX} J = 0$. In particular, the function $\iota_{JX}(Jh) = -h(X)$ is constant on M, and thus it can only be identically zero as $h(X)$ tends to zero at infinity. This, together with integration by parts, in turn gives

$$\begin{aligned}
\langle h, h \rangle_{L^2} &= \langle \iota_{JX}\omega + d\tau, h \rangle_{L^2} \\
&= \langle \iota_{JX}\omega, h \rangle_{L^2} + \langle \tau, d^*h \rangle_{L^2} \\
&= -\int_M h(X) \, \mathrm{dV}_g \\
&= 0,
\end{aligned}$$

where we used in the penultimate line that $\iota_{JX}\omega$ is the negative g-dual of X and $d^*h = 0$. We conclude $h \equiv 0$, and consequently

$$\iota_{JX}\omega = d\tilde{f} - d\tau$$

with $\tilde{f} \in C^\infty_\varepsilon(M)$. It remains to improve the decay rate of $\tilde{f}$, i.e. we need to show $\tilde{f} \in C^\infty_\delta(M)$ instead of merely $\tilde{f} \in C^\infty_\varepsilon(M)$. It clearly suffices to prove $\tilde{f} \in C^0_\delta(M)$ because we already know from (17) that $df \in C^\infty_\delta(\Lambda^1 M)$.

Working on the cylindrical end, we write $\tilde{f}(t,x) := \tilde{f} \circ \Phi(t,x)$ for $t \in \mathbb{R}_+$ and $x \in L$, and express $\tilde{f}$ as an integral of the radial derivative as follows:

$$\tilde{f}(t,x) = -\int_t^\infty \partial_s \tilde{f}(s,x) ds.$$

This, together with $d\tilde{f}(X) = O(e^{-\delta t})$, implies $\tilde{f} = O(e^{-\delta t})$ as required. Proposition 3.5 now follows by setting $f := -\tilde{f} + \tau$. □

4. The existence theorem

The goal of this section is to show the main result of this article (Theorem 1.2). We begin by introducing a more general setup and discussing the existence of ACyl Kähler metrics on the considered manifolds. Step by step, we then add further assumptions and point out their importance for Theorem 1.2. This discussion will also be accompanied by a simple, but illustrative example.

Throughout this section, let $D = D^{n-1}$ be a compact Kähler manifold of complex dimension $n-1$ and assume that $\gamma : D \to D$ is a biholomorphism of order $m > 1$. Consider the orbifold $M_{orb} := (\mathbb{C} \times D) / \Gamma$, where we set $\Gamma := \langle \gamma \rangle \cong \mathbb{Z}_m$ and let γ act on the product via

$$\gamma(z, w) := \left(e^{\frac{2\pi i}{m}} z, \gamma(w)\right). \tag{20}$$

The singular part M_{orb}^{sing} of M_{orb} is clearly contained in the slice $(\{0\} \times D)/\, \Gamma$ and corresponds to the fixed points of γ on D.

Let $\pi : M \to M_{orb}$ be a resolution of M_{orb}, with exceptional set $E = \pi^{-1}(M_{orb}^{sing})$. Then we use π to identify $M \setminus E \cong M_{orb} \setminus M_{orb}^{sing}$ and, in particular, we view $(\mathbb{C}^* \times D) / \Gamma$ as an (open) complex submanifold of M.

It is instructive to keep the following example in mind.

Example 4.1 (A first example)**.** Let $D = \mathbb{T}$ be the (real) 2-torus and define $\gamma = -\operatorname{Id}$. Then consider the orbifold $(\mathbb{C} \times D) / \langle \gamma \rangle$ with four isolated singular points contained inside the slice $\{0\} \times D / \langle \gamma \rangle$ and locally isomorphic to a neighborhood of the origin in $\mathbb{C}^2/\mathbb{Z}_2$. Blowing-up each of these rational double points then yields a resolution $\pi : M \to (\mathbb{C} \times D) / \langle \gamma \rangle$.

We point out that this complex manifold M does admit Kähler metrics, and in fact, certain Calabi-Yau metrics (so-called ALG gravitational instantons) were constructed on M in [BM11][Theorem 2.3].

Before finding ACyl Kähler metrics on a resolution $\pi : M \to M_{orb}$, we have to fix an asymptotic model g_{cyl} on $(\mathbb{C}^* \times D) / \Gamma$. For this, we choose a γ-invariant Kähler metric g_D on D and define the cylindrical parameter $t : \mathbb{C}^* \times D \to \mathbb{R}$ to be

$$t(z, w) := \log |z|. \tag{21}$$

If $g_{\mathbb{C}}$ denotes the standard flat metric on $\mathbb{C}$, then the product metric

$$g_{cyl} := e^{-2t} g_{\mathbb{C}} + g_D \tag{22}$$

is Γ-invariant and can thus be viewed as a metric on the quotient $(\mathbb{C}^* \times D) / \Gamma$. Note that if we let

$$\begin{aligned} \Phi : \mathbb{R} \times \mathbb{S}^1 \times D &\to \mathbb{C}^* \times D, \\ (t, e^{2\pi i \theta}, w) &\mapsto (e^{t + 2\pi i \theta}, w) \end{aligned} \tag{23}$$

then $\Phi^*(g_{cyl}) = dt^2 + g_{\mathbb{S}^1} + g_D$, where $g_{\mathbb{S}^1}$ denotes the metric on $\mathbb{S}^1$ of length 1. So g_{cyl} is indeed a Γ-invariant cylinder with cross-section $(\mathbb{S}^1 \times D, g_{\mathbb{S}^1} + g_D)$. The corresponding Kähler form ω_{cyl} on $\mathbb{C}^* \times D$ is given by

$$\omega_{cyl} = |z|^{-2}\frac{i}{2}dz \wedge d\bar{z} + \omega_D = i\partial\bar{\partial}t^2 + \omega_D, \tag{24}$$

where ω_D is the Kähler form associated to g_D.

We would like to understand how to construct ACyl Kähler metris on M that are asymptotic to g_{cyl} as in (22) for some choice of Kähler metric g_D on D. Moreover, we wish to know which de Rham cohomology classes contain the corresponding Kähler forms.

To simplify notation, we introduce the following notion of Kähler class.

Definition 4.2. Let $\pi : M \to M_{orb}$ be as above. A class $\kappa \in H^2(M, \mathbb{R})$ is said to be *Kähler* if there exists a Kähler form $\omega \in \kappa$.

A Kähler class is called *ACyl* if it contains a Kähler form whose metric g is ACyl and satisfies

$$|\left(\nabla^{g_{cyl}}\right)^k (g - g_{cyl})|_{g_{cyl}} = O(e^{-\delta t}) \quad \text{as} \quad t \to \infty, \tag{25}$$

for some $\delta > 0$ and all $k \in \mathbb{N}_0$, where g_{cyl} is given by (22) for some γ-invariant Kähler metric g_D on D.

We point out that this notion of ACyl Kähler classes is quite restrictive since we only allow ACyl metrics with ACyl diffeomorphism Φ defined by (23). In particular, the ACyl Kähler metric g and its asymptotic cylinder are Kähler with respect to the *same* complex structure since $M \setminus E$ is biholomorphic to $(\mathbb{C}^* \times D)/\Gamma$.

One way to describe ACyl classes is by introducing a complex compactification $\overline{M}$ of M, whose construction we now describe.

Recall that $\mathbb{C}$ can naturally be compactified to the Riemann sphere $\mathbb{CP}^1$ by adding one point 'at infinity'. We denote this point by ∞, i.e. $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Consequently, the orbifold M_{orb} is naturally compactified by $\left(\mathbb{CP}^1 \times D\right)/\Gamma$ and, since $(\mathbb{C}^* \times D)/\Gamma$ and M are biholomorphic outside of the exceptional set E, we also obtain a compactification $\overline{M}$ of M.

In other words, $\overline{M}$ is constructed from M by gluing in the orbifold divisor $\overline{D} := (\{\infty\} \times D)/\Gamma$ at 'infinity'. We emphasize this by writing $\overline{M} = M \cup \overline{D}$. Then the following theorem provides equivalent characterisations of ACyl Kähler classes.

Theorem 4.3. *Let $\pi : M \to M_{orb}$ be as introduced at the beginning of Section 4, and suppose that $\overline{M} = M \cup \overline{D}$ is the compacification obtained by adding an orbifold divisor $\overline{D}$ at infinity. For a given $\kappa \in H^2(M, \mathbb{R})$, the following are equivalent:*

(i) κ is an ACyl Kähler class.

(ii) $\kappa = \kappa_{\overline{M}}|_M$ *for some orbifold Kähler class* $\kappa_{\overline{M}}$ *on* $\overline{M}$.

Moreover, if the $\mathbb{C}^*$*-action* $(\mathbb{C} \times D)/\langle\gamma\rangle$ *given by*

$$\lambda * (z, w) = (\lambda z, w), \quad \lambda \in \mathbb{C}^*, \tag{26}$$

extends π*-equivariantly to a holomorphic action of* $\mathbb{C}^*$ *on* M*, then* (i) *is equivalent to the following:*

(iii) There exists some Kähler form $\omega_0 \in \kappa$ *on* M *such that the* 1*-form* $\iota_{J\frac{\partial}{\partial t}}\omega_0$ *is defined on* M *and the restriction of* $\iota_{J\frac{\partial}{\partial t}}\omega_0$ *to the open set* $(\mathbb{C}^* \times D)/\langle\gamma\rangle$ *is exact, where* J *denotes the complex structure on* M *and* t *is defined in (21).*

The equivalence of (i) and (ii) originates in work on ACyl Calabi-Yau manifolds [HHN15], however, it is impractical to verify in concrete examples. This is why we introduce criterion (iii). In fact, this condition allows us to prove:

Corollary 4.4. *Let* $\pi : M \to M_{orb}$ *be as introduced at the beginning of Section 4 and assume that the* $\mathbb{C}^*$*-action given by (26) extends* π*-equivariantly to a holomorphic action on* M*.*

If every closed, γ*-invariant 1-form on* D *is exact, then each Kähler class is ACyl.*

The proof of this corollary also partly justifies extending the $\mathbb{C}^*$-action (26) to the resolution.

Proof. Let ω_0 a Kähler form on M. Since $\mathbb{S}^1$ is compact and connected, we can average ω_0 over this group to obtain a new closed 2-form $\hat{\omega}_0$ such that $[\hat{\omega}_0] = [\omega_0] \in H^2(M, \mathbb{R})$. In fact, $\hat{\omega}_0$ is a positive (1,1)-form because $\mathbb{S}^1$ acts by biholomorphisms and the averaging does not affect the positivity.

As the $\mathbb{C}^*$-action (26) extends to M, the radial vector field $\partial/\partial t$ also extends to a real holomorphic vector field Y on M. In particular,

$$Y = \frac{\partial}{\partial t} \quad \text{on} \quad (\mathbb{C}^* \times D)/\Gamma \subset M \tag{27}$$

and JY is a generator of the corresponding $\mathbb{S}^1$-action, so that

$$\mathcal{L}_{JY}(\hat{\omega}_0) = 0.$$

Hence, the 1-form $\iota_{JY}(\hat{\omega}_0)$ is closed and to apply (iii) of Theorem 4.3, we need to show that its restriction to $M \setminus E \cong (\mathbb{C}^* \times D)/\langle\gamma\rangle$ is exact.

Observe that it is sufficient for the lift of $\iota_{JY}(\hat{\omega}_0)$ to $\mathbb{C}^* \times D$ to be exact. This lift, in turn, is clearly exact if its restriction to a slice $\{0\} \times \mathbb{S}^1 \times D \subset \mathbb{R} \times S^1 \times D \cong \mathbb{C}^* \times D$ is exact. Since $\hat{\omega}_0$ is $\mathbb{S}^1$-invariant and we have $\iota_{JY}(\hat{\omega}_0)(JY) = 0$, this restriction, however, is of the form $p_D^*\alpha$ for some 1-form α on D, where $p_D : \mathbb{S}^1 \times D \to D$ denotes the projection. Using that $\iota_{JY}(\hat{\omega}_0)$ is also closed and γ-invariant, we conclude that α must be closed and γ-invariant as well, and hence exact by assumption. $\square$

The proof of Theorem 4.3 is postponed to Section 4.1.

Remark 4.5. Let us examine the usefulness of this corollary by considering Example 4.1. Recall that in this case, the resolution $\pi : M \to (\mathbb{C}^* \times \mathbb{T})/\langle\gamma\rangle$ is obtained by blowing-up the four fixed points of $\gamma = -\operatorname{Id}$ on $\mathbb{C} \times \mathbb{T}$. For showing that the $\mathbb{C}^*$-action given by (26) extends to the blow-up M, it suffices to do so locally near each singularity because these are isolated points. This, however, is clearly true because the blow-up

$$\mathcal{O}_{\mathbb{CP}^1}(-2) \to \mathbb{C}^2/\{\pm \operatorname{Id}_{\mathbb{C}^2}\} \tag{28}$$

is a toric resolution (with respect to the standard action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2$).

Verifying the condition in Corollary 4.4 is also straight forward. Indeed, denoting the holomorphic coordinate of the universal cover $\mathbb{C}$ of $\mathbb{T}^1$ by $w = u + iv$, we see that the translation-invariant 1-forms du and dv are clearly *not* fixed by the action of $-\operatorname{Id}$ on $\mathbb{C}$. Thus, *every* Kähler class of the blow-up M admits an ACyl Kähler metric.

Having understood when a resolution $\pi : M \to M_{orb}$ admits ACyl Kähler metrics, we may continue adding further assumptions in order to find steady Kähler-Ricci solitons on M. Namely, assume that D^{n-1} admits a nowhere-vanishing holomorphic $(n-1,0)$-form Ω_D such that

$$\gamma^*\Omega_D = e^{-\frac{2\pi i}{m}}\Omega_D.$$

This, together with (20), implies that the holomorphic $(n,0)$-form $\Omega := dz \wedge \Omega_D$ is γ-invariant and descends to M_{orb}. Thus, we may require the resolution $\pi : M \to M_{orb}$ to be *crepant*, i.e. we assume that Ω extends to a nowhere-vanishing form on M.

As in Theorem 4.3, we additionally assume the extension of the $\mathbb{C}^*$-action (26) from M_{orb} to M. This guarantees that the infinitessimal generator Y of the corresponding $\mathbb{R}_+$-action is a real holomorphic vector field and thus, multiples of Y are candidates for the soliton field of the desired solitons.

With these conditions, we recall the main result of this article.

Theorem 4.6. *Let D^{n-1} be a compact Kähler manifold with nowhere-vanishing holomorphic $(n-1,0)$-form Ω_D. Suppose $\gamma : D \to D$ is a complex automorphism of order $m > 1$ such that*

$$\gamma^*\Omega_D = e^{-\frac{2\pi i}{m}}\Omega_D, \tag{29}$$

and consider the orbifold $(\mathbb{C} \times D)/\langle\gamma\rangle$, where γ acts on the product via

$$\gamma(z,w) = \left(e^{\frac{2\pi i}{m}}z, \gamma(w)\right). \tag{30}$$

Let $\pi : M \to (\mathbb{C} \times D)/\langle\gamma\rangle$ be a crepant resolution such that the $\mathbb{C}^*$-action on $(\mathbb{C} \times D)/\langle\gamma\rangle$ given by

$$\lambda * (z,w) = (\lambda z, w), \quad \lambda \in \mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action of $\mathbb{C}^*$ on M.

Then every ACyl Kähler class admits a gradient steady Kähler-Ricci soliton. Moreover, the soliton metric is ACyl of rate ε for each $0 < \varepsilon < 2$ and with asymptotic cylinder given by

$$g_{cyl} = e^{-2t} g_{\mathbb{C}} + g_{RF},$$

where g_{RF} is a Ricci-flat Kähler metric on D.

Looking back at our Example 4.1, we see that the resolution $\pi : M \to (\mathbb{C} \times \mathbb{T})/\{\pm \mathrm{Id}\}$ satisfies all requirements because the blow-up (28) of each singularity is indeed crepant, and $\gamma = -\mathrm{Id}$ acts on the holomorphic 1-form on $\mathbb{T}^1$ by multiplication with -1. Hence, Theorem 4.6, together with Remark 4.5, imply the existence of a steady Kähler-Ricci soliton in *each* Kähler class on M.

Following ideas of Conlon and Deruelle developed in [CD20b][Section 4.2], the strategy for proving Theorem 4.6 is reducing it to a complex Monge-Ampère equation. As explain before Theorem 4.6, the assumptions ensure the existence of a nowhere-vanishing holomorphic $(n, 0)$-form as well as suitable real holomorphic vector fields, so that Lemma 3.3 may indeed be used to set up a Monge-Ampère equation for finding a steady Kähler-Ricci soliton. The technical argument for solving the resulting equation is then provided by Theorem 4.7 below, whose proof we postpone to Section 5.

Theorem 4.7. *Let (M, g, J) be an ACyl Kähler manifold of complex dimension n with Kähler form ω. Suppose that M admits a real holomorphic vector field X such that*

$$X = 2\Phi_* \frac{\partial}{\partial t}$$

outside some compact domain, where Φ is the ACyl map and t the cylindrical coordinate function. Moreover, assume that JX is Killing for g.

If $1 < \varepsilon < 2$ and $F \in C^\infty_\varepsilon(M)$ is JX-invariant, then there exists a unique, JX-invariant $\varphi \in C^\infty_\varepsilon(M)$ such that $\omega + i\partial\bar{\partial}\varphi > 0$ and

$$\big(\omega + i\partial\bar{\partial}\varphi\big)^n = e^{F - \frac{X}{2}(\varphi)} \omega^n \tag{31}$$

The remainder of this section is structured as follows. In Section 4.1, we focus on proving Theorem 4.3. In fact, we provide a detailed construction of the ACyl metrics, and thus obtain more precise statements than those in Theorem 4.3.

Having derived the necessary tools, we then present the proof of Theorem 4.6 by reducing it to Theorem 4.7. Further examples to which Theorem 4.6 may be applied are then discussed in Section 4.3.

4.1. **Constructing ACyl Kähler metrics.** The goal is to prove Theorem 4.3, and we use the notation introduced at the beginning of Section 4.

Let $\pi : M \to M_{orb} := (\mathbb{C} \times D)/\Gamma$ be a resolution, where D denotes some compact Kähler manifold, and the action of $\Gamma = \langle\gamma\rangle \cong \mathbb{Z}_m$ is given by (20). Also, recall that the cylindrical parameter $t : \mathbb{C}^* \times D \to \mathbb{R}$ is defined as $t(z, w) = \log|z|$.

We begin by focusing on the equivalence of Conditions (i) and (iii) in Theorem 4.3 as this is most relevant to our purpose. That (iii) implies (i) is settled in the next proposition.

Proposition 4.8. *Let $\pi : M \to M_{orb}$ be as introduced at the beginning of Section 4 and let the function t be defined by (21). Suppose that g_0 is a Kähler metric on M, whose Kähler form ω_0 satisfies*

$$\iota_{J\frac{\partial}{\partial t}}\omega_0 = df \quad \textit{on} \quad \{t \geq 0\} \subset (\mathbb{C}^* \times D)/\Gamma, \tag{32}$$

for some smooth $f : \{t \geq 0\} \to \mathbb{R}$, where J denotes the complex structure on M. Then there exists an ACyl Kähler metric g on M, with Kähler form ω, such that $[\omega] = [\omega_0] \in H^2(M, \mathbb{R})$.

Moreover, if g is lifted to $\mathbb{C}^* \times D$, it is explicitly given by

$$g = g_{cyl} = e^{-2t}g_{\mathbb{C}} + g_D \quad \textit{on} \quad \{t \geq t_0\} \subset \mathbb{C}^* \times D \tag{33}$$

for some $t_0 > 1$, where $g_{\mathbb{C}}$ denotes the Euclidean metric on $\mathbb{C}$ and g_D is the restriction of g_0 to the slice $\{1\} \times D \subset \mathbb{C}^* \times D$.

Interestingly, the ACyl metrics obtained by the previous proposition are of *optimal rate*, i.e. they are *equal* to its asymptotic model g_{cyl} outside some compact domain. This is an even stronger statement than claimed in Theorem 4.3.

Proof. Analogously to [HHN15][Section 4.2], the idea is to glue the Kähler form ω_0 to a certain cylindrical Kähler form ω_{cyl} outside of some compact domain. Doing so, however, requires that the difference of these two (1,1)-forms is $\partial\bar{\partial}$-exact.

Thus, before we can perform any gluing, we need to have a description of ω_0 in terms of a Kähler potential, at least on the set $\{t \geq 0\}$. We begin by explaining the construction of such a potential function.

Suppose that ω_0 is a Kähler form satisfying

$$\iota_{J\frac{\partial}{\partial t}}\omega_0 = df \quad \text{on} \quad \{t \geq 0\} \subset (\mathbb{C}^* \times D)/\Gamma, \tag{34}$$

for some smooth function f. Working on $\mathbb{C}^* \times D$, we lift ω_0 and f to Γ-invariant forms denoted by the same letters. We view $\mathbb{C}^* \times D$ as a (trivial) fibre bundle over D, and introduce two holomorphic maps

$$j : D \to \mathbb{C}^* \times D \quad \text{and} \quad p : \mathbb{C}^* \times D \to D,$$

where j is the inclusion of the slice $\{1\} \times D \subset \mathbb{C}^* \times D$, and p the projection onto D. Then we *define* a Kähler form ω_D on D by setting

$$\omega_D := j^*\omega_0.$$

Using the cylindrical parameter t as defined in (21), we identify $\mathbb{C}^* \cong \mathbb{R} \times \mathbb{S}^1$ and define a new function φ by

$$\varphi(t, y) := 2\int_0^t f(s, y)ds \quad \text{for } t \in \mathbb{R}_{\geq 0} \text{ and } y \in \mathbb{S}^1 \times D.$$

Then we claim that

$$\omega_0 = i\partial\bar{\partial}\varphi + p^*\omega_D \quad \text{on} \quad \{t \geq 0\} \cong \mathbb{R}_{\geq 0} \times \mathbb{S}^1 \times D. \tag{35}$$

In other words, we have to show that the $(1, 1)$-form $\alpha := \omega_0 - i\partial\bar{\partial}\varphi$ is a basic form for the fibre bundle $p : \mathbb{C}^* \times D \to D$. This means that

$$\mathcal{L}_V\alpha = 0 \quad \text{and} \quad \iota_V\alpha = 0 \tag{36}$$

for all vector fields V on $\mathbb{C}^* \times D$ which are tangent to the fibres of the projection p. However, since α is d-closed, it suffices to show the second condition in (36), and thus we only have to prove that

$$\iota_{\frac{\partial}{\partial t}}\alpha = 0 \quad \text{and } \iota_{J\frac{\partial}{\partial t}}\alpha = 0 \tag{37}$$

since any vector field tangent to fibres of p can be written in terms of $\partial/\partial t$ and $J\partial/\partial t$.

Let us begin by considering the first equation in (37). Keeping in mind that $(J\partial/\partial t)(f) = 0$ by (34), we split $df = d_t f + d_D f$, where d_t and d_D are the differentials in direction of the $\mathbb{R}$- and D-factor, respectively. Using the definition of φ and the fact that $\partial\bar{\partial}t = 0$, we observe

$$2i\partial\bar{\partial}\varphi = dJd\varphi = 2df \wedge Jdt + d_t Jd_D\varphi + d_D Jd_D\varphi,$$

so that we conclude from (34)

$$\iota_{\frac{\partial}{\partial t}}\left(i\partial\bar{\partial}\varphi\right) = \frac{\partial}{\partial t}f Jdt + \frac{1}{2}Jd_D\frac{\partial}{\partial t}\varphi = Jdf = \iota_{\frac{\partial}{\partial t}}\omega_0,$$

as claimed. The second equation in (37) follows similarly:

$$\iota_{J\frac{\partial}{\partial t}}\left(i\partial\bar{\partial}\varphi\right) = -df \cdot (Jdt)\left(J\frac{\partial}{\partial t}\right) = df = \iota_{J\frac{\partial}{\partial t}}\omega_0.$$

This finishes the proof of (35).

Let us define the cylindrical Kähler form ω_{cyl} on $\mathbb{C}^* \times D$ to be

$$\omega_{cyl} := i\partial\bar{\partial}t^2 + p^*\omega_D.$$

The goal is to construct a new Kähler form ω, cohomologous to ω_0, such that

$$\omega = \begin{cases} \omega_{cyl} & \text{on } \{t \geq t_2\}, \\ \omega_0 & \text{on } \{t \leq t_1\} \end{cases} \tag{38}$$

for some positive numbers $t_1 < t_2$. The following gluing procedure is an adaptation of the one contained on [HHN15][p. 247]. For this construction, we first fix $t_0 > 1$ and choose a cut-off function $\chi = \chi(t)$ satisfying

$$\chi(t) = \begin{cases} 1 & \text{if } t \geq t_0, \\ 0 & \text{if } t \leq 1, \end{cases}$$

and then we define a Γ-invariant $(1,1)$-form ω on $\{t \geq 0\}$ by

$$\omega := i\partial\bar{\partial}\left(\chi(t) \cdot t^2 + (1 - \chi(t)) \cdot \varphi\right) + \rho(t)dt \wedge d^c t + p^*\omega_D,$$

where $\rho(t)dt \wedge d^c t$ is an exact bump-form supported inside a neighborhood of $[1, t_0]$, say $[1/2, t_0 + 1/2]$. Clearly, $\omega - \omega_0$ is exact and ω agrees with ω_0 inside the region $\{t \leq 1/2\}$, so that ω extends to a (1,1)-form on M.

Moreover, we notice that $\omega = \omega_{cyl}$ if $t \geq t_0 + 1/2$, and thus, the only thing left to show is the positivity of ω on the region $\{1/2 \leq t \leq t_0 + 1/2\}$. For $t \in [1/2, t_0 + 1/2] \setminus [1, t_0]$, this is clear because we have

$$\omega = \begin{cases} \omega_{cyl} + \rho dt \wedge d^c t & \text{on } \{t \geq t_0\}, \\ \omega_0 + \rho dt \wedge d^c t & \text{on } \{t \leq 1\} \end{cases}$$

and $\rho \geq 0$, so we only need to focus on the case $t \in [1, t_0]$.

To show that $\omega > 0$ on this region, it suffices to check that ω is positive in the direction of the D-factor since we can then compensate for potentially negative terms by choosing ρ sufficiently large inside $[1, t_0]$. Hence, consider $0 \neq v \in T_{\mathbb{C}}D$ and observe

$$\begin{aligned} \omega(v, \overline{v}) &= (1 - \chi(t)) \cdot \left(i\partial\bar{\partial}\varphi\right)(v, \overline{v}) + p^*\omega_D(v, \overline{v}) \\ &= (1 - \chi(t)) \cdot \omega_0(v, \overline{v}) + \chi(t) \cdot p^*\omega_D(v, \overline{v}) \\ &> 0, \end{aligned}$$

where we used in the first line that χ only depends on t, and the second equation follows from (35). As explain before, ω is positive on $\{1 \leq t \leq t_0\}$ once we choose $\rho \gg 1$ on $[1, t_0]$, and so we constructed a Kähler form ω on M in the same cohomology class as ω_0, which also satisfies (38). The corresponding ACyl metric g then fulfills (33), since both g and g_{cyl} are Kähler with respect to the same complex structure. □

For the converse to Proposition 4.8, i.e. that (i) of Theorem 4.3 implies (iii), we additionally assume that the $\mathbb{C}^*$-action on M_{orb} given by (26) extends π-equivariantly to a holomorphic action on the resolution $\pi : M \to M_{orb}$. Hence, the infinitesimal generators of this action extend to real holomorphic vector fields on all of M. Let Y denote the generator of the induced $\mathbb{R}_+$-action (corresponding to translation in the cylindrial parameter t), i.e.

$$Y = \frac{\partial}{\partial t} \quad \text{on} \quad (\mathbb{C}^* \times D)/\Gamma \subset M. \tag{39}$$

Note that if J is the complex structure on M, then JY is generating the $\mathbb{S}^1$-action on M.

Next, we show that Condition (iii) in Theorem 4.3 is in fact necessary for a Kähler class to be ACyl.

Proposition 4.9. *Let $\pi : M \to M_{orb}$ be as introduced at the beginning of Section 4 and assume that the $\mathbb{C}^*$-action given by (26) extends π-equivariantly to a holomorphic action on M.*

Then every ACyl Kähler class contains an ACyl Kähler form $\hat{\omega}$ such that

$$\iota_{JY}\hat{\omega} = df,$$

where JY is the infinitessimal generator of the $\mathbb{S}^1$-action.

Proof. Let g be an ACyl Kähler metric, with Kähler form ω, such that (25) holds. First, average ω over the $\mathbb{S}^1$-action to obtain a Kähler form $\hat{\omega}$ in the same cohomology class. Then observe that the averaging does not change the asymptotic behavior since both g_{cyl} and t are $\mathbb{S}^1$-invariant, so that the corresponding metric $\hat{g}$ is ACyl and satisfies (25). In particular, the function $t = \log|z|$ is also the cylindrical parameter for $\hat{g}$.

Then we notice that JY, for Y given by (39), is a Killing field for $\hat{g}$ because $\mathcal{L}_{JY}\hat{\omega} = 0$. Thus, Proposition 3.5 implies that Y is the gradient field of some function on M, or equivalently that $\iota_{JY}\hat{\omega}$ is exact. □

It only remains to show the equivalence of (i) and (ii) in Theorem 4.3, i.e. that each ACyl Kähler class is the restriction of some orbifold Kähler class on the complex compactification $\overline{M}$.

This goes back to a construction of Haskins, Hein and Nordström [HHN15]. In fact, their ideas can be used to prove the following

Proposition 4.10. *Let $\pi : M \to M_{orb}$ be as introduced at the beginning of Section 4, and suppose that $\overline{M} = M \cup \overline{D}$ is the compacification obtained by adding the orbifold divisor $\overline{D} = D/\Gamma$ at infinity.*

For a given $\kappa \in H^2(M, \mathbb{R})$, the following are equivalent:

(i) κ is an ACyl Kähler class.

(ii) $\kappa = \kappa_{\overline{M}}|_M$ for some orbifold Kähler class $\kappa_{\overline{M}}$ on $\overline{M}$.

Proof. That (i) implies (ii) is a direct consequence of [HHN15][Theorem 3.2], which can be applied here since g and g_{cyl} are Kähler with respect to the same complex structure.

The construction required for the converse implication can be found on [HHN15][p. 247], so we only briefly sketch the idea.

If $\omega_{\overline{M}}$ is a Kähler form on $\overline{M}$, then we define ω_D to be the restriction of $\omega_{\overline{M}}$ to the orbifold divisor $\overline{D} = \{\infty\} \times D/\Gamma$. Note that ω_D lifts to a smooth Γ-invariant form on D, and so we can define the asymptotic model ω_{cyl} on $\mathbb{C}^* \times D$ to be

$$\omega_{cyl} := i\partial\partial t^2 + \omega_D.$$

The new ACyl Kähler form asymptotic to ω_{cyl} is then constructed as

$$\omega := \omega_{\overline{M}} + i\partial\partial\left(\chi \cdot t^2\right) + \rho dt \wedge d^c t,$$

for some cut-off function χ and a bump-function ρ. The cut-off χ is equal to 1 in a neighborhood of $\overline{D}$ and 0 if $t \leq 0$, and ρ is chosen sufficiently large to ensure positivity. □

This concludes the proof of Theorem 4.3, and so we focus on proving Theorem 4.6 next.

4.2. **Proof of Theorem 4.6.** Let $D^{n-1}, \Omega_D, \Gamma = \langle\gamma\rangle$ and $\pi : M \to (\mathbb{C} \times D)/\langle\gamma\rangle$ be defined as in Theorem 4.6. In particular, the discussion of the previous subsection applies and we use the same notation as introduced at the beginning of Section 4. We also assume that the $\mathbb{C}^*$-action on M_{orb} defined by

$$\lambda * (z, w) := (\lambda z, w), \quad \lambda \in \mathbb{C}^*,$$

extends π-equivariantly to a holomorphic action on M. As a consequence, the infinitesimal generators of this action extend to real holomorphic vector fields on M. Let X be two-times the generator of the induced $\mathbb{R}$-action (corresponding to translation in the cylindrical parameter t), i.e.

$$X = 2\frac{\partial}{\partial t} \quad \text{on} \quad \left(\mathbb{C}^* \times D\right)/\Gamma \subset M.$$

Then JX is two-times the generator of the $\mathbb{S}^1$-action, where J is the complex structure on M.

Moreover, we point out that the action of γ given by (30) preserves the holomorphic $(n, 0)$ form Ω on $\mathbb{C}^* \times D$ defined as

$$\Omega := dz \wedge \Omega_D$$

since γ satisfies (29). In particular, Ω descends to M_{orb} and, because the resolution $\pi : M \to M_{orb}$ is crepant, Ω then extends to a holomorphic $(n, 0)$-form on M, which we also denote by Ω.

Let $\kappa \in H^2(M, \mathbb{R})$ be an ACyl Kähler class, i.e. there exists an ACyl metric g satisfying (25) and with Kähler form $\omega \in \kappa$. We need to find a different ACyl metric g_0 with Kähler form ω_0 also contained in the given class κ, such that $X = \nabla^{g_0} f$ and

$$\left(\omega_0 + i\partial\bar{\partial}\varphi\right)^n = \alpha e^{-f-\frac{X}{2}(\varphi)} i^{n^2} \Omega \wedge \overline{\Omega}, \tag{40}$$

for some JX-invariant functions $f, \varphi : M \to \mathbb{R}$ and some constant $\alpha \in \mathbb{R}$. According to Lemma 3.3, $\omega_0 + i\partial\bar{\partial}\varphi$ is then a gradient steady Kähler-Ricci soliton, as required. To achieve this, we begin by modifying ω near infinity to improve the convergence rate and to ensure that it is asymptotic to a *Ricci-flat* cylinder.

First, we improve the asymptotic behavior of ω by applying Proposition 4.8, so that there exists an ACyl Kähler form $\omega_1 \in [\omega]$ which, if lifted to $\mathbb{C}^* \times D$, is of the form

$$\omega_1 = i\partial\bar{\partial}t^2 + \omega_D \quad \text{on} \quad \{t \geq t_0\}$$

for some $t_0 > 0$. Here, ω_D denotes the restriction of ω to the slice $\{1\} \times D$.

In a second step, we modify ω_0 so that it becomes Ricci-flat if restricted to $\{t\} \times D$ for $t \gg t_0$. Recall that by Yau's Theorem [Yau78], there exists $u_D : D \to \mathbb{R}$ such that $\omega_{RF} := \omega_D + i\partial\bar{\partial}u_D > 0$ and

$$(\omega_{RF})^{n-1} = ci^{(n-1)^2}\Omega_D \wedge \overline{\Omega}_D. \tag{41}$$

Moreover, the uniqueness of solutions to (41) implies that u_D is γ-invariant, because γ preserves both ω_D and $\Omega_D \wedge \overline{\Omega}_D$.

Choosing a cut-off function χ with

$$\chi(t) = \begin{cases} 1 & \text{if } t \geq t_0 + 2 \\ 0 & \text{if } t \leq t_0 + 1, \end{cases}$$

we then define a Γ-invariant $(1,1)$-form by

$$\omega_0 := \omega_1 + i\partial\bar{\partial}\left(\chi \cdot u_D\right) + \rho dt \wedge d^c t,$$

where ρ is a bump-function supported in a small neighborhood of $[t_0 + 1, t_0 + 2]$. By the same reasoning as in the proof of Proposition 4.8, ω_0 is positive if ρ is sufficiently large and thus, ω_0 defines a Kähler metric on M in the class $\kappa = [\omega]$. Note that by construction we have

$$\omega_0 = i\partial\bar{\partial}t^2 + \omega_{RF} \tag{42}$$

on the region $\{t \geq t_0 + 3\}$.

The next step is to further modify ω_0 so that it satisfies the requirements of Theorem 4.7. Note that after averaging ω_0 over the compact and connected group $\mathbb{S}^1$ we can assume that ω_0 is invariant under the $\mathbb{S}^1$-action because averaging neither affects the cohomology class, nor the positivity of ω_0. Hence, JX is a Killing field for the corresponding Kähler metric g_0 and by Proposition 3.5, there exists a function f such that

$$X = \nabla^{g_0} f \quad \text{and} \quad f - 2t \in C^\infty_\delta(M),$$

for each $\delta > 0$. In fact, we conclude from (42) that

$$f = 2t \quad \text{on} \quad \{t \geq t_0 + 3\}. \tag{43}$$

In particular, we notice that (M, g_0) satisfies the assumptions of Theorem 4.7.

Let us define a JX-invariant function $F : M \to \mathbb{R}$ by

$$F := \log \frac{\alpha i^{n^2} \Omega \wedge \overline{\Omega}}{\omega_0^n} - f$$

for some constant α to be fixed later. For an appropriate choice of α, we claim that F has compact support. To see this, first observe from (41) and (42) that the cylindrical volume form of ω_{cyl} can be computed as

$$\omega_{cyl}^n = \frac{c\,n}{2} |z|^{-2} i^{n^2} dz \wedge \Omega_D \wedge d\bar{z} \wedge \overline{\Omega}_D,$$

so we set $\alpha := cn/2$, and obtain

$$\begin{aligned} F &= \log \frac{\alpha i^{n^2} \Omega \wedge \overline{\Omega}}{\omega_{cyl}^n} + \log \frac{\omega_{cyl}^n}{\omega_0^n} - f \\ &= 2t - f \\ &= 0, \end{aligned}$$

if $t \geq t_0 + 3$. Thus, F is compactly supported.

If we fix *some* $0 < \varepsilon < 2$, Theorem 4.7 yields a JX-invariant $\varphi \in C_\varepsilon^\infty(M)$ such that

$$(44) \qquad \left(\omega_0 + i\partial\bar{\partial}\varphi\right)^n = e^{F - \frac{X}{2}(\varphi)} \omega_0^n = \frac{cn}{2} e^{-f - \frac{X}{2}(\varphi)} i^{n^2} \Omega \wedge \overline{\Omega},$$

which is precisely (40), so that $\omega_0 + i\partial\bar{\partial}\varphi$ defines a gradient steady Kähler-Ricci soliton. The underlying Kähler metric is clearly ACyl of rate ε.

However, since $F \in C_\varepsilon^\infty(M)$ for *all* $0 < \varepsilon < 2$ and since solutions to (44) contained in $C_\varepsilon^\infty(M)$ are *unique*, we may conclude that indeed $\varphi \in C_\varepsilon^\infty(M)$ for all $0 < \varepsilon < 2$, finishing the proof.

4.3. **Examples.** We begin by providing further examples in complex dimension two. The manifolds M_k considered below are defined as in [BM11][Section 2.2], and their construction is similar to Example 4.1.

Example 4.11. For $k = 2, 3, 4, 6$ we consider the maps $\gamma_k : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$\gamma_k(z_1, z_2) := \left(e^{\frac{2\pi i}{k}} z_1, e^{-\frac{2\pi i}{k}} z_2\right)$$

If we let $\mathbb{T}$ be the (real) 2-torus, then γ_k descends to $\mathbb{C} \times \mathbb{T}$, provided the lattice in $\mathbb{C}$ is chosen appropriately: For $k = 2, 4$, let $\mathbb{T}$ be obtained from the square-lattice, and for $k = 3, 6$ use the hexagonal one instead.

In any case, we may define orbifolds $M_{orb}^k := (\mathbb{C} \times \mathbb{T}) / \langle \gamma_k \rangle$ with isolated singular points which are locally modelled on a neighborhood of the origin in $\mathbb{C}^2/\mathbb{Z}_j$, with $\mathbb{Z}_j$-action induced by the map

$$(z_1, z_2) \mapsto (e^{\frac{2\pi i}{j}} z_1, e^{-\frac{2\pi i}{j}} z_2)$$

for $j \in \{2, 3, 4, 6\}$. More precisely,

- If $k = 2$, M^2_{orb} has four singularities, all isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$.
- If $k = 4$, the corresponding orbifold M^4_{orb} has one $\mathbb{C}^2/\mathbb{Z}_2$ and two $\mathbb{C}^2/\mathbb{Z}_4$ singularities.
- If $k = 3$, there are three singular points in M^3_{orb} and all are isomorphic to $\mathbb{C}^2/\mathbb{Z}_3$.
- If $k = 6$, M^6_{orb} also has three singularities: one $\mathbb{C}^2/\mathbb{Z}_2$, one $\mathbb{C}^2/\mathbb{Z}_3$ and one $\mathbb{C}^2/\mathbb{Z}_6$ singularity.

In each case, condition (29) is fulfilled and the blow-up of all singularities results in a complex manifold denoted by M_k. The corresponding resolution is indeed crepant since all singularities are isolated points and because blowing-up the origin in the local models $\mathbb{C}^2/\mathbb{Z}_j$ yields in fact a crepant resolution. Similar to the reasoning in Example 4.1 and Remark 4.5, one can show that M_k satisfies the requirements of both Theorem 4.6 and Corollary 4.4.

Thus, there is a steady Kähler-Ricci soliton in *each* Kähler class of M_k. Interestingly, these manifolds also admit ALG gravitational instantons by [BM11][Theorem 2.3], for instance.

For finding examples of complex dimension 3, we may take D to be a product $\mathbb{T} \times \mathbb{T}$, but then we consider a different resolution, as the next example shows.

Example 4.12. Let $\mathbb{T}$ be constructed from the hexagonal lattice in $\mathbb{C}$. By setting $D := \mathbb{T} \times \mathbb{T}$ we define $\gamma : \mathbb{C} \times D \to \mathbb{C} \times D$ by

$$\gamma(z_1, z_2, z_3) = e^{\frac{2\pi i}{3}} (z_1, z_2, z_3)$$

and note that $\gamma^*(dz_2 \wedge dz_3) = e^{-\frac{2\pi i}{3}} dz_2 \wedge dz_3$, i.e. (29) is satisfied. Each of the $3^2 = 9$ singularities of $(\mathbb{C} \times D)/\mathbb{Z}_3$ is modelled on $\mathbb{C}^3/\mathbb{Z}_3$, and so we may consider the blow-up M of all singular points.

As before, this resolution is crepant and the $\mathbb{C}^*$-action on the first factor extends, because the same is true for the resolution

$$\mathcal{O}_{\mathbb{CP}^2}(-3) \to \mathbb{C}^3/\mathbb{Z}_3.$$

Moreover, the only closed, γ-invariant 1-forms on D are clearly exact, so that again *each* Kähler class admits a steady Kähler-Ricci soliton.

We conclude this section by discussing another class of examples with D a K3-surface and γ an antisymplectic involution. Explicit examples of such K3-surfaces can for instance be obtain form the Kummer's construction.

Example 4.13. Let D be a K3-surface together with a trivialisation Ω_D of the canonical bundle. Suppose that γ_D is a holomorphic involution on D such that

$$\gamma_D^*\Omega_D = -\Omega_D.$$

Also assume that the fixed point set $\mathrm{Fix}(\gamma_D)$ is non-empty. This implies that $\mathrm{Fix}(\gamma_D)$ is the disjoint union of smooth, complex curves. (In fact, there is a classification for all possibilities of $\mathrm{Fix}(\gamma_D)$, compare [Nik83].)

At any $p \in \mathrm{Fix}(\gamma_D)$, we may linearise γ_D so that its action in a suitable chart is given by

$$\begin{aligned} \mathbb{C}^2 &\to \mathbb{C}^2 \\ (z_1, z_2) &\to (-z_1, z_2) \end{aligned} \tag{45}$$

In particular, the singular set of the orbifold $D/\langle\gamma_D\rangle$ locally corresponds to $\{z_1 = 0\}$ inside $\mathbb{C}^2/\mathbb{Z}_2$, with $\mathbb{Z}_2$-action defined by (45).

As in Theorem 4.6, we let $\gamma : \mathbb{C} \times D \to \mathbb{C} \times D$ be

$$\gamma(z_0, z) := (-z_0, \gamma_D(z)).$$

Then the singularities of $M_{orb} = (\mathbb{C} \times D)/\langle\gamma\rangle$ are locally isomorphic to $\mathbb{C}^3/\mathbb{Z}_2 \cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, where $\mathbb{Z}_2$ acts by -1 in the first two factors, and trivially in the third one. This orbifold, however, admits a *unique* crepant resolution

$$\mathcal{O}_{\mathbb{CP}^1}(-2) \times \mathbb{C} \to \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}, \tag{46}$$

so that the local resolutions may be patched together to yield a crepant resolution $M \to M_{orb}$. Moreover, the $\mathbb{C}^*$-action by multiplication in the first factor extends to M, because this is clearly true for the local model (46).

Since $H^1(D, \mathbb{R}) = 0$, we deduce that each Kähler class on M admits a steady Kähler-Ricci soliton, thanks to Theorem 4.6 and Corollary 4.4.

5. The Monge-Ampère equation

In this section, we present the proof of Theorem 4.7. We consider a more general setting as in Theorem 4.6 in order to clarify which assumptions are used for the a priori estimates below. The following list of properties is assumed throughout this section:

Assumption 5.1. Let (M, g) be an ACyl manifold of (real) dimension $2n$ in the sense of Definition 2.1.

A.1 Suppose there exists a complex structure J on M, so that (M, g, J) is Kähler and denote the Kähler form by ω.

A.2 There exists a real holomorphic vector field X on M such that

$$X = 2\Phi_* \frac{\partial}{\partial t},$$

where Φ denotes the ACyl map and t the cylindrical coordinate function of (M, g).

A.3 JX is a Killing field of g. In particular, $\mathcal{L}_{JX}\omega = 0$ and according to Proposition 3.5, there exists a smooth $\tilde{f} : M \to \mathbb{R}$ such that $X = \nabla^g \tilde{f}$ and

$$\tilde{f} - 2t \in C^\infty_\delta(M),$$

where $\delta > 0$ is the convergence rate of (M, g) to its asymptotic model. We normalise the proper function $\tilde{f}$ by choosing a $c > 0$ such that $f := \tilde{f} + c \geq 1$ so that we still have $X = \nabla^g f$.

The reader may recall that the ACyl metric constructed in Section 4.2 satisfies all of these requirements.

We define new function spaces $C^\infty_{\varepsilon,JX}(M)$ consisting of all elements in $C^\infty_\varepsilon(M)$ which are JX-invariant, i.e.

$$C^\infty_{\varepsilon, JX}(M) := \{u \in C^\infty_\varepsilon(M) \mid JX(u) = 0\}\,.$$

Using this notation, the main result of this section is the next

Theorem 5.2. *Let (M, g) be an ACyl manifold of real dimension $2n$ satisfying the assumptions A.1, A.2 and A.3. Given $F \in C^\infty_{\varepsilon, JX}(M)$ for some $1 < \varepsilon < 2$, there exists a unique $\varphi \in C^\infty_{\varepsilon, JX}(M)$ such that $\omega + i\partial\bar{\partial}\varphi$ is Kähler and satisfies*

$$\left(\omega + i\partial\bar{\partial}\varphi\right)^n = e^{F - \frac{X}{2}(\varphi)}\omega^n. \tag{47}$$

This theorem is analogue to [CD20b][Theorem 7.1], and we also follow the same strategy as in [CD20b][Section 7] to prove it, i.e. we set up a continuity method.

For given $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$ and $F \in C^\infty_{\varepsilon, JX}(M)$ with $1 < \varepsilon < 2$, we define the Monge-Ampère operator on the set $\mathcal{U}$ containing all $\varphi \in C^{k+2,\alpha}_{\varepsilon, JX}(M)$ with $\omega + i\partial\bar{\partial}\varphi > 0$ as follows:

$$\begin{aligned} \mathcal{M} : \mathcal{U} \times [0, 1] &\to C^{k,\alpha}_{\varepsilon, JX}(M) \\ (\varphi, s) &\mapsto \log \frac{(\omega + i\partial\bar{\partial}\varphi)^n}{\omega^n} + \frac{X}{2}(\varphi) - sF \end{aligned} \tag{48}$$

It is worth mentioning that the function $\mathcal{M}(\varphi, s)$ is indeed JX-invariant since F is assumed to be invariant under JX, and also $\mathcal{L}_{JX}\omega = 0$ by A.3. Before applying the implicit function theorem, we need to compute the linearization of $\mathcal{M}$, i.e. the derivative at the point (φ, s) in direction of $(u, 0)$:

$$D\mathcal{M}_{(\varphi,s)}(u, 0) = \frac{1}{2}\Delta_{g_\varphi}(u) + \frac{X}{2}(u). \tag{49}$$

Here Δ_{g_φ} denotes the Riemannian Laplace operator of the metric g_φ associated to the Kähler form $\omega + i\partial\bar{\partial}\varphi$.

As in [CD20b], the first step is to show that the linearized operator is an isomorphism $C^{k+2,\alpha}_{\varepsilon,JX}(M) \to C^{k,\alpha}_{\varepsilon,JX}(M)$, which is covered in the next

Proposition 5.3. *Let (M, g) be an ACyl manifold of real dimension $2n$ satisfying the assumptions A.1, A.2 and A.3. Given $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and $0 < \varepsilon < 2$, the operator*

$$\Delta_g + X : C^{k+2,\alpha}_{\varepsilon,JX}(M) \to C^{k,\alpha}_{\varepsilon,JX}(M)$$

is an isomorphism.

Here, our arguments differ from those in [CD20b][Theorem 6.6], because the metrics we consider have a different asymptotic behavior. Instead, we reduce the proof to Theorem 2.11.

Proof. First, we observe by Assumption A.3, that $\Delta_g + X$ is an ACyl drift operator in the sense of Definition 2.10. Thus, according to Theorem 2.11, the map

$$\Delta_g + X : C^{k+2,\alpha}_{\varepsilon}(M) \to C^{k,\alpha}_{\varepsilon}(M)$$

is an isomorphism for $k \in \mathbb{N}_0, \alpha \in (0,1)$ and $0 < \varepsilon < 2$. Consequently, it only remains to show that $u \in C^{k+2,\alpha}_{\varepsilon}(M)$ is JX-invariant, provided $(\Delta_g + X)(u)$ is invariant under JX. To see this, we use that X is real holomorphic and obtain

$$[X, JX] = J[X, X] = 0,$$

so that $JX(X(u)) = X(JX(u))$. Moreover, we have $JX(\Delta_g u) = \Delta_g(JX(u))$ which follows directly from the relation

$$\frac{1}{2}\Delta_g u\, \omega^n = n\, i\partial\bar{\partial}u \wedge \omega^{n-1}$$

by applying $\mathcal{L}_{JX}\omega = 0$. Hence, we conclude that if $(\Delta_g + X)(u)$ is JX-invariant for some $u \in C^{k+2,\alpha}_{\varepsilon}(M)$, then

$$0 = JX((\Delta_g + X)(u)) = (\Delta_g + X)(JX(u)).$$

As $|X|_g$ is bounded, $JX(u)$ tends to 0 as $t \to \infty$, and the maximum principle yields $JX(u) = 0$, as desired. $\square$

Remark 5.4 (on the decay rate ε)**.** The reader may notice that Proposition 5.3 holds for all $0 < \varepsilon < 2$, whereas Theorem 5.2 only includes the case $F \in C^{\infty}_{\varepsilon,JX}$ with $1 < \varepsilon < 2$. This is because Conlon and Deruelle's approach to the uniform C^0-estimate requires the convergence of certain weighted functionals, compare Definition 5.8 below.

However, it seems plausible to use Theorem 5.2 together with ideas contained in [CD20b][Section 9] to cover the case $0 < \varepsilon \leq 1$ as well, but we do not pursue this further in this article.

We also obtain the following regularity statement for the Monge-Ampère operator.

Proposition 5.5 (Regularity). *Let* $(M,g), F \in C^{\infty}_{\varepsilon,JX}(M)$ *and* $1 < \varepsilon < 2$ *be as in Theorem 5.2. Suppose that* $\varphi \in C^{3,\alpha}_{\varepsilon',JX}(M)$ *for some* $0 < \varepsilon' \leq \varepsilon$ *satisfies*

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^{F - \frac{X}{2}(\varphi)}\omega^n.$$

Then $\varphi \in C^{\infty}_{\varepsilon,JX}(M)$.

Note that this statement only gives *qualitative* information about the function φ, i.e. it does *not* provide uniform estimates for the $C^{\infty}_{\varepsilon}(M)$-norm of φ. The crucial part of the continuity method, however, is precisely to obtain uniform a priori bounds on $||\varphi||_{C^{k,\alpha}_{\varepsilon}}$. This is achieved in the next

Theorem 5.6 (A priori estimates). *Let* (M,g), $F \in C^{\infty}_{\varepsilon,JX}(M)$ *and* $1 < \varepsilon < 2$ *be as in Theorem 5.2. Suppose that* $(\varphi_s)_{0\leq s\leq 1}$ *is a family in* $C^{\infty}_{\varepsilon,JX}(M)$ *such that* $\omega + i\partial\bar{\partial}\varphi_s$ *is Kähler for each* $s \in [0,1]$ *and satisfies*

$$\big(\omega + i\partial\bar{\partial}\varphi_s\big)^n = e^{s\cdot F - \frac{X}{2}(\varphi_s)}\omega^n. \tag{50}$$

Then, for given $k \in \mathbb{N}_0, \alpha \in (0,1)$, *there exists a constant* $C > 0$ *such that*

$$\sup_{s\in[0,1]} ||\varphi_s||_{C^{k,\alpha}_{\varepsilon}} \leq C,$$

where C *only depends on* k, α, F *and the geometry of* (M,g).

The strategy for proving Proposition 5.5 and Theorem 5.6 is to follow, up to some minor adjustments, the arguments provided by Conlon and Deruelle ([CD20b][Section 7]). In particular, we use their idea to achieve the uniform C^0-bound, but we present a variation of their arguments which allows us to immediately assume $F \in C^{\infty}_{\varepsilon,JX}(M)$ with $1 < \varepsilon < 2$, instead of first considering functions F with compact support as in [CD20b][Theorem 7.1].

We postpone the proofs of both Proposition 5.5 and Theorem 5.6 to subsequent sections and for now assume these results to conclude Theorem 5.2.

Proof of Theorem 5.2. First, we point out that we only need to show the existence statement since the uniqueness part is a direct consequence of the maximum principle, see [BM17][Proposition 1.2].

For the proof of existence, assume we are given $F \in C^{\infty}_{\varepsilon,JX}(M)$, and consider the set

$$S := \{s \in [0,1] \,|\, \text{there exists a } \varphi_s \in C^{\infty}_{\varepsilon,JX}(M) \text{ satisfying (50)}\}.$$

Clearly, $0 \in S$ and so it is sufficient to show that S is both open and closed.

The openness is a consequence of Proposition 5.3. To see this, let $\mathcal{U}$ be the set of all $\psi \in C^{3,\alpha}_{\varepsilon,JX}(M)$ such that $\omega + i\partial\bar{\partial}\psi > 0$ and consider the Monge-Ampère operator $\mathcal{M}$ defined by

$$\begin{aligned} \mathcal{M} : \mathcal{U} \times [0,1] &\to C^{1,\alpha}_{\varepsilon, JX}(M) \\ (\psi, s) &\mapsto \log \frac{(\omega + i\partial\bar{\partial}\psi)^n}{\omega^n} + \frac{X}{2}(\psi) - sF \end{aligned}$$

Suppose we are given $s_0 \in S$, i.e. $\varphi_{s_0} \in C^{\infty}_{\varepsilon, JX}(M)$ solving (50). Since φ_{s_0} is JX-invariant and $\varphi_{s_0} \in C^{\infty}_{\varepsilon}(M)$, the Riemannian metric $g_{\varphi_{s_0}}$ corresponding to $\omega + i\partial\bar{\partial}\varphi_{s_0}$ is ACyl, with the same ACyl map as g, and satisfies Assumptions A.1, A.2 and A.3. Hence, the linearization of $\mathcal{M}$ at the point (φ_s, s), which is given by (49), is injective if restricted to the subspace $C^{3,\alpha}_{\varepsilon, JX}(M)$ and also surjective according to Proposition 5.3. Thus, the implicit function theorem implies the existence of a $\delta_0 > 0$ such that for all $s \in (s_0 - \delta_0, s_0 + \delta_0)$ there exists a $\varphi_s \in C^{3,\alpha}_{\varepsilon}(M)$ solving (50). But then $\varphi_s \in C^{\infty}_{\varepsilon, JX}(M)$ by Proposition 5.5, and consequently $(s_0 - \delta_0, s_0 + \delta_0) \cap [0,1] \subset S$.

That S is closed follows from Theorem 5.6. Indeed, consider a sequence $(s_k)_{k\in\mathbb{N}}$ in S which converges to some $s_\infty \in [0,1]$, and denote the corresponding sequence in $C^{\infty}_{\varepsilon, JX}(M)$ of solutions to (50) by (φ_{s_k}). According to Theorem 5.6, this sequence (φ_{s_k}) is uniformly bounded in $C^{3,\alpha}_{\varepsilon}(M)$. Choosing $\varepsilon' \in (0, \varepsilon)$ and $\beta \in (0, \alpha)$, the inclusion $C^{3,\alpha}_{\varepsilon}(M) \subset C^{3,\beta}_{\varepsilon'}(M)$ is compact (by [Mar02][Theorem 4.3] for instance), so that we can extract a subsequence of (φ_{s_k}) converging in $C^{3,\beta}_{\varepsilon'}(M)$ to some limit $\varphi_{s_\infty} \in C^{3,\beta}_{\varepsilon'}(M)$. Note that we must have $JX(\varphi_{s_\infty}) = 0$ and that φ_{s_∞} satisfies

$$(\omega + i\partial\bar{\partial}\varphi_{s_\infty})^n = e^{s_\infty F - \frac{X}{2}(\varphi_{s_\infty})}\omega^n,$$

as we can take the point-wise limit $k \to \infty$ in (50). From this, we immediately see that $\omega + i\partial\bar{\partial}\varphi_{s_\infty}$ is a Kähler form, and applying Proposition 5.5 then implies $\varphi_{s_\infty} \in C^{\infty}_{\varepsilon, JX}(M)$, i.e. $s_\infty \in S$. This concludes the proof. $\square$

The rest of this section is devoted to proving Proposition 5.5 and Theorem 5.6. We begin in Section 5.1 by deriving the C^0-estimate which is the key part of the proof. Then we move on to higher-order estimates in Section 5.2 to finish the proof of Theorem 5.6. Afterwards, we conclude by verifying Proposition 5.5.

5.1. **The C^0-estimate.** Throughout this section, let (M, g) satisfy Assumptions A.1, A.2 and A.3. The goal is to obtain uniform estimates for solutions $(\varphi_s)_{0\leq s\leq 1}$ to (50), among which the C^0-bound is the most difficult one to achieve.

The proof of the C^0-estimate is split into three parts: First, we obtain a weighted upper bound on φ_s, then an L^2-bound with a certain weight and finally, we can conclude a lower bound on $\inf_M \varphi_s$. The last two steps closely follow the ideas developed in [CD20b][Section 7.1]. Before beginning with the preparations, let us fix some notation.

Notation. We denote the metric associated with $\omega + i\partial\bar{\partial}\varphi_s$ by g_{φ_s}, and $\nabla^{g_{\varphi_s}}$, $\Delta_{g_{\varphi_s}}$, etc. denote the Levi-Civita connection, the Laplace operator, etc. of g_{φ_s}. We point out that $\Delta_{g_{\varphi_s}}$ is the *Riemannian* Laplace operator, i.e. it satisfies

$$\frac{1}{2}\Delta_{g_{\varphi_s}} u\,\omega_{\varphi_s}^n = n\, i\partial\bar{\partial}u \wedge \omega_{\varphi_s}^{n-1} \tag{51}$$

for each C^2-function u.

5.1.1. *An upper bound on φ_s.* We begin by estimating φ_s from above:

Proposition 5.7 (Weighted upper bound on φ_s)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s\in[0,1]}\sup_M e^{\varepsilon t}\varphi_s \leq C,$$

where C only depends on $F \in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M, g).

We present a proof based on the use of a barrier function, so our argument differs from the one given in [CD20b][Proposition 7.9].

Proof. We begin by observing that φ_s satisfies

$$\frac{1}{2}\Delta_g(\varphi_s) + \frac{X}{2}(\varphi_s) \geq sF. \tag{52}$$

Indeed, consider any $p \in M$ and holomorphic coordinates $(z_1, \dots, z_n)$ such that

$$g_{i\bar{j}} = \delta_{i\bar{j}} \quad \text{and} \quad \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} = \lambda_i \delta_{i\bar{j}} \quad \text{at} \quad p$$

for some $\lambda_i \in \mathbb{R}$ with $1 + \lambda_i > 0$, where $g_{i\bar{j}}$ are the local components of g and $\delta_{i\bar{j}}$ denotes the Kronecker delta. Starting from (50), we compute at p that

$$\begin{aligned} sF - \frac{X}{2}(\varphi_s) &= \log\frac{\left(\omega + i\partial\bar{\partial}\varphi_s\right)^n}{\omega^n} \\ &= \log(1+\lambda_1)\cdots(1+\lambda_n) \\ &= \sum_{j=1}^n \log(1+\lambda_j) \\ &\leq \sum_{j=1}^n \lambda_j \\ &= \operatorname{tr}_\omega(i\partial\bar{\partial}\varphi_s) = \frac{1}{2}\Delta_g(\varphi_s), \end{aligned}$$

where $\mathrm{tr}_\omega(i\partial\bar{\partial}\varphi_s)$ denotes the trace of $i\partial\bar{\partial}\varphi_s$ with respect to ω and we used $\log(1+\tau) \leq \tau$ if $\tau > -1$ to obtain the inequality in the fourth line. This finishes the proof of (52).

Moreover, since $F \in C^\infty_\varepsilon(M)$ with $0 < \varepsilon < 2$ and because of Assumption A.3, Theorem 2.11 implies the existence of a function $u_F \in C^\infty_\varepsilon(M)$ such that

$$\frac{1}{2}\Delta_g(u_F) + \frac{X}{2}(u_F) = F,$$

which, in combination with (52), leads to

$$(\Delta_g + X)(\varphi_s - su_F) \geq 2s(F - F) = 0.$$

Choosing a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \to \infty$ and applying Hopf's maximum principle to a sequence of domains of the form $\{t \leq t_k\} \subset M$ then yields

$$\sup_M(\varphi_s - su_F) \leq \lim_{t\to\infty}(\varphi_s - su_F) = 0,$$

i.e. $\varphi_s \leq su_F$ holds on all of M. In particular, we observe that

$$e^{\varepsilon t}\varphi_s \leq su_F e^{\varepsilon t} \leq ||u_F||_{C^0_\varepsilon} =: C,$$

which proves the claim. $\square$

For obtaining a lower bound on φ_s, we need to work considerably harder. The important idea in [CD20b] is to first obtain a weighted L^2-bound.

5.1.2. *A weighted L^2-bound.* As in [CD20b][Subsection 7.1.1.], we consider two functionals which were used by Tian and Zhu [TZ00] to study shrinking Kähler-Ricci solitons on compact Fano manifolds.

Definition 5.8. Consider $1 < \varepsilon < 2$ and let $(\psi_\tau)_{0\leq\tau\leq 1}$ be a C^1-path in $C^\infty_{\varepsilon,JX}(M)$ from $\psi_0 = 0$ to $\psi_1 = \psi$ and assume for each $\tau \in [0,1]$ that $\omega_{\psi_\tau} := \omega + i\partial\bar{\partial}\psi_\tau > 0$. Define:

$$I_{\omega,X}(\psi) := \int_M \psi\left(e^f\omega^n - e^{f+\frac{X}{2}(\psi)}\omega^n_\psi\right),$$
$$J_{\omega,X}(\psi) := \int_0^1\int_M \dot{\psi}_\tau\left(e^f\omega^n - e^{f+\frac{X}{2}(\psi_\tau)}\omega^n_{\psi_\tau}\right)\wedge d\tau,$$

where $\dot{\psi}_\tau = \frac{\partial}{\partial\tau}\psi_\tau$.

Since M is non-compact, we need to show that $I_{\omega,X}$ and $J_{\omega,X}$ are well-defined, i.e. that the resulting integrals are finite. Given $\psi \in C^\infty_{\varepsilon,JX}(M)$ with $1 < \varepsilon < 2$, we deduce from (A.3) that $\psi\, e^f = O(e^{(2-\varepsilon)t})$, so it suffices to show

$$|\omega^n - e^{\frac{X}{2}(\psi)}\omega^n_\psi|_g = O(e^{-\varepsilon t}) \tag{53}$$

since $\varepsilon > 1$. To see that this is true, we expand ω_ψ^n and obtain

$$\omega^n - e^{\frac{X}{2}(\psi)}\omega_\psi^n = \left(1 - e^{\frac{X}{2}(\psi)}\right)\omega^n - e^{\frac{X}{2}(\psi)}\sum_{k=1}^n \binom{n}{k}\left(i\partial\bar\partial\psi\right)^k \wedge \omega^{n-k}$$

from which (53) follows because $\frac{X}{2}(\psi) = O(e^{-\varepsilon t})$ and $|i\partial\bar\partial\psi|_g = O(e^{-\varepsilon t})$ by definition of $C^\infty_\varepsilon(M)$. Thus $I_{\omega,X}(\psi)$ is finite, and the same argument also proves that $J_{\omega,X}$ is well-defined. The crucial starting point is the next

Theorem 5.9. *Let $(\psi_\tau)_{0\le\tau\le1}$ be a C^1-path as in Definition 5.8. Then the first variation of the difference $I_{\omega,X} - J_{\omega,X}$ is given by*

$$\frac{d}{d\tau}(I_{\omega,X} - J_{\omega,X})(\psi_\tau) = -\int_M \psi_\tau\left(\frac{1}{2}\Delta_{g_{\psi_\tau}}(\dot\psi_\tau) + \frac{X}{2}(\dot\psi_\tau)\right)e^{f+\frac{X}{2}(\psi_\tau)}\omega^n_{\psi_\tau},$$

where g_{ψ_τ} is the metric with Kähler form $\omega_{\psi_\tau} = \omega + i\partial\bar\partial\psi_\tau$. Moreover, $J_{\omega,X}$ does not depend on the choice of path $(\psi_\tau)_{0\le\tau\le1}$, but only on the end points $\psi_0 = 0$ and $\psi_1 = \psi$.

Proof. This is [CD20b][Theorem 7.5], whose proof in turn relies on [TZ00]. The reader may observe that this proof is a completely formal calculation, which applies word-by-word to our case if Stokes theorem holds. This, however, is only used once on [CD20b][p. 50]. Given our asymptotics, it is clear from Lemma 2.8 that we as well can integrate by parts because the integrands decay exponentially in the parameter t. □

Before we can continue with the weighted L^2 bounds, we need another lemma as preparation.

Lemma 5.10 (A first bound on $\inf_M X(\varphi_s)$)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0\le s\le1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\inf_{s\in[0,1]}\inf_M\left(f + \frac{X}{2}(\varphi_s)\right) \ge 1, \tag{54}$$

where C only depends on the geometry of (M, g).

Proof. Since both f and φ_s are JX-invariant, the argument [CD20b][(7.6)] applies and we obtain that

$$X = \nabla^{g_{\varphi_s}}\left(f + \frac{X}{2}(\varphi_s)\right). \tag{55}$$

Also observe that $\frac{X}{2}(\varphi_s) \to 0$ as $t \to \infty$ because X is bounded with respect to the norm g_{φ_s}. Thus, we conclude from (A.3) that $f + \frac{X}{2}(\varphi_s)$ converges to the function $2t+c$ with $c > 0$ and consequently, $f + \frac{X}{2}(\varphi_s)$

attains a global minimum at some point $p \in M$. By (55), we see that X must vanish at p, so we conclude that

$$\inf_M \left(f + \frac{X}{2}(\varphi_s) \right) = \min_{\{X=0\}} \left(f + \frac{X}{2}(\varphi_s) \right) = \min_{\{X=0\}} f$$

holds for all $s \in [0,1]$. In particular, (54) follows since we normalised f such that $f \geq 1$ on M. $\square$

Proposition 5.11 (A priori bound on weighted energy)**.** *Let* $1 < \varepsilon < 2$ *and suppose* $(\varphi_s)_{0\leq s\leq 1}$ *is a family in* $C^\infty_{\varepsilon,JX}(M)$ *solving (50). Then there exists a constant* $C > 0$ *such that*

$$\sup_{0\leq s\leq 1} \int_M |\varphi_s|^2 \frac{e^f}{f^2}\, \mathrm{dV}_g \leq C, \tag{56}$$

where C *only depends on* $F \in C^\infty_{\varepsilon,JX}(M)$ *and on the geometry of* (M,g)*.*

Proof. We follow [CD20b][Proposition 7.7]. The idea is to consider two different paths in $C^\infty_{\varepsilon,JX}(M)$ with $1 < \varepsilon < 2$ and to use Theorem 5.9 for obtaining the required bound.

We begin by considering a linear path from 0 to φ_s. Given $s \in [0,1]$, define this path $(\psi_\tau)_{0\leq\tau\leq 1}$ by $\psi_\tau := \tau\varphi_s$. Since $\omega + i\partial\bar{\partial}\psi_\tau > 0$, Theorem 5.9 implies that

(57)

$$(I_{\omega,X} - J_{\omega,X})(\varphi_s) = -\int_0^1 \int_M \frac{\tau\varphi_s}{2} \left(\Delta_{g_{\tau\varphi_s}} + X\right)(\varphi_s) e^{f+\tau\frac{X}{2}(\varphi_s)} \omega^n_{\tau\varphi_s} \wedge d\tau$$

Recalling that $X = \nabla^{g_{\tau\varphi_s}}(f + \frac{X}{2}(\varphi_s))$, we integrate by parts and obtain

$$\begin{aligned}
(I_{\omega,X} - J_{\omega,X})(\varphi_s) &= n\int_0^1 \int_M \tau e^{f+\tau\frac{X}{2}(\varphi_s)} i\partial\varphi_s \wedge \bar{\partial}\varphi_s \wedge \omega^{n-1}_{\tau\varphi_s} \wedge d\tau \qquad (58)\\
&= n\int_0^1 \int_M \tau e^{f+\tau\frac{X}{2}(\varphi_s)} i\partial\varphi_s \wedge \bar{\partial}\varphi_s \wedge ((1-\tau)\omega + \tau\omega_{\varphi_s})^{n-1} \wedge d\tau\\
&\geq n\int_0^1 \int_M \tau(1-\tau)^{n-1} e^{f+\tau\frac{X}{2}(\varphi_s)} i\partial\varphi_s \wedge \bar{\partial}\varphi_s \wedge \omega^{n-1} \wedge d\tau\\
&\geq n\int_0^1 \int_M \tau(1-\tau)^{n-1} e^{(1-\tau)f} i\partial\varphi_s \wedge \bar{\partial}\varphi_s \wedge \omega^{n-1} \wedge d\tau\\
&= n\int_M \left(\int_0^1 \tau(1-\tau)^{n-1} e^{(1-\tau)f} d\tau \right) \wedge i\partial\varphi_s \wedge \bar{\partial}\varphi_s \wedge \omega^{n-1},
\end{aligned}$$

where the penultimate line holds since $\frac{X}{2}(\varphi_s) \geq -f$ by Lemma 5.10. Thanks to [CD20b][Claim 7.8], there exists a constant $C > 0$ such that

$$n\int_0^1 \tau(1-\tau)^{n-1} e^{(1-\tau)f} d\tau \geq C\frac{e^f}{f^2}, \tag{59}$$

which, in combination with (58), then leads to

$$(I_{\omega,X} - J_{\omega,X})(\varphi_s) \geq C \int_M \frac{e^f}{f^2} i\partial\varphi_s \wedge \bar\partial\varphi_s \wedge \omega^{n-1}. \tag{60}$$

To estimate $(I_{\omega,X} - J_{\omega,X})(\varphi_s)$ from above, we recall from Theorem 5.9 that $J_{\omega,X}$ is independent of the choice of path from 0 to φ_s. Thus, we can compute $(I_{\omega,X} - J_{\omega,X})(\varphi_s)$ by defining a new path $(\psi_\tau)_{0\leq\tau\leq 1}$ as $\psi_\tau := \varphi_{\tau s}$. We point out that $\psi_0 = \varphi_0 \equiv 0$ follows from the maximum principle applied to the Monge-Ampère equation (50). For calculating $\dot\psi_\tau$, differentiate (50) with respect to s and obtain

$$n\, i\partial\bar\partial\dot\varphi_s \wedge \omega_{\varphi_s}^{n-1} = \left(F - \frac{X}{2}(\dot\varphi_s)\right)\omega_{\varphi_s}^n.$$

Combining with (51) and using $\dot\psi_\tau = s\dot\varphi_{\tau s}$, we arrive at

$$\frac{1}{2}\Delta_{\psi_\tau}\dot\psi_\tau + \frac{X}{2}(\dot\psi_\tau) = sF,$$

to which we further apply Theorem (5.9) and continue:

$$\begin{aligned}
(I_{\omega,X} - J_{\omega,X})(\varphi_s) &= -\int_0^1 \int_M \psi_\tau \cdot sFe^{f+\frac{X}{2}(\psi_\tau)}\omega_{\psi_\tau}^n \wedge d\tau \\
&= -\int_0^1 \int_M \psi_\tau \cdot sFe^{f+\tau sF}\omega^n \wedge d\tau \\
&\leq \int_0^1 \int_M |\psi_\tau||F|e^{f+|F|}\omega^n \wedge d\tau \\
&= \int_0^1 \int_M f|F|e^{\frac{f}{2}+|F|} \cdot |\psi_\tau|\frac{e^{\frac{f}{2}}}{f}\omega^n \wedge d\tau \\
&\leq C\int_0^1 \left(\int_M |\psi_\tau|^2 \frac{e^f}{f^2}\omega^n\right)^{\frac{1}{2}} d\tau.
\end{aligned}$$

Here, we applied (50) in the second line, Cauchy-Schwarz in the last one and the uniform constant $C > 0$ is given by $C^2 = \int_M f^2|F|^2 e^{f+2|F|}\omega^n$, which is finite since $f^2e^f = O(t^2e^{2t})$ and $F^2 = O(e^{-2\varepsilon t})$ with $\varepsilon > 1$. From the previous estimate together with (60), we thus conclude

$$\begin{aligned}
\int_M |\nabla^g\varphi_s|_g^2 \frac{e^f}{f^2}\,\mathrm{dV}_g &\leq C\int_0^1 \left(\int_M |\varphi_{\tau s}|^2 \frac{e^f}{f^2}\,\mathrm{dV}_g\right)^{\frac{1}{2}} d\tau \\
&= \frac{C}{s}\int_0^s \left(\int_M |\varphi_\tau|^2 \frac{e^f}{f^2}\,\mathrm{dV}_g\right)^{\frac{1}{2}} d\tau.
\end{aligned}$$

Together with Proposition 2.12, we finally arrive at

$$\lambda \int_M |\varphi_s|^2 \frac{e^f}{f^2}\,\mathrm{dV}_g \leq \frac{C}{s}\int_0^s \left(\int_M |\varphi_\tau|^2 \frac{e^f}{f^2}\,\mathrm{dV}_g\right)^{\frac{1}{2}} d\tau. \tag{61}$$

As observed by Conlon and Deruelle [CD20b][Proposition 7.7], this is a Grönwall-type differential inequality for the function $U : (0,1] \to \mathbb{R}_+$ defined by

$$U(s) := \int_0^s \left(\int_M |\varphi_\tau|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \right)^{\frac{1}{2}} d\tau.$$

Indeed, it is immediate that (61) becomes

$$\frac{\dot{U}(s)}{\sqrt{U(s)}} \leq \frac{C}{\sqrt{s}},$$

so that we integrate to obtain $\sqrt{U(s)} \leq C\sqrt{s}$ with $s \in (0,1]$. Hence,

$$\left(\int_M |\varphi_s|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \right)^{\frac{1}{2}} d\tau = \dot{U}(s) \leq C,$$

where $C = C(M, g, F)$ is independent of $s \in [0,1]$, as claimed. $\square$

5.1.3. *A lower bound on φ_s.* For proving a uniform bound on $\sup_M |\varphi_s|$, it remains to bound $\inf_M \varphi_s$ from below. This is the main result of this subsection:

Proposition 5.12 (Lower bound on $\inf_M \varphi_s$)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\inf_{s \in [0,1]} \inf_M \varphi_s \geq -C,$$

where C only depends $F \in C^\infty_{\varepsilon, JX}(M)$ and on the geometry of (M, g).

If we assumed that F was compactly supported, the same argument as in [CD20b][Proposition 7.10] would go through verbatim and provide the required bound on $\inf_M \varphi_s$, since we already obtained uniform bounds on $\sup_M \varphi_s$ (Proposition 5.7) and on the weighted L^2-norm (Proposition 5.11).

In our situation, however, we do *not* assume that F has compact support, but merely $F \in C^\infty_{\varepsilon, JX}(M)$ with $1 < \varepsilon < 2$. Thus, we proceed as follows.

First, we construct a compact domain $K \subset M$ so that we obtain a suitable barrier function on its complement $M \setminus K$, which will be useful for arguments relying on the maximum principle. In a second step, the argument in [CD20b][Proposition 7.10] gives a lower bound on $\inf_K \varphi_s$. And finally, we will see that the maximum principle yields a lower bound on $\inf_{M \setminus K} \varphi_s$.

In other words, our strategy is to prove the following lemma, as well as the next two propositions:

Lemma 5.13 (Construction of K)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant*

$0 < \varepsilon_0 < 1$ *and a compact domain* $K \subset M$ *such that for all* $s \in [0,1]$, *we have*

(62)
$$\left(\Delta_{g_{\varphi_s}} + X\right)\left(e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right) \leq -\frac{\varepsilon_0}{2}e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} < 0 \quad on \quad M \setminus K,$$

where both ε_0 *and* K *only depend on* $F \in C^{\infty}_{\varepsilon,JX}(M)$ *and the geometry of* (M, g).

Proposition 5.14 (Lower bound on a compact set)**.** *Let* $1 < \varepsilon < 2$ *and suppose* $(\varphi_s)_{0\leq s\leq 1}$ *is a family in* $C^{\infty}_{\varepsilon,JX}(M)$ *solving (50). For the compact domain* $K \subset M$ *given by Lemma 5.13, there exists a constant* $C > 0$ *such that*

$$\inf_{s\in[0,1]} \inf_K \varphi_s \geq -C,$$

where C *only depends on* K, $F \in C^{\infty}_{\varepsilon,JX}(M)$ *and the geometry of* (M, g).

Proposition 5.15 (Lower bound outside of a compact set)**.** *Let* $1 < \varepsilon < 2$ *and suppose* $(\varphi_s)_{0\leq s\leq 1}$ *is a family in* $C^{\infty}_{\varepsilon,JX}(M)$ *solving (50). For the compact domain* $K \subset M$ *constructed in Lemma 5.13, there exists a constant* $C > 0$ *such that*

$$\inf_{s\in[0,1]} \inf_{M\setminus K} \varphi_s \geq -C,$$

where C *only depends on* K, $F \in C^{\infty}_{\varepsilon,JX}(M)$ *and the geometry of* (M, g).

Clearly, Proposition 5.15, together with Proposition 5.14, yield a uniform lower bound on $\inf_M \varphi_s$, as claimed in Proposition 5.12.

Since Lemma 5.13 requires some preparation, let us for the moment assume that we are given the compact set $K \subset M$ from Lemma 5.13 and see how this implies the lower bound on $\inf_K \varphi_s$, i.e. Proposition 5.14.

Proof of Proposition 5.14. We follow the proof of [CD20b][Proposition 7.10], which in turn relies on Błocki's local argument [Bło05].

Let $K \subset M$ be the compact domain constructed in Lemma 5.13. For each $p \in K$, let V be a chart around p so that ω can be written as $\omega = i\partial\bar{\partial}G$. According to the proof of [Bło05][Theorem 4], there are constants $a, r > 0$ only depending on the local geometry of (M, g) around p such that $G < 0$ on $B_g(p, 2r)$, G is minimal at p and $G \geq G(p) + a$ on $B_g(p, 2r) \setminus B_g(p, r)$, where $B_g(p, 2r) \subset V$ is the geodesic ball of radius $2r$ around p. Since K is compact, we can cover K by a *finite* number of such balls $B_g(p, 2r)$.

For a given $s \in [0, 1]$, we consider φ_s solving (50) and point out that there exists a $p_s \in K$ such that $\varphi_s(p_s) = \inf_K \varphi_s$. Then $p_s \in B_g(p, 2r)$

for one of the balls constructed above. Define a plurisubharmonic function $u : B_g(p, 2r) \to \mathbb{R}_{\leq 0}$ by

$$u = \begin{cases} \varphi_s + G & \text{if } \sup_M \varphi_s \leq 0, \\ \varphi_s - \sup_M \varphi_s + G & \text{otherwise,} \end{cases}$$

so that [Bło05][Proposition 3] implies the following estimate

$$\sup_{B_g(p,2r)} |u| \leq a + \left(c_n \cdot 2r \cdot a^{-1}\right)^{2n} \int_{B_g(p,2r)} |u| \, \mathrm{dV}_g \cdot \left(\sup_{B_g(p,2r)} \frac{\omega_{\varphi_s}^n}{\omega^n} \right)^2 \tag{63}$$

where $\omega_{\varphi_s} = \omega + i\partial\bar{\partial}\varphi_s$ and $c_n > 0$ is a constant only depending on the dimension n of M.

We now explain how to estimate the terms appearing on the right hand side of (63). We begin by using (50) together with Lemma 5.10 to obtain

$$\sup_{B_g(p,2r)} \frac{\omega_{\varphi_s}^n}{\omega^n} = \sup_{B_g(p,2r)} e^{s \cdot F - \frac{X}{2}(\varphi_s)} \leq \sup_{N_{2r}(K)} e^{|F| + f} =: C_1. \tag{64}$$

Here $N_{2r}(K)$ denotes the tabular neighborhood of radius $2r$ around K. Note that since K is compact, the constant C_1 is indeed finite.

Next, we focus on the integral appearing in (63) and first consider the case $\sup_M \varphi_s \leq 0$. We continue:

$$\begin{aligned} &\int_{B_g(p,2r)} |u| \, \mathrm{dV}_g \\ &\leq \int_{B_g(p,2r)} |\varphi_s| \, \mathrm{dV}_g + \sup_M \varphi_s - G(p) \\ &\leq \max\{1, \mathrm{Vol}(B_g(p, 2r))\} \left(\left(\int_{B_g(p,2r)} |\varphi_s|^2 \, \mathrm{dV}_g \right)^{\frac{1}{2}} + C - G(p) \right) \\ &\leq \max\{1, \mathrm{Vol}(N_{2r}(K))\} \left(\sup_M \frac{e^{-f}}{f^2} \cdot \left(\int_M |\varphi_s|^2 \frac{e^f}{f^2} \, \mathrm{dV}_g \right)^{\frac{1}{2}} + C - G(p) \right) \\ &\leq \max\{1, \mathrm{Vol}(N_{2r}(K))\} \left(\sup_M \frac{e^{-f}}{f^2} \cdot C + C - G(p) \right) =: C_2, \end{aligned}$$

where we used Cauchy-Schwarz and Proposition 5.7 in the second line and Proposition 5.11 in the last one. Combining this estimate with (63) and (64) then leads to

$$\begin{aligned} - \inf_K \varphi_s = -\varphi_s(p_s) &= -u(p_s) - \sup_M \varphi_s + G(p_s) \\ &\leq \sup_{B_g(p,2r)} |u| \\ &\leq a + \left(c_n \cdot 2r \cdot a^{-1}\right)^{2n} \cdot C_2 \cdot C_1^2. \end{aligned} \tag{65}$$

Note that a priori, the constants in the last line of (65) depend on the ball $B_g(p, 2r)$ containing the point in which φ_s attains its minimum inside K. However, since K is covered by only *finitely* many of such balls $B_g(p, 2r)$, (65) does indeed prove the required uniform lower bound on $\inf_K \varphi_s$. Observing that the above estimates hold in the case $\sup_M \varphi_s > 0$ as well then finishes the proof. $\square$

Thus, it only remains to show Lemma 5.13 and Proposition 5.15. We begin with the following crucial observation.

Lemma 5.16 (Uniform bound on $X^2(\varphi_s)$)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s\in[0,1]} \sup_M |X(X(\varphi_s))| \leq C,$$

where C only depends on $F \in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M, g).

Proof. The idea is to obtain a differential equality to which the maximum principle applies, so that the desired estimate follows.

First, we differentiate (50) in the direction of $\frac{X}{2}$, i.e. apply $\mathcal{L}_{\frac{X}{2}}$, which leads to

$$ni\partial\bar\partial\left(f + \frac{X}{2}(\varphi_s)\right) \wedge \omega_{\varphi_s}^{n-1} = \left(\frac{X}{2}(sF) - \frac{X^2}{4}(\varphi_s) + \frac{1}{2}\Delta_g f\right)\omega_{\varphi_s}^n \tag{66}$$

where we abbreviated $X(X(\cdot)) = X^2(\cdot)$. Here, we also used two formulas, $\mathcal{L}_{\frac{X}{2}}\omega = i\partial\bar\partial f$ and $\mathcal{L}_{\frac{X}{2}}\omega_{\varphi_s} = i\partial\bar\partial f + \frac{X}{2}(\varphi_s)$, whose computations can be found in the proof Lemma 3.3. Next, recall that for any real $(1, 1)$-form α, we have

$$n(n-1)\alpha^2 \wedge \omega_{\varphi_s}^{n-2} = \left((\mathrm{tr}_{\omega_{\varphi_s}}(\alpha))^2 - |\alpha|^2_{g_{\varphi_s}}\right)\omega_{\varphi_s}^n, \tag{67}$$

where $\mathrm{tr}_{\omega_{\varphi_s}}(\alpha)$ is defined by

$$n\alpha \wedge \omega_{\varphi_s}^{n-1} = \mathrm{tr}_{\omega_{\varphi_s}}(\alpha)\,\omega_{\varphi_s}^n. \tag{68}$$

Setting $\alpha := \mathcal{L}_{\frac{X}{2}}\omega_{\varphi_s} = i\partial\bar\partial f + \frac{X}{2}(\varphi_s)$ and applying $\mathcal{L}_{\frac{X}{2}}$ to the left-hand side of (68) then yields

$$\begin{aligned}\mathcal{L}_{\frac{X}{2}}\left(n\alpha \wedge \omega_{\varphi_s}^{n-1}\right) =& n\left(\mathcal{L}_{\frac{X}{2}}\alpha\right) \wedge \omega_{\varphi_s}^{n-1} + n(n-1)\alpha^2 \wedge \omega_{\varphi_s}^{n-2}\\ =& \frac{n}{2}i\partial\bar\partial\left(X(f) + \frac{X^2}{2}(\varphi_s)\right) \wedge \omega_{\varphi_s}^{n-1}\\ &+ \left((\mathrm{tr}_{\omega_{\varphi_s}}(\alpha))^2 - |\alpha|^2_{g_{\varphi_s}}\right)\omega_{\varphi_s}^n,\end{aligned} \tag{69}$$

where we used $\mathcal{L}_X\alpha = i\partial\bar\partial X(f) + \frac{X^2}{2}(\varphi_s)$ and (67) to conclude the second inequality.

If we differentiate the right-hand side of (68) in direction of $\frac{X}{2}$, we obtain

$$\begin{aligned}\mathcal{L}_{\frac{X}{2}}\left(\operatorname{tr}_{\omega_{\varphi_s}}(\alpha)\omega_{\varphi_s}^n\right) =&\frac{X}{2}\left(\operatorname{tr}_{\omega_{\varphi_s}}(\alpha)\right)\omega_{\varphi_s}^n + \operatorname{tr}_{\omega_{\varphi_s}}(\alpha) n\alpha\wedge\omega_{\varphi_s}^{n-1}\\ =&\left(\frac{X^2}{4}(sF) - \frac{X^3}{8}(\varphi_s) + \frac{X}{4}(\Delta_g f)\right)\omega_{\varphi_s}^n\\ &+\left(\operatorname{tr}_{\omega_{\varphi_s}}(\alpha)\right)^2\omega_{\varphi_s}^n,\end{aligned} \tag{70}$$

where the second equality follows from (68) together with the expression of $\operatorname{tr}_{\omega_{\varphi_s}}(\alpha)$ provided by (66).

Since (69) equals (70), we see that the $\operatorname{tr}_{\omega_{\varphi_s}}(\alpha)^2$-term is canceled and, after dividing by $\omega_{\varphi_s}^n$, we conclude that

$$\operatorname{tr}_{\omega_{\varphi_s}} i\partial\bar\partial\left(\frac{X}{2}(f) + \frac{X^2}{4}(\varphi_s)\right) - |\alpha|^2_{g_{\varphi_s}} = \frac{X^2}{4}(sF) - \frac{X^3}{8}(\varphi_s) + \frac{X}{4}(\Delta_g f).$$

Multiplying by 4, adding $X^2(f)$ on both sides and keeping in mind that $2\operatorname{tr}_{\omega_{\varphi_s}} i\partial\bar\partial = \Delta_{g_{\varphi_s}}$, we may rearrange the previous equation to finally arrive at

$$(\Delta_{g_{\varphi_s}} + X)\left(X(f) + \frac{X^2}{2}(\varphi_s)\right) = H_1 + 4\left|\partial\bar\partial f + \frac{X}{2}(\varphi_s)\right|^2_{g_{\varphi_s}} \tag{71}$$

with $H_1 := X^2(sF) + X(\Delta_g f) + X^2(f)$. We continue to estimate the right-hand side of (71) from below :

$$\begin{aligned}4\left|\partial\bar\partial f + \frac{X}{2}(\varphi_s)\right|^2_{g_{\varphi_s}} &\geq \frac{1}{n}\left(\Delta_{g_{\varphi_s}}\left(f + \frac{X}{2}(\varphi_s)\right)\right)^2\\ &= \frac{1}{n}\left(X(f) + \frac{X^2}{2}(\varphi_s) - H_2\right)^2,\end{aligned}$$

where $H_2 := X(f) + X(sF) + \Delta_g f$ and we made use of (66) in the second line. Combining the previous inequality with (71), we then obtain

$$(\Delta_{g_{\varphi_s}} + X)\left(X(f) + \frac{X^2}{2}(\varphi_s)\right) \geq H_1 + \frac{1}{n}\left(X(f) + \frac{X^2}{2}(\varphi_s) - H_2\right)^2.$$

Note that by Assumption A.3, the function $X(f) + \frac{X^2}{2}(\varphi_s)$ tends to 1 as $t\to\infty$, and so either $X(f) + \frac{X^2}{2}(\varphi_s) \leq 1$, or $X(f) + \frac{X^2}{2}(\varphi_s)$ attains its maximum at some point. In the first case, we are done so we assume that $X(f) + \frac{X^2}{2}(\varphi_s)$ is maximal at $p_{\max}\in M$. Then we observe that the previous inequality gives at $p_{\max}$

$$X(f) + \frac{X^2}{2}(\varphi_s) \leq \sqrt{n\sup_M|H_1|} + \sup_M|H_2| < \infty,$$

i.e. $X(f) + \frac{X^2}{2}(\varphi_s)$ is uniformly bounded from above. This, in turn, implies the required uniform upper bound on $X^2(\varphi_s)$ since $X(f)$ is bounded.

For the lower bound on $X^2(\varphi_s)$ we recall from (55) that

$$X = \nabla^{g_{\varphi_s}}\left(f + \frac{X}{2}(\varphi_s)\right),$$

so that we can estimate as follows:

$$X\left(\frac{X}{2}(\varphi_s)\right) = -X(f) + |X|^2_{g_{\varphi_s}} \geq -\sup_M |X(f)|,$$

which is finite. This completes the proof.

□

With the previous lemma, we can finish the proof of Lemma 5.13.

Proof of Lemma 5.13. Let us define the barrier function $v := e^{\varepsilon_0\left(f + \frac{X}{2}(\varphi_s)\right)}$ for some $0 < \varepsilon_0 < 1$ to be chosen later on. Since $X = \nabla^{g_{\varphi_s}}\left(f + \frac{X}{2}(\varphi_s)\right)$, we compute

$$\begin{aligned}(\Delta_{g_{\varphi_s}} + X)\left(v^{-1}\right) &= \varepsilon_0 v^{-1}\left((\varepsilon_0 - 1)|X|^2_{g_{\varphi_s}} - \Delta_{g_{\varphi_s}}\left(f + \frac{X}{2}(\varphi_s)\right)\right)\\ &= \varepsilon_0 v^{-1}\left((\varepsilon_0 - 1)|X|^2_{g_{\varphi_s}} + \frac{X^2}{2}(\varphi_s) - X(sF) - \Delta_g f\right)\end{aligned}$$

where we used (66) in the second line. Recalling the identity

$$|X|^2_{g_{\varphi_s}} = \frac{X^2}{2}(\varphi_s) + X(f), \tag{72}$$

we may further simplify the previous equation to

$$\begin{aligned}(\Delta_{g_{\varphi_s}} + X)\left(v^{-1}\right) &= \varepsilon_0 v^{-1}\left(\varepsilon_0|X|^2_{g_{\varphi_s}} - X(f) - X(sF) - \Delta_g f\right)\\ &\leq \varepsilon_0 v^{-1}\left(\varepsilon_0 C - X(f) - X(sF) - \Delta_g f\right)\end{aligned} \tag{73}$$

for some uniform constant $C > 0$ only depending on $\sup_M X(f)$ and the uniform bound on $X^2(\varphi_s)$ from Lemma 5.16. Note that this estimate again uses (72).

Since $X(f) \to 1$, and $X(F), \Delta_g f \to 0$ as $t \to \infty$, there exists a compact domain $K \subset M$ such that

$$X(f) \geq \frac{3}{4} \quad \text{and} \quad |\Delta_g f| + |X(F)| \leq \frac{1}{8} \quad \text{on} \quad M \setminus K.$$

Moreover, we can assume that K is of the form

$$K = \{x \in M \,|\, t(x) \leq t_0\} \tag{74}$$

for some $t_0 > 0$. Choosing $\varepsilon_0 > 0$ sufficiently small so that

$$\varepsilon_0 C \leq \frac{1}{8},$$

we thus obtain from (73) that

$$(\Delta_{g_{\varphi_s}} + X)\left(v^{-1}\right) \leq -\frac{\varepsilon_0}{2} v^{-1} \quad \text{on} \quad M \setminus K,$$

as claimed. $\square$

Before obtaining Proposition 5.15, we require yet another

Lemma 5.17 (Bounding $X(\varphi_s)$ on a compact set)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^{\infty}_{\varepsilon, JX}(M)$ solving (50). For the compact domain $K \subset M$ given by Lemma 5.13, there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} \sup_K X(\varphi_s) \leq C \quad \text{and} \quad \sup_{s \in [0,1]} X(\varphi_s) \leq Ct + C \text{ on } M \setminus K,$$

where C only depends on K, $F \in C^{\infty}_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Proof. For the first part of the statement, we essentially follow the argument in [CD20b][Proposition 7.11].

Consider the flow $(\Phi_\tau)_{\tau \in \mathbb{R}}$ of the complete vector field $\frac{X}{2}$. In particular, the map Φ_τ corresponds to translation by τ in the radial parameter t on the cylindrical end $[0, \infty) \times L$. Then we let $\psi_x(\tau) := \varphi_s(\Phi_\tau(x))$ for $(x, \tau) \in M \times \mathbb{R}$ and observe that for each fixed $x \in M$, the limit $\lim_{\tau \to \pm\infty} \psi_x(\tau)$ always exists because φ_s tends to zero as $t \to \infty$. Keeping this in mind, we consider $\eta_-(\tau) := e^{-\tau}$ and integrate by parts as follows

$$\begin{aligned} \int_0^{\infty} \eta_-''(\tau)\psi_x(\tau) d\tau &= -\int_0^{\infty} \eta_-'(\tau)\psi_x'(\tau) d\tau + \psi_x(0) \\ &= \int_0^{\infty} \eta_-(\tau)\psi_x''(\tau) d\tau + \psi_x'(0) + \psi_x(0). \end{aligned} \tag{75}$$

By choosing $x \in K$, rearranging (75) and using $\frac{X}{2}(\varphi_s)(x) = \psi_x'(0)$, we consequently estimate

$$\begin{aligned} \frac{X}{2}(\varphi_s)(x) &\leq -\inf_K \varphi_s + \sup_M \varphi_s \int_0^{\infty} e^{-\tau} d\tau - \inf_M \frac{X^2}{4}(\varphi_s) \int_0^{\infty} e^{-\tau} d\tau \\ &\leq C_1, \end{aligned}$$

where $C_1 > 0$ only depends on F and the geometry of (M, g), thanks to Propositions 5.14 and 5.7 as well as Lemma 5.16. This shows the first part of this lemma.

For the second part, recall from (74), that we can identify $M \setminus K \cong (t_0, \infty) \times L$ for some $t_0 > 0$. To emphasize this splitting, we write $x = (t, y)$ for points $x \in M \setminus K$. Under this identification, $X = 2\partial/\partial t$ and so we can write

$$\begin{aligned} X(\varphi_s)(t, y) &= \int_{t_0}^{t} \frac{X^2}{2}(\varphi_s)(\sigma, y) d\sigma + X(\varphi_s)(t_0, y) \\ &\leq C_2(t - t_0) + C_1, \end{aligned} \tag{76}$$

since $(0, y) \in K$ and $X^2(\varphi_s) \leq C_2$ for some uniform constant $C_2 > 0$ given by Lemma 5.16. As the right-hand side of (76) is independent of $s \in [0, 1]$, the lemma follows. □

Now we can deduce Proposition 5.15.

Proof of Proposition 5.15. As in [CD20b][Proposition 7.20], we use a barrier function to show the claim. Let $0 < \varepsilon_0 < 1$ and $K \subset M$ be given by Lemma 5.13, i.e. on $M \setminus K$, we have

$$(\Delta_{g_{\varphi_s}} + X)\left(e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right) \leq -\frac{\varepsilon_0}{2} e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} < 0. \tag{77}$$

The reader may observe from the proof that (77) holds as long as $0 < \varepsilon_0 \ll 1$ is sufficient small. In particular, we are free to choose $\varepsilon_0 > 0$ as small was we require and (77) is still valid.

Similar to (52), which was used for proving the upper bound, the Monge-Ampère equation (50) implies

$$(\Delta_{g_{\varphi_s}} + X)(\varphi_s) \leq |F|,$$

and so for some $A > 0$ to be specified later on, we obtain

$$(\Delta_{g_{\varphi_s}} + X)\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right) \leq |F| - A\frac{\varepsilon_0}{2} e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}. \tag{78}$$

The idea is to choose $A \gg 1$ sufficiently large so that the right term in (78) becomes negative.

Note that by Lemma 5.17 and Assumption A.3 there exists a constant $C > 0$ only depending on F and the geometry of (M, g) such that

$$\varepsilon_0\left(f + \frac{X}{2}(\varphi_s)\right) \leq \varepsilon_0 Ct + C \leq \varepsilon t + C, \tag{79}$$

where the second inequality holds if we fix some $\varepsilon_0 > 0$ with

$$C\varepsilon_0 < \varepsilon.$$

Applying (79) to the right-hand side of (78) then yields

$$\begin{aligned}
|F| - A\frac{\varepsilon_0}{2} e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} &\leq e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\left(e^{\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)-\varepsilon t}||F||_{C^0_\varepsilon} - A\frac{\varepsilon_0}{2}\right) \\
&\leq e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\left(e^C||F||_{C^0_\varepsilon} - A\frac{\varepsilon_0}{2}\right).
\end{aligned}$$

Thus, choosing $A > 0$ sufficiently large so that

$$A > \frac{2}{\varepsilon_0} e^C ||F||_{C^0_\varepsilon},$$

and plugging this back into (78), we arrive at

$$(\Delta_{g_{\varphi_s}} + X)\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right) \leq 0 \quad \text{on} \quad M \setminus K.$$

Hence, Hopf's maximum principle states that

$$
\begin{aligned}
\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} &\geq \min\left\{0, \min_{\partial K}\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right)\right\} \\
&\geq \min\left\{0, \min_{K}\varphi_s\right\} \qquad (80)\\
&\geq -C
\end{aligned}
$$

holds on $M \setminus K$ because $\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}$ goes to 0 as $t \to \infty$ and $\min_K \varphi_s$ is, according to Lemma 5.17, uniformly bounded from below by some constant $-C < 0$. To conclude the Proposition, we observe that

$$
f + \frac{X}{2}(\varphi_s) \geq 1
$$

by Lemma 5.10 and consequently,

$$
\varphi_s \geq -C - Ae^{-\varepsilon_0} \quad \text{on} \quad M \setminus K,
$$

as claimed.

□

Having finally finished the proof of Proposition 5.12, we can now strengthen the estimates in Lemma 5.17, i.e. achieve a uniform bound on the radial derivative of φ_s.

Corollary 5.18 (Uniform bound on $X(\varphi_s)$)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$
\sup_{s\in[0,1]}\sup_{M}|X(\varphi_s)| \leq C,
$$

where C only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Proof. We apply the same idea as in the proof of Lemma 5.17. Namely, by (75), we can estimate for each $x \in M$:

$$
\frac{X}{2}(\varphi_s)(x) \leq -\inf_M \varphi_s + \sup_M \varphi_s - \inf_M \frac{X^2}{4}(\varphi_s),
$$

so that the uniform upper bound follows from Propositions 5.7, 5.12 and Lemma 5.16. The lower bound is similar. Using $\eta_+ = e^\tau$ instead of $\eta_- = e^{-\tau}$ leads to

$$
\int_{-\infty}^0 \eta_+''(\tau)\psi_x(\tau)d\tau = \int_{-\infty}^0 \eta_+(\tau)\psi_x''(\tau)d\tau - \psi_x'(0) + \psi_x(0),
$$

and estimating as before then yields

$$
\frac{X}{2}(\varphi_s)(x) \geq \inf_M \varphi_s - \sup_M \varphi_s + \inf_M \frac{X^2}{4}(\varphi_s),
$$

finishing the proof. □

This new bound on $X(\varphi_s)$ enables us to conclude a weighted lower bound on φ_s, at least for *some* $\varepsilon_0 < \varepsilon$.

Proposition 5.19 (A first weighted lower bound on φ_s)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^{\infty}_{\varepsilon,JX}(M)$ solving (50). Then there exist two constants $0 < \varepsilon_0 < 1$ and $C > 0$ such that*

$$\inf_{s\in[0,1]} \inf_M e^{\varepsilon_0 t}\varphi_s \geq -C,$$

where both ε_0 and C only depend on $F \in C^{\infty}_{\varepsilon,JX}(M)$ and the geometry of (M,g).

Proof. Using Corollary 5.18, the proof of Proposition 5.12 can be refined by following the argument in [CD20b][Proposition 7.20].

We repeat the proof until arriving at (80), so that we have on $M\setminus K$:

$$\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} \geq \min\left\{0, \min_{\partial K}\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right)\right\}. \tag{81}$$

By Corollary 5.18 and Assumption A.3, there is a uniform constant $C > 0$ such that

$$C^{-1}e^{-2\varepsilon_0 t} \leq e^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} \leq Ce^{-2\varepsilon_0 t} \tag{82}$$

holds on M. In particular, since $\inf_K \varphi_s$ is uniformly bounded from below by Proposition 5.14, we can choose $A \gg 1$ even larger, so that

$$\min_{\partial K}\left(\varphi_s + Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)}\right) \geq \inf_K \varphi_s + A\inf_K C^{-1}e^{-2\varepsilon_0 t} \geq 0.$$

Consequently, we arranged that (81) becomes

$$\varphi_s \geq -Ae^{-\varepsilon_0\left(f+\frac{X}{2}(\varphi_s)\right)} \geq -ACe^{-2\varepsilon_0 t} \quad \text{on} \quad M\setminus K,$$

because of (82). This is precisely what we wanted to prove. □

Before improving the weighted bound from ε_0 to ε, we require uniform bounds on all derivatives of φ_s, which is the content of the subsequent section.

5.2. **Higher order estimates.** In the previous section, we obtained uniform bounds on φ_s and its radial derivative up to second order. Using these results, we begin by deriving bounds on the C^2- and C^3-norms of φ_s, which then leads to estimates for all derivatives. We purse essentially the same strategy as in [CD20b][Section 7], but occasionally we present different computations.

5.2.1. *The C^2-estimate.* The C^2-estimate for φ_s is equivalent to bounding the associated metric g_{φ_s} uniformly in terms of g.

Proposition 5.20 (Uniform bound on the metric)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^{\infty}_{\varepsilon,JX}(M)$ solving (50). If g_{φ_s} denotes* *the metric associated to the Kähler form $\omega + i\partial\bar{\partial}\varphi_s$, then there exists a constant $C > 0$ such that*

$$C^{-1}g \leq g_{\varphi_s} \leq Cg, \tag{83}$$

where C only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Before proceeding with the proof, we immediately obtain a uniform bound on the volume form by looking at (50) and applying Corollary 5.18.

Corollary 5.21 (Uniform bound on volume form)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$C^{-1}\omega^n \leq (\omega + i\partial\bar{\partial}\varphi_s)^n \leq C\omega^n,$$

where $C > 0$ only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Proof of Proposition 5.20. We argue as in [CD20b][Proposition 7.14], but present different calculations. The bound (83) amounts to bounding both $\operatorname{tr}_\omega \omega_{\varphi_s}$ and $\operatorname{tr}_{\omega_{\varphi_s}} \omega$ uniformly from above. However, there is the well-known formula

$$\operatorname{tr}_{\omega_{\varphi_s}} \omega \leq n \cdot \frac{\omega^n}{\omega^n_{\varphi_s}} (\operatorname{tr}_\omega \omega_{\varphi_s})^{n-1}$$

compare for example ([BEG13][Lemma 4.1.1]). Thus, it suffices to estimate $\operatorname{tr}_\omega \omega_{\varphi_s}$ since the volume form $\omega^n_{\varphi_s}$ is uniformly bounded by Corollary 5.21.

In this proof, $C > 0$ denotes a uniform constant, which may increase from line to line but only depends on the geometry of (M, g) and the C^∞-norm of F.

Recall that a standard computation yields the following inequality

$$\frac{1}{2}\Delta_{g_{\varphi_s}} \log \operatorname{tr}_\omega \omega_{\varphi_s} \geq -\frac{\operatorname{tr}_\omega \operatorname{Ric}(\omega_{\varphi_s})}{\operatorname{tr}_\omega \omega_{\varphi_s}} - C \operatorname{tr}_{\omega_{\varphi_s}} \omega, \tag{84}$$

where $\operatorname{Ric}(\omega_{\varphi_s})$ is the Ricci form of ω_{φ_s} and $C > 0$ a constant such that the holomorphic bisectional curvature of g is bounded from below by $-C$. For a proof of this inequality, we refer the reader to [BEG13][Proposition 4.1.2]. Also observe that in our case the bisectional curvature of g is bounded since g is asymptotically cylindrical. Starting from (50), we compute the Ricci form of ω_{φ_s} :

$$\operatorname{Ric}(\omega_{\varphi_s}) = -i\partial\bar{\partial} \log \omega^n_{\varphi_s} = \operatorname{Ric}(\omega) - i\partial\bar{\partial} sF + i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right). \tag{85}$$

As both $||F||_{C^2}$ and the curvature of g are uniformly bounded, we continue to estimate

$$-\operatorname{tr}_\omega \operatorname{Ric}(\omega_{\varphi_s}) \geq -C - \operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right). \tag{86}$$

Also recall from [BEG13][Lemma 4.1.1] that

$$\operatorname{tr}_\omega \omega_{\varphi_s} \geq n \cdot \left(\frac{\omega_{\varphi_s}^n}{\omega^n}\right)^{\frac{1}{n}} \geq C^{-1} > 0, \tag{87}$$

where the lower bound again follows from Corollary 5.21. Combining (87) and (86) with (84), we consequently arrive at

$$\frac{1}{2}\Delta_{g_{\varphi_s}} \log \operatorname{tr}_\omega \omega_{\varphi_s} \geq -\frac{\operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right)}{\operatorname{tr}_\omega \omega_{\varphi_s}} - C - C \operatorname{tr}_{\omega_{\varphi_s}} \omega. \tag{88}$$

Next, we calculate the radial derivative of $\operatorname{tr}_\omega \omega_{\varphi_s}$ by considering its defining equation:

$$\operatorname{tr}_\omega \omega_{\varphi_s} \cdot \omega^n = n \cdot \omega_{\varphi_s} \wedge \omega^{n-1}.$$

Taking the Lie derivative in direction $\frac{X}{2}$ on both sides of this equation and then dividing by ω^n leads to

$$\begin{aligned}
&\frac{X}{2}(\operatorname{tr}_\omega \omega_{\varphi_s}) + \operatorname{tr}_\omega \omega_{\varphi_s} \cdot \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega) \\
&= \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega_{\varphi_s}) + n(n-1) \cdot \frac{\omega_{\varphi_s} \wedge \mathcal{L}_{\frac{X}{2}}(\omega) \wedge \omega^{n-2}}{\omega^n} \\
&= \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega_{\varphi_s}) + \operatorname{tr}_\omega \omega_{\varphi_s} \cdot \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega) - \langle \omega_{\varphi_s}, \mathcal{L}_{\frac{X}{2}}(\omega)\rangle_g,
\end{aligned} \tag{89}$$

or equivalently,

$$\frac{X}{2}(\operatorname{tr}_\omega \omega_{\varphi_s}) = \operatorname{tr}_\omega \mathcal{L}_{\frac{X}{2}}(\omega_{\varphi_s}) - \langle \omega_{\varphi_s}, \mathcal{L}_{\frac{X}{2}}(\omega)\rangle_g \tag{90}$$

Here, the last equation in (89) is a straight forward computation, which can be found in [Szé14][Lemma 4.6], and $\langle , \rangle_g$ denotes the metric on 2-forms induced by g. We recall that $\mathcal{L}_{\frac{X}{2}}(\omega) = i\partial\bar{\partial} f$ since $X = \nabla^g f$ and also that the norm $|i\partial\bar{\partial} f|_g \leq C$ is uniformly bounded by some $C > 0$ because of A.3. Applying these observations to the previous equation, we obtain

$$\begin{aligned}
\frac{X}{2}(\log \operatorname{tr}_\omega \omega_{\varphi_s}) &= \frac{\operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right)}{\operatorname{tr}_\omega \omega_{\varphi_s}} + \frac{\operatorname{tr}_\omega i\partial\bar{\partial} f}{\operatorname{tr}_\omega \omega_{\varphi_s}} - \frac{\langle \omega_{\varphi_s}, \mathcal{L}_{\frac{X}{2}}(\omega)\rangle_g}{\operatorname{tr}_\omega \omega_{\varphi_s}} \\
&\geq \frac{\operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right)}{\operatorname{tr}_\omega \omega_{\varphi_s}} - \frac{C}{\operatorname{tr}_\omega \omega_{\varphi_s}} - \frac{C \cdot |\omega_{\varphi_s}|_g}{\operatorname{tr}_\omega \omega_{\varphi_s}} \\
&\geq \frac{\operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right)}{\operatorname{tr}_\omega \omega_{\varphi_s}} - C,
\end{aligned}$$

where we used the bound on $|i\partial\bar{\partial} f|_g$ in the second line and (87) in the last one. Altogether, we finally arrive at

$$\frac{1}{2}\left(X + \Delta_{g_{\varphi_s}}\right) \log \operatorname{tr}_\omega \omega_{\varphi_s} \geq -C - C \operatorname{tr}_{\omega_{\varphi_s}} \omega. \tag{91}$$

From there, it is standard to conclude an upper bound on $\mathrm{tr}_\omega \omega_{\varphi_s}$. We begin by considering the following inequality

$$\frac{1}{2}\left(X + \Delta_{g_{\varphi_s}}\right)\varphi_s \leq C + n - \mathrm{tr}_{\omega_{\varphi_s}} \omega$$

where we used the upper bound on $X(\varphi_s)$ from Proposition 5.18 and the definition of ω_{φ_s}. In combination with (91), we then obtain

$$\frac{1}{2}\left(X + \Delta_{g_{\varphi_s}}\right)(\log \mathrm{tr}_\omega \omega_{\varphi_s} - (C+1)\varphi_s) \geq -C + \mathrm{tr}_{\omega_{\varphi_s}} \omega. \tag{92}$$

Applying the maximum principle to this equation, yields the desired estimate for $\mathrm{tr}_\omega \omega_{\varphi_s}$ as follows. Note that we can assume $\log \mathrm{tr}_\omega \omega_{\varphi_s} - (C+1)\varphi_s > n$ at least somewhere on M, because otherwise we are done by the uniform upper bound on φ_s (Proposition 5.7). Thus, there exists $p_{\max} \in M$ such that $\log \mathrm{tr}_\omega \omega_{\varphi_s} - (C+1)\varphi_s$ is maximal at $p_{\max}$. Then at this point, we obtain from (92) that $\mathrm{tr}_{\omega_{\varphi_s}} \omega \leq C$, so that at p_{max}:

$$\mathrm{tr}_\omega \omega_{\varphi_s} \cdot e^{-(C+1)\varphi_s} \leq n e^{-(C+1)\varphi_s} \cdot \frac{\omega_{\varphi_s}^n}{\omega^n} (\mathrm{tr}_{\omega_{\varphi_s}} \omega)^{n-1} \leq C.$$

Hence, $\log \mathrm{tr}_\omega \omega_{\varphi_s} - (C+1)\varphi_s$ is uniformly bounded from above, and so is $\mathrm{tr}_\omega \omega_{\varphi_s}$, finishing the proof. □

5.2.2. *The C^3-estimate.*

Proposition 5.22 (Uniform C^3-estimate)**.** *Let $1 < \varepsilon < 2$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon, JX}(M)$ solving (50). If g_{φ_s} denotes the metric associated to the Kähler form $\omega + i\partial\bar{\partial}\varphi_s$, then there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} \sup_M |\nabla^g g_{\varphi_s}|_g \leq C,$$

where the constant C only depends on $F \in C^\infty_{\varepsilon, JX}(M)$ and the geometry of (M, g).

Proof. We define

$$S := |\nabla^g g_{\varphi_s}|_g^2,$$

and then the computation in [CD20b][Proposition 7.16] goes through verbatim. In particular, if $\mathrm{Rm}(g)$ denotes the curvature tensor of g, there exists a constant $C > 0$, which only depends on the constant in Proposition 5.20 as well as on bounds for covariant derivatives of both F and $\mathrm{Rm}(g)$, such that

$$\frac{1}{2}\left(\Delta_{g_{\varphi_s}} - X\right) S \geq -C(S+1). \tag{93}$$

Moreover, recall that the standard Schwarz-Lemma calculation in holomorphic coordinates yields

(94)
$$\frac{1}{2}\Delta_{g_{\varphi_s}}\operatorname{tr}_\omega\omega_{\varphi_s} = -\operatorname{tr}_\omega\operatorname{Ric}(\omega_{\varphi_s}) + g_{\varphi_s}^{\bar{l}k}R_{k\bar{l}}{}^{j\bar{i}}g_{i\bar{j}}^{\varphi_s} + g^{\bar{j}i}g_{\varphi_s}^{\bar{q}p}g_{\varphi_s}^{\bar{l}k}\nabla_i^g g_{p\bar{l}}^{\varphi_s}\nabla_{\bar{j}}^g g_{k\bar{q}}^{\varphi_s},$$

where $g_{i\bar{j}}^{\varphi_s}$ denotes the components of g_{φ_s} in coordinates, with inverse $g_{\varphi_s}^{\bar{j}i}$, and $R_{k\bar{l}i\bar{j}}$ is the local expression of $\operatorname{Rm}(g)$. For the computation, we refer the reader to [BEG13][(3.67)], for example. Starting from (94), and keeping Proposition 5.20 as well as (86) in mind, we estimate

$$\frac{1}{2}\Delta_{g_{\varphi_s}}\operatorname{tr}_\omega\omega_{\varphi_s} \geq -\operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right) - C + C^{-1}S.$$

Proposition 5.20 applied to (90) also leads to

$$\frac{X}{2}(\operatorname{tr}_\omega\omega_{\varphi_s}) \geq \operatorname{tr}_\omega i\partial\bar{\partial}\left(\frac{X}{2}(\varphi_s)\right) - C,$$

and hence,

$$\frac{1}{2}\left(\Delta_{g_{\varphi_s}} + X\right)\operatorname{tr}_\omega\omega_{\varphi_s} \geq -C + C^{-1}S \tag{95}$$

for some constant $C > 0$ only depending on $||\operatorname{Rm}(g)||_{C^0(M)}$, $||\partial\partial f||_{C^0(M)}$, $||F||_{C^2(M)}$ and the constant in Proposition 5.20. If we choose a sufficiently large constant $C_1 > 0$ and then add (93) to C_1-times (95), we can arrange that

$$\frac{1}{2}(\Delta_{g_{\varphi_s}} - X)\left(S + C_1\operatorname{tr}_\omega\omega_{\varphi_s}\right) \geq -C + S. \tag{96}$$

Again, there are two cases to consider. If $S + C_1\operatorname{tr}_\omega\omega_{\varphi_s} \leq \lim_{t\to\infty} S + C_1\operatorname{tr}_\omega\omega_{\varphi_s} = n$, there is nothing to show, so we can assume $S + C_1\operatorname{tr}_\omega\omega_{\varphi_s} > n$. Thus, there exists a point $p_{\max} \in M$, where $S + C_1\operatorname{tr}_\omega\omega_{\varphi_s}$ is maximal. Applying the maximum principle to (96), we have at $p_{\max}$:

$$S + C_1\operatorname{tr}_\omega\omega_{\varphi_s} \leq C + C_1\sup_M\operatorname{tr}_\omega\omega_{\varphi_s} \leq C. \tag{97}$$

This implies a uniform upper bound on S, as claimed. □

Since the C^1-norm of g_{φ_s} is uniformly bounded, we obtain a uniform $C^{0,\alpha}$-bound on g_{φ_s}, as in [CD20b][Corollary 7.17].

Corollary 5.23. *Let $1 < \varepsilon < 2$, $\alpha \in (0,1)$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). If g_{φ_s} denotes the Riemannian metric corresponding to ω_{φ_s}, and $g_{\varphi_s}^{-1}$ the induced metric on 1-forms, then there exists a constant $C > 0$ such that*

$$\sup_{s\in[0,1]}\left(||g_{\varphi_s}||_{C^{0,\alpha}} + ||g_{\varphi_s}^{-1}||_{C^{0,\alpha}}\right) \leq C,$$

where C only depends on α, $F \in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M,g).

Proof. Recall that the natural embedding

$$C^1(TM \otimes TM) \subseteq C^{0,\alpha}(TM \otimes TM)$$

is continuous, so that the operator norm of this inclusion only depends on (M, g) and α. Hence, the $C^{0,\alpha}$-norm of g_{φ_s} is uniformly bounded from above by $||g_{\varphi_s}||_{C^1}$, which in turn is uniformly bounded according to Proposition 5.20 and 5.22.

Similarly, we find a uniform $C > 0$, only depending on (M, g) and α, such that

$$||g_{\varphi_s}^{-1}||_{C^{0,\alpha}} \leq C \left(||g_{\varphi_s}^{-1}||_{C^0} + ||\nabla^g g_{\varphi_s}^{-1}||_{C^0} \right). \tag{98}$$

Moreover, there is the following point-wise estimate

$$|\nabla^g g_{\varphi_s}^{-1}|_g \leq |g_{\varphi_s}|_g^2 \cdot |\nabla^g g_{\varphi_s}|_g. \tag{99}$$

Indeed, this inequality follows immediately by using holomorphic normal coordinates and differentiating the relation

$$g_{\varphi_s}^{\bar{j}i} \cdot g_{k\bar{j}}^{\varphi_s} = \delta_k^i,$$

where $g_{\varphi_s}^{\bar{j}i}$ and $g_{i\bar{j}}^{\varphi_s}$ are the components of $g_{\varphi_s}^{-1}$ and g_{φ_s}, respectively.

Thus, we conclude the required uniform bound on $||g_{\varphi_s}^{-1}||_{C^{0,\alpha}}$ from (98) and (99), together with Proposition 5.20 and 5.22. □

The standard Schauder theory for ACyl metrics then implies uniform $C^{2,\alpha}$-bounds for φ_s.

Proposition 5.24 (Uniform $C^{2,\alpha}$-bound on φ_s)**.** *Let $1 < \varepsilon < 2$, $\alpha \in (0, 1)$ and suppose $(\varphi_s)_{0 \leq s \leq 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s \in [0,1]} ||\varphi_s||_{C^{2,\alpha}} \leq C,$$

where C only depends on α, $F \in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M, g).

Proof. Recall that Δ_g is an asymptotically translation-invariant operator of order 2, and so the Schauder estimates (Theorem 2.4) apply and yield a constant $C > 0$, only depending on (M, g), such that

$$||\varphi_s||_{C^{2,\alpha}} \leq C \left(||\varphi_s||_{C^0} + ||\Delta_g \varphi_s||_{C^{0,\alpha}} \right).$$

As $||\Delta_g \varphi_s||_{C^{0,\alpha}}$ can be bounded from above in terms of $||g_{\varphi_s}||_{C^{0,\alpha}}$, the claim then follows immediately from Corollary 5.23 and the uniform bound on $\sup_M |\varphi_s|$ given by Propositions 5.7 and 5.12. □

5.2.3. *Local $C^{k,\alpha}$-estimates.* Uniform higher order estimates can be obtained similarly to the compact case considered by Yau [Yau78]. The idea is to use *local* Schauder estimates to conclude higher regularity in a uniform way. Since our manifold (M, g) is non-compact, we require the use of special coordinates in which the metric g, and all its derivatives, are uniformly bounded. This is provided by the following

Theorem 5.25. *Let (M,g) be an n-dimensional ACyl Kähler manifold and $i(g)>0$ the corresponding injectivity radius. For each $q\in\mathbb{N}_0$, suppose that $C_q>0$ is a constant such that the curvature tensor* $\mathrm{Rm}(g)$ *satisfies*

$$\sup_M |\nabla^q \mathrm{Rm}(g)|_g \leq C_q.$$

Then there are two constants $r_2>r_1>0$, depending only on $n, i(g), C_q$, such that for each $x\in M$, there exists a chart $\phi: U\subset\mathbb{C}^n\to M$ satisfying the following properties:

(i) $B_{\mathbb{C}^n}(0,r_1)\subset U\subset B_{\mathbb{C}^n}(0,r_2)$ and $\phi(0)=x$, where $B_{\mathbb{C}^n}(0,r_i)$ denotes the Euclidean ball of radius r_i around the origin.

(ii) There exists a constant $C>0$, depending only on r_1,r_2 such that the Euclidean metric $g_{\mathbb{C}^n}$ satisfies

$$C^{-1}g_{\mathbb{C}^n}\leq \phi^*g\leq C g_{\mathbb{C}^n} \quad \text{on } U.$$

(iii) For each $l\in\mathbb{N}_0$, there exist constants $A_l>0$, depending only on l,r_1,r_2, such that

$$\sup_U \left|\frac{\partial^{|\mu|+|\nu|}g_{i\bar{j}}}{\partial z^\mu\partial\bar{z}^\nu}\right|\leq A_l \quad \text{for all } |\mu|+|\nu|\leq l,$$

where $g_{i\bar{j}}$ are the components of g in the holomorphic coordinates $(z_1,\dots,z_n)$ induced by ϕ, and μ,ν are multi-indices with $|\mu|=\mu_1+\cdots+\mu_n$.

This theorem follows because the asymptotic cylinder is given explicitly. Similar results have previously been used to solve complex Monge-Ampère equations on non-compact Kähler manifolds, see for instance [CY80] and [TY90]. More generally, Theorem 5.25 is also valid for every non-compact Kähler manifold of positive injectivity radius and bounded geometry, compare [WY20][Theorem 9].

Using the coordinates given by Theorem 5.25, we can apply the local Schauder theory and conclude estimates on the $C^{k,\alpha}_{\mathrm{loc}}$-norm of φ_s. This argument is by induction on k starting at $k=3$.

Proposition 5.26 (Local $C^{3,\alpha}$-bound on φ_s)**.** *Let $1<\varepsilon<2$, $\alpha\in(0,1)$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). Then there exists a constant $C>0$ such that*

$$\sup_{s\in[0,1]}||\varphi_s||_{C^{3,\alpha}_{\mathrm{loc}}}\leq C,$$

where C only depends on α, $F\in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M,g).

Proof. As in [CD20b][Proposition 7.19], we follow the argument given in the compact case [Yau78].

We consider $x\in M$ and work in the holomorphic chart $\phi:U\to M$ as in Theorem 5.25. To simplify notation, we suppress ϕ and simply view U as a subset of M. The conditions (ii) and (iii) ensure that the

Euclidean Hölder norm $||\cdot||_{C^{k,\alpha}(B_x)}$ on the ball $B_x := B(0,r_1) \subset M$ is uniformly equivalent to $||\cdot||_{C^{k,\alpha}(B_x,g)}$, the Hölder norm on B_x induced by the ACyl metric g. In other words, there exists a constant $C_1 > 0$, only depending on k, α and the constants in Theorem 5.25, such that

$$C_1^{-1}||\cdot||_{C^{k,\alpha}(B_x)} \leq ||\cdot||_{C^{k,\alpha}(B_x,g)} \leq C_1||\cdot||_{C^{k,\alpha}(B_x)}. \tag{100}$$

In particular, the interior Schauder estimates ([GT01][Theorem 6.2, 6.17]) on B_x are valid for the norms $||\cdot||_{C^{k,\alpha}(B_x,g)}$. The goal is to apply these estimates to the equation

$$\frac{1}{2}\Delta_{g_{\varphi_s}}(\partial_j\varphi_s) = \partial_j\left(sF - \frac{X}{2}(\varphi_s)\right) + (\mathrm{tr}_\omega - \mathrm{tr}_{\omega_{\varphi_s}})\mathcal{L}_{\partial_j}(\omega), \tag{101}$$

where ∂_j denotes the coordinate field $\partial/\partial z_j$ induced by the chart ϕ and $j = 1,\ldots,n$. Observe that (101) is obtained by applying the Lie derivative $\mathcal{L}_{\partial_j}$ to the Monge-Ampère equation (50) and dividing by $\omega^n_{\varphi_s}$.

Recall that in holomorphic coordinates, we have

$$\Delta_{g_{\varphi_s}} = g^{\bar{j}i}_{\varphi_s}\partial_i\partial_{\bar{j}},$$

so that applying Schauder requires to bound the coefficients of $\Delta_{g_{\varphi_s}}$ uniformly in $C^{0,\alpha}(B_x)$, i.e. we have to find a constant $D > 0$, only depending on α, F and the geometry of (M,g), such that

$$||g^{\bar{j}i}_{\varphi_s}||_{C^{0,\alpha}(B_x)} \leq D \quad \text{and} \quad g^{-1}_{\varphi_s} \geq D\, g_{\mathbb{C}^n}. \tag{102}$$

The first inequality is clear by (100) together with Corollary 5.23 and the second bound follows immediately from Proposition 5.20 and condition (ii) in Theorem 5.25. Thus, interior Schauder estimates provide a constant $C_2 > 0$, only depending on n, α and D, such that

$$\begin{aligned} ||\partial_j\varphi_s||_{C^{2,\alpha}(B_x)} \leq& C_2\left(||\Delta_{g_{\varphi_s}}\partial_j\varphi_s||_{C^{0,\alpha}(B_x)} + ||\partial_j\varphi_s||_{C^0(B_x)}\right) \\ \leq& C_2\left(||\Delta_{g_{\varphi_s}}\partial_j\varphi_s||_{C^{0,\alpha}(B_x)} + C_1||\varphi_s||_{C^{2,\alpha}(M,g)}\right), \end{aligned} \tag{103}$$

where we used (100) for the second inequality. We continue to estimate the first term on the right-hand side of (103) as follows

$$\begin{aligned} &||\Delta_{g_{\varphi_s}}\partial_j\varphi_s||_{C^{0,\alpha}(B_x)} \\ \leq& ||\varphi_s||_{C^{2,\alpha}(B_x)} + ||F||_{C^{1,\alpha}(B_x)} + ||(\mathrm{tr}_\omega - \mathrm{tr}_{\omega_{\varphi_s}})\mathcal{L}_{\partial_j}(\omega)||_{C^{0,\alpha}(B_x)} \\ \leq& C_1\left(||\varphi_s||_{C^{2,\alpha}(M,g)} + ||F||_{C^{1,\alpha}(M,g)} + A\cdot||g^{-1} - g^{-1}_{\varphi_s}||_{C^{0,\alpha}(M,g)}\right), \end{aligned} \tag{104}$$

for some constant $A > 0$ determined by condition (iii) of Theorem 5.25. Here, the first inequality is a consequence of (101) and the second one is obtained from (100). In combination with (104), inequality (103) then becomes

$$||\partial_j\varphi_s||_{C^{2,\alpha}(B_x)} \leq C_3 \tag{105}$$

for some constant $C_3 > 0$ which only depends on C_1, C_2, A, F and the uniform bounds on $||\varphi_s||_{C^{2,\alpha}}$ and $||g^{-1}_{\varphi_s}||_{C^{0,\alpha}}$ given by Proposition 5.24 and Corollary 5.23, respectively.

To conclude the proof, we point out that the same arguments for (105) also yield

$$||\partial_{\bar{j}}\varphi_s||_{C^{2,\alpha}(B_x)} \leq C_3,$$

and hence

$$\begin{aligned}||\varphi_s||_{C^{3,\alpha}(B_x)} &\leq \sum_{j=1}^{n} ||\partial_j\varphi_s||_{C^{2,\alpha}(B_x)} + ||\partial_{\bar{j}}\varphi_s||_{C^{2,\alpha}(B_x)} + ||\varphi_s||_{C^0(B_x)}, \\ &\leq 2nC_3 + C_1||\varphi_s||_{C^{2,\alpha}(M,g)} \\ &\leq C_4\end{aligned}$$

with $C_4 > 0$ only depending on n, C_1, C_3 and the uniform bound on $||\varphi_s||_{C^{2,\alpha}}$. In particular, the constant C_4 is independent of both $x \in M$ and $s \in [0,1]$, so that the proposition then follows.

□

The standard bootstrapping argument then leads to uniform $C^{k,\alpha}$-estimates.

Proposition 5.27 (Local $C^{k,\alpha}$-bounds on φ_s)**.** *Let $1 < \varepsilon < 2$, $\alpha \in (0,1)$, $k \in \mathbb{N}_{\geq 1}$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s\in[0,1]} ||\varphi_s||_{C^{k+2,\alpha}_{\mathrm{loc}}} \leq C, \tag{106}$$

where C only depends on k, α, $F \in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M,g).

Proof. As in [CD20b][Proposition 7.19], the proof is by induction on $k \geq 1$, with the $k = 1$ case being settled by Proposition 5.26. Thus, we consider $k \geq 2$ and can assume that the statement holds for $k-1$, i.e. that there is a $C_{k-1} > 0$, only depending on k, α, F and the geometry of (M,g), such that

$$||\varphi_s||_{C^{k+1,\alpha}_{\mathrm{loc}}} \leq C_{k-1}. \tag{107}$$

Using the same notation as in the previous proof, we work near a given $x \in M$ in the chart $\phi : U \to M$ given by Theorem 5.25. Because of (100), it suffices to show (106) for the Euclidean ball $B_x := B(0,r_1)$ and the Euclidean Hölder norm $||\cdot||_{C^{k,\alpha}(B_x)}$.

This time, we aim at applying interior Schauder estimates (of higher order) to equation (101), for which we require a constant D_{k-1}, depending only on k, α, F and the geometry of (M,g), such that

$$||g^{\bar{j}i}_{\varphi_s}||_{C^{k-1,\alpha}(B_x)} \leq D_{k-1} \quad \text{and} \quad g^{-1}_{\varphi_s} \geq D_{k-1}\, g_{\mathbb{C}}. \tag{108}$$

The second inequality is again clear by Proposition 5.20 and condition (ii) in Theorem 5.25, and for the first, recall that

$$g^{\varphi_s}_{i\bar{j}} = g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi_s.$$

Together with condition (iii) in Theorem 5.25, we obtain

$$\begin{aligned} ||g^{\varphi_s}_{i\bar{j}}||_{C^{k-1,\alpha}(B_x)} &\leq ||\partial\bar{\partial}\varphi_s||_{C^{k-1,\alpha}}(B_x) + A_{k-1} \\ &\leq ||\varphi_s||_{C^{k+1,\alpha}(B_x)} + A_{k-1} \\ &\leq C_{k-1} + A_{k-1}, \end{aligned}$$

where we used the induction hypothesis (107) in the last line. Consequently, the entries of the inverse matrix can be bounded as well since there exists a $C_0 > 0$, depending only on the uniform bound on $||g^{-1}_{\varphi_s}||_{C^0(M)}$ from Proposition 5.20, such that

$$||g^{\bar{j}i}_{\varphi_s}||_{C^{k-1,\alpha}(B_x)} \leq C_0||g^{\varphi_s}_{i\bar{j}}||_{C^{k-1,\alpha}(B_x)}.$$

Note that this follows by differentiating the identity

$$g^{\bar{j}i}_{\varphi_s}g^{\varphi_s}_{l\bar{j}} = \delta^i_l$$

and using the fact that for functions u with $\inf u > 0$, one has

$$||u||_{C^{0,\alpha}} \leq (\inf u)^{-1}\left(1 + ||u||_{C^{0,\alpha}}(\inf u)^{-1}\right).$$

Thus, (108) holds if $D_{k-1} := C_0(C_k + A_{k-1})$. Then the interior Schauder estimates [GT01][Theorem 6.17] provide a constant $E_{k-1} > 0$, depending only on n, k, α and D_{k-1}, such that

$$\begin{aligned} &||\partial_j\varphi_s||_{C^{k+1,\alpha}(B_x)} \\ \leq &E_{k-1}\left(||\Delta_{g_{\varphi_s}}(\partial_j\varphi_s)||_{C^{k-1,\alpha}(B_x)} + ||\partial_j\varphi_s||_{C^0(B_x)}\right) \\ \leq &E_{k-1}\left(||\varphi_s||_{C^{k+1,\alpha}(B_x)} + ||F||_{C^{k,\alpha}(B_x)} + ||(\mathrm{tr}_\omega - \mathrm{tr}_{\omega_{\varphi_s}})\mathcal{L}_{\partial_j}(\omega)||_{C^{k-1,\alpha}(B_x)}\right) \\ \leq &E_{k-1}\left(||\varphi_s||_{C^{k+1,\alpha}(B_x)} + ||F||_{C^{k,\alpha}(B_x)} + A_{k-1}||g^{-1} - g^{-1}_{\varphi_s}||_{C^{k-1,\alpha}(B_x)}\right) \end{aligned}$$

where (101) implies the second inequality, and for the third one, we used the bounds in (iii) of Theorem 5.25. Hence, we conclude from this, together with the induction hypothesis (107) and (108), that

$$||\partial_j\varphi_s||_{C^{k+1,\alpha}(B_x)} \leq C_k$$

for some $C_k > 0$ only depending on E_{k-1}, C_{k-1}, F and the constants in Theorem 5.25. As in the previous proof, we finally arrive at

$$\begin{aligned} ||\varphi_s||_{C^{k+2,\alpha}(B_x)} &\leq \sum_{j=1}^n ||\partial_j\varphi_s||_{C^{k+1,\alpha}(B_x)} + ||\partial_{\bar{j}}\varphi_s||_{C^{k+1,\alpha}(B_x)} + ||\varphi_s||_{C^0(B_x)} \\ &\leq 3nC_k, \end{aligned}$$

as required. □

5.2.4. *Weighted $C^{k,\alpha}$-estimates.* Recall from Propositions 5.7 and 5.19 that $|\varphi_s|$ is uniformly bounded from above by $e^{-\varepsilon_0 t}$ for some $0 < \varepsilon_0 \ll 1$. First, we will see that the $C^{k,\alpha}_{\varepsilon_0}$-norms of φ_s are also uniformly bounded and, in a second step, we explain how to improve the decay from ε_0 to ε.

Proposition 5.28 (Weighted $C^{k,\alpha}$-bounds on φ_s). *Let $1 < \varepsilon < 2$, $\alpha \in (0,1)$, $k \in \mathbb{N}_0$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^\infty_{\varepsilon,JX}(M)$ solving (50). For the constant $0 < \varepsilon_0 < 1$ given by Proposition 5.19, exists a $C > 0$ such that*

$$\sup_{s\in[0,1]} ||e^{\varepsilon_0 t}\varphi_s||_{C^{k,\alpha}} \leq C,$$

where C only depends on ε_0, k, α, $F \in C^\infty_{\varepsilon,JX}(M)$ and the geometry of (M,g).

Proof. We follow the argument given in [CD20b][Proposition 7.22]. For $\tau \in [0,1]$, consider the function

$$H(\tau) := \log \frac{\left(\omega + i\partial\bar{\partial}(\tau\cdot\varphi_s)\right)^n}{\omega^n}, \tag{109}$$

so that

$$H'(\tau) = \frac{1}{2}\Delta_{g_{\tau\varphi_s}}(\varphi_s),$$

where $g_{\tau\varphi_s}$ denotes the metric with Kähler form $\omega + i\partial\bar{\partial}(\tau\varphi_s)$. By using (50) and $H(0) = 0$, we can write

$$sF - \frac{X}{2}(\varphi_s) = H(1) = \int_0^1 H'(\tau)d\tau = \frac{1}{2}\int_0^1 \Delta_{g_{\tau\varphi_s}}(\varphi_s)d\tau. \tag{110}$$

The goal is to apply local Schauder estimates to this differential equation. For any $x \in M$, let $\phi : U \to M$ be the holomorphic chart with $\phi(0) = x$ given by Theorem 5.25. Then (110) becomes

$$sF = \left(\int_0^1 g^{\bar{j}i}_{\tau\varphi_s} d\tau\right)\partial_i\partial_{\bar{j}}\varphi_s + \frac{X}{2}(\varphi_s) =: a^{\bar{j}i}\partial_i\partial_{\bar{j}}\varphi_s + b_j\partial_j\varphi_s,$$

where we use Einstein's sum convention, and the fact that X is real-holomorphic as well as $JX(\varphi_s) = 0$.

Let $k \geq 0$ be an integer and $\alpha \in (0,1)$. Recall that by conditions (ii), (iii) of Theorem 5.25 and Proposition 5.27, there exists a constant $C_1 > 0$ such that

$$a^{\bar{j}i} \geq C_1^{-1}\delta_{i\bar{j}}, \quad \text{and} \quad ||a^{\bar{j}i}||_{C^{k,\alpha}(B_x,g)} \leq C_1,$$

where B_x is the holomorphic ball of radius r_1 around x and $||\cdot||_{C^{k,\alpha}(B_x,g)}$ the Hölder norm on B_x induced by the restriction of g. Moreover, we can arrange that $||b_j||_{C^{k,\alpha}(B_x,g)} \leq C_1$ since X and all its covariant derivatives (w.r.t. g) are uniformly bounded. Recall from (100) that the norms on B_x induced by g are uniformly equivalent to the Euclidean Hölder norms, so that interior Schauder estimates ([GT01][Theorem 6.17]) can be applied. Hence, there exists a constant $C_2 > 0$, depending only on n, k, α and C_1, such that

$$\begin{aligned} ||\varphi_s||_{C^{k+2,\alpha}(B_x,g)} &\leq C_2\left(||\varphi_s||_{C^0(B_x)} + ||F||_{C^{k,\alpha}(B_x,g)}\right) \\ &\leq C_2\left(C_3 + C_3||F||_{C^{k,\alpha}_{\varepsilon_0}(M,g)}\right)e^{-\varepsilon_0 t(x)}, \end{aligned} \tag{111}$$

for some $C_3 > 0$ only depending on the radius r_1 of the ball B_x and the bounds from Propositions 5.7 and 5.19. Note that in the last inequality, we also used that the function t is uniformly equivalent to the distance function of (M, g) to some fixed point.

As the constants in (111) are independent of the considered point $x \in M$, we conclude the desired estimate for $||e^{\varepsilon_0 t}\varphi_s||_{C^{k,\alpha}}$ as follows. Let $0 \leq l \leq k+1$ and notice that (111) implies

$$\begin{aligned} |(\nabla^g)^l \varphi_s|_g(x) \leq ||\varphi_s||_{C^{k,\alpha}(B_x,g)} &\leq C_2 C_3 \left(1 + ||F||_{C^{k,\alpha}_{\varepsilon_0}(M,g)}\right) e^{-\varepsilon_0 t(x)} \\ &=: Ce^{-\varepsilon_0 t(x)} \end{aligned}$$

holds for all $x \in M$, or equivalently,

$$||e^{\varepsilon_0 t}\varphi_s||_{C^{k+1}(M,g)} \leq C.$$

This finishes the proof because the inclusion $C^{k+1}_{\varepsilon_0}(M) \subset C^{k,\alpha}_{\varepsilon_0}(M)$ is continuous.

□

It remains to improve the uniform decay rate of φ_s from $e^{-\varepsilon_0 t}$ to $e^{-\varepsilon t}$, which is achieved in the next

Proposition 5.29 (Improved weighted $C^{k,\alpha}$-bounds on φ_s)**.** *Let $1 < \varepsilon < 2$, $\alpha \in (0,1)$, $k \in \mathbb{N}_0$ and suppose $(\varphi_s)_{0\leq s\leq 1}$ is a family in $C^{\infty}_{\varepsilon,JX}(M)$ solving (50). Then there exists a constant $C > 0$ such that*

$$\sup_{s\in[0,1]} ||e^{\varepsilon t}\varphi_s||_{C^{k,\alpha}} \leq C,$$

where C only depends on k, α, $F \in C^{\infty}_{\varepsilon,JX}(M)$ and the geometry of (M, g).

Proof. This improvement of the rate based on [CD20b][p. 63]. We begin by noting that $H(\tau)$ define by (109) satisfies

$$H''(\tau) = -|\partial\bar{\partial}\varphi_s|^2_{g_{\tau\varphi_s}},$$

so that we can write

$$\begin{aligned} sF + \int_0^1 \int_0^\tau |\partial\bar{\partial}\varphi_s|^2_{g_{\tau\varphi_s}} d\sigma\, d\tau &= sF + H'(0) - H(1) + H(0) \\ &= \frac{1}{2}\left(\Delta_g + X\right)(\varphi_s), \end{aligned} \tag{112}$$

where we used (50) for the second inequality. From (112) and Proposition 5.28, we conclude that there exists a uniform constant $C > 0$ such that

$$\left(\Delta_g + X\right)(\varphi_s) \leq Ce^{-\varepsilon_1 t} \quad \text{with} \quad \varepsilon_1 := \min\{2\varepsilon_0, \varepsilon\} > \varepsilon_0. \tag{113}$$

Starting from this equation, we can obtain a uniform lower bound on $e^{\varepsilon_1 t}\varphi_s$ by using the maximum principle and arguing as in Proposition

5.7. Let $v \in C^\infty_{\varepsilon_1}(M)$ be the unique solution to

$$(\Delta_g + X)(v) = Ce^{-\varepsilon_1 t},$$

so that we have

$$(\Delta_g + X)(\varphi_s - v) \leq 0 \quad \text{on} \quad M.$$

Thus, the maximum principle implies

$$\varphi_s - v \geq \lim_{t\to\infty}(\varphi_s - v) = 0 \quad \text{on} \quad M \tag{114}$$

which is a uniform weighted lower bound on φ_s since $v \in C^\infty_{\varepsilon_1}(M)$ only depends on ε_1, C and (M, g). Combining (114) with the upper bound in Proposition 5.7, the term $||e^{\varepsilon_1 t}\varphi_s||_{C^0}$ is uniformly bound from above.

The next step is to prove that for each $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$, there exists a uniform constant $C > 0$ such that

$$||e^{\varepsilon_1 t}\varphi_s||_{C^{k,\alpha}} \leq C. \tag{115}$$

Indeed, the same argument as in Proposition 5.28 goes through verbatim, starting this time from the uniform bound on $||e^{\varepsilon_1 t}\varphi_s||_{C^0}$ instead of merely $||e^{\varepsilon_0 t}\varphi_s||_{C^0}$. Hence, we improved the uniform decay from ε_0 to ε_1.

If $\varepsilon_1 = \varepsilon$, we are done, so we assume $\varepsilon_1 = 2\varepsilon_0 < \varepsilon$. Notice that (115) and (112) can then be used to further improve the uniform decay of $(\Delta_g + X)\varphi_s$ in (113) to

$$\varepsilon_2 := \min\{2\varepsilon_1, \varepsilon\} > \varepsilon_1 = 2\varepsilon_0$$

so that repeating the entire argument then gives a uniform bound on $||e^{\varepsilon_2 t}\varphi_s||_{C^{k,\alpha}}$.

After iterating this process a bounded number of times, we finally conclude the required uniform estimate on $||e^{\varepsilon t}\varphi_s||_{C^{k,\alpha}}$. □

Since the previous Proposition is precisely the content of Theorem 5.6, the only statement left to show is the regularity result in Proposition 5.5.

Proof of Proposition 5.5. Let $F \in C^\infty_{\varepsilon, JX}(M)$ for some $1 < \varepsilon < 2$ and suppose that $\varphi \in C^{3,\alpha}_{\varepsilon', JX}(M)$ solves

$$(\omega + i\partial\bar{\partial}\varphi)^n = e^{F - \frac{X}{2}(\varphi)}\omega^n \tag{116}$$

with $0 < \varepsilon' \leq \varepsilon$. We have essentially seen all required arguments in Propositions 5.27, 5.28 and 5.29, but the difference is that we now only require *qualitative* information on the solution φ, i.e. all the constants below a priori *do* depend on φ.

First, we improve the regularity and claim that $\varphi \in C^{k,\alpha}_{\text{loc}}(M)$ for each integer $k \geq 3$ and $\alpha \in (0, 1)$. As in Proposition 5.27, we work around some $x \in M$ in the holomorphic chart $\phi : B_x = B(0, r_1) \to M$ given by Theorem 5.25. Differentiating (116) in direction of $\partial_j = \partial/\partial z_j$, we obtain

$$\frac{1}{2}\Delta_{g_\varphi}(\partial_j\varphi) = \partial_j\left(F - \frac{X}{2}(\varphi)\right) + \left(\mathrm{tr}_\omega - \mathrm{tr}_{\omega_\varphi}\right)\mathcal{L}_{\partial_j}(\omega) \tag{117}$$

Then we notice that the coefficients $g_\varphi^{\bar{j}i}$ of Δ_{g_φ}, as well as the right-hand side of (117), are in $C^{1,\alpha}(B_x)$, so that the local regularity for elliptic equations ([GT01][Theorem 6.17]) implies $\partial_j\varphi \in C^{3,\alpha}(B_x)$ for all $j = 1,\dots,n$. Similarly, one can show that each $\partial_{\bar{j}}\varphi$ is also in $C^{3,\alpha}(B_x)$, implying $\varphi \in C^{4,\alpha}(B_x)$. Hence, the standard bootstrapping gives $\varphi \in C^{k,\alpha}(B_x)$ for any given $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Indeed, using $\varphi \in C^{4,\alpha}(B_x)$, we observe that the coefficients $g_\varphi^{\bar{j}i}$ and the right-hand side of (117) are in $C^{2,\alpha}(B_x)$, so that $\partial_j\varphi, \partial_{\bar{j}}\varphi \in C^{4,\alpha}(B_x)$. This implies φ is $C^{5,\alpha}(B_x)$, and so forth, until we finally arrive that $\varphi \in C^{k,\alpha}(B_x)$, as claimed.

In the second step, we show that $\varphi \in C^{k,\alpha}_{\varepsilon'}(M)$, i.e. that higher order derivatives of φ decay as $e^{-\varepsilon' t}$. In the same notation as in Proposition 5.28, consider the following equation on B_x:

$$F = \left(\int_0^1 g^{\bar{j}i}_{\tau\varphi}d\tau\right)\partial_i\partial_{\bar{j}}\varphi + \frac{X}{2}(\varphi) =: a^{\bar{j}i}\partial_i\partial_{\bar{j}}\varphi + b_j\partial_j\varphi.$$

Then, by local Schauder estimates, there exists a constant $C > 0$, depending on k, α, $||g_\varphi||_{C^{k,\alpha}(M)}$ and $||X||_{C^{k,\alpha}(M)}$, such that

$$||\varphi||_{C^{k+2,\alpha}(B_x)} \leq C\left(||\varphi||_{C^0(B_x)} + ||F||_{C^{k,\alpha}(B_x)}\right).$$

Since $\varphi = O(e^{-\varepsilon' t})$, $F \in C^\infty_\varepsilon(M)$ and because the Euclidean Hölder norms on B_x are uniformly equivalent to the ones induced by the restriction of g, we conclude from this equation that $\varphi \in C^{k,\alpha}_{\varepsilon'}(M)$ by the same argument used in Proposition 5.28.

Finally, it remains to show $\varphi \in C^{k,\alpha}_\varepsilon(M)$, i.e. to improve the decay rate from ε' to ε. Similarly to Proposition 5.29, consider the equation

$$F + \int_0^1\int_0^\tau |\partial\bar{\partial}\varphi|^2_{g_{\tau\varphi}}d\sigma\, d\tau = \frac{1}{2}\left(\Delta_g + X\right)(\varphi), \tag{118}$$

and deduce that

$$\left(\Delta_g + X\right)(\varphi) \in C^\infty_{\varepsilon_1}(M) \quad \text{for} \quad \varepsilon_1 = \min\{2\varepsilon', \varepsilon\}.$$

Applying Theorem 2.11, we find a unique $v \in C^\infty_{\varepsilon_1}(M)$ such that

$$(\Delta_g + X)(v) = (\Delta_g + X)(\varphi),$$

but then the maximum principle implies $\varphi = v \in C^\infty_{\varepsilon_1}(M)$. If $\varepsilon_1 < \varepsilon$, iterate this process starting from (118) a bounded number of times, and conclude that $\varphi \in C^\infty_\varepsilon(M)$, finishing the proof.

□

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SUMMARY

In this thesis we constructed new examples of gradient steady Kähler-Ricci solitons in a given Kähler class. The underlying complex manifolds that we considered are divided into two classes. First, we found new examples on vector bundles $E \to B$ over certain Kähler manifolds of non-negative Ricci curvature and studied the uniqueness of these examples. And second, we focused on crepant resolutions of certain orbifolds $(\mathbb{C} \times D)/\Gamma$ and obtained a precise characterization of the Kähler classes that admit asymptotically cylindrical solitons.