Symplectic Automorphic Forms and Kloosterman Sums

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Abstract

In this thesis, we study automorphic forms on the rank 2 symplectic group Sp(4), in the context of analytic number theory. While much of the abstract theory is described in Langlands' theory, one needs more explicit formulae for applications in analytic number theory. The thesis consists of three parts.

In the first part of the thesis, we first give explicit formulations for Sp(4) Eisenstein series. Then we compute explicit formulae for constant terms and Fourier coefficients of Sp(4) Eisenstein series, in terms of Whittaker functions.

In the second part of the thesis, we study Sp(4) Kloosterman sums, and evaluate non-trivial bounds for these sums, using a stratification argument, and p-adic stationary phase method. We also compute explicitly the Fourier coefficients of Sp(4) Poincaré series, using Kloosterman sums.

In the third part of the thesis, we construct an Sp(4) analogue of the Kuznetsov trace formulae. We also obtain explicit relations between Fourier coefficients of Sp(4) automorphic forms and Hecke eigenvalues. Using these results, and estimates of Sp(4) Kloosterman sums, we establish strong bounds for the number of automorphic forms of level q violating the Ramanujan conjecture at any given unramified place, which go beyond Sarnak's density hypothesis.

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Contents

Chapter 1

Introduction

The theory of automorphic forms has its origin in the study of modular forms. In the classical sense, a modular form for the group $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is a holomorphic function f defined on the complex upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ satisfying the transformation property

$$f(\gamma z) := f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

where k is called the weight of f. We also require that f is "holomorphic at the cusp", that is, f satisfies the growth condition $f(x+iy) \ll y^N$ for some fixed N.

An important example of modular forms is the holomorphic Eisenstein series E_{2k} of weight $2k$ for $2 \leq k \in \mathbb{Z}$, given by

$$E_{2k}(z) = \frac{1}{2\zeta(2k)} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d) \neq (0,0)}} \frac{1}{(cz+d)^{2k}}.$$

Maaß [Maa49] extended the study to functions that are not holomorphic, but only real-analytic, and introduced the notion of Maaß forms. A Maaß form for Γ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following properties:

(i) $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$;

(ii) f is an eigenfunction for the hyperbolic Laplacian $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$;

(iii) f has moderate growth at the cusp, that is, $f(x+iy) \ll y^N$ for some fixed N.

Furthermore, if f satisfies $\int_0^1 f(x+iy)dx = 0$, then f is called a Maaß cusp form.

An important example of Maaß form is the non-holomorphic Eisenstein series

$$E(z,s) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d) \neq (0,0)}} \frac{\mathrm{Im}(z)^{s+\frac{1}{2}}}{|cz+d|^{2s+1}}, \qquad \mathrm{Re}(s) > \frac{1}{2}.$$

This function is real-analytic in z, and holomorphic in s. Since

$$E(z+1,s) = E(z),$$

the Eisenstein series has a Fourier expansion

$$E(z,s) = \sum_{n \in \mathbb{Z}} a_n(y) e(nx),$$

where $z = x + iy$, and $e(x) := e^{2\pi i x}$. One may compute that the Fourier expansion of $E(z, s)$ is given by

$$E(z,s) = y^{s+\frac{1}{2}} + \frac{\Lambda(2s)}{\Lambda(2s+1)} y^{-s+\frac{1}{2}} + \frac{2\sqrt{y}}{\Lambda(2s+1)} \sum_{0 \neq n \in \mathbb{Z}} \sigma_{-2s}(n) \left|n\right|^s K_s(2\pi \left|n\right| y) e(nx),$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed zeta function, $\sigma_s(n) = \sum_{d|n} d^s$ is the divisor function, and

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}y(u+u^{-1})} u^s \frac{du}{u}$$

is the K-Bessel function. It follows from the Fourier expansion that $E(z, s)$ can be continued into a meromorphic function on $\mathbb{C}$ as a function in s.

The Eisenstein series is also of great importance in the spectral decomposition of automorphic functions. Precisely, we have the Selberg spectral decomposition [Sel56]

$$L^2(\Gamma\backslash\mathbb{H}) = \mathbb{C} \oplus L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H}) \oplus L^2_{\text{cont}}(\Gamma\backslash\mathbb{H}),$$

where $L^2_{\text{cusp}}(\Gamma\backslash\mathbb{H})$ denotes the cuspidal spectrum, spanned by Maaß cusp forms, and $L^2_{\text{cont}}(\Gamma\backslash\mathbb{H})$ denotes the continuous spectrum, spanned by Eisenstein series.

In the monumental theory of Langlands [Lan76], we have a description of Eisenstein series on adelic quotients $G(F)\backslash G(\mathbb{A})$, where G is a suitable reductive Lie group, F is a number field, and $\mathbb{A}$ is the ring of adeles of F. This then gives a spectral decomposition of the L^2-space of the locally symmetric space $\Gamma\backslash G(\mathbb{R})/K$ for a congruence subgroup Γ and a maximal compact subgroup K of the real group $G(\mathbb{R})$. The Selberg spectral decomposition then corresponds to the case where $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, $G(\mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$, and $K = \mathrm{SO}(2, \mathbb{R})$. While the spectral decomposition is known in general, its application in analytic number theory remains limited in other cases, because the constant terms and Fourier coefficients of Eisenstein series are only known explicitly for few cases, such as $\mathrm{GL}(2, \mathbb{R})$ and $\mathrm{GL}(3, \mathbb{R})$.

1.1 Symplectic Eisenstein series

While much of the theory was already worked out implicitly in the work of Langlands [Lan76, Art79], relatively little is known about the explicit formulations for Eisenstein series for $G = \mathrm{Sp}(4)$. For applications in analytic number theory, we often require explicit formulae. This applies in particular to trace formulae and relative trace formulae (à la Kuznetsov) whose use in analytic number theory is based on its explicit shape [Blo19b]. Such formulae are only worked out for few groups. Besides the classical case $\mathrm{GL}(2)$, such explicit computations have only been done for $\mathrm{GL}(3)$ by Bump, Goldfeld and others [Bum84, BFG88, Thi04, Gol06, Bal15], with hints on how to generalise to $\mathrm{GL}(n)$, and are not known for other classical groups. The group $\mathrm{Sp}(4)$ is a natural candidate as the first step for the generalisation of these computations to a group besides $\mathrm{GL}(n)$. It is worth noting that some work has been done on the exceptional group G_2 [Xio17].

Eisenstein series find many applications in number theory. Langlands [Lan76] introduced in his spectral theory the notion of constant terms along a parabolic subgroup. This generalises the notion of constant Fourier coefficient in the classical theory, and is essential to the spectral decomposition of automorphic forms. The Fourier coefficients of Eisenstein series are featured in the construction of automorphic L-functions by Langlands-Shahidi method [Sha10]. Eisenstein series are also connected to algebraic objects, such as quadratic forms [Blo20] and algebraic varieties [FMT89]. Through the construction of the Eisenstein series, we see that their Fourier coefficients feature a generalised version of exponential sums and divisor-type functions, which are worthy of investigating by their own.

For $G = \mathrm{Sp}(4)$, there are three types of Eisenstein series, corresponding to the three parabolic subgroups of $\mathrm{Sp}(4)$: those associated to the Siegel maximal parabolic subgroup, those associated to the non-Siegel maximal parabolic subgroup, and those associated to the Borel subgroup. Since the Levi factor for the maximal parabolic subgroups is $\mathrm{GL}(2)$, it is also possible to twist the corresponding Eisenstein series by classical Maaß cusp forms. However, we shall only focus on Eisenstein series with trivial twist. Such Eisenstein series correspond to the residual spectrum. The residual Eisenstein series are special in the sense that their properties can be inferred from those of the minimal Eisenstein series with relative ease.

It follows from the general theory [Lan76, MW95] that the Eisenstein series, while originally defined on an open subset of the complex space where the series converges absolutely, can be extended meromorphically to functions defined on the whole complex space.

The objective of Chapter 2 is to compute the constant terms and the Fourier coefficients of the minimal Eisenstein series $E_0(g,\nu)$, and the residual Eisenstein series $E_\alpha(g,\nu,1)$, $E_\beta(g,\nu,1)$. The precise notations for the Eisenstein series are given in Section 2.1. We outline our approach here.

We recall the definition of Eisenstein series in general. Let G be a reductive group, Γ a discrete subgroup, and $P = NM$ be a standard parabolic subgroup of G, with Levi subgroup M. Let A be the maximal torus of the identity component of its centre, which we assume to be split. Let $M' = A\backslash M$. Then we have decompositions $M = AM'$ and $P = NAM'$. Let K be a fixed maximal compact subgroup of G. By Iwasawa decomposition, we have $P = NMK = NAM'K$.

Let $\mathfrak{a}_P$ be the real Lie algebra of A, and $\mathfrak{a}_P^*$ its dual. Let $\mathfrak{a}_{P\mathbb{C}} = \mathfrak{a}_P \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{a}_{P\mathbb{C}}^* = \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}$ be the complexifications of $\mathfrak{a}_P$ and $\mathfrak{a}_P^*$ respectively. This gives a natural pairing $\langle -,- \rangle : \mathfrak{a}_{P\mathbb{C}}^* \times \mathfrak{a}_{P\mathbb{C}} \to \mathbb{C}$. There is a homomorphism $H_P : G \to \mathfrak{a}_P$, which takes $g \in G$ to $H_P(g)$, for $g \in N\exp(H_P(g))MK$. It is easily checked that this is well-defined.

Let f be an automorphic form on M. The Eisenstein series associated to the parabolic P and twist f is

$$E_P(g,\nu,f) = \sum_{\gamma \in P\cap\Gamma\backslash\Gamma} f(m_P(\gamma g)) \exp\left\langle \lambda + \rho_P, H_P(\gamma g)\right\rangle, \tag{1.1}$$

where $g \in G$, $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$, ρ_P the half-sum of positive roots of $\mathfrak{n}$, the Lie algebra of N, and $m_P : G \to M/(K \cap M)$ the projection map with respect to the decomposition $G = NMK$. We see that $E_P(g,\nu,f)$ defines a function on $\Gamma\backslash G/K$, whenever the sum converges.

To obtain explicit formulations for the Eisenstein series, it is necessary to obtain a system of representatives for the quotient $P\cap\Gamma\backslash\Gamma$. This is done for $G = \mathrm{Sp}(4)$ in Section 2.2, by introducing parameters known as Plücker coordinates, cf. [BFH90] and [Gol06, Ch. 11]. We also give a partition of the coset representatives with respect to Bruhat decomposition $G = BWB$, where B is a standard Borel subgroup of G, and W is the Weyl group of G, with each piece corresponding to a Weyl element $w \in W$. This is useful for the computation of the constant terms and the Fourier coefficients.

In Section 2.3 we compute the constant terms of the Eisenstein series. While we have explicit systems of coset representatives, and hence explicit expressions for the constant term integrals, these integrals are complicated, and it is difficult to evaluate them using elementary methods. To evaluate the integrals, we switch to the adelic side, and make use of the intertwining operators for automorphic forms [Lan76, MW95]. Through a functional equation of Langlands, we can relate the constant term integrals for different Bruhat pieces. In this way, we can express the constant terms for all Bruhat pieces using the constant terms for pieces corresponding to simple reflections in the Weyl group, which are easy to compute. In this way we obtain the constant term of the minimal Eisenstein series $E_0(g,\nu)$ along the minimal parabolic P_0. The constant term consists of 8 terms, the size of the Weyl group W of Sp(4). Deferring the notations to Chapter 2, the constant term is given as follows.

Theorem 1.1. The constant term of the minimal Eisenstein series $E_0(g,\nu)$ along the minimal parabolic subgroup P_0 is given by

$$\int_{N_0(\mathbb{Z})\backslash N_0(\mathbb{R})} E_0(ug,\nu)du = \sum_{w\in W} C_{0,w}(g,\nu),$$

where

$$\begin{aligned}
C_{0,\mathrm{id}}(g,\nu) &= y_1^{\nu_1+2}y_2^{2\nu_2-\nu_1+1},\\
C_{0,s_\alpha}(g,\nu) &= \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{2\nu_2-\nu_1+2}y_2^{\nu_1+1},\\
C_{0,s_\beta}(g,\nu) &= \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}y_1^{\nu_1+2}y_2^{\nu_1-2\nu_2+1},\\
C_{0,s_\alpha s_\beta}(g,\nu) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{2\nu_2-\nu_1+2}y_2^{-\nu_1+1},\\
C_{0,s_\beta s_\alpha}(g,\nu) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}y_1^{\nu_1-2\nu_2+2}y_2^{\nu_1+1},\\
C_{0,s_\alpha s_\beta s_\alpha}(g,\nu) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{-\nu_1+2}y_2^{2\nu_2-\nu_1+1},\\
C_{0,s_\beta s_\alpha s_\beta}(g,\nu) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}y_1^{\nu_1-2\nu_2+2}y_2^{-\nu_1+1},\\
C_{0,s_\alpha s_\beta s_\alpha s_\beta}(g,\nu) &= \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}\frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{-\nu_1+2}y_2^{\nu_1-2\nu_2+1}.
\end{aligned}$$

A more precise version of the theorem, as well as the constant terms of the minimal Eisenstein series $E_0(g,\nu)$ along other parabolic subgroups are given in Section 2.3, in Theorems 2.21, 2.23 and 2.24.

Since the residual Eisenstein series $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$ are residues of the minimal Eisenstein series $E_0(g,\nu)$, one can obtain their constant terms simply by taking the residues of the constant term of $E_0(g,\nu)$. These constant terms are given in Corollaries 2.25 to 2.30.

In Section 2.5 we compute the Fourier coefficients of the Eisenstein series, in terms of Whittaker functions. To state the result, we need to evaluate the Dirichlet series for a Sp(4) Ramanujan sum. This is treated separately in Section 2.4. The Fourier coefficients of $E_0(g,\nu)$ is given in Theorem 2.36. By taking residues, we obtain the Fourier coefficients of $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$, in Corollaries 2.37 and 2.38.

1.2 Symplectic Kloosterman sums and Poincaré series

We first give a brief review of classical Kloosterman sums. A Kloosterman sum is given by

$$S(m,n,q) := \sum_{\substack{x,y\in\mathbb{Z}/q\mathbb{Z}\\ xy\equiv 1 (\mathrm{mod}\ q)}} e\left(\frac{mx+ny}{q}\right).$$

Such sums naturally appear in the Fourier expansion of GL(2) Poincaré series

$$P_m(z;\nu) = \sum_{\gamma \in P^2 \cap \Gamma \backslash \Gamma} \operatorname{Im}(\gamma z)^{\nu} e\left(m(\gamma z)\right), \tag{1.2}$$

where $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $P^2 = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \right\} \subseteq \mathrm{SL}(2,\mathbb{R})$, $z \in \mathbb{H}$, $m > 0$, $\operatorname{Re}(\nu) > 1$.

To look for generalisations of Kloosterman sums, it is helpful to reformulate the definition of Kloosterman sums in the context of automorphic forms. We start by noting that Kloosterman sums satisfies a multiplicativity relation. Let $q = q_1 q_2$, with $(q_1, q_2) = 1$. Choose r_1, r_2 such that $r_1 q_1 \equiv 1 \pmod{q_2}$, and $r_2 q_2 \equiv 1 \pmod{q_1}$. Then

$$S(m,n;q) = S(r_2 m, r_2 n; q_1) S(r_1 m, r_1 n; q_2).$$

So it is sufficient to consider the case where $q = p^k$ is a prime power.

Let

$$T := \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Q}) \right\}, \qquad U := \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Q}) \right\}$$

be the standard torus and the standard unipotent subgroup of $\mathrm{SL}(2,\mathbb{Z})$ respectively. We denote by N the normaliser of T in $\mathrm{SL}(2,\mathbb{Q})$. Then the Weyl group of $\mathrm{SL}(2,\mathbb{Q})$ is given by $W := N/T$. The Weyl group $W = \{\mathrm{id}, w_0\}$ consists of two elements, where the non-identity element w_0 is represented by the matrix

$$w_0 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}.$$

Let $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. We consider a Bruhat decomposition $\gamma = uwtu'$ of γ, where $u, u' \in U(\mathbb{Q})$, $w \in W$, and $t \in T$. Let $C(p^k)$ denote the set of $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ with Bruhat decomposition $\gamma = u w_0 t_{p^k} u'$, where

$$t_{p^k} = \begin{pmatrix} p^k & \\ & p^{-k} \end{pmatrix} \in T,$$

and let $X(p^k) = U(\mathbb{Z}) \backslash C(p^k) / U(\mathbb{Z})$. Now we give an explicit characterisation of $X(p^k)$. An element $\gamma \in X(p^k)$ has the form

$$\gamma = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} p^k & \\ & p^{-k} \end{pmatrix} \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} = \begin{pmatrix} xp^k & xyp^k - p^{-k} \\ p^k & yp^k \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}).$$

The resultant matrix having integral entries implies $x, y \in p^{-k}\mathbb{Z}/\mathbb{Z}$, and $xyp^k - p^{-k} \in \mathbb{Z}$.

For $m \in \mathbb{Z}$, let $\chi_m : \mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times$ be the character defined by $x \mapsto e(mx)$. We define projection maps

$$\begin{aligned} u :& X(p^k) \to U(\mathbb{Z}) \backslash U(\mathbb{Q}), \\ u' :& X(p^k) \to U(\mathbb{Q}) / U(\mathbb{Z}) \end{aligned}$$

by the relation $\gamma = u(\gamma) w_0 t_{p^k} u'(\gamma)$ for $\gamma \in X(p^k)$. We consider for $m, n \in \mathbb{Z}$ the character sum

$$\mathrm{Kl}(p^k, \chi_m, \chi_n) := \sum_{\gamma \in X(p^k)} \chi_m(u(\gamma)) \chi_n(u'(\gamma)).$$

Using the characterisation above, we see that

$$\mathrm{Kl}(p^k, \chi_m, \chi_n) = \sum_{\substack{x,y \in p^{-k}\mathbb{Z}/\mathbb{Z} \\ xyp^k - p^{-k} \in \mathbb{Z}}} \chi_m(x) \chi_n(y) = \sum_{\substack{x,y \in \mathbb{Z}/p^k\mathbb{Z} \\ xy \equiv 1 (\mathrm{mod}\ p^k)}} e\left(\frac{mx + ny}{p^k} \right) = S(m,n;p^k)$$

returns a Kloosterman sum.

It is apparent from this formulation that one can construct generalised Kloosterman sums over arbitrary reductive groups; in particular, we can define Kloosterman sums for every element in the Weyl group. However, in the classical $\mathrm{SL}(2,\mathbb{Z})$ case, the Kloosterman sums for $w = \mathrm{id}$ is trivial, so we only see one kind of Kloosterman sums, corresponding to $w = w_0$ the non-trivial Weyl element, in the classical theory.

In [BFG88], Bump, Friedberg and Goldfeld introduced $\mathrm{GL}(r)$ Poincaré series for $r \geq 2$, and gave a generalisation of Kloosterman sums to $\mathrm{GL}(3)$. The notion of Kloosterman sums was then generalised to $\mathrm{GL}(r)$ for $r \geq 2$ by Friedberg [Fri87], and then to arbitrary simply connected Chevalley groups by Dąbrowski [Dąb93].

By methods in algebraic geometry, Weil [Wei48] obtained a bound for $\mathrm{GL}(2)$ Kloosterman sums

$$|S(m,n;q)| \ll \tau(q)\,(m,n,q)^{1/2}\,q^{1/2}, \tag{1.3}$$

where τ denotes the divisor function. However, it remains a major open problem to give non-trivial bounds for Kloosterman sums in general, and currently only a small set of examples can be treated. Bounds for $\mathrm{GL}(3)$ Kloosterman sums were first obtained by Larsen [BFG88, Appendix] and Stevens [Ste87], and were improved by Dąbrowski and Fisher [DF97]. Bounds for some $\mathrm{GL}(4)$ Kloosterman sums were given by Huang [GSW19, Appendix]. Friedberg [Fri87] generalised the results to $\mathrm{GL}(r)$ Kloosterman sums attached to certain Weyl elements. On reductive groups, Dąbrowski and Reeder [DR98] gave the size of Kloosterman sets, establishing a trivial bound for Kloosterman sums on reductive groups.

Poincaré series can be considered as a generalisation of Eisenstein series, by introducing an extra twist by a character; this is apparent from the definition (1.2) of the classical Poincaré series. And as in the classical case, the Fourier coefficients of symplectic Poincaré series features symplectic Kloosterman sums. Therefore, having a good bound for the Kloosterman sums leads to information on the Poincaré series.

Poincaré series play an important role in number theory. Beside being examples of automorphic functions, they are also involved in various trace formulae, the most prominent of which being the Petersson/Kuznetsov trace formulae, which have great importance in the context of analytic number theory [Blo19b]. Indeed, in Chapter 4 we obtain a density theorem for symplectic automorphic forms using a Kuznetsov-type trace formula.

The main objective of Chapter 3 is to prove non-trivial bounds for $\mathrm{Sp}(4)$ Kloosterman sums. Let $N(\mathbb{Q})$ be the set of rational matrices which normalise the diagonal torus T of the symplectic group $G = \mathrm{Sp}(4)$ (see Section 3.1 for details). For $w \in W$, and $c_1, c_2 \in \mathbb{N}$, we set

$$n_w(c_1,c_2) := \begin{pmatrix} 1/c_1 & & & \\ & c_1/c_2 & & \\ & & c_1 & \\ & & & c_2/c_1 \end{pmatrix} w \in N(\mathbb{Q}).$$

Then we have the following theorem.

Theorem 1.2. Let $c_1, c_2 \in \mathbb{N}$. Then we have

$$\begin{aligned}
\mathrm{Kl}\left(n_{\mathrm{id}}(c_1,c_2),\psi,\psi'\right) &= 1 && \text{if } c_1=c_2=1,\\
\left|\mathrm{Kl}\left(n_{s_\alpha}(c_1,c_2),\psi,\psi'\right)\right| &\ll_{\psi,\psi',\varepsilon} c_1^{1/2+\varepsilon} && \text{if } c_2=1,\\
\left|\mathrm{Kl}\left(n_{s_\beta}(c_1,c_2),\psi,\psi'\right)\right| &\ll_{\psi,\psi',\varepsilon} c_2^{1/2+\varepsilon} && \text{if } c_1=1,\\
\left|\mathrm{Kl}\left(n_{s_\alpha s_\beta}(c_1,c_2),\psi,\psi'\right)\right| &\ll_{\psi,\psi',\varepsilon} (c_2^2,c_1)(c_1c_2)^{\varepsilon} && \text{if } c_2 \mid c_1,\\
\left|\mathrm{Kl}\left(n_{s_\beta s_\alpha}(c_1,c_2),\psi,\psi'\right)\right| &\ll_{\psi,\psi',\varepsilon} (c_1^3,c_2)(c_1c_2)^{\varepsilon} && \text{if } c_1^2 \mid c_2,\\
\left|\mathrm{Kl}\left(n_{s_\alpha s_\beta s_\alpha}(c_1,c_2),\psi,\psi'\right)\right| &\ll_{\psi,\psi',\varepsilon} (c_1,c_2)(c_1c_2)^{1/3+\varepsilon} && \text{if } c_2 \mid c_1^2,\\
\left|\mathrm{Kl}\left(n_{s_\beta s_\alpha s_\beta}(c_1,c_2),\psi,\psi'\right)\right| &\ll_{\psi,\psi',\varepsilon} (c_1^2,c_2)c_1^{-1/2}c_2^{1/2}(c_1c_2)^{\varepsilon} && \text{if } c_1 \mid c_2,\\
\left|\mathrm{Kl}\left(n_{w_0}(c_1,c_2),\psi,\psi'\right)\right| &\ll_{\psi,\psi',\varepsilon} (c_1,c_2)^{1/2}c_1^{1/2}c_2^{3/4}(c_1c_2)^{\varepsilon},
\end{aligned}$$

and the Kloosterman sum $\mathrm{Kl}\left(n_w(c_1,c_2),\psi,\psi'\right)$ vanishes if the condition on the right is not satisfied.

For comparison, the trivial bound of Dąbrowski and Reeder [DR98] says

$$\left|\mathrm{Kl}\left(n_w(c_1,c_2),\psi,\psi'\right)\right| \le c_1c_2,$$

and we can check explicitly that the bounds given above are always non-trivial.

We outline the content of Chapter 3 below. In Section 3.1, we follow the notations of Stevens [Ste87] and Dąbrowski [Dąb93], and define Kloosterman sums for $\mathrm{Sp}(2n)$ in general. While the classical Kloosterman sums are defined globally (over $\mathbb{Q}$), and multiplicativity is proven as a theorem, we define the Kloosterman sums locally (over $\mathbb{Q}_p$), and define global Kloosterman sums as the product of local Kloosterman sums for all primes p. So, under this construction, multiplicativity holds by definition. We also make explicit formulations for local $\mathrm{Sp}(4)$ Kloosterman sums $\mathrm{Kl}_p(n,\psi,\psi')$, using the coset representatives obtained in Section 2.2.

In Section 3.2, we introduce a decomposition for $\mathrm{Sp}(2n)$ Kloosterman sums. This generalises the treatment in [Ste87] for $\mathrm{GL}(n)$ Kloosterman sums. Each piece in the decomposition is an exponential sum of classical Kloosterman sums, or a product of classical Kloosterman sums. Then we can bound each piece individually.

However, in general it is not sufficient to just use the classical bound (1.3) to obtain non-trivial bounds for $\mathrm{Sp}(2n)$ Kloosterman sums. Briefly, a local Kloosterman sum is an exponential sum of the form

$$\sum_{x\in S} e\left(\frac{f(x)}{p^k}\right).$$

for some $k \in \mathbb{N}$. To obtain non-trivial bounds for $\mathrm{Sp}(4)$ Kloosterman sums, we adopt two different approaches, depending on the size of k:

(i) when $k \ge 2$, we use the p-adic stationary phase method [DF97];

(ii) when $k = 1$, the stationary phase method fails, and one has to resort to algebro-geometric arguments. Known results of Deligne [Del77], and Adolphson and Sperber [AS89] are manipulated to give the bounds we need.

In Section 3.3, using these two approaches, we obtain power-saving bounds for local Kloosterman sums for all Weyl elements, given in Theorems 3.9 to 3.13. The bounds for global Kloosterman sums are then obtained by combining the local Kloosterman sums. The end results are given in Theorems 3.14 to 3.18, in Section 3.4. These theorems entail Theorem 1.2, and also describe the behaviour of the Kloosterman sums in relation to the characters ψ, ψ'.

Finally, in Section 3.5, we give an introduction to symplectic Poincaré series, and relate the symplectic Kloosterman sums to the Fourier coefficients of the Poincaré series. We also give explicit expressions for the Fourier coefficients of Sp(4) Poincaré series.

1.3 Kuznetsov trace formula and density theorems

We introduce the problem of density estimates in the context of automorphic forms. We first recall the Ramanujan conjecture. In the context of automorphic forms, the conjecture says that cuspidal automorphic representations of the group $\mathrm{GL}(n)$ over a number field F are tempered. However, this conjecture in its full generality is far out of reach, even for GL(2), the simplest case. Instead, we can consider approximations to the conjecture as a substitute, and try to bound the number of members in a given family of automorphic representations violating the conjecture relative to the amount by which they violate the conjecture. Such results are known collectively as density theorems. Clearly, these density results do not prove the conjecture, but they are often sufficient in applications.

On the other hand, it is natural to consider the generalisation of the Ramanujan conjecture to reductive groups other than $\mathrm{GL}(n)$. It is however well-known that the naive generalisation of the Ramanujan conjecture is false for Sp(4), because of the presence of Saito-Kurokawa lifts, which are not tempered. This is not the end of the investigation, however. It is also known that Saito-Kurokawa lifts are not generic, i.e. do not have a Whittaker model. So it is natural to rephrase the question, and ask whether generic cuspidal automorphic representations of Sp(4) are tempered. This problem is also open, and currently far out of reach. Density theorems in this context have numerous applications as well.

Because of the importance of density theorems, they have attracted much attention in history, and many strong density results are known for various automorphic families on GL(2) with different settings [Hux86, Sar87, Iwa90, BM98, BM03, BBR14]. Via Kuznetsov-type trace formulae on GL(3), strong density results on GL(3) were obtained in [Blo13, BBR14, BBM17]. Blomer [Blo19a] further generalised the technique to obtain results on $\mathrm{GL}(n)$ beyond Sarnak's density hypothesis. However, relatively little is known for general reductive groups. Finis and Matz [FM19] give as by-products some density results for the family of Maaß forms of Laplace eigenvalue up to a height T and fixed level. However, these bounds are weak, and even the "convexity bound" cannot be obtained.

We describe the problem of density estimates in detail, for $G = \mathrm{Sp}(4)$. Fix a place v of $\mathbb{Q}$. For an automorphic representation $\pi = \bigotimes_v \pi_v$ of Sp(4), we denote by $\mu_\pi(v) = (\mu_\pi(v,1), \mu_\pi(v,2))$ its local Langlands spectral parameter, which we define precisely in Section 4.1. We write

$$\sigma_\pi(v) = \max\left\{|\mathrm{Re}\,\mu_\pi(v,1)|, |\mathrm{Re}\,\mu_\pi(v,2)|\right\}. \tag{1.4}$$

The representation π is tempered at v if $\sigma_\pi(v) = 0$, and the size of $\sigma_\pi(v)$ gives a measure on how far π is from being tempered at v. An example of a non-tempered representation is the trivial representation, which satisfies $\sigma_{\mathrm{triv}}(v) = 3/2$ for all places v.

For a finite family $\mathcal{F}$ of automorphic representations of Sp(4) and $\sigma \geq 0$ we define

$$N_v(\sigma, \mathcal{F}) = |\{\pi \in \mathcal{F} \mid \sigma_\pi(v) \geq \sigma\}|.$$

Trivially, we have $N_v(0, \mathcal{F}) = |\mathcal{F}|$, and if $\mathcal{F}$ contains the trivial representation, then we have $N_v(3/2, \mathcal{F}) \geq 1$. One may hope to interpolate linearly between the two extreme cases, and obtain a bound of the form

$$N_v(\sigma, \mathcal{F}) \ll_{v,\varepsilon} |\mathcal{F}|^{1-\frac{\sigma}{a}+\varepsilon} \tag{1.5}$$

with $a = 3/2$. In the context of groups G of real rank 1, for the principal congruence subgroup $\Gamma(q) = \{\gamma \in G(\mathbb{Z}) \mid \gamma = \mathrm{id} \pmod q\}$ and $v = \infty$, this is known as Sarnak's density hypothesis [Sar90].

In this chapter, we consider the family $\mathcal{F}_I(q)$ of generic cuspidal automorphic representations for the group $\Gamma_0(q) \subseteq \mathrm{Sp}(4, \mathbb{Z})$ for a large prime q, and Laplace eigenvalue λ in a fixed interval I. A simple application of Weyl's law shows that $|\mathcal{F}_I(q)| \asymp_I q^3$ when the size of I is sufficiently large, noting that the contribution from the continuous spectrum has size $O(q)$. The main result of the chapter is that for the family $\mathcal{F}_I(q)$ and any place $v \neq q$ of $\mathbb{Q}$, we go beyond the density hypothesis and obtain obtain a density estimate with $a = 3/4$, which is halfway between the density hypothesis and the Ramanujan conjecture.

Theorem 1.3. Let q be a prime, and v a place of $\mathbb{Q}$ different from q, $I \subseteq [0, \infty)$ a fixed interval, $\varepsilon > 0$, and $\sigma \geq 0$. Then

$$N_v(\sigma, \mathcal{F}_I(q)) \ll_{I,v,n,\varepsilon} q^{3-4\sigma+\varepsilon}.$$

The proof is based on a careful analysis on the arithmetic side of the Kuznetsov formula, and on the spectral side through a relation of Fourier coefficients between automorphic forms and Hecke eigenvalues. Let $\lambda(m, \pi)$ be the Hecke eigenvalue of $\pi \in \mathcal{F}_I(q)$ for the m-th standard Hecke operator $T(m)$. It is convenient to adopt the normalisation $\lambda'(m, \pi) := m^{-3/2}\lambda(m, \pi)$.

Theorem 1.4. Keep the notations above. Let $m \in \mathbb{N}$ be coprime to q and $Z \geq 1$. Then

$$\sum_{\pi \in \mathcal{F}_I(q)} \left|\lambda'(m, \pi)\right|^2 Z^{2\sigma_\pi(\infty)} \ll_{I,\varepsilon} q^{3+\varepsilon}$$

uniformly in $mZ \ll q^2$ for a sufficiently small implied constant depending on I.

Let us roughly sketch the proof of Theorem 1.4. We denote by $\{\varpi\}$ an orthonormal basis of right K-invariant automorphic forms for $\Gamma_0(q)$, cuspidal or Eisenstein series, where K is a maximal compact subgroup of $\mathrm{Sp}(4, \mathbb{R})$. We denote by $\int_{(q)} d\varpi$ the integral over the complete spectrum of $L^2(\Gamma_0(q) \backslash \mathrm{Sp}(4, \mathbb{R})/K)$. Very roughly, the Kuznetsov formula takes the form

$$\int_{(q)} |A_\varpi(M)|^2 Z^{2\sigma_\pi(\infty)} \delta_{\lambda_\varpi \in I} d\varpi \text{ “} \approx \text{” } 1 + \sum_{\mathrm{id} \neq w \in W} \sum_{c_1, c_2} \frac{\mathrm{Kl}_{q,w}(c, M, M)}{c_1 c_2}, \tag{1.6}$$

where $M = (1, m) \in \mathbb{Z}^2$, $A_\varpi(M)$ is the M-th Fourier coefficient of ϖ, defined in (4.8), W is the Weyl group of $\mathrm{Sp}(4)$, and $\mathrm{Kl}_{q,w}(c, M, M)$ is a generalised Kloosterman sum of level q, defined in (4.22), associated with the Weyl element w, and moduli $c = (c_1, c_2)$. Note that the Kuznetsov formula only extracts the generic spectrum.

However, the situation here is very different from $\mathrm{GL}(n)$ case found in [Blo19a]. In the symplectic case, there are no simple relations between the Fourier coefficients $A_\varpi(M)$ of a cuspidal newform ϖ and Hecke eigenvalues $\lambda'(m, \pi)$ of the corresponding automorphic representation (i.e. $\varpi \in V_\pi$). This is in stark contrast with the $\mathrm{GL}(n)$ case, where the Fourier coefficients and Hecke eigenvalues are proportional [Gol06, Theorem 9.3.11]. It is because of this obstacle that the Kuznetsov formula is not yet a standard tool for the group $\mathrm{GSp}(4)$, and the present paper seems to be the first application of the Kuznetsov formula that is seen in action for a group other than $\mathrm{GL}(n)$.

While the Fourier coefficients in principle contain the information on Hecke eigenvalues, it is not obvious how to extract it. A detailed analysis of the relations between them is found in Section 4.4. In Theorem 4.10 we establish a recursive formula of $\lambda(p^r, \pi)$ in terms of Fourier coefficients.

Using Theorem 4.10, we deduce from Lemma 4.12 that for a prime $p \nmid q$ and $r \in \mathbb{N}$, the size of Fourier coefficients $A_\varpi(1, p^r)$ of an L^2-normalised generic cuspidal form ϖ is often as big as $q^{-3/2-\varepsilon} p^{r\sigma_\pi(p)}$. Through this relation, we are able to use the Kuznetsov formula to derive information on $\sigma_\pi(p)$ from an analysis of the Kloosterman sums. Meanwhile, the factor $Z^{2\sigma_\pi(\infty)}$ deals with the infinite place, so the test function $|A_\varpi(M)|^2 Z^{2\sigma_\pi(\infty)}$ treats the finite places and the infinite place essentially on the same footing.

When $mZ \ll q$, the Kloosterman sums associated to non-trivial Weyl elements are empty, hence the off-diagonal terms vanish completely. We will use this observation to prove Theorem 1.5 below. To obtain stronger density results, we have to deal with the Kloosterman sums appearing in the off-diagonal terms, and improve the trivial bound $|S_{q,w}(c, M, N)| \leq c_1 c_2$. In our scenario, the Kloosterman sums we need can be evaluated explicitly, and there is no need to rely on the general bounds in Chapter 3.

Now we give applications of Theorem 1.4, for a large sieve inequality analogous to the GL(n) case [Blo19a].

Theorem 1.5. Let q be prime and $\{\alpha(m)\}_{m\in\mathbb{N}}$ any sequence of complex numbers. Then

$$\sum_{\pi\in\mathcal{F}_I(q)} \Big|\sum_{m\leq x} \alpha(m)\lambda'(m,\pi)\Big|^2 \ll_{I,\varepsilon} q^{3+\varepsilon} \sum_{m\leq x} |\alpha(m)|^2$$

uniformly in $x \ll q$ for a sufficiently small implied constant depending on I.

As a corollary, we establish a bound for the second moment of spinor L-functions on the critical line. Precisely, let $L(s,\pi)$ be the spinor L-function associated to π, normalised such that its critical strip is $0 < \operatorname{Re} s < 1$.

Corollary 1.6. For q prime and $t \in \mathbb{R}$, we have

$$\sum_{\pi\in\mathcal{F}_I(q)} |L(1/2+it,\pi)|^2 \ll_{I,t,\varepsilon} q^{3+\varepsilon}.$$

Finally, in the appendix (Section 4.8), we outline an algorithm for computing arbitrary Fourier coefficients of a cuspidal form in terms of its Hecke eigenvalues. While this is not needed for the proof of the theorems, such results serve an independent interest in number theory, in laying the groundwork for further applications of the Kuznetsov formula on Sp(4), as well as Fourier analysis of automorphic forms on Sp(4) in general.

Chapter 2

Symplectic Eisenstein series

2.1 The setup

Let $G = \mathrm{Sp}(4, \mathbb{R})$ be the real symplectic group of degree 2, namely

$$G = \mathrm{Sp}(4, \mathbb{R}) = \left\{ g \in \mathrm{GL}(4, \mathbb{R}) \;\middle|\; g^T \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where g^T denotes the matrix transpose of g as usual. Let T and U be a maximal split torus and a maximal unipotent subgroup of G respectively, defined as follows:

$$T = \left\{ \mathrm{diag}\left(y_1, y_2, y_1^{-1}, y_2^{-1}\right) \in G \right\},$$
$$U = \left\{ \begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ & 1 & n_4 & n_5 \\ & & 1 & \\ & & -n_1 & 1 \end{pmatrix} \in G \;\middle|\; n_3 = n_1 n_5 + n_4 \right\}.$$

Then $B = UT$ is a Borel subgroup of G. We also define

$$T^+ = \left\{ \mathrm{diag}\left(y_1, y_2, y_1^{-1}, y_2^{-1}\right) \in G \mid y_1, y_2 > 0 \right\} \subseteq T,$$
$$V = \left\{ \mathrm{diag}\left(\varepsilon_1, \varepsilon_2, \varepsilon_1^{-1}, \varepsilon_2^{-1}\right) \mid \varepsilon_1, \varepsilon_2 = \pm 1 \right\} \subseteq G.$$

Let $X(T)$ and $X^*(T)$ be the character group and the cocharacter group of T respectively, with a natural pairing $\langle -, - \rangle : X(T) \times X^*(T) \mapsto \mathbb{Z}$. Let $\alpha, \beta \in X(T)$ such that $\alpha\left(\mathrm{diag}\left(y_1, y_2, y_1^{-1}, y_2^{-1}\right)\right) = y_1 y_2^{-1}$ and $\beta\left(\mathrm{diag}\left(y_1, y_2, y_1^{-1}, y_2^{-1}\right)\right) = y_2^2$. Then $\Delta = \Delta(T, G) = \{\alpha, \beta\}$ is a set of simple roots, and $R^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$ is a set of positive roots with respect to (B, T). We denote by s_α and s_β the simple reflections in the hyperplane orthogonal to α and β respectively. Then the Weyl group $W = W(T, G)$ is given by

$$W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta, s_\alpha s_\beta s_\alpha s_\beta\}.$$

We often write $w_0 := s_\alpha s_\beta s_\alpha s_\beta$ for the long Weyl element. The generators s_α and s_β can be represented by matrices

$$s_\alpha = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}, \quad s_\beta = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}. \tag{2.1}$$

Definition 2.1. A parabolic subgroup of G is a closed subgroup P such that G/P is a projective variety. It is known that a parabolic subgroup contains a Borel subgroup [Bor97, Corollary 11.2]. We say P is stardard if $P \supseteq B = UT$.

Let P be a standard parabolic subgroup, and N the unipotent radical of P. The projection $P \to N\backslash P$ splits, giving a reductive subgroup M of P such that $P = NM$. A splitting M is called a Levi subgroup of P, and the decomposition $P = NM$ is called a Levi decomposition. If we fix a maximal torus $T \subseteq P$, then the condition $M \supseteq T$ determines M uniquely. There is a bijective correspondence between standard parabolic subgroups of G and subsets of $\Delta(T,G)$, the simple roots of G [Sha10, Chapter 1.2]. Let $P = MN$ be a standard parabolic subgroup of G. Then P corresponds to $\Delta_M = \Delta(T,M)$, the set of simple roots of M with respect to T, which is a subset of $\Delta(T,G)$.

For $G = \mathrm{Sp}(4,\mathbb{R})$, we have standard parabolic subgroups P_0, P_α, P_β, corresponding to the subsets $\emptyset$, $\{\alpha\}$, $\{\beta\}$ of Δ respectively. Explicitly, the standard parabolic subgroups of G are given by

$$B = P_0 = \left\{\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & & * & * \end{pmatrix}\right\} \cap G, \quad \text{(minimal parabolic subgroup)}$$

$$P_\alpha = \left\{\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}\right\} \cap G, \quad \text{(Siegel parabolic subgroup)}$$

$$P_\beta = \left\{\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & \\ & * & * & * \end{pmatrix}\right\} \cap G. \quad \text{(non-Siegel maximal parabolic subgroup)}$$

The Levi decompositions $P_j = N_j M_j$, $j \in \{0,\alpha,\beta\}$ are given by

$$N_0 = \left\{\begin{pmatrix} 1 & n_1 & n_2 & n_1n_5+n_4 \\ & 1 & n_4 & n_5 \\ & & 1 & \\ & & -n_1 & 1 \end{pmatrix}\right\}, \quad M_0\left\{\mathrm{diag}\left(y_1,y_2,y_1^{-1},y_2^{-1}\right) \in G \;\middle|\; y_1,y_2 \in \mathbb{R}^\times\right\},$$

$$N_\alpha = \left\{\begin{pmatrix} I_2 & S \\ & I_2 \end{pmatrix} \;\middle|\; S^T = S\right\}, \quad M_\alpha = \left\{\begin{pmatrix} A & \\ & (A^{-1})^T \end{pmatrix} \;\middle|\; A \in \mathrm{GL}_2(\mathbb{R})\right\},$$

$$N_\beta = \left\{\begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ & 1 & n_3 & \\ & & 1 & \\ & & -n_1 & 1 \end{pmatrix}\right\}, \quad M_\beta = \left\{\begin{pmatrix} y_1 & & & \\ & a & & b \\ & & y_1^{-1} & \\ & c & & d \end{pmatrix} \;\middle|\; y_1 \in \mathbb{R}^\times, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})\right\}.$$

Let K be the maximal compact subgroup of G given by

$$K = \left\{\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \;\middle|\; A + Bi \in U(2)\right\}.$$

By Iwasawa decomposition, elements in G/K can be represented by matrices of the form

$$g = \begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ & 1 & n_4 & n_5 \\ & & 1 & \\ & & -n_1 & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & \\ & y_2 & & \\ & & y_1^{-1} & \\ & & & y_2^{-1} \end{pmatrix} \in UT^+, \tag{2.2}$$

with $n_3 = n_1 n_5 + n_4$. So we may assume that y_1, y_2 are positive.

We now give explicit characterisations for Eisenstein series for parabolic subgroups of G, using the general definition in (1.1).

Notation. For symbols indexed with a parabolic subgroup P, we often replace the parabolic subgroup with the index of the parabolic subgroup. So we write E_0 for E_{P_0}, ρ_α for ρ_{P_α}, and so on.

For the minimal parabolic subgroup P_0, the automorphic form f is a constant function. Parametrising $\mathfrak{a}_{0\mathbb{C}}^*$ by $\nu_1\alpha + \nu_2\beta$ for $\nu_1, \nu_2 \in \mathbb{C}$, we have $\rho_0 = (2, 3/2)$. So the minimal Eisenstein series is given by

$$E_0(g, \nu) = \sum_{\gamma \in P_0 \cap \Gamma \backslash \Gamma} I_0(\gamma g, \nu),$$

where $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, and $I_0(g, \nu) = y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1}$.

For the Siegel parabolic subgroup P_α, an automorphic form f on M_α is simply an automorphic function on $\mathrm{GL}(2, \mathbb{R})$. Parametrising $\mathfrak{a}_{\alpha\mathbb{C}}^*$ by $\nu(\alpha + \beta)$ for $\nu \in \mathbb{C}$, we have $\rho_\alpha = 3/2$. So the Siegel Eisenstein series is given by

$$E_\alpha(g, \nu, f) = \sum_{\gamma \in P_\alpha \cap \Gamma \backslash \Gamma} f(m_\alpha(\gamma g)) I_\alpha(\gamma g, \nu),$$

where $\nu \in \mathbb{C}$, and $I_\alpha(g, \nu) = (y_1 y_2)^{\nu+3/2}$.

For the non-Siegel maximal parabolic subgroup P_β, an automorphic form f on M_β is also an automorphic function on $\mathrm{GL}(2, \mathbb{R})$. Parametrising $\mathfrak{a}_{\beta\mathbb{C}}^*$ by $\nu(\alpha + \beta/2)$ for $\nu \in \mathbb{C}$, we have $\rho_\beta = 2$. So the non-Siegel Eisenstein series is given by

$$E_\beta(g, \nu, f) = \sum_{\gamma \in P_\beta \cap \Gamma \backslash \Gamma} f(m_\beta(\gamma g)) I_\beta(\gamma g, \nu),$$

where $\nu \in \mathbb{C}$, and $I_\beta(g, \nu) = y_1^{\nu+2}$.

2.2 Coset representatives

The Eisenstein series E_P is defined as a sum over $P \cap \Gamma \backslash \Gamma$. Hence, for explicit computations, we need explicit characterisations of the coset representatives for $P \cap \Gamma \backslash \Gamma$.

2.2.1 Minimal parabolic

Let P_0 be the standard minimal parabolic subgroup of G. We denote by $U = U_0 \subseteq P_0$ the subgroup of unipotent matrices, and $\Gamma_0 = U \cap \Gamma$. We compute the coset representatives of $U \backslash G$ and $\Gamma_0 \backslash \Gamma$. Note that we have $P_0 \cap \Gamma \backslash \Gamma = (V \cdot \Gamma_0) \backslash \Gamma$.

Let

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in G.$$

We define the following quantities, known as Plücker coordinates, associated to a:

$$v_1 = a_{31}, \quad v_2 = a_{32}, \quad v_3 = a_{33}, \quad v_4 = a_{34},$$

$$v_{12} = a_{31}a_{42} - a_{32}a_{41}, \quad v_{13} = a_{31}a_{43} - a_{33}a_{41}, \quad v_{14} = a_{31}a_{44} - a_{34}a_{41},$$
$$v_{23} = a_{32}a_{43} - a_{33}a_{42}, \quad v_{24} = a_{32}a_{44} - a_{34}a_{42}, \quad v_{34} = a_{33}a_{44} - a_{34}a_{43}.$$

It is well-known that these quantities are invariant under left action by U. The following relations come immediately from the definition:

$$\begin{aligned} v_1v_{23} - v_2v_{13} + v_3v_{12} = 0, \quad v_1v_{24} - v_2v_{14} + v_4v_{12} = 0, \\ v_1v_{34} - v_3v_{14} + v_4v_{13} = 0, \quad v_2v_{34} - v_3v_{24} + v_4v_{23} = 0. \end{aligned} \tag{2.3}$$

And symplecticity implies

$$v_{13} + v_{24} = 0. \tag{2.4}$$

Define

$$V_0 = \left\{ v = (v_1, \cdots, v_4, v_{12}, \cdots, v_{34}) \in \mathbb{R}^{10} \;\middle|\; v \text{ satisfies (2.3) and (2.4)} \right\}. \tag{2.5}$$

Proposition 2.2. Via the Plücker coordinates, there is a bijection between $U\backslash G$ and $V_0\backslash\{0\}$.

Proof. As the coordinates are invariant under left action by U, the map $U\backslash G \to V_0\backslash\{0\}$ is well-defined.

Now we show injectivity. Suppose $a = (a_{ij}), b = (b_{ij}) \in G$ have the same Plücker coordinates. We want to show that there exists $\gamma \in U$ such that $\gamma a = b$. Fix the following parameterisation of γ:

$$\gamma = \begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ & 1 & n_4 & n_5 \\ & & 1 & \\ & & -n_1 & 1 \end{pmatrix} \in U,$$

subject to the condition $n_3 = n_1n_5 + n_4$.

Firstly, we show that there exists $n_1 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & \\ -n_1 & 1 \end{pmatrix} \begin{pmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}. \tag{2.6}$$

Clearly, we have

$$a_{3j} = v_j = b_{3j}, \quad j \in \{1, 2, 3, 4\}.$$

By permuting the columns, we may assume without loss of generality that $a_{31} \neq 0$. By comparing secondary Plücker coordinates we obtain

$$a_{31}(a_{4j} - b_{4j}) = a_{3j}(a_{41} - b_{41}), \quad j \in \{2, 3, 4\}. \tag{2.7}$$

Then we solve $n_1 = (a_{41} - b_{41})/a_{31}$. The relations (2.7) then imply (2.6).

Again by permuting columns, we may assume $v_{12} \neq 0$. So the vectors (a_{31}, a_{32}) and (a_{41}, a_{42}) are linearly independent, and we can find n_4, n_5 such that

$$\begin{pmatrix} 1 & n_4 & n_5 \end{pmatrix} \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} b_{21} & b_{22} \end{pmatrix}.$$

By symplecticity of b we have

$$b_{21}b_{33} + b_{22}b_{34} = b_{23}b_{31} + b_{24}b_{32}, \quad b_{21}b_{43} + b_{22}b_{44} = b_{23}b_{41} + b_{24}b_{42} + 1,$$

from which we solve

$$\begin{aligned}
b_{23} &= \frac{b_{21}b_{33}b_{42}+b_{22}b_{34}b_{42}-b_{21}b_{32}b_{43}-b_{22}b_{32}b_{44}+b_{32}}{b_{31}b_{42}-b_{41}b_{32}}, \\
&= \frac{-(a_{21}+n_4a_{31}+n_5a_{41})v_{23}-(a_{22}+n_4a_{32}+n_5a_{42})v_{24}+a_{32}}{v_{12}} \\
&= \frac{-a_{21}v_{23}-a_{22}v_{24}+a_{32}}{v_{12}}+\frac{-(n_4a_{31}+n_5a_{41})v_{23}+(n_4a_{32}+n_5a_{42})v_{13}}{v_{12}} \\
&= a_{23}+n_4a_{33}+n_5a_{43}.
\end{aligned}$$

Analogously we solve $b_{24}=a_{24}+n_4a_{34}+n_5a_{44}$.

Noting that $n_3=n_4+n_1n_5$, it remains to show that there exists n_2 such that

$$\begin{pmatrix}1 & n_1 & n_2 & n_3\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}=\begin{pmatrix}b_{11} & b_{12} & b_{13} & b_{14}\end{pmatrix}.$$

Again, we may assume $v_1\neq 0$. Then we solve $n_2=(b_{11}-a_{11}-n_1a_{21}-n_3a_{41})/a_{31}$. By symplecticity of b we have

$$\begin{aligned}
b_{12}b_{31}+b_{22}b_{41}&=b_{11}b_{32}+b_{21}b_{42},\\
b_{12}b_{33}+b_{22}b_{43}&=b_{32}b_{13}+b_{42}b_{23},\\
b_{14}b_{33}+b_{24}b_{43}&=b_{13}b_{34}+b_{23}b_{44},
\end{aligned}$$

from which we solve

$$\begin{aligned}
b_{12}&=a_{12}+n_1a_{22}+n_2a_{32}+n_3a_{42},\\
b_{13}&=a_{13}+n_1a_{23}+n_2a_{33}+n_3a_{43},\\
b_{14}&=a_{14}+n_1a_{24}+n_2a_{34}+n_3a_{44}.
\end{aligned}$$

So we have injectivity.

Now we show surjectivity. Let $v\in V_0\backslash\{0\}$. Put

$$a_{31}=v_1,\quad a_{32}=v_2,\quad a_{33}=v_3,\quad v_{34}=v_4.$$

Again, we may assume $v_1\neq 0$. Then there exists $\xi_1,\xi_2,\xi_3\in\mathbb{R}$ such that

$$\xi_1v_1+\xi_2v_2+\xi_3v_3=1.$$

Now put

$$\begin{aligned}
a_{41}&=-\xi_3v_{13}-\xi_2v_{12}, & a_{42}&=\xi_1v_{12}-\xi_3v_{23},\\
a_{43}&=\xi_2v_{23}+\xi_1v_{13}, & a_{44}&=\left(v_{14}-v_4(\xi_3v_{13}+\xi_2v_{12})\right)/v_1.
\end{aligned}$$

We check that the bottom two rows

$$\begin{pmatrix}a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{pmatrix}$$

have the correct Plücker coordinates. By completing remaining rows, we obtain surjectivity. □

Proposition 2.3. A coset of $U\backslash G$ contains an element of Γ if and only if its corresponding Plücker coordinates are such that $(v_1,\cdots,v_4)$ are coprime integers, and $(v_{12},\cdots,v_{34})$ are coprime integers.

Proof. (⇒) Trivial.

(⇐) The case $v_1 = v_2 = v_3 = 0$ is trivial. Now suppose $(v_1, v_2, v_3) = d > 0$. Then there exist $\xi_1, \xi_2, \xi_3 \in d^{-1}\mathbb{Z}$ such that

$$\xi_1 v_1 + \xi_2 v_2 + \xi_3 v_3 = 1.$$

By the relation

$$v_1 v_{24} - v_2 v_{14} + v_4 v_{12} = 0,$$

we deduce that $d \mid v_4 v_{12}$. But $(d, v_4) = 1$, so $d \mid v_{12}$. Similarly, we have $d \mid v_{13}, v_{23}$. Construct $a_{41}, a_{42}, a_{43}, a_{44}$ as in the proof of surjectivity in Proposition 2.2. Note that a_{41}, a_{42}, a_{43} are constructed as integers. For a_{44}, observe that

$$\begin{aligned} a_{44} &= v_1^{-1}\left(v_{14} - v_4(\xi_3 v_{13} + \xi_2 v_{12})\right) \\ &= v_1^{-1}\left(v_{14} + \xi_3 v_1 v_{34} - \xi_3 v_3 v_{14} + \xi_2 v_1 v_{24} - \xi_2 v_2 v_{14}\right) \\ &= v_1^{-1}\left(\xi_1 v_1 v_{14} + \xi_3 v_1 v_{34} + \xi_2 v_1 v_{24}\right) \\ &= \xi_1 v_{14} + \xi_2 v_{24} + \xi_3 v_{34}. \end{aligned}$$

So $a_{44} \in d^{-1}\mathbb{Z}$. As $(d, v_4) = 1$, there exists $n \in \mathbb{Z}$ such that $d \mid da_{44} + nv_4$. Then

$$\begin{pmatrix} 1 & \\ \frac{n}{d} & 1 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ a_{41} + \frac{nv_1}{d} & a_{42} + \frac{nv_2}{d} & a_{43} + \frac{nv_3}{d} & a_{44} + \frac{nv_4}{d} \end{pmatrix}$$

is integral and has the correct Plücker coordinates. It is then straightforward to show that this can be completed to a symplectic matrix with integral entries. □

2.2.2 Siegel parabolic

Let P_α be the Siegel parabolic subgroup. Let $U_\alpha \subseteq G$ be the subgroup of matrices of the form

$$g = \begin{pmatrix} X & Y \\ & \left(X^{-1}\right)^T \end{pmatrix} \in G, \quad X \in \mathrm{SL}_2(\mathbb{R}),$$

and $\Gamma_\alpha = U_\alpha \cap \Gamma$. We compute the coset representatives of $U_\alpha \backslash G$ and $\Gamma_\alpha \backslash \Gamma$. Note that $P_\alpha \cap \Gamma \backslash \Gamma = (V' \cdot \Gamma_\alpha) \backslash \Gamma$, where

$$V' = \{\mathrm{diag}\,(\varepsilon, 1, \varepsilon, 1) \mid \varepsilon = \pm 1\} \subseteq V. \tag{2.8}$$

It is clear that the Plücker coordinates v_{ij} are invariant under left action of U_α. We know that [Gol06, Ch. 11.3]

$$v_{12} v_{34} - v_{24} v_{13} + v_{14} v_{23} = 0. \tag{2.9}$$

Again, by symplecticity we have

$$v_{13} + v_{24} = 0 \tag{2.10}$$

We define

$$V_\alpha = \left\{ v = (v_{12}, \cdots, v_{34}) \in \mathbb{R}^6 \;\middle|\; v \text{ satisfies (2.9) and (2.10)} \right\}. \tag{2.11}$$

Proposition 2.4. Via the Plücker coordinates, there is a bijection between $U_\alpha \backslash G$ and $V_\alpha \backslash \{0\}$.

Proof. As the coordinates are invariant under left action by U_α, the map $U_\alpha\backslash G \to V_\alpha\backslash\{0\}$ is well-defined.

Now suppose $a = (a_{ij}), b = (b_{ij}) \in G$ have the same Plücker coordinates. We want to show that there exists $\gamma \in U_\alpha$ such that $\gamma a = b$. Firstly, we show that there exists $h \in \mathrm{SL}_2(\mathbb{R})$ such that

$$h\begin{pmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}.$$

Assume $v_{12} \neq 0$. Then there is a unique $h \in \mathrm{SL}_2(\mathbb{R})$ such that

$$h\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}.$$

Now note that

$$\begin{pmatrix} a_{33} \\ a_{43} \end{pmatrix} = \frac{v_{13}}{v_{12}}\begin{pmatrix} a_{32} \\ a_{42} \end{pmatrix} - \frac{v_{23}}{v_{12}}\begin{pmatrix} a_{31} \\ a_{41} \end{pmatrix}, \begin{pmatrix} b_{33} \\ b_{43} \end{pmatrix} = \frac{v_{13}}{v_{12}}\begin{pmatrix} b_{32} \\ b_{42} \end{pmatrix} - \frac{v_{23}}{v_{12}}\begin{pmatrix} b_{31} \\ b_{41} \end{pmatrix}.$$

Hence

$$h\begin{pmatrix} a_{33} \\ a_{43} \end{pmatrix} = \frac{v_{13}}{v_{12}}h\begin{pmatrix} a_{32} \\ a_{42} \end{pmatrix} - \frac{v_{23}}{v_{12}}h\begin{pmatrix} a_{31} \\ a_{41} \end{pmatrix} = \frac{v_{13}}{v_{12}}\begin{pmatrix} b_{32} \\ b_{42} \end{pmatrix} - \frac{v_{23}}{v_{12}}\begin{pmatrix} b_{31} \\ b_{41} \end{pmatrix} = \begin{pmatrix} b_{33} \\ b_{43} \end{pmatrix}.$$

The same argument gives

$$h\begin{pmatrix} a_{34} \\ a_{44} \end{pmatrix} = \begin{pmatrix} b_{34} \\ b_{44} \end{pmatrix}.$$

By the same argument, for any $X \in \mathrm{SL}_2(\mathbb{R})$, we can find a 2×2 matrix Y such that

$$X\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + Y\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

So we obtain a matrix $X \in \mathrm{SL}_2(\mathbb{R})$ and a 2×2 matrix Y such that

$$\begin{pmatrix} X & Y \\ & (X^{-1})^T \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & * & * \\ b_{21} & b_{22} & * & * \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}.$$

Denote

$$\gamma = \begin{pmatrix} X & Y \\ & (X^{-1})^T \end{pmatrix}.$$

Symplecticity of b says

$$\begin{aligned} b_{23} &= \frac{b_{21}b_{33}b_{42} + b_{22}b_{34}b_{42} - b_{21}b_{32}b_{43} - b_{22}b_{32}b_{44} + b_{32}}{b_{31}b_{42} - b_{41}b_{32}} \\ &= \frac{-(x_{21}a_{11} + x_{22}a_{21} + y_{21}a_{31} + y_{22}a_{41})v_{23} - (x_{21}a_{12} + x_{22}a_{22} + y_{21}a_{32} + y_{22}a_{42})v_{24}}{v_{12}} \\ &\quad + \frac{x_{22}a_{32} - x_{21}a_{42}}{v_{12}} \\ &= \frac{x_{21}(-a_{11}v_{23} - a_{12}v_{24} - a_{42})}{v_{12}} + \frac{x_{22}(-a_{21}v_{23} - a_{22}v_{24} + a_{32})}{v_{12}} + \frac{y_{21}(a_{32}v_{13} - a_{31}v_{23})}{v_{12}} \\ &\quad + \frac{y_{22}(a_{42}v_{13} - a_{41}v_{23})}{v_{12}} \\ &= x_{21}a_{13} + x_{22}a_{23} + y_{21}a_{33} + y_{22}a_{43}. \end{aligned}$$

Analogous results hold for b_{13}, b_{14} and b_{24}. Thus we have $\gamma a = b$. Finally, note that $\gamma = ba^{-1}$ is necessarily symplectic. So $\gamma \in U_\alpha$.

Surjectivity follows immediately from Proposition 2.2. ☐

Proposition 2.5. A coset of $U_\alpha \backslash G$ contains an element of Γ if and only if its corresponding Plücker coordinates $(v_{12}, \cdots, v_{34})$ are coprime integers.

Proof. The statement follows immediately from Proposition 2.3. ☐

2.2.3 Non-Siegel parabolic

Let P_β be the non-Siegel parabolic subgroup. Let $U_\beta \subseteq G$ be the group of matrices of the form

$$g = \begin{pmatrix} 1 & * & * & * \\ & * & * & * \\ & & 1 & \\ & * & * & * \end{pmatrix} \in G,$$

and $\Gamma_\beta = U_\beta \cap \Gamma$. We compute the coset representatives of $U_\beta \backslash G$ and $\Gamma_\beta \backslash \Gamma$. We have $P_\beta \cap \Gamma \backslash \Gamma = (V' \cdot \Gamma_\beta) \backslash \Gamma$, with V' as in (2.8).

We define

$$V_\beta = \left\{ v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4 \right\}. \tag{2.12}$$

Proposition 2.6. Via the Plücker coordinates, there is a bijection between $U_\beta \backslash G$ and $V_\beta \backslash \{0\}$.

Proof. As the coordinates are invariant under left action by U_β, the map $U_\beta \backslash G \to V_\beta \backslash \{0\}$ is well-defined.

Suppose $a = (a_{ij}), b = (b_{ij}) \in G$ have the same Plücker coordinates (i.e. the same third row). We want to show that there exists $\gamma \in U_\beta$ such that $\gamma a = b$. Consider the columns

$$\begin{pmatrix} a_{21} \\ a_{31} \\ a_{41} \end{pmatrix}, \quad \begin{pmatrix} a_{22} \\ a_{32} \\ a_{42} \end{pmatrix}, \quad \begin{pmatrix} a_{23} \\ a_{33} \\ a_{43} \end{pmatrix}, \quad \begin{pmatrix} a_{24} \\ a_{34} \\ a_{44} \end{pmatrix}.$$

As a is symplectic, it has nonzero determinant. By permuting columns, we may assume that

$$\det \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \neq 0,$$

and $a_{32} \neq 0$. Then we can find $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \eta_1, \eta_2, \eta_3$ such that

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} = \begin{pmatrix} b_{21} & b_{22} & b_{23} \end{pmatrix},$$

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} = \begin{pmatrix} b_{41} & b_{42} & b_{43} \end{pmatrix},$$

$$\begin{pmatrix} 1 & \eta_1 & \eta_2 & \eta_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \end{pmatrix}.$$

Symplecticity of b says

$$\begin{aligned}
b_{24} &= \frac{b_{21}b_{33} + b_{22}b_{34} - b_{23}b_{31}}{b_{32}} \\
&= \frac{(\lambda_1 a_{21} + \lambda_2 a_{31} + \lambda_3 a_{41})a_{33} + (\lambda_1 a_{22} + \lambda_2 a_{32} + \lambda_3 a_{42})a_{34} - (\lambda_1 a_{23} + \lambda_2 a_{33} + \lambda_3 a_{43})a_{31}}{a_{32}} \\
&= \lambda_1 a_{24} + \lambda_2 a_{34} + \lambda_3 a_{44}.
\end{aligned}$$

Analogously, we have

$$b_{44} = \mu_1 a_{24} + \mu_2 a_{34} + \mu_3 a_{44}, \quad b_{14} = a_{14} + \eta_1 a_{24} + \eta_2 a_{34} + \eta_3 a_{44}.$$

Thus, denote by γ the matrix

$$\gamma = \begin{pmatrix} 1 & \eta_1 & \eta_2 & \eta_3 \\ & \lambda_1 & \lambda_2 & \lambda_3 \\ & & 1 & \\ & \mu_1 & \mu_2 & \mu_3 \end{pmatrix},$$

we have $\gamma a = b$. Again, as $\gamma = ba^{-1}$ is necessarily symplectic, we have $\gamma \in U_\beta$.

Surjectivity follows immediately from Proposition 2.2. □

Proposition 2.7. A coset of $U_\beta \backslash G$ contains an element of Γ if and only if its corresponding Plücker coordinates $(v_1, \cdots, v_4)$ are coprime integers.

Proof. The statement follows immediately from Proposition 2.3. □

2.2.4 Bruhat decomposition

By Proposition 2.2, we can enumerate the cosets $U\backslash G$ using Plücker coordinates. Now it remains to find representatives with given coordinates.

Bruhat decomposition says

$$G = \coprod_{w \in W} G_w := \coprod_{w \in W} UwTU.$$

Hence a coset $\gamma \in U\backslash G$ can be represented by a matrix in $wTU = wP_0$ for some $w \in W$; such Weyl element is unique, and depends on the corresponding Plücker coordinates of the coset. For example, let $\gamma \in U\backslash G$ correspond to Plücker coordinates v_γ, and suppose γ lies in G_{s_α}, then γ has a representative of the form

$$\gamma \sim \begin{pmatrix} & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & \end{pmatrix} \in G.$$

This says v_γ satisfies $v_4, v_{34} \neq 0$, and $v_1, v_2, v_{12}, v_{13}, v_{14}, v_{23}, v_{24} = 0$.

We define an equivalence of Plücker coordinates $(v_1, v_2, v_3, v_4; v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34})$ by

$$(v_1, \cdots, v_4; v_{12}, \cdots, v_{34}) \sim (k_1 v_1, \cdots, k_1 v_4; k_2 v_{12}, \cdots, k_2 v_{34}) \text{ for } k_1, k_2 \in \mathbb{R}^\times.$$

Then we have $v_\gamma \sim (0, 0, *, 1; 0, 0, 0, 0, 0, 1)$, where the entries marked by $*$ are arbitrary.

Now we give representatives of $\gamma \in U\backslash G$ with corresponding Plücker coordinates v_γ, classified by the Weyl element $w \in W$:

(i) $w = \mathrm{id}$: This says $v_\gamma \sim (0,0,1,0;0,0,0,0,0,1)$. In this case, the matrix

$$\begin{pmatrix} 1/v_3 & & & \\ & v_3/v_{34} & & \\ & & v_3 & \\ & & & v_{34}/v_3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1/v_3 & & & \\ & v_3/v_{34} & & \\ & & v_3 & \\ & & & v_{34}/v_3 \end{pmatrix}$$

has the given coordinates.

(ii) $w = s_\alpha$: This says $v_\gamma \sim (0,0,*,1;0,0,0,0,0,1)$. In this case, the matrix

$$\begin{pmatrix} & 1/v_4 & & \\ -v_4/v_{34} & v_3/v_{34} & & \\ & & v_3 & v_4 \\ & & -v_{34}/v_4 & \end{pmatrix} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} \begin{pmatrix} -v_4/v_{34} & v_3/v_{34} & & \\ & 1/v_4 & & \\ & & -v_{34}/v_4 & \\ & & v_3 & v_4 \end{pmatrix}$$

has the given coordinates.

(iii) $w = s_\beta$: This says $v_\gamma \sim (0,0,1,0;0,0,0,1,0,*)$. In this case, the matrix

$$\begin{pmatrix} 1/v_3 & & & \\ & & & v_3/v_{23} \\ & & v_3 & \\ & -v_{23}/v_3 & & v_{34}/v_3 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix} \begin{pmatrix} 1/v_3 & & & \\ & v_{23}/v_3 & & -v_{34}/v_3 \\ & & v_3 & \\ & & & v_3/v_{23} \end{pmatrix}$$

has the given coordinates.

(iv) $w = s_\alpha s_\beta$: This says $v_\gamma \sim (0,1,*,*;0,0,0,1,0,*)$. In this case, the matrix

$$\begin{pmatrix} & & & -1/v_2 \\ v_2/v_{23} & & & v_3/v_{23} \\ & v_2 & v_3 & v_4 \\ & & v_{23}/v_2 & \end{pmatrix} = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & -1 & & \\ & & 1 & \end{pmatrix} \begin{pmatrix} v_2/v_{23} & & & v_3/v_{23} \\ & -v_2 & -v_3 & -v_4 \\ & & v_{23}/v_2 & \\ & & & -1/v_2 \end{pmatrix}$$

has the given coordinates.

(v) $w = s_\beta s_\alpha$: This says $v_\gamma \sim (0,0,*,1;0,*,1,*,*,*)$. In this case, the matrix

$$\begin{pmatrix} & 1/v_4 & & \\ & & v_4/v_{14} & \\ & & v_3 & v_4 \\ -v_{14}/v_4 & -v_{24}/v_4 & -v_{34}/v_4 & \end{pmatrix} = \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ -1 & & & \end{pmatrix} \begin{pmatrix} v_{14}/v_4 & v_{24}/v_4 & v_{34}/v_4 & \\ & 1/v_4 & & \\ & & v_4/v_{14} & \\ & & v_3 & v_4 \end{pmatrix}$$

has the given coordinates.

(vi) $w = s_\alpha s_\beta s_\alpha$: This says $v_\gamma \sim (1,*,*,*;0,*,1,*,*,*)$. In this case, the matrix

$$\begin{pmatrix} & & -1/v_1 & \\ & v_1/v_{14} & v_4/v_{14} & \\ v_1 & v_2 & v_3 & v_4 \\ & & v_{13}/v_1 & v_{14}/v_1 \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} -v_1 & -v_2 & -v_3 & -v_4 \\ & v_1/v_{14} & v_4/v_{14} & \\ & & -1/v_1 & \\ & & v_{13}/v_1 & v_{14}/v_1 \end{pmatrix}$$

has the given coordinates.

(vii) $w = s_\beta s_\alpha s_\beta$: This says $v_\gamma \sim (0,1,*,*;1,*,*,*,*,*)$. In this case, the matrix

$$\begin{pmatrix} & & & -1/v_2 \\ & & v_2/v_{12} & \\ & v_2 & v_3 & v_4 \\ -v_{12}/v_2 & & v_{23}/v_2 & v_{24}/v_2 \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \begin{pmatrix} v_{12}/v_2 & & -v_{23}/v_2 & -v_{24}/v_2 \\ & -v_2 & -v_3 & -v_4 \\ & & v_2/v_{12} & \\ & & & -1/v_2 \end{pmatrix}$$

has the given coordinates.

(viii) $w = w_0$: This says $v_\gamma \sim (1,*,*,*;1,*,*,*,*,*)$. In this case, the matrix

$$\begin{pmatrix} & & -1/v_1 & \\ & & v_2/v_{12} & -v_1/v_{12} \\ v_1 & v_2 & v_3 & v_4 \\ & v_{12}/v_1 & v_{13}/v_1 & v_{14}/v_1 \end{pmatrix} = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \begin{pmatrix} -v_1 & -v_2 & -v_3 & -v_4 \\ & -v_{12}/v_1 & -v_{13}/v_1 & -v_{14}/v_1 \\ & & -1/v_1 & \\ & & v_2/v_{12} & -v_1/v_{12} \end{pmatrix}$$

has the given coordinates.

Now we find coset representatives for $P_0 \cap \Gamma \backslash \Gamma$. For $w \in W$, let $\Gamma_w = \Gamma_0 \cap w^{-1}\Gamma_0^T w$. We also let $U_w = U \cap w^{-1}U^T w$, and $\overline{U}_w = U \cap w^{-1}Uw$. Clearly we have $U = U_w\overline{U}_w = \overline{U}_w U_w$.

Lemma 2.8. Γ_w acts freely on $P_0 \cap \Gamma \backslash \Gamma \cap G_w$ on the right.

Proof. This is [Fri87, Lemma 1.2] when stated for $\mathrm{GL}(n)$, but the proof works in our settings as well. For convenience we repeat the short argument. Suppose $\gamma \in \Gamma_w$ fixes the left coset $(P_0 \cap \Gamma)\delta$. By Bruhat decomposition, we can write $\delta = b_1 w t b_2$, with $b_1, b_2 \in U$, $t \in T$. By choosing different Bruhat decompositions, we may assume $b_2 \in U_w$. It is then obvious that $b_2\gamma b_2^{-1} \in U_w$. On the other hand, we have

$$(P_0 \cap \Gamma)b_1 w t b_2 \gamma = (P_0 \cap \Gamma)b_1 w t b_2.$$

So we have

$$(P_0 \cap \Gamma)b_1 w t b_2 \gamma b_2^{-1} t^{-1} w^{-1} = (P_0 \cap \Gamma)b_1,$$

which implies $wtb_2\gamma b_2^{-1}t^{-1}w^{-1} \in \Gamma_w \subseteq U_w$. This says $b_2\gamma b_2^{-1} \in \overline{U}_w$. So $b_2\gamma b_2^{-1} \in U_w \cap \overline{U}_w = \{I\}$, and $\gamma = I$. ☐

By Lemma 2.8, it suffices to give a complete set of coset representatives for the quotient $P_0 \cap \Gamma \backslash \Gamma \cap G_w / \Gamma_w$, which we denote by R_w.

Note. For a coset in $P_0 \cap \Gamma \backslash \Gamma \cap G_w$, the matrix representative given above does not necessarily have integral entries, but it can always be converted to one under left action by U.

Now we compute R_w in terms of the corresponding coordinates v, using the coprimality condition given in Proposition 2.3.

(i) $w = \mathrm{id}$: We have $v \sim (0,0,1,0;0,0,0,0,0,1)$. Coprimality condition gives $v_3 = v_{34} = 1$. Hence

$$R_{\mathrm{id}} = \{(0,0,1,0;0,0,0,0,0,1)\}.$$

(ii) $w = s_\alpha$: We have $v \sim (0,0,*,1;0,0,0,0,0,1)$. Right action by Γ_w says that v_3 is defined modulo v_4. Coprimality condition gives $v_{34} = 1$, and $(v_3, v_4) = 1$. Hence

$$R_{s_\alpha} = \{(0,0,v_3,v_4;0,0,0,0,0,1)\},$$

where $v_4 \geq 1$, and $v_3 \pmod{v_4}$ such that $(v_3, v_4) = 1$.

(iii) $w = s_\beta$: We have $v \sim (0,0,1,0;0,0,0,1,0,*)$. Right action by Γ_w says that v_{34} is defined modulo v_{23}. Coprimality condition gives $v_3 = 1$, and $(v_{23}, v_{34}) = 1$. Hence

$$R_{s_\beta} = \{(0,0,1,0;0,0,0,v_{23},0,v_{34})\},$$

where $v_{23} \geq 1$, and $v_{34} \pmod{v_{23}}$ such that $(v_{23}, v_{34}) = 1$.

(iv) $w = s_\alpha s_\beta$: We have $v \sim (0,1,*,*;0,0,0,1,0,*)$. From the matrix representative, we deduce $\frac{v_{34}}{v_{23}} = -\frac{v_4}{v_2}$. Coprimality condition gives $(v_2, v_3, v_4) = 1$ and $(v_{23}, v_{34}) = 1$. Then we solve $v_{23} = \frac{v_2}{(v_2,v_4)}$. Finally, right action by Γ_w says v_3, v_4 are defined modulo v_2. Hence

$$R_{s_\alpha s_\beta} = \left\{ \left(0, v_2, v_3, v_4; 0, 0, 0, \frac{v_2}{d}, 0, -\frac{v_4}{d}\right) \right\},$$

where $v_2 \geq 1$, $v_3, v_4 \pmod{v_2}$ such that $(v_2, v_3, v_4) = 1$, and $d = (v_2, v_4)$.

(v) $w = s_\beta s_\alpha$: We have $v \sim (0,0,*,1;0,*,1,*,*,*)$. Recall the symplectic relation $v_{13} + v_{24} = 0$. From the matrix representative, we deduce $\frac{v_{24}}{v_{14}} = -\frac{v_3}{v_4}$. Coprimality condition gives $(v_3, v_4) = 1$ and $(v_{14}, v_{23}, v_{24}, v_{34}) = 1$. Fix v_{14}, v_{24}. Let $d = (v_{14}, v_{24})$, and write $v_{14} = dv'_{14}$, $v_{24} = dv'_{24}$. Then we have $v_3 = -v'_{24}$, $v_4 = v'_{14}$. From the matrix representative, we deduce $v_{23} = -\frac{{v'_{24}}^2 d}{v'_{14}}$. Since v_{23} is an integer, and $(v'_{14}, v'_{24}) = 1$, this implies $v'_{14} \mid d$. Write $d = v'_{14} d'$. Then the coprimality condition becomes

$$\left(d'{v'_{14}}^2, -d'{v'_{24}}^2, d'v'_{14}v'_{24}, v_{34}\right) = 1.$$

Since $(v'_{14}, v'_{24}) = 1$, the coprimality condition simplifies to $(d', v_{34}) = 1$. Finally, right action by Γ_w says v_{24}, v_{34} are defined modulo v_{14}. Hence

$$R_{s_\beta s_\alpha} = \left\{ \left(0, 0, -\frac{v_{24}}{d}, \frac{v_{14}}{d}; 0, -v_{24}, v_{14}, -\frac{v_{24}^2}{v_{14}}, v_{24}, v_{34}\right) \right\},$$

where $v_{14} \geq 1$, $v_{24}, v_{34} \pmod{v_{14}}$, $d = (v_{14}, v_{24})$, such that $v_{14} \mid d^2$ and $\left(\frac{d^2}{v_{14}}, v_{34}\right) = 1$.

(vi) $w = s_\alpha s_\beta s_\alpha$: We have $v \sim (1,*,*,*;0,*,1,*,*,*)$. Again, the symplectic relation says $v_{13} + v_{24} = 0$. From the matrix representative, we deduce $\frac{v_{13}}{v_{14}} = -\frac{v_2}{v_1}$. Coprimality condition gives $(v_1, v_2, v_3, v_4) = 1$ and $(v_{13}, v_{14}, v_{23}, v_{34}) = 1$. So we can write $v_{14} = rv_1^2$, $v_{13} = -rv_1v_2$ for some $r \in \mathbb{Q} \setminus \{0\}$. Then the coprimality condition can be rewritten as

$$1 = (v_{13}, v_{14}, v_{23}, v_{34}) = \big(-rv_1v_2, rv_1^2, -rv_2^2, r(v_1v_3 + v_2v_4)\big).$$

Writing $d = (v_1, v_2)$, the condition simplifies to $\big(rd^2, r(v_1v_3 + v_2v_4)\big) = 1$, so we solve $r = (d^2, v_1v_3 + v_2v_4)^{-1}$. Finally, right action by Γ_w says v_2, v_3, v_4 are defined modulo v_1. Hence

$$R_{s_\alpha s_\beta s_\alpha} = \left\{ \left(v_1, v_2, v_3, v_4; 0, -\frac{v_1v_2}{\delta}, \frac{v_1^2}{\delta}, -\frac{v_2^2}{\delta}, \frac{v_1v_2}{\delta}, \frac{v_1v_3 + v_2v_4}{\delta}\right) \right\},$$

where $v_1 \geq 1$, $v_2, v_3, v_4 \pmod{v_1}$, such that $(v_1, v_2, v_3, v_4) = 1$, and $d = (v_1, v_2)$, $\delta = (d^2, v_1v_3 + v_2v_4)$.

(vii) $w = s_\beta s_\alpha s_\beta$: We have $v \sim (0,1,*,*;1,*,*,*,*,*)$. Symplectic relation says $v_{13} + v_{24} = 0$. From the matrix representative, we deduce $\frac{v_3}{v_2} = \frac{v_{13}}{v_{12}}$, $\frac{v_4}{v_2} = \frac{v_{14}}{v_{12}}$. Let $d_0 = (v_{12}, v_{13}, v_{14})$. Coprimality condition says $(v_2, v_3, v_4) = 1$ and $(v_{12}, v_{13}, v_{14}, v_{23}, v_{34}) = 1$. This implies

$$v_2 = \frac{v_{12}}{d_0}, \quad v_3 = \frac{v_{13}}{d_0}, \quad v_4 = \frac{v_{14}}{d_0}.$$

Let $d_1 = (v_{12}, v_{14})$. The relations

$$v_{13} + v_{24} = 0, \quad v_{12}v_{34} - v_{13}v_{24} + v_{14}v_{23} = 0$$

imply $d_1 \mid v_{13}^2$. Write $v_{13}^2 = d_1 k$, $v_{12} = d_1 v'_{12}$, $v_{14} = d_1 v'_{14}$. Then we require that

$$v_{34} = -\frac{v_{13}^2 + v_{14}v_{23}}{v_{12}} = -\frac{k + v'_{14}v_{23}}{v'_{12}}$$

is an integer, and satisfies the coprimality condition. Since $(v_{12}, v_{13}, v_{14}) = d_1$, the coprimality condition simplifies to

$$(d, v_{23}, v_{34}) = 1.$$

Since v_{34} is an integer, we have $v'_{14}v_{23} \equiv -k \pmod{v'_{12}}$. Since $(v'_{12}, v'_{14}) = 1$, we can write $v_{23} = a + rv'_{12}$, where a is a particular solution for the congruence equation

$$av'_{14} \equiv -k \pmod{v'_{12}}, \tag{2.13}$$

and $r \in \mathbb{Z}$. Let $t = (k, d_1)$. We claim that a can be chosen such that a and $\frac{av'_{14}+k}{v'_{12}}$ are both divisible by t. Let a be an arbitrary solution to the congruence (2.13). Then

$$a \equiv -k\overline{v'_{14}} \pmod{v'_{12}} \iff a + k\overline{v'_{14}} = uv'_{12} \text{ for some } u \in Z.$$

Then $a - uv'_{12}$ is a solution to the congruence, which is divisible by t. So we may assume a is divisible by t. Again we write $a + k\overline{v'_{14}} = uv'_{12}$. Let $f = (t, v'_{12})$. Then uv'_{12} is divisible by t, so u is divisible by t/f. Now consider the equation

$$\left(a + \frac{nt}{f}v'_{12}\right) + k\overline{v'_{14}} = \left(u + \frac{nt}{f}\right)v'_{12}.$$

Pick $n \in \mathbb{Z}$ such that $u + \frac{nt}{f}$ is divisible by t. Then $a' := a + \frac{nt}{f}v'_{12}$ is a solution to the congruence (2.13) divisible by t, and $\frac{a'+k\overline{v'_{14}}^{-1}}{v'_{12}}$ is also divisible by t. Multiplying by v'_{14}, we see that $\frac{a'v'_{14}+k}{v'_{12}}$ is divisible by t. This finishes the proof of the claim. Now the coprimality condition becomes

$$\left(d_1, a + rv'_{12}, -\frac{av'_{14}+k}{v'_{12}} - rv'_{14}\right) = 1,$$

which holds if and only if $(r, t) = 1$.

Now we give an alternative expression for t. Let $d' = d_1/d_0$. Then the conditions $d_1 \mid v_{13}^2$ and $(v_{12}, v_{13}, v_{14}) = d_0$ imply $d' \mid d_0$. Write $v_{13} = d_0v'_{13}$. Then we see that

$$\left(\frac{v_{12}}{d_0}, v'_{13}, \frac{v_{14}}{d_0}\right) = 1,$$

which implies $(d', v'_{13}) = 1$. Now define $t := d_0/d'$. Then

$$t = t\left({v'_{13}}^2, d'^2\right) = \left({v'_{13}}^2 t, d'^2 t\right) = \left(\frac{{v_{13}}^2}{d_1}, d_1\right) = (k, d_1)$$

returns the original definition.

Finally, right action by Γ_w says v_{13}, v_{14}, v_{23} are defined modulo v_{12}. Hence

$$R_{s_\beta s_\alpha s_\beta} = \left\{\left(0, \frac{v_{12}}{d_0}, \frac{v_{13}}{d_0}, \frac{v_{14}}{d_0}; v_{12}, v_{13}, v_{14}, v_{23}, -v_{13}, -\frac{v_{13}^2 + v_{14}v_{23}}{v_{12}}\right)\right\}.$$

where $v_{12} \geq 1$, $v_{13}, v_{14}, v_{23} \pmod{v_{12}}$, with the following conditions. Let $d_1 = (v_{12}, v_{14})$, and $d_0 = (v_{12}, v_{13}, v_{14})$. Then we require $d_1 \mid d_0^2$. Write $v_{12} = d_1v'_{12}$, $v_{14} = d_1v'_{14}$, $v_{13} = d_1k$, and $d' = d_1/d_0$, $t = d_0/d'$. Let a be a solution to $av'_{14} \equiv -k \pmod{v'_{12}}$, such that a and $\frac{av'_{14}+k}{v'_{12}}$ are divisible by t. Then we require v_{23} to be of the form $v_{23} = a + rv'_{12}$ with $(r, t) = 1$.

(viii) $w = w_0$: We have $v \sim (1,*,*,*;1,*,*,*,*,*)$. Recall the relation $v_1v_{24} - v_2v_{14} + v_4v_{12} = 0$. By the symplectic relation $v_{13} + v_{24} = 0$, this translates to $v_1v_{13} + v_2v_{14} - v_4v_{12} = 0$. From the matrix representative, we compute

$$v_{23} = \frac{v_2v_{13} - v_3v_{12}}{v_1}, \quad v_{34} = \frac{v_3v_{14} - v_4v_{13}}{v_1}.$$

Coprimality condition says $(v_1, v_2, v_3, v_4) = 1$, and $(v_{12}, v_{13}, v_{14}, v_{23}, v_{34}) = 1$. However, no good simplification to these conditions is found. Right action by Γ_w says v_2, v_3, v_4 are defined modulo v_1, and v_{13}, v_{14} are defined modulo v_{12}. Hence

$$R_{w_0} = \left\{ \left(v_1, v_2, v_3, v_4; v_{12}, v_{13}, v_{14}, \frac{v_2v_{13} - v_3v_{12}}{v_1}, -v_{13}, \frac{v_3v_{14} - v_4v_{13}}{v_1} \right) \right\},$$

where $v_1, v_{12} \geq 1$, and $v_2, v_3, v_4 \pmod{v_1}$, $v_{13}, v_{14} \pmod{v_{12}}$, such that $v_1v_{13} + v_2v_{14} - v_4v_{12} = 0$, $(v_1, v_2, v_3, v_4) = 1$, and $\left(v_{12}, v_{13}, v_{14}, \frac{v_2v_{13} - v_3v_{12}}{v_1}, \frac{v_3v_{14} - v_4v_{13}}{v_1}\right) = 1$.

2.2.5 Residual Eisenstein series

Recall the definition of GL(2) Eisenstein series

$$E(z, s) = \sum_{\delta \in P^2 \cap \Gamma^2 \backslash \Gamma^2} I(\delta z, s),$$

where $\Gamma^2 = \mathrm{SL}(2, \mathbb{Z})$, $P^2 = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \right\} \subseteq \mathrm{SL}(2, \mathbb{R})$ the standard parabolic subgroup of $\mathrm{SL}(2, \mathbb{R})$, and $I(z, s) = \mathrm{Im}(z)^{s+1/2}$.

Recall the maximal parabolic Eisenstein series $E_\alpha(g, \nu, f)$ and $E_\beta(g, \nu, f)$. We show that if f is a GL(2) Eisenstein series, then $E_\alpha(g, \nu, f)$ and $E_\beta(g, \nu, f)$ become minimal Eisenstein series.

Proposition 2.9. We have

$$\begin{aligned} E_\alpha(g, \nu, E(*, s)) &= E_0\left(g, (\nu + s, \nu)\right), \\ E_\beta(g, \nu, E(*, s)) &= E_0\left(g, (\nu, \nu/2 + s)\right). \end{aligned}$$

Proof. First we assume $\mathrm{Re}\, \nu \gg 0$, and $\mathrm{Re}\, s > \frac{1}{2}$. For the Siegel Eisenstein series, we have

$$E_\alpha\left(g, \nu, E(*, s)\right) = \sum_{\gamma \in P_\alpha \cap \Gamma \backslash \Gamma} \sum_{\delta \in P^2 \cap \Gamma^2 \backslash \Gamma^2} I(\delta m_\alpha(\gamma g), s) I_\alpha(\gamma g, \nu).$$

Recall the formula

$$\mathrm{Im}(\delta z) = \frac{\mathrm{Im}(z)}{|cz + d|} \text{ for } \delta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Assume g has the form as in (2.2). Then we set

$$m_\alpha(g) = \begin{pmatrix} 1 & n_1 \\ & 1 \end{pmatrix} \begin{pmatrix} y_1 & \\ & y_2 \end{pmatrix} \sim n_1 + \frac{y_1}{y_2} i =: z_0.$$

Evaluate the inner sum:

$$\begin{aligned} \sum_{\delta \in P^2 \cap \Gamma^2 \backslash \Gamma^2} I(\delta m_\alpha(g), s) I_\alpha(g, \nu) &= \frac{1}{2} \sum_{\substack{c, d \in \mathbb{Z} \\ (c,d) = 1}} \left(\frac{y_1}{|cz_0 + d|} \right)^{\nu + s + 2} \left(y_2 \left|cz_0 + d\right|\right)^{\nu - s + 1} \\ &= \sum_{\delta \in P^2 \cap \Gamma^2 \backslash \Gamma^2} I_0 \left(\begin{pmatrix} \delta & \\ & (\delta^{-1})^T \end{pmatrix} g, (\nu + s, \nu) \right). \end{aligned}$$

We observe that there is an isomorphism $P^2 \cap \Gamma^2 \backslash \Gamma^2 \simeq (P_0 \cap \Gamma) \backslash (P_\alpha \cap \Gamma)$ via the map $\delta \mapsto \begin{pmatrix} \delta & \\ & (\delta^{-1})^T \end{pmatrix}$. Hence

$$
\begin{aligned}
E_\alpha(g, \nu, E(*, s)) &= \sum_{\gamma \in P_\alpha \cap \Gamma \backslash \Gamma} \sum_{\delta \in (P_0 \cap \Gamma) \backslash (P_\alpha \cap \Gamma)} I_0(\delta\gamma g, (\nu + s, \nu)) \\
&= \sum_{\gamma \in P_0 \cap \Gamma \backslash \Gamma} I_0(\gamma g, (\nu + s, \nu)) \\
&= E_0(g, (\nu + s, \nu))
\end{aligned}
$$

is a minimal Eisenstein series.

For the non-Siegel Eisenstein series, we have

$$
E_\beta(g, \nu, E(*, s)) = \sum_{\gamma \in P_\beta \cap \Gamma \backslash \Gamma} \sum_{\delta \in P^2 \cap \Gamma^2 \backslash \Gamma^2} I(\delta m_\beta(\gamma g), s) I_\beta(\gamma g, \nu).
$$

Then we set

$$
m_\beta(g) = \begin{pmatrix} 1 & n_5 \\ & 1 \end{pmatrix} \begin{pmatrix} y_2 & \\ & y_2^{-1} \end{pmatrix} \sim n_5 + y_2^2 i =: z_0.
$$

Evaluate the inner sum:

$$
\begin{aligned}
\sum_{\delta \in P^2 \cap \Gamma^2 \backslash \Gamma^2} I(\delta m_\beta(\gamma g), s) I_\beta(\gamma g, \nu) &= \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \left(\frac{y_2^2}{|cz_0 + d|^2} \right)^{s+1/2} y_1^{\nu+2} \\
&= \sum_{\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in P^2 \cap \Gamma^2 \backslash \Gamma^2} I_0\left(\begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix} g, (\nu, \nu/2 + s) \right).
\end{aligned}
$$

Again, we have $P^2 \cap \Gamma^2 \backslash \Gamma^2 \simeq (P_0 \cap \Gamma) \backslash (P_\beta \cap \Gamma)$ via the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\begin{smallmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{smallmatrix}\right)$. Hence

$$
\begin{aligned}
E_\beta(g, \nu, E(*, s)) &= \sum_{\gamma \in P_\beta \cap \Gamma \backslash \Gamma} \sum_{\delta \in (P_0 \cap \Gamma) \backslash (P_\beta \cap \Gamma)} I_0(\delta\gamma g, (\nu, \nu/2 + s)) \\
&= \sum_{\gamma \in P_0 \cap \Gamma \backslash \Gamma} I_0(\gamma g, (\nu, \nu/2 + s)) \\
&= E_0(g, (\nu, \nu/2 + s))
\end{aligned}
$$

is a minimal Eisenstein series.

It is well-known from the general theory [Lan76] that these Eisenstein series can be continued into meromorphic functions in ν and s respectively. So we deduce that the equalities hold for all ν and s. $\square$

Recall that the GL(2) Eisenstein series $E(z, s)$ has a pole at $s = 1/2$ with residue $3/\pi$. Taking the residue of $E_\alpha(g, \nu, E(*, s))$ and $E_\beta(g, \nu, E(*, s))$ at $s = 1/2$ gives the following:

Proposition 2.10. We have

$$
\begin{aligned}
\operatorname{Res}_{s=1/2} E_0(g, (\nu + s, \nu)) &= \frac{3}{\pi} E_\alpha(g, \nu, 1), \\
\operatorname{Res}_{s=1/2} E_0(g, (\nu, \nu/2 + s)) &= \frac{3}{\pi} E_\beta(g, \nu, 1).
\end{aligned}
$$

2.2.6 Alternative expressions for Eisenstein series

We end the section by giving alternative expressions for Eisenstein series, directly in terms of Plücker coordinates. Recall the definition of the minimal Eisenstein series

$$E_0(g,\nu) = \sum_{\gamma\in P_0\cap\Gamma\backslash\Gamma} I_0(\gamma g,\nu),$$

where $I_0(g,\nu) = y_1^{\nu_1+2}y_2^{2\nu_2-\nu_1+1}$. Let $\gamma\in\Gamma$ be fixed. Let $\gamma g = nak$ be the Iwasawa decomposition of γg, with $n\in U$, $a\in T^+$, and $k\in K$. If we write

$$a = \operatorname{diag}\left(a_1, a_2, a_1^{-1}, a_2^{-1}\right) \in T^+,$$

then $I_0(\gamma g,\nu) = a_1^{\nu_1+2}a_2^{2\nu_2-\nu_1+1}$. So it suffices to find expressions for a_1 and a_2 in terms of Plücker coordinates of γ.

Suppose γ has Plücker coordinates $v = (v_1,\cdots,v_4; v_{12},\cdots,v_{34})$. Define

$$v_\alpha = (v_{12}, v_{13}, v_{14}, v_{23}, v_{24}, v_{34})^T, \quad v_\beta = (v_1, v_2, v_3, v_4)^T.$$

Suppose γg has the form

$$\gamma g = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} = nak.$$

Then $\gamma g(\gamma g)^T = nak(nak)^T = na^2n^T$. Since $n\in U$ has the form

$$n = \begin{pmatrix} 1 & u & * & * \\ & 1 & * & * \\ & & 1 & \\ & & -u & 1 \end{pmatrix} \in N,$$

we compute

$$na^2n^T = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & a_1^{-2} & -ua_1^{-2} \\ * & * & -ua_1^{-2} & u^2a_1^{-2}+a_2^{-2} \end{pmatrix}.$$

Evaluating $\gamma g(\gamma g)^T = \gamma g g^T\gamma^T$ yields

$$\begin{aligned} a_1^{-2} &= b_{31}^2+b_{32}^2+b_{33}^2+b_{34}^2, \\ -ua_1^{-2} &= b_{31}b_{41}+b_{32}b_{42}+b_{33}b_{43}+b_{34}b_{44}, \\ u^2a_1^{-2}+a_2^{-2} &= b_{41}^2+b_{42}^2+b_{43}^2+b_{44}^2, \end{aligned}$$

from which we solve

$$a_2^{-2} = b_{41}^2+b_{42}^2+b_{43}^2+b_{44}^2 - \frac{(b_{31}b_{41}+b_{32}b_{42}+b_{33}b_{43}+b_{34}b_{44})^2}{b_{31}^2+b_{32}^2+b_{33}^2+b_{34}^2}.$$

In particular, we have

$$\begin{aligned} a_1^{-2}a_2^{-2} &= \left(b_{31}^2+b_{32}^2+b_{33}^2+b_{34}^2\right)\left(b_{41}^2+b_{42}^2+b_{43}^2+b_{44}^2\right) - (b_{31}b_{41}+b_{32}b_{42}+b_{33}b_{43}+b_{34}b_{44})^2 \\ &= \sum_{1\le i<j\le 4} (b_{3i}b_{4j}-b_{3j}b_{4i})^2. \end{aligned}$$

Meanwhile, expanding γg, we see that

$$\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \end{pmatrix} = v_\beta^T g.$$

Let $g \wedge g$ be the exterior square of g, that is, $g \wedge g = \left(g_{ij,kl}\right)_{\substack{1\le i<j\le 4\\ 1\le k<l\le 4}}$, where $g_{ij,kl} = g_{ik}g_{jl} - g_{il}g_{jk}$. Then we have

$$\left(b_{3i}b_{4j} - b_{3j}b_{4i}\right)_{1\le i<j\le 4} = v_\alpha^T \left(g \wedge g\right),$$

where we consider $\left(b_{3i}b_{4j} - b_{3j}b_{4i}\right)_{1\le i<j\le 4}$ as a row vector. So we can write

$$a_1^{-2} = v_\beta^T g g^T v_\beta, \tag{2.14}$$

$$a_1^{-2}a_2^{-2} = v_\alpha^T (g\wedge g)(g\wedge g)^T v_\alpha. \tag{2.15}$$

Hence we have

$$I_0(\gamma g,\nu) = a_1^{\nu_1+2}a_2^{2\nu_2-\nu_1+1} = \left(v_\alpha^T (g\wedge g)(g\wedge g)^T v_\alpha\right)^{\nu_1/2-\nu_2-1/2}(v_\beta^T g g^T v_\beta)^{\nu_2-\nu_1-1/2}.$$

To conclude, we see that $E_0(g,\nu)$ can be expressed as a height zeta function associated with a bi-projective quadratic variety.

Proposition 2.11. Let V_0 be defined as in (2.5). Then we have

$$E_0(g,\nu) = \frac{1}{4}\sum_{v\in V_0(\mathbb{Z}) \text{ primitive}} \left(v_\alpha^T (g\wedge g)(g\wedge g)^T v_\alpha\right)^{\nu_1/2-\nu_2-1/2}(v_\beta^T g g^T v_\beta)^{\nu_2-\nu_1-1/2},$$

where $v_\alpha := (v_{12},\cdots,v_{34})^T$ and $v_\beta := (v_1,\cdots,v_4)^T$ for $v = (v_1,\cdots,v_4;v_{12},\cdots,v_{34}) \in V_0(\mathbb{Z})$.

By the same argument, we can show that $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$ can be expressed as Epstein zeta functions.

Proposition 2.12. Let V_α, V_β be defined as in (2.11) and (2.12) respectively. Then we have

$$E_\alpha(g,\nu,1) = \frac{1}{2}\sum_{v_\alpha\in V_\alpha(\mathbb{Z}) \text{ primitive}} \left(v_\alpha^T (g\wedge g)(g\wedge g)^T v_\alpha\right)^{-\nu/2-3/4},$$

$$E_\beta(g,\nu,1) = \frac{1}{2}\sum_{v_\beta\in V_\beta(\mathbb{Z}) \text{ primitive}} (v_\beta^T g g^T v_\beta)^{-\nu/2-1}.$$

Proof. By definition, we have

$$E_\alpha(g,\nu,1) = \sum_{\gamma\in P_\alpha\cap\Gamma\backslash\Gamma} I_\alpha(\gamma g,\nu), \qquad E_\beta(g,\nu,1) = \sum_{\gamma\in P_\beta\cap\Gamma\backslash\Gamma} I_\beta(\gamma g,\nu),$$

where $I_\alpha(g,\nu) = (y_1y_2)^{\nu+3/2}$, and $I_\beta(g,\nu) = y_1^{\nu+2}$. Then the statement follows from expressions (2.14) and (2.15). □

2.3 Constant terms

Definition 2.13. Let $E_P(g,\nu,f)$ be an Eisenstein series for a standard parabolic subgroup $P = MN \subseteq G$. Let $P' = M'N'$ be another standard parabolic subgroup. The constant term of $E_P(g,\nu,f)$ along the parabolic P' is defined as

$$C_P^{P'}(g,\nu,f) := \int_{N'(\mathbb{Z})\backslash N'(\mathbb{R})} E_P(\eta g,\nu,f)d\eta,$$

where $N'(\mathbb{Z}) = \Gamma\cap N'(\mathbb{R})$.

Notation. When $P = P'$, the superscript P' is omitted from the notation.

For the computation of constant terms, we make use of intertwining operators, introduced by Langlands [Lan76], in the theory of automorphic forms. The intertwining operators are usually defined in adelic settings. Instead of translating the notion into classical settings, we simply establish a relation between classical and adelic objects, and use the adelic theory.

To state the functional equation of Langlands, we follow the setup in [MW95]. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. Let G be a reductive group, and $P = NM$ a standard parabolic subgroup of G, with respect to a fixed Borel subgroup $B \subseteq G$.

Definition 2.14. Let π be an irreducible automorphic representation of $M(\mathbb{A})$, and ϕ_π an element in $A(N(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A}))_\pi$, the π-isotypic part of the space of automorphic forms on $N(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A})$. The Eisenstein series associated to ϕ_π is a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$, given by

$$E(\phi_\pi, \pi)(g) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_\pi(\gamma g)$$

whenever it converges. The constant term of $E(\phi_\pi, \pi)$ along another standard parabolic subgroup $P' = N'M'$ is given by

$$E_{P'}(\phi_\pi, \pi)(g) := \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} E(\phi_\pi, \pi)(\eta g) d\eta.$$

Definition 2.15. Let $P' = N'M'$ be another standard parabolic subgroup of G, π an irreducible automorphic representation of $M(\mathbb{A})$, and $\phi_\pi \in A(N(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A}))_\pi$. Let $w \in G(\mathbb{Q})$ be such that $wMw^{-1} = M'$. For $g \in G(\mathbb{A})$, we set

$$M(w, \pi)\phi_\pi(g) = \int_{(N'(\mathbb{Q}) \cap wN(\mathbb{Q})w^{-1})\backslash N'(\mathbb{A})} \phi_\pi(w^{-1}\eta g) d\eta$$

whenever the integral is convergent. This defines an intertwining operator

$$M(w, \pi) : A(N(\mathbb{A})M(\mathbb{Q})\backslash G(\mathbb{A}))_\pi \to A(N'(\mathbb{A})M'(\mathbb{Q})\backslash G(\mathbb{A}))_{w\pi}.$$

Now we are able to state the functional equation of Langlands.

Theorem 2.16. (Langlands [Lan76]) Assume the settings above. Then we have

$$M(w', w\pi) \circ M(w, \pi) = M(w'w, \pi).$$

Let $G = \mathrm{Sp}(4)$. By strong approximation, for $g \in G(\mathbb{A})$, we can decompose $g = \delta g_\infty k_0$, with $\delta \in G(\mathbb{Q})$, $g_\infty \in G(\mathbb{R})$, and $k_0 \in K$, the maximal compact subgroup of $G(\mathbb{A})$. Let $P_0 = N_0 M_0$ be the minimal parabolic subgroup of $\mathrm{Sp}(4)$, with Levi component $M_0 = T$. For $\nu \in \mathbb{C}^2$, let π_ν be the character on $M_0(\mathbb{A})$ defined by

$$\pi_\nu(\mathrm{diag}(y_1, y_2, y_1^{-1}, y_2^{-1})) = |y_1|^{\nu_1+2} |y_2|^{2\nu_2-\nu_1+1}.$$

By parabolic induction, we see that $\phi_\nu(g) := |y_1|^{\nu_1+2} |y_2|^{2\nu_2-\nu_1+1}$ lies in $A(N_0(\mathbb{A})M_0(\mathbb{Q})\backslash \mathbb{G}(A))_{\pi_\nu}$.

Proposition 2.17. Assume the setup above. Then $E(\phi_\nu, \pi_\nu)(g) = E_0(g_\infty, \nu)$.

Proof. Write $\Gamma = \mathrm{Sp}(4, \mathbb{Z})$ as usual. Unfolding the definitions, the equation says

$$\sum_{\gamma \in P_0(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_\nu(\gamma g) = \sum_{\gamma \in P_0(\mathbb{R}) \cap \Gamma\backslash\Gamma} I_0(\gamma g_\infty, \nu).$$

First we observe a bijection between $P_0(\mathbb{Q})\backslash G(\mathbb{Q})$ and $P_0(\mathbb{R})\cap\Gamma\backslash\Gamma$. Let $\gamma \in G(\mathbb{Q})$. Via left action by $T(\mathbb{Q}) \subseteq P_0(\mathbb{Q})$, we may assume that γ has Plücker coordinates $v = (v_1, \cdots, v_4; v_{12}, \cdots, v_{34})$ such that $(v_1, \cdots, v_4)$ are coprime integers, and $(v_{12}, \cdots, v_{34})$ are coprime integers. By Proposition 2.2, we see that γ is equivalent to a matrix in Γ under left action by $N_0(\mathbb{Q}) \subseteq N_0(\mathbb{R})$. So $\gamma \in P_0(\mathbb{Q})\backslash G(\mathbb{Q})$ corresponds to a unique element in $P_0(\mathbb{R}) \cap \Gamma\backslash\Gamma$.

As $E(\phi_\nu, \nu)$ is left $G(\mathbb{Q})$ and right K-invariant, we may assume $g = (g_\infty, 1, 1, \cdots)$. Let $\gamma \in P_0(\mathbb{Q})\backslash G(\mathbb{Q})$. From the bijection above, we may assume that $\gamma \in \Gamma$ has integral entries. Then γ is integrally invertible, that is, $\gamma \in K_{\mathrm{fin}}$, the maximal compact subgroup of the finite adele $G(\mathbb{A}_{\mathrm{fin}})$. This implies the $P_0(\mathbb{Q})\backslash G(\mathbb{Q})$-action at finite places is trivial. Hence

$$\sum_{\gamma\in P_0(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_\nu(\gamma g) = \sum_{\gamma\in P_0(\mathbb{R})\cap\Gamma\backslash\Gamma} \phi_{\nu,\infty}(\gamma g_\infty) = \sum_{\gamma\in P_0(\mathbb{R})\cap\Gamma\backslash\Gamma} I_0(\gamma g_\infty, \nu). \qquad \square$$

We also have a correspondence between constant terms.

Proposition 2.18. Let $g = (g_\infty, 1, 1, \cdots) \in G(\mathbb{A})$. Then $E_{P'}(\phi_\nu, \pi_\nu)(g) = C_0^{P'}(g_\infty, \nu)$.

Proof. We expand

$$E_{P'}(\phi_\nu, \pi_\nu)(g) = \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \sum_{\gamma\in P_0(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_\nu(\gamma\eta g)d\eta = \int_{N'(\mathbb{Q})\backslash N'(\mathbb{A})} \sum_{\gamma\in P_0(\mathbb{R})\cap\Gamma\backslash\Gamma} \phi_\nu(\gamma\eta g)d\eta.$$

At finite places, since both Γ and $N'(\mathbb{Z}_p)$ lie in $K_p := G(\mathbb{Z}_p)$ the maximal compact subgroup at p, it follows that the integral at finite places is trivial. So only the archimedean place remains, and hence

$$E_{P'}(\phi_\nu, \pi_\nu)(g) = \int_{N'(\mathbb{Z})\backslash N'(\mathbb{R})} \sum_{\gamma\in P_0(\mathbb{R})\cap\Gamma\backslash\Gamma} \phi_{\nu,\infty}(\gamma\eta g_\infty)d\eta = C_0^{P'}(g_\infty, \nu). \qquad \square$$

2.3.1 Minimal Eisenstein series

We consider the minimal Eisenstein series

$$E_0(g, \nu) = \sum_{\gamma\in P_0\cap\Gamma\backslash\Gamma} I_0(\gamma g, \nu),$$

where $I_0(g, \nu) = y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1}$. The constant term of $E_0(g, \nu)$ along P_0 is

$$C_0(g, \nu) := \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \sum_{\gamma\in P_0\cap\Gamma\backslash\Gamma} I_0(\gamma\eta g, \nu)d\eta.$$

It is clear from the definition of the constant term that $C_0(g, \nu)$ is invariant under left action by $N_0(\mathbb{R})$. So we may assume that $g = \mathrm{diag}(y_1, y_2, y_1^{-1}, y_2^{-1})$ is diagonal. Write

$$\eta = \begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ & 1 & n_4 & n_5 \\ & & 1 & \\ & & -n_1 & 1 \end{pmatrix} \in N_0(\mathbb{R}),$$

with the relation $n_3 = n_4 + n_1 n_5$. So the constant term can be rewritten as

$$C_0(g, \nu) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \sum_{\gamma\in P_0\cap\Gamma\backslash\Gamma} I_0(\gamma\eta g, \nu)dn_1 dn_2 dn_4 dn_5.$$

We break down the summation over $P_0 \cap \Gamma \backslash \Gamma$ via Bruhat decomposition

$$E_0(g,\nu) = \sum_{w \in W} E_{0,w}(g,\nu), \quad \text{where } E_{0,w}(g,\nu) := \sum_{\gamma \in P_0 \cap \Gamma \backslash (\Gamma \cap G_w)} I_0(\gamma g, \nu).$$

This gives a decomposition of the constant term

$$C_0(g,\nu) = \sum_{w \in W} C_{0,w}(g,\nu), \quad \text{where } C_{0,w}(g,\nu) := \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E_{0,w}(\eta g, \nu) d\eta.$$

Proposition 2.19. For $g = (g_\infty, 1, 1, \cdots) \in G(\mathbb{A})$, we have $M(w,\nu)\phi_\nu(g) = C_{0,w^{-1}}(g_\infty, \nu)$.

Proof. We expand

$$\begin{aligned} M(w,\pi_\nu)\phi_\nu(g) &= \int_{(N_0(\mathbb{Q}) \cap w N_0(\mathbb{Q}) w^{-1})\backslash N_0(\mathbb{A})} \phi_\nu(w^{-1}\eta g) d\eta \\ &= \int_{N_0(\mathbb{Q})\backslash N_0(\mathbb{A})} \sum_{u \in (N_0(\mathbb{Q}) \cap w N_0(\mathbb{Q}) w^{-1})\backslash N_0(\mathbb{Q})} \phi_\nu(w^{-1} u \eta g) d\eta. \end{aligned}$$

As $\phi_\pi(w^{-1}\eta g)$ is trivial at finite places, we only have to consider the archimedean place:

$$\begin{aligned} M(w,\pi_\nu)\phi_\nu(g) &= \int_{N_0(\mathbb{Z})\backslash N_0(\mathbb{R})} \sum_{u \in (N_0(\mathbb{Q}) \cap w N_0(\mathbb{Q}) w^{-1})\backslash N_0(\mathbb{Q})} \phi_{\nu,\infty}(w^{-1} u \eta g) d\eta \\ &= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \sum_{\gamma \in P_0(\mathbb{R}) \cap \Gamma \backslash (\Gamma \cap G_{w^{-1}})} I_0(\gamma g_\infty, \nu) = C_{0,w^{-1}}(g_\infty, \nu). \end{aligned}$$

□

For $g \in G(\mathbb{R})$, we abuse notation and also write g to denote the corresponding element $(g, 1, 1, \cdots) \in G(\mathbb{A})$. Via the functional equation, it suffices to just compute $C_{0,w}(g,\nu)$ for $w = \mathrm{id}, s_\alpha, s_\beta$.

Lemma 2.20. For $w \in W$, we have

$$C_{0,w}(g,\nu) = \sum_{\gamma \in R_w} \int_{U_w(\mathbb{R})} I_0(\gamma \eta g, \nu) d\eta.$$

Proof. By Lemma 2.8, we expand

$$\begin{aligned} C_{0,w}(g,\nu) &= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \sum_{\gamma \in P_0 \cap \Gamma \backslash (\Gamma \cap G_w)} I_0(\gamma \eta g, \nu) d\eta \\ &= \sum_{\gamma \in R_w} \sum_{\delta \in \Gamma_w} \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} I_0(\gamma \delta \eta g, \nu) d\eta. \end{aligned}$$

Recall that $\Gamma_w = \Gamma_0 \cap w^{-1}\Gamma_0^T w = U_w(\mathbb{Z})$. Through the decomposition $U = \overline{U}_w U_w$, the constant term can be rewritten as

$$C_{0,w}(g,\nu) = \sum_{\gamma \in R_w} \int_{\overline{U}_w(\mathbb{Z})\backslash \overline{U}_w(\mathbb{R})} \int_{U_w(\mathbb{R})} I_0(\gamma \eta' \eta g, \nu) d\eta d\eta'.$$

Now we show that the integral is independent of η'. Consider a Bruhat decomposition $\gamma = b_1 w t b_2$, with $b_1, b_2 \in U(\mathbb{R})$, and $t \in T$. Without loss of generality, we may assume $b_2 \in U_w(\mathbb{R})$. Then

$$\gamma = (b_1 w t \eta' t^{-1} w^{-1}) w t (\eta'^{-1} b_2)$$

is another Bruhat decomposition of γ. Hence

$$\int_{U_w(\mathbb{R})} I_0(\gamma\eta'\eta g,\nu)d\eta = \int_{U_w(\mathbb{R})} I_0\big((b_1wt\eta't^{-1}w^{-1})wt(\eta'^{-1}b_2)\eta'\eta g,\nu\big).$$

Noting that $\eta'^{-1}b_2\eta' \in U_w(\mathbb{R})$, the change of variables $\eta'^{-1}b_2\eta'\eta \mapsto \eta$ gives

$$\int_{U_w(\mathbb{R})} I_0(\gamma\eta'\eta g,\nu)d\eta = \int_{U_w(\mathbb{R})} I_0\big((b_1wt\eta't^{-1}w^{-1})wt\eta g,\nu\big) = \int_{U_w(\mathbb{R})} I_0(wt\eta g,\nu),$$

which is independent of η'. The lemma then follows from that $\overline{U}_w(\mathbb{Z})\backslash\overline{U}_w(\mathbb{R})$ has unit measure. □

We also need the following integration formula [GR07, 3.251.2]:

$$\int_{\mathbb{R}} |x|^{\mu}\left(a^2+x^2\right)^{\nu}dx = a^{\mu+2\nu+1}B\left(\frac{\mu+1}{2}, -\nu-\frac{\mu+1}{2}\right) \tag{2.16}$$

for $\operatorname{Re}\mu > -1$, $\operatorname{Re}\left(\nu+\frac{\mu}{2}\right) < \frac{1}{2}$, where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

stands for the beta function.

Now we compute the constant terms $C_{0,w}(g,\nu)$. Let $\gamma \in P_0 \cap \Gamma\backslash\Gamma$. Consider the Iwasawa decomposition of $\gamma\eta g$:

$$\gamma\eta g \equiv \begin{pmatrix} 1 & n_1' & n_2' & n_3' \\ & 1 & n_4' & n_5' \\ & & 1 & \\ & & -n_1' & 1 \end{pmatrix} \begin{pmatrix} y_1' & & & \\ & y_2' & & \\ & & {y_1'}^{-1} & \\ & & & {y_2'}^{-2} \end{pmatrix} \pmod{K}.$$

Since I_0 is left $N_0(\mathbb{R})$-invariant, we may assume that γ takes the form given in Section 2.2.4.

(i) $w = \mathrm{id}$: In this case, $R_{\mathrm{id}} = \{I\}$ is a singleton. So the constant term is simply

$$C_{0,\mathrm{id}}(g,\nu) = I_0(g,\nu) = y_1^{\nu_1+2}y_2^{2\nu_2-\nu_1+1}.$$

(ii) $w = s_\alpha$: The constant term is given by

$$C_{0,s_\alpha}(g,\nu) = \sum_{\gamma\in R_{s_\alpha}} \int_{U_{s_\alpha}(\mathbb{R})} I_0(\gamma\eta g)d\eta.$$

By linear algebra, we solve

$$y_1' = \frac{y_1y_2}{v_4\sqrt{s_1^2y_2^2+y_1^2}}, \quad y_2' = v_4\sqrt{s_1^2y_2^2+y_1^2}, \tag{2.17}$$

where $s_1 = n_1 - \frac{v_3}{v_4}$. So

$$C_{0,s_\alpha}(g,\nu) = \sum_{v_4\geq 1}\sum_{\substack{v_3 \pmod{v_4} \\ (v_3,v_4)=1}} \int_{\mathbb{R}} y_1^{\nu_1+2}y_2^{\nu_1+2}v_4^{2\nu_2-2\nu_1-1}\left(s_1^2y_2^2+y_1^2\right)^{\nu_2-\nu_1-1/2}dn_1.$$

By (2.16), we evaluate the integral and obtain for $\mathrm{Re}(\nu_1-\nu_2)>1$

$$\begin{aligned}
C_{0,s_\alpha}(g,\nu) &= y_1^{2\nu_2-\nu_1+2}y_2^{\nu_1+1}\sum_{v_4\geq 1}\sum_{\substack{v_3 \pmod{v_4}\\(v_3,v_4)=1}} v_4^{2\nu_2-2\nu_1-1}B\left(\frac{1}{2},\nu_1-\nu_2\right)\\
&= y_1^{2\nu_2-\nu_1+2}y_2^{\nu_1+1}\frac{\zeta(2\nu_1-2\nu_2)}{\zeta(2\nu_1-2\nu_2+1)}B\left(\frac{1}{2},\nu_1-\nu_2\right)\\
&= y_1^{2\nu_2-\nu_1+2}y_2^{\nu_1+1}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)},
\end{aligned}$$

where $\Lambda(s)=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is the completed zeta function as usual.

(iii) $w=s_\beta$: The constant term is given by

$$C_{0,s_\beta}(g,\nu)=\sum_{\gamma\in R_{s_\beta}}\int_{U_{s_\beta}(\mathbb{R})} I_0(\gamma\eta g)d\eta.$$

By linear algebra, we solve

$$y_1'=y_1,\quad y_2'=\frac{y_2}{v_{23}\sqrt{y_2^4+s_5^2}}, \tag{2.18}$$

where $s_5=n_5-\frac{v_{34}}{v_{23}}$. So

$$C_{0,s_\beta}(g,\nu)=\sum_{v_{23}\geq 1}\sum_{\substack{v_{34}\pmod{v_{23}}\\(v_{23},v_{34})=1}}\int_{\mathbb{R}} y_1^{\nu_1+2}y_2^{2\nu_2-2\nu_1-1}v_{23}^{\nu_1-2\nu_2-1}(y_2^4+s_5^2)^{\nu_1/2-\nu_2-1/2}dn_5.$$

By (2.16), we evaluate the integral and obtain for $\mathrm{Re}(2\nu_2-\nu_1)>1$

$$\begin{aligned}
C_{0,s_\beta}(g,\nu) &= y_1^{\nu_1+2}y_2^{\nu_1-2\nu_2+1}\sum_{v_{23}\geq 1}\sum_{\substack{v_{34}\pmod{v_{23}}\\(v_{23},v_{34})=1}} v_{23}^{\nu_1-2\nu_2-1}B\left(\frac{1}{2},\nu_2-\frac{\nu_1}{2}\right)\\
&= y_1^{\nu_1+2}y_2^{\nu_1-2\nu_2+1}\frac{\zeta(2\nu_2-\nu_1)}{\zeta(2\nu_2-\nu_1+1)}B\left(\frac{1}{2},\nu_2-\frac{\nu_1}{2}\right)\\
&= y_1^{\nu_1+2}y_2^{\nu_1-2\nu_2+1}\frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}.
\end{aligned}$$

Remark. The constant terms $C_{0,s_\alpha}(g,\nu)$ and $C_{0,s_\beta}(g,\nu)$ are originally defined on an open subset of $\mathfrak{a}_{0\mathbb{C}}^*$, but it follows readily from the expressions that they can be continued into meromorphic functions on $\mathfrak{a}_{0\mathbb{C}}^*$.

By Proposition 2.19, we obtain the expressions for the intertwining operators:

$$C_{0,s_\alpha}(g,\nu)=M(s_\alpha,\pi_{(\nu_1,\nu_2)})\phi_{(\nu_1,\nu_2)}=\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}\phi_{(2\nu_2-\nu_1,\nu_2)}, \tag{2.19}$$

$$C_{0,s_\beta}(g,\nu)=M(s_\beta,\pi_{(\nu_1,\nu_2)})\phi_{(\nu_1,\nu_2)}=\frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}\phi_{(\nu_1,\nu_1-\nu_2)}. \tag{2.20}$$

By the functional equation of Langlands, we compute the constant terms for other Weyl elements:

$$\begin{aligned} C_{0,s_\alpha s_\beta}(g,\nu) &= M(s_\beta s_\alpha,\pi_\nu)\phi_\nu(g) = M(s_\beta,\pi_{s_\alpha\nu})M(s_\alpha,\phi_\nu)\phi_\nu(g) \\ &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{2\nu_2-\nu_1+2}y_2^{-\nu_1+1}, \quad (2.21) \\ C_{0,s_\beta s_\alpha}(g,\nu) &= M(s_\alpha s_\beta,\pi_\nu)\phi_\nu(g) = M(s_\alpha,\pi_{s_\beta\nu})M(s_\beta,\phi_\nu)\phi_\nu(g) \\ &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}y_1^{\nu_1-2\nu_2+2}y_2^{\nu_1+1}, \quad (2.22) \\ C_{0,s_\alpha s_\beta s_\alpha}(g,\nu) &= M(s_\alpha s_\beta s_\alpha,\pi_\nu)\phi_\nu(g) = M(s_\alpha,\pi_{s_\beta s_\alpha\nu})M(s_\beta s_\alpha,\pi_\nu)\phi_\nu(g) \\ &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{-\nu_1+2}y_2^{2\nu_2-\nu_1+1}, \quad (2.23) \\ C_{0,s_\beta s_\alpha s_\beta}(g,\nu) &= M(s_\beta s_\alpha s_\beta,\pi_\nu)\phi_\nu(g) = M(s_\beta,\pi_{s_\alpha s_\beta\nu})M(s_\alpha s_\beta,\pi_\nu)\phi_\nu(g) \\ &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}y_1^{\nu_1-2\nu_2+2}y_2^{-\nu_1+1}, \quad (2.24) \\ C_{0,w_0}(g,\nu) &= M(s_\beta s_\alpha s_\beta s_\alpha,\pi_\nu)\phi_\nu(g) = M(s_\beta,\pi_{s_\alpha s_\beta s_\alpha})M(s_\alpha s_\beta s_\alpha,\pi_\nu)\phi_\nu(g) \\ &= \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}\frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{-\nu_1+2}y_2^{\nu_1-2\nu_2+1}. \quad (2.25) \end{aligned}$$

Again, these constant terms are originally defined on an open subset of $\mathfrak{a}_{0\mathbb{C}}^*$, but they can be continued into meromorphic functions on $\mathfrak{a}_{0\mathbb{C}}^*$.

The computations above thus summarise into the following theorem:

Theorem 2.21. The constant term of the minimal Eisenstein series $E_0(g,\nu)$ along the minimal parabolic subgroup P_0 is given by

$$C_0(g,\nu) = \sum_{w\in W} C_{0,w}(g,\nu),$$

where

$$\begin{aligned} C_{0,\mathrm{id}}(g,\nu) &= y_1^{\nu_1+2}y_2^{2\nu_2-\nu_1+1}, \\ C_{0,s_\alpha}(g,\nu) &= \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{2\nu_2-\nu_1+2}y_2^{\nu_1+1}, \\ C_{0,s_\beta}(g,\nu) &= \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}y_1^{\nu_1+2}y_2^{\nu_1-2\nu_2+1}, \\ C_{0,s_\alpha s_\beta}(g,\nu) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{2\nu_2-\nu_1+2}y_2^{-\nu_1+1}, \\ C_{0,s_\beta s_\alpha}(g,\nu) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}y_1^{\nu_1-2\nu_2+2}y_2^{\nu_1+1}, \\ C_{0,s_\alpha s_\beta s_\alpha}(g,\nu) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{-\nu_1+2}y_2^{2\nu_2-\nu_1+1}, \\ C_{0,s_\beta s_\alpha s_\beta}(g,\nu) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}y_1^{\nu_1-2\nu_2+2}y_2^{-\nu_1+1}, \\ C_{0,s_\alpha s_\beta s_\alpha s_\beta}(g,\nu) &= \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)}\frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)}\frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)}\frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)}y_1^{-\nu_1+2}y_2^{\nu_1-2\nu_2+1}. \end{aligned}$$

The constant term $C_0(g,\nu)$ is originally defined on an open subset of $\mathfrak{a}_{0\mathbb{C}}^*$, but it can be continued into a meromorphic function on $\mathfrak{a}_{0\mathbb{C}}^*$.

For the computation of the constant term of $E_0(g,\nu)$ along other parabolic subgroups, we need some more adelic theory. Let $W = W(T,G)$ be the Weyl group of G. For a standard parabolic subgroup $P = NM$ of G, we denote by $W_M := W(T,M)$ the Weyl group of M. We write

$$W_M^\bullet := \{w \in W \mid w^{-1}(\alpha) > 0 \, \forall \alpha \in R^+(T,M)\}$$

where $R^+(T,M)$ is the set of positive roots of M with respect to torus T.

Proposition 2.22. Let π be an irreducible automorphic representation of $M_0(\mathbb{A})$, and ϕ_π an element in $A(N_0(\mathbb{A})M_0(\mathbb{Q})\backslash G(\mathbb{A}))_\pi$. Then the constant term of $E(\phi_\pi,\pi)$ along P is

$$E_P(\phi_\pi,\pi)(g) = \sum_{w\in W_M^\bullet} \sum_{m\in M(\mathbb{Q})\cap wP_0(\mathbb{Q})w^{-1}\backslash M(\mathbb{Q})} M(w,\pi)\phi_\pi(mg).$$

Proof. First we consider a Bruhat decomposition

$$G(\mathbb{Q}) = \coprod_{w\in W} P_0(\mathbb{Q})w^{-1}P_0(\mathbb{Q}) = \coprod_{w\in W_M\backslash W} P_0(\mathbb{Q})w^{-1}P(\mathbb{Q}).$$

Now observe that $W_M^\bullet$ is a system of representatives for $W_M\backslash W$. It then follows from the definition that

$$E_P(\phi_\pi,\pi)(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\gamma\in P_0(\mathbb{Q})\backslash G(\mathbb{Q})} \phi_\pi(\gamma\eta g)d\eta.$$

Since the Eisenstein series $E(\phi_\pi,\pi)$ is absolutely convergent, and the constant term integral is over a compact set, we may exchange the order of sums and integrals, and deduce that

$$\begin{aligned} E_P(\phi_\pi,\pi)(g) = & \sum_{w\in W_M^\bullet} \sum_{m\in(M(\mathbb{Q})\cap wP_0(\mathbb{Q})w^{-1})\backslash M(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \\ & \sum_{\eta'\in(N(\mathbb{Q})\cap m^{-1}wP_0(\mathbb{Q})w^{-1}m)\backslash N(\mathbb{Q})} \phi_\pi(w^{-1}m\eta'\eta g)d\eta \\ = & \sum_{w\in W_M^\bullet} \sum_{m\in(M(\mathbb{Q})\cap wP_0(\mathbb{Q})w^{-1})\backslash M(\mathbb{Q})} \int_{N(\mathbb{Q})\cap wP_0(\mathbb{Q})w^{-1}\backslash N(\mathbb{A})} \phi_\pi(w^{-1}\eta mg)d\eta. \end{aligned}$$

Observe that $N(\mathbb{Q}) \cap wP_0(\mathbb{Q})w^{-1} = N(\mathbb{Q}) \cap wN_0(\mathbb{Q})w^{-1}$. So

$$E_P(\phi_\pi,\pi)(g) = \sum_{w\in W_M^\bullet} \sum_{m\in(M(\mathbb{Q})\cap wP_0(\mathbb{Q})w^{-1})\backslash M(\mathbb{Q})} M(w,\pi)\phi_\pi(mg). \qquad \square$$

We compute the constant term of $E_0(g,\nu)$ along the Siegel parabolic subgroup $P_\alpha = N_\alpha M_\alpha$. From Proposition 2.18, we have

$$C_0^\alpha(g,\nu) = E_{P_\alpha}(\phi_\nu,\pi_\nu)(g) = \sum_{w\in W_{M_\alpha}^\bullet} \sum_{m\in M_\alpha(\mathbb{Q})\cap wP_0(\mathbb{Q})w^{-1}\backslash M_\alpha(\mathbb{Q})} M(w,\pi_\nu)\phi_\nu(mg).$$

Since $R^+(T,M_\alpha) = \{\alpha\}$, we compute that $W_{M_\alpha}^\bullet = \{\mathrm{id}, s_\beta, s_\beta s_\alpha, s_\beta s_\alpha s_\beta\}$. Recall (2.20), (2.21), (2.24):

$$\begin{aligned} \phi_\nu(g) &= y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1}, \\ M(s_\beta,\nu)\phi_\nu(g) &= \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)} y_1^{\nu_1+2} y_2^{\nu_1-2\nu_2+1}, \\ M(s_\beta s_\alpha,\nu)\phi_\nu(g) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)} \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)} y_1^{2\nu_2-\nu_1+2} y_2^{-\nu_1+1}, \\ M(s_\beta s_\alpha s_\beta,\nu)\phi_\nu(g) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)} \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)} \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)} y_1^{\nu_1-2\nu_2+2} y_2^{-\nu_1+1}. \end{aligned}$$

Now we compute $M(w,\nu)(mg)$ for $w \in W^{\bullet}_{M_\alpha}$ and $m \in M_\alpha(\mathbb{Q}) \cap wP_0(\mathbb{Q})w^{-1}\backslash M_\alpha(\mathbb{Q})$. For $w \in W^{\bullet}_{M_\alpha}$, a set of coset representatives of $M_\alpha(\mathbb{Q}) \cap wP_0(\mathbb{Q})w^{-1}\backslash M_\alpha(\mathbb{Q})$ is given by

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cup \left\{ m^{\alpha}_{\kappa_1,\kappa_2} := \begin{pmatrix} & \kappa_2^{-1} & & \\ \kappa_2 & -\kappa_1 & & \\ & & \kappa_1 & \kappa_2 \\ & & \kappa_2^{-1} & \end{pmatrix} \,\middle|\, \kappa_2 \in \mathbb{N}, (\kappa_1,\kappa_2)=1 \right\},$$

which is independent of the choice of $w \in W^{\bullet}_{M_\alpha}$. Note that $m^{\alpha}_{\kappa_1,\kappa_2}$ are simply matrix representatives given in Section 2.2.4 with Plücker coordinates $v = (0,0,\kappa_1,\kappa_2;0,0,0,0,0,1)$, which lies in the class $w = s_\alpha$ in Bruhat decomposition. As the Plücker coordinates of these representatives satisfy the conditions in Proposition 2.3, they are equivalent to integral matrices with unit determinant under left action by $P_0(\mathbb{Q})$. So the contribution from the finite places is trivial, and we only have to consider the archimedean place.

By (2.17), we see that

$$\phi_\nu(m^{\alpha}_{\kappa_1,\kappa_2}g) = y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1} Q_{\alpha,g}(\kappa_1,\kappa_2)^{\nu_2-\nu_1-1/2},$$

where $Q_{\alpha,g}(\kappa_1,\kappa_2)$ is the quadratic form defined by

$$Q_{\alpha,g}(\kappa_1,\kappa_2) := \kappa_1^2 - 2n_1\kappa_1\kappa_2 + \left(n_1^2 + \frac{y_1^2}{y_2^2}\right)\kappa_2^2 = |\kappa_2 z_\alpha + \kappa_1|^2,$$

where $z_\alpha := -n_1 + \frac{y_1}{y_2} i$. Then, summing $\phi_\nu(mg)$ gives a GL(2) Eisenstein series:

$$\begin{aligned} \phi_\nu(g) + \sum_{\substack{\kappa_2 \in \mathbb{N} \\ (\kappa_1,\kappa_2)=1}} \phi_\nu(m^{\alpha}_{\kappa_1,\kappa_2}g) &= \frac{1}{2} y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1} \sum_{\substack{\kappa_1,\kappa_2 \in \mathbb{Z} \\ (\kappa_1,\kappa_2)=1}} |\kappa_2 z_\alpha + \kappa_1|^{2\nu_2-2\nu_1-1} \\ &= E(z_\alpha, \nu_1-\nu_2)\, y_1^{\nu_2+3/2} y_2^{\nu_2+3/2}. \end{aligned}$$

Finally, through the intertwining operators (2.19), (2.20), we compute

$$\begin{aligned} \sum_m M(s_\beta,\nu)\phi_\nu(mg) &= \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)} E(z_\alpha,\nu_2)\, y_1^{\nu_1-\nu_2+3/2} y_2^{\nu_1-\nu_2+3/2}, \\ \sum_m M(s_\beta s_\alpha,\nu)\phi_\nu(mg) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)} \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)} E(z_\alpha,\nu_2)\, y_1^{\nu_2-\nu_1+3/2} y_2^{\nu_2-\nu_1+3/2}, \\ \sum_m M(s_\beta s_\alpha s_\beta,\nu)\phi_\nu(mg) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)} \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)} \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)} E(z_\alpha,\nu_1-\nu_2)\, y_1^{-\nu_2+3/2} y_2^{-\nu_2+3/2}. \end{aligned}$$

Since the GL(2) Eisenstein series $E(z,s)$ can be continued into a meromorphic function on $\mathbb{C}$ as a function in s, all the sums above can be continued into meromorphic functions on $\mathfrak{a}^*_{0\mathbb{C}}$. So we conclude:

Theorem 2.23. The constant term of the minimal Eisenstein series $E_0(g,\nu)$ along the Siegel parabolic subgroup P_α is given by

$$C_0^\alpha(g,\nu) = \sum_{w \in W^{\bullet}_{M_\alpha}} C^\alpha_{0,w}(g,\nu),$$

where $W^{\bullet}_{M_\alpha} = \{\mathrm{id}, s_\beta, s_\beta s_\alpha, s_\beta s_\alpha s_\beta\}$, and

$$\begin{aligned}
C^{\alpha}_{0,\mathrm{id}}(g,\nu) &= E\left(z_\alpha, \nu_1-\nu_2\right) y_1^{\nu_2+3/2} y_2^{\nu_2+3/2},\\
C^{\alpha}_{0,s_\beta}(g,\nu) &= \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)} E\left(z_\alpha,\nu_2\right) y_1^{\nu_1-\nu_2+3/2} y_2^{\nu_1-\nu_2+3/2},\\
C^{\alpha}_{0,s_\beta s_\alpha}(g,\nu) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)} \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)} E\left(z_\alpha,\nu_2\right) y_1^{\nu_2-\nu_1+3/2} y_2^{\nu_2-\nu_1+3/2},\\
C^{\alpha}_{0,s_\beta s_\alpha s_\beta}(g,\nu) &= \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)} \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)} \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)} E\left(z_\alpha,\nu_1-\nu_2\right) y_1^{-\nu_2+3/2} y_2^{-\nu_2+3/2},
\end{aligned}$$

with $z_\alpha := -n_1 + \frac{y_1}{y_2} i$. Moreover, the constant term $C^{\alpha}_0(g,\nu)$ can be continued into a meromorphic function on $\mathfrak{a}^*_{0\mathbb{C}}$.

We compute the constant term of $E_0(g,\nu)$ along the non-Siegel maximal parabolic subgroup $P_\beta = N_\beta M_\beta$. From Proposition 2.18, we have

$$C^{\beta}_0(g,\nu) = E_{P_\beta}(\phi_\nu,\pi_\nu)(g) = \sum_{w\in W^{\bullet}_{M_\beta}} \sum_{m\in M_\beta(\mathbb{Q})\cap wP_0(\mathbb{Q})w^{-1}\backslash M_\beta(\mathbb{Q})} M(w,\pi_\nu)\phi_\nu(mg).$$

Since $R^+(T, M_\beta) = \{\beta\}$, we compute that $W^{\bullet}_{M_\beta} = \{\mathrm{id}, s_\alpha, s_\alpha s_\beta, s_\alpha s_\beta s_\alpha\}$. Recall (2.19), (2.22), (2.23):

$$\begin{aligned}
\phi_\nu(g) &= y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1},\\
M(s_\alpha,\nu)\phi_\nu(g) &= \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)} y_1^{2\nu_2-\nu_1+2} y_2^{\nu_1+1},\\
M(s_\alpha s_\beta,\nu)\phi_\nu(g) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)} \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)} y_1^{\nu_1-2\nu_2+2} y_2^{\nu_1+1},\\
M(s_\alpha s_\beta s_\alpha,\nu)\phi_\nu(g) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)} \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)} \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)} y_1^{-\nu_1+2} y_2^{2\nu_2-\nu_1+1}.
\end{aligned}$$

Now we compute $M(w,\nu)(mg)$ for $w \in W^{\bullet}_{M_\beta}$ and $m \in M_\beta(\mathbb{Q}) \cap wP_0(\mathbb{Q})w^{-1}\backslash M_\beta(\mathbb{Q})$. For $w \in W^{\bullet}_{M_\beta}$, a set of coset representatives of $M_\beta(\mathbb{Q}) \cap wP_0(\mathbb{Q})w^{-1}\backslash M_\alpha(\mathbb{Q})$ is given by

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cup \left\{ m^{\beta}_{\kappa_1,\kappa_2} := \begin{pmatrix} 1 & & & \\ & & & \kappa_1^{-1} \\ & & 1 & \\ & -\kappa_1 & & \kappa_2 \end{pmatrix} \,\middle|\, \kappa_1\in\mathbb{N}, (\kappa_1,\kappa_2)=1 \right\},$$

which is independent of the choice of $w \in W^{\bullet}_{M_\beta}$. Note that $m^{\beta}_{\kappa_1,\kappa_2}$ are simply matrix representatives given in Section 2.2.4 with Plücker coordinates $v = (0,0,1,0;0,0,0,\kappa_1,0,\kappa_2)$, which lies in the class $w = s_\beta$ in Bruhat decomposition. As the Plücker coordinates of these representatives satisfy the conditions in Proposition 2.3, they are equivalent to integral matrices with unit determinant under left action by $P_0(\mathbb{Q})$. So the contribution from the finite places is trivial, and we only have to consider the archimedean place.

By (2.18), we see that

$$\phi_\nu(m^{\beta}_{\kappa_1,\kappa_2} g) = y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1} Q_{\beta,g}(\kappa_1,\kappa_2)^{\nu_1/2-\nu_2-1/2},$$

where $Q_{\beta,g}(\kappa_1,\kappa_2)$ is the quadratic form defined by

$$Q_{\beta,g}(\kappa_1,\kappa_2) := (n_5^2 + y_2^4)\kappa_1^2 - 2n_5\kappa_1\kappa_2 + \kappa_2^2 = \left|\kappa_1 z_\beta + \kappa_2\right|^2,$$

where $z_\beta := -n_5 + y_2^2 i$. Then, summing $\phi_\nu(mg)$ gives a GL(2) Eisenstein series:

$$\begin{aligned}\phi_\nu(g) + \sum_{\substack{\kappa_1\in\mathbb{N}\\(\kappa_1,\kappa_2)=1}} \phi_\nu(m^\beta_{\kappa_1,\kappa_2} g) &= \frac{1}{2} y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1} \sum_{\substack{\kappa_1,\kappa_2\in\mathbb{Z}\\(\kappa_1,\kappa_2)=1}} |\kappa_1 z_\beta + \kappa_2|^{\nu_1-2\nu_2-1}\\ &= E(z_\beta, \nu_2 - \nu_1/2) y_1^{\nu_1+2}.\end{aligned}$$

Finally, through the intertwining operators (2.19), (2.20), we compute

$$\begin{aligned}\sum_m M(s_\alpha,\nu)\phi_\nu(mg) &= \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)} E(z_\beta,\nu_1/2) y_1^{2\nu_2-\nu_1+2},\\ \sum_m M(s_\alpha s_\beta,\nu)\phi_\nu(mg) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)} \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)} E(z_\beta,\nu_1/2) y_1^{\nu_1-2\nu_2+2},\\ \sum_m M(s_\alpha s_\beta s_\alpha,\nu)\phi_\nu(mg) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)} \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)} \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)} E(z_\beta,\nu_2-\nu_1/2) y_1^{-\nu_1+2}.\end{aligned}$$

Again, the sums above can be continued into meromorphic functions on $\mathfrak{a}^*_{0\mathbb{C}}$. So we conclude:

Theorem 2.24. The constant term of the minimal Eisenstein series $E_0(g,\nu)$ along the non-Siegel maximal parabolic subgroup P_β is given by

$$C_0^\beta(g,\nu) = \sum_{w\in W^\bullet_{M_\beta}} C^\beta_{0,w}(g,\nu),$$

where

$$\begin{aligned}C^\beta_{0,\mathrm{id}}(g,\nu) &= E(z_\beta,\nu_2-\nu_1/2) y_1^{\nu_1+2},\\ C^\beta_{0,s_\alpha}(g,\nu) &= \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)} E(z_\beta,\nu_1/2) y_1^{2\nu_2-\nu_1+2},\\ C^\beta_{0,s_\alpha s_\beta}(g,\nu) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)} \frac{\Lambda(2\nu_2-\nu_1)}{\Lambda(2\nu_2-\nu_1+1)} E(z_\beta,\nu_1/2) y_1^{\nu_1-2\nu_2+2},\\ C^\beta_{0,s_\alpha s_\beta s_\alpha}(g,\nu) &= \frac{\Lambda(2\nu_2)}{\Lambda(2\nu_2+1)} \frac{\Lambda(\nu_1)}{\Lambda(\nu_1+1)} \frac{\Lambda(2\nu_1-2\nu_2)}{\Lambda(2\nu_1-2\nu_2+1)} E(z_\beta,\nu_2-\nu_1/2) y_1^{-\nu_1+2},\end{aligned}$$

with $z_\beta := -n_5+y_2^2 i$. Moreover, the constant term $C_0^\beta(g,\nu)$ can be continued into a meromorphic function on $\mathfrak{a}^*_{0\mathbb{C}}$.

2.3.2 Maximal Eisenstein series $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$

Now we compute the constant terms of the maximal Eisenstein series $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$. In Proposition 2.10 we showed that they can be expressed as residues of the minimal Eisenstein series $E_0(g,\nu)$. Since the constant terms are an integral over a compact set, we may find the constant terms of $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$ by taking the residues of the constant terms of the minimal Eisenstein series. It is hence straightforward to obtain the following statements.

Corollary 2.25. The constant term of $E_\alpha(g,\nu,1)$ along the minimal parabolic is given by

$$C^0_\alpha(g,\nu,1) = C^0_{\alpha,\mathrm{id}}(g,\nu,1) + C^0_{\alpha,s_\beta}(g,\nu,1) + C^0_{\alpha,s_\beta s_\alpha}(g,\nu,1) + C^0_{\alpha,s_\beta s_\alpha s_\beta}(g,\nu,1),$$

where

$$
\begin{aligned}
C^0_{\alpha,\mathrm{id}}(g,\nu,1) &= y_1^{\nu+3/2} y_2^{\nu+3/2},\\
C^0_{\alpha,s_\beta}(g,\nu,1) &= \frac{\Lambda(\nu+\frac{1}{2})}{\Lambda(\nu+\frac{3}{2})} y_1^{\nu+3/2} y_2^{-\nu+1/2},\\
C^0_{\alpha,s_\beta s_\alpha}(g,\nu,1) &= \frac{\Lambda(2\nu)}{\Lambda(2\nu+1)} \frac{\Lambda(\nu+\frac{1}{2})}{\Lambda(\nu+\frac{3}{2})} y_1^{-\nu+3/2} y_2^{\nu+1/2},\\
C^0_{\alpha,s_\beta s_\alpha s_\beta}(g,\nu,1) &= \frac{\Lambda(\nu-\frac{1}{2})}{\Lambda(\nu+\frac{1}{2})} \frac{\Lambda(2\nu)}{\Lambda(2\nu+1)} \frac{\Lambda(\nu+\frac{1}{2})}{\Lambda(\nu+\frac{3}{2})} y_1^{-\nu+3/2} y_2^{-\nu+3/2}.
\end{aligned}
$$

Corollary 2.26. The constant term of $E_\alpha(g,\nu,1)$ along the Siegel parabolic is given by

$$
C_\alpha\left(g,\nu,1\right) = C_{\alpha,\mathrm{id}}(g,\nu,1) + C_{\alpha,s_\beta s_\alpha}(g,\nu,1) + C_{\alpha,s_\beta s_\alpha s_\beta}(g,\nu,1),
$$

where

$$
\begin{aligned}
C_{\alpha,\mathrm{id}}(g,\nu,1) &= y_1^{\nu+3/2} y_2^{\nu+3/2},\\
C_{\alpha,s_\beta s_\alpha}(g,\nu,1) &= \frac{\Lambda(\nu+\frac{1}{2})}{\Lambda(\nu+\frac{3}{2})} E(z_\alpha,\nu) y_1 y_2,\\
C_{\alpha,s_\beta s_\alpha s_\beta}(g,\nu,1) &= \frac{\Lambda(\nu-\frac{1}{2})}{\Lambda(\nu+\frac{1}{2})} \frac{\Lambda(2\nu)}{\Lambda(2\nu+1)} \frac{\Lambda(\nu+\frac{1}{2})}{\Lambda(\nu+\frac{3}{2})} y_1^{-\nu+3/2} y_2^{-\nu+3/2},
\end{aligned}
$$

with $z_\alpha := -n_1 + \frac{y_1}{y_2} i$.

Corollary 2.27. The constant term of $E_\alpha(g,\nu,1)$ along the non-Siegel parabolic is given by

$$
C^\beta_\alpha\left(g,\nu,1\right) = C^\beta_{\alpha,s_\alpha}(g,\nu,1) + C^\beta_{\alpha,s_\alpha s_\beta s_\alpha}(g,\nu,1),
$$

where

$$
\begin{aligned}
C^\beta_{\alpha,s_\alpha}(g,\nu,1) &= E\left(z_\beta,\nu/2+1/4\right) y_1^{\nu+3/2},\\
C^\beta_{\alpha,s_\alpha s_\beta s_\alpha}(g,\nu,1) &= \frac{\Lambda(2\nu)}{\Lambda(2\nu+1)} \frac{\Lambda(\nu+\frac{1}{2})}{\Lambda(\nu+\frac{3}{2})} E\left(z_\beta,\nu/2-1/4\right) y_1^{-\nu+3/2},
\end{aligned}
$$

with $z_\beta := -n_5 + y_2^2 i$.

Corollary 2.28. The constant term of $E_\beta(g,\nu,1)$ along the minimal parabolic is given by

$$
C^0_\beta\left(g,\nu,1\right) = C^0_{\beta,\mathrm{id}}(g,\nu,1) + C^0_{\beta,s_\alpha}(g,\nu,1) + C^0_{\beta,s_\alpha s_\beta}(g,\nu,1) + C^0_{\beta,s_\alpha s_\beta s_\alpha}(g,\nu,1),
$$

where

$$
\begin{aligned}
C^0_{\beta,\mathrm{id}}(g,\nu,1) &= y_1^{\nu+2},\\
C^0_{\beta,s_\alpha}(g,\nu,1) &= \frac{\Lambda(\nu+1)}{\Lambda(\nu+2)} y_1 y_2^{\nu+1},\\
C^0_{\beta,s_\alpha s_\beta}(g,\nu,1) &= \frac{\Lambda(\nu)}{\Lambda(\nu+1)} \frac{\Lambda(\nu+1)}{\Lambda(\nu+2)} y_1 y_2^{-\nu+1},\\
C^0_{\beta,s_\alpha s_\beta s_\alpha}(g,\nu,1) &= \frac{\Lambda(\nu-1)}{\Lambda(\nu)} \frac{\Lambda(\nu)}{\Lambda(\nu+1)} \frac{\Lambda(\nu+1)}{\Lambda(\nu+2)} y_1^{-\nu+2}.
\end{aligned}
$$

Corollary 2.29. The constant term of $E_\beta\left(g,\nu,1\right)$ along the Siegel parabolic is given by

$$
C^\alpha_\beta\left(g,\nu,1\right) = C^\alpha_{\beta,s_\beta}(g,\nu,1) + C^\alpha_{\beta,s_\beta s_\alpha s_\beta}(g,\nu,1),
$$

where

$$
\begin{aligned}
C^\alpha_{\beta,s_\beta}(g,\nu,1) &= E\left(z_\alpha,(\nu+1)/2\right) y_1^{\nu/2+1} y_2^{\nu/2+1},\\
C^\alpha_{\beta,s_\beta s_\alpha s_\beta}(g,\nu,1) &= \frac{\Lambda(\nu)}{\Lambda(\nu+1)} \frac{\Lambda(\nu+1)}{\Lambda(\nu+2)} E\left(z_\alpha,(\nu-1)/2\right) y_1^{-\nu/2+1} y_2^{-\nu/2+1},
\end{aligned}
$$

with $z_\alpha := -n_1 + \frac{y_1}{y_2} i$.

Corollary 2.30. The constant term of $E_\beta(g,\nu,1)$ along the non-Siegel parabolic is given by

$$C_\beta\left(g,\nu,1\right)=C_{\beta,\mathrm{id}}(g,\nu,1)+C_{\beta,s_\alpha s_\beta}(g,\nu,1)+C_{\beta,s_\alpha s_\beta s_\alpha}(g,\nu,1),$$

where

$$\begin{aligned}C_{\beta,\mathrm{id}}(g,\nu,1)&=y_1^{\nu+2},\\ C_{\beta,s_\alpha s_\beta}(g,\nu,1)&=\frac{\Lambda(\nu+1)}{\Lambda(\nu+2)}E\left(z_\beta,\nu/2\right)y_1,\\ C_{\beta,s_\alpha s_\beta s_\alpha}(g,\nu,1)&=\frac{\Lambda(\nu-1)}{\Lambda(\nu)}\frac{\Lambda(\nu)}{\Lambda(\nu+1)}\frac{\Lambda(\nu+1)}{\Lambda(\nu+2)}y_1^{-\nu+2},\end{aligned}$$

with $z_\beta:=-n_5+y_2^2i$.

2.4 Sp(4) Ramanujan sums

The aim of this section is to give an explicit characterisation for Sp(4) Ramanujan sums. Ramanujan sums naturally arises in the theory of Eisenstein series. We start with a brief review for classical Ramanujan sums on GL(2). A detailed exposition can be found in [Gol06, Bum84].

A Ramanujan sum is an exponential sum of the following form:

$$c_q(n)=\sum_{\substack{a=1\\(a,q)=1}}^{q}e\left(\frac{an}{q}\right),$$

where $e(x):=e^{2\pi ix}$ as usual. To find the Fourier expansion of the GL(2) Eisenstein series $E(z,s)$, we need the following identity:

$$\zeta(s)\sum_{q=1}^{\infty}c_q(n)q^{-s}=\sigma_{1-s}(n),$$

where $\sigma_\nu(n):=\sum_{d|n}d^\nu$ is the divisor function. This is not difficult to prove. First observe that

$$\sum_{d|q}c_d(n)=\sum_{a=1}^{q}e\left(\frac{an}{q}\right)=\begin{cases}q & \text{if } q\mid n,\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\zeta(s)\sum_{q=1}^{\infty}c_q(n)q^{-s}=\sum_{q=1}^{\infty}\Big(\sum_{d|q}c_d(n)\Big)q^{-s}=\sum_{q|n}q^{1-s}=\sigma_{1-s}(n).$$

So Ramanujan sums and divisor sums are related by an identity of the form above. This actually holds for more general cases, with appropriately defined Ramanujan sums and divisor sums. For example, an explicit characterisation for GL(3) Ramanujan sums is found in [Bum84].

We start by defining Sp(4) Ramanujan sums. Recall from Section 2.2.4 the set of representatives R_{w_0} for $P_0\cap\Gamma\backslash\Gamma\cap G_{w_0}/\Gamma_{w_0}$. For fixed $v_1,v_{12}\in\mathbb{N}$, we denote by $R_{w_0}(v_1,v_{12})$ the subset of R_{w_0} with Plücker coordinates v_1,v_{12} as given. Now we define the Sp(4) Ramanujan sum:

$$R_{v_1,v_{12}}(n_1,n_2)=\sum_{v\in R_{w_0}(v_1,v_{12})}e\left(\frac{n_1v_2}{v_1}+\frac{n_2v_{14}}{v_{12}}\right). \tag{2.26}$$

We define an appropriate analogue of the divisor function on Sp(4). We start with the symplectic Schur function for $\mathrm{Sp}(4,\mathbb{C})$:

$$\mathrm{Sp}_{\lambda_1,\lambda_2}(x_1,x_2) = \frac{\begin{vmatrix} x_1^{\lambda_1+2} - x_1^{-(\lambda_1+2)} & x_2^{\lambda_1+2} - x_2^{-(\lambda_1+2)} \\ x_1^{\lambda_2+1} - x_1^{-(\lambda_2+1)} & x_2^{\lambda_2+1} - x_2^{-(\lambda_2+1)} \end{vmatrix}}{\begin{vmatrix} x_1^2 - x_1^{-2} & x_2^2 - x_2^{-2} \\ x_1 - x_1^{-1} & x_2 - x_2^{-1} \end{vmatrix}} \quad (\lambda_1 \geq \lambda_2 \geq 0).$$

Remark. Terms in $\mathrm{Sp}_{e_1+e_2,e_2}(x_1,x_2)$ correspond to the dimensions of weight spaces of the irreducible representation $V((e_1\omega_1 + e_2\omega_2))$ of $\mathfrak{sp}(4,\mathbb{C})$, and is a special instance of the Weyl character formula (see [FH04, Ch. 24]).

Now we define a multiplicative function σ_{ν_1,ν_2} by setting for p prime

$$\sigma_{\nu_1,\nu_2}(p^{e_1},p^{e_2}) = p^{(e_1+e_2)\nu_1+e_1\nu_2}\,\mathrm{Sp}_{e_1+e_2,e_1}(p^{\nu_1},p^{\nu_2}).$$

We state the main result of the section. Let

$$\mathcal{R}_{\nu_1,\nu_2}(n_1,n_2) = \sum_{v_1,v_{12}\geq 1} R_{v_1,v_{12}}(n_1,n_2) v_1^{-\nu_1} v_{12}^{-\nu_2}.$$

Proposition 2.31. For $\mathrm{Re}\,\nu_1, \mathrm{Re}\,\nu_2 > 2$, the sum $\mathcal{R}_{\nu_1,\nu_2}(n_1,n_2)$ evaluates as

$$\mathcal{R}_{\nu_1,\nu_2}(n_1,n_2) = \begin{cases} \dfrac{\sigma_{\frac{3}{2}-\frac{\nu_1}{2}-\nu_2,\frac{1}{2}-\frac{\nu_1}{2}}(n_1,n_2)}{\zeta(\nu_1)\zeta(\nu_2)\zeta(\nu_1+\nu_2-1)\zeta(\nu_1+2\nu_2-2)} & \text{if } n_1,n_2 \neq 0, \\ \dfrac{\sigma_{1-\nu_1}(n_1)\zeta(\nu_2-1)\zeta(\nu_1+\nu_2-2)\zeta(\nu_1+2\nu_2-3)}{\zeta(\nu_1)\zeta(\nu_2)\zeta(\nu_1+\nu_2-1)\zeta(\nu_1+2\nu_2-2)} & \text{if } n_1 \neq 0, n_2 = 0, \\ \dfrac{\sigma_{1-\nu_2}(n_2)\zeta(\nu_1-1)\zeta(\nu_1+\nu_2-2)\zeta(\nu_1+2\nu_2-3)}{\zeta(\nu_1)\zeta(\nu_2)\zeta(\nu_1+\nu_2-1)\zeta(\nu_1+2\nu_2-2)} & \text{if } n_1 = 0, n_2 \neq 0, \\ \dfrac{\zeta(\nu_1-1)\zeta(\nu_2-1)\zeta(\nu_1+\nu_2-2)\zeta(\nu_1+2\nu_2-3)}{\zeta(\nu_1)\zeta(\nu_2)\zeta(\nu_1+\nu_2-1)\zeta(\nu_1+2\nu_2-2)} & \text{if } n_1 = n_2 = 0. \end{cases}$$

For a proof of the proposition, we need to study an auxiliary sum. We define

$$r_{v_1,v_{12}}(n_1,n_2) := \sum_{\substack{u_1|v_1 \\ u_{12}|v_{12}}} R_{u_1,u_{12}}(n_1,n_2).$$

Expanding the definition, we see that

$$r_{v_1,v_{12}}(n_1,n_2) \sum_{\substack{u_2 (\mathrm{mod}\ u_1) \\ u_{14} (\mathrm{mod}\ u_{12})}} \sum_{\substack{u_1|v_1 \\ u_{12}|v_{12}}} \sum_{\substack{u_3,u_4 (\mathrm{mod}\ u_1) \\ u_{13} (\mathrm{mod}\ u_2) \\ u_1u_{13}+u_2u_{14}-u_4u_{12}=0 \\ (u_1,u_2,u_3,u_4)=1 \\ (u_{12},u_{13},u_{14},u_{23},u_{34})=1}} e\left(\frac{n_1u_2}{u_1} + \frac{n_2u_{14}}{u_{12}}\right).$$

Let d_1, d_2 be such that $v_1 = u_1d_1$, and $v_{12} = u_{12}d_2$. We also write $v_i := u_id_1$, $v_{ij} := u_{ij}d_2$ for

$1 \le i < j \le 4$. Then the sum becomes

$$r_{v_1,v_{12}}(n_1,n_2) = \sum_{\substack{d_1|v_1 \\ d_2|v_{12}}} \sum_{\substack{v_2 (\mathrm{mod}\ v_1) \\ v_{14} (\mathrm{mod}\ v_{12})}} \sum_{\substack{v_3,v_4 (\mathrm{mod}\ v_1) \\ v_{13} (\mathrm{mod}\ v_{12}) \\ v_1v_{13}+v_2v_{14}-v_4v_{12}=0 \\ (v_1,v_2,v_3,v_4)=d_1 \\ (v_{12},v_{13},v_{14},v_{23},v_{34})=d_2}} e\left(\frac{n_1v_2}{v_1} + \frac{n_2v_{14}}{v_{12}}\right)$$

$$= \sum_{\substack{v_2 (\mathrm{mod}\ v_1) \\ v_{14} (\mathrm{mod}\ v_{12})}} \sum_{\substack{v_3,v_4 (\mathrm{mod}\ v_1) \\ v_{13} (\mathrm{mod}\ v_{12}) \\ v_1v_{13}+v_2v_{14}-v_4v_{12}=0 \\ v_{23},v_{34}\in\mathbb{Z}}} e\left(\frac{n_1v_2}{v_1} + \frac{n_2v_{14}}{v_{12}}\right). \tag{2.27}$$

So we get rid of the coprimality conditions appearing in the definition of $R_{v_1,v_{12}}(n_1,n_2)$. Evidently, the following equality of Dirichlet series holds:

$$\sum_{v_1,v_{12}\ge 1} r_{v_1,v_{12}}(n_1,n_2) v_1^{-\nu_1} v_{12}^{-\nu_2} = \zeta(\nu_1)\zeta(\nu_2)\mathcal{R}_{\nu_1,\nu_2}(n_1,n_2). \tag{2.28}$$

Now we determine the sum $r_{v_1,v_{12}}(n_1,n_2)$. For fixed v_1, v_{12}, v_2, v_{14}, we define

$$S(v_1,v_{12},v_2,v_{14}) = \#\left\{ \begin{array}{cc} v_3,v_4 & (\mathrm{mod}\ v_1) \\ v_{13} & (\mathrm{mod}\ v_{12}) \end{array} \,\middle|\, \begin{array}{l} v_1v_{13}+v_2v_{14}-v_4v_{12}=0 \\ v_1 \mid v_2v_{13}-v_3v_{12} \\ v_1 \mid v_3v_{14}-v_4v_{13} \end{array} \right\}.$$

Note that the conditions $v_1 \mid v_2v_{13}-v_3v_{12}$ and $v_1 \mid v_3v_{14}-v_4v_{13}$ are equivalent to that $v_{23}, v_{34} \in \mathbb{Z}$. Then we may rewrite (2.27) as

$$r_{v_1,v_{12}}(n_1,n_2) = \sum_{\substack{v_2 (\mathrm{mod}\ v_1) \\ v_{14} (\mathrm{mod}\ v_{12})}} S(v_1,v_{12},v_2,v_{14}) e\left(\frac{n_1v_2}{v_1} + \frac{n_2v_{14}}{v_{12}}\right).$$

From the definition, we see that $S(v_1,v_{12},v_2,v_{14})$ is multiplicative. Precisely, let u_1,u_{12},v_1,v_{12} be such that $(u_1u_{12},v_1v_{12})=1$, and let $t_2 \pmod{u_1v_1}$, $t_{14} \pmod{u_{12}v_{12}}$. Let u_2,u_{14},v_2,v_{14} be such that

$$t_2 \equiv u_1v_2+v_1u_2 \pmod{u_1v_1}, \qquad t_{14} \equiv u_{12}v_{14}+v_{12}u_{14} \pmod{u_{12}v_{12}}.$$

Then we have

$$S(u_1v_1,u_{12}v_{12},t_2,t_{14}) = S(u_1,u_{12},v_1u_2,v_{12}u_{14})S(v_1,v_{12},u_1v_2,u_{12}v_{14}).$$

Hence, we can reduce the task of finding $S(v_1,v_{12},v_2,v_{14})$ to a local problem, and it suffices to determine the quantities

$$S_p(w_1,w_{12},w_2,w_{14}) := S(p^{w_1},p^{w_{12}},p^{w_2},p^{w_{14}}).$$

The evaluation of quantities $S_p(w_1,w_{12},w_2,w_{14})$ is straightforward. We simply state the results.

Proposition 2.32. Let p be a prime, and let $0 \le w_2 \le w_1$, $0 \le w_{14} \le w_{12}$ be integers. Let $d = \min\{w_1,w_{14}\}$. Then $S_p(w_1,w_{12},w_2,w_{14})$ is given as follows:

Case 1. If $w_1 \le w_{12}$,

Case 1.1. if $2w_2 + w_{14} < 2w_1$,

Case 1.1.1. if $w_2 + w_{14} < w_{12}$, then $S_p(w_1,w_{12},w_2,w_{14}) = 0$;

Case 1.1.2. if $w_2 + w_{14} \geq w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = p^{w_2+w_{14}}$;

Case 1.2. if $2w_2 + w_{14} \geq 2w_1$ and $w_2 \leq 2w_1 - w_{12}$,

Case 1.2.1. if $w_1 + w_{12} - 2w_2 - 2w_{14} + d \geq 1$, then $S_p(w_1, w_{12}, w_2, w_{14}) = 2p^{w_2+w_{14}}$;

Case 1.2.2. if $w_1+w_{12}-2w_2-2w_{14}+d = 0$ or -1, then $S_p(w_1, w_{12}, w_2, w_{14}) = p^{w_1+w_{12}-w_2-w_{14}+d}$;

Case 1.2.3. if $w_1 + w_{12} - 2w_2 - 2w_{14} + d \leq -2$, then $S_p(w_1, w_{12}, w_2, w_{14}) = p^{\lfloor \frac{w_1+w_{12}+d}{2} \rfloor}$;

Case 1.3. if $2w_2 + w_{14} \geq 2w_1$ and $w_2 > 2w_1 - w_{12}$,

Case 1.3.1. if $w_1 + w_{12} - 2w_2 - 2w_{14} + d \geq 1$,

Case 1.3.1.1. if $w_2 + w_{14} < w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = \min\left\{p^{w_2+w_{14}}, p^{w_1+d}\right\}$;

Case 1.3.1.2. if $w_2 + w_{14} \geq w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = 2p^{w_2+w_{14}}$;

Case 1.3.2. if $w_1 + w_{12} - 2w_2 - 2w_{14} + d = 0$ or -1,

Case 1.3.2.1. if $w_2 + w_{14} < w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = p^{w_1+d}$;

Case 1.3.2.2. if $w_2 + w_{14} \geq w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = p^{w_1+w_{12}-w_2-w_{14}+d}$;

Case 1.3.3. if $w_1 + w_{12} - 2w_2 - 2w_{14} + d \leq -2$,

Case 1.3.3.1. if $w_2 + w_{14} < w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = p^{w_1+d}$;

Case 1.3.3.2. if $w_2 + w_{14} = w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = p^{w_1+w_{12}-w_2-w_{14}+d}$;

Case 1.3.3.3. if $w_2 + w_{14} > w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = \min\left\{p^{\lfloor \frac{w_1+w_{12}+d}{2} \rfloor}, p^{w_1+d}\right\}$;

Case 2. if $w_1 > w_{12}$,

Case 2.1. if $w_2 + w_{14} < w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = 0$;

Case 2.2. if $w_2 + w_{14} \geq w_{12}$,

Case 2.2.1. if $w_2 \leq w_{12}$ and $2w_2 + w_{14} \leq 2w_{12}$, then $S_p(w_1, w_{12}, w_2, w_{14}) = p^{w_2+w_{14}}$;

Case 2.2.2. otherwise, $S_p(w_1, w_{12}, w_2, w_{14}) = p^{w_{12}+\lfloor \frac{w_{14}}{2} \rfloor}$.

Multiplicativity of $S(v_1, v_{12}, v_2, v_{14})$ also implies the multiplicativity of $r_{v_1,v_{12}}(n_1, n_2)$, that is, if $(u_1 u_{12} m_1 m_2, v_1 v_{12} n_1 n_2) = 1$, then

$$r_{u_1 v_1, u_{12} v_{12}}(m_1 n_1, m_2 n_2) = r_{u_1,u_{12}}(m_1, m_2) r_{v_1,v_{12}}(n_1, n_2).$$

Indeed, we see that

$$\begin{aligned}
& r_{u_1 v_1, u_{12} v_{12}}(m_1 n_1, m_2 n_2) \\
&= \sum_{\substack{t_2 (\text{mod } u_1 v_1) \\ t_{14} (\text{mod } u_{12} v_{12})}} S(u_1 v_1, u_{12} v_{12}, t_2, t_{14}) e\left(\frac{m_1 n_1 t_2}{u_1 v_1} + \frac{m_2 n_2 t_{14}}{u_{12} v_{12}}\right) \\
&= \sum_{\substack{u_2 (\text{mod } u_1) \\ u_{14} (\text{mod } u_{12})}} \sum_{\substack{v_2 (\text{mod } v_1) \\ v_{14} (\text{mod } v_{12})}} S(u_1 v_1, u_{12} v_{12}, u_1 v_2 + v_1 u_2, u_{12} v_{14} + v_{12} u_{14}) e\left(\frac{m_1 u_2}{u_1} + \frac{n_1 v_2}{v_1} + \frac{m_2 u_{14}}{u_{12}} + \frac{n_2 v_{14}}{v_{12}}\right) \\
&= \sum_{\substack{u_2 (\text{mod } u_1) \\ u_{14} (\text{mod } u_{12})}} S(u_1, u_{12}, v_1 u_2, v_{12} u_{14}) e\left(\frac{m_1 u_2}{u_1} + \frac{m_2 u_{14}}{u_{12}}\right) \sum_{\substack{v_2 (\text{mod } v_1) \\ v_{14} (\text{mod } v_{12})}} S(v_1, v_{12}, u_1 v_2, u_{12} v_{14}) e\left(\frac{n_1 v_2}{v_1} + \frac{n_2 v_{14}}{v_{12}}\right) \\
&= r_{u_1,u_{12}}(m_1, m_2) r_{v_1,v_{12}}(n_1, n_2).
\end{aligned}$$

Hence, it suffices to consider local sums $r_{p^{w_1},p^{w_{12}}}(p^{e_1},p^{e_2})$. We have

$$\begin{aligned} r_{p^{w_1},p^{w_{12}}}(p^{e_1},p^{e_2}) &= \sum_{\substack{v_2 (\text{mod } p^{w_1}) \\ v_{14} (\text{mod } p^{w_{12}})}} S_p\left(w_1, w_{12}, v_p(v_2), v_p(v_{14})\right) e\left(v_2 p^{e_1 - w_1} + v_{14} p^{e_2 - w_{12}}\right) \\ &= \sum_{w_2=0}^{w_1} \sum_{w_{14}=0}^{w_{12}} S_p(w_1, w_{12}, w_2, w_{14}) \sum_{\substack{v_2 (\text{mod } p^{w_1}) \\ v_p(v_2) = w_2}} \sum_{\substack{v_{14} (\text{mod } p^{w_{12}}) \\ v_p(v_{14}) = w_{14}}} e\left(v_2 p^{e_1 - w_1} + v_{14} p^{e_2 - w_{12}}\right). \end{aligned} \tag{2.29}$$

Note that this also covers the degenerate cases where $n_1 = 0$ or $n_2 = 0$. Indeed, it is clear from definition that we have

$$\begin{aligned} r_{p^{w_1},p^{w_{12}}}(p^{e_1},0) &= r_{p^{w_1},p^{w_{12}}}(p^{e_1},p^{w_{12}}), \\ r_{p^{w_1},p^{w_{12}}}(0,p^{e_2}) &= r_{p^{w_1},p^{w_{12}}}(p^{w_1},p^{e_2}), \\ r_{p^{w_1},p^{w_{12}}}(0,0) &= r_{p^{w_1},p^{w_{12}}}(p^{w_1},p^{w_{12}}). \end{aligned}$$

The inner sums of (2.29) can be evaluated as follows:

$$\sum_{\substack{v (\text{mod } p^{w}) \\ v_p(v) = w'}} e\left(v p^{e-w}\right) = \begin{cases} 1 & \text{if } w' = w, \\ p^{w-w'-1}(p-1) & \text{if } w > w' \geq w - e, \\ -p^{w-w'-1} & \text{if } w' = w - e - 1, \\ 0 & \text{if } w' \leq w - e - 2. \end{cases} \tag{2.30}$$

Using (2.30) and Proposition 2.32, we can compute $r_{p^{w_1},p^{w_{12}}}(p^{e_1},p^{e_2})$ explicitly. By comparing the coefficients of the power series, we obtain the following identities:

$$\sum_{w_1 \geq 0} \sum_{w_{12} \geq 0} r_{p^{w_1},p^{w_{12}}}(p^{e_1},p^{e_2}) p^{-w_1\nu_1 - w_{12}\nu_2} = \sigma_{\frac{3}{2}-\frac{\nu_1}{2}-\nu_2, \frac{1}{2}-\frac{\nu_1}{2}}(p^{e_1},p^{e_2})(1-p^{1-\nu_1-\nu_2})(1-p^{2-\nu_1-2\nu_2}), \tag{2.31}$$

$$\sum_{w_1 \geq 0} \sum_{w_{12} \geq 0} r_{p^{w_1},p^{w_{12}}}(p^{e_1},0) p^{-w_1\nu_1 - w_{12}\nu_2} = \frac{\sigma_{1-\nu_1}(p^{e_1})(1-p^{1-\nu_1-\nu_2})(1-p^{2-\nu_1-2\nu_2})}{(1-p^{1-\nu_2})(1-p^{2-\nu_1-\nu_2})(1-p^{3-\nu_1-2\nu_2})}, \tag{2.32}$$

$$\sum_{w_1 \geq 0} \sum_{w_{12} \geq 0} r_{p^{w_1},p^{w_{12}}}(0,p^{e_2}) p^{-w_1\nu_1 - w_{12}\nu_2} = \frac{\sigma_{1-\nu_2}(p^{e_2})(1-p^{1-\nu_1-\nu_2})(1-p^{2-\nu_1-2\nu_2})}{(1-p^{1-\nu_1})(1-p^{2-\nu_1-\nu_2})(1-p^{3-\nu_1-2\nu_2})}, \tag{2.33}$$

$$\sum_{w_1 \geq 0} \sum_{w_{12} \geq 0} r_{p^{w_1},p^{w_{12}}}(0,0) p^{-w_1\nu_1 - w_{12}\nu_2} = \frac{(1-p^{1-\nu_1-\nu_2})(1-p^{2-\nu_1-2\nu_2})}{(1-p^{1-\nu_1})(1-p^{1-\nu_2})(1-p^{2-\nu_1-\nu_2})(1-p^{3-\nu_1-2\nu_2})}. \tag{2.34}$$

Combining the Euler factors in (2.31) - (2.34) yields for $n_1, n_2 \neq 0$ the following identities:

$$\sum_{v_1, v_{12} \geq 1} r_{v_1,v_{12}}(n_1,n_2) v_1^{-\nu_1} v_{12}^{-\nu_2} = \frac{\sigma_{\frac{3}{2}-\frac{\nu_1}{2}-\nu_2, \frac{1}{2}-\frac{\nu_1}{2}}(n_1,n_2)}{\zeta(\nu_1+\nu_2-1)\zeta(\nu_1+2\nu_2-2)}, \tag{2.35}$$

$$\sum_{v_1, v_{12} \geq 1} r_{v_1,v_{12}}(n_1,0) v_1^{-\nu_1} v_{12}^{-\nu_2} = \frac{\sigma_{1-\nu_1}(n_1)\zeta(\nu_2-1)\zeta(\nu_1+\nu_2-2)\zeta(\nu_1+2\nu_2-3)}{\zeta(\nu_1+\nu_2-1)\zeta(\nu_1+2\nu_2-2)}, \tag{2.36}$$

$$\sum_{v_1, v_{12} \geq 1} r_{v_1,v_{12}}(0,n_2) v_1^{-\nu_1} v_{12}^{-\nu_2} = \frac{\sigma_{1-\nu_2}(n_2)\zeta(\nu_1-1)\zeta(\nu_1+\nu_2-2)\zeta(\nu_1+2\nu_2-3)}{\zeta(\nu_1+\nu_2-1)\zeta(\nu_1+2\nu_2-2)}, \tag{2.37}$$

$$\sum_{v_1, v_{12} \geq 1} r_{v_1,v_{12}}(0,0) v_1^{-\nu_1} v_{12}^{-\nu_2} = \frac{\zeta(\nu_1-1)\zeta(\nu_2-1)\zeta(\nu_1+\nu_2-2)\zeta(\nu_1+2\nu_2-3)}{\zeta(\nu_1+\nu_2-1)\zeta(\nu_1+2\nu_2-2)}. \tag{2.38}$$

Finally, applying (2.28) yields the expressions in Proposition 2.31, and finishes the proof.

2.5 Fourier coefficients of Eisenstein series

2.5.1 Invariant differential operators

We consider the Siegel upper half space

$$H_2 = \left\{ Z = X + iY \in M_2(\mathbb{C}) \;\middle|\; Z^t = Z, Y \text{ positive definite} \right\}.$$

An element of H_2 is denoted by a matrix $\begin{pmatrix} Z_1 & Z_2 \\ Z_2 & Z_3 \end{pmatrix}$, with $Z_j = X_j + iY_j$, $j = 1, 2, 3$. It is well-known that $G = \mathrm{Sp}(4, \mathbb{R})$ acts on H_2 as a group of biholomorphic automorphisms by

$$\mathrm{Sp}(4, \mathbb{R}) \times H_2 \to H_2, \quad \left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z \right) \mapsto M \langle Z \rangle := (AZ + B)(CZ + D)^{-1} \in H_2. \tag{2.39}$$

There is a canonical bijection between G/K and H_2, given by the map

$$gK \mapsto g \left\langle \begin{pmatrix} i & \\ & i \end{pmatrix} \right\rangle \tag{2.40}$$

for $g \in G$. Let $\mathscr{D}$ be the algebra of differential operators on H_2 that is invariant under the $\mathrm{Sp}(4, \mathbb{R})$-action given in (2.39). The algebra $\mathscr{D}$ is generated by Δ_1, Δ_2 [Niw91], where

$$\begin{aligned}
\Delta_1 &= \sum_{i,j=1} Y_i Y_j \partial_i \overline{\partial}_j - D \left(\partial_1 \overline{\partial}_3 + \overline{\partial}_1 \partial_3 - \frac{1}{2} \partial_2 \overline{\partial_2} \right), \\
\Delta_2 &= D^2 \left(\partial_1 \partial_3 - \frac{1}{4} \partial_2^2 \right) \left(\overline{\partial}_1 \overline{\partial}_3 - \frac{1}{4} \overline{\partial}_2^2 \right) + \frac{i}{4} D \Big(\sum_{i=1}^{3} Y_i \partial_i \Big) \left(\overline{\partial}_1 \overline{\partial}_3 - \frac{1}{4} \overline{\partial}_2^2 \right) \\
&\quad + \frac{i}{4} D \Big(\sum_{i=1}^{3} Y_i \overline{\partial}_i \Big) \left(\partial_1 \partial_3 - \frac{1}{4} \partial_2^2 \right) + \frac{1}{16} D \left(\partial_1 \overline{\partial}_3 + \overline{\partial}_1 \partial_3 - \frac{1}{2} \partial_2 \overline{\partial}_2 \right).
\end{aligned}$$

Here, we write $D = Y_1 Y_3 - Y_2^2$, and for $1 \leq i \leq 3$, differential operators

$$\partial_i = \frac{\partial}{\partial Z_i} = \frac{1}{2} \left(\frac{\partial}{\partial X_i} - i \frac{\partial}{\partial Y_i} \right), \qquad \overline{\partial}_i = \frac{\partial}{\partial \overline{Z}_i} = \frac{1}{2} \left(\frac{\partial}{\partial X_i} + i \frac{\partial}{\partial Y_i} \right).$$

Through the isomorphism (2.40), we can also view $\mathscr{D}$ as the algebra of invariant differential operators on G/K, with the same generators. It is straightforward to verify that for $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, the function

$$I_0(g, \nu) = y_1^{\nu_1 + 2} y_2^{2\nu_2 - \nu_1 + 1}$$

is an eigenfunction for Δ_1 and Δ_2, with eigenvalues given by

$$\begin{aligned}
\lambda_1 &= \frac{1}{16} \left(2\nu_1^2 - 4\nu_1\nu_2 + 4\nu_2^2 - 5 \right), \\
\lambda_2 &= \frac{1}{256} \left(\nu_1^2 - 2\nu_1 - 2 \right) \left(\nu_1 - 2\nu_2 - 2 \right) \left(\nu_1 + 2 \right).
\end{aligned}$$

This says $I_0(g, \nu)$ is an eigenfunction for all differential operators in $\mathscr{D}$.

2.5.2 Jacquet's Whittaker functions

For $m_1, m_2 \in \mathbb{Z}$, let $\psi = \psi_{m_1,m_2}$ be the character on $U(\mathbb{Z})\backslash U(\mathbb{R})$ given by

$$\psi_{m_1,m_2}\begin{pmatrix} 1 & n_1 & n_2 & n_3 \\ & 1 & n_4 & n_5 \\ & & 1 & \\ & & -n_1 & 1 \end{pmatrix} = e\left(m_1 n_1 + m_2 n_5\right).$$

Definition 2.33. A Whittaker function on $G = \mathrm{Sp}(4,\mathbb{R})$ of type ν associated to a character ψ of $U(\mathbb{Z})\backslash U(\mathbb{R})$ is a smooth function $f : G/K \to \mathbb{C}$ such that

(i) f is an eigenfunction for differential operators Δ_1 and Δ_2, with same eigenvalues as $I_0(g,\nu)$; and

(ii) $f(\eta g) = \psi(\eta) f(g)$ for all $\eta \in U(\mathbb{R})$.

The space of Whittaker functions of type ν associated to ψ is denoted by $\mathcal{W}(\nu,\psi)$. We also denote by $\mathcal{W}(\nu,\psi)^{\mathrm{mod}}$ the space of Whittaker functions of moderate growth. It is well-known that $\mathcal{W}(\nu,\psi)$ is finite dimensional. More precisely, we have the following "multiplicity-one" theorem.

Theorem 2.34. (Shalika [Sha74], Wallach [Wal83]) The dimension of the space $\mathcal{W}(\nu,\psi)$ is equal to 8, the order of the Weyl group $W = W(T,G)$ of G. Furthermore, we have

$$\dim \mathcal{W}(\nu,\psi)^{\mathrm{mod}} \le 1.$$

Recall from (2.1) that elements in the Weyl group W can be identified with matrices in $\mathrm{Sp}(4,\mathbb{Z})$. As Δ_1, Δ_2 are $\mathrm{Sp}(4,\mathbb{R})$-invariant differential operators, it follows that for $w \in W$, $I_0(wg,\nu)$ is also an eigenfunction for Δ_1 and Δ_2, with the same eigenvalues as $I_0(g,\nu)$. For a fixed Weyl element $w \in W$, let ψ be a character of $U(\mathbb{Z})\backslash U(\mathbb{R})$ which is trivial on $\overline{U}_w(\mathbb{R})$. Then

$$W_w(g,\nu,\psi) := \int_{U_w(\mathbb{R})} I_0(w\eta g,\nu)\overline{\psi}(\eta)d\eta \in \mathcal{W}(\nu,\psi)$$

is a Whittaker function. If ψ is not trivial on $\overline{U}_w(\mathbb{R})$, we define $W_w(g,\nu,\psi)$ to be zero.

Using $W_w(g,\nu,\psi)$, we construct eight functions in $\mathcal{W}(\nu,\psi)$, one for each $w \in W$.

(i) $w = \mathrm{id}$: We have

$$W_{\mathrm{id}}(g,\nu,\psi_{0,0}) = y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1},$$

and $W_{\mathrm{id}}(g,\nu,\psi_{m_1,m_2}) = 0$ if $(m_1,m_2) \neq (0,0)$.

(ii) $w = s_\alpha$: We have

$$W_{s_\alpha}(g,\nu,\psi_{m_1,0}) = y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}} \left(n_1^2 y_2^2 + y_1^2\right)^{\nu_2-\nu_1-1/2} e\left(-m_1 n_1\right) dn_1,$$

and $W_{s_\alpha}(g,\nu,\psi_{m_1,m_2}) = 0$ if $m_2 \neq 0$.

(iii) $w = s_\beta$: We have

$$W_{s_\beta}(g,\nu,\psi_{0,m_2}) = y_1^{\nu_1+2} y_2^{2\nu_2-\nu_1+1} \int_{\mathbb{R}} \left(y_2^4 + n_5^2\right)^{\nu_1/2-\nu_2-1/2} e\left(-m_2 n_5\right) dn_5,$$

and $W_{s_\beta}(g,\nu,\psi_{m_1,m_2}) = 0$ if $m_1 \neq 0$.

(iv) $w = s_\alpha s_\beta$: We have

$$W_{s_\alpha s_\beta}(g,\nu,\psi_{0,m_2}) = y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}}\int_{\mathbb{R}} \left(y_2^4+n_5^2\right)^{\nu_1/2-\nu_2-1/2} \left(\left(y_2^4+n_5^2\right)y_1^2+y_2^2n_4^2\right)^{\nu_2-\nu_1-1/2} e\left(-m_2n_5\right)dn_4dn_5,$$

and $W_{s_\alpha s_\beta}(g,\nu,\psi_{m_1,m_2}) = 0$ if $m_1 \neq 0$.

(v) $w = s_\beta s_\alpha$: We have

$$W_{s_\beta s_\alpha}(g,\nu,\psi_{m_1,0}) = y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}}\int_{\mathbb{R}} \left(n_1^2y_2^2+y_1^2\right)^{\nu_2-\nu_1-1/2} \left(n_2^2+\left(n_1^2y_2^2+y_1^2\right)^2\right)^{\nu_1/2-\nu_2-1/2} e\left(-m_1n_1\right)dn_1dn_2,$$

and $W_{s_\beta s_\alpha}(g,\nu,\psi_{m_1,m_2}) = 0$ if $m_2 \neq 0$.

(vi) $w = s_\alpha s_\beta s_\alpha$: We have

$$W_{s_\alpha s_\beta s_\alpha}(g,\nu,\psi_{m_1,0}) = y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\left(y_1^2+n_1^2y_2^2\right)^2+\left(n_2+n_1n_4\right)^2\right)^{\nu_1/2-\nu_2-1/2} \left(y_1^4y_2^2+n_2^2y_2^2+n_1^2y_1^2y_2^4+n_4^2y_1^2\right)^{\nu_2-\nu_1-1/2} e\left(-m_1n_1\right)dn_1dn_2dn_4,$$

and $W_{s_\alpha s_\beta s_\alpha}(g,\nu,\psi_{m_1,m_2}) = 0$ if $m_2 \neq 0$.

(vii) $w = s_\beta s_\alpha s_\beta$: We have

$$W_{s_\beta s_\alpha s_\beta}(g,\nu,\psi_{0,m_2}) = y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(y_1^2y_2^4+n_5^2y_1^2+n_4^2y_2^2\right)^{\nu_2-\nu_1-1/2} \left(y_1^4y_2^4+n_5^2y_1^4+2n_4^2y_1^2y_2^2+\left(n_1n_4-n_2\right)^2y_2^4+\left(n_2n_5-n_4^2-n_1n_4n_5\right)^2\right)^{\nu_1/2-\nu_2-1/2} e\left(-m_2n_5\right)dn_2dn_4dn_5,$$

and $W_{s_\beta s_\alpha s_\beta}(g,\nu,\psi_{m_1,m_2}) = 0$ if $m_1 \neq 0$.

(viii) $w = s_\alpha s_\beta s_\alpha s_\beta$: We have

$$W_{s_\alpha s_\beta s_\alpha s_\beta}(g,\nu,\psi_{m_1,m_2}) = y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(n_1^2n_4^2y_2^4+y_1^4y_2^4-2n_1n_5n_4y_1^2y_2^2-2n_1n_2n_4y_2^4 +n_5^2y_1^4+2n_3n_4y_1^2y_2^2+n_2^2y_2^4+n_2^2n_5^2-2n_3n_2n_5n_4+n_3^2n_4^2\right)^{\nu_1/2-\nu_2-1/2} \left(n_1^2y_1^2y_2^4+y_1^4y_2^2+n_3^2y_1^2+n_2^2y_2^2\right)^{\nu_2-\nu_1-1/2} e\left(-m_1n_1-m_2n_5\right)dn_1dn_2dn_4dn_5.$$

With the exception of the long Weyl element $w = w_0$, the functions W_w can be expressed in terms of classical Whittaker function

$$W(y,\nu,\chi) = \int_{\mathbb{R}} \left(\frac{y}{y^2+u^2}\right)^{\nu+1/2} \overline{\chi}(u)du,$$

where $\chi = \chi_t$ is the additive character of $\mathbb{R}$ given by $\chi_t(u) = e(tu)$, for $t \in \mathbb{R}$.

Proposition 2.35. We have

$$\begin{aligned}
W_{\mathrm{id}}\left(g,\nu,\psi_{0,0}\right) &= y_1^{\nu_1+2}y_2^{2\nu_2-\nu_1+1},\\
W_{s_\alpha}\left(g,\nu,\psi_{m_1,0}\right) &= y_1^{\nu_2+3/2}y_2^{\nu_1+1}W\left(y_1,\nu_1-\nu_2,\chi_{m_1/y_2}\right),\\
W_{s_\beta}\left(g,\nu,\psi_{0,m_2}\right) &= y_1^{\nu_1+2}W\left(y_2^2,\nu_2-\frac{\nu_1}{2},\chi_{m_2}\right),\\
W_{s_\alpha s_\beta}\left(g,\nu,\psi_{0,m_2}\right) &= y_1^{2\nu_2-\nu_1+2}B\left(\frac{1}{2},\nu_1-\nu_2\right)W\left(y_2^2,\frac{\nu_1}{2},\chi_{m_2}\right),\\
W_{s_\beta s_\alpha}\left(g,\nu,\psi_{m_1,0}\right) &= y_1^{\nu_1-\nu_2+3/2}y_2^{\nu_1+1}B\left(\frac{1}{2},\nu_2-\frac{\nu_1}{2}\right)W\left(y_1,\nu_2,\chi_{m_1/y_2}\right),\\
W_{s_\alpha s_\beta s_\alpha}\left(g,\nu,\psi_{m_1,0}\right) &= y_1^{\nu_2-\nu_1+3/2}y_2^{2\nu_2-\nu_1+1}B\left(\frac{1}{2},\frac{\nu_1}{2}\right)B\left(\frac{1}{2},\nu_1-\nu_2\right)W\left(y_1,\nu_2,\chi_{m_1/y_2}\right),\\
W_{s_\beta s_\alpha s_\beta}\left(g,\nu,\psi_{0,m_2}\right) &= y_1^{\nu_1-2\nu_2+2}B\left(\frac{1}{2},\nu_2-\frac{\nu_1}{2}\right)B\left(\frac{1}{2},\nu_2\right)W\left(y_2^2,\frac{\nu_1}{2},\chi_{m_2}\right).
\end{aligned}$$

Proof. (i) The statement for $w=\mathrm{id}$ is obvious.

(ii) For W_{s_α}, we have

$$W_{s_\alpha}\left(g,\nu,\psi_{m_1,0}\right) = y_1^{\nu_1+2}y_2^{\nu_1+2}\int_{\mathbb{R}}\left(n_1^2y_2^2+y_1^2\right)^{\nu_2-\nu_1-1/2}e\left(-m_1n_1\right)dn_1.$$

Change of variables $n_1y_2\mapsto n_1'$ gives

$$\begin{aligned}
&y_1^{\nu_1+2}y_2^{\nu_1+1}\int_{\mathbb{R}}\left({n_1'}^2+y_1^2\right)^{\nu_2-\nu_1-1/2}e\left(-\frac{m_1}{y_2}n_1'\right)dn_1'\\
&= y_1^{\nu_2+3/2}y_2^{\nu_1+1}\int_{\mathbb{R}}\left(\frac{y_1}{{n_1'}^2+y_1^2}\right)^{\nu_2-\nu_1-1/2}e\left(-\frac{m_1}{y_2}n_1'\right)dn_1'\\
&= y_1^{\nu_2+3/2}y_2^{\nu_1+1}W\left(y_1,\nu_1-\nu_2,\chi_{m_1/y_2}\right).
\end{aligned}$$

(iii) For W_{s_β}, we have

$$\begin{aligned}
W_{s_\beta}\left(g,\nu,\psi_{0,m_2}\right) &= y_1^{\nu_1+2}y_2^{2\nu_2-\nu_1+1}\int_{\mathbb{R}}\left(y_2^4+n_5^2\right)^{\nu_1/2-\nu_2-1/2}e\left(-m_2n_5\right)dn_5\\
&= y_1^{\nu_1+2}\int_{\mathbb{R}}\left(\frac{y_2^2}{y_2^4+n_5^2}\right)^{\nu_2-\nu_1/2+1/2}e\left(-m_2n_5\right)dn_5\\
&= y_1^{\nu_1+2}W\left(y_2^2,\nu_2-\frac{\nu_1}{2},\chi_{m_2}\right).
\end{aligned}$$

(iv) For $W_{s_\alpha s_\beta}$, we have

$$\begin{aligned}
W_{s_\alpha s_\beta}\left(g,\nu,\psi_{0,m_2}\right) = y_1^{\nu_1+2}y_2^{\nu_1+2}&\int_{\mathbb{R}}\int_{\mathbb{R}}\left(y_2^4+n_5^2\right)^{\nu_1/2-\nu_2-1/2}\left(\left(y_2^4+n_5^2\right)y_1^2+y_2^2n_4^2\right)^{\nu_2-\nu_1-1/2}\\
&e\left(-m_2n_5\right)dn_4dn_5.
\end{aligned}$$

Change of variables $n_4 y_2 \mapsto n_4'$ gives

$$y_1^{\nu_1+2} y_2^{\nu_1+1} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(y_2^4+n_5^2\right)^{\nu_1/2-\nu_2-1/2} \left(\left(y_2^4+n_5^2\right) y_1^2+{n_4'}^2\right)^{\nu_2-\nu_1-1/2} e\left(-m_2 n_5\right) dn_4' dn_5$$
$$= y_1^{2\nu_2-\nu_1+2} y_2^{\nu_1+1} B\left(\frac{1}{2}, \nu_1-\nu_2\right) \int_{\mathbb{R}} \left(y_2^4+n_5^2\right)^{-\nu_1/2-1/2} e\left(-m_2 n_5\right) dn_5$$
$$= y_1^{2\nu_2-\nu_1+2} B\left(\frac{1}{2}, \nu_1-\nu_2\right) \int_{\mathbb{R}} \left(\frac{y_2^2}{y_2^4+n_5^2}\right)^{\nu_1/2+1/2} e\left(-m_2 n_5\right) dn_5$$
$$= y_1^{2\nu_2-\nu_1+2} B\left(\frac{1}{2}, \nu_1-\nu_2\right) W\left(y_2^2, \frac{\nu_1}{2}, \chi_{m_2}\right).$$

(v) For $W_{s_\beta s_\alpha}$, we have

$$W_{s_\beta s_\alpha}\left(g, \nu, \psi_{m_1,0}\right) = y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(n_1^2 y_2^2+y_1^2\right)^{\nu_2-\nu_1-1/2} \left(n_2^2+\left(n_1^2 y_2^2+y_1^2\right)^2\right)^{\nu_1/2-\nu_2-1/2}$$
$$e\left(-m_1 n_1\right) dn_1 dn_2$$
$$= y_1^{\nu_1+2} y_2^{\nu_1+2} B\left(\frac{1}{2}, \nu_2-\frac{\nu_1}{2}\right) \int_{\mathbb{R}} \left(n_1^2 y_2^2+y_1^2\right)^{-\nu_2-1/2} e\left(-m_1 n_1\right) dn_1.$$

Change of variables $n_1 y_2 \mapsto n_1'$ gives

$$y_1^{\nu_1+2} y_2^{\nu_1+1} B\left(\frac{1}{2}, \nu_2-\frac{\nu_1}{2}\right) \int_{\mathbb{R}} \left({n_1'}^2+y_1^2\right)^{-\nu_2-1/2} e\left(-\frac{m_1}{y_2} n_1'\right) dn_1'$$
$$= y_1^{\nu_1-\nu_2+3/2} y_2^{\nu_1+1} B\left(\frac{1}{2}, \nu_2-\frac{\nu_1}{2}\right) \int_{\mathbb{R}} \left(\frac{y_1}{{n_1'}^2+y_1^2}\right)^{\nu_2+1/2} e\left(-\frac{m_1}{y_2} n_1'\right) dn_1'$$
$$= y_1^{\nu_1-\nu_2+3/2} y_2^{\nu_1+1} B\left(\frac{1}{2}, \nu_2-\frac{\nu_1}{2}\right) W\left(y_1, \nu_2, \chi_{m_1/y_2}\right).$$

(vi) For $W_{s_\alpha s_\beta s_\alpha}$, we have

$$W_{s_\alpha s_\beta s_\alpha}\left(g, \nu, \psi_{m_1,0}\right) = y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\left(y_1^2+n_1^2 y_2^2\right)^2+\left(n_2+n_1 n_4\right)^2\right)^{\nu_1/2-\nu_2-1/2}$$
$$\left(y_1^4 y_2^2+n_2^2 y_2^2+n_1^2 y_1^2 y_2^4+n_4^2 y_1^2\right)^{\nu_2-\nu_1-1/2} e\left(-m_1 n_1\right) dn_1 dn_2 dn_4.$$

Change of variables $n_2+n_1 n_4 \mapsto n_2'$ gives

$$y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\left(y_1^2+n_1^2 y_2^2\right)^2+{n_2'}^2\right)^{\nu_1/2-\nu_2-1/2}$$
$$\left(y_1^4 y_2^2+{n_2'}^2 y_2^2-2 n_1 n_2' n_4 y_2^2+n_1^2 n_4^2 y_2^2+n_1^2 y_1^2 y_2^4+n_4^2 y_1^2\right)^{\nu_2-\nu_1-1/2} e\left(-m_1 n_1\right) dn_1 dn_2' dn_4.$$

Completing square with respect to n_4 followed by change of variables $n_4-\frac{n_1 n_2' y_2^2}{\left(n_1^2 y_2^2+y_1^2\right)} \mapsto n_4'$ gives

$$y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\left(y_1^2+n_1^2 y_2^2\right)^2+{n_2'}^2\right)^{\nu_1/2-\nu_2-1/2}$$
$$\left(n_1^2 y_2^2+y_1^2\right)^{\nu_2-\nu_1-1/2} \left({n_4'}^2+\frac{y_1^2 y_2^2\left(\left(n_1^2 y_2^2+y_1^2\right)^2+{n_2'}^2\right)}{\left(n_1^2 y_2^2+y_1^2\right)^2}\right)^{\nu_2-\nu_1-1/2} e\left(-m_1 n_1\right) dn_1 dn_2' dn_4'$$
$$= y_1^{2\nu_2-\nu_1+2} y_2^{2\nu_2-\nu_1+2} B\left(\frac{1}{2}, \nu_1-\nu_2\right) \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\left(y_1^2+n_1^2 y_2^2\right)^2+{n_2'}^2\right)^{-\nu_1/2-1/2} \left(n_1^2 y_2^2+y_1^2\right)^{\nu_1-\nu_2-1/2}$$
$$e\left(-m_1 n_1\right) dn_1 dn_2'$$
$$= y_1^{2\nu_2-\nu_1+2} y_2^{2\nu_2-\nu_1+2} B\left(\frac{1}{2}, \frac{\nu_1}{2}\right) B\left(\frac{1}{2}, \nu_1-\nu_2\right) \int_{\mathbb{R}} \left(n_1^2 y_2^2+y_1^2\right)^{-\nu_2-1/2} e\left(-m_1 n_1\right) dn_1.$$

Change of variables $n_1 y_2 \mapsto n_1'$ gives

$$y_1^{2\nu_2-\nu_1+2} y_2^{2\nu_2-\nu_1+1} B\left(\frac{1}{2}, \frac{\nu_1}{2}\right) B\left(\frac{1}{2}, \nu_1-\nu_2\right) \int_{\mathbb{R}} \left({n_1'}^2 + y_1^2\right)^{-\nu_2-1/2} e\left(-\frac{m_1}{y_2} n_1'\right) dn_1'$$

$$= y_1^{\nu_2-\nu_1+3/2} y_2^{2\nu_2-\nu_1+1} B\left(\frac{1}{2}, \frac{\nu_1}{2}\right) B\left(\frac{1}{2}, \nu_1-\nu_2\right) \int_{\mathbb{R}} \left(\frac{y_1}{{n_1'}^2 + y_1^2}\right)^{\nu_2+1/2} e\left(-\frac{m_1}{y_2} n_1'\right) dn_1'$$

$$= y_1^{\nu_2-\nu_1+3/2} y_2^{2\nu_2-\nu_1+1} B\left(\frac{1}{2}, \frac{\nu_1}{2}\right) B\left(\frac{1}{2}, \nu_1-\nu_2\right) W\left(y_1, \nu_2, \chi_{m_1/y_2}\right).$$

(vii) For $W_{s_\beta s_\alpha s_\beta}$ we have

$$W_{s_\beta s_\alpha s_\beta}(g, \nu, \psi_{0,m_2}) = y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(y_1^2 y_2^4 + n_5^2 y_1^2 + n_4^2 y_2^2\right)^{\nu_2-\nu_1-1/2}$$
$$\left(y_1^4 y_2^4 + n_5^2 y_1^4 + 2n_4^2 y_1^2 y_2^2 + (n_1 n_4 - n_2)^2 y_2^4 + \left(n_2 n_5 - n_4^2 - n_1 n_4 n_5\right)^2\right)^{\nu_1/2-\nu_2-1/2}$$
$$e(-m_2 n_5)\, dn_2 dn_4 dn_5.$$

We first simplify the expression by setting $n_2' = n_2 - n_1 n_4$:

$$y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(y_1^4 y_2^4 + n_5^2 y_1^4 + 2n_4^2 y_1^2 y_2^2 + {n_2'}^2 y_2^4 + \left(n_2' n_5 - n_4^2\right)^2\right)^{\nu_1/2-\nu_2-1/2}$$
$$\left(y_1^2 y_2^4 + n_5^2 y_1^2 + n_4^2 y_2^2\right)^{\nu_2-\nu_1-1/2} e(-m_2 n_5)\, dn_2' dn_4 dn_5.$$

Completing square with respect to n_2' followed by change of variable $n_2' - \frac{n_4^2 n_5}{y_2^4 + n_5^2} \mapsto n_2''$ gives

$$y_1^{\nu_1+2} y_2^{\nu_1+2} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(y_2^4 + n_5^2\right)^{\nu_1/2-\nu_2-1/2} \left({n_2''}^2 + \left(\frac{y_1^2 y_2^4 + n_5^2 y_1^2 + n_4^2 y_2^2}{y_2^4 + n_5^2}\right)^2\right)^{\nu_1/2-\nu_2-1/2}$$
$$\left(y_1^2 y_2^4 + n_5^2 y_1^2 + n_4^2 y_2^2\right)^{\nu_2-\nu_1-1/2} e(-m_2 n_5)\, dn_2'' dn_4 dn_5$$
$$= y_1^{\nu_1+2} y_2^{\nu_1+2} B\left(\frac{1}{2}, \nu_2 - \frac{\nu_1}{2}\right) \int_{\mathbb{R}}\int_{\mathbb{R}} \left(y_2^4 + n_5^2\right)^{\nu_2-\nu_1/2-1/2} \left(y_1^2 y_2^4 + n_5^2 y_1^2 + n_4^2 y_2^2\right)^{-\nu_2-1/2}$$
$$e(-m_2 n_5)\, dn_4 dn_5.$$

Change of variables $n_4 y_2 \mapsto n_4'$ gives

$$y_1^{\nu_1+2} y_2^{\nu_1+1} B\left(\frac{1}{2}, \nu_2 - \frac{\nu_1}{2}\right) \int_{\mathbb{R}}\int_{\mathbb{R}} \left(y_2^4 + n_5^2\right)^{\nu_2-\nu_1/2-1/2} \left(y_1^2\left(y_2^4 + n_5^2\right) + {n_4'}^2\right)^{-\nu_2-1/2}$$
$$e(-m_2 n_5)\, dn_4 dn_5$$
$$= y_1^{\nu_1-2\nu_2+2} y_2^{\nu_1+1} B\left(\frac{1}{2}, \nu_2 - \frac{\nu_1}{2}\right) B\left(\frac{1}{2}, \nu_2\right) \int_{\mathbb{R}} \left(y_2^4 + n_5^2\right)^{-\nu_1/2-1/2} e(-m_2 n_5)\, dn_5$$
$$= y_1^{\nu_1-2\nu_2+2} B\left(\frac{1}{2}, \nu_2 - \frac{\nu_1}{2}\right) B\left(\frac{1}{2}, \nu_2\right) \int_{\mathbb{R}} \left(\frac{y_2^2}{y_2^4 + n_5^2}\right)^{\nu_1/2+1/2} e(-m_2 n_5)\, dn_5$$
$$= y_1^{\nu_1-2\nu_2+2} B\left(\frac{1}{2}, \nu_2 - \frac{\nu_1}{2}\right) B\left(\frac{1}{2}, \nu_2\right) W\left(y_2^2, \frac{\nu_1}{2}, \chi_{m_2}\right). \qquad \square$$

Remark. If ψ is a non-degenerate character of $U(\mathbb{Z})\backslash U(\mathbb{R})$, that is, $\psi = \psi_{m_1,m_2}$ with $m_1, m_2 \neq 0$, it is shown in [Ish05] that $W_{w_0}(g, \nu, \psi)$ has moderate growth, i.e. $W_{w_0}(g, \nu, \psi) \in \mathcal{W}(\nu, \psi)^{\mathrm{mod}}$. Hence, by Theorem 2.34, $W_{w_0}(g, \nu, \psi)$ is the unique function (up to a constant multiple) in $\mathcal{W}(\nu, \psi)^{\mathrm{mod}}$. This function is studied extensively by Ishii [Ish05].

2.5.3 Minimal Eisenstein series

Let $\psi = \psi_{m_1,m_2}$ be a character of $U(\mathbb{Z})\backslash U(\mathbb{R})$. The ψ-th Fourier coefficient of the minimal Eisenstein series $E_0(g,\nu)$ is given by

$$E_{0,\psi}(g,\nu) := \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E_0(\eta g,\nu)\overline{\psi}(\eta)d\eta.$$

Remark. In principle, one may consider the Fourier coefficients along other subgroups. For example, for Siegel modular forms, one usually considers the Fourier coefficients along the upper right block, which forms an abelian group. Here, we consider the Fourier coefficients along the unipotent part U of G. As U is not abelian, we are not guaranteed a Fourier expansion from these Fourier coefficients. Indeed, on Sp(4), there exist automorphic forms that do not admit a Whittaker model, and all the Fourier coefficients along U vanish. Nevertheless, these Fourier coefficients find applications for instance in the constructions of L-functions via Langlands-Shahidi method [Sha10].

To compute the Fourier coefficients $E_{0,\psi}(g,\nu)$, we break down the expression via Bruhat decomposition, and express them in terms of Whittaker functions. We have

$$\begin{aligned} E_{0,\psi}(g,\nu) &= \sum_{w\in W}\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E_{0,w}(\eta g,\nu)\overline{\psi}(\eta)d\eta \\ &= \sum_{w\in W}\sum_{\gamma\in R_w}\sum_{\delta\in\Gamma_w}\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} I_0(\gamma\delta\eta g,\nu)\overline{\psi}(\eta)d\eta \\ &= \sum_{w\in W}\sum_{\gamma\in R_w}\int_{\overline{U}_w(\mathbb{Z})\backslash\overline{U}_w(\mathbb{R})}\int_{U_w(\mathbb{R})} I_0(\gamma\eta\eta' g,\nu)\overline{\psi}(\eta\eta')d\eta d\eta'. \end{aligned}$$

Let $\gamma = b_1wtb_2$ be a Bruhat decomposition, with $b_1, b_2 \in U$, $t \in T$. Again we may assume that $b_2 \in U_w$. Then we have

$$E_{0,\psi}(g,\nu) = \sum_{w\in W}\sum_{\substack{\gamma\in R_w\\ \gamma=b_1wtb_2}}\int_{\overline{U}_w(\mathbb{Z})\backslash\overline{U}_w(\mathbb{R})}\int_{U_w(\mathbb{R})} I_0(b_1wtb_2\eta\eta' g,\nu)\overline{\psi}(\eta\eta')d\eta d\eta'.$$

Then change of variables $b_2\eta \mapsto \eta$ gives

$$E_{0,\psi}(g,\nu) = \sum_{w\in W}\sum_{\substack{\gamma\in R_w\\ \gamma=b_1wtb_2}}\psi(b_2)\int_{\overline{U}_w(\mathbb{Z})\backslash\overline{U}_w(\mathbb{R})}\int_{U_w(\mathbb{R})} I_0(wt\eta\eta' g,\nu)\overline{\psi}(\eta\eta')d\eta d\eta'.$$

Now observe that

$$I_0(wtg,\nu) = I_0\left((wtw^{-1})wg,\nu\right) = I_0(wtw^{-1},\nu)I_0(wg,\nu).$$

So the Fourier coefficient becomes

$$\begin{aligned} E_{0,\psi}(g,\nu) &= \sum_{w\in W}\sum_{\substack{\gamma\in R_w\\ \gamma=b_1wtb_2}}\psi(b_2)I_0(wtw^{-1},\nu)\int_{\overline{U}_w(\mathbb{Z})\backslash\overline{U}_w(\mathbb{R})}\int_{U_w(\mathbb{R})} I_0(w\eta\eta' g,\nu)\overline{\psi}(\eta\eta')d\eta d\eta' \\ &= \sum_{w\in W}\sum_{\substack{\gamma\in R_w\\ \gamma=b_1wtb_2}}\psi(b_2)I_0(wtw^{-1},\nu)\int_{\overline{U}_w(\mathbb{Z})\backslash\overline{U}_w(\mathbb{R})} W_w(\eta' g,\nu,\psi)\overline{\psi}(\eta')d\eta'. \end{aligned}$$

Recall that $W_w(g,\nu,\psi)=0$ unless ψ is trivial on $\overline{U}_w(\mathbb{R})$. If ψ is trivial on $\overline{U}_w(\mathbb{R})$, then it follows from the definition of a Whittaker function that $W_w(\eta' g,\nu,\psi)=W_w(g,\nu,\psi)$ for $\eta'\in\overline{U}_w(\mathbb{R})$. So the Fourier coefficient becomes

$$\sum_{w\in W}\sum_{\substack{\gamma\in R_w\\ \gamma=b_1wtb_2}}\psi(b_2)I_0(wtw^{-1},\nu)\int_{\overline{U}_w(\mathbb{Z})\backslash\overline{U}_w(\mathbb{R})}W_w(g,\nu,\psi).$$

Hence, to obtain the Fourier coefficients of $E_0(g,\nu)$, it suffices to evaluate for $w\in W$ the sum

$$E_{0,\psi,w}(g,\nu):=\sum_{\substack{\gamma\in R_w\\ \gamma=b_1wtb_2}}\psi(b_2)I_0(wtw^{-1},\nu)W_w(g,\nu,\psi).$$

(i) For $w=\mathrm{id}$, we have $R_{\mathrm{id}}=\{I_4\}$. So we immediately obtain

$$E_{0,\psi,\mathrm{id}}(g,\nu)=W_{\mathrm{id}}(g,\nu,\psi).$$

(ii) For $w=s_\alpha$, from Bruhat decomposition in Section 2.2.4, we deduce that if $\gamma\in R_{s_\alpha}$ has Plücker coordinates $v=(0,0,v_3,v_4;0,0,0,0,0,1)$, then we have

$$I_0(wtw^{-1},\nu)=v_4^{2\nu_2-2\nu_1-1},$$

and $\psi_{m_1,m_2}(b_2)=e\left(-\frac{m_1v_3}{v_4}\right)$. Hence

$$\begin{aligned}E_{0,\psi,s_\alpha}(g,\nu)&=\sum_{v_4\ge1}\sum_{\substack{v_3 \,(\mathrm{mod}\ v_4)\\(v_3,v_4)=1}}v_4^{2\nu_2-2\nu_1-1}e\left(-\frac{m_1v_3}{v_4}\right)W_{s_\alpha}(g,\nu,\psi)\\&=\sum_{v_4\ge1}v_4^{2\nu_2-2\nu_1-1}c_{v_4}(m_1)W_{s_\alpha}(g,\nu,\psi).\end{aligned}$$

So the inner sum is actually a classical Ramanujan sum. Using the well-known identity

$$\sum_{n\ge1}c_n(m)n^{-k-1}=\begin{cases}\dfrac{\sigma_{-k}(m)}{\zeta(k+1)} & \text{if } m\neq0,\\ \dfrac{\zeta(k)}{\zeta(k+1)} & \text{if } m=0,\end{cases}\tag{2.41}$$

we see that

$$E_{0,\psi,s_\alpha}(g,\nu)=\begin{cases}\dfrac{\sigma_{2\nu_2-2\nu_1}(m_1)}{\zeta(2\nu_1-2\nu_2+1)}W_{s_\alpha}(g,\nu,\psi) & \text{if } m_1\neq0,\\ \dfrac{\zeta(2\nu_1-2\nu_2)}{\zeta(2\nu_1-2\nu_2+1)}W_{s_\alpha}(g,\nu,\psi) & \text{if } m_1=0.\end{cases}$$

(iii) For $w=s_\beta$, if $\gamma\in R_{s_\beta}$ has Plücker coordinates v, then we have

$$I_0(wtw^{-1},\nu)=v_{23}^{\nu_1-2\nu_2-1},$$

and $\psi_{m_1,m_2}(b_2)=e\left(-\frac{m_2v_{34}}{v_{23}}\right)$. Hence

$$\begin{aligned}E_{0,\psi,s_\beta}(g,\nu)&=\sum_{v_{23}\ge1}\sum_{\substack{v_{34}\,(\mathrm{mod}\ v_{23})\\(v_{23},v_{34})=1}}v_{23}^{\nu_1-2\nu_2-1}e\left(-\frac{m_2v_{34}}{v_{23}}\right)W_{s_\beta}(g,\nu,\psi)\\&=\sum_{v_{23}\ge1}v_{23}^{\nu_1-2\nu_2-1}c_{v_{23}}(m_2)W_{s_\beta}(g,\nu,\psi).\end{aligned}$$

By (2.41), we obtain that

$$E_{0,\psi,s_\beta}(g,\nu)=\begin{cases}\dfrac{\sigma_{\nu_1-2\nu_2}(m_2)}{\zeta(2\nu_2-\nu_1+1)}W_{s_\beta}(g,\nu,\psi) & \text{if } m_2\neq 0,\\ \dfrac{\zeta(2\nu_2-\nu_1)}{\zeta(2\nu_2-\nu_1+1)}W_{s_\beta}(g,\nu,\psi) & \text{if } m_2=0.\end{cases}$$

(iv) For $w=s_\alpha s_\beta$, recall that

$$R_{s_\alpha s_\beta}=\left\{\left(0,v_2,v_3,v_4;0,0,0,\frac{v_2}{d},0,-\frac{v_4}{d}\right)\right\},$$

where $v_2\geq 1$, $v_3,v_4 \pmod{v_2}$, such that $(v_2,v_3,v_4)=1$, and $d=(v_2,v_4)$. If $\gamma\in R_{s_\alpha s_\beta}$ has Plücker coordinates v, then we have

$$I_0(wtw^{-1},\nu)=v_2^{2\nu_2-2\nu_1-1}v_{23}^{\nu_1-2\nu_2-1}=v_2^{-\nu_1-2}d^{2\nu_2-\nu_1+1},$$

and $\psi_{m_1,m_2}(b_2)=e\left(\frac{m_2v_4}{v_2}\right)$. Hence

$$E_{0,\psi,s_\alpha s_\beta}(g,\nu)=\sum_{v_2\geq 1}\sum_{\substack{v_3,v_4 \pmod{v_2}\\(v_2,v_3,v_4)=1}}v_2^{-\nu_1-2}d^{2\nu_2-\nu_1+1}e\left(\frac{m_2v_4}{v_2}\right)W_{s_\alpha s_\beta}(g,\nu,\psi).$$

Write $v_2=dv_2'$, $v_4=dv_4'$. Then the sum can be rewritten as

$$\begin{aligned}E_{0,\psi,s_\alpha s_\beta}(g,\nu)&=\sum_{d\geq 1}d^{2\nu_2-2\nu_1-1}\sum_{v_2'\geq 1}{v_2'}^{-\nu_1-2}\sum_{\substack{v_4' \pmod{v_2'}\\(v_2',v_4')=1}}e\left(\frac{m_2v_4'}{v_2'}\right)\sum_{\substack{v_3 \pmod{dv_2'}\\(d,v_3)=1}}W_{s_\alpha s_\beta}(g,\nu,\psi)\\&=\sum_{d\geq 1}\varphi(d)d^{2\nu_2-2\nu_1-1}\sum_{v_2'\geq 1}{v_2'}^{-\nu_1-1}c_{v_2'}(m_2)W_{s_\alpha s_\beta}(g,\nu,\psi),\end{aligned}$$

where φ stands for the Euler totient function. By (2.41), we obtain that

$$E_{0,\psi,s_\alpha s_\beta}(g,\nu)=\begin{cases}\dfrac{\zeta(2\nu_1-2\nu_2)}{\zeta(2\nu_1-2\nu_2+1)}\dfrac{\sigma_{-\nu_1}(m_2)}{\zeta(\nu_1+1)}W_{s_\alpha s_\beta}(g,\nu,\psi) & \text{if } m_2\neq 0,\\ \dfrac{\zeta(2\nu_1-2\nu_2)}{\zeta(2\nu_1-2\nu_2+1)}\dfrac{\zeta(\nu_1)}{\zeta(\nu_1+1)}W_{s_\alpha s_\beta}(g,\nu,\psi) & \text{if } m_2=0.\end{cases}$$

(v) For $w=s_\beta s_\alpha$, recall that

$$R_{s_\beta s_\alpha}=\left\{\left(0,0,-\frac{v_{24}}{d},\frac{v_{14}}{d};0,-v_{24},v_{14},-\frac{v_{24}^2}{v_{14}},v_{24},v_{34}\right)\right\},$$

where $v_{14}\geq 1$, $v_{24},v_{34} \pmod{v_{14}}$, $d=(v_{14},v_{24})$, such that $v_{14}\mid d^2$ and $\left(\frac{d^2}{v_{14}},v_{34}\right)=1$. If $\gamma\in R_{s_\beta s_\alpha}$ has Plücker coordinates v, then we have

$$I_0(wtw^{-1},\nu)=v_4^{2\nu_2-2\nu_1-1}v_{14}^{\nu_1-2\nu_2-1}=v_{14}^{-\nu_1-2}d^{2\nu_1-2\nu_2+1},$$

and $\psi_{m_1,m_2}(b_2)=e\left(\frac{m_1v_{24}}{v_{14}}\right)$. Hence

$$E_{0,\psi,s_\beta s_\alpha}(g,\nu)=\sum_{v_{14}\geq 1}\sum_{\substack{v_{24} \pmod{v_{14}}\\v_{14}\mid d^2}}\sum_{\substack{v_{34} \pmod{v_{14}}\\\left(\frac{d^2}{v_{14}},v_{34}\right)=1}}v_{14}^{-\nu_1-2}d^{2\nu_1-2\nu_2+1}e\left(\frac{m_1v_{24}}{v_{14}}\right)W_{s_\beta s_\alpha}(g,\nu,\psi).$$

Write $v_{14} = dv'_{14}$, $v_{24} = dv'_{24}$, and $d' = d^2/v_{14}$. Recall that we have $d = v'_{14}d'$. Then we have $v_{14} = d'{v'_{14}}^2$, and the sum can be rewritten as

$$\begin{aligned} E_{0,\psi,s_\beta s_\alpha}(g,\nu) &= \sum_{d'\geq 1} d'^{\nu_1-2\nu_2-1} \sum_{v'_{14}\geq 1} {v'_{14}}^{-2\nu_2-3} \sum_{\substack{v'_{24} (\mathrm{mod}\ v'_{14}) \\ (v'_{14},v'_{24})=1}} e\left(\frac{m_1 v'_{24}}{v'_{14}}\right) \sum_{\substack{v_{34} (\mathrm{mod}\ d'{v'_{14}}^2) \\ (d',v_{34})=1}} W_{s_\beta s_\alpha}(g,\nu,\psi) \\ &= \sum_{d'\geq 1} \varphi(d') d'^{\nu_1-2\nu_2-1} \sum_{v'_{14}\geq 1} {v'_{14}}^{-2\nu_2-1} c_{v'_{14}}(m_1) W_{s_\beta s_\alpha}(g,\nu,\psi). \end{aligned}$$

By (2.41), we obtain that

$$E_{0,\psi,s_\beta s_\alpha}(g,\nu) = \begin{cases} \dfrac{\zeta(2\nu_2-\nu_1)}{\zeta(2\nu_2-\nu_1+1)} \dfrac{\sigma_{-2\nu_2}(m_1)}{\zeta(2\nu_2+1)} W_{s_\beta s_\alpha}(g,\nu,\psi) & \text{if } m_1 \neq 0, \\ \dfrac{\zeta(2\nu_2-\nu_1)}{\zeta(2\nu_2-\nu_1+1)} \dfrac{\zeta(2\nu_2)}{\zeta(2\nu_2+1)} W_{s_\beta s_\alpha}(g,\nu,\psi) & \text{if } m_1 = 0. \end{cases}$$

(vi) For $w = s_\alpha s_\beta s_\alpha$, recall that

$$R_{s_\alpha s_\beta s_\alpha} = \left\{ \left(v_1, v_2, v_3, v_4; 0, -\frac{v_1 v_2}{\delta}, \frac{v_1^2}{\delta}, -\frac{v_2^2}{\delta}, \frac{v_1 v_2}{\delta}, \frac{v_1 v_3 + v_2 v_4}{\delta} \right) \right\},$$

where $v_1 \geq 1$, $v_2, v_3, v_4 \pmod{v_1}$, such that $(v_1, v_2, v_3, v_4) = 1$, and $d = (v_1, v_2)$, $\delta = (d^2, v_1 v_3 + v_2 v_4)$. If $\gamma \in R_{s_\alpha s_\beta s_\alpha}$ has Plücker coordinates v, then we have

$$I_0(wtw^{-1},\nu) = v_1^{2\nu_2-2\nu_1-1} v_{14}^{\nu_1-2\nu_2-1} = v_1^{-2\nu_2-3} \delta^{2\nu_2-\nu_1+1},$$

and $\psi_{m_1,m_2}(b_2) = e\left(\frac{m_1 v_2}{v_1}\right)$. Hence

$$E_{0,\psi,s_\alpha s_\beta s_\alpha}(g,\nu) = \sum_{v_1\geq 1} \sum_{\substack{v_2,v_3,v_4 (\mathrm{mod}\ v_1) \\ (v_1,v_2,v_3,v_4)=1}} v_1^{-2\nu_2-3} \delta^{2\nu_2-\nu_1+1} e\left(\frac{m_1 v_2}{v_1}\right) W_{s_\alpha s_\beta s_\alpha}(g,\nu,\psi).$$

Write $v_1 = dv'_1$, $v_2 = dv'_2$. Since $d \mid \delta$, so we may also write $\delta = d\delta'$. Note that $\delta' = (d, v'_1 v_3 + v'_2 v_4)$ divides d. Then the sum can be rewritten as

$$E_{0,\psi,s_\alpha s_\beta s_\alpha}(g,\nu) = \sum_{d\geq 1} d^{-\nu_1-2} \sum_{v'_1\geq 1} {v'_1}^{-2\nu_2-3} \sum_{\substack{v'_2 (\mathrm{mod}\ v'_1) \\ (v'_1,v'_2)=1}} e\left(\frac{m_1 v'_2}{v'_1}\right) \sum_{\substack{v_3,v_4 (\mathrm{mod}\ dv'_1) \\ (d,v_3,v_4)=1}} \delta'^{2\nu_2-\nu_1+1} W_{s_\alpha s_\beta s_\alpha}(g,\nu,\psi).$$

For fixed $l \mid d$, we find the number of pairs (v_3, v_4) modulo d satisfying $(d, v_3, v_4) = 1$, and $(d, v'_1 v_3 + v'_2 v_4) = l$. We first observe that for every residue class (v_3, v_4) modulo d, we can find representatives such that $0 \leq v'_1 v_3 + v'_2 v_4 < d$. As $(v'_1, v'_2) = 1$, we can find $u_3, u_4 \in \mathbb{Z}$ such that $v'_1 u_3 + v'_2 u_4 = 1$. Then for $0 \leq n < d$, the equation

$$v'_1 v_3 + v'_2 v_4 \equiv n \pmod{d} \tag{2.42}$$

has d distinct solutions, given by $(v_3, v_4) = (nu_3 + kv'_2, nu_4 - kv'_1)$ for $0 \leq k < d$. A residue class (v_3, v_4) modulo d satisfies $(d, v'_1 v_3 + v'_2 v_4) = l$ if and only if $l = (n, d)$. Let $0 \leq n < d$ be such that $(n, d) = l$. Then the number of solutions to (2.42) satisfying $(d, v_3, v_4) = 1$ is given by $d\varphi(l)/l$. Meanwhile, the number of integers $0 \leq n < d$ with $(n, d) = l$ is given by $\varphi(d/l)$. Hence, there are in total $d\varphi(d/l)\varphi(l)/l$ solutions for (v_3, v_4) modulo d such that $(d, v'_1 v_3 + v'_2 v_4) = l$. Hence the sum becomes

$$E_{0,\psi,s_\alpha s_\beta s_\alpha}(g,\nu) = \sum_{d\geq 1} d^{-\nu_1-1} \sum_{v'_1\geq 1} {v'_1}^{-2\nu_2-3} c_{v'_1}(m_1) \sum_{l\mid d} \varphi\left(\frac{d}{l}\right) \varphi(l) l^{2\nu_2-\nu_1} W_{s_\alpha s_\beta s_\alpha}(g,\nu,\psi).$$

Writing $d = d'l$ gives

$$E_{0,\psi,s_\alpha s_\beta s_\alpha}(g,\nu) = \sum_{d\geq 1} \varphi(d')d'^{-\nu_1-1} \sum_{l\geq 1} \varphi(l) l^{2\nu_2-2\nu_1-1} \sum_{v_1'\geq 1} {v_1'}^{-2\nu_2-3} c_{v_1'}(m_1) W_{s_\alpha s_\beta s_\alpha}(g,\nu,\psi).$$

By (2.41), we obtain that

$$E_{0,\psi,s_\alpha s_\beta s_\alpha}(g,\nu) = \begin{cases} \dfrac{\zeta(\nu_1)}{\zeta(\nu_1+1)} \dfrac{\zeta(2\nu_1-2\nu_2)}{\zeta(2\nu_1-2\nu_2+1)} \dfrac{\sigma_{-2\nu_2}(m_1)}{\zeta(2\nu_2+1)} W_{s_\alpha s_\beta s_\alpha}(g,\nu,\psi) & \text{if } m_1 \neq 0, \\ \dfrac{\zeta(\nu_1)}{\zeta(\nu_1+1)} \dfrac{\zeta(2\nu_1-2\nu_2)}{\zeta(2\nu_1-2\nu_2+1)} \dfrac{\zeta(2\nu_2)}{\zeta(2\nu_2+1)} W_{s_\alpha s_\beta s_\alpha}(g,\nu,\psi) & \text{if } m_1 = 0. \end{cases}$$

(vii) For $w = s_\beta s_\alpha s_\beta$, recall that

$$R_{s_\beta s_\alpha s_\beta} = \left\{ \left(0, \frac{v_{12}}{d_0}, \frac{v_{13}}{d_0}, \frac{v_{14}}{d_0}; v_{12}, v_{13}, v_{14}, v_{23}, -v_{13}, -\frac{v_{13}^2 + v_{14}v_{23}}{v_{12}} \right) \right\}.$$

where $v_{12} \geq 1$, $v_{13}, v_{14}, v_{23} \pmod{v_{12}}$, with the following conditions. Let $d_1 = (v_{12}, v_{14})$, and $d_0 = (v_{12}, v_{13}, v_{14})$. Then we require $d_1 \mid d_0^2$. Write $v_{12} = d_1 v_{12}'$, $v_{14} = d_1 v_{14}'$, $v_{13} = d_1 k$, and $d' = d_1/d_0$, $t = d_0/d'$. Let a be a solution to $a v_{14}' \equiv -k \pmod{v_{12}'}$, such that a and $\frac{a v_{14}' + k}{v_{12}'}$ are divisible by t. Then we require v_{23} to be of the form $v_{23} = a + r v_{12}'$ with $(r,t) = 1$.

If $\gamma \in R_{s_\beta s_\alpha s_\beta}$ has Plücker coordinates v, then we have

$$I_0(wtw^{-1},\nu) = v_2^{2\nu_2-2\nu_1-1} v_{12}^{\nu_1-2\nu_2-1} = v_{12}^{-\nu_1-2} d_0^{2\nu_1-2\nu_2+1},$$

and $\psi_{m_1,m_2}(b_2) = e\left(\frac{m_2 v_{14}}{v_{12}}\right)$. Hence

$$E_{0,\psi,s_\beta s_\alpha s_\beta}(g,\nu) = \sum_{v_{12}\geq 1} \sum_{\substack{v_{13},v_{14},v_{23} \pmod{v_{12}} \\ \text{conditions}}} v_{12}^{-\nu_1-2} d_0^{2\nu_1-2\nu_2+1} e\left(\frac{m_2 v_{14}}{v_{12}}\right) W_{s_\beta s_\alpha s_\beta}(g,\nu,\psi).$$

Expanding the conditions above, we rewrite the sum in terms of d', t and v_{12}':

$$\begin{aligned} E_{0,\psi,s_\beta s_\alpha s_\beta}(g,\nu) &= \sum_{d'\geq 1} d'^{-2\nu_2-3} \sum_{t\geq 1} t^{\nu_1-2\nu_2-1} \sum_{v_{12}'\geq 1} {v_{12}'}^{-\nu_1-2} \sum_{\substack{v_{14}' \pmod{v_{12}'} \\ (v_{12}',v_{14}')=1}} e\left(\frac{m_2 v_{14}'}{v_{12}'}\right) \\ &\qquad \sum_{\substack{v_{13}' \pmod{d' v_{12}'} \\ (d',v_{13}')=1}} \sum_{\substack{v_{23} \pmod{d'^2 t v_{12}'} \\ v_{23}=a+r v_{12}' \\ (r,t)=1}} W_{s_\beta s_\alpha s_\beta}(g,\nu,\psi) \\ &= \sum_{d'\geq 1} \varphi(d') d'^{-2\nu_2-1} \sum_{t\geq 1} \varphi(t) t^{\nu_1-2\nu_2-1} \sum_{v_{12}'\geq 1} {v_{12}'}^{-\nu_1-1} c_{v_{12}'}(m_2) W_{s_\beta s_\alpha s_\beta}(g,\nu,\psi). \end{aligned}$$

By (2.41), we obtain that

$$E_{0,\psi,s_\beta s_\alpha s_\beta}(g,\nu) = \begin{cases} \dfrac{\zeta(2\nu_2)}{\zeta(2\nu_2+1)} \dfrac{\zeta(2\nu_2-\nu_1)}{\zeta(2\nu_2-\nu_1+1)} \dfrac{\sigma_{-\nu_1}(m_2)}{\zeta(\nu_1+1)} W_{s_\beta s_\alpha s_\beta}(g,\nu,\psi) & \text{if } m_2 \neq 0, \\ \dfrac{\zeta(2\nu_2)}{\zeta(2\nu_2+1)} \dfrac{\zeta(2\nu_2-\nu_1)}{\zeta(2\nu_2-\nu_1+1)} \dfrac{\zeta(\nu_1)}{\zeta(\nu_1+1)} W_{s_\beta s_\alpha s_\beta}(g,\nu,\psi) & \text{if } m_2 = 0. \end{cases}$$

(viii) For $w = w_0$, if $\gamma \in R_{w_0}$ has Plücker coordinates v, then

$$I_0(wtw^{-1}, \nu) = v_1^{2\nu_2 - 2\nu_1 - 1} v_{12}^{\nu_1 - 2\nu_2 - 1},$$

and $\psi_{m_1,m_2}(b_2) = e\left(\frac{m_1 v_2}{v_1} + \frac{m_2 v_{14}}{v_{12}}\right)$. Hence

$$E_{0,\psi,w_0}(g,\nu) = \sum_{\gamma \in R_{w_0}} v_1^{2\nu_2 - 2\nu_1 - 1} v_{12}^{\nu_1 - 2\nu_2 - 1} e\left(\frac{m_1 v_2}{v_1} + \frac{m_2 v_{14}}{v_{12}}\right) W_{w_0}(g,\nu,\psi).$$

Note that this is actually a Dirichlet series of Sp(4) Ramanujan sums. Indeed, we have

$$E_{0,\psi,w_0}(g,\nu) = \sum_{v_1, v_{12} \geq 1} R_{v_1,v_{12}}(m_1,m_2) v_1^{2\nu_2 - 2\nu_1 - 1} v_{12}^{\nu_1 - 2\nu_2 - 1} W_{w_0}(g,\nu,\psi),$$

where $R_{v_1,v_{12}}(m_1,m_2)$ is an Sp(4) Ramanujan sum, defined in (2.26). By Proposition 2.31, we obtain

$$E_{0,\psi,w_0}(g,\nu) = \begin{cases} \dfrac{\sigma_{-\nu_2,\nu_2-\nu_1}(m_1,m_2)}{\zeta(2\nu_1 - 2\nu_2 + 1)\zeta(2\nu_2 - \nu_1 + 1)\zeta(\nu_1 + 1)\zeta(2\nu_2 + 1)} W_{w_0}(g,\nu,\psi) & \text{if } m_1, m_2 \neq 0, \\ \dfrac{\sigma_{2\nu_2 - 2\nu_1}(m_1)\zeta(2\nu_2 - \nu_1)\zeta(\nu_1)\zeta(2\nu_2)}{\zeta(2\nu_1 - 2\nu_2 + 1)\zeta(2\nu_2 - \nu_1 + 1)\zeta(\nu_1 + 1)\zeta(2\nu_2 + 1)} W_{w_0}(g,\nu,\psi) & \text{if } m_1 \neq 0, m_2 = 0, \\ \dfrac{\sigma_{\nu_1 - 2\nu_2}(m_2)\zeta(2\nu_1 - 2\nu_2)\zeta(\nu_1)\zeta(2\nu_2)}{\zeta(2\nu_1 - 2\nu_2 + 1)\zeta(2\nu_2 - \nu_1 + 1)\zeta(\nu_1 + 1)\zeta(2\nu_2 + 1)} W_{w_0}(g,\nu,\psi) & \text{if } m_1 = 0, m_2 \neq 0, \\ \dfrac{\zeta(2\nu_1 - 2\nu_2)\zeta(2\nu_2 - \nu_1)\zeta(\nu_1)\zeta(2\nu_2)}{\zeta(2\nu_1 - 2\nu_2 + 1)\zeta(2\nu_2 - \nu_1 + 1)\zeta(\nu_1 + 1)\zeta(2\nu_2 + 1)} W_{w_0}(g,\nu,\psi) & \text{if } m_1 = m_2 = 0. \end{cases}$$

Remark. The Fourier coefficients $E_{0,\psi,w}(g,\nu)$ are originally defined on an open subset of $\mathfrak{a}_{0\mathbb{C}}^*$, but it follows readily from the expressions that they can be continued into meromorphic functions on $\mathfrak{a}_{0\mathbb{C}}^*$.

Recalling that $W_w(g,\nu,\psi)$ is nonzero only if ψ is trivial on $\overline{U}_w(\mathbb{R})$, we conclude the computations with the following theorem.

Theorem 2.36. For $\psi = \psi_{m_1,m_2}$, the ψ-th Fourier coefficient of the minimal Eisenstein series $E_0(g,\nu)$ is given as follows:

(i) If $m_1, m_2 \neq 0$, then

$$E_{0,\psi}(g,\nu) = \frac{\sigma_{-\nu_2,\nu_2-\nu_1}(m_1,m_2)}{\zeta(2\nu_1 - 2\nu_2 + 1)\zeta(2\nu_2 - \nu_1 + 1)\zeta(\nu_1 + 1)\zeta(2\nu_2 + 1)} W_{w_0}(g,\nu,\psi);$$

(ii) if $m_1 \neq 0$, $m_2 = 0$, then

$$\begin{aligned} E_{0,\psi}(g,\nu) = {} & \frac{\sigma_{2\nu_2 - 2\nu_1}(m_1)}{\zeta(2\nu_1 - 2\nu_2 + 1)} W_{s_\alpha}(g,\nu,\psi) + \frac{\zeta(2\nu_2 - \nu_1)}{\zeta(2\nu_2 - \nu_1 + 1)} \frac{\sigma_{-2\nu_2}(m_1)}{\zeta(2\nu_2 + 1)} W_{s_\beta s_\alpha}(g,\nu,\psi) \\ & + \frac{\zeta(\nu_1)}{\zeta(\nu_1 + 1)} \frac{\zeta(2\nu_1 - 2\nu_2)}{\zeta(2\nu_1 - 2\nu_2 + 1)} \frac{\sigma_{-2\nu_2}(m_1)}{\zeta(2\nu_2 + 1)} W_{s_\alpha s_\beta s_\alpha}(g,\nu,\psi) \\ & + \frac{\zeta(2\nu_2 - \nu_1)}{\zeta(2\nu_2 - \nu_1 + 1)} \frac{\zeta(\nu_1)}{\zeta(\nu_1 + 1)} \frac{\zeta(2\nu_2)}{\zeta(2\nu_2 + 1)} \frac{\sigma_{2\nu_2 - 2\nu_1}(m_1)}{\zeta(2\nu_1 - 2\nu_2 + 1)} W_{w_0}(g,\nu,\psi); \end{aligned}$$

(iii) if $m_1 = 0$, $m_2 \neq 0$, then

$$\begin{aligned} E_{0,\psi}(g,\nu) = &\frac{\sigma_{\nu_1-2\nu_2}(m_2)}{\zeta(2\nu_2-\nu_1+1)} W_{s_\beta}(g,\nu,\psi) + \frac{\zeta(2\nu_1-2\nu_2)}{\zeta(2\nu_1-2\nu_2+1)} \frac{\sigma_{-\nu_1}(m_2)}{\zeta(\nu_1+1)} W_{s_\alpha s_\beta}(g,\nu,\psi) \\ &+ \frac{\zeta(2\nu_2)}{\zeta(2\nu_2+1)} \frac{\zeta(2\nu_2-\nu_1)}{\zeta(2\nu_2-\nu_1+1)} \frac{\sigma_{-\nu_1}(m_2)}{\zeta(\nu_1+1)} W_{s_\beta s_\alpha s_\beta}(g,\nu,\psi) \\ &+ \frac{\zeta(2\nu_1-2\nu_2)}{\zeta(2\nu_1-2\nu_2+1)} \frac{\zeta(\nu_1)}{\zeta(\nu_1+1)} \frac{\zeta(2\nu_2)}{\zeta(2\nu_2+1)} \frac{\sigma_{\nu_1-2\nu_2}(m_2)}{\zeta(2\nu_2-\nu_1+1)} W_{w_0}(g,\nu,\psi); \end{aligned}$$

(iv) if $m_1 = m_2 = 0$, then the Fourier coefficient $E_{0,\psi_{0,0}}(g,\nu)$ is precisely the constant term $C_0(g,\nu)$ of the minimal Eisenstein series along the minimal parabolic, and the expression is given in Theorem 2.21.

The Fourier coefficients $E_{0,\psi}(g,\nu)$ are originally defined on an open subset of $\mathfrak{a}_{0\mathbb{C}}^*$, but they can be continued into meromorphic functions on $\mathfrak{a}_{0\mathbb{C}}^*$.

2.5.4 Maximal Eisenstein series $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$

In Proposition 2.10 we showed that the maximal Eisenstein series $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$ are actually residues of the minimal Eisenstein series $E_0(g,\nu)$. Since the Fourier coefficients are integrals over a compact set, we obtain the Fourier coefficients of $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$ by taking the residues of the Fourier coefficients of the minimal Eisenstein series. Let

$$E_{\alpha,\psi}(g,\nu,1) := \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E_\alpha(\eta g,\nu,1)\overline{\psi}(\eta)d\eta, \quad E_{\beta,\psi}(g,\nu,1) := \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E_\beta(\eta g,\nu,1)\overline{\psi}(\eta)d\eta$$

denote the ψ-th Fourier coefficients of $E_\alpha(g,\nu,1)$ and $E_\beta(g,\nu,1)$ respectively. Then we have the following corollaries.

Corollary 2.37. For $\psi = \psi_{m_1,m_2}$, the ψ-th Fourier coefficient of $E_\alpha(g,\nu,1)$ is given as follows:

(i) If $m_1, m_2 \neq 0$, then $E_{\alpha,\psi}(g,\nu,1) = 0$;

(ii) if $m_1 \neq 0$, $m_2 = 0$, then

$$E_{\alpha,\psi}(g,\nu,1) = \frac{1}{\pi} \frac{\zeta(\nu+1/2)}{\zeta(\nu+3/2)} \frac{\sigma_{-2\nu}(m_1)}{\zeta(2\nu+1)} W_{s_\alpha s_\beta s_\alpha}(g,(\nu+1/2,\nu),\psi);$$

(iii) if $m_1 = 0$, $m_2 \neq 0$, then

$$\begin{aligned} E_{\alpha,\psi}(g,\nu,1) = &\frac{1}{\pi} \frac{\sigma_{-\nu-1/2}(m_2)}{\zeta(\nu+3/2)} W_{s_\alpha s_\beta}(g,(\nu+1/2,\nu),\psi) \\ &+ \frac{1}{\pi} \frac{\zeta(\nu+1/2)}{\zeta(\nu+3/2)} \frac{\zeta(2\nu)}{\zeta(2\nu+1)} \frac{\sigma_{-\nu+1/2}(m_2)}{\zeta(\nu+1/2)} W_{w_0}(g,(\nu+1/2,\nu),\psi); \end{aligned}$$

(iv) if $m_1 = m_2 = 0$, then the Fourier coefficient $E_{\alpha,\psi_{0,0}}(g,\nu,1)$ is precisely the constant term $C_\alpha^0(g,\nu,1)$ of $E_\alpha(g,\nu,1)$ along the minimal parabolic, and the expression is given in Corollary 2.25.

Corollary 2.38. For $\psi = \psi_{m_1,m_2}$, the ψ-th Fourier coefficient of $E_\beta(g,\nu,1)$ is given as follows:

(i) If $m_1, m_2 \neq 0$, then $E_{\beta,\psi}(g,\nu,1) = 0$;

(ii) if $m_1 \neq 0$, $m_2 = 0$, then

$$E_{\beta,\psi}(g,\nu,1) = \frac{1}{\pi}\frac{\sigma_{-\nu-1}(m_1)}{\zeta(\nu+2)} W_{s_\beta s_\alpha}(g,(\nu,(\nu+1)/2),\psi) + \frac{1}{\pi}\frac{\zeta(\nu)}{\zeta(\nu+1)}\frac{\zeta(\nu+1)}{\zeta(\nu+2)}\frac{\sigma_{-\nu+1}(m_1)}{\zeta(\nu)} W_{w_0}(g,(\nu,(\nu+1)/2),\psi);$$

(iii) if $m_1 = 0$, $m_2 \neq 0$, then

$$E_{\beta,\psi}(g,\nu,1) = \frac{1}{\pi}\frac{\zeta(\nu+1)}{\zeta(\nu+2)}\frac{\sigma_{-\nu}(m_2)}{\zeta(\nu+1)} W_{s_\beta s_\alpha s_\beta}(g,(\nu,(\nu+1)/2),\psi);$$

(iv) if $m_1 = m_2 = 0$, then the Fourier coefficient $E_{\beta,\psi_{0,0}}(g,\nu,1)$ is precisely the constant term $C^0_\beta(g,\nu,1)$ of $E_\beta(g,\nu,1)$ along the minimal parabolic, and the expression is given in Corollary 2.28.

Chapter 3

Symplectic Kloosterman sums

3.1 Construction of symplectic Kloosterman sums

Let

$$G = \mathrm{Sp}(2r) = \left\{ M \in \mathrm{GL}(2r) \mid M^T J M = J \right\}, \quad J = \begin{pmatrix} & I_r \\ -I_r & \end{pmatrix}$$

be the standard symplectic group, with the standard torus and the standard unipotent subgroup given by

$$T = \left\{ \begin{pmatrix} * & & & & & \\ & * & & & & \\ & & * & & & \\ & & & \ddots & & \\ & & & & * & \\ & & & & & * \end{pmatrix} \right\} \subseteq G, \quad U = \left\{ \begin{pmatrix} 1 & \cdots & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & 1 & * & \cdots & * \\ & & & 1 & & \\ & & & \vdots & \ddots & \\ & & & * & \cdots & 1 \end{pmatrix} \right\} \subseteq G$$

respectively. Let $N = N_G(T)$ be the normaliser of T in G. The Weyl group is given by $W := N_G(T)/T$. Let $w : N \to W$ be the canonical quotient map. For $w \in W$, we define $U_w := U \cap w^{-1} U^T w$, and $\overline{U}_w := U \cap w^{-1} U w$.

Let p be a rational prime. We have a Bruhat decomposition

$$G(\mathbb{Q}_p) = U(\mathbb{Q}_p) N(\mathbb{Q}_p) U(\mathbb{Q}_p).$$

For $n \in N(\mathbb{Q}_p)$, we define

$$\begin{aligned} C(n) &:= U(\mathbb{Q}_p) n U(\mathbb{Q}_p) \cap G(\mathbb{Z}_p), \\ X(n) &:= U(\mathbb{Z}_p) \backslash C(n) / U_{w(n)}(\mathbb{Z}_p), \end{aligned}$$

and projection maps

$$\begin{aligned} u &: X(n) \to U(\mathbb{Z}_p) \backslash U(\mathbb{Q}_p), \\ u' &: X(n) \to U(\mathbb{Q}_p) / U_{w(n)}(\mathbb{Z}_p) \end{aligned}$$

by the relation $x = u(x) n u'(x)$ for $x \in X(n)$.

For $n \in N(\mathbb{Q}_p)$, let ψ_p be a character of $U(\mathbb{Q}_p)$ which is trivial on $U(\mathbb{Z}_p)$, and ψ'_p a character of $U_{w(n)}(\mathbb{Q}_p)$ trivial on $U_{w(n)}(\mathbb{Z}_p)$, such that ψ'_p is the restriction of some character of $U(\mathbb{Q}_p)$ trivial on $U(\mathbb{Z}_p)$. The local Kloosterman sum is then given by

$$\mathrm{Kl}_p(n, \psi_p, \psi'_p) = \sum_{x \in X(n)} \psi_p\left(u(x)\right) \psi'_p\left(u'(x)\right).$$

Usually, ψ'_p is given as a character of $U(\mathbb{Q}_p)$ trivial on $U(\mathbb{Z}_p)$, and we write $\mathrm{Kl}_p(n, \psi_p, \psi'_p)$ to mean $\mathrm{Kl}_p(n, \psi_p, \psi'_p|_{U_{w(n)}(\mathbb{Q}_p)})$.

Now we give a global construction of Kloosterman sums. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$. Let $n \in N(\mathbb{Q})$, $\psi = \prod_p \psi_p$ a character of $U(\mathbb{A})$ trivial on $\prod_p U(\mathbb{Z}_p)$, and $\psi' = \prod_p \psi'_p$ a character of $U_{w(n)}(\mathbb{A})$ trivial on $\prod_p U_{w(n)}(\mathbb{Z}_p)$, such that ψ' is the restriction of some character of $U(\mathbb{A})$ trivial on $\prod_p U(\mathbb{Z}_p)$. The global Kloosterman sum is then given by

$$\mathrm{Kl}(n, \psi, \psi') = \prod_p \mathrm{Kl}_p(n, \psi_p, \psi'_p).$$

Remark. For characters ψ, ψ' of $U(\mathbb{Q})/U(\mathbb{Z})$, we can also define $\mathrm{Kl}(n, \psi, \psi')$ by considering ψ, ψ' as characters of $U(\mathbb{A})/\prod_p U(\mathbb{Z}_p)$. In fact, this is how global Kloosterman sums are usually defined in practice, for instance in Sections 3.4 and 3.5.

Remark. This definition of Kloosterman sums is different from the symplectic Kloosterman sums introduced by Kitaoka [Kit84], which are more relevant for classical Sp(4) Fourier expansions with respect to the upper right 2-by-2 block. Tóth [Tót13] proved some properties and estimates of such Kloosterman sums. The Kloosterman sums introduced here fit into the general framework of Kloosterman sums defined on reductive groups, see e.g. Dąbrowski [Dąb93].

Proposition 3.1 ([Ste87, Theorem 3.2]). Let $n \in N(\mathbb{Q}_p)$, and ψ, ψ' characters of $U(\mathbb{Q}_p)$ trivial on $U(\mathbb{Z}_p)$. If $t \in T(\mathbb{Z}_p^\times)$, then

$$\begin{aligned} \mathrm{Kl}_p(tn, \psi, \psi') &= \mathrm{Kl}_p(n, \psi_t, \psi'), \\ \mathrm{Kl}_p(nt^{-1}, \psi, \psi') &= \mathrm{Kl}_p(n, \psi, \psi'_t), \end{aligned}$$

where $\psi_t(x) = \psi(txt^{-1})$.

Proof. If $x \in C(n)$ has decomposition $x = u(x)nu'(x)$, then tx has decomposition $tu(x)t^{-1}nu'(x)$. As t is invertible, this shows that $C(tn) = t \cdot C(n)$. Hence

$$\mathrm{Kl}_p(n, \psi_t, \psi') = \sum_{x \in C(n)} \psi\left(tu(x)t^{-1}\right)\psi'\left(u'(x)\right) = \sum_{x \in C(tn)} \psi\left(u(x)\right)\psi'\left(u'(x)\right) = \mathrm{Kl}_p(tn, \psi, \psi').$$

The second statement is proved analogously, using $C(nt^{-1}) = C(n) \cdot t^{-1}$. □

By Proposition 3.1, we can reduce a local Kloosterman sum into a Kloosterman sum $\mathrm{Kl}_p(n, \psi, \psi')$ where the entries of $n \in N(\mathbb{Q}_p)$ are powers of p.

3.1.1 Sp(4) Kloosterman sums

For the rest of the section, we restrict our attention to the case $G = \mathrm{Sp}(4)$. A description of the Weyl group W of G is given in Section 2.1. For $m_1, m_2 \in \mathbb{Z}$, let ψ_{m_1,m_2} be the character of $U(\mathbb{Q}_p)$ given by

$$\psi_{m_1,m_2}\begin{pmatrix} 1 & x_1 & * & * \\ & 1 & * & x_2 \\ & & 1 & \\ & & -x_1 & 1 \end{pmatrix} = e\left(m_1x_1 + m_2x_2\right). \tag{3.1}$$

Then ψ_{m_1,m_2} is trivial on $U(\mathbb{Z}_p)$, and it is easy to verify that every character of $U(\mathbb{Q}_p)$ trivial on $U(\mathbb{Z}_p)$ is of this form.

Fix $\psi = \psi_{m_1,m_2}$, and $\psi' = \psi_{n_1,n_2}$. We give an explicit characterisation of local Kloosterman sums for $G = \mathrm{Sp}(4)$. By Proposition 3.1, it suffices to consider the Kloosterman sums $\mathrm{Kl}_p(n, \psi, \psi')$ for which the entries of n are powers of p. It is also natural to just consider $n \in N(\mathbb{Q}_p)$ such that $X(n)$ is nonempty. The Kloosterman sums are classified by the Weyl element $w(n)$, and the elements in $X(n)$ are identified by their Plücker coordinates, introduced in Section 2.2.

(i) If $w(n) = \mathrm{id}$, then $X(n)$ is nonempty when $n = n_{\mathrm{id}} := I_4$. In this case, the Kloosterman sum is trivial:

$$\mathrm{Kl}_p(n_{\mathrm{id}}, \psi, \psi') = 1.$$

(ii) If $w(n) = s_\alpha$, then $X(n)$ is nonempty when

$$n = n_{s_\alpha,r} := \begin{pmatrix} & p^{-r} & & \\ -p^r & & & \\ & & & p^r \\ & & -p^{-r} & \end{pmatrix}$$

for $r \geq 0$. We identify $X(n_{s_\alpha,r})$ by the Plücker coordinates

$$X(n_{s_\alpha,r}) = \{(0,0,v_3,p^r;0,0,0,0,0,1)\},$$

where $v_3 \pmod{p^r}$, such that $(v_3, p^r) = 1$. Bruhat decomposition gives

$$\begin{aligned} x &= \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} & p^{-r} & & \\ -p^r & & & \\ & & & p^r \\ & & -p^{-r} & \end{pmatrix} \begin{pmatrix} 1 & -v_3p^{-r} & & \\ & 1 & & \\ & & 1 & \\ & & v_3p^{-r} & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\beta_1 p^r & \beta_1 v_3 + p^{-r} & \beta_2 v_3 - \beta_3 p^{-r} & \beta_2 p^r \\ -p^r & v_3 & \beta_4 v_3 - \beta_5 p^{-r} & \beta_4 p^r \\ 0 & 0 & v_3 & p^r \\ 0 & 0 & -\beta_1 v_3 - p^{-r} & -\beta_1 p^r \end{pmatrix}. \end{aligned}$$

The entry $\beta_1 v_3 + p^{-r}$ being an integer says $\beta_1 \equiv -\overline{v_3}p^{-r} \pmod 1$. The entry $\beta_4 p^r$ being an integer says $\beta_4 \in p^{-r}\mathbb{Z}$. So, the entry $\beta_4 v_3 - \beta_5 p^{-r}$ being an integer says $\beta_5 \equiv 0 \pmod 1$. So the Kloosterman sum is given by

$$\mathrm{Kl}_p(n_{s_\alpha,r}, \psi, \psi') = \sum_{\substack{v_3 \pmod{p^r} \\ (v_3,p^r)=1}} e\left(\frac{-m_1\overline{v_3} - n_1 v_3}{p^r}\right) = S(m_1, n_1; p^r).$$

So this is actually a classical Kloosterman sum.

(iii) If $w(n) = s_\beta$, then $X(n)$ is nonempty when

$$n = n_{s_\beta,s} := \begin{pmatrix} 1 & & & \\ & & & p^{-s} \\ & & 1 & \\ & -p^s & & \end{pmatrix}$$

for $s \geq 0$. We identify $X(n_{s_\beta,s})$ by the Plücker coordinates

$$X(n_{s_\beta,s}) = \{(0,0,1,0;0,0,0,p^s,0,v_{34})\},$$

where $v_{34} \pmod{p^s}$, such that $(v_{34}, p^s) = 1$. Bruhat decomposition gives

$$x = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & & p^{-s} \\ & & 1 & \\ & -p^{-s} & & \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & -v_{34}p^{-s} \\ & & 1 & \\ & & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -\beta_3 p^s & \beta_2 & \beta_3 v_{34} + \beta_1 p^{-s} \\ 0 & -\beta_5 p^s & \beta_4 & \beta_5 v_{34} + p^{-s} \\ 0 & 0 & 1 & 0 \\ 0 & -p^{-s} & -\beta_1 & v_{34} \end{pmatrix}.$$

The entry $\beta_5 v_{34} + p^{-s}$ being an integer says $\beta_5 \equiv -\overline{v_{34}}p^{-s} \pmod 1$. The entry $-\beta_1$ being an integer says $\beta_1 \equiv 0 \pmod 1$. So the Kloosterman sum is given by

$$\mathrm{Kl}_p(n_{s_\beta,s}, \psi, \psi') = \sum_{\substack{v_{34} (\mathrm{mod}\ p^s) \\ (v_{34}, p^s)=1}} e\left(\frac{-m_2\overline{v_{34}} - n_2 v_{34}}{p^s}\right) = S(m_2, n_2; p^s).$$

So this is actually a classical Kloosterman sum.

(iv) If $w(n) = s_\alpha s_\beta$, then $X(n)$ is nonempty when

$$n = n_{s_\alpha s_\beta, r, s} := \begin{pmatrix} & & & -p^{-r} \\ p^{r-s} & & & \\ & p^r & & \\ & & p^{s-r} & \end{pmatrix}$$

for $r \geq s \geq 0$. We identify $X(n_{s_\alpha s_\beta, r,s})$ by the Plücker coordinates

$$X(n_{s_\alpha s_\beta,r,s}) = \left\{ \left(0, p^r, v_3, v_4; 0, 0, 0, p^s, 0, -v_4 p^{s-r}\right) \right\},$$

where $v_3, v_4 \pmod{p^r}$, such that $(v_4, p^r) = p^{r-s}$, and $(v_3, p^{r-s}) = 1$. Write $v_4 = v_4' p^{r-s}$, so $(v_4', p^s) = 1$. Bruhat decomposition gives

$$x = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} & & & -p^{-r} \\ p^{r-s} & & & \\ & p^r & & \\ & & p^{s-r} & \end{pmatrix} \begin{pmatrix} 1 & & & v_3 p^{-r} \\ & 1 & v_3 p^{-r} & v_4' p^{-s} \\ & & 1 & \\ & & & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \beta_1 p^{r-s} & \beta_2 p^r & \beta_2 v_3 + \beta_3 p^{s-r} & \beta_2 v_4' p^{r-s} + \beta_1 v_3 p^{-s} - p^{-r} \\ p^{r-s} & \beta_4 p^r & \beta_4 v_3 + \beta_5 p^{s-r} & \beta_4 v_4' p^{r-s} + v_3 p^{-s} \\ 0 & p^r & v_3 & v_4' p^{r-s} \\ 0 & -\beta_1 p^r & -\beta_1 v_3 + p^{s-r} & -\beta_1 v_4' p^{r-s} \end{pmatrix}.$$

The entry $-\beta_1 v_3 + p^{s-r}$ being an integer says $\beta_1 \equiv \overline{v_3} p^{s-r} \pmod 1$. The entry $\beta_4 v_4' p^{r-s} + v_3 p^{-s}$ being an integer says $\beta_4 \equiv -\overline{v_4'} v_3 p^{-r} \pmod{p^{s-r}}$. Write $\beta_4 = -\overline{v_4'} v_3 p^{-r} + \gamma_4 p^{s-r}$ for some $\gamma_4 \in \mathbb{Z}$. The entry $\beta_4 v_3 + \beta_5 p^{s-r}$ being an integer says $\gamma_4 v_3 + \beta_5 \equiv \overline{v_4'} v_3^2 p^{-s} \pmod{p^{r-s}}$, hence $\beta_5 \equiv \overline{v_4'} v_3^2 p^{-s} \pmod 1$. So the Kloosterman sum is given by

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,r,s}, \psi, \psi'\right) = \sum_{\substack{v_4 (\mathrm{mod}\ p^s) \\ (v_4, p^s)=1}} \sum_{\substack{v_3 (\mathrm{mod}\ p^r) \\ (v_3, p^{r-s})=1}} e\left(\frac{m_1 \overline{v_3} p^s}{p^r}\right) e\left(\frac{m_2 \overline{v_4} v_3^2 + n_2 v_4}{p^s}\right).$$

(v) If $w(n) = s_\beta s_\alpha$, then $X(n)$ is nonempty when

$$n = n_{s_\beta s_\alpha, r,s} := \begin{pmatrix} & p^{-r} & & \\ & & p^{r-s} & \\ & & & p^r \\ -p^{s-r} & & & \end{pmatrix}$$

for $s \geq 2r \geq 0$. We identify $X(n_{s_\beta s_\alpha,r,s})$ by the Plücker coordinates

$$X(n_{s_\beta s_\alpha,r,s}) = \left\{ \left(0, 0, -v_{24}p^{r-s}, p^r; 0, -v_{24}, p^s, -v_{24}p^{-s}, v_{24}, v_{34}\right) \right\},$$

where $v_{24}, v_{34} \pmod{p^s}$, such that $(v_{24}, p^s) = p^{s-r}$, and $(v_{34}, p^{s-2r}) = 1$. Write $v_{24} = v_{24}'p^{s-r}$, so $(v_{24}', p^r) = 1$. Bruhat decomposition gives

$$\begin{aligned} x &= \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} & p^{-r} & & \\ & & p^{r-s} & \\ & & & p^r \\ -p^{s-r} & & & \end{pmatrix} \begin{pmatrix} 1 & v_{24}'p^{-r} & v_{34}p^{-s} & \\ & 1 & & \\ & & 1 & \\ & & -v_{24}'p^{-r} & 1 \end{pmatrix} \\ &= \begin{pmatrix} -\beta_3 p^{s-r} & -\beta_3 v_{24}' p^{s-2r} + p^{-r} & -\beta_2 v_{24}' - \beta_3 v_{34} p^{-r} + \beta_1 p^{r-s} & \beta_2 p^r \\ -\beta_5 p^{s-r} & -\beta_5 v_{24}' p^{s-2r} & -\beta_4 v_{24}' - \beta_5 v_{34} p^{-r} + p^{r-s} & \beta_4 p^r \\ 0 & 0 & -v_{24}' & p^r \\ -p^{s-r} & -v_{24}' p^{s-2r} & \beta_1 v_{24}' - v_{34} p^{-r} & -\beta_1 p^r \end{pmatrix}. \end{aligned}$$

The entry $\beta_1 v_{24}' - v_{34}p^{-r}$ being an integer says $\beta_1 \equiv \overline{v_{24}'} v_{34} p^{-r} \pmod 1$. The entry $\beta_4 p^r$ being an integer says $\beta_4 = \beta_4' p^{-r}$ for some $\beta_4' \in \mathbb{Z}$. The entry $-\beta_4 v_{24}' - \beta_5 v_{34} p^{-r} + p^{r-s}$ being an integer says $\beta_4' v_{24}' + \beta_5 v_{34} \equiv p^{2r-s} \pmod{p^r}$, hence $\beta_5 \equiv \overline{v_{34}} p^{2r-s} \pmod 1$. So the Kloosterman sum is given by

$$\mathrm{Kl}_p\left(n_{s_\beta s_\alpha,r,s}, \psi, \psi'\right) = \sum_{\substack{v_{24} (\mathrm{mod}\ p^r) \\ (v_{24}, p^r) = 1}} \sum_{\substack{v_{34} (\mathrm{mod}\ p^s) \\ (v_{34}, p^{s-2r}) = 1}} e\left(\frac{m_1 \overline{v_{24}} v_{34} + n_1 v_{24}}{p^r}\right) e\left(\frac{m_2 \overline{v_{34}} p^{2r}}{p^s}\right).$$

(vi) If $w(n) = s_\alpha s_\beta s_\alpha$, then $X(n)$ is nonempty when

$$n = n_{s_\alpha s_\beta s_\alpha, r,s} := \begin{pmatrix} & & -p^{-r} & \\ & p^{r-s} & & \\ p^r & & & \\ & & & p^{s-r} \end{pmatrix},$$

for $2r \geq s \geq 0$. We identify $X(n_{s_\alpha s_\beta s_\alpha,r,s})$ by the Plücker coordinates

$$X(n_{s_\alpha s_\beta s_\alpha,r,s}) = \left\{ \left(p^r, v_2, v_3, v_4; 0, -v_2 p^{s-r}, p^s, -v_2^2 p^{s-2r}, v_2 p^{s-r}, (p^r v_3 + v_2 v_4) p^{s-2r}\right) \right\},$$

where $v_2, v_3, v_4 \pmod{p^r}$, such that $(v_2, v_3, v_4, p^r) = 1$, and $d = (v_2, p^r)$, $(d^2, p^r v_3 + v_2 v_4) = p^{2r-s}$. Let $d = p^{r-a}$. Then a satisfies $s - r \leq a \leq s/2$. Write $v_2 = v_2' p^{r-a}$, so $(v_2', p^a) = 1$. Bruhat decomposition gives

$$\begin{aligned} x &= \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} & & -p^{-r} & \\ & p^{r-s} & & \\ p^r & & & \\ & & & p^{s-r} \end{pmatrix} \begin{pmatrix} 1 & v_2' p^{-a} & v_3 p^{-r} & v_4 p^{-r} \\ & 1 & v_4 p^{-r} & \\ & & 1 & \\ & & -v_2' p^{-a} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta_2 p^r & \beta_2 v_2' p^{r-a} + \beta_1 p^{r-s} & \beta_2 v_3 - \beta_3 v_2' p^{s-a-r} + \beta_1 v_4 p^{-s} - p^{-r} & \beta_2 v_4 + \beta_3 p^{s-r} \\ \beta_4 p^r & \beta_4 v_2' p^{r-a} + p^{r-s} & \beta_4 v_3 - \beta_5 v_2' p^{s-a-r} + v_4 p^{-s} & \beta_4 v_4 + \beta_5 p^{s-r} \\ p^r & v_2' p^{r-a} & v_3 & v_4 \\ -\beta_1 p^r & -\beta_1 v_2' p^{r-a} & -\beta_1 v_3 - v_2' p^{s-a-r} & -\beta_1 v_4 + p^{s-r} \end{pmatrix}. \end{aligned}$$

The entry $-\beta_1 v_2' p^{r-a}$ being an integer says $\beta_1 = \beta_1' p^{a-r}$ for some $\beta_1' \in \mathbb{Z}$. Entries $-\beta_1 v_3 - v_2' p^{s-a-r}$ and $-\beta_1 v_4 + p^{s-r}$ being integers says

$$\beta_1' v_3 \equiv -v_2' p^{s-2a} \pmod{p^{r-a}}, \qquad \beta_1' v_4 \equiv p^{s-a} \pmod{p^{r-a}}. \tag{3.2}$$

As $(v_3, v_4, p^{r-a}) = 1$, these equations determine β_1 uniquely modulo 1.

The entry $\beta_4 v_2' p^{r-a} + p^{r-s}$ being an integer says $\beta_4 \equiv -\overline{v_2'} p^{a-s} \pmod{p^{a-r}}$. Write $\beta_4 = -\overline{v_2'} p^{a-s} + \gamma_4 p^{a-r}$ for some $\gamma_4 \in \mathbb{Z}$. Then $\beta_4 v_3 - \beta_5 v_2' p^{s-a-r} + v_4 p^{-s}$ being an integer says

$$-\overline{v_2'} v_3 p^a + \gamma_4 v_3 p^{s+a-r} - \beta_5 v_2' p^{2s-a-r} + v_4 \equiv 0 \pmod{p^s}. \tag{3.3}$$

Write $\beta_5 = \beta_5' p^{a+r-2s}$ for some $\beta_5' \in \mathbb{Z}$. Then we solve

$$\beta_5' \equiv -\overline{v_2'}^2 v_3 p^a + \gamma_4 \overline{v_2'} v_3 p^{s+a-r} + \overline{v_2'} v_4 \pmod{p^s}. \tag{3.4}$$

Then $\beta_4 v_4 + \beta_5 p^{s-r}$ being an integer says

$$\gamma_4 \left(p^a v_3 + v_2' v_4\right) p^{s+a-r} \equiv v_3 p^{2a} \pmod{p^s}. \tag{3.5}$$

Recall that $(p^{r-a}, p^a v_3 + v_2' v_4) = p^{r+a-s}$. Hence, unless $a = \frac{s}{2}$, we can write $p^a v_3 + v_2' v_4 = V' p^{r+a-s}$, with $(V', p) = 1$. Then we solve (3.5):

$$\gamma_4 \equiv \overline{V'} v_3 \pmod{p^{s-2a}}.$$

Putting back to (3.4) gives

$$\beta_5' \equiv -\overline{v_2'}^2 v_3 p^a + \overline{V' v_2'} v_3^2 p^{s+a-r} + \overline{v_2'} v_4 \pmod{p^{2s-a-r}},$$

hence β_5 is uniquely determined modulo 1.

When $a = \frac{s}{2}$, γ_4 can be arbitrary, and we have

$$\beta_5' \equiv -\overline{v_2'}^2 v_3 p^a + \overline{v_2'} v_4 \pmod{p^{2s-a-r}},$$

hence β_5 is also uniquely determined modulo 1 in this case.

So the Kloosterman sum is given by

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta s_\alpha, r, s}, \psi, \psi'\right) = \sum_{s-r \le a \le s/2} \sum_{\substack{v_2, v_3, v_4 \,(\mathrm{mod}\ p^r) \\ v_2 = v_2' p^{r-a},\ (v_2', p^a) = 1 \\ (v_3, v_4, p^{r-a}) = 1 \\ \left(p^{r-a}, p^a v_3 + v_2' v_4\right) = p^{r+a-s}}} e\left(\frac{m_1 \hat{v}_2 + n_1 v_2}{p^r}\right) e\left(\frac{m_2 u}{p^s}\right),$$

where $\hat{v}_2$ is chosen modulo p^r such that

$$\hat{v}_2 v_3 \equiv -v_2' p^{s-a} \pmod{p^r}, \qquad \hat{v}_2 v_4 \equiv p^s \pmod{p^r}, \tag{3.6}$$

and

$$u \equiv \begin{cases} -\overline{v_2'}^2 v_3 p^{2a+r-s} + \overline{V' v_2'} v_3^2 p^{2a} + \overline{v_2'} v_4 p^{a+r-s} \pmod{p^s} & \text{if } a < \frac{s}{2}, \\ -\overline{v_2'}^2 v_3 p^{2a+r-s} + \overline{v_2'} v_4 p^{a+r-s} \pmod{p^s} & \text{if } a = \frac{s}{2}, \end{cases} \tag{3.7}$$

where $V' = p^{s-r-a}\left(p^a v_3 + v_2' v_4\right)$.

(vii) If $w(n) = s_\beta s_\alpha s_\beta$, then $X(n)$ is nonempty when

$$n = n_{s_\beta s_\alpha s_\beta,r,s} := \begin{pmatrix} & & & -p^{-r} \\ & & p^{r-s} & \\ & p^r & & \\ -p^{s-r} & & & \end{pmatrix}$$

for $s \geq r \geq 0$. We identify $X(n_{s_\beta s_\alpha s_\beta,r,s})$ by the Plücker coordinates

$$X(n_{s_\beta s_\alpha s_\beta,r,s}) = \left\{ \left(0, p^r, v_{13}p^{r-s}, v_{14}p^{r-s}; p^s, v_{13}, v_{14}, v_{23}, -v_{13}, -(v_{13}^2 + v_{14}v_{23})p^{-s}\right) \right\},$$

where $v_{13}, v_{14}, v_{23} \pmod{p^s}$, such that $(v_{13}, v_{14}, p^s) = p^{s-r}$, $(v_{14}, p^s) \mid v_{13}^2$, and $(p^{s-r}, v_{23}, v_{34}) = 1$. Recall that $v_{34} = -(v_{13}^2 + v_{14}v_{23})p^{-s}$. Write $v_{13} = v_{13}'p^{s-r}$, $v_{14} = v_{14}'p^{s-r}$, so $(v_{13}', v_{14}', p^r) = 1$. Bruhat decomposition gives

$$x = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} & & & -p^{-r} \\ & & p^{r-s} & \\ & p^r & & \\ -p^{s-r} & & & \end{pmatrix} \begin{pmatrix} 1 & & -v_{23}p^{-s} & v_{13}'p^{-r} \\ & 1 & v_{13}'p^{-r} & v_{14}'p^{-r} \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\beta_3 p^{s-r} & \beta_2 p^r & \beta_2 v_{13}' + \beta_1 p^{r-s} + \beta_3 v_{23} p^{-r} & \beta_2 v_{14}' - \beta_3 v_{13}' p^{s-2r} - p^{-r} \\ -\beta_5 p^{s-r} & \beta_4 p^r & \beta_4 v_{13}' + \beta_5 v_{23} p^{-r} + p^{r-s} & \beta_4 v_{14}' - \beta_5 v_{13}' p^{s-2r} \\ 0 & p^r & v_{13}' & v_{14}' \\ -p^{s-r} & -\beta_1 p^r & -\beta_1 v_{13}' + v_{23} p^{-r} & -\beta_1 v_{14}' - v_{13}' p^{s-2r} \end{pmatrix}.$$

The entry $-\beta_1 p^r$ being an integer says $\beta_1 = \beta_1' p^{-r}$ for $\beta_1' \in \mathbb{Z}$. Entries $-\beta_1 v_{13}' + v_{23}p^{-r}$ and $-\beta_1 v_{14}' - v_{13}'p^{s-2r}$ being integers says

$$\beta_1' v_{13}' \equiv v_{23} \pmod{p^r}, \qquad \beta_1' v_{14}' \equiv -v_{13}' p^{s-r} \pmod{p^r}. \tag{3.8}$$

As $(v_{13}', v_{14}', p^r) = 1$, this determines β_1 uniquely modulo 1.

Entries $\beta_4 p^r$ and $-\beta_5 p^{s-r}$ being integers says $\beta_4 = \beta_4' p^{-r}$ and $\beta_5 = \beta_5' p^{r-s}$ for some $\beta_4', \beta_5' \in \mathbb{Z}$. The entry $\beta_4 v_{13}' + \beta_5 v_{23} p^{-r} + p^{r-s}$ being an integer says

$$\beta_4' v_{13}' p^{s-r} + \beta_5' v_{23} + p^r \equiv 0 \pmod{p^s}, \tag{3.9}$$

which implies

$$\beta_5' v_{23} \equiv -p^r \pmod{p^{s-r}}. \tag{3.10}$$

The entry $\beta_4 v_{14}' - \beta_5 v_{13}' p^{s-2r}$ being an integer says

$$\beta_4' v_{14}' p^{s-r} - \beta_5' v_{13}' p^{s-r} \equiv 0 \pmod{p^s}. \tag{3.11}$$

Then, v_{13}' times (3.11) minus v_{14}' times (3.9) gives

$$\begin{aligned} \beta_5'(-{v_{13}'}^2 p^{s-r} - v_{14}' v_{23}) &\equiv p^r v_{14}' \pmod{p^s} \\ \beta_5' p^r v_{34} &\equiv p^r v_{14}' \pmod{p^s} \\ \beta_5' v_{34} &\equiv v_{14}' \pmod{p^{s-r}}. \end{aligned} \tag{3.12}$$

As $(p^{s-r}, v_{23}, v_{34}) = 1$, (3.10) and (3.12) determine β_5 uniquely modulo 1.

So the Kloosterman sum is given by

$$\mathrm{Kl}_p\left(n_{s_\beta s_\alpha s_\beta,r,s}, \psi, \psi'\right) = \sum_{\substack{v_{13}, v_{14}, v_{23} \pmod{p^s} \\ (p^s, v_{13}, v_{14}) = p^{s-r} \\ (p^s, v_{14}) \mid v_{13}^2 \\ \left(p^{s-r}, v_{23}, v_{34}\right) = 1}} e\left(\frac{m_1 u}{p^r}\right) e\left(\frac{m_2 \hat{v}_{14} + n_2 v_{14}}{p^s}\right),$$

where u is chosen modulo p^r such that

$$uv_{13}p^{r-s} \equiv v_{23} \pmod{p^r}, \qquad uv_{14}p^{r-s} \equiv -v_{13} \pmod{p^r}, \tag{3.13}$$

and $\hat{v}_{14}$ is chosen modulo p^s such that

$$\hat{v}_{14}v_{23} \equiv -p^{2r} \pmod{p^s}, \qquad \hat{v}_{14}v_{34} \equiv v_{14}p^{2r-s} \pmod{p^s}. \tag{3.14}$$

(viii) If $w(n) = w_0$, then $X(n)$ is nonempty when

$$n = n_{w_0,r,s} := \begin{pmatrix} & & -p^{-r} & \\ & & & -p^{r-s} \\ p^r & & & \\ & p^{s-r} & & \end{pmatrix}$$

for $r, s \geq 0$. We identify $X(n_{w_0,r,s})$ by the Plücker coordinates

$$X(n_{w_0,r,s}) = \left\{ \left(p^r, v_2, v_3, v_4; p^s, v_{13}, v_{14}, (v_2v_{13} - v_3p^s)p^{-r}, -v_{13}, (v_3v_{14} - v_4v_{13})p^{-r}\right) \right\},$$

where $v_2, v_3, v_4 \pmod{p^r}$, $v_{13}, v_{14} \pmod{p^s}$, such that $v_{13}p^r + v_2v_{14} - v_4p^s = 0$, $(v_2, v_3, v_4, p^r) = 1$, and $(v_{13}, v_{14}, v_{23}, v_{34}, p^s) = 1$. Recall that

$$v_{23} = (v_2v_{13} - v_3p^s)p^{-r}, \quad v_{34} = (v_3v_{14} - v_4v_{13})p^{-r}.$$

Bruhat decomposition gives

$$x = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} & & -p^{-r} & \\ & & & -p^{r-s} \\ p^r & & & \\ & p^{s-r} & & \end{pmatrix} \begin{pmatrix} 1 & v_2p^{-r} & v_3p^{-r} & v_4p^{-r} \\ & 1 & v_{13}p^{-s} & v_{14}p^{-s} \\ & & 1 & \\ & & -v_2p^{-r} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \beta_2p^r & \beta_2v_2 + \beta_3p^{s-r} & \beta_2v_3 + \beta_3v_{13}p^{-r} + \beta_1v_2p^{-s} - p^{-r} & \beta_2v_4 - \beta_1p^{r-s} + \beta_3v_{14}p^{-r} \\ \beta_4p^r & \beta_4v_2 + \beta_5p^{s-r} & \beta_4v_3 + \beta_5v_{13}p^{-r} + v_2p^{-s} & \beta_4v_4 + \beta_5v_{14}p^{-r} - p^{r-s} \\ p^r & v_2 & v_3 & v_4 \\ -\beta_1p^r & -\beta_1v_2 + p^{s-r} & -\beta_1v_3 + v_{13}p^{-r} & -\beta_1v_4 + v_{14}p^{-r} \end{pmatrix}.$$

The entry $-\beta_1p^r$ being an integer says $\beta_1 = \beta_1'p^{-r}$ for some $\beta_1' \in \mathbb{Z}$. The last row of γ being integral gives

$$\beta_1'v_2 \equiv p^s \pmod{p^r}, \qquad \beta_1'v_3 \equiv v_{13} \pmod{p^r}, \qquad \beta_1'v_4 \equiv v_{14} \pmod{p^r}. \tag{3.15}$$

As $(p^r, v_2, v_3, v_4) = 1$, these equations determine β_1 uniquely modulo 1.

The entry β_4p^r being an integer says $\beta_4 = \beta_4'p^{-r}$ for some $\beta_4' \in \mathbb{Z}$. Then $\beta_4v_2 + \beta_5p^{s-r}$ being an integer says

$$\beta_4'v_2 + \beta_5p^s \equiv 0 \pmod{p^r}. \tag{3.16}$$

In particular, this means $\beta_5 = \beta_5'p^{-s}$ for some $\beta_5' \in \mathbb{Z}$. The entries $\beta_4v_3 + \beta_5v_{13}p^{-r} + v_2p^{-s}$ and $\beta_4v_4 + \beta_5v_{14}p^{-r} - p^{r-s}$ being integers says

$$\beta_4'v_3p^s + \beta_5'v_{13} + v_2p^r \equiv 0 \pmod{p^{r+s}}, \tag{3.17}$$
$$\beta_4'v_4p^s + \beta_5'v_{14} - p^{2r} \equiv 0 \pmod{p^{r+s}}. \tag{3.18}$$

In particular we deduce

$$\beta_5'v_{13} + v_2p^r \equiv 0 \pmod{p^s}, \tag{3.19}$$
$$\beta_5'v_{14} - p^{2r} \equiv 0 \pmod{p^s}. \tag{3.20}$$

Then, v_2 times (3.17) minus v_3p^s times (3.16) gives

$$\beta_5'(v_2v_{13} - v_3p^s) + v_2^2p^r \equiv 0 \pmod{p^{r+s}} \implies \beta_5'v_{23} + v_2^2 \equiv 0 \pmod{p^s}. \tag{3.21}$$

Similarly, v_3 times (3.17) minus v_4 times (3.18) gives

$$\begin{aligned} &\beta_5'(v_3v_{14} - v_4v_{13}) - p^r(v_3p^r + v_2v_4) \equiv 0 \pmod{p^{r+s}} \\ \implies &\beta_5'v_{34} \equiv (v_3p^r + v_2v_4) \pmod{p^s}. \end{aligned} \tag{3.22}$$

In summary, β_5' satisfies the following equations:

$$\begin{aligned} &\beta_5'v_{13} \equiv -v_2p^r \pmod{p^s}, \quad &\beta_5'v_{14} \equiv p^{2r} \pmod{p^s}, \\ &\beta_5'v_{23} \equiv -v_2^2 \pmod{p^s}, \quad &\beta_5'v_{34} \equiv v_3p^r + v_2v_4 \pmod{p^s}. \end{aligned}$$

As $(p^s, v_{13}, v_{14}, v_{23}, v_{34}) = 1$, these equations determine β_5 uniquely modulo 1.

So the Kloosterman sum is given by

$$\mathrm{Kl}_p(n_{w_0,r,s}, \psi, \psi') = \sum_{\substack{v_2,v_3,v_4 (\mathrm{mod}\ p^r) \\ v_{13},v_{14} (\mathrm{mod}\ p^s) \\ v_{13}p^r + v_2v_{14} - v_4p^s = 0 \\ (p^r,v_2,v_3,v_4)=1 \\ (p^s,v_{13},v_{14},v_{23},v_{34})=1}} e\left(\frac{m_1\hat{v}_2 + n_1v_2}{p^r}\right) e\left(\frac{m_2\hat{v}_{14} + n_2v_{14}}{p^s}\right),$$

where $\hat{v}_2$ is chosen modulo p^r such that

$$\hat{v}_2v_2 \equiv p^s \pmod{p^r}, \quad \hat{v}_2v_3 \equiv v_{13} \pmod{p^r}, \quad \hat{v}_2v_4 \equiv v_{14} \pmod{p^r}; \tag{3.23}$$

and $\hat{v}_{14}$ chosen modulo p^s such that

$$\begin{aligned} &\hat{v}_{14}v_{13} \equiv -v_2p^r \pmod{p^s}, \quad &\hat{v}_{14}v_{14} \equiv p^{2r} \pmod{p^s}, \\ &\hat{v}_{14}v_{23} \equiv -v_2^2 \pmod{p^s}, \quad &\hat{v}_{14}v_{34} \equiv v_3p^r + v_2v_4 \pmod{p^s}. \end{aligned} \tag{3.24}$$

3.1.2 Properties of $\mathrm{Sp}(4)$ Kloosterman sums

Proposition 3.2. Let $n \in N(\mathbb{Q}_p)$, such that $w(n) = w_0$ is the long Weyl element. Let ψ, ψ' be characters of $U(\mathbb{Q}_p)$ trivial on $U(\mathbb{Z}_p)$. Then

$$\mathrm{Kl}_p(n, \psi, \psi') = \mathrm{Kl}_p(n, \psi', \psi).$$

Proof. By Proposition 3.1, it suffices to consider the case where

$$n = n_{w_0} = \begin{pmatrix} & & -p^{-r} & \\ & & & -p^{r-s} \\ p^r & & & \\ & p^{s-r} & & \end{pmatrix}.$$

Let $x = unu' \in X(n)$. Write

$$u = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ & 1 & \alpha_4 & \alpha_5 \\ & & 1 & \\ & & -\alpha_1 & 1 \end{pmatrix} \in U(\mathbb{Z}_p)\backslash U(\mathbb{Q}_p), \quad u' = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \in U(\mathbb{Q}_p)/U(\mathbb{Z}_p).$$

Then

$$x=\begin{pmatrix}\alpha_2p^r & \alpha_2\beta_1p^r+\alpha_3p^{s-r} & \alpha_1\beta_1p^{r-s}+\alpha_2\beta_2p^r+\alpha_3\beta_4p^{s-r}-p^{-r} & \alpha_2\beta_3p^r+\alpha_3\beta_5p^{s-r}-\alpha_1p^{r-s}\\ \alpha_4p^r & \alpha_4\beta_1p^r+\alpha_5p^{s-r} & \alpha_4\beta_2p^r+\alpha_5\beta_4p^{s-r}+\beta_1p^{r-s} & \alpha_4\beta_3p^r+\alpha_5\beta_5p^{s-r}-p^{r-s}\\ p^r & \beta_1p^r & \beta_2p^r & \beta_3p^r\\ -\alpha_1p^r & -\alpha_1\beta_1p^r+p^{s-r} & -\alpha_1\beta_2p^r+\beta_4p^{s-r} & -\alpha_1\beta_3p^r+\beta_5p^{s-r}\end{pmatrix}\in G\left(\mathbb{Z}_p\right).$$

Now let

$$\tilde{u}=\begin{pmatrix}1 & \beta_1 & \beta_2 & -\beta_4\\ & 1 & -\beta_3 & \beta_5\\ & & 1 & \\ & & -\beta_1 & 1\end{pmatrix},\quad \tilde{u}'=\begin{pmatrix}1 & \alpha_1 & \alpha_2 & -\alpha_4\\ & 1 & -\alpha_3 & \alpha_5\\ & & 1 & \\ & & -\alpha_1 & 1\end{pmatrix}.$$

Then we see that

$$\begin{aligned}\tilde{x}&=\tilde{u}n\tilde{u}'\\ &=\begin{pmatrix}\beta_2p^r & \alpha_1\beta_2p^r-\beta_4p^{s-r} & \alpha_1\beta_1p^{r-s}+\alpha_2\beta_2p^r+\alpha_3\beta_4p^{s-r}-p^{-r} & -\alpha_4\beta_2p^r-\alpha_5\beta_4p^{s-r}-\beta_1p^{r-s}\\ -\beta_3p^r & -\alpha_1\beta_3p^r+\beta_5p^{s-r} & -\alpha_2\beta_3p^r-\alpha_3\beta_5p^{s-r}+\alpha_1p^{r-s} & \alpha_4\beta_3p^r+\alpha_5\beta_5p^{s-r}-p^{r-s}\\ p^r & \alpha_1p^r & \alpha_2p^r & -\alpha_4p^r\\ -\beta_1p^r & -\alpha_1\beta_1p^r+p^{s-r} & -\alpha_2\beta_1p^r-\alpha_3p^{s-r} & \alpha_4\beta_1p^r+\alpha_5p^{s-r}\end{pmatrix}\in G\left(\mathbb{Z}_p\right).\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Kl}_p\left(n,\psi,\psi'\right)&=\sum_{x\in X(n)}\psi\left(u(x)\right)\psi'\left(u'(x)\right)=\sum_{x\in X(n)}\psi\left(u'(\tilde{x})\right)\psi'\left(u(\tilde{x})\right)\\ &=\sum_{x\in X(n)}\psi'\left(u(x)\right)\psi\left(u'(x)\right)=\mathrm{Kl}_p\left(n,\psi',\psi\right).\end{aligned}$$ □

We give a few reduction formulae for Kloosterman sums $\mathrm{Kl}_p(n_{w,r,s},\psi,\psi')$, when one of r,s equals zero.

Proposition 3.3. Let $\psi=\psi_{m_1,m_2}$, $\psi'=\psi_{n_1,n_2}$. Then

$$\mathrm{Kl}_p\left(n_{w_0,r,0},\psi,\psi'\right)=S\left(m_1,n_1;p^r\right),\quad \mathrm{Kl}_p\left(n_{w_0,0,s},\psi,\psi'\right)=S\left(m_2,n_2;p^s\right).$$

Proof. For the first statement, we have by explicit construction

$$\mathrm{Kl}_p\left(n_{w_0,r,0},\psi,\psi'\right)=\sum_{\substack{v_2,v_3,v_4(\mathrm{mod}\ p^r)\\ v_{13},v_{14}(\mathrm{mod}\ 1)\\ v_{13}p^r+v_2v_{14}-v_4=0\\ (p^r,v_2,v_3,v_4)=1}}e\left(\frac{m_1\hat{v}_2+n_1v_2}{p^r}\right).$$

The condition $v_{13}p^r+v_2v_{14}-v_4=0$ reduces to $v_4=0$, and $v_{23}=-v_3p^{-r}$ being an integer implies $v_3\equiv 0 \pmod{p^r}$. Finally we solve $\hat{v}_2\equiv\overline{v_2} \pmod{p^r}$. Hence the sum reduces to

$$\mathrm{Kl}_p\left(n_{w_0,r,0},\psi,\psi'\right)=\sum_{\substack{v_2(\mathrm{mod}\ p^r)\\ (v_2,p^r)=1}}e\left(\frac{m_1\overline{v_2}+n_1v_2}{p^r}\right)=S\left(m_1,n_1;p^r\right).$$

For the second statement, we have

$$\mathrm{Kl}_p\left(n_{w_0,0,s},\psi,\psi'\right)=\sum_{\substack{v_2,v_3,v_4(\mathrm{mod}\ 1)\\ v_{13},v_{14}(\mathrm{mod}\ p^s)\\ v_{13}+v_2v_{14}-v_4p^s=0\\ (p^s,v_{13},v_{14},v_{23},v_{34})=1}}e\left(\frac{m_2\hat{v}_{14}+n_2v_{14}}{p^s}\right).$$

The condition $v_{13} + v_2 v_{14} - v_4 p^s = 0$ reduces to $v_{13} = 0$. We also have $v_{23} = v_{34} = 0$. Finally we solve $\hat{v}_{14} \equiv \overline{v_{14}} \pmod{p^s}$. Hence the sum reduces to

$$\mathrm{Kl}_p\left(n_{w_0,0,s}, \psi, \psi'\right) = \sum_{\substack{v_{14} (\mathrm{mod}\ p^s) \\ (v_{14}, p^s) = 1}} e\left(\frac{m_2 \overline{v_{14}} + n_2 v_{14}}{p^s}\right) = S\left(m_2, n_2; p^s\right). \qquad \square$$

Proposition 3.4. Let $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$. Then

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta s_\alpha, r, 0}, \psi, \psi'\right) = c_{p^r}\left(m_1\right), \quad \mathrm{Kl}_p\left(n_{s_\beta s_\alpha s_\beta, 0, s}, \psi, \psi'\right) = c_{p^s}\left(m_2\right).$$

Proof. For the first statement, we have by explicit construction

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta s_\alpha, r, 0}, \psi, \psi'|_{U_{s_\alpha s_\beta s_\alpha}}\right) = \sum_{\substack{v_2, v_3, v_4 (\mathrm{mod}\ p^r) \\ v_2 = v_2' p^r \\ (v_3, v_4, p^r) = 1 \\ \left(p^r, v_3 + v_2' v_4\right) = p^r}} e\left(\frac{m_1 \hat{v}_2 + n_1 v_2}{p^r}\right).$$

We may set $v_2 = 0$. The condition $(p^r, v_3 + v_2' v_4) = p^r$ implies $p^r \mid v_3$, so we may also let $v_3 = 0$. Then we solve $\hat{v}_2 \equiv \overline{v_4} \pmod{p^r}$. Hence the sum reduces to

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta s_\alpha, r, 0}, \psi, \psi'|_{U_{s_\alpha s_\beta s_\alpha}}\right) = \sum_{\substack{v_4 (\mathrm{mod}\ p^r) \\ (v_4, p) = 1}} e\left(\frac{m_1 \overline{v_4}}{p^r}\right) = c_{p^r}\left(m_1\right).$$

For the second statement, we have

$$\mathrm{Kl}_p\left(n_{s_\beta s_\alpha s_\beta, 0, s}, \psi, \psi'|_{U_{s_\beta s_\alpha s_\beta}}\right) = \sum_{\substack{v_{13}, v_{14}, v_{23} (\mathrm{mod}\ p^s) \\ (p^s, v_{13}, v_{14}) = p^s \\ (p^s, v_{14}) | v_{13}^2 \\ \left(p^s, v_{23}, -p^{-s}\left(v_{13}^2 + v_{14} v_{23}\right)\right) = 1}} e\left(\frac{m_2 \hat{v}_{14} + n_2 v_{14}}{p^s}\right).$$

We may set $v_{13} = v_{14} = 0$. Then we solve $\hat{v}_{14} \equiv -\overline{v_{23}} \pmod{p^s}$. Hence the sum reduces to

$$\mathrm{Kl}_p\left(n_{s_\beta s_\alpha s_\beta, 0, s}, \psi, \psi'|_{U_{s_\beta s_\alpha s_\beta}}\right) = \sum_{\substack{v_{23} (\mathrm{mod}\ p^s) \\ (v_{23}, p) = 1}} e\left(-\frac{m_2 \overline{v_{23}}}{p^s}\right) = c_{p^s}\left(m_2\right). \qquad \square$$

Proposition 3.5. Let $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$. Then

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta, r, 0}, \psi, \psi'|_{U_{s_\alpha s_\beta}}\right) = c_{p^r}\left(m_1\right), \quad \mathrm{Kl}_p\left(n_{s_\beta s_\alpha, 0, s}, \psi, \psi'|_{U_{s_\beta s_\alpha}}\right) = c_{p^s}\left(m_2\right).$$

Proof. Trivial. $\square$

3.2 Stratification of symplectic Kloosterman sums

In this section, we again consider symplectic Kloosterman sums on $G = \mathrm{Sp}(2r)$ in general, and develop a stratification of $\mathrm{Sp}(2r)$ Kloosterman sums $\mathrm{Kl}_p(n, \psi, \psi')$.

We first recall some facts about the Lie algebra of $G = \mathrm{Sp}(2r)$. Let T be the standard maximal torus of G, and let

$$t = (a_1, \cdots, a_r, a_1^{-1}, \cdots, a_r^{-1}) \in T.$$

A set of simple roots $\Delta = \Delta(T,G)$ of G is given by $\{\alpha_1, \cdots, \alpha_r\}$, where $\alpha_i t = a_t a_{t+1}^{-1}$ for $1 \leq i \leq r-1$, and $\alpha_r t = a_r^2$. The Weyl group $W = W(T,G)$ of G is generated by reflections s_{α_i} for $1 \leq i \leq r$, which are represented by matrices

$$s_{\alpha_i} = \begin{pmatrix} A_i & \\ & A_i \end{pmatrix}, \qquad A_i = \begin{pmatrix} I_{i-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{r-i-1} \end{pmatrix}, \qquad 1 \leq i \leq r-1,$$

and

$$s_{\alpha_r} = \begin{pmatrix} I_{r-1} & & & \\ & & & 1 \\ & & I_{r-1} & \\ & -1 & & \end{pmatrix}.$$

Consider the set of diagonal matrices

$$\mathcal{T} := \left\{ \begin{pmatrix} A & \\ & cA^{-1} \end{pmatrix} \in \mathrm{GL}(2r, \mathbb{Z}_p) \;\middle|\; A = \mathrm{diag}(a_1, \cdots, a_r),\ a_1, \cdots, a_r, c \in \mathbb{Z}_p^\times \right\}.$$

Note that elements of $\mathcal{T}$ are in general not symplectic. We have the following simple lemma.

Lemma 3.6. Let $n \in N(\mathbb{Q}_p)$, and $t \in \mathcal{T}$. Then $n^{-1}tn \in \mathcal{T}$.

Proof. We show that $w^{-1}tw \in \mathcal{T}$ for $w \in W$. It suffices to just check the generators s_{α_i} of the Weyl group. Suppose $t = \mathrm{diag}\left(a_1, \cdots, a_r, ca_1^{-1}, \cdots, ca_r^{-1}\right)$. Then we check

$$s_{\alpha_i}^{-1} t s_{\alpha_i} = \mathrm{diag}\left(a_1, \cdots, a_{i+1}, a_i, \cdots, a_r, ca_1^{-1}, \cdots, ca_{i+1}^{-1}, ca_i^{-1}, \cdots, ca_r^{-1}\right) \in \mathcal{T}, \quad 1 \leq i \leq r-1,$$

and

$$s_{\alpha_r}^{-1} t s_{\alpha_r} = \mathrm{diag}\left(a_1, \cdots, a_{r-1}, ca_r^{-1}, ca_1^{-1}, \cdots, ca_{r-1}^{-1}, a_r\right) \in \mathcal{T}.$$

Finally, for $n = wa \in N(\mathbb{Q}_p)$, with $w \in W$, $a \in T(\mathbb{Q}_p)$, we have

$$n^{-1}tn = a^{-1}w^{-1}twa = w^{-1}tw \in \mathcal{T}.$$

□

Let $x = unu' \in C(n)$, and $t \in \mathcal{T}$. By Lemma 3.6, $s := n^{-1}tn \in \mathcal{T}$. Hence

$$txs^{-1} = \left(tut^{-1}\right) n \left(su's^{-1}\right) \in U(\mathbb{Q}_p)\, nU(\mathbb{Q}_p) \cap G(\mathbb{Z}_p) = C(n).$$

As conjugation by t and s preserves $U(\mathbb{Z}_p)$ and $U_n(\mathbb{Z}_p)$, this induces an action on $X(n)$:

$$\mathcal{T} \times X(n) \to X(n), \quad (t, x) \mapsto t * x := txs^{-1}.$$

Let $n \in N(\mathbb{Q}_p)$, and ψ, ψ' characters of $U(\mathbb{Q}_p)$ trivial on $U(\mathbb{Z}_p)$. Partition of $X(n)$ into $\mathcal{T}$-orbits gives a decomposition of Kloosterman sums

$$\mathrm{Kl}_p(n, \psi, \psi') = \sum_{x \in \mathcal{T} \backslash X(n)} \sum_{y \in \mathcal{T} * x} \psi\left(u(y)\right) \psi'\left(u'(y)\right).$$

Characters of $U(\mathbb{Q}_p)$ trivial on $U(\mathbb{Z}_p)$ are of the form $\psi_{n_1, \cdots, n_r}$ with $n_i \in \mathbb{Z}$, $1 \leq i \leq r$, where

$$\psi_{n_1, \cdots, n_r} \begin{pmatrix} 1 & x_1 & \cdots & * & * & \cdots & \cdots & * \\ & 1 & \ddots & \vdots & \vdots & & & \vdots \\ & & \ddots & x_{r-1} & \vdots & & & * \\ & & & 1 & * & \cdots & * & x_r \\ & & & & 1 & & & \\ & & & & -x_1 & 1 & & \\ & & & & \vdots & \ddots & \ddots & \\ & & & & * & \cdots & -x_{r-1} & 1 \end{pmatrix} = \prod_{i=1}^{r} e\left(n_i x_i\right).$$

For $w \in W$, let $\Delta_w := \{\alpha \in \Delta \mid w(\alpha) < 0\}$. For $x \in X$, suppose

$$u(x) = \begin{pmatrix} 1 & x_1 & \cdots & * & * & \cdots & \cdots & * \\ & 1 & \ddots & \vdots & \vdots & & & \vdots \\ & & \ddots & x_{r-1} & \vdots & & & * \\ & & & 1 & * & \cdots & * & x_r \\ & & & & 1 & & & \\ & & & & -x_1 & 1 & & \\ & & & & \vdots & \ddots & \ddots & \\ & & & & * & \cdots & -x_{r-1} & 1 \end{pmatrix}, \quad u'(x) = \begin{pmatrix} 1 & x'_1 & \cdots & * & * & \cdots & \cdots & * \\ & 1 & \ddots & \vdots & \vdots & & & \vdots \\ & & \ddots & x'_{r_1} & \vdots & & & * \\ & & & 1 & * & \cdots & * & x'_r \\ & & & & 1 & & & \\ & & & & -x'_1 & 1 & & \\ & & & & \vdots & \ddots & \ddots & \\ & & & & * & \cdots & -x'_{r-1} & 1 \end{pmatrix}.$$

Note that $x'_i = 0$ unless $\alpha_i \in \Delta_w$. For $x = u(x)nu'(x)$, we define projections

$$\kappa_i(x) = x_i, \quad \kappa'_i(x) = x'_i, \quad 1 \le i \le r.$$

Let $t = \operatorname{diag}\left(a_1, \cdots, a_r, ca_1^{-1}, \cdots, ca_r^{-1}\right) \in \mathcal{T}$, and $s := n^{-1}tn = \operatorname{diag}(a'_1, \cdots, a'_r, c{a'_1}^{-1}, \cdots, c{a'_r}^{1}) \in \mathcal{T}$. Note from the proof of Lemma 3.6 that we have the same c. Then

$$tu(x)t^{-1} = \begin{pmatrix} 1 & a_1a_2^{-1}x_1 & \cdots & * & * & \cdots & \cdots & * \\ & 1 & \ddots & \vdots & \vdots & & & \vdots \\ & & \ddots & a_{r-1}a_r^{-1}x_{r_1} & \vdots & & & * \\ & & & 1 & * & \cdots & * & c^{-1}a_r^2x_r \\ & & & & 1 & & & \\ & & & & -a_1a_2^{-1}x_1 & 1 & & \\ & & & & \vdots & \ddots & \ddots & \\ & & & & * & \cdots & -a_{r-1}a_r^{-1}x_{r-1} & 1 \end{pmatrix},$$

and hence

$$\begin{aligned} \kappa_i\left(t * x\right) &= a_ia_{i+1}^{-1}\kappa_i(x), \quad 1 \le i \le r-1, \\ \kappa_r\left(t * x\right) &= c^{-1}a_r^2\kappa_r(x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \kappa'_i\left(t * x\right) &= a'_i{a'_{i+1}}^{-1}\kappa'_i(x), \quad 1 \le i \le r-1, \\ \kappa'_r\left(t * x\right) &= c^{-1}{a'_r}^2\kappa'_r(x). \end{aligned}$$

For $\ell \in \mathbb{N}$, we define

$$\begin{aligned} A_w(\ell) &:= \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^{\Delta} \times \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^{\Delta_w}, \\ V_w(\ell) &:= \left\{ \lambda \times \lambda' \in A_w(\ell) \;\middle|\; \begin{array}{l} \lambda_i, \lambda'_j \in \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^\times, \text{ such that } \exists t \in \mathcal{T} \text{ with} \\ \kappa_i(t * x) = \lambda_i\kappa_i(x), \kappa'_j(t * x) = \lambda'_j\kappa'_j(x) \\ \text{ for } x \in X(n),\ 1 \le i, j \le r,\ \alpha_j \in \Delta_w \end{array} \right\}. \end{aligned}$$

Note that $|V_w(\ell)| = \left(p^\ell\left(1 - p^{-1}\right)\right)^r$. For a character $\theta : A_w(\ell) \to \mathbb{C}^\times$, we define

$$S_w\left(\theta; \ell\right) := \sum_{v \in V_w(\ell)} \theta(v).$$

Now we state the main result of this section.

Theorem 3.7. Let $n \in N\left(\mathbb{Q}_p\right)$, and suppose ℓ is large enough such that the matrix entries of $u(x), u'(x)$ lie in $p^{-\ell}\mathbb{Z}_p/\mathbb{Z}_p$ for every $x \in X(n)$. Let $\psi = \psi_{n_1, \cdots, n_r}$ and $\psi' = \psi_{n'_1, \cdots, n'_r}$ be characters of $U(\mathbb{Q}_p)$ trivial on $U(\mathbb{Z}_p)$. Define a character $\theta_x : A_w(\ell) \to \mathbb{C}^\times$ by

$$\theta_x\left(\lambda \times \lambda'\right) = \prod_{i=1}^{r} e\left(\lambda_in_i\kappa_i(x)\right) \prod_{\substack{i=1 \\ w(\alpha_i)<0}}^{r} e\left(\lambda'_in'_i\kappa'_i(x)\right).$$

Then

$$\mathrm{Kl}_p\left(n,\psi,\psi'\right)=\left(p^{\ell}\left(1-p^{-1}\right)\right)^{-r}\sum_{x\in\mathcal{T}\backslash X(n)}\mathfrak{N}(x)S_w\left(\theta_x;\ell\right),$$

where $\mathfrak{N}(x):=|\mathcal{T}*x|$ is the size of $\mathcal{T}$-orbit of $x\in X(n)$.

Proof. Rewrite the Kloosterman sum

$$\begin{aligned}
\mathrm{Kl}_p\left(n,\psi,\psi'\right)&=\sum_{x\in\mathcal{T}\backslash X(n)}\sum_{y\in\mathcal{T}*x}\psi\left(u(y)\right)\psi'\left(u'(y)\right)\\
&=\sum_{x\in\mathcal{T}\backslash X(n)}\sum_{y\in\mathcal{T}*x}\prod_{i=1}^{r}e\left(n_i\kappa_i(y)\right)\prod_{\substack{i=1\\ w(\alpha_i)<0}}^{r}e\left(n_i'\kappa_i'(y)\right)\\
&=|V_w(\ell)|^{-1}\sum_{x\in\mathcal{T}\backslash X(n)}\sum_{y\in\mathcal{T}*x}\sum_{\lambda\times\lambda'\in V_w(\ell)}\prod_{i=1}^{r}e\left(\lambda_in_i\kappa_i(y)\right)\prod_{\substack{i=1\\ w(\alpha_i)<0}}^{r}e\left(\lambda_i'n_i'\kappa_i'(y)\right)\\
&=|V_w(\ell)|^{-1}\sum_{x\in\mathcal{T}\backslash X(n)}\mathfrak{N}(x)\sum_{\lambda\times\lambda'\in V_w(\ell)}\prod_{i=1}^{r}e\left(\lambda_in_i\kappa_i(y)\right)\prod_{\substack{i=1\\ w(\alpha_i)<0}}^{r}e\left(\lambda_i'n_i'\kappa_i'(y)\right)\\
&=\left(p^{\ell}\left(1-p^{-1}\right)\right)^{-r}\sum_{x\in\mathcal{T}\backslash X(n)}\mathfrak{N}(x)S_w\left(\theta_x;\ell\right).
\end{aligned}$$

□

3.3 Bounds for local Kloosterman sums

A trivial bound for Kloosterman sums is given by Dąbrowski and Reeder [DR98, Theorem 0.3], by counting the number of terms. In our context, the trivial bound says

$$\left|\mathrm{Kl}_p(n_{w,r,s},\psi,\psi')\right|\le p^{r+s}.$$

In this section, we establish non-trivial bounds for the Kloosterman sums $\mathrm{Kl}_p(n_{w,r,s},\psi,\psi')$, for fixed characters $\psi=\psi_{m_1,m_2}$, $\psi'=\psi_{n_1,n_2}$ of $U(\mathbb{Q}_p)/U(\mathbb{Z}_p)$.

We first recall from Section 3.1.1 that $\mathrm{Kl}_p(n_{\mathrm{id}},\psi,\psi')=1$ is trivial, and

$$\begin{aligned}
\mathrm{Kl}_p(n_{s_\alpha,r},\psi,\psi')&=S(m_1,n_1;p^r),\\
\mathrm{Kl}_p(n_{s_\beta,s},\psi,\psi')&=S(m_2,n_2;p^s)
\end{aligned}$$

are just classical Kloosterman sums. It is well-known that the classical Kloosterman sums are bounded by [Wei48, Smi80]

$$\left|S(m,n;p^k)\right|\le 2p^{k/2}(|m|_p^{-1},|n|_p^{-1},p^k)^{1/2},\tag{3.25}$$

where $|m|_p$ stands for the p-adic norm of m. We immediately obtain the bounds

$$\begin{aligned}
\left|\mathrm{Kl}_p\left(n_{s_\alpha,r},\psi,\psi'\right)\right|&\ll p^{r/2}(|m_1|_p^{-1},|n_1|_p^{-1},p^r)^{1/2},\\
\left|\mathrm{Kl}_p\left(n_{s_\beta,s},\psi,\psi'\right)\right|&\ll p^{s/2}(|m_2|_p^{-1},|n_2|_p^{-1},p^s)^{1/2}.
\end{aligned}$$

For Kloosterman sums $\mathrm{Kl}_p(n_{w,r,s},\psi,\psi')$ attached to other Weyl elements, we apply Theorem 3.7, and decompose the Kloosterman sum into sums of classical Kloosterman sums. Then one may apply the bound (3.25) for classical Kloosterman sums. However, applying the classical bound alone is in general insufficient to give a non-trivial bound for $\mathrm{Kl}_p(n_{w,r,s}, \psi, \psi')$. To obtain non-trivial bounds, we use two different approaches. Note that $\mathrm{Kl}_p(n_{w,r,s}, \psi, \psi')$ is in general an exponential sum of the form

$$\sum_{x \in S} e\left(\frac{f(x)}{p^k}\right)$$

for some $k \in \mathbb{N}$. The approach we use then depends on the value of k:

(i) when $k \geq 2$, we use the p-adic stationary phase method [DF97];

(ii) when $k = 1$, the stationary phase method fails, and we instead apply known results for exponential sums, which are derived using algebro-geometric arguments.

We now give an overview of the p-adic stationary phase method, following [DF97]. Let us first consider a simple case. Let f be a polynomial with coefficients in $\mathbb{Z}$. For $m \in \mathbb{N}$ we consider the exponential sum

$$S_m(f) := \sum_{x \in \mathbb{Z}/p^m\mathbb{Z}} e\left(\frac{f(x)}{p^m}\right).$$

Consider the Taylor expansion of f

$$f(x + p^{m-j}y) = f(x) + p^{m-j} f'(x) y + \frac{1}{2} p^{2(m-j)} f''(x) y^2 + \cdots .$$

If $2(m-j) \geq m$ (or $2(m-j) - 1 \geq m$ if $p = 2$), then we see that

$$\begin{aligned} S_m(f) &= p^{-j} \sum_{x \in \mathbb{Z}/p^m\mathbb{Z}} \sum_{y \in \mathbb{Z}/p^j\mathbb{Z}} e\left(\frac{f(x + p^{m-j}y)}{p^m}\right) \\ &= \sum_{x \in \mathbb{Z}/p^m\mathbb{Z}} e\left(\frac{f(x)}{p^m}\right) \cdot p^{-j} \sum_{y \in \mathbb{Z}/p^j\mathbb{Z}} e\left(\frac{f'(x)y}{p^j}\right). \end{aligned}$$

The inner sum vanishes unless $f'(x) \equiv 0 \pmod{p^j}$, hence the sum becomes

$$S_m(f) = \sum_{\substack{x \in \mathbb{Z}/p^m\mathbb{Z} \\ f'(x) \equiv 0 \pmod{p^j}}} e\left(\frac{f(x)}{p^m}\right).$$

This generalises easily to higher-dimensional cases. Let V be a smooth scheme of dimension n, and $f : V \to \mathbb{A}^1 = \mathbb{A}^1_{\mathbb{Z}_p}$ a $\mathbb{Z}_p$-morphism. We consider the exponential sum

$$S = S_m(f) := \sum_{x \in V(\mathbb{Z}/p^m\mathbb{Z})} e\left(\frac{f(x)}{p^m}\right). \tag{3.26}$$

Let $j \leq m$ be a positive integer. We write

$$D(\mathbb{Z}/p^j\mathbb{Z}) := \left\{ x \in V(\mathbb{Z}/p^j\mathbb{Z}) \;\middle|\; \nabla f(x) \equiv 0 \pmod{p^j} \right\} \tag{3.27}$$

to denote the "approximate critical points" of f. For $\overline{x} \in (\mathbb{Z}/p^j\mathbb{Z})^n$, we define

$$S_{\overline{x}} = \sum_{\substack{x \in V(\mathbb{Z}/p^m\mathbb{Z}) \\ x \equiv \overline{x} (\mathrm{mod}\ p^j)}} e\left(\frac{f(x)}{p^m}\right).$$

Clearly we have

$$S = \sum_{\overline{x} \in (\mathbb{Z}/p^j\mathbb{Z})^n} S_{\overline{x}}.$$

Theorem 3.8. [DF97, Theorem 1.8(a)] If $2j \le m$, then $S_{\overline{x}} = 0$ unless $\overline{x} \in D(\mathbb{Z}/p^j\mathbb{Z})$. Now suppose $m = 2j$ or $2j+1$, and let $x \in (\mathbb{Z}/p^m\mathbb{Z})^n$ map to $\overline{x} \in D(\mathbb{Z}/p^j\mathbb{Z})$. If $m = 2j$, then we have

$$S_{\overline{x}} = p^{mn/2} e\left(\frac{f(x)}{p^m}\right).$$

If $m = 2j+1$, then we have

$$S_{\overline{x}} = p^{(m-1)n/2} e\left(\frac{f(x)}{p^m}\right) \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^n} e\left(\frac{\frac{1}{2} y^T H_x y + p^{-j} \nabla f(x) \cdot y}{p}\right),$$

where H_x is the Hessian matrix of f at x. In particular, if we let t denote the maximum value of $n - \operatorname{rank}_{\mathbb{F}_p} H_{\overline{x}}$ for $\overline{x} \in D(\mathbb{Z}/p^j\mathbb{Z})$, then $|S| \le \left|D(\mathbb{Z}/p^j\mathbb{Z})\right| p^{(mn+t)/2}$.

Proof. We give a proof to the special case where $V = \mathbb{A}^n$ is the affine space. Then f is a polynomial with coefficients in $\mathbb{Z}_p$. The general case follows from a reduction lemma [DF97, Lemma 1.18], which reduces the general case into this special case.

Consider the Taylor expansion of f

$$f(x + p^{m-j} y) = f(x) + p^{m-j} \nabla f(x) \cdot y + \frac{1}{2} p^{2(m-j)} y^T H_x y + \cdots$$

Since $2j \le m$, we have

$$f(x + p^{m-j} y) = f(x) + p^{m-j} \nabla f(x) \cdot y \in \mathbb{Z}/p^m\mathbb{Z}.$$

This is obvious when p is odd, and when $p = 2$, the diagonal entries of the Hessian H_x are even, so the second-order term vanishes as well. Hence

$$\begin{aligned} S_{\overline{x}} &= p^{-nj} \sum_{\substack{x \in (\mathbb{Z}/p^m\mathbb{Z})^n \\ x \equiv \overline{x} (\text{mod } p^j)}} \sum_{y \in (\mathbb{Z}/p^j\mathbb{Z})^n} e\left(\frac{f(x + p^{(m-j)} y)}{p^m}\right) \\ &= \sum_{\substack{x \in (\mathbb{Z}/p^m\mathbb{Z})^n \\ x \equiv \overline{x} (\text{mod } p^j)}} e\left(\frac{f(x)}{p^m}\right) \cdot p^{-nj} \sum_{y \in (\mathbb{Z}/p^j\mathbb{Z})^n} e\left(\frac{\nabla f(x) \cdot y}{p^j}\right). \end{aligned}$$

The inner sum vanishes unless $\nabla f(x) \equiv 0 \pmod{p}^j$, that is, $\overline{x} \in D(\mathbb{Z}/p^j\mathbb{Z})$. Assuming this is the case, we continue

$$S_{\overline{x}} = \sum_{y \in (\mathbb{Z}/p^{m-j}\mathbb{Z})^n} e\left(\frac{f(x + p^j y)}{p^m}\right).$$

If $m = 2j$, then $f(x + p^j y) = f(x) + p^j \nabla f(x) \cdot y = f(x) \in \mathbb{Z}/p^m\mathbb{Z}$, so

$$S_{\overline{x}} = p^{mn/2} e\left(\frac{f(x)}{p^m}\right).$$

If $m = 2j+1$, then we have

$$f(x + p^j y) = f(x) + p^j \nabla f(x) \cdot y + \frac{1}{2} p^{2j} y^T H_x y \in \mathbb{Z}/p^m\mathbb{Z}.$$

Hence

$$\begin{aligned} S_{\overline{x}} &= e\left(\frac{f(x)}{p^m}\right) \sum_{y \in (\mathbb{Z}/p^{m-j}\mathbb{Z})^n} e\left(\frac{\frac{1}{2} p^{2j} y^T H_x y + p^j \nabla f(x) \cdot y}{p^m}\right) \\ &= p^{(m-1)n/2} e\left(\frac{f(x)}{p^m}\right) \sum_{y \in (\mathbb{Z}/p\mathbb{Z})^n} e\left(\frac{\frac{1}{2} y^T H_x y + p^{-j} \nabla f(x) \cdot y}{p}\right). \end{aligned}$$

Finally, we observe that the inner sum is an n-dimensional Gauß sum, and it follows from straightforward computations that the Gauß sum is bounded by $p^{n-\operatorname{rank}_{\mathbb{F}_p} H_x/2}$. The bound for S then follows. □

Theorem 3.9. Let $0 \leq s \leq r$ be integers, and $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$ characters of $U(\mathbb{Q}_p)/U(\mathbb{Z}_p)$. Then

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,r,s},\psi,\psi'\right)\right| \ll \min\left\{p^{2s}\left(|m_1|_p^{-1},p^{r-s}\right),p^r\left(|m_2|_p^{-1},p^s\right)^{1/2}\left(|n_2|_p^{-1},p^s\right)^{1/2}\right\}.$$

Proof. We may assume $v_p(m_1) \leq r-s$, and $v_p(m_2), v_p(n_2) \leq s$. Observe that

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,r,s},\psi_{m_1,m_2},\psi_{n_1,n_2}\right) = p^{k+2l}\,\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,r-k-l,s-l}\psi_{m_1p^{-k},m_2p^{-l}},\psi_{n_1,n_2p^{-l}}\right)$$

whenever $p^k \mid (m_1,p^{r-s})$ and $p^l \mid (m_2,n_2,p^s)$. So we may assume $s=0$, $r=s$, or $p \nmid m_1\,(m_2,n_2)$.

If $s=0$, then

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,r,0},\psi,\psi'\right)\right| = \left|\sum_{\substack{v_3(\mathrm{mod}\ p^r)\\(v_3,p^r)=1}} e\left(\frac{m_1\overline{v_3}}{p^r}\right)\right| \leq p^{v_p(m_1)}.$$

If $r=s$, then

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,r,r},\psi,\psi'\right)\right| = \left|\sum_{\substack{v_4(\mathrm{mod}\ p^r)\\(v_4,p^r)=1}}\ \sum_{v_3(\mathrm{mod}\ p^r)} e\left(\frac{m_2\overline{v_4}v_3^2+n_2v_4}{p^r}\right)\right| \leq p^{r+\frac{v_p(m_2)}{2}+\frac{v_p(n_2)}{2}}$$

is just a summation of quadratic Gauss sums, and is easily evaluated.

Now suppose $p \nmid m_1\,(m_2,n_2)$. If $p \mid m_2$ and $s>1$, then

$$\begin{aligned}\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,r,s},\psi,\psi'\right) &= \sum_{\substack{v_4'(\mathrm{mod}\ p^{s-1})\\(v_4',p)=1}}\ \sum_{\substack{v_3(\mathrm{mod}\ p^r)\\(v_3,p^{r-s})=1}}\ \sum_{k=0}^{p-1} e\left(\frac{m_1\overline{v_3}}{p^{r-s}}\right)e\left(\frac{m_2\overline{v_4'}v_3^2+n_2\left(v_4'+kp^{s-1}\right)}{p^s}\right)\\ &= p\sum_{k=0}^{p-1} e\left(\frac{n_2k}{p}\right)\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,r-1,s-1},\psi_{m_1,m_2/p},\psi'\right) = 0.\end{aligned}$$

If $p \mid m_2$ and $s=1$, the same argument shows that the sum is either 0 or p. Similarly, if $p \mid n_2$, the sum is also either 0 or p. So we may assume $p \nmid m_1m_2n_2$.

If $r>2s$, we write $r=2s+l$, for $l>0$. Then

$$\begin{aligned}&\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2s+l,s},\psi,\psi'\right)\\ &= \sum_{\substack{v_4(\mathrm{mod}\ p^s)\\(v_4,p)=1}}\ \sum_{\substack{v_3'=0\\(v_3',p)=1}}^{p^{s+l-1}-1}\ \sum_{k=0}^{p-1} e\left(\frac{m_1(\overline{v_3'}+\overline{k}p^{s+l-1})+p^lm_2\overline{v_4}(v_3'+kp^{s+l-1})^2+p^ln_2v_4}{p^{s+l}}\right),\end{aligned}$$

where $\overline{k}$ (mod p) is chosen such that $(v_3'+kp^{s+l-1})(\overline{v_3'}+\overline{k}p^{s+l-1}) \equiv 1 \pmod{p^{s+l}}$. Then the sum becomes

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2s+l,s},\psi,\psi'\right) = \sum_{\substack{v_4(\mathrm{mod}\ p^s)\\(v_4,p)=1}}\ \sum_{\substack{v_3'=0\\(v_3',p)=1}}^{p^{s+l-1}-1} e\left(\frac{m_1\overline{v_3'}+p^lm_2\overline{v_4}{v_3'}^2+p^ln_2v_4}{p^{s+l}}\right)\sum_{k=0}^{p-1} e\left(\frac{m_1\overline{k}}{p}\right) = 0.$$

If $r < 2s$, we write $r = 2s - l$, for $0 < l < s$. Then

$$\begin{aligned}\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2s-l,s},\psi,\psi'\right) &= \sum_{\substack{v_4 \pmod{p^s}\\(v_4,p)=1}} \sum_{\substack{v_3 \pmod{p^{2s-l}}\\(v_3,p)=1}} e\left(\frac{p^l m_1\overline{v_3} + m_2\overline{v_4}v_3^2 + n_2 v_4}{p^s}\right)\\ &= p^{s-l} \sum_{\substack{v_4 \pmod{p^s}\\(v_4,p)=1}} \sum_{\substack{v_3 \pmod{p^s}\\(v_3,p)=1}} e\left(\frac{p^l m_1\overline{v_3} + m_2\overline{v_4}v_3^2 + n_2 v_4}{p^s}\right).\end{aligned}$$

When p is odd, we apply the same argument and see that

$$\begin{aligned}\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2s-l,s},\psi,\psi'\right) &= p^{s-l} \sum_{\substack{v_4 \pmod{p^s}\\(v_4,p)=1}} \sum_{\substack{v_3'=0\\(v_3',p)=1}}^{p^{s-1}-1} \sum_{k=0}^{p-1} e\left(\frac{p^l m_1(\overline{v_3'} + \overline{k}p^{s-1}) + m_2\overline{v_4}(v_3' + kp^{s-1})^2 + n_2 v_4}{p^s}\right)\\ &= p^{s-l} \sum_{\substack{v_4 \pmod{p^s}\\(v_4,p)=1}} \sum_{\substack{v_3'=0\\(v_3',p)=1}}^{p^{s-1}-1} e\left(\frac{p^l m_1\overline{v_3'} + m_2\overline{v_4}{v_3'}^2 + n_2 v_4}{p^s}\right) \sum_{k=0}^{p-1} e\left(\frac{2m_2\overline{v_4}v_3'k}{p}\right) = 0.\end{aligned}$$

When $p = 2$, if we further assume $l \geq 2$, then we have

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2s-l,s},\psi,\psi'\right) = p^{s-l} \sum_{\substack{v_4 \pmod{p^s}\\(v_4,p)=1}} \sum_{\substack{v_3'=0\\(v_3',p)=1}}^{p^{s-2}-1} \sum_{k=0}^{p^2-1} e\left(\frac{p^l m_1(\overline{v_3'} + \overline{k}p^{s-2}) + m_2\overline{v_4}(v_3' + kp^{s-2})^2 + n_2 v_4}{p^s}\right),$$

where now $\overline{k} \pmod{p^2}$ is chosen such that $(v_3' + kp^{s+l-2})(\overline{v_3'} + \overline{k}p^{s+l-2}) \equiv 1 \pmod{p^{s+l}}$. Then the sum becomes

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2s-l,s},\psi,\psi'\right) = p^{s-l} \sum_{\substack{v_4 \pmod{p^s}\\(v_4,p)=1}} \sum_{\substack{v_3'=0\\(v_3',p)=1}}^{p^{s-2}-1} e\left(\frac{p^l m_1\overline{v_3'} + m_2\overline{v_4}{v_3'}^2 + n_2 v_4}{p^s}\right) \sum_{k=0}^{p^2-1} e\left(\frac{2m_2\overline{v_4}v_3'k}{p^2}\right) = 0.$$

Therefore, it remains to consider the case $r = 2s$, and, if $p = 2$, the case $r = 2s - 1$.

Now suppose $r = 2s$. When $s = 1$, we have

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2,1},\psi,\psi'\right) = p \sum_{\substack{v_4 \pmod{p}\\(v_4,p)=1}} \sum_{\substack{v_3 \pmod{p}\\(v_3,p)=1}} e\left(\frac{m_1\overline{v_3} + m_2\overline{v_4}v_3^2 + n_2 v_4}{p}\right).$$

When $p = 2$, there is nothing to prove. When p is odd, this exponential sum is estimated by Adolphson and Sperber [AS89, Corollary 4.3] to be of $O(p^2)$ as well. So we conclude that

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2,1},\psi,\psi'\right)\right| \ll p^2.$$

So the theorem holds for this case.

If $s > 1$, we apply the stationary phase method. Let $f(x,y) = \frac{m_1}{x} + \frac{m_2 x^2}{y} + n_2 y$. Consider the sum

$$S = \sum_{x,y \in (\mathbb{Z}/p^s\mathbb{Z})^\times} e\left(\frac{f(x,y)}{p^s}\right) = p^{-s}\,\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2s,s},\psi,\psi'\right).$$

Let $j \geq 1$ be such that $2j \leq s$. Define as in (3.27)

$$\begin{aligned}D\left(\mathbb{Z}/p^j\mathbb{Z}\right) &= \left\{(x,y) \in \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \times \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \;\middle|\; \nabla f(x,y) \equiv 0 \pmod{p^j}\right\}\\ &= \left\{(x,y) \in \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \times \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \;\middle|\; \begin{array}{l} 2m_2x^3 \equiv m_1 y \pmod{p^j},\\ m_2x^2 \equiv n_2y^2 \pmod{p^j}\end{array}\right\}.\end{aligned}$$

It is straightforward to check that $\left|D\left(\mathbb{Z}/p^j\mathbb{Z}\right)\right| \leq 4$, and $H_{x,y}$ is invertible over $\mathbb{F}_p$ for all $(x,y) \in D\left(\mathbb{Z}/p^j\mathbb{Z}\right)$, so $\operatorname{rank}_{\mathbb{F}_p} H_{x,y} = 2$. So we deduce from Theorem 3.8 that

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,r,s},\psi,\psi'\right)\right| \leq 4p^{2s}.$$

Now suppose $p=2$, and $r = 2s-1$. It suffices to prove the bound for sufficiently large s, so we can always use stationary phase method. Let $f(x,y) = \frac{2m_1}{x} + \frac{m_2x^2}{y} + n_2y$. Consider the sum

$$S = \sum_{x,y\in(\mathbb{Z}/p^s\mathbb{Z})^\times} e\left(\frac{f(x,y)}{p^s}\right) = p^{-s+1}\,\mathrm{Kl}_p(n_{s_\alpha s_\beta,2s-1,s},\psi,\psi').$$

Let $j \geq 1$ be such that $2j \leq s$. Define as in (3.27)

$$\begin{aligned} D\left(\mathbb{Z}/p^j\mathbb{Z}\right) &= \left\{(x,y)\in\left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times\times\left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \,\middle|\, \nabla f(x,y)\equiv 0 \pmod{p^j}\right\} \\ &= \left\{(x,y)\in\left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times\times\left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \,\middle|\, \begin{array}{l} 2m_2x^3\equiv 2m_1y \pmod{p^j}, \\ m_2x^2\equiv n_2y^2 \pmod{p^j} \end{array}\right\}. \end{aligned}$$

Then we have $\left|D\left(\mathbb{Z}/p^j\mathbb{Z}\right)\right| \leq 16$. The Hessian $H_{x,y}$ is not invertible, but nevertheless we have from Theorem 3.8 that

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta,2s-1,s},\psi,\psi'\right)\right| \leq 64p^{2s-1}.$$

This finishes the proof of the theorem. □

Theorem 3.10. Let $0 \leq 2r \leq s$ be integers, and $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$ characters of $U(\mathbb{Q}_p)/U(\mathbb{Z}_p)$. Then

$$\left|\mathrm{Kl}_p\left(n_{s_\beta s_\alpha,r,s},\psi,\psi'\right)\right| \ll \min\left\{p^{3r}\left(|m_2|_p^{-1},p^{s-2r}\right), p^s\left(|m_1|_p^{-1},|n_1|_p^{-1},p^r\right)\right\}.$$

Remark. Up to multiplication by a constant, this Kloosterman sum can also be considered as a GL(3) Kloosterman sum. Precisely, following the notation in [BFG88, (4.3)], we have

$$\mathrm{Kl}_p\left(n_{s_\beta s_\alpha,r,s},\psi,\psi'\right) = p^r S\left(n_1,m_1,m_2;p^r,p^{s-r}\right).$$

A non-trivial bound for $\mathrm{Kl}_p\left(n_{s_\beta s_\alpha,r,s},\psi,\psi'\right)$ then follows from Larsen [BFG88, Appendix]. For sake of completeness, we still give a proof below.

Proof. We may assume that $v_p(m_2) \leq s-2r$, and $v_p(m_1), v_p(n_1) \leq r$. Observe that

$$\mathrm{Kl}_p\left(n_{s_\beta s_\alpha,r,s},\psi_{m_1,m_2},\psi_{n_1,n_2}\right) = p^{3k+l}\,\mathrm{Kl}_p\left(n_{s_\beta s_\alpha,r-k,s-2k-l},\psi_{m_1p^{-k},m_2p^{-l}},\psi_{n_1p^{-k},n_2}\right)$$

whenever $p^k \mid (m_1,n_1,p^r)$ and $p^l \mid (m_2,p^{s-2r})$. So we may assume $r=0$, $s=2r$, or $p \nmid m_2(m_1,n_1)$.

If $r=0$, then

$$\left|\mathrm{Kl}_p\left(n_{s_\beta s_\alpha,0,s},\psi,\psi'\right)\right| = \left|\sum_{\substack{v_{34} \,(\mathrm{mod}\ p^s)\\ (v_{34},p^s)=1}} e\left(\frac{m_2\overline{v_{34}}}{p^s}\right)\right| \leq p^{v_p(m_2)}.$$

If $s=2r$, then

$$\left|\mathrm{Kl}_p\left(n_{s_\beta s_\alpha,r,2r},\psi,\psi'\right)\right| = \left|\sum_{\substack{v_{24} \,(\mathrm{mod}\ p^r)\\ (v_{24},p^r)=1}}\sum_{v_{34}\,(\mathrm{mod}\ p^{2r})} e\left(\frac{m_1\overline{v_{24}}v_{34}+n_1v_{24}}{p^r}\right)\right| \leq p^{2r+\min\{v_p(m_1),v_p(n_1)\}}.$$

Now suppose $p \nmid m_2(m_1, n_1)$. If $p \mid m_1$ and $r > 1$, then

$$\begin{aligned}\mathrm{Kl}_p(n_{s_\beta s_\alpha, r, s}, \psi, \psi') &= \sum_{\substack{v'_{24} (\mathrm{mod}\ p^{r-1}) \\ (v'_{24}, p)=1}} \sum_{\substack{v_{34} (\mathrm{mod}\ p^{s}) \\ (v_{34}, p^{s-2r})=1}} \sum_{k=0}^{p-1} e\left(\frac{m_1 \overline{v'_{24}} v_{34} + n_1 (v'_{24} + kp^{r-1})}{p^r} \right) e\left(\frac{m_2 \overline{v_{34}}}{p^{s-2r}} \right) \\ &= p^2 \sum_{k=0}^{p-1} e\left(\frac{n_1 k}{p} \right) \mathrm{Kl}_p\left(n_{s_\beta s_\alpha, r-1, s-2}, \psi_{m_1/p, m_2}, \psi' \right) = 0.\end{aligned}$$

If $p \mid m_1$ and $r = 1$, the same argument shows that the sum is either 0 or p. Similarly, if $p \mid n_1$, the sum is also either 0 or p. So we may assume $p \nmid m_1 m_2 n_1$.

If $s > 3r$, we write $s = 3r + l$, for $l > 0$. Then

$$\begin{aligned}&\mathrm{Kl}_p\left(n_{s_\beta s_\alpha, r, 3r+l}, \psi, \psi' \right) \\ &= p^{2r} \sum_{\substack{v_{24} (\mathrm{mod}\ p^{r}) \\ (v_{24}, p)=1}} \sum_{\substack{v'_{34}=0 \\ (v'_{34}, p)=1}}^{p^{r+l-1}-1} \sum_{k=0}^{p-1} e\left(\frac{p^l m_1 \overline{v_{24}} (v'_{34} + kp^{r+l-1}) + p^l n_1 v_{24} + m_2 (\overline{v'_{34}} + \overline{k} p^{r+l-1})}{p^{r+l}} \right),\end{aligned}$$

where $\overline{k} \pmod p$ is chosen such that $(v'_{34} + kp^{r+l-1})(\overline{v'_{34}} + \overline{k} p^{r+l-1}) \equiv 1 \pmod{p^{r+l}}$. Then the sum becomes

$$\mathrm{Kl}_p\left(n_{s_\beta s_\alpha, r, 3r+l}, \psi, \psi' \right) = p^{2r} \sum_{\substack{v_{24} (\mathrm{mod}\ p^{r}) \\ (v_{24}, p)=1}} \sum_{\substack{v'_{34}=0 \\ (v'_{34}, p)=1}}^{p^{r+l-1}-1} e\left(\frac{p^l m_1 \overline{v_{24}} v'_{34} + p^l n_1 v_{24} + m_2 \overline{v'_{34}}}{p^{r+l}} \right) \sum_{k=0}^{p-1} \sum_{k=0}^{p-1} e\left(\frac{m_2 \overline{k}}{p} \right) = 0.$$

If $s < 3r$, we write $s = 3r - l$, for $0 < l < r$. We apply the same argument, and obtain

$$\begin{aligned}\mathrm{Kl}_p\left(n_{s_\beta s_\alpha, r, 3r-l}, \psi, \psi' \right) &= p^{2r-l} \sum_{\substack{v_{24} (\mathrm{mod}\ p^{r}) \\ (v_{24}, p)=1}} \sum_{\substack{v'_{34}=0 \\ (v'_{34}, p)=1}}^{p^{r-1}-1} \sum_{k=0}^{p-1} e\left(\frac{m_1 \overline{v_{24}} (v'_{34} + kp^{r-1}) + n_1 v_{24} + p^l m_2 (\overline{v'_{34}} + kp^{r-1})}{p^r} \right) \\ &= p^{2r-l} \sum_{\substack{v_{24} (\mathrm{mod}\ p^{r}) \\ (v_{24}, p)=1}} \sum_{\substack{v'_{34}=0 \\ (v'_{34}, p)=1}}^{p^{r-1}-1} e\left(\frac{m_1 \overline{v_{24}} v'_{34} + n_1 v_{24} + p^l m_2 \overline{v'_{34}}}{p^r} \right) \sum_{k=0}^{p-1} e\left(\frac{m_1 \overline{v_{24}} k}{p} \right) = 0.\end{aligned}$$

So it remains to consider the case $s = 3r$. When $r = 1$, we have

$$\mathrm{Kl}_p\left(n_{s_\beta s_\alpha, 1, 3}, \psi, \psi' \right) = p^2 \sum_{\substack{v_{24} (\mathrm{mod}\ p) \\ (v_{24}, p)=1}} \sum_{\substack{v_{34} (\mathrm{mod}\ p) \\ (v_{34}, p)=1}} e\left(\frac{m_1 \overline{v_{24}} v_{34} + n_1 v_{24} + m_2 \overline{v_{34}}}{p} \right).$$

Let $x = m_1 \overline{v_{24}} v_{34}$, $y = n_1 v_{24}$, and $z = m_2 \overline{v_{34}}$. After this change of variables, the sum becomes

$$p^2 \sum_{\substack{x, y, z \in \mathbb{F}_p \\ xyz = m_2 m_1 n_1}} e\left(\frac{x + y + z}{p} \right),$$

which is known as a generalised Kloosterman sum in the sense of Deligne [Del77]. By a theorem of Deligne [Del77, Sommes. trig., 7.1.3], this sum is bounded by $3p^3$. So the theorem holds for this case.

For $r > 1$, we apply the stationary phase method. Let $f(x, y) = \frac{m_1 y}{x} + n_1 x + \frac{m_2}{y}$. Consider the sum

$$S = \sum_{x, y \in (\mathbb{Z}/p^r\mathbb{Z})^\times} e\left(\frac{f(x, y)}{p^r} \right) = p^{-2r} \mathrm{Kl}_p\left(n_{s_\beta s_\alpha, r, 3r}, \psi, \psi' \right).$$

Let $j \geq 1$ be such that $2j \leq r$. Define as in (3.27)

$$D\left(\mathbb{Z}/p^j\mathbb{Z}\right) = \left\{(x,y) \in \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \times \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \;\middle|\; \nabla f(x,y) \equiv 0 \pmod{p^j}\right\}$$
$$= \left\{(x,y) \in \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \times \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \;\middle|\; \begin{array}{l} m_1 y \equiv n_1 x^2 \pmod{p^j}, \\ m_1 y^2 \equiv m_2 x \pmod{p^j} \end{array}\right\}.$$

We have $\left|D(\mathbb{Z}/p^j\mathbb{Z})\right| \leq 3$. The Hessian $H_{x,y}$ is invertible unless $p = 3$. So we conclude from Theorem 3.8 that

$$\left|\mathrm{Kl}_p(n_{s_\beta s_\alpha, r, 3r}, \psi, \psi')\right| \ll p^{3r}.$$

This finishes the proof of the theorem. □

Theorem 3.11. Let $0 \leq s \leq 2r$ be integers, and $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$ characters of $U(\mathbb{Q}_p)/U(\mathbb{Z}_p)$. Then

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta s_\alpha, r, s}, \psi, \psi'\right)\right| \ll \begin{cases} p^{\frac{r}{3} + \frac{2s}{3} + \frac{2}{3}\min\{v_p(m_1)+s, v_p(n_1)+r\} + \frac{1}{3}v_p(m_2)} & \text{if } s \leq r, \\ p^{r+\min\{v_p(m_2), r+v_p(n_1)\}} + p^{r+\min\left\{\frac{s}{2}+v_p(m_1), r-\frac{s}{2}+v_p(n_1)\right\}} & \text{if } r < s < 2r, \\ p^{r+\min\{v_p(m_2), r+v_p(n_1)\}} & \text{if } s = 2r. \end{cases}$$

Proof. We make use of the stratification of Kloosterman sums in Section 3.2. For $w = s_\alpha s_\beta s_\alpha$, we have $\Delta_w = \{\alpha\}$. Hence, for $\ell \in \mathbb{N}$, we have

$$A_w(\ell) = (\mathbb{Z}/p^\ell\mathbb{Z})^2 \times (\mathbb{Z}/p^\ell\mathbb{Z}).$$

Let $t = \operatorname{diag}\left(a_1, a_2, ca_1^{-1}, ca_2^{-1}\right) \in \mathcal{T}$. Then $s := n^{-1}tn = \operatorname{diag}\left(ca_1^{-1}, a_2, a_1, ca_2^{-1}\right)$. We compute

$$\kappa_1'(t * x) = ca_1^{-1}a_2^{-1}\kappa_1'(x).$$

So

$$V_w(\ell) = \left\{\lambda \times \lambda' \in A_w(\ell) \;\middle|\; \begin{array}{l} \lambda_1, \lambda_2, \lambda_1' \in (\mathbb{Z}/p^\ell\mathbb{Z})^\times, \\ \lambda_1\lambda_2\lambda_1' = 1 \end{array}\right\}.$$

Let $\theta : A_w(\ell) \to \mathbb{C}^\times$ be a character given by

$$\theta(\lambda \times \lambda') = e\left(\frac{n_1\lambda_1 + n_2\lambda_2}{p^\ell}\right) e\left(\frac{n_1'\lambda_1'}{p^\ell}\right)$$

for $n_1, n_2, n_1' \in \mathbb{Z}$, then

$$S_w(\theta, \ell) = \sum_{\lambda_2 \in (\mathbb{Z}/p^\ell\mathbb{Z})^\times} e\left(\frac{n_2\lambda_2}{p^\ell}\right) S(n_1\overline{\lambda}_2, n_1'; p^\ell). \tag{3.28}$$

Let $n = n_{s_\alpha s_\beta s_\alpha, r, s}$. In terms of Plücker coordinates (see Section 2.2.4), this says $v_1 = p^r$ and $v_{14} = p^s$. Suppose $x_{a,b}^{v_3} \in X(n)$ has coordinates

$$(v_1, v_2, v_3, v_4; v_{14}) = (p^r, p^{r-a}, v_3, p^{r-b}; p^s).$$

Let $\delta' = (p^{r-a}, p^a v_3 + p^{r-b})$. Then $v_{14} = p^{r+a}/\delta'$. This says $s - r \leq a \leq s/2$, $b \leq r$, and $\delta' = p^{r+a-s}$. From Bruhat decomposition, we have

$$u'\left(x_{a,b}^{v_3}\right) = \begin{pmatrix} 1 & p^{-a} & v_3p^{-r} & p^{-b} \\ & 1 & p^{-b} & \\ & & 1 & \\ & & -p^{-a} & 1 \end{pmatrix} \pmod{U(\mathbb{Z}_p)}.$$

Let $X^{v_3}_{a,b}(n) = \mathcal{T} * x^{v_3}_{a,b}$, and define

$$S^{v_3}_{a,b}\left(n, \psi, \psi'\right) = \sum_{x \in X^{v_3}_{a,b}(n)} \psi\left(u(x)\right) \psi'\left(u'(x)\right).$$

We also set

$$X_{a,b}(n) = \bigcup_{\substack{v_3 \pmod{p^r} \\ \left(p^{r-a}, p^a v_3 + p^{r-b}\right) = p^{r+a-s}}} X^{v_3}_{a,b}(n),$$

and

$$S_{a,b}\left(n, \psi, \psi'\right) = \sum_{x \in X_{a,b}(n)} \psi\left(u(x)\right) \psi'\left(u'(x)\right).$$

It is easy to see that

$$X\left(n\right) = \coprod_{\substack{s-r \le a \le s/2 \\ 0 \le b \le r}} X_{a,b}(n).$$

As $r \ge s/2 \ge a$, $r \ge b$, we see that $u(x), u'(x)$ have entries in $p^{-2r}\mathbb{Z}_p/\mathbb{Z}_p$ for all $x \in X(n)$. Let $\mathcal{S}_{a,b}$ be a finite subset of $\mathbb{Z}_p$ such that

$$X_{a,b}(n) = \coprod_{v_3 \in \mathcal{S}_{a,b}} X^{v_3}_{a,b}(n).$$

By Theorem 3.7, we have

$$S_{a,b}\left(n, \psi, \psi'\right) = p^{-4r}\left(1 - p^{-1}\right)^{-2} \sum_{v_3 \in \mathcal{S}_{a,b}} \left|X^{v_3}_{a,b}(n)\right| S_w\left(\theta^{v_3}_{a,b}; 2r\right),$$

where

$$\theta^{v_3}_{a,b}\left(\lambda \times \lambda'\right) = e\left(\frac{m_2 u \lambda_2}{p^s}\right) e\left(\frac{m_1 \hat{v}_2 \lambda_1 + n_1 p^{r-a} \lambda'_1}{p^r}\right),$$

with $\hat{v}_2$ and u given as in (3.6) and (3.7). By (3.28), we have

$$S_w\left(\theta^{v_3}_{a,b}; 2r\right) = \sum_{x,y \in (\mathbb{Z}/p^{2r}\mathbb{Z})^\times} e\left(\frac{m_2 u x}{p^s}\right) e\left(\frac{m_1 \hat{v}_2 \overline{x} y + n_1 p^{r-a} \overline{y}}{p^r}\right), \tag{3.29}$$

and we easily deduce that

$$\sum_{v_3 \in \mathcal{S}_{a,b}} \left|X^{v_3}_{a,b}(n)\right| \le \left|\mathcal{S}_{a,b}\right| p^{a+b} \le p^{r+a+b}. \tag{3.30}$$

We estimate the size of $S_w\left(\theta^{v_3}_{a,b}; 2r\right)$ below. We start by computing $v_p(\hat{v}_2)$ and $v_p(u)$. From (3.6), it is clear that $v_p(\hat{v}_2) = s - a$. Now we consider $v_p(u)$. If $a \ne s/2$, then we have (after putting $v'_2 = \overline{v'_2} = 1$)

$$\begin{aligned}
u &= p^{a+r-s}\left(-p^a v_3 + v_4\right) + \overline{V'} v_3^2 p^{2a} \\
&= p^{a+r-s}\left(p^a v_3 + v_4\right) - 2 v_3 p^{2a+r-s} + \overline{V'} v_3^2 p^{2a} \\
&= p^{2a+2r-2s} V' - 2 v_3 p^{2a+r-s} + \overline{V'} v_3^2 p^{2a} \\
&= p^{2a}\overline{V'}\left(p^{2r-2s} V'^2 - 2p^{r-s} v_3 V' + v_3^2\right) \\
&= p^{2a}\overline{V'}\left(p^{r-s} V' - v_3\right)^2 \\
&= p^{2a}\overline{V'}\left(p^{-a} v_4\right)^2 \\
&= v_4^2 \overline{V'}.
\end{aligned}$$

So $v_p(u) = 2(r-b)$. If $a = s/2$, then (again we set $v_2' = \overline{v_2'} = 1$)

$$u = -v_3 p^{2a+r-s} + v_4 p^{a+r-s} = p^{a+r-s}\left(2v_4 - (p^a v_3 + v_4)\right). \tag{3.31}$$

These expressions will be useful in computing $v_p(u)$, when more conditions are given.

Case I: Suppose $s < r$. We deduce from (3.6) that $v_p(v_3) = 0, v_p(v_4) = a$, so only terms with $r = a + b$ contribute. When $a \neq s/2$, we have $v_p(u) = 2(r-b) = 2a$. When $a = s/2$, we can still take $v_p(u) = s = 2a$. So $v_p(u) = 2a$ always holds.

(i) Suppose $a \leq \frac{2s-r}{3}$. Write $u = p^{2a}u'$. Let

$$t = \min\left\{v_p(m_2), v_p(m_1) + 2s - r - 3a, v_p(n_1) + s - 3a\right\},$$

and

$$f(x,y) = p^{-t}\left(m_2 u' y + \frac{m_1 \hat{v}_2 p^{s-r-2a} x}{y} + \frac{n_1 p^{s-3a}}{x}\right) = m_2' y + \frac{m_1' x}{y} + \frac{n_1'}{x},$$

where $m_1' = m_1 \hat{v}_2 p^{s-r-2a-t}$, $m_2' = m_2 u' p^{-t}$, $n_1' = n_1 p^{s-3a-t}$. Consider the sum

$$S = \sum_{x,y \in (\mathbb{Z}/p^{s-2a-t}\mathbb{Z})^\times} e\left(\frac{f(x,y)}{p^{s-2a-t}}\right) = p^{2s-4a-4r-2t} S_w\left(\theta_{a,b}^{v_3}; 2r\right).$$

When $s - 2a - t > 1$, let $j \geq 1$ be such that $2j \leq s - 2a - t$. Define as in (3.27)

$$\begin{aligned} D\left(\mathbb{Z}/p^j\mathbb{Z}\right) &= \left\{(x,y) \in \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \times \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \;\middle|\; \nabla f(x,y) \equiv 0 \pmod{p^j}\right\} \\ &= \left\{(x,y) \in \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \times \left(\mathbb{Z}/p^j\mathbb{Z}\right)^\times \;\middle|\; \begin{array}{l} m_1' x^2 \equiv n_1' y \pmod{p^j}, \\ m_2' y^2 \equiv m_1' x \pmod{p^j} \end{array}\right\}. \end{aligned}$$

Note that at least one of m_1', m_2' and n_1' is not divisible by p. It then follows that $D\left(\mathbb{Z}/p^j\mathbb{Z}\right)$ is empty unless $v_p(m_2) = v_p(m_1) + 2s - r - 3a = v_p(n_1) + s - 3a$. Then this reduces to the situation seen in the proof of Theorem 3.10, and we obtain a bound

$$\left|S_w\left(\theta_{a,b}^{v_3}; 2r\right)\right| \ll p^{4r+2a-s+t}. \tag{3.32}$$

Now suppose $s - 2a - t = 1$. If $p \nmid m_1' m_2' n_1'$, then it again follows by the theorem of Deligne [Del77, Sommes. trig., 7.1.3] that $S \ll p$. When p divides some (but not all) of m_1', m_2', n_1', then the sum reduces to a Ramanujan sum, and is easily evaluated that $S \ll p$ as well. So the bound (3.32) also holds for this case.

The bounds for $S_w\left(\theta_{a,b}^{v_3}; 2r\right)$ in other cases are obtained analogously, and we shall omit the repetitive computations thereafter.

(ii) Suppose $a > \frac{2s-r}{3}$. Write $\hat{v}_2 = p^{s-a}\hat{v}_2'$. Let

$$t = \min\left\{v_p(m_2) + r + 3a - 2s, v_p(m_1), v_p(n_1) + r - s\right\},$$

and

$$f(x,y) = p^{-t}\left(m_2 u p^{r+a-2s} y + \frac{m_1 \hat{v}_2' x}{y} + \frac{n_1 p^{r-s}}{x}\right) = m_2' y + \frac{m_1' x}{y} + \frac{n_1'}{x},$$

where $m_1' \hat{v}_2' p^{-t}$, $m_2' = m_2 u p^{r+a-2s-t}$, $n_1' = n_1 p^{r-s-t}$. Then we have

$$S = \sum_{x,y \in (\mathbb{Z}/p^{r+a-s-t}\mathbb{Z})^\times} e\left(\frac{f(x,y)}{p^{r+a-s-t}}\right) = p^{2a-2r-2s-2t} S_w\left(\theta_{a,b}^{v_3}; 2r\right).$$

Then we obtain analogously

$$\left|S_w\left(\theta_{a,b}^{v_3}; 2r\right)\right| \ll p^{3r-a+s+t}.$$

Recall that we have $\delta' = (p^{r-a}, p^a(v_3+1)) = p^{r+a-s}$. A necessary condition for this to hold is that $p^{r-s} \mid v_3 + 1$. So $|\mathcal{S}_{a,b}| \le p^s$. So, from (3.30) we actually have

$$\sum_{v_3 \in \mathcal{S}_{a,b}} \left| X_{a,b}^{v_3}(n) \right| \le p^{s+a+b}.$$

Hence

$$\begin{aligned}
\left|\mathrm{Kl}_p(n,\psi,\psi')\right| &\le \sum_{\substack{0\le a\le s/2\\ b=r-a}} \left|S_{a,b}(n,\psi,\psi')\right| \\
&\ll \sum_{\substack{0\le a\le s/2\\ b=r-a}} p^{-4r} p^{s+a+b} S_w\left(\theta_{a,b}^{v_3}; 2r\right) \\
&\ll \sum_{0\le a\le s/2} \min\left\{p^{r+2a+v_p(m_2)}, p^{s-a+\min\{s+v_p(m_1), r+v_p(n_1)\}}\right\} \\
&\ll p^{\frac{r}{3}+\frac{2s}{3}+\frac{2}{3}\min\{v_p(m_1)+s, v_p(n_1)+r\}+\frac{1}{3}v_p(m_2)}.
\end{aligned}$$

Case II: Suppose $s = r$. We deduce from (3.6) that when $a \ne 0$, then $v_p(v_3) = 0, v_p(v_4) \ge a$. So, only terms with $r \ge a+b$ contribute. When $a \ne s/2$, we have $v_p(u) = 2(r-b)$. When $a = s/2$, we can still take $v_p(u) = s = 2(r-b)$. So $v_p(u) = 2(r-b)$ always holds. We compute

$$\left|S_w\left(\theta_{a,b}^{v_3}; 2r\right)\right| \ll p^{2r} \min\left\{p^{3r-2b+v_p(m_2)}, p^{2r-a+\min\{v_p(m_1), v_p(n_1)\}}\right\}.$$

Hence

$$\begin{aligned}
\left|\mathrm{Kl}_p(n,\psi,\psi')\right| &\le \sum_{\substack{0\le a\le r/2\\ b\le r-a}} \left|S_{a,b}(n,\psi,\psi')\right| \\
&\ll \sum_{\substack{0\le a\le s/2\\ b\le r-a}} p^{-4r} p^{r+a+b} \left(p^{2r} \min\left\{p^{3r-2b+v_p(m_2)}, p^{2r-a+\min\{v_p(m_1), v_p(n_1)\}}\right\}\right) \\
&\ll \sum_{\substack{0\le a\le s/2\\ b\le r-a}} p^{-r+a+b} \min\left\{p^{3r-2b+v_p(m_2)}, p^{2r-a+\min\{v_p(m_1), v_p(n_1)\}}\right\} \\
&\ll p^{\frac{5r}{3}+\frac{2}{3}\min\{v_p(m_1), v_p(n_1)\}+\frac{1}{3}v_p(m_2)}.
\end{aligned}$$

Case III: $2r > s > r$. We consider the following subcases:

(a) Suppose $a = s-r$. Then the condition $\left(p^{r-a}, p^a v_3 + p^{r-b}\right) = 1$ implies $b = r$. So $v_p(u) = 0$. We deduce from (3.6) that $\hat{v}_2 = 0$. So

$$\left|S_w\left(\theta_{a,b}^{v_3}; 2r\right)\right| \ll p^{3r-s} \min\left\{p^{r+v_p(m_2)}, p^{2r+v_p(n_1)}\right\}.$$

(b) Suppose $s - r < a < s/2$. Then we deduce from (3.6) that $v_p(v_3) = 0$, $v_p(v_4) \ge a$. So $a + b \le r$. Meanwhile, as $r + a - s < a$, the condition $\left(p^{r-a}, p^a v_3 + p^{r-b}\right) = p^{r+a-s}$ says $r - b = r + a - s$, which implies $a + b = s > r$, a contradiction. So there is no contribution from this case.

(c) Suppose $a = s/2$. Again, we deduce from (3.6) that $v_p(v_3) = 0$, $v_p(v_4) \ge a$. So, only terms with $r \ge a+b$ contribute. In this case, we do not have a good bound for $v_p(u)$. So

$$\left|S_w\left(\theta_{a,b}^{v_3}; 2r\right)\right| \ll p^{3r+\min\left\{\frac{s}{2}+v_p(m_1), r-\frac{s}{2}+v_p(n_1)\right\}}.$$

Hence

$$\begin{aligned}\left|\mathrm{Kl}_p\left(n,\psi,\psi'\right)\right| &\le \sum_{\substack{s-r\le a\le s/2\\ b\le r-a}} \left|S_{a,b}\left(n,\psi,\psi'\right)\right| \\ &\ll \sum_{\substack{a=s-r\\ b=r}} p^{-4r}p^{r+a+b}\left(p^{3r-s}\min\left\{p^{r+v_p(m_2)},p^{2r+v_p(n_1)}\right\}\right) \\ &\quad+\sum_{\substack{a=s/2\\ b\le r-s/2}} p^{-4r}p^{r+a+b}\left(p^{3r+\min\left\{\frac{s}{2}+v_p(m_1),r-\frac{s}{2}+v_p(n_1)\right\}}\right) \\ &\ll p^{r+\min\{v_p(m_2),r+v_p(n_1)\}}+p^{r+\min\left\{\frac{s}{2}+v_p(m_1),r-\frac{s}{2}+v_p(n_1)\right\}}.\end{aligned}$$

Case IV: $s=2r$. In this case, we have $a=r$, and $v_3,v_4=p^{r-b}$ is arbitrary. We deduce from (3.6) that $\hat{v}_2=0$. We consider the following subcases:

(a) Suppose $b=0$. We may assume $v_4=0$. Then $v_p(u)=r+v_p(v_3)$. We compute

$$\left|S_w\left(\theta_{a,b}^{v_3};2r\right)\right| \ll p^r\min\left\{p^{2r+v_p(v_3)+v_p(m_2)},p^{2r+v_p(n_1)}\right\}.$$

Fix $c\le r$. Then

$$\left|\{v_3\in\mathcal{S}_{a,b}\mid v_p(v_3)=c\}\right|\le p^{r-c}.$$

(b) Suppose $b>0$. Then $v_p(u)=r-b$. We compute

$$\left|S_w\left(\theta_{a,b}^{v_3};2r\right)\right| \ll p^r\min\left\{p^{2r-b+v_p(m_2)},p^{2r+v_p(n_1)}\right\}.$$

Hence

$$\begin{aligned}\left|\mathrm{Kl}_p\left(n,\psi,\psi'\right)\right| &\le \sum_{\substack{a=r/2\\ b\le r}} \left|S_{a,b}\left(n,\psi,\psi'\right)\right| \\ &\ll \sum_{\substack{a=r/2\\ b=0\\ c\le r}} p^{-4r}p^{r-c+a+b}\left(p^r\min\left\{p^{2r+c+v_p(m_2)},p^{2r+v_p(n_1)}\right\}\right) \\ &\quad+\sum_{\substack{a=r/2\\ b>0}} p^{-4r}p^{r+a+b}\left(p^r\min\left\{p^{2r-b+v_p(m_2)},p^{2r+v_p(n_1)}\right\}\right) \\ &\ll p^{r+\min\{v_p(m_2),r+v_p(n_1)\}}.\end{aligned}$$

This finishes the proof of the theorem. ☐

Theorem 3.12. Let $0\le r\le s$ be integers, and $\psi=\psi_{m_1,m_2}$, $\psi'=\psi_{n_1,n_2}$ characters of $U(\mathbb{Q}_p)/U(\mathbb{Z}_p)$. Then

$$\left|\mathrm{Kl}_p\left(n_{s_\beta s_\alpha s_\beta,r,s},\psi,\psi'\right)\right| \ll \begin{cases} p^{\frac{s}{2}+\frac{r}{2}+\frac{1}{2}v_p(m_1)+\frac{1}{2}\min\{2r+v_p(m_2),s+v_p(n_2)\}} & \text{if } r\le s/2,\\ p^{s-\frac{r}{2}+\frac{1}{2}v_p(m_1)+\frac{1}{2}\min\{2r+v_p(m_2),s+v_p(n_2)\}} & \text{if } s/2<r<s,\\ p^{s+\min\{v_p(m_1),v_p(n_2)\}}. & \text{if } r=s.\end{cases}$$

Proof. We make use of the stratification of Kloosterman sums in Section 3.2. For $w=s_\beta s_\alpha s_\beta$, we have $\Delta_w=\{\beta\}$. Hence, for $\ell\in\mathbb{N}$, we have

$$A_w(\ell)=\left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^2\times\left(\mathbb{Z}/p^\ell\mathbb{Z}\right).$$

Let $t = \mathrm{diag}\left(a_1, a_2, ca_1^{-1}, ca_2^{-1}\right) \in \mathcal{T}$. Then $s = n^{-1}tn = \mathrm{diag}\left(ca_2^{-1}, ca_1^{-1}, a_2, a_1\right)$. We compute

$$\kappa_2'(t * x) = ca_1^{-2}\kappa_2'(x).$$

So

$$V_w(\ell) = \left\{ \lambda \times \lambda' \in A_w(\ell) \;\middle|\; \begin{array}{l} \lambda_1, \lambda_2, \lambda_2' \in \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^\times, \\ \lambda_1^2\lambda_2\lambda_2' = 1 \end{array} \right\}.$$

Let $\theta : A_w(\ell) \to \mathbb{C}^\times$ be a character given by

$$\theta\left(\lambda \times \lambda'\right) = e\left(\frac{n_1\lambda_1 + n_2\lambda_2}{p^\ell}\right) e\left(\frac{n_2'\lambda_2'}{p^\ell}\right)$$

for $n_1, n_2, n_2' \in \mathbb{Z}$, then

$$S_w(\theta, \ell) = \sum_{\lambda_1 \in \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^\times} e\left(\frac{n_1\lambda_1}{p^\ell}\right) S\left(n_2\lambda_1^{-2}, n_2'; p^\ell\right). \tag{3.33}$$

Let $n = n_{s_\beta s_\alpha s_\beta, r, s}$. In terms of Plücker coordinates (see Section 2.2.4), this says $v_2 = p^r$, and $v_{12} = p^s$. Suppose $x_{a,b}^{v_{23}} \in X(n)$ has coordinates

$$(v_{12}, v_{13}, v_{14}, v_{23}) = \left(p^s, p^{s-a}, p^{s-b}, v_{23}\right).$$

The condition $(v_{12}, v_{14}) \mid v_{13}^2$ says $s - b \le 2(s-a)$, that is, $2a - b \le s$. We also have $\max\{a,b\} = r$. From Bruhat decomposition, we have

$$u'\left(x_{a,b}^{v_{23}}\right) = \begin{pmatrix} 1 & & -v_{23}p^{-s} & p^{-a} \\ & 1 & p^{-a} & p^{-b} \\ & & 1 & \\ & & & 1 \end{pmatrix} \pmod{U(\mathbb{Z}_p)}.$$

Let $X_{a,b}^{v_{23}}(n) = \mathcal{T} * x_{a,b}^{v_{23}}$, and define

$$S_{a,b}^{v_{23}}\left(n, \psi, \psi'\right) = \sum_{x \in X_{a,b}^{v_{23}}(n)} \psi(u(x))\, \psi'\left(u'(x)\right).$$

We also set

$$X_{a,b}(n) = \coprod_{\substack{v_{23} \pmod{p^s} \\ \left(p^{s-r}, v_{23}, p^{-b}v_{23} - p^{s-2a}\right) = 1}} X_{a,b}^{v_{23}}(n),$$

and

$$S_{a,b}\left(n, \psi, \psi'\right) = \sum_{x \in X_{a,b}(n)} \psi(u(x))\, \psi'\left(u'(x)\right).$$

It is easy to see that

$$X(n) = \coprod_{\substack{0 \le a, b \le r \\ \max\{a,b\} = r \\ 2a - b \le s}} X_{a,b}(n).$$

It is clear that $u(x), u'(x)$ have entries in $p^{-s}\mathbb{Z}_p/\mathbb{Z}_p$ for all $x \in X(n)$. Let $\mathcal{S}_{a,b}$ be a finite subset of $\mathbb{Z}_p$ such that

$$X_{a,b}(n) = \coprod_{v_{23} \in \mathcal{S}_{a,b}} X_{a,b}^{v_{23}}(n).$$

By Theorem 3.7, we have

$$S_{a,b}\left(n,\psi,\psi'\right)=p^{-2s}\left(1-p^{-1}\right)^{-2}\sum_{v_{23}\in\mathcal{S}_{a,b}}\left|X_{a,b}^{v_{23}}(n)\right|S_w\left(\theta_{a,b}^{v_{23}};s\right),$$

where

$$\theta_{a,b}^{v_{23}}\left(\lambda\times\lambda'\right)=e\left(\frac{m_1u\lambda_1}{p^r}\right)e\left(\frac{m_2\hat{v}_{14}\lambda_2+n_2p^{s-b}\lambda_2'}{p^s}\right).$$

with $\hat{v}_{14}$ and u given as in (3.13) and (3.14). By (3.33), we have

$$S_w\left(\theta_{a,b}^{v_{23}};s\right)=\sum_{x,y\in(\mathbb{Z}/p^s\mathbb{Z})^\times}e\left(\frac{m_1u\overline{x}}{p^r}\right)e\left(\frac{m_2\hat{v}_{14}x^2\overline{y}+n_2p^{s-b}y}{p^s}\right),\tag{3.34}$$

and we easily deduce that

$$\sum_{v_{23}\in\mathcal{S}_{a,b}}\left|X_{a,b}^{v_{23}}(n)\right|\leq\left|\mathcal{S}_{a,b}\right|p^{a+b}\leq p^{s+a}.\tag{3.35}$$

We estimate the size of $S_w\left(\theta_{a,b}^{v_{23}};s\right)$. We start by computing $v_p(\hat{v}_{14})$ and $v_p(u)$ in (3.34). From (3.13), we see that

$$up^{r-a}\equiv v_{23}\pmod{p^r},\qquad\qquad up^{r-b}\equiv-p^{s-a}\pmod{p^r}.\tag{3.36}$$

So, if $a=r$, then $u\equiv v_{23}\pmod{p^r}$, and if $b=r$, then $u\equiv-p^{s-a}\pmod{p^r}$. (Recall that $\max\{a,b\}=r$.) Also, we know that

$$v_{23}=-p^{s-2a+b}+\beta p^b\tag{3.37}$$

for some $\beta\in\mathbb{Z}$ such that $\left(\beta,p^{s-2r+b}\right)=1$ (see Section 2.2.4). Meanwhile, from (3.14), we see that unless $r=s$, we have $v_p\left(\hat{v}_{14}\right)=2r-b$.

Case I: Suppose $r<s/2$. We deduce from (3.37) that $v_p(v_{23})=b$. From (3.36), we deduce $a\geq b$. So we actually have $a=r$, and then $v_p(u)=b$.

(i) Suppose $b\leq\frac{3r-s}{2}$. Write $u=p^bu'$. Let

$$t=\min\left\{v_p(m_1),v_p(m_2)+3r-2b-s,v_p(n_2)+r-2b\right\}$$

and

$$f(x,y)=p^{-t}\left(\frac{m_1u'}{x}+\frac{m_2\hat{v}_{14}p^{r-b-s}x^2}{y}+n_2p^{r-2b}y\right)=\frac{m_1'}{x}+\frac{m_2'x^2}{y}+n_2'y,$$

where $m_1'=m_1u'p^{-t}$, $m_2'=m_2\hat{v}_{14}p^{r-b-s-t}$, $n_2'=n_2p^{r-2b-t}$. Consider the sum

$$S=\sum_{x,y\in(\mathbb{Z}/p^{r-b-t}\mathbb{Z})^\times}e\left(\frac{f(x,y)}{p^{r-b-t}}\right)=p^{2r-2s-2b-2t}S_w\left(\theta_{a,b}^{v_{23}};s\right).$$

When $r-b-t>1$, let $j\geq1$ be such that $2j\leq r-b-t$. Define as in (3.27)

$$\begin{aligned}D(\mathbb{Z}/p^j\mathbb{Z})&=\left\{(x,y)\in(\mathbb{Z}/p^j\mathbb{Z})^\times\times(\mathbb{Z}/p^j\mathbb{Z})^\times\;\middle|\;\nabla f(x,y)\equiv0\pmod{p^j}\right\}\\&=\left\{(x,y)\in(\mathbb{Z}/p^j\mathbb{Z})^\times\times(\mathbb{Z}/p^j\mathbb{Z})^\times\;\middle|\;\begin{array}{l}2m_2'x^3\equiv m_1'y\pmod{p^j}\\m_2'x^2\equiv n_2'y^2\pmod{p^j}\end{array}\right\}.\end{aligned}$$

Note that at least one of m_1', m_2' and n_2' is not divisible by p. It then follows that when p is odd, $D(\mathbb{Z}/p^j\mathbb{Z})$ is empty unless $v_p(m_1) = v_p(m_2) + 3r - 2b - s = v_p(n_2) + r - 2b$. Then this reduces to the situation seen in the proof of Theorem 3.9 (see the case $r = 2s$). When $p = 2$, $D(\mathbb{Z}/p^j\mathbb{Z})$ is empty unless $v_p(m_1) - 1 = v_p(m_2) + 3r - 2b - s = v_p(n_2) + r - 2b$. This is also dealt with in the proof of Theorem 3.9 (see the case $r = 2s - 1$). In either case, we obtain a bound

$$\left|S_w\left(\theta_{a,b}^{v_{23}}; s\right)\right| \ll p^{2s-r+b+t}. \tag{3.38}$$

Now suppose $r - b - t = 1$. If $p \nmid m_1'm_2'n_2'$, then it again follows from the argument in the proof of Theorem 3.9 that $|S| \ll p$. When p divides some (but not all) of m_1', m_2', n_2', then the sum reduces to Gauß sums or Ramanujan sums, and is easily evaluated that $|S| \ll p$ as well. So the bound (3.38) also holds for this case.

The bounds for $S_w\left(\theta_{a,b}^{v_{23}}; s\right)$ in other cases are obtained analogously, and we shall omit the repetitive computations thereafter.

(ii) Suppose $b > \frac{3r-s}{2}$. Write $\hat{v}_{14} = p^{2r-b}\hat{v}_{14}'$. Let

$$t = \min\left\{v_p(m_1) + s + 2b - 3r, v_p(m_2), v_p(n_2) + s - 2r\right\},$$

and

$$f(x, y) = p^{-t}\left(\frac{m_1 u p^{s+b-3r}}{x} + \frac{m_2 \hat{v}_{14}' x^2}{y} + n_2 p^{s-2r} y\right) = \frac{m_1'}{x} + \frac{m_2' x^2}{y} + n_2' y,$$

where $m_1' = m_1 u p^{s+b-3r-t}$, $m_2' = m_2 \hat{v}_{14}' p^{-t}$, $n_2' = n_2 p^{s-2r-t}$. Then we have

$$S = \sum_{x,y \in (\mathbb{Z}/p^{s+b-2r-t}\mathbb{Z})^\times} e\left(\frac{f(x,y)}{p^{s+b-2r-t}}\right) = p^{2b-4r-2t} S_w\left(\theta_{a,b}^{v_{23}}; s\right).$$

Then we obtain analogously

$$\left|S_w\left(\theta_{a,b}^{v_{23}}; s\right)\right| \ll p^{s+2r-b+t}.$$

Hence

$$\begin{aligned}
\left|\mathrm{Kl}_p\left(n, \psi, \psi'\right)\right| &\leq \sum_{\substack{a=r \\ 0 \leq b \leq r}} \left|S_{a,b}\left(n, \psi, \psi'\right)\right| \\
&\ll \sum_{\substack{a=r \\ 0 \leq b \leq r}} p^{-2s} p^{s+a} \left|S_w\left(\theta_{a,b}^{v_{23}}; s\right)\right| \\
&\ll \sum_{\substack{a=r \\ 0 \leq b \leq r}} p^{-2s} p^{s+a} \left(p^{s-r} \min\left\{p^{s+b+v_p(m_1)}, p^{r-b+\min\{2r+v_p(m_2), s+v_p(n_2)\}}\right\}\right) \\
&\ll p^{\frac{s}{2}+\frac{r}{2}+\frac{1}{2}\min\{2r+v_p(m_2), s+v_p(n_2)\}+\frac{1}{2}v_p(m_1)}.
\end{aligned}$$

Case II: Suppose $r = s/2$. We consider the following subcases:

(a) Suppose $b = r$. From (3.36), we may assume $u = 0$. We compute

$$\left|S_w\left(\theta_{a,b}^{v_{23}}; s\right)\right| \ll p^{\frac{3s}{2}+\min\{v_p(m_2), v_p(n_2)\}}.$$

(b) Suppose $b < r$. Then $a = r$. From (3.37), we see that $v_{23} = (\beta - 1)\, p^b$ for some $\beta \in \mathbb{Z}$ such that $\left(\beta, p^b\right) = 1$. So $v_p(v_{23}) \geq b$. And from (3.36), we deduce that $v_p(u) = v_p(v_{23})$. We compute

$$\left|S_w\left(\theta_{a,b}^{v_{23}}; s\right)\right| \ll p^{s/2} \min\left\{p^{s+v_p(v_{23})+v_p(m_1)}, p^{\frac{3s}{2}-b+\min\{v_p(m_2),v_p(n_2)\}}\right\}.$$

Fix $c \geq b$. Then

$$|\{v_{23} \in \mathcal{S}_{a,b} \mid v_p(v_{23}) = c\}| \leq p^{s-c}.$$

Hence

$$\begin{aligned}\left|\mathrm{Kl}_p\left(n, \psi, \psi'\right)\right| &\leq \sum_{\substack{a,b\leq r\\ \max\{a,b\}=r}} \left|S_{a,b}\left(n, \psi, \psi'\right)\right| \\ &\ll \sum_{\substack{b=r\\ a\leq r}} p^{-2s}p^{s+a}\left(p^{\frac{3s}{2}+\min\{v_p(m_2),v_p(n_2)\}}\right) \\ &\quad + \sum_{\substack{a=r\\ b<r\\ b\leq c\leq r}} p^{-2s}p^{s-c+a+b}\left(p^{s/2}\min\left\{p^{s+v_p(v_{23})+v_p(m_1)}, p^{\frac{3s}{2}-b+\min\{v_p(m_2),v_p(n_2)\}}\right\}\right) \\ &\ll p^{\frac{5s}{4}+\frac{1}{2}v_p(m_1)+\frac{1}{2}\min\{v_p(m_2),v_p(n_2)\}}.\end{aligned}$$

Case III: Suppose $s > r > s/2$. We consider the following subcases:

(a) Suppose $b = r$. Then $v_p(u) = s - a$, and $v_p(\hat{v}_{14}) = r$. We compute

$$\left|S_w\left(\theta_{a,b}^{v_{23}}; s\right)\right| \ll p^{s-r} \min\left\{p^{2s-a+v_p(m_1)}, p^{r+\min\{r+v_p(m_2)\},s-r+v_p(n_2)}\right\}.$$

(b) Suppose $b < r$. Then $a = r$. Then from (3.37) we deduce that $v_p(v_{23}) = p^{s-2r+b}$, and hence $v_p(u) = p^{s-2r+b}$. We compute

$$\left|S_w\left(\theta_{a,b}^{v_{23}}; s\right)\right| \ll p^{s-r} \min\left\{p^{2s-2r+b+v_p(m_1)}, p^{r-b+\min\{2r+v_p(m_2),s+v_p(n_2)\}}\right\}.$$

Hence

$$\begin{aligned}\left|\mathrm{Kl}_p\left(n, \psi, \psi'\right)\right| &\leq \sum_{\substack{a,b\leq r\\ \max\{a,b\}=r\\ 2a-b\leq s}} \left|S_{a,b}\left(n, \psi, \psi'\right)\right| \\ &\ll \sum_{\substack{b=r\\ a\leq r}} p^{-2s}p^{s+a}\left(p^{s-r}\min\left\{p^{2s-a+v_p(m_1)}, p^{r+\min\{r+v_p(m_2)\},s-r+v_p(n_2)}\right\}\right) \\ &\quad + \sum_{\substack{a=r\\ 2r-s\leq b<r}} p^{-2s}p^{s+a}\left(p^{s-r}\min\left\{p^{2s-2r+b+v_p(m_1)}, p^{r-b+\min\{2r+v_p(m_2),s+v_p(n_2)\}}\right\}\right) \\ &\ll p^{s-\frac{r}{2}+\frac{1}{2}v_p(m_1)+\frac{1}{2}\min\{2r+v_p(m_2),s+v_p(n_2)\}}.\end{aligned}$$

Case IV: $r = s$. In this case we only have to consider terms with $b = r$. Indeed, if $b < r$, then $a = r$, and then by (3.36), we see that $up^{r-b} \equiv -1 \pmod{p^r}$, which says $b = r$, a contradiction. When $b = r$, we have $v_p(u) = s - a$, and from (3.14) we may assume $\hat{v}_{14} = 0$. We compute

$$S_w\left(\theta_{a,b}^{v_{23}}; s\right) \ll \min\left\{p^{2s-a+v_p(m_1)}, p^{s+v_p(n_2)}\right\}.$$

Hence

$$\begin{aligned}\left|\mathrm{Kl}_p\left(n,\psi,\psi'\right)\right| &\leq \sum_{\substack{b=s\\a\leq s}}\left|S_{a,b}\left(n,\psi,\psi'\right)\right|\\ &\ll \sum_{\substack{b=s\\a\leq s}} p^{-2s}p^{s+a}\left(\min\left\{p^{2s-a+v_p(m_1)},p^{s+v_p(n_2)}\right\}\right)\\ &\ll p^{s+\min\{v_p(m_1),v_p(n_2)\}}.\end{aligned}$$

This finishes the proof of the theorem. □

Theorem 3.13. Let $\psi=\psi_{m_1,m_2}$, $\psi'=\psi_{n_1,n_2}$ be characters of $U(\mathbb{Q}_p)/U(\mathbb{Z}_p)$. Then

$$\left|\mathrm{Kl}_p\left(n_{w_0,r,s},\psi,\psi'\right)\right| \ll \left(\left|m_1m_2\right|_p^{-1},\left|n_1n_2\right|_p^{-1}\right)^{1/2}(s+1)\,p^{\frac{r}{2}+\frac{3s}{4}+\frac{1}{2}\min\{r,s\}}.$$

Proof. We make use of the stratification of Kloosterman sums in Section 3.2. For $w=w_0$, we have $\Delta_{w_0}=\Delta$. Hence, for $\ell\in\mathbb{N}$, we have

$$A_{w_0}(\ell)=\left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^2\times\left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^2.$$

Let $t=\mathrm{diag}\left(a_1,a_2,ca_1^{-1},ca_2^{-1}\right)\in\mathcal{T}$. Then $s=n^{-1}tn=\mathrm{diag}\left(ca_1^{-1},ca_2^{-1},a_1,a_2\right)$. We compute

$$\kappa_1'(t*x)=a_2a_1^{-1}\kappa_1'(x),\quad \kappa_2'(t*x)=ca_2^{-2}\kappa_2'(x).$$

So

$$V_{w_0}(\ell)=\left\{\lambda\times\lambda'\in A_{w_0}(\ell)\;\middle|\;\lambda_1\lambda_1'=1,\lambda_2\lambda_2'=1\right\}.$$

Let $\theta: A_{w_0}(\ell)\to\mathbb{C}^\times$ be a character given by

$$\theta\left(\lambda\times\lambda'\right)=\prod_{i=1}^{2}e\left(\frac{n_i\lambda_i}{p^\ell}\right)\prod_{i=1}^{2}e\left(\frac{n_i'\lambda_i'}{p^\ell}\right)$$

for $n_1,n_2,n_1',n_2'\in\mathbb{Z}$, then

$$S_{w_0}(\theta;\ell)=S\left(n_1,n_1';p^\ell\right)S\left(n_2,n_2';p^\ell\right). \tag{3.39}$$

Let $n=n_{w_0,r,s}$. In terms of Plücker coordinates (see Section 2.2.4), this says $v_1=p^r$, and $v_{12}=p^s$. Suppose $x_{a,b}^{v_3,v_4,v_{13}}\in X(n)$ has coordinates

$$(v_1,v_2,v_3,v_4;v_{12},v_{13},v_{14})=\left(p^r,p^{r-a},v_3,v_4;p^s,v_{13},p^{s-b}\right).$$

Note that this also says $r\geq a, s\geq b$. From Bruhat decomposition, we have

$$u'\left(x_{a,b}^{v_3,v_4,v_{13}}\right)=\begin{pmatrix}1 & p^{-a} & v_3p^{-r} & v_4p^{-r}\\ & 1 & v_{13}p^{-s} & p^{-b}\\ & & 1 & \\ & & -p^{-a} & 1\end{pmatrix}\pmod{U(\mathbb{Z}_p)}.$$

Let $X_{a,b}^{v_3,v_4,v_{13}}(n)=\mathcal{T}*x_{a,b}^{v_3,v_4,v_{13}}$, and define

$$S_{a,b}^{v_3,v_4,v_{13}}\left(n,\psi,\psi'\right)=\sum_{x\in X_{a,b}^{v_3,v_4,v_{13}}(n)}\psi\left(u(x)\right)\psi'\left(u'(x)\right).$$

We also set

$$X_{a,b}(n) = \bigcup_{\substack{v_3,v_4 (\text{mod } p^r)\\ v_{13} (\text{mod } p^s)\\ \text{conditions}}} X_{a,b}^{v_3,v_4,v_{13}}(n),$$

and

$$S_{a,b}\left(n,\psi,\psi'\right) = \sum_{x \in X_{a,b}(n)} \psi\left(u(x)\right)\psi'\left(u'(x)\right).$$

It is easy to see that

$$X(n) = \coprod_{\substack{0\le a\le r\\ 0\le b\le s}} X_{a,b}(n).$$

Now we consider cases $r \geq s$ and $r < s$ separately.

(i) Suppose $r > s$. As $r \geq a, r \geq s \geq b$, we see that $u(x), u'(x)$ have entries in $p^{-r}\mathbb{Z}_p/\mathbb{Z}_p$ for all $x \in X(n)$. Let $\mathcal{S}_{a,b}$ be a finite subset of $\mathbb{Z}_p^3$ such that

$$X_{a,b}(n) = \coprod_{(v_3,v_4,v_{13})\in\mathcal{S}_{a,b}} X_{a,b}^{v_3,v_4,v_{13}}(n).$$

By Theorem 3.7, we have

$$S_{a,b}\left(n,\psi,\psi'\right) = p^{-2r}\left(1-p^{-1}\right)^{-2} \sum_{(v_3,v_4,v_{13})\in\mathcal{S}_{a,b}} \left|X_{a,b}^{v_3,v_4,v_{13}}(n)\right| S_{w_0}\left(\theta_{a,b}^{v_3,v_4,v_{13}}; r\right),$$

where

$$\theta_{a,b}^{v_3,v_4,v_{13}}\left(\lambda\times\lambda'\right) = e\left(\frac{m_1\hat{v}_2\lambda_1 + n_1p^{r-a}\lambda_1'}{p^r}\right) e\left(\frac{m_2\hat{v}_{14} + n_2p^{s-b}}{p^s}\right).$$

By (3.39), we have

$$S_{w_0}\left(\theta_{a,b}^{v_3,v_4,v_{13}}; r\right) = S\left(m_1\hat{v}_2, n_1\hat{p}^{r-a}; p^r\right) S\left(m_2\hat{v}_{14}p^{r-s}, n_2p^{r-b}; p^r\right).$$

And we obtain a bound by applying (3.25):

$$\left|S_{w_0}\left(\theta_{a,b}^{v_3,v_4,v_{13}}; r\right)\right| \leq 4p^r\left(\gcd\left(m_1\hat{v}_2, n_1p^{r-a}, p^r\right)\gcd\left(m_2\hat{v}_{14}p^{r-s}, n_2p^{r-b}, p^r\right)\right)^{1/2}.$$

(ii) Suppose $s \geq r$. Then $u(x), u'(x)$ has entries in $p^{-s}\mathbb{Z}_p/\mathbb{Z}_p$ for all $x \in X(n)$. Again, by Theorem 3.7 we have

$$S_{a,b}\left(n,\psi,\psi'\right) = p^{-2s}\left(1-p^{-1}\right)^{-2} \sum_{(v_3,v_4,v_{13})\in\mathcal{S}_{a,b}} \left|X_{a,b}^{v_3,v_4,v_{13}}(n)\right| S_{w_0}\left(\theta_{a,b}^{v_3,v_4,v_{13}}; s\right),$$

where

$$\theta_{a,b}^{v_3,v_4,v_{13}}\left(\lambda\times\lambda'\right) = e\left(\frac{\left(m_1\hat{v}_2p^{s-r}\right)\lambda_1 + \left(m_2\hat{v}_{14}\right)\lambda_2 + \left(n_1p^{s-a}\right)\lambda_1' + \left(n_2p^{s-b}\right)\lambda_2'}{p^s}\right).$$

By (3.39), we have

$$S_{w_0}\left(\theta_{a,b}^{v_3,v_4,v_{13}}; s\right) = S\left(m_1\hat{v}_2p^{s-r}, n_1p^{s-a}; p^s\right) S\left(m_2\hat{v}_{14}, n_2p^{s-b}; p^s\right).$$

Applying (3.25) gives

$$\left|S_{w_0}\left(\theta_{a,b}^{v_3,v_4,v_{13}}; s\right)\right| \leq 4p^s\left(\gcd\left(m_1\hat{v}_2p^{s-r}, n_1p^{s-a}, p^s\right), \gcd\left(m_2\hat{v}_{14}, n_2p^{s-b}, p^s\right)\right)^{1/2}.$$

Now we give a bound to the size of $\mathrm{Kl}_p(n,\psi,\psi')$. To ease computations, we consider a relaxed bound by ignoring $\hat{v}_2$ and $\hat{v}_{14}$.

Suppose $r > s$. Then the bound says

$$\begin{aligned}\left|S_{w_0}\left(\theta_{a,b}^{v_3,v_4,v_{13}};r\right)\right| &\leq 4p^r\left(\gcd\left(m_1\hat{v}_2, n_1p^{r-a}, p^r\right)\gcd\left(m_2\hat{v}_{14}p^{r-s}, n_2p^{r-b}, p^r\right)\right)^{1/2}\\ &\leq 4p^r\left(|n_1n_2|_p^{-1}p^{2r-a-b}\right)^{1/2}\\ &= 4p^{2r-\frac{a+b}{2}}|n_1n_2|_p^{-1/2}.\end{aligned}$$

Note that

$$\sum_{(v_3,v_4,v_{13})\in\mathcal{S}_{a,b}}\left|X_{a,b}^{v_3,v_4,v_{13}}(n)\right| \leq |\mathcal{S}_{a,b}|\,p^{a+b}.$$

Hence

$$\begin{aligned}\left|\mathrm{Kl}_p(n,\psi,\psi')\right| &\leq \sum_{\substack{a\leq r\\ b\leq s}}\left|S_{a,b}(n,\psi,\psi')\right|\\ &\leq \sum_{\substack{a\leq r\\ b\leq s}}p^{-2r}\left(1-p^{-1}\right)^{-2}4\,|n_1n_2|_p^{-1/2}|\mathcal{S}_{a,b}|\,p^{2r+\frac{a+b}{2}}\\ &\ll |n_1n_2|_p^{-1/2}\sum_{\substack{a\leq r\\ b\leq s}}|\mathcal{S}_{a,b}|\,p^{\frac{a+b}{2}}.\end{aligned}$$

So it suffices to give an upper bound to $|\mathcal{S}_{a,b}|$. Such bounds were computed in Section 2.4. Note that we require $r \geq a+b$ in order to have $\mathcal{S}_{a,b}$ nonempty.

Case I: Suppose $s-r+a \geq 0$.

(a) If $s-2r+2a+b \geq 0$, then $|\mathcal{S}_{a,b}| \leq p^{r+s-a-b}$.

(b) If $s-2r+2a+b < 0$, then $|\mathcal{S}_{a,b}| \leq p^{2s-b-\lceil\frac{s-b}{2}\rceil} \leq p^{3s/2-b/2}$.

Case II: Suppose $s-r+a < 0$. Then $|\mathcal{S}_{a,b}| \leq p^{2s-b-\lceil\frac{s-b}{2}\rceil} \leq p^{3s/2-b/2}$.

Combining the cases, we obtain

$$\begin{aligned}\sum_{\substack{a\leq r\\ b\leq s}}|\mathcal{S}_{a,b}|\,p^{\frac{a+b}{2}} &\leq \sum_{\substack{r-s\leq a\leq r\\ 2r-2a-s\leq b\leq r-a}}p^{r+s-\frac{a}{2}-\frac{b}{2}} + \sum_{\substack{r-s\leq a\leq r\\ b<2r-2a-s}}p^{\frac{3s}{2}+\frac{a}{2}} + \sum_{\substack{a<r-s\\ b\leq s}}p^{\frac{3s}{2}+\frac{a}{2}}\\ &\ll (s+1)\,p^{\frac{r}{2}+\frac{5s}{4}}.\end{aligned}$$

Hence, we have for $r > s$

$$\left|\mathrm{Kl}_p(n,\psi,\psi')\right| \ll |n_1n_2|_p^{-1/2}(s+1)\,p^{\frac{r}{2}+\frac{5s}{4}}. \tag{3.40}$$

For $r \leq s$, applying the same argument gives

$$\left|\mathrm{Kl}_p(n,\psi,\psi')\right| \ll |n_1n_2|_p^{-1/2}(s-r+1)\,p^{r+\frac{3s}{4}}. \tag{3.41}$$

Combining (3.40) and (3.41), we get

$$\left|\mathrm{Kl}_p(n,\psi,\psi')\right| \ll |n_1n_2|_p^{-1/2}(s+1)\,p^{\frac{r}{2}+\frac{3s}{4}+\frac{1}{2}\min\{r,s\}}. \tag{3.42}$$

By Proposition 3.2, we can swap the characters, so

$$\left|\mathrm{Kl}_p(n,\psi,\psi')\right| \ll |m_1m_2|_p^{-1/2}(s+1)\,p^{\frac{r}{2}+\frac{3s}{4}+\frac{1}{2}\min\{r,s\}} \tag{3.43}$$

as well. Combining (3.42) and (3.43) yields the theorem. □

3.4 Bounds for global Kloosterman sums

By combining the bounds for local Kloosterman sums $\mathrm{Kl}_p(n_{w,r,s}, \psi, \psi')$, we obtain bounds for global Kloosterman sums. For $w \in W$, let

$$n_w(c_1, c_2) = \begin{pmatrix} 1/c_1 & & & \\ & c_1/c_2 & & \\ & & c_1 & \\ & & & c_2/c_1 \end{pmatrix} w \in N(\mathbb{Q}).$$

For $c_1 = p^r$, $c_2 = p^s$, then we have $n_w(c_1, c_2) = n_{w,r,s}$. Again we fix $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$, as characters of $U(\mathbb{Q})/U(\mathbb{Z})$.

We recall that $\mathrm{Kl}_p(n_{\mathrm{id}}, \psi, \psi) = 1$ is trivial, so $\mathrm{Kl}(n_{\mathrm{id}}(1,1), \psi, \psi') = 1$. Meanwhile,

$$\mathrm{Kl}_p(n_{s_\alpha,r}, \psi, \psi') = S(m_1, n_1; p^r),$$
$$\mathrm{Kl}_p(n_{s_\beta,s}, \psi, \psi') = S(m_2, n_2; p^s)$$

are just classical Kloosterman sums. So it follows from the global bounds for classical Kloosterman sums that

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha}(c_1, 1), \psi, \psi'\right)\right| \ll_\varepsilon (m_1, n_1, c_1)^{1/2} c_1^{1/2+\varepsilon},$$
$$\left|\mathrm{Kl}_p\left(n_{s_\beta}(1, c_2), \psi, \psi'\right)\right| \ll_\varepsilon (m_2, n_2, c_2)^{1/2} c_2^{1/2+\varepsilon}.$$

For $w = s_\alpha s_\beta$ and $s_\beta s_\alpha$, the global bounds are easily derived from the local bounds, given in Theorems 3.9 and 3.10.

Theorem 3.14. Let $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$ be characters of $U(\mathbb{Q})/U(\mathbb{Z})$. The Kloosterman sum $\mathrm{Kl}\left(n_{s_\alpha s_\beta}(c_1, c_2), \psi, \psi'\right)$ vanishes unless $c_2 \mid c_1$. When $c_2 \mid c_1$, we have

$$\left|\mathrm{Kl}\left(n_{s_\alpha s_\beta}(c_1, c_2), \psi, \psi'\right)\right| \ll_\varepsilon \left(c_2^2(m_1, c_1/c_2), c_1(m_2, c_2)^{1/2}(n_2, c_2)^{1/2}\right)(c_1 c_2)^\varepsilon$$

for every $\varepsilon > 0$.

Theorem 3.15. Let $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$ be characters of $U(\mathbb{Q})/U(\mathbb{Z})$. The Kloosterman sum $\mathrm{Kl}\left(n_{s_\beta s_\alpha}(c_1, c_2), \psi, \psi'\right)$ vanishes unless $c_1^2 \mid c_2$. When $c_1^2 \mid c_2$, we have

$$\left|\mathrm{Kl}\left(n_{s_\beta s_\alpha}(c_1, c_2), \psi, \psi'\right)\right| \ll_\varepsilon \left(c_1^3(m_2, c_2/c_1^2), c_2(m_1, n_1, c_1)\right)(c_1 c_2)^\varepsilon$$

for every $\varepsilon > 0$.

For $w = s_\alpha s_\beta s_\alpha$ and $s_\beta s_\alpha s_\beta$, the situation is more complicated, since the shapes of the local bounds depend on the relative size of r, s. Therefore, in order to obtain a global bound, we have to find an expression for the local bound that works for all values of r, s.

Theorem 3.16. Let $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$ be characters of $U(\mathbb{Q})/U(\mathbb{Z})$. The Kloosterman sum $\mathrm{Kl}\left(n_{s_\alpha s_\beta s_\alpha}(c_1, c_2), \psi, \psi'\right)$ vanishes unless $c_2 \mid c_1^2$. When $c_2 \mid c_1^2$, we have

$$\left|\mathrm{Kl}\left(n_{s_\alpha s_\beta s_\alpha}(c_1, c_2), \psi, \psi'\right)\right| \ll_\varepsilon (m_1, n_1, c_1)(m_2, c_2)(c_1, c_2)(c_1 c_2)^{1/3+\varepsilon}$$

for every $\varepsilon > 0$.

Proof. For $s \le r$, we have

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta s_\alpha, r, s}, \psi, \psi'\right)\right| \ll p^{\frac{r}{3}+\frac{4s}{3}+\frac{2}{3}\min\{v_p(m_1), v_p(n_1)\}+\frac{1}{3}v_p(m_2)} \le p^{\frac{4r}{3}+\frac{s}{3}+\frac{2}{3}\min\{v_p(m_1), v_p(n_1)\}+\frac{1}{3}v_p(m_2)}.$$

For $r < s < 2r$, we have

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta s_\alpha,r,s},\psi,\psi'\right)\right| \ll p^{r+v_p(m_2)} + p^{r+\frac{s}{2}+\min\{v_p(m_1),v_p(n_1)\}},$$

and we have inequalities

$$p^{r+v_p(m_2)} + p^{r+\frac{s}{2}+\min\{v_p(m_1),v_p(n_1)\}} \le p^{r+v_p(m_2)} + p^{\frac{4r}{3}+\frac{s}{3}+\min\{v_p(m_1),v_p(n_1)\}},$$
$$p^{r+v_p(m_2)} + p^{r+\frac{s}{2}+\min\{v_p(m_1),v_p(n_1)\}} \le p^{s+v_p(m_2)} + p^{\frac{r}{6}+\frac{4s}{3}+\min\{v_p(m_1),v_p(n_1)\}}.$$

For $s = 2r$, we have

$$\mathrm{Kl}_p\left(n_{s_\alpha s_\beta s_\alpha,r,s},\psi,\psi'\right) \ll p^{r+v_p(m_2)} = p^{\frac{s}{2}+v_p(m_2)}.$$

So we can conclude for $0 \le s \le 2r$ that

$$\left|\mathrm{Kl}_p\left(n_{s_\alpha s_\beta s_\alpha,r,s},\psi,\psi'\right)\right| \ll p^{\min\left\{\frac{4r}{3}+\frac{s}{3},\frac{r}{3}+\frac{4s}{3}\right\}+v_p(m_2)+\min\{v_p(m_1),v_p(n_1)\}}.$$

Since we may assume from (3.29) that $v_p(m_1), v_p(n_1) \le r$, and $v_p(m_2) \le s$, we have

$$\left|\mathrm{Kl}\left(n_{s_\alpha s_\beta s_\alpha}(c_1,c_2),\psi,\psi'\right)\right| \ll_\varepsilon (m_1,n_1,c_1)(m_2,c_2)(c_1,c_2)(c_1c_2)^{1/3+\varepsilon}$$

for every $\varepsilon > 0$. □

Theorem 3.17. Let $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$ be characters of $U(\mathbb{Q})/U(\mathbb{Z})$. The Kloosterman sum $\mathrm{Kl}\left(n_{s_\beta s_\alpha s_\beta}(c_1,c_2),\psi,\psi'\right)$ vanishes unless $c_1 \mid c_2$. When $c_1 \mid c_2$, we have

$$\left|\mathrm{Kl}\left(n_{s_\beta s_\alpha s_\beta}(c_1,c_2),\psi,\psi'\right)\right| \ll_\varepsilon (m_1,c_1)(m_2,n_2,c_2)(c_1^2,c_2)c_1^{-1/2}c_2^{1/2}(c_1c_2)^\varepsilon$$

for every $\varepsilon > 0$.

Proof. For $r \le s/2$, we have

$$\left|\mathrm{Kl}_p\left(n_{s_\beta s_\alpha s_\beta,r,s},\psi,\psi'\right)\right| \ll p^{\frac{3r}{2}+\frac{s}{2}+\frac{1}{2}v_p(m_1)+\frac{1}{2}\min\{v_p(m_2),v_p(n_2)\}} \le p^{-\frac{r}{2}+\frac{3s}{2}+\frac{1}{2}v_p(m_1)+\frac{1}{2}\min\{v_p(m_2),v_p(n_2)\}}.$$

For $s/2 < r < s$, we have

$$\left|\mathrm{Kl}_p\left(n_{s_\beta s_\alpha s_\beta,r,s},\psi,\psi'\right)\right| \ll p^{-\frac{r}{2}+\frac{3s}{2}+\frac{1}{2}v_p(m_1)+\frac{1}{2}\min\{v_p(m_2),v_p(n_2)\}} \le p^{\frac{3r}{2}+\frac{s}{2}+\frac{1}{2}v_p(m_1)+\frac{1}{2}\min\{v_p(m_2),v_p(n_2)\}}.$$

For $s = r$, we have

$$\left|\mathrm{Kl}_p\left(n_{s_\beta s_\alpha s_\beta,r,s},\psi,\psi'\right)\right| \ll p^{s+\min\{v_p(m_1),v_p(n_2)\}} = p^{r+\min\{v_p(m_1),v_p(n_2)\}}.$$

So we can conclude for $0 \le r \le s$ that

$$\left|\mathrm{Kl}_p\left(n_{s_\beta s_\alpha s_\beta,r,s},\psi,\psi'\right)\right| \ll p^{\min\left\{\frac{3r}{2}+\frac{s}{2},-\frac{r}{2}+\frac{3s}{2}\right\}+v_p(m_1)+\frac{1}{2}\min\{v_p(m_2),v_p(n_2)\}}.$$

Since we may assume from (3.34) that $v_p(m_1) \le r$, and $v_p(m_2), v_p(n_2) \le s$, we have

$$\left|\mathrm{Kl}\left(n_{s_\beta s_\alpha s_\beta}(c_1,c_2),\psi,\psi'\right)\right| \ll_\varepsilon (m_1,c_1)(m_2,n_2,c_2)(c_1^2,c_2)c_1^{-1/2}c_2^{1/2}(c_1c_2)^\varepsilon$$

for every $\varepsilon > 0$. □

Theorem 3.18. Let $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$ be characters of $U(\mathbb{Q})/U(\mathbb{Z})$. Then we have

$$\left|\mathrm{Kl}\left(n_{w_0}(c_1,c_2),\psi,\psi'\right)\right| \ll_\varepsilon (m_1m_2,n_1n_2,c_1c_2)^{1/2}(c_1,c_2)^{1/2}c_1^{1/2}c_2^{3/4}(c_1c_2)^\varepsilon$$

for every $\varepsilon > 0$.

Proof. This follows immediately from the local bound given in Theorem 3.13, noting that $s+1 \ll (p^s)^\varepsilon$. □

3.5 Symplectic Poincaré series

We start by defining Poincaré series on $\mathrm{Sp}(2r)$. Let $G = \mathrm{Sp}(2r, \mathbb{R})$. Again, we denote by K the standard maximal compact subgroup of G. Let $F : T(\mathbb{R}^+) \to \mathbb{C}$ be a smooth function with rapid decay. Let ψ, ψ' be characters of $U(\mathbb{R})$ trivial on $U(\mathbb{Z})$. For $g = uy \in G/K$, where $u \in U(\mathbb{R})$, $y \in T(\mathbb{R}^+)$, we define $\mathcal{F}_\psi(g) := \psi(u)F(y)$. The symplectic Poincaré series associated to F is given by

$$P_\psi(g) := \sum_{\gamma \in P_0 \cap \Gamma \backslash \Gamma} \mathcal{F}_\psi(\gamma g),$$

where $\Gamma = \mathrm{Sp}(2r, \mathbb{Z})$, and P_0 is the standard minimal parabolic subgroup of G. The ψ'-th Fourier coefficient of $P_\psi(g)$ is given by

$$P_{\psi,\psi'}(g) := \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} P_\psi(ug)\overline{\psi'}(u)du.$$

The aim of this section is to compute the Fourier coefficients of symplectic Poincaré series, using Kloosterman sums. However, for this purpose, it is more convenient to use a slightly different definition for Kloosterman sums, denoted by $\underline{\mathrm{Kl}}(n, \psi, \psi')$. To motivate the alternative definition, we start with the following proposition.

Proposition 3.19. [Fri87, Proposition 1.3] Let $G = \mathrm{Sp}(2r, \mathbb{Q}_p)$, $n \in N(\mathbb{Q}_p)$, and $x \in X(n)$, with Bruhat decomposition $x = b_1 n b_2$, with $b_1, b_2 \in U(\mathbb{Q}_p)$. Let ψ, ψ' be characters of $U(\mathbb{Q}_p)$ which are trivial on $U(\mathbb{Z}_p)$. Then the quantity $\psi(b_1)\psi'(b_2)$ is well-defined as a function on $X(n)$ if $\psi\left(nun^{-1}\right) = \psi'(u)$ for $u \in \overline{U}_n(\mathbb{Q}_p)$.

Proof. Suppose $\psi\left(nun^{-1}\right) = \psi'(u)$ for all $u \in \overline{U}_n(\mathbb{Q}_p)$. Let $x = b_1 n b_2 = b_1' n b_2'$ be two Bruhat decompositions. This says $b_1' = \gamma b_1$ for some $\gamma \in U(\mathbb{Z}_p)$, and $b_2' = b_2\delta$ for some $\delta \in U_n(\mathbb{Z}_p)$. Then we have

$$U(\mathbb{Z}_p)\, b_1 n b_2 \delta^{-1} = U(\mathbb{Z}_p)\, b_1 n b_2,$$

which implies $b_2 {b_2'}^{-1} = b_2 \delta^{-1} b_2^{-1} \in \overline{U}_n(\mathbb{Q}_p)$. Now, from the equivalence of Bruhat decompositions, we deduce that

$$U(\mathbb{Z}_p)\, n b_2 {b_2'}^{-1} n^{-1} U_n(\mathbb{Z}_p) = U(\mathbb{Z}_p)\, b_1^{-1} b_1' U_n(\mathbb{Z}_p),$$

which implies $\psi'\left(b_2 {b_2'}^{-1}\right) = \psi\left(n b_2 {b_2'}^{-1} n^{-1}\right) = \psi\left(b_1^{-1} b_1'\right)$. ☐

Now we give the definition for $\underline{\mathrm{Kl}}(n, \psi, \psi')$. Let $n \in N(\mathbb{Q}_p)$, and ψ_p, ψ_p' be characters of $U(\mathbb{Q}_p)$ which are trivial on $U(\mathbb{Z}_p)$. We consider the sum

$$\sum_{\substack{x \in X(n) \\ x = b_1 n b_2}} \psi_p(b_1)\, \psi_p'(b_2).$$

By Proposition 3.19, this sum is well-defined as a function in $n \in N(\mathbb{Q}_p)$ if if $\psi_p\left(nun^{-1}\right) = \psi_p'(u)$ for $u \in \overline{U}_n(\mathbb{Q}_p)$. Now we define

$$\underline{\mathrm{Kl}}_p(n, \psi_p, \psi_p') := \sum_{\substack{x \in X(n) \\ x = b_1 n b_2}} \psi_p(b_1)\, \psi_p'(b_2).$$

if this condition is satisfied, and we say the sum is well-defined. When this is not the case, we set $\underline{\mathrm{Kl}}_p(n, \psi_p, \psi_p')$ to be zero.

For the global version, let $n \in N(\mathbb{Q})$, and $\psi = \prod_p \psi_p$, $\psi' = \prod_p \psi'_p$ be characters of $U(\mathbb{A})$ which are trivial on $\prod_p U(\mathbb{Z}_p)$. Then we define

$$\underline{\mathrm{Kl}}\left(n, \psi, \psi'\right) = \prod_p \underline{\mathrm{Kl}}_p\left(n, \psi_p, \psi'_p\right).$$

Now we give a relation between $\underline{\mathrm{Kl}}_p\left(n, \psi_p, \psi'_p\right)$, and the Kloosterman sum $\mathrm{Kl}_p\left(n, \psi_p, \psi'_p\right)$ introduced in Section 3.1.

Proposition 3.20. If $\underline{\mathrm{Kl}}_p\left(n, \psi_p, \psi'_p\right)$ is well-defined, then $\underline{\mathrm{Kl}}_p\left(n, \psi_p, \psi'_p\right) = \mathrm{Kl}_p\left(n, \psi_p, \psi'_p\right)$.

Proof. Trivial. □

Let $G = \mathrm{Sp}(4, \mathbb{Q}_p)$, and $\psi = \psi_{m_1,m_2}$, $\psi' = \psi_{n_1,n_2}$. We make Proposition 3.19 explicit, and list the conditions for Sp(4) Kloosterman sums $\underline{\mathrm{Kl}}_p(n_{w,r,s}, \psi, \psi')$ to be well-defined.

(i) If $w = \mathrm{id}$, then $\underline{\mathrm{Kl}}_p(n_{w,0,0}, \psi, \psi')$ is well-defined if $m_1 = n_1$, $m_2 = n_2$;

(ii) if $w = s_\alpha$, then $\underline{\mathrm{Kl}}_p(n_{w,r,0}, \psi, \psi')$ is well-defined if $m_2 = n_2 = 0$;

(iii) if $w = s_\beta$, then $\underline{\mathrm{Kl}}_p(n_{w,0,s}, \psi, \psi')$ is well-defined if $m_1 = n_1 = 0$;

(iv) if $w = s_\alpha s_\beta$, then $\underline{\mathrm{Kl}}_p(n_{w,r,s}, \psi, \psi')$ is well-defined if $m_2 = n_1 = 0$;

(v) if $w = s_\beta s_\alpha$, then $\underline{\mathrm{Kl}}_p(n_{w,r,s}, \psi, \psi')$ is well-defined if $m_1 = n_2 = 0$;

(vi) if $w = s_\alpha s_\beta s_\alpha$, then $\underline{\mathrm{Kl}}_p(n_{w,r,s}, \psi, \psi')$ is well-defined if $n_2 = m_2 p^{2r-2s}$;

(vii) if $w = s_\beta s_\alpha s_\beta$, then $\underline{\mathrm{Kl}}_p(n_{w,r,s}, \psi, \psi')$ is well-defined if $n_1 = m_1 p^{s-2r}$;

(viii) if $w = w_0$, then $\underline{\mathrm{Kl}}_p(n_{w,r,s}, \psi, \psi')$ is always well-defined.

Remark. From the list above, we see that not all Kloosterman sums $\mathrm{Kl}_p(n, \psi, \psi')$ correspond to a well-defined sum $\underline{\mathrm{Kl}}_p(n, \psi, \psi')$.

The Fourier coefficients $P_{\psi,\psi'}(g)$ can be evaluated using the following theorem of Friedberg:

Theorem 3.21. [Fri87, Theorem A] The Fourier coefficient $P_{\psi,\psi'}(g)$ of $\mathrm{Sp}(2r)$ Poincaré series is given by

$$P_{\psi,\psi'}(g) = \sum_{\substack{n \in N(\mathbb{Q}) \\ w(n)=w}} \underline{\mathrm{Kl}}\left(n, \psi, \psi'\right) \int_{U_w(\mathbb{R})} \mathcal{F}_\psi\left(n u_1 y\right) \overline{\psi'}\left(u_1\right) du_1.$$

Remark. In [Fri87], the statement concerns $\mathrm{GL}(r)$ Poincaré series, but the proof also works for $\mathrm{Sp}(2r)$ Poincaré series.

Proof. We start with

$$\begin{aligned} P_\psi^{\psi'}(g) &= \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} P_\psi\left(ug\right) \overline{\psi'}(u) du = \sum_{\gamma \in P_0 \cap \Gamma \backslash \Gamma} \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \mathcal{F}_\psi\left(\gamma u g\right) \overline{\psi'}(u) du \\ &= \sum_{w \in W} \sum_{\gamma \in R_w} \sum_{\ell \in \Gamma_w} \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \mathcal{F}_\psi\left(\gamma \ell u g\right) \overline{\psi'}(u) du. \end{aligned}$$

For $\gamma \in R_w$, write $\gamma = b_1 d w b_2$, with $b_1, b_2 \in U$, $d \in D$. Write $u = u_1 u_2$, with $u_1 \in U_w$, and $u_2 \in \overline{U}_w$. Let $\mathfrak{F}$ be a fundamental domain of $\overline{U}_w(\mathbb{Z})\backslash\overline{U}_w(\mathbb{R})$. We then have

$$P_\psi^{\psi'}(g) = \sum_{w\in W} \sum_{\substack{\gamma\in R_w \\ \gamma=b_1dwb_2}} \sum_{\ell\in\Gamma_w} \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \mathcal{F}_\psi\left(b_1 d w b_2 \ell u_1 u_2 y\right) \overline{\psi'}(u_1u_2) du_1 du_2$$
$$= \sum_{w\in W} \sum_{\substack{\gamma\in R_w \\ \gamma=b_1dwb_2}} \psi\left(b_1\right) \int_{U_w(\mathbb{R})} \int_{\mathfrak{F}} \mathcal{F}_\psi\left(dwb_2u_1u_2y\right) \overline{\psi'}\left(u_1u_2\right) du_1 du_2.$$

Write $b_2 = b_2' b_2''$, with $b_2' \in U_w$, $b_2'' \in \overline{U}_w$. After change of variables $b_2'' u_1 \mapsto u_1$, $u_2 \mapsto u_1^{-1} u_2 u_1$, $b_2' u_2 \mapsto u_2$, we have

$$P_\psi^{\psi'}(g) = \sum_{w\in W} \sum_{\substack{\gamma\in R_w \\ \gamma=b_1dwb_2}} \psi\left(b_1\right) \psi'\left(b_2\right) \int_{U_w(\mathbb{R})} \int_{b_2'\mathfrak{F}} \mathcal{F}_\psi\left(dwu_2u_1y\right) \overline{\psi'}\left(u_2u_1\right) du_1 du_2.$$

Since $\mathcal{F}_\psi\left(dug\right) = \psi\left(dud^{-1}\right) \mathcal{F}_\psi\left(dg\right)$, and $wu_2u_1y = \left(wu_2w^{-1}\right)\left(wu_1y\right)$,

$$P_\psi^{\psi'}(g) = \sum_{w\in W} \sum_{\substack{\gamma\in R_w \\ \gamma=b_1dwb_2}} \psi\left(b_1\right) \psi'\left(b_2\right) \int_{U_w(\mathbb{R})} \mathcal{F}_\psi\left(dwu_1y\right) \overline{\psi'}\left(u_1\right) du_1 \int_{b_2'\mathfrak{F}} \psi\left(dwu_2w^{-1}d^{-1}\right) \overline{\psi'}\left(u_2\right) du_2.$$

Observe that

$$\int_{b_2'\mathfrak{F}} \psi\left(dwu_2w^{-1}d^{-1}\right) \overline{\psi'}\left(u_2\right) du_2 = \begin{cases} 1 & \text{if } \psi'\left(u_2\right) = \psi\left(dwu_2w^{-1}d^{-1}\right) \ \forall u_2 \in \overline{U}_w, \\ 0 & \text{otherwise.} \end{cases}$$

The condition for this integral to be nonzero is exactly the same as the condition for the Kloosterman sum to be well-defined. Hence

$$P_\psi^{\psi'}(g) = \sum_{w\in W} \sum_{n\in\mathcal{N}_w} \underline{\mathrm{Kl}}\left(n, \psi, \psi'\right) \int_{U_w(\mathbb{R})} \mathcal{F}_\psi\left(nu_1y\right) \overline{\psi'}\left(u_1\right) du_1. \qquad \square$$

3.5.1 Sp(4) Poincaré series

Let P_0 be the standard minimal parabolic subgroup of $G = \mathrm{Sp}(4)$. For $w \in W$, let $G_w = UwDU$, $\Gamma_w = U(\mathbb{Z}) \cap w^{-1} U(\mathbb{Z})^T w$, and R_w be a complete set of coset representatives for $P_0 \cap \Gamma \backslash \Gamma \cap G_w / \Gamma_w$, as in Section 2.2.4. Define

$$\mathcal{N}_w = \left\{ n \in N(\mathbb{R}) \mid \exists \gamma \in R_w \text{ such that } \gamma = b_1 n b_2 \text{ for } b_1, b_2 \in U(\mathbb{R}) \right\}.$$

For $\psi = \psi_{m_1,m_2}$, and $u_1, u_2 \in \mathbb{R}$, we denote the exponential $e\left(m_1u_1 + m_2u_2\right)$ by $\psi\left(u_1, u_2\right)$.

Now we compute the Fourier coefficients $P_{\psi,\psi'}(g)$ for $P_\psi(g)$, making use of Theorem 3.21. Since for $g = uy \in G/K$ we have $\mathcal{F}_\psi(g) = \psi(u)F(y)$, it suffices to just compute $P_{\psi,\psi'}(y)$, for $y = \mathrm{diag}\left(y_1, y_2, y_1^{-1}, y_2^{-1}\right) \in T(\mathbb{R}^+)$.

(i) For $w = \mathrm{id}$, we have $n = I$, and the integral just gives $F\left(y_1, y_2\right)$. Hence

$${}_{\mathrm{id}}P_{\psi,\psi'} = \underline{\mathrm{Kl}}\left(I, \psi, \psi'\right) F\left(y_1, y_2\right).$$

(ii) For $w = s_\alpha$, we have

$$\mathcal{N}_{s_\alpha} = \left\{ \begin{pmatrix} & 1/v_4 & & \\ -v_4 & & & \\ & & & v_4 \\ & & -1/v_4 & \end{pmatrix} \middle| \, v_4 \geq 1 \right\}, \quad U_{s_\alpha}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & u_1 & & \\ & 1 & & \\ & & 1 & \\ & & -u_1 & 1 \end{pmatrix} \middle| \, u_1 \in \mathbb{R} \right\}.$$

Meanwhile, through Iwasawa decomposition, we obtain that

$$
\begin{aligned}
&\int_{U_{s_\alpha}(\mathbb{R})} \mathcal{F}_\psi\left(nu_1 y\right) \overline{\psi'}\left(u_1\right) du_1 \\
&= \int_{\mathbb{R}} \psi\left(-\frac{u_1 y_2^2}{v_4^2\left(u_1^2 y_2^2+y_1^2\right)}, 0\right) F\left(\frac{y_1 y_2}{v_4\sqrt{u_1^2 y_2^2+y_1^2}}, v_4\sqrt{u_1^2 y_2^2+y_1^2}\right) \overline{\psi'}\left(u_1, 0\right) du_1 .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
{}_{s_\alpha}P_{\psi,\psi'}(g) = \sum_{v_4 \geq 1} \underline{\mathrm{Kl}}\left(\begin{pmatrix} & 1/v_4 & & \\ -v_4 & & & \\ & & & v_4 \\ & & -1/v_4 & \end{pmatrix}, \psi, \psi'\right) \int_{\mathbb{R}} \psi\left(-\frac{u_1 y_2^2}{v_4^2\left(u_1^2 y_2^2+y_1^2\right)}, 0\right) \\
F\left(\frac{y_1 y_2}{v_4\sqrt{u_1^2 y_2^2+y_1^2}}, v_4\sqrt{u_1^2 y_2^2+y_1^2}\right) \overline{\psi'}\left(u_1, 0\right) du_1 .
\end{aligned}
$$

(iii) For $w = s_\beta$, we have

$$
\mathcal{N}_{s_\beta} = \left\{ \begin{pmatrix} 1 & & & \\ & & & 1/v_{23} \\ & & 1 & \\ & -v_{23} & & \end{pmatrix} \middle| \; v_{23} \geq 1 \right\}, \quad U_{s_\beta}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & u_5 \\ & & 1 & \\ & & & 1 \end{pmatrix} \middle| \; u_1 \in \mathbb{R} \right\}.
$$

Hence,

$$
\begin{aligned}
{}_{s_\beta}P_{\psi,\psi'}(g) = \sum_{v_{23} \geq 1} \underline{\mathrm{Kl}}\left(\begin{pmatrix} 1 & & & \\ & & & 1/v_{23} \\ & & 1 & \\ & -v_{23} & & \end{pmatrix}, \psi, \psi'\right) \int_{\mathbb{R}} \psi\left(0, -\frac{u_5}{v_{23}^2(y_2^4+u_5^2)}\right) \\
F\left(y_1, \frac{y_2}{v_{23}\sqrt{y_2^4+u_5^2}}\right) \overline{\psi'}\left(0, u_5\right) du_5 .
\end{aligned}
$$

(iv) For $w = s_\alpha s_\beta$, we have

$$
\mathcal{N}_{s_\alpha s_\beta} = \left\{ \begin{pmatrix} & & & -1/v_2 \\ v_2/v_{23} & & & \\ & v_2 & & \\ & & v_{23}/v_2 & \end{pmatrix} \middle| \; \begin{matrix} v_2, v_{23} \geq 1 \\ v_{23} \mid v_2 \end{matrix} \right\}, \quad U_{s_\alpha s_\beta}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & & & u_4 \\ & 1 & u_4 & u_5 \\ & & 1 & \\ & & & 1 \end{pmatrix} \middle| \; u_4, u_5 \in \mathbb{R} \right\}.
$$

Hence,

$$
\begin{aligned}
{}_{s_\alpha s_\beta}P_{\psi,\psi'}(g) = \sum_{v_2 \geq 1} \sum_{v_{23} \mid v_2} \underline{\mathrm{Kl}}\left(\begin{pmatrix} & & & -1/v_2 \\ v_2/v_{23} & & & \\ & v_2 & & \\ & & v_{23}/v_2 & \end{pmatrix}, \psi, \psi'\right) \\
\int_{\mathbb{R}}\int_{\mathbb{R}} \psi\left(-\frac{u_4 v_{23} y_2^2}{v_2^2 \eta}, -\frac{u_4^2 u_5 v_2^2}{v_{23}^2\left(y_2^4+u_5^2\right)}\right) F\left(\frac{y_1 y_2}{v_2\sqrt{\eta}}, \frac{v_2}{v_{23}}\sqrt{\frac{\eta}{y_2^4+u_5^2}}\right) \overline{\psi'}\left(0, u_5\right) du_4 du_5,
\end{aligned}
$$

where $\eta = y_1^2 y_2^4 + u_5^2 y_1^2 + u_4^2 y_2^2$.

(v) For $w = s_\beta s_\alpha$, we have

$$
\mathcal{N}_{s_\beta s_\alpha} = \left\{ \begin{pmatrix} & 1/v_4 & & \\ & & v_4/v_{14} & \\ & & & v_4 \\ -v_{14}/v_4 & & & \end{pmatrix} \middle| \; \begin{matrix} v_4, v_{14} \geq 1 \\ v_4^2 \mid v_{14} \end{matrix} \right\}, \quad U_{s_\beta s_\alpha}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & u_1 & u_2 & \\ & 1 & & \\ & & 1 & \\ & & -u_1 & 1 \end{pmatrix} \middle| \; u_1, u_2 \in \mathbb{R} \right\}.
$$

Hence,

$$
{}_{s_\beta s_\alpha}P_{\psi,\psi'}(g) = \sum_{v_{14}\geq 1}\sum_{v_4^2|v_{14}} \underline{\mathrm{Kl}}\left(\begin{pmatrix} & 1/v_4 & & \\ & & v_4/v_{14} & \\ & & & v_4 \\ -v_{14}/v_4 & & & \end{pmatrix}, \psi, \psi'\right)
$$

$$
\int_{\mathbb{R}}\int_{\mathbb{R}} \psi\left(-\frac{u_1u_2v_{14}y_2^2}{v_4^2\left(u_1^2y_2^2+y_1^2\right)}, -\frac{u_2v_4^2}{v_{14}^2\eta}\right) F\left(\frac{y_1y_2}{v_4\sqrt{u_1^2y_2^2+y_1^2}}, \frac{v_4}{v_{14}}\sqrt{\frac{u_1^2y_2^2+y_1^2}{\eta}}\right)\overline{\psi'}\left(u_1,0\right)du_1du_2,
$$

where $\eta = \left(u_1^2y_2^2+y_1^2\right)^2 + u_2^2$.

(vi) For $w = s_\alpha s_\beta s_\alpha$, we have

$$
\mathcal{N}_{s_\alpha s_\beta s_\alpha} = \left\{\begin{pmatrix} & & -1/v_1 & \\ & v_1/v_{14} & & \\ v_1 & & & \\ & & & v_{14}/v_1 \end{pmatrix} \,\middle|\, \begin{matrix} v_1, v_{14} \geq 1 \\ v_{14} \mid v_1^2 \end{matrix}\right\}, \quad U_{s_\alpha s_\beta s_\alpha}(\mathbb{R}) = \left\{\begin{pmatrix} 1 & u_1 & u_2 & u_4 \\ & 1 & u_4 & \\ & & 1 & \\ & & -u_1 & 1 \end{pmatrix} \,\middle|\, u_i \in \mathbb{R}\right\}.
$$

Hence,

$$
{}_{s_\alpha s_\beta s_\alpha}P_{\psi,\psi'}(g) = \sum_{v_1\geq 1}\sum_{v_{14}|v_1^2} \underline{\mathrm{Kl}}\left(\begin{pmatrix} & & -1/v_1 & \\ & v_1/v_{14} & & \\ v_1 & & & \\ & & & v_{14}/v_1 \end{pmatrix}, \psi, \psi'\right)
$$

$$
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi\left(\frac{v_{14}(u_1u_2y_2^2-u_4y_1^2)}{v_1^2\eta_2}, \frac{v_1^2\eta_3}{v_{14}^2\eta_1}\right) F\left(\frac{y_1y_2}{v_1\sqrt{\eta_2}}, \frac{v_1}{v_{14}}\sqrt{\frac{\eta_2}{\eta_1}}\right)\overline{\psi'}\left(u_1,0\right)du_1du_2du_4,
$$

where

$$
\begin{aligned}
\eta_1 &= \left(u_1^2y_2^2+y_1^2\right)^2 + \left(u_1u_4+u_2\right)^2, \\
\eta_2 &= u_1^2y_1^2y_2^4 + y_1^4y_2^2 + u_4^2y_1^2 + u_2^2y_2^2, \\
\eta_3 &= u_1^2u_2y_2^4 - u_1^3u_4y_2^4 - 2u_1u_4y_1^2y_2^2 - u_1u_4^3 - u_2u_4^2.
\end{aligned}
$$

(vii) For $w = s_\beta s_\alpha s_\beta$, we have

$$
\mathcal{N}_{s_\beta s_\alpha s_\beta} = \left\{\begin{pmatrix} & & & -1/v_2 \\ & & v_2/v_{12} & \\ & v_2 & & \\ -v_{12}/v_2 & & & \end{pmatrix} \,\middle|\, \begin{matrix} v_{12}, v_2 \geq 1 \\ v_2 \mid v_{12} \end{matrix}\right\}, \quad U_{s_\beta s_\alpha s_\beta}(\mathbb{R}) = \left\{\begin{pmatrix} 1 & & u_2 & u_4 \\ & 1 & u_4 & u_5 \\ & & 1 & \\ & & & 1 \end{pmatrix} \,\middle|\, u_i \in \mathbb{R}\right\}.
$$

Hence,

$$
{}_{s_\beta s_\alpha s_\beta}P_{\psi,\psi'}(g) = \sum_{v_{12}\geq 1}\sum_{v_2|v_{12}} \underline{\mathrm{Kl}}\left(\begin{pmatrix} & & & -1/v_2 \\ & & v_2/v_{12} & \\ & v_2 & & \\ -v_{12}/v_2 & & & \end{pmatrix}, \psi, \psi'\right)
$$

$$
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \psi\left(\frac{v_{12}\eta_3}{v_2^2\eta_1}, \frac{v_2^2\eta_4}{v_{12}^2\eta_2}\right) F\left(\frac{y_1y_2}{v_2\sqrt{\eta_1}}, \frac{v_2}{v_{12}}\sqrt{\frac{\eta_1}{\eta_2}}\right)\overline{\psi'}\left(0,u_5\right)du_2du_4du_5,
$$

where

$$
\begin{aligned}
\eta_1 &= y_1^2y_2^4 + u_5^2y_1^2 + u_4^2y_2^2, \\
\eta_2 &= y_1^4y_2^4 + u_5^2y_1^4 + 2u_4^2y_1^2y_2^2 + u_2^2y_2^4 + u_4^4 - 2u_2u_4^2u_5 + u_2^2u_5^2, \\
\eta_3 &= u_4u_5y_1^2 + u_2u_4y_2^2, \\
\eta_4 &= u_4^2u_5 - u_2y_2^4 - u_2u_5^2.
\end{aligned}
$$

(viii) For $w = s_\alpha s_\beta s_\alpha s_\beta$, we have

$$\mathcal{N}_{s_\alpha s_\beta s_\alpha s_\beta} = \left\{ \begin{pmatrix} & & -1/v_1 & \\ & & & -v_1/v_{12} \\ v_1 & & & \\ & v_{12}/v_1 & & \end{pmatrix} \middle| \, v_1, v_{12} \geq 1 \right\}, \quad U_{s_\alpha s_\beta s_\alpha s_\beta}(\mathbb{R}) = U(\mathbb{R}).$$

Hence,

$$\begin{aligned} {}_{s_\alpha s_\beta s_\alpha s_\beta} P_{\psi,\psi'}(g) = \sum_{v_1 \geq 1} \sum_{v_{12} \geq 1} \underline{\mathrm{Kl}} \left(\begin{pmatrix} & & -1/v_1 & \\ & & & -v_1/v_{12} \\ v_1 & & & \\ & v_{12}/v_1 & & \end{pmatrix}, \psi, \psi' \right) \\ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \psi \left(-\frac{v_{12}\eta_3}{v_1^2 \eta_2}, \frac{v_1^2 \eta_4}{v_{12}^2 \eta_1} \right) F \left(\frac{y_1 y_2}{v_1 \sqrt{\eta_2}}, \frac{v_1}{v_{12}} \sqrt{\frac{\eta_2}{\eta_1}} \right) \overline{\psi'} \left(u_1, u_5 \right) du_1 du_2 du_4 du_5, \end{aligned}$$

where

$$\begin{aligned} \eta_1 &= u_1^2 u_4^2 y_2^4 + y_1^4 y_2^4 - 2u_1 u_2 u_4 y_2^4 + u_1^2 u_4^2 u_5^2 + u_5^2 y_1^4 + 2u_4^2 y_1^2 y_2^2 + u_2^2 y_2^4 + 2u_1 u_4^3 u_5 - 2u_1 u_2 u_4 u_5^2 \\ &\quad + u_4^4 - 2u_2 u_4^2 u_5 + u_2^2 u_5^2 \\ \eta_2 &= u_1^2 y_1^2 y_2^4 + u_1^2 u_5^2 y_1^2 + y_1^4 y_2^2 + 2u_1 u_4 u_5 y_1^2 + u_4^2 y_1^2 + u_2^2 y_2^2, \\ \eta_3 &= u_1 y_1^2 y_2^4 + u_1 u_5^2 y_1^2 + u_4 u_5 y_1^2 + u_2 u_4 y_2^2, \\ \eta_4 &= u_1^3 u_4 y_2^4 - u_1^2 u_2 y_2^4 + u_1^3 u_4 u_5^2 + 2u_1 u_4 y_1^2 y_2^2 + 2u_1^2 u_4^2 u_5 - u_1^2 u_2 u_5^2 - u_5 y_1^4 + u_1 u_4^3 + u_2 u_4^2 - u_2^2 u_5. \end{aligned}$$

Chapter 4

Density theorem for Sp(4)

4.1 Preliminaries

Let $U(\mathbb{R}) \subseteq \mathrm{Sp}(4,\mathbb{R})$ be the standard unipotent subgroup

$$U(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ & 1 & x_{23} & x_{24} \\ & & 1 & \\ & & x_{43} & 1 \end{pmatrix} \middle| \begin{array}{l} x_{ij} \in \mathbb{R}, \\ x_{12} = -x_{43}, \\ x_{14} = x_{23} + x_{12}x_{24} \end{array} \right\}.$$

Remark. In order to better express the arguments in this chapter, the variables are named differently than in Section 2.1.

For $N = (N_1, N_2) \in \mathbb{R}^2$ we define a character $\psi_N : U(\mathbb{R}) \to \mathbb{C}^\times$ by

$$\psi_N(x) = e\left(N_1 x_{12} + N_2 x_{24}\right). \tag{4.1}$$

Note that if $N \in \mathbb{Z}^2$, this defines a character of $U(\mathbb{Z})\backslash U(\mathbb{R})$.

Let $T \subseteq \mathrm{Sp}(4,\mathbb{R})$ be the diagonal torus. The standard minimal parabolic subgroup is given by $P_0 = TU$. We embed $y = (y_1, y_2) \in \mathbb{R}_+^2$ into $T(\mathbb{R})$ via the map

$$\iota(y) = (y_1 y_2^{1/2}, y_2^{1/2}, 1/y_1 y_2^{1/2}, 1/y_2^{1/2}).$$

We denote the image of $\mathbb{R}_+^2$ in $T(\mathbb{R})$ by $T(\mathbb{R}_+)$. An element $g \in \mathrm{Sp}(4,\mathbb{R})$ admits Iwasawa decomposition $g = xyk$, with $x = U(\mathbb{R})$, $y \in T(\mathbb{R}_+)$ and $k \in K$, where $K = \mathrm{SO}(4,\mathbb{R}) \cap \mathrm{Sp}(4,\mathbb{R})$ is the maximal compact subgroup of $\mathrm{Sp}(4,\mathbb{R})$. We denote by $\mathrm{y}(g) = \iota^{-1}(y)$ the Iwasawa y-coordinates of g. For $w \in W$, $y \in \mathbb{R}_+^2$ we write ${}^w y = \mathrm{y}(w\iota(y)^{-1}w^{-1})$.

For $\alpha \in \mathbb{C}^2$, $y \in \mathbb{R}_+^2$, we write $y^\alpha = y_1^{\alpha_1} y_2^{\alpha_2}$. Let $\eta = (2, 3/2)$. We define measures

$$dx = dx_{12}dx_{13}dx_{23}dx_{24}, \quad d^*y = y^{-2\eta}\frac{dy_1}{y_1}\frac{dy_2}{y_2}$$

on $U(\mathbb{R})$ and $\mathbb{R}_+^2$ respectively. We denote the pushforward of d^*y to $T(\mathbb{R}_+)$ by ι also by d^*y. Then dx is the Haar measure on $U(\mathbb{R})$, and dxd^*y is a left $\mathrm{Sp}(4,\mathbb{R})$-invariant measure on $\mathrm{Sp}(4,\mathbb{R})/K$.

We define another embedding of $\mathbb{R}_+^2$ into $T(\mathbb{R}_+)$ by

$$c = (c_1, c_2) \mapsto c^* = \mathrm{diag}(1/c_1, c_1/c_2, c_1, c_2/c_1).$$

A simple calculation shows that $\mathrm{y}(c^*)^\eta = (c_1c_2)^{-1}$.

Let $\pi = \bigotimes \pi_v$ be a globally generic irreducible spherical representation of GSp(4) with trivial central character. Using notations in [RS07], π_v is induced from the character $\chi_1 \times \chi_2 \rtimes \sigma$, given by

$$\operatorname{diag}\left(t_1, t_2, t_1^{-1}v, t_2^{-1}v\right) \mapsto \chi_1(t_1)\chi_2(t_2)\sigma(v).$$

As π_v is right K_v-invariant, we may assume that χ_1, χ_2, σ are unramified, and we may write $\chi_1 = |\cdot|^{\alpha_1}$, $\chi_2 = |\cdot|^{\alpha_2}$ and $\sigma = |\cdot|^{\beta}$. As π_v has trivial central character, we have $\alpha_1 + \alpha_2 + 2\beta = 0$. So the L-parameter is given by

$$(\chi_1\chi_2\sigma, \chi_1\sigma, \chi_2\sigma, \sigma) = \left(\frac{\alpha_1+\alpha_2}{2}, \frac{\alpha_1-\alpha_2}{2}, \frac{-\alpha_1+\alpha_2}{2}, \frac{-\alpha_1-\alpha_2}{2}\right).$$

We then take

$$\mu_\pi(v) = \left(\frac{\alpha_1+\alpha_2}{2}, \frac{\alpha_1-\alpha_2}{2}\right), \tag{4.2}$$

so the L-parameter becomes $(\mu_\pi(v,1), \mu_\pi(v,2), -\mu_\pi(v,1), -\mu_\pi(v,2))$. When π_v is lifted to a self-dual representation of GL(4), this is precisely the natural Langlands parameter of the lift. In terms of simple roots coordinates $(\nu_1, \nu_2) \in \mathfrak{a}^*_{\mathbb{C}}$, introduced in Section 2.1, we have $\alpha_1 = \nu_1$, $\alpha_2 = 2\nu_2 - \nu_1$.

4.2 Auxiliary results

In this section, we prove several technical results about Sp(4), which will be used in the proofs in later sections.

Since the Iwasawa decomposition $\mathrm{Sp}(4,\mathbb{R}) = U(\mathbb{R})T(\mathbb{R}_+)K$ is actually the Gram-Schmidt orthogonalisation of rows, we can compute $\mathrm{y}(g)$ explicitly. Let Δ_1 be the norm of the third row of g, and Δ_2 be the area of the parallelogram spanned by the bottom two rows of g. Then we have

$$g \equiv \begin{pmatrix} 1/\Delta_1 & * & * & * \\ & \Delta_1/\Delta_2 & * & * \\ & & \Delta_1 & \\ & & * & \Delta_2/\Delta_1 \end{pmatrix} \pmod{K}.$$

In particular, we have $\mathrm{y}(g) = (\Delta_2/\Delta_1^2, \Delta_1^2/\Delta_2^2)$. Conversely, if $\mathrm{y}(g) = (Y_1, Y_2)$, then $\Delta_1(g) = Y_1^{-1}Y_2^{-1/2}$ and $\Delta_2(g) = Y_1^{-1}Y_2^{-1}$.

Lemma 4.1. Let $w \in W$, $x \in U_w(\mathbb{R})$, and $y, c, B \in \mathbb{R}_+^2$. Write $\mathrm{y}(\iota(B)c^*wx\iota(y)) = Y \in \mathbb{R}_+^2$ and $A = \iota(B)c^*$. Then we have

$$c_1 \ll_{y,Y} B_1B_2^{1/2}, \quad c_2 \ll_{y,Y} B_1B_2,$$

and

$$1 \leq \Delta_1(wx) \ll_{y,Y} \mathrm{y}(A)_1\,\mathrm{y}(A)_2^{1/2}, \quad 1 \leq \Delta_2(wx) \ll_{y,Y} \mathrm{y}(A)_1\,\mathrm{y}(A)_2.$$

Proof. Note that $\Delta_i(wx) \geq 1$ as one of its minors is always 1. For the first statement, we compute

$$\Delta_1(\iota(B)c^*wx\iota(y)) = \frac{c_1}{B_1B_2^{1/2}}\Delta_1(wx\iota(y)), \quad \Delta_2(\iota(B)c^*wx\iota(y)) = \frac{c_2}{B_1B_2}\Delta_2(wx\iota(y)).$$

Then we obtain

$$\begin{aligned} c_1 &\leq c_1 \Delta_1(wx) \ll_y c_1 \Delta_1(wx\iota(y)) = \Delta_1(\iota(B)c^* wx\iota(y)) B_1 B_2^{1/2} \ll_Y B_1 B_2^{1/2}, \\ c_2 &\leq c_2 \Delta_2(wx) \ll_y c_2 \Delta_2(wx\iota(y)) = \Delta_2(\iota(B)c^* wx\iota(y)) B_1 B_2 \ll_Y B_1 B_2. \end{aligned}$$

For the second statement, we observe that

$$c_1^{-1} = \mathrm{y}(c^*)_1 \, \mathrm{y}(c^*)_2^{1/2}, \quad c_2^{-1} = \mathrm{y}(c^*)_1 \, \mathrm{y}(c^*)_2.$$

Hence

$$\begin{aligned} \Delta_1(wx) &\ll_{y,Y} B_1 B_2^{1/2} c_1^{-1} = (B_1 \, \mathrm{y}(c^*)_1)(B_2 \, \mathrm{y}(c^*)_2)^{1/2} = \mathrm{y}(A)_1 \, \mathrm{y}(A)_2^{1/2}, \\ \Delta_2(wx) &\ll_{y,Y} B_1 B_2 c_2^{-1} = (B_1 \, \mathrm{y}(c^*)_1)(B_2 \, \mathrm{y}(c^*)_2) = \mathrm{y}(A)_1 \, \mathrm{y}(A)_2. \end{aligned}$$

□

Lemma 4.2. Let $N \in \mathbb{N}^2$ and $w \in W$. For $x \in U_w(\mathbb{R})$, define $x' = \iota(N) x \iota(N)^{-1}$. Then

$$dx' = ({}^w N)^\eta N^\eta dx.$$

Proof. This is direct computation. For $w = w_0$, we have

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ & 1 & x_{23} & x_{24} \\ & & 1 & \\ & & -x_{12} & 1 \end{pmatrix} \rightsquigarrow x' = \begin{pmatrix} 1 & N_1 x_{12} & N_1^2 N_2 x_{13} & N_1 N_2 x_{14} \\ & 1 & N_1 N_2 x_{23} & N_2 x_{24} \\ & & 1 & \\ & & -N_1 x_{12} & 1 \end{pmatrix}.$$

It follows that $dx' = N_1^4 N_2^3 dx = ({}^{w_0} N)^\eta N^\eta dx$. The proof is similar for other Weyl elements. □

Lemma 4.3. Let $B \in \mathbb{R}_+^2$, and $w = s_\beta s_\alpha s_\beta$. Then

$$\mathrm{vol}\,\{x \in U_w(\mathbb{R}) \mid \Delta_j(wx) \leq B_j, j = 1, 2\} \ll (B_1 B_2)^{1+\varepsilon}$$

for any $\varepsilon > 0$.

Proof. We can assume without loss of generality that $B_j \geq 1$, otherwise we deduce from Lemma 4.1 that the volume is 0. We have

$$x = \begin{pmatrix} 1 & & x_{13} & x_{23} \\ & 1 & x_{23} & x_{24} \\ & & 1 & \\ & & & 1 \end{pmatrix} \in U_w(\mathbb{R}) \rightsquigarrow wx = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & x_{23} & x_{24} \\ -1 & & -x_{13} & -x_{23} \end{pmatrix}.$$

Then we obtain bounds

$$|x_{23}| \leq B_1, \left|x_{13} x_{24} - x_{23}^2\right| \leq B_2.$$

We also have $|x_{13}|, |x_{24}| \leq b := 1 + \max\{B_1, B_2\}$. If $I \subseteq \mathbb{R}$ is any interval of length $|I| \geq 1$, then

$$\mathrm{vol}\left\{(x, y) \in [-b, b]^2 \;\middle|\; xy \in I\right\} \leq \int_{|y| \leq b} \min\left\{\frac{|I|}{|y|}, 2b\right\} dy \leq 4\,|I|\,(1 + \log b).$$

Hence, if $|x_{23}| \leq B_1$ is fixed, the volume of (x_{13}, x_{24}) is $O(B_2 \log b)$. This establishes the bound. □

4.3 Whittaker functions and automorphic forms

In this section, we prove some relations for Jacquet's Whittaker function on Sp$(2n)$. Let $G = \mathrm{Sp}(2n, \mathbb{R})$, and

$$T = \left\{ \begin{pmatrix} * & & & & & \\ & * & & & & \\ & & * & & & \\ & & & \ddots & & \\ & & & & * & \\ & & & & & * \end{pmatrix} \right\} \subseteq G, \quad U = \left\{ \begin{pmatrix} 1 & \cdots & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & 1 & * & \cdots & * \\ & & & 1 & & \\ & & & \vdots & \ddots & \\ & & & * & \cdots & 1 \end{pmatrix} \right\} \subseteq G,$$

the standard torus and the standard unipotent subgroup respectively. For $m = (m_1, \cdots, m_n) \in \mathbb{Z}^n$, we define a character $\psi_m : U(\mathbb{Z})\backslash U(\mathbb{R}) \to \mathbb{C}^\times$ by

$$\psi_m \begin{pmatrix} 1 & x_1 & \cdots & * & * & \cdots & \cdots & * \\ & 1 & \ddots & \vdots & \vdots & & & \vdots \\ & & \ddots & x_{n-1} & \vdots & & & * \\ & & & 1 & * & \cdots & * & x_n \\ & & & & 1 & & & \\ & & & & -x_1 & 1 & & \\ & & & & \vdots & \ddots & \ddots & \\ & & & & * & \cdots & -x_{n-1} & 1 \end{pmatrix} = \prod_{i=1}^{n} e\left(m_i x_i\right). \tag{4.3}$$

It is easy to see that all characters of $U(\mathbb{Z})\backslash U(\mathbb{R})$ are of this form. The character ψ_m is called non-degenerate if $m_1 \cdots m_n \neq 0$.

Let K be the maximal compact subgroup of G, and $\mathfrak{a}$ the real Lie algebra of $T(\mathbb{R})$. Define a homomorphism $H_0 : G \to \mathfrak{a}$, which takes $g \in G$ to $H_0(g)$, for $g \in U \exp(H_0(g))K$. Let $\mathfrak{a}_{\mathbb{C}}$ be the complexification of $\mathfrak{a}$, and $\mathfrak{a}_{\mathbb{C}}^*$ the dual of $\mathfrak{a}_{\mathbb{C}}$. Let $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, and $\psi : U(\mathbb{Z})\backslash U(\mathbb{R}) \to \mathbb{C}^\times$ a non-degenerate character. Then the Jacquet-Whittaker function associated to ψ is given by

$$W(g, \nu, \psi) = \int_{U(\mathbb{R})} I_\nu(w_0 u g) \overline{\psi}(u) du,$$

where $I_\nu(g) = \exp \langle \nu + \rho, H_0(g) \rangle$, ρ is the half-sum of positive roots, and $w_0 \in W = W(T, G)$ is the long element of the Weyl group.

We rephrase more explicitly. Let $g \in G/K$. By Iwasawa decomposition, we may assume $g = uy$, with $u \in U(\mathbb{R})$, and $t \in T(\mathbb{R}^+)$. Let

$$y = \mathrm{diag}\left(y_1, \cdots, y_n, y_1^{-1}, \cdots, y_n^{-1}\right) \in T(\mathbb{R}^+).$$

A set of simple roots of $W = W(T, G)$ is given by $\Delta = \{\alpha_1, \cdots, \alpha_n\}$, where $\alpha_i y = y_i y_{i+1}^{-1}$ for $1 \le i \le n-1$, and $\alpha_n y = y_n^2$. Then ρ is given by

$$\rho = \sum_{i=1}^{n} r_i \alpha_i, \quad \text{where } r_i = \begin{cases} \frac{(2n-i+1)i}{2} & \text{for } 1 \le i \le n-1, \\ \frac{n(n+1)}{4} & \text{for } i = n. \end{cases}$$

For $\nu = (\nu_1, \cdots, \nu_n) \in \mathfrak{a}_{\mathbb{C}}^*$, we have

$$I_\nu(g) = y_1^{\nu_1 + n} y_2^{\nu_2 - \nu_1 + n - 1} \cdots y_{n-1}^{\nu_{n-1} - \nu_{n-2} + 2} y_n^{2\nu_n - \nu_{n-1} + 1}.$$

Recall from Section 3.2 that the Weyl group W can be realised as matrices in $\mathrm{Sp}(2n)$, and W is generated by reflections s_{α_i} for $1 \le i \le n$, represented by the matrices

$$s_{\alpha_i} = \begin{pmatrix} A_i & \\ & A_i \end{pmatrix}, \qquad A_i = \begin{pmatrix} I_{i-1} & & & \\ & & 1 & \\ & 1 & & \\ & & & I_{n-i-1} \end{pmatrix}, \qquad 1 \le i \le n-1,$$

and

$$s_{\alpha_n} = \begin{pmatrix} I_{n-1} & & & \\ & & & 1 \\ & & I_{n-1} & \\ & -1 & & \end{pmatrix}.$$

Lemma 4.4. We have

$$W(g, \nu, \psi_m) = c_{\nu,m} W\left(Mg, \nu, \psi_{1,\cdots,1,\frac{m_n}{|m_n|}}\right),$$

where $c_{\nu,m} = \prod_{i=1}^{n} |m_i|^{\nu_i - r_i}$, and

$$M = \begin{pmatrix} M_0 & \\ & M_0^{-1} \end{pmatrix}, \quad M_0 = |m_n|^{1/2} \begin{pmatrix} m_1 \cdots m_{n-1} & & & & \\ & \ddots & & & \\ & & m_{n-2} m_{n-1} & & \\ & & & m_{n-1} & \\ & & & & 1 \end{pmatrix}.$$

Proof. We expand

$$W\left(Mg, \nu, \psi_{1,\cdots,1,\frac{m_n}{|m_n|}}\right) = \int_{U(\mathbb{R})} I_\nu (w_0 \eta M g)\, e\left(-x_1 - \cdots - x_{n-1} - \frac{m_n}{|m_n|} x_n\right) d\eta,$$

where x_i is as in (4.3). After the change of variables

$$x_1 \mapsto m_1 x_1, \quad \cdots, \quad x_{n-1} \mapsto m_{n-1} x_{n-1}, \quad x_n \mapsto |m_n|\, x_n,$$

we obtain

$$\begin{aligned}
&\prod_{i=1}^{n} |m_i|^{2r_i} \int_{U(\mathbb{R})} I_\nu (w_0 M \eta g)\, e(-m_1 x_1 - \cdots - m_n x_n)\, d\eta \\
=&\prod_{i=1}^{n} |m_i|^{2r_i} \int_{U(\mathbb{R})} I_\nu (w_0 M w_0 \cdot w_0 \eta g)\, e(-m_1 x_1 - \cdots - m_n x_n)\, d\eta \\
=&\prod_{i=1}^{n} |m_i|^{-\nu_i + r_i} \int_{U(\mathbb{R})} I_\nu (w_0 \eta g)\, e(-m_1 x_1 - \cdots - m_n x_n)\, d\eta \\
=&\prod_{i=1}^{n} |m_i|^{-\nu_i + r_i} W(g, \nu, \psi_m).
\end{aligned}$$

□

It is well-known that $W(g, \nu, \psi)$ satisfies a functional equation. To state the functional equation, we introduce some notations. Let $\tau = (\tau_1, \cdots, \tau_n)$ be given by

$$\tau_i = \begin{cases} \frac{i(i-1)}{2} - in & \text{if } i \ne n, \\ -\frac{n(n+1)}{4} & \text{if } i = n. \end{cases}$$

Then for $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{C}^n$, we have

$$I_{\nu+\tau}(y) = y_1^{\nu_1} y_2^{\nu_2-\nu_1} \cdots y_{n-1}^{\nu_{n-1}-\nu_{n_2}} y_n^{2\nu_n-\nu_n}.$$

For $w \in W$, we define $w\nu = (\nu_1', \cdots, \nu_n') \in \mathbb{C}^n$ by

$$I_{\nu+\tau}(y) = I_{w\nu+\tau}(wy).$$

It is more convenient to consider a renormalisation of $W(g, \nu, \psi)$. For $\nu = (\nu_1, \cdots, \nu_n) \in \mathfrak{a}_{\mathbb{C}}^*$, we write $\hat{\nu} = \sum_{i=1}^{n} \nu_i$. Let

$$W^*(g, \nu, \psi) = W(g, \nu, \psi) \pi^{-\hat{\nu}-n} \prod_{i=1}^{n} \Gamma\left(\frac{1+e_i}{2}\right) \prod_{i<j\le n} \Gamma\left(\frac{1+e_i-e_j}{2}\right) \Gamma\left(\frac{1+e_i+e_j}{2}\right),$$

where

$$e_i = \begin{cases} \nu_1 & \text{if } i = 1, \\ \nu_i - \nu_{i-1} & \text{if } 2 \le i \le n-1, \\ 2\nu_n - \nu_{n-1} & \text{if } i = n. \end{cases}$$

Then we have the following functional equation (cf. [Gol06, Theorem 5.9.8], where an analogous statement for $\mathrm{GL}(n)$ Whittaker functions is given).

Theorem 4.5. The equation

$$W^*(g, \nu, \psi) = W^*(g, w\nu, \psi)$$

holds for all $w \in W$.

Proof. It suffices to prove the statement for $w = s_{\alpha_i}$ for $1 \le i \le n$.

(i) Let $1 \le i \le n-1$. Let

$$N_i := \left\{ \begin{pmatrix} N_{i,0} & \\ & (N_{i,0}^{-1})^T \end{pmatrix} \in U(\mathbb{R}) \;\middle|\; N_{i,0} = \begin{pmatrix} I_{i-1} & & & \\ & 1 & * & \\ & & 1 & \\ & & & I_{n-i-1} \end{pmatrix} \right\} \subseteq U(\mathbb{R}),$$

and $N_i' := U(\mathbb{R})/N_i$. We may rewrite

$$W(g, \nu, \psi) = \int_{N_i'} \int_{N_i} I_\nu(s_{\alpha_i} n_i w_i n_i' g) \overline{\psi}(n_i) dn_i \overline{\psi}(n_i') dn_i', \tag{4.4}$$

where $w_i := s_{\alpha_i}^{-1} w_0$. Consider the Iwasawa decomposition $w_i n_i' g = hk$, with $k \in K$, and $h \in U(\mathbb{R}) T(\mathbb{R}^+)$. We further decompose $h = h_0 h'$, with

$$h_0 = \begin{pmatrix} H_0 & \\ & (H_0^{-1})^T \end{pmatrix}, \quad H_0 = \begin{pmatrix} I_{i-1} & & & \\ & y_i/y_{i+1} & x & \\ & & 1 & \\ & & & I_{n-i-1} \end{pmatrix},$$

$$h' = \begin{pmatrix} \ddots & * & * & * & \cdots & * \\ & y_{i+1} I_2 & * & \vdots & \ddots & \vdots \\ & & \ddots & * & \cdots & * \\ & & & \ddots & & \\ & & & * & y_{i+1}^{-1} I_2 & \\ & & & * & * & \ddots \end{pmatrix} \in U(\mathbb{R}) T(\mathbb{R}^+).$$

Now consider the Iwasawa decomposition $s_{\alpha_i} n_i h_0 = n'' a'' k''$, with $n'' \in U(\mathbb{R})$, $a'' \in T(\mathbb{R}^+)$ and $k'' \in K$. Then we have

$$I_\nu(s_{\alpha_i} n_i w_i n_i' g) = I_\nu(n'' a'' k'' h') = I_\nu(n'' a'' h' k'') = I_\nu(a'') I_\nu(h').$$

We compute

$$a'' = \begin{pmatrix} a_0'' & \\ & (a_0''^{-1})^T \end{pmatrix}, \quad a_0'' = \begin{pmatrix} I_{i-1} & & & \\ & \frac{y_i}{y_{i+1}\sqrt{(y_i/y_{i+1})^2+(x+n_i)^2}} & & \\ & & \sqrt{(y_i/y_{i+1})^2+(x+n_i)^2} & \\ & & & I_{n-i-1} \end{pmatrix}.$$

Hence

$$I_\nu(a'') = \begin{cases} \left(\frac{y_i}{y_{i+1}}\right)^{\nu_i - \nu_{i-1} + n + 1 - i} \left(\left(\frac{y_i}{y_{i+1}}\right)^2 + (x+n_i)^2\right)^{\frac{1}{2}(\nu_{i-1} - 2\nu_i + \nu_{i+1} - 1)} & \text{if } i \neq n-1, \\ \left(\frac{y_{n-1}}{y_n}\right)^{\nu_{n-1} - \nu_{n-2} + 2} \left(\left(\frac{y_{n-1}}{y_n}\right)^2 + (x+n_i)^2\right)^{\frac{1}{2}(\nu_{n-2} - 2\nu_{n-1} + 2\nu_n - 1)} & \text{if } i = n-1. \end{cases}$$

Write

$$\nu_0 := \begin{cases} \frac{1}{2}(-\nu_{i-1} + 2\nu_i - \nu_{i+1}) & \text{if } i \neq n-1, \\ \frac{1}{2}(-\nu_{n-2} + 2\nu_{n-1} - 2\nu_n) & \text{if } i = n-1. \end{cases}$$

Then

$$\begin{aligned} \int_{N_i} I_\nu(s_{\alpha_i} n_i w_i n_i' g) \overline{\psi}(n_i) dn_i &= I_\nu(h') \int_{N_i} I_\nu(a'') \overline{\psi}(n_i) dn_i \\ &= I_\nu(h') \left(\frac{y_i}{y_{i+1}}\right)^{\nu_i - \nu_{i+1} + n + 1 - i - \nu_0} W_2\left(x + \frac{y_i}{y_{i+1}} i, \nu_0, \psi|_{N_i}\right), \end{aligned}$$

where

$$W_2(z, \nu, \psi) = \int_{\mathbb{R}} \left(\frac{y}{(u+x)^2 + y^2}\right)^{\nu + \frac{1}{2}} \psi(-u) du,$$

denotes the classical GL(2) Whittaker function. Through the functional equation for GL(2) Whittaker function

$$W_2(z, -\nu, \psi) = \pi^{-2\nu} \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma\left(-\nu + \frac{1}{2}\right)} W_2(z, \nu, \psi), \tag{4.5}$$

We deduce

$$\int_{N_i} I_\nu(s_{\alpha_i} n_i w_i n_i' g) \overline{\psi}(n_i) dn_i = \pi^{2\nu_0} \frac{\Gamma\left(-\nu_0 + \frac{1}{2}\right)}{\Gamma\left(\nu_0 + \frac{1}{2}\right)} \int_{N_i} I_{s_{\alpha_i}\nu}(s_{\alpha_i} n_i w_i n_i' g) \overline{\psi}(n_i) dn_i.$$

Putting back into (4.4) gives

$$W(g, \nu, \psi) = \pi^{2\nu_0} \frac{\Gamma\left(-\nu_0 + \frac{1}{2}\right)}{\Gamma\left(\nu_0 + \frac{1}{2}\right)} W(g, s_{\alpha_i}\nu, \psi)$$

and

$$W^*(g, \nu, \psi) = W^*(g, s_{\alpha_i}\nu, \psi).$$

(ii) Let $i = n$. The argument is similar. Let

$$N_n := \left\{ \begin{pmatrix} I_{n-1} & & & \\ & 1 & & * \\ & & I_{n-1} & \\ & & & 1 \end{pmatrix} \right\} \subseteq U(\mathbb{R}),$$

and $N'_n := U(\mathbb{R})/N_n$. We write

$$W(g,\nu,\psi) = \int_{N'_n} \int_{N_n} I_\nu(s_{\alpha_n} n_n w_i n'_n g) \overline{\psi}(n_i) dn_n \overline{\psi}(n'_i) dn'_n, \tag{4.6}$$

where $w_n := s_{\alpha_n}^{-1} w_0$. Consider the Iwasawa decomposition $w_n n'_n g = hk$, with $k \in K$, and $h \in U(\mathbb{R})T(\mathbb{R}^+)$. We write $h = h_0 h'$, with

$$h_0 = \begin{pmatrix} I_{n-1} & & & \\ & y_n & & x y_n^{-1} \\ & & I_{n-1} & \\ & & & y_n^{-1} \end{pmatrix}, \qquad h' = \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & 1 & * & \cdots & 0 \\ & & & * & & \\ & & & \vdots & \ddots & \\ & & & * & \cdots & 1 \end{pmatrix} \in U(\mathbb{R})T(\mathbb{R}^+).$$

Let $s_{\alpha_n} n_n h_0 = n'' a'' k''$. Then $I_\nu(s_{\alpha_n} n_n w_n n'_n g) = I_\nu(a'') I_\nu(h')$. We compute

$$I_\nu(a'') = y_n^{2\nu_n - \nu_{n-1} + 1} \left(y_n^4 + (x + n_n)^2 \right)^{-\frac{1}{2}(2\nu_n - \nu_{n-1} + 1)}.$$

Writing $\nu_0 := \frac{1}{2}(2\nu_n - \nu_{n-1})$, we have

$$\int_{N_n} I_\nu(s_{\alpha_n} n_n w_n n'_n g) \overline{\psi}(n_n) dn_n = I_\nu(h') W_2 \left(x + y_n^2 i, \nu_0, \psi|_{N_n} \right).$$

From the functional equation (4.5), again we have

$$W(g,\nu,\psi) = \pi^{2\nu_0} \frac{\Gamma\left(-\nu_0 + \frac{1}{2}\right)}{\Gamma\left(\nu_0 + \frac{1}{2}\right)} W(g, s_{\alpha_n}\nu, \psi)$$

and

$$W^*(g,\nu,\psi) = W^*(g, s_{\alpha_n}\nu, \psi).$$

□

Example 4.6. For Sp(4), the explicit normalisation is given by

$$W^*(g,\nu,\psi) = W(g,\nu,\psi)\pi^{-(\nu_1+\nu_2+2)} \Gamma\left(\frac{1+\nu_1}{2}\right) \Gamma\left(\frac{1+2\nu_2}{2}\right) \Gamma\left(\frac{1+2\nu_1-2\nu_2}{2}\right) \Gamma\left(\frac{1-\nu_1+2\nu_2}{2}\right).$$

The functional equation says that $W^*(g,\nu,\psi)$ is invariant under transformations

$$\begin{array}{ccccccc} (\nu_1,\nu_2) & \xleftrightarrow{s_\alpha} & (2\nu_2-\nu_1,\nu_2) & \xleftrightarrow{s_\beta} & (2\nu_2-\nu_1,\nu_2-\nu_1) & \xleftrightarrow{s_\alpha} & (-\nu_1,\nu_2-\nu_1) \\ \updownarrow{\scriptstyle s_\beta} & & & & & & \updownarrow{\scriptstyle s_\beta} \\ (\nu_1,\nu_1-\nu_2) & \xleftrightarrow{s_\alpha} & (\nu_1-2\nu_2,\nu_1-\nu_2) & \xleftrightarrow{s_\beta} & (\nu_1-2\nu_2,-\nu_2) & \xleftrightarrow{s_\alpha} & (-\nu_1,-\nu_2). \end{array}$$

A Whittaker function is determined by its value on $T(\mathbb{R}^+)$, and the character ψ. If we define

$$W(-,\nu) = W(-,\nu,\psi)|_{T(\mathbb{R}^+)} : T(\mathbb{R}^+) \to \mathbb{C}, \tag{4.7}$$

then for $g = xyk \in G$ with $x \in U(\mathbb{R})$, $y \in T(\mathbb{R}^+)$ and $k \in K$, we have $W(g, \nu, \psi) = \psi(x) W(y, \nu)$.

Now let q be a prime, and

$$\Gamma_0(q) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4, \mathbb{Z}) \,\middle|\, C \equiv 0 \pmod{q} \right\} \subseteq \mathrm{Sp}(4, \mathbb{Z})$$

be the Siegel congruence subgroup of level q. We denote by $\{\varpi\}$ an orthonormal basis of right K-invariant automorphic forms for $\Gamma_0(q)$, cuspidal or Eisenstein series. Then we equip $L^2(\Gamma_0(q)\backslash \mathrm{Sp}(4, \mathbb{R})/K)$ with the standard inner product

$$\langle f, g \rangle = \int_{\Gamma_0(q)\backslash \mathrm{Sp}(4,\mathbb{R})/K} f(xy)\overline{g(xy)} dx d^*y.$$

An integral over the complete spectrum of $L^2(\Gamma_0(q)\backslash \mathrm{Sp}(4, \mathbb{R})/K)$ is denoted by $\int_{(q)} d\varpi$. All the automorphic forms ϖ belong to representations π of level $q' \mid q$, and we assume that $\{\varpi\}$ contains all cuspidal newvectors of level $q' \mid q$. For simplicity in notations, we denote the local archimedean spectral parameter $\mu_\pi(\infty)$ by $\mu = (\mu_1, \mu_2)$.

Let ϖ be an automorphic form for $\Gamma_0(q)$, with spectral parameter μ. We suppose ϖ is generic throughout the section. For $M = (M_1, M_2) \in \mathbb{Z}^2$, the M-th Fourier coefficient of ϖ is given by

$$\varpi_M(g) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varpi(xg)\overline{\psi_M(x)} dx.$$

The Fourier coefficients $\varpi_M(g)$ are actually Whittaker functions. For $g = xyk \in \mathrm{Sp}(4, \mathbb{R})$, we have

$$\varpi_M(g) = \frac{A_\varpi(M)}{M^\eta} \psi_M(x) \cdot W_\mu(\iota(M)y), \tag{4.8}$$

where $A_\varpi(M) \in \mathbb{C}$ is a constant, also called the M-th Fourier coefficient of ϖ, and

$$W_\mu(y) := W\left(y, (\mu_1 + \mu_2, \mu_2)\right).$$

Note the change of coordinates between parameters μ and ν. As in the $\mathrm{GL}(n)$ case, the size of $\sigma_\pi(\infty)$ captures the growth of W_μ near the origin. Precisely, for a function E on $\mathbb{R}_+^2$ and $X \in \mathbb{R}_+^2$, we define

$$E^{(X)}(y_1, y_2) = E(X_1 y_1, X_2 y_2). \tag{4.9}$$

For $\beta \in \mathbb{C}$, let $\mathcal{D}_\beta = -y\partial_y + \beta$. This is a commutative family of differential operators, which correspond to multiplication by $s + \beta$ under Mellin transform. In the proof of Lemma 4.8, we need the following technical lemma, found in [Blo19a].

Lemma 4.7. Let $\alpha \geq 0$, and $\beta \in \mathbb{C}$ such that $\mathrm{Re}\,\beta \leq \alpha$. Let $I = [a, b] \subset (0, 1)$ be an interval with $a < b \leq 2a$, and $w : I \to \mathbb{C}$ a smooth function satisfying

$$|\mathcal{D}_\beta w(y)| \geq c_1 y^{-\alpha}, \quad |\partial_y(\mathcal{D}_\beta w)(y)| \leq c_2 \left|\left|\mathcal{D}_\beta w\right|\right| y^{-1} \tag{4.10}$$

for $y \in I$ and some $c_1, c_2 > 0$, where $||w||$ denotes the sup-norm of w. Then there exists constants $a', b', c_1', c_2' > 0$, depending only on c_1, c_2, α, β (not on a, b) such that $a \leq a' < b \leq b$, and

$$|w(y)| \geq c_1' y^{-\alpha}, \quad \left|w'(y)\right| \leq c_2' \left|\left|w\right|\right| y^{-1} \tag{4.11}$$

for $y \in [a', b']$.

Proof. Let $\tilde{w}(y) = w(y)y^{-\beta}$. Then $\tilde{w}' = -y^{-\beta-1}\mathcal{D}_\beta w$. We deduce that (4.10) implies

$$\left|y\tilde{w}'(y)\right| \geq c_1 y^{-\tilde{\alpha}}, \quad \left|\left|\tilde{w}''\right|\right| \leq \tilde{c}_2 \left|\left|\tilde{w}'\right|\right| y^{-1}$$

for some constant $\tilde{c}_2 \geq c_2$, and $\tilde{\alpha} = \alpha + \operatorname{Re}\beta$. Let $y_0 = \max_{y\in I} |\tilde{w}'(y)|$. Changing $\tilde{w}$ by a fourth root of unity if necessary, we may assume that

$$\operatorname{Re}\tilde{w}'(y_0) \geq \frac{1}{\sqrt{2}} \max\left\{c_1 y_0^{-\alpha-1}, \left|\left|\tilde{w}'\right|\right|\right\}.$$

Meanwhile, the condition $|\tilde{w}''(y)| \leq c_2 \left|\left|\tilde{w}'\right|\right| y^{-1}$ implies that the following inequality

$$\operatorname{Re}\tilde{w}'(y) \geq \frac{1}{2\sqrt{2}} \max\left\{c_1 y_0^{-\tilde{\alpha}-1}, \left|\left|\tilde{w}'\right|\right|\right\} \asymp \operatorname{Re}\tilde{w}'(y_0)$$

holds for $y \in I_0 = [a', b']$, for some $a' < b'$ such that $y_0 \in I_0$. Now we show that $y\,|\tilde{w}'(y)| \ll ||\tilde{w}||_{[a',b']}$ for $y \in [a', b']$. Let $c_3 > 0$ be a sufficiently small constant. We distinguish two cases:

(i) Suppose $\operatorname{Re}\tilde{w}(a') \leq -c_3 y_0 \operatorname{Re}\tilde{w}'(y_0)$. Then $||w||_{[a',b']} \geq c_3 y_0 \operatorname{Re}\tilde{w}'(y_0) \gg y\,|\tilde{w}'(y)|$ for $y \in [a', b']$.

(ii) Suppose $\operatorname{Re}\tilde{w}(a') > -c_3 y_0 \operatorname{Re}\tilde{w}'(y_0)$. When $c_3 > 0$ is sufficiently small, we have $\operatorname{Re}\tilde{w}(b') \gg y_0 \operatorname{Re}\tilde{w}'(y_0)$, and hence $||w||_{[a',b']} \gg y \operatorname{Re}\tilde{w}'(y_0) \gg y\,|\tilde{w}'(y)|$ for $y \in [a', b']$.

From the bound $y\,|\tilde{w}'(y)| \ll ||\tilde{w}||$, it follows immediately that $|\tilde{w}(y)| \gg y^{-\tilde{\alpha}}$ on $[a', b']$. Reverting back to w yields (4.11). □

Lemma 4.8. Assume that $\mu = (\mu_1, \mu_2)$ varies in some compact set Ω, and let $Z \geq 1$. There exists $r \in \mathbb{N}$ and a compact set $S \subseteq \mathbb{R}_+^2$ depending only on Ω (independent of Z), and a finite collection of functions $E_1, \cdots, E_r : \mathbb{R}_+^2 \to \mathbb{R}$ depending on Ω and Z that are uniformly bounded and supported in a compact subset of S such that

$$\sum_{j=1}^{r} \left|\left\langle E_j^{(1,Z)}, W_\mu \right\rangle\right|^2 \gg_\Omega Z^{2\eta_2+2\sigma_\pi(\infty)} = Z^{3+2\sigma_\pi(\infty)}.$$

Proof. The case $Z \ll 1$ is proved in [BBM17, Blo19a]. For each $\mu \in \Omega$, choose an open set $S_\mu \subseteq \mathbb{R}_+^2$ such that $\operatorname{Re} W_\mu(y) \neq 0$ for all $y \in S_\mu$ or $\operatorname{Im} W_\mu(y) \neq 0$ for all $y \in S_\mu$. Now choose open neighbourhoods U_μ about μ such that $\operatorname{Re} W_{\mu^*}(y) \neq 0$ for all $y \in S_\mu$ and $\mu^* \in U_\mu$, or $\operatorname{Im} W_{\mu^*}(y) \neq 0$ for all $y \in S_\mu$ and $\mu^* \in U_\mu$. By compactness, Ω is covered by a finite collection of neighbourhoods $U_{\mu_1}, \cdots, U_{\mu_r}$, and we may pick corresponding E_j to be real-valued functions with supports on S_{μ_j} and non-vanishing on the interior $S_{\mu_j}^\circ$.

Now suppose $Z \gg 1$ is sufficiently large. Consider the following renormalisation of the Whittaker function:

$$W_\mu^*(y) := y^{-\eta} W_\mu(y). \tag{4.12}$$

The Mellin transform $M_\mu^*(s) = \int_{\mathbb{R}_+^2} W_\mu^*(y) y^s \frac{dy_1}{y_1}\frac{dy_2}{y_2}$ is given in [Ish05] (where ν_1, ν_2 in [Ish05] are $\mu_1 + \mu_2$ and $-\mu_1 + \mu_2$ in our notation)

$$\begin{aligned} M_\mu^*(s) =& 2^{-4}\Gamma\left(\frac{s_1+\mu_1+\mu_2}{2}\right)\Gamma\left(\frac{s_1+\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{s_1-\mu_1+\mu_2}{2}\right)\Gamma\left(\frac{s_1-\mu_1-\mu_2}{2}\right) \\ & \Gamma\left(\frac{s_2+\mu_1}{2}\right)\Gamma\left(\frac{s_2-\mu_1}{2}\right)\Gamma\left(\frac{s_2+\mu_2}{2}\right)\Gamma\left(\frac{s_2-\mu_2}{2}\right) \\ & \left\{\Gamma\left(\frac{s_1+s_2+\mu_2}{2}\right)\Gamma\left(\frac{s_1+s_2-\mu_2}{2}\right)\right\}^{-1} {}_3F_2\left(\begin{matrix}\frac{s_1}{2}, \frac{s_2+\mu_1}{2}, \frac{s_2-\mu_1}{2} \\ \frac{s_1+s_2+\mu_2}{2}, \frac{s_1+s_2-\mu_2}{2}\end{matrix}\,\middle|\, 1\right). \end{aligned}$$

By Weyl group symmetry, we may assume without loss of generality that $\sigma_\pi(\infty) = \operatorname{Re}\mu_2$. For $\operatorname{Re}(s_1)$ sufficiently large, $M_\mu^*(s)$ is holomorphic for $\operatorname{Re}(s_2) > \sigma_\pi(\infty)$, as poles can only occur at $s_2 = \pm\mu_1 - k$, $\pm\mu_2 - k$ for $k \in \mathbb{N}_0$. Hence, for $\operatorname{Re}(s_1)$ sufficiently large, the function

$$M_\mu^\dagger(s) := M_\mu^*(s)(s_2+\mu_1)(s_2-\mu_1)(s_2+\mu_2)(s_2-\mu_2)$$

is holomorphic for $\operatorname{Re}(s_2) > \sigma_\pi(\infty) - 1$. Now let

$$\hat{M}_\mu(s) := \frac{M_\mu^\dagger(s)}{s_2-\mu_2} = M_\mu^*(s)(s_2+\mu_1)(s_2-\mu_1)(s_2+\mu_2).$$

The inverse Mellin transform of $\hat{M}_\mu(s)$ is then given by

$$\hat{W}_\mu(y) = \mathcal{D}_{\mu_1}\mathcal{D}_{-\mu_1}\mathcal{D}_{\mu_2}W_\mu^*(y),$$

where the differential operators are applied to y_2. On the other hand, we compute the inverse Mellin transform directly, and by shifting the contour to $\operatorname{Re}(s_2) = \sigma_\pi(\infty) - \frac{1}{2}$, we obtain the estimate

$$\hat{W}_\mu(y) = y_2^{-\mu_2}W_\mu^{**}(y_1) + \mathcal{O}_{y_1,\mu}(y_2^{-\sigma_\pi(\infty)+\frac{1}{2}})$$

for $y_2 \to 0$, where

$$W_\mu^{**}(y_1) = \Gamma(\mu_2+1)\,\Gamma\left(\frac{\mu_1+\mu_2}{2}+1\right)\Gamma\left(\frac{-\mu_1+\mu_2}{2}+1\right)W_{\mu_1}^*(y_1)y_1^{-\mu_2},$$

where $W_{\mu_1}^*(y_1) = y_1^{-1/2}W_{\mu_1}(y_1)$ is a normalised GL(2)-Whittaker function.

The rest of the proof follows the argument in [Blo19a]. Applying Lemma 4.7 repeatedly, we obtain constants $\frac{1}{2} < \gamma_1 < \gamma_2 < 1$ such that the bound

$$\left|W_\mu^*(y)\right| \gg y_2^{-\sigma_\pi(\infty)}\left|W_\mu^{**}(y_1)\right|$$

holds for $y_2 \in [\gamma_1/Z, \gamma_2/Z]$, when y_1 and μ vary in some fixed compact domain. Now choose functions $E_j^{**} : \mathbb{R}^+ \to \mathbb{C}$, depending on Ω but not Z, such that $\sum_j \left|\left\langle E_j^{**}, W_\mu^{**}\right\rangle_{\mathbb{R}_+}\right|^2 \gg 1$ for $\mu \in \Omega$, where $\langle -,-\rangle_{\mathbb{R}_+}$ denotes the inner product with respect to the Haar measure on $\mathbb{R}_+$. Now define $E_j^*(y_1,y_2) = \delta_{\gamma_1\le y_2\le\gamma_2}E_j^{**}(y_1)$. This choice depends on Z, but the support of E_j^* varies inside some interval depending only on Ω. Using the relation (4.12), and upon setting $E_j(y) = y^\eta E_j^*(y)$, we obtain

$$\sum_j \left|\left\langle E_j^{(1,Z)}, W_\mu\right\rangle\right|^2 \gg Z^{2\eta_2+2\sigma_\pi(\infty)}$$

as desired. □

4.4 Hecke eigenvalues and Fourier coefficients

Let $\mathcal{M}$ be a set of matrices in $\mathrm{GSp}(4,\mathbb{Q})^+$ that is left- and right-invariant under $\Gamma = \mathrm{Sp}(4,\mathbb{Z})$ and is a finite union $\bigcup_j \Gamma\mathcal{M}_j$ of left cosets. Then $\mathcal{M}$ defines a Hecke operator $T_\mathcal{M}$ on the space of cuspidal automorphic forms by

$$T_\mathcal{M}\varpi(g) = \sum_j \varpi(\mathcal{M}_j g).$$

For a matrix $g \in \mathrm{GSp}(4,\mathbb{Q})^+$, we denote by T_g the Hecke operator $T_{\Gamma g \Gamma}$. For $m \in \mathbb{N}$, let

$$S(m) := \left\{ M \in \mathrm{GSp}(4,\mathbb{Z})^+ \,\middle|\, M^\top J M = mJ \right\}, \quad J = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}.$$

The m-th standard Hecke operator is then given by $T(m) = T_{S(m)}$. The set of matrices

$$\mathcal{H}(m) = \left\{ \begin{pmatrix} A & m^{-1}BD \\ & D \end{pmatrix} \in S(m) \,\middle|\, A = \begin{pmatrix} a_1 & a \\ & a_2 \end{pmatrix}, B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}, \begin{array}{l} a_1, a_2 > 0,\ 0 \le a < a_2, \\ 0 \le b_i < m, AD^\top = mI_2, \\ BD \equiv 0 \pmod m \end{array} \right\}. \tag{4.13}$$

gives a complete system of left coset representatives for $\Gamma \backslash S(m)$ [Spe72]. For $r \in \mathbb{N}_0$, $0 \le a \le b \le r/2$ and any prime p, define

$$T_{a,b}^{(r)}(p) := T_{\mathrm{diag}(p^a, p^b, p^{r-a}, p^{r-b})}.$$

When the context is clear, we suppress p from the notation, and write $T_{a,b}^r$ instead. Then $T(p^r)$ admits a decomposition

$$T(p^r) = \sum_{0 \le a \le b \le r/2} T_{a,b}^{(r)}(p).$$

It is well-known that the Hecke algebra $\mathscr{H}$ of $\mathrm{Sp}(4,\mathbb{R})$ is generated by $T(p) = T_{0,0}^{(1)}(p)$ and $T_{0,1}^{(2)}(p)$ for primes p, along with the identity.

We also define involutions T_{ε_1}, T_{ε_2} on the space of cuspidal automorphic forms by

$$T_{\varepsilon_1}\varpi(g) = \varpi(\varepsilon_1 g), \quad T_{\varepsilon_2}\varpi\left(\begin{pmatrix} Y & X \\ & (Y^{-1})^\top \end{pmatrix}\right) = \varpi\left(\begin{pmatrix} Y & -X \\ & (Y^{-1})^\top \end{pmatrix}\right),$$

where $\varepsilon_1 = \mathrm{diag}(-1,1,-1,1)$. It is clear that

$$(T_{\varepsilon_1}\varpi)_{(M_1,M_2)}(g) = \varpi_{(-M_1,M_2)}(g), \qquad (T_{\varepsilon_2}\varpi)_{(M_1,M_2)}(g) = \varpi_{(M_1,-M_2)}(g). \tag{4.14}$$

It is also straightforward to check that T_{ε_1}, T_{ε_2} commute with the Hecke operators and the invariant differential operators. So we may assume a cuspidal automorphic form ϖ is also an eigenfunction of T_{ε_1}, T_{ε_2}.

Let π be the irreducible automorphic representation corresponding to ϖ. We write $\lambda(m,\pi)$ and $\lambda_{a,b}^{(r)}(p,\pi)$ to denote the eigenvalue of ϖ with respect to $T(m)$ and $T_{a,b}^{(r)}$ respectively, and write $\lambda'(m,\pi) := m^{-3/2}\lambda(m,\pi)$. Again, we omit π from the notation when the context is clear.

It is known that if ϖ is generic and L^2-normalised, then by [CI19, Theorem 1.1] and [Li10, Theorem 3], we have

$$|A_\varpi(1,1)|^2 \asymp_\mu \frac{1}{[\mathrm{Sp}(4,\mathbb{Z}) : \Gamma_0(q)] L(1,\pi,\mathrm{Ad})} \gg_\mu q^{-3-\varepsilon}. \tag{4.15}$$

In particular, $A_\varpi(1,1) \neq 0$.

Notation. Let ϖ be an L^2-normalised generic cuspidal newform. For the rest of the section, it is however instructive to have an alternative normalisation, such that the $(1,1)$-st Fourier coefficient is 1. To avoid confusion, we always denote by ϖ an L^2-normalised form, and by ϖ_1 a scalar multiple of ϖ such that $A_{\varpi_1}(1,1) = 1$. From (4.15), we see that $\varpi_1 = k\varpi$ for some $|k| \ll q^{3/2+\varepsilon}$.

Now fix a prime $p \nmid q$. Let $M = (M_1, M_2)$, and $0 \le c, d \le r$ such that $p^{d-c} \mid M_1$ and $p^{r-2d} \mid M_2$. Write

$$\Gamma \operatorname{diag}(p^a, p^b, p^{r-a}, p^{r-b})\Gamma = \bigcup_i \Gamma h_i$$

as a finite union of left cosets. We can assume that $h_i \in U(\mathbb{Q})T(\mathbb{Q}_+)$. Consider the decomposition $h_i = \hat{y}_i \hat{x}_i$, with $\hat{y}_i \in T(\mathbb{Q}^+)$, $\hat{x}_i \in U(\mathbb{Q}^+)$. We define exponential sums

$$\mathfrak{S}^{(r)}_{a,b,M}(c,d) := \sum_{\substack{\Gamma h_i \subseteq \Gamma \operatorname{diag}(p^a,p^b,p^{r-a},p^{r-b})\Gamma \\ \hat{y}_i = \operatorname{diag}(p^c,p^d,p^{r-c},p^{r-d})}} \psi_M(\hat{x}_i),$$

and

$$\mathfrak{S}^{(r)}(c,d) := \sum_{0 \le a,b \le r/2} \mathfrak{S}^{(r)}_{a,b,(1,1)}(c,d) = \sum_{\substack{\Gamma h_i \subseteq S(p^r) \\ \hat{y}_i = \operatorname{diag}(p^c,p^d,p^{r-c},p^{r-d})}} \psi(\hat{x}_i).$$

Proposition 4.9. We have

$$\lambda^{(r)}_{a,b}(p) A_\varpi(M) = \sum_{\substack{0 \le c,d \le r \\ p^{c-d} \mid M_1, p^{2d-r} \mid M_2}} \mathfrak{S}^{(r)}_{a,b,M}(c,d) p^{2c+d-\frac{3r}{2}} A_\varpi(M_1 p^{d-c}, M_2 p^{r-2d}).$$

Proof. We compute the Fourier coefficient of $T^{(r)}_{a,b}\varpi$ in two ways. On one hand, we have

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} T_h \varpi(xy)\overline{\psi_M(x)}dx = \lambda^{(r)}_{a,b}(p) \frac{A_\varpi(M)}{M^\eta} W_\mu(\iota(M)y). \tag{4.16}$$

On the other hand, we expand the Hecke operator

$$\begin{aligned}\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} T_h \varpi(xy)\overline{\psi_M(x)}dx =& \sum_{\Gamma h_i} \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varpi(h_i xy)\overline{\psi_M(x)}dx \\ =& p^{-4r} \sum_{\Gamma h_i} \int_{U(p^r\mathbb{Z})\backslash U(\mathbb{R})} \varpi(h_i xy)\overline{\psi_M(x)}dx.\end{aligned}$$

Write $h_i x = x' \hat{y}_i$, with $x' \in U(\mathbb{R})$, and $\hat{y}_i = \operatorname{diag}(c_1, \cdots, c_4)$. A simple calculation shows that

$$x'_{kl} = c_l \sum_j (h_i)_{kj} x_{jl}.$$

In particular, we have

$$x_{12} = \frac{c_2}{c_1} x'_{12} - \frac{(h_i)_{12}}{c_1} = \frac{c_2}{c_1} x'_{12} - (\hat{x}_i)_{12}, \quad x_{24} = \frac{c_4}{c_2} x'_{24} - \frac{(h_i)_{24}}{c_2} = \frac{c_4}{c_2} x'_{24} - (\hat{x}_i)_{24}.$$

Making this substitution, the expression becomes

$$p^{-4r} \sum_{\Gamma h_i} \prod_{k,l} \int_{\sum_j (h_i)_{kj} x_{jl}}^{\frac{c_k}{c_l} p^r + \sum_j (h_i)_{kj} x_{jl}} \varpi(x' \hat{y}_i y) e\left(M_1 (\hat{x}_i)_{12} + M_2 (\hat{x}_i)_{24}\right) e\left(-\frac{c_2}{c_1} M_1 x'_{12} - \frac{c_4}{c_2} M_2 x'_{24}\right) \frac{c_l}{c_k} dx'_{k,l},$$

where (k,l) runs through the indices $(1,2), (1,3), (2,3), (2,4)$. By periodicity, we shift the integral and get

$$p^{-4r} \sum_{\Gamma h_i} \prod_{k,l} \int_0^{\frac{c_k}{c_l} p^r} \varpi(x' \hat{y}_i y) e\left(M_1 (\hat{x}_i)_{12} + M_2 (\hat{x}_i)_{24}\right) e\left(-\frac{c_2}{c_1} M_1 x'_{12} - \frac{c_4}{c_2} M_2 x'_{24}\right) \frac{c_l}{c_k} dx'_{k,l},$$

Since $\varpi(x'\hat{y}_i y)$ is 1-periodic with respect to x'_{kl}, this integral vanishes unless $c_1 \mid c_2 M_1$ and $c_2 \mid c_4 M_2$. We sum over the terms with the same $\hat{y}_i = \mathrm{diag}(p^c, p^d, p^{r-c}, p^{r-d})$ and get

$$\sum_{\substack{0 \le c,d \le r \\ p^{c-d} \mid M_1, p^{2d-r} \mid M_2}} \mathfrak{S}^{(r)}_{a,b,M}(c,d) \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varpi(x'\hat{y}_i y) e\left(-p^{d-c} M_1 x'_{12} - p^{r-2d} M_2 x'_{24}\right) dx'.$$

Evaluating the integral gives

$$\sum_{\substack{0 \le c,d \le r \\ p^{c-d} \mid M_1, p^{2d-r} \mid M_2}} \mathfrak{S}^{(r)}_{a,b,M}(c,d) p^{2c+d-\frac{3r}{2}} \frac{A_\varpi(p^{d-c}M_1, p^{r-2d}M_2)}{M^\eta} W_\mu(\iota(M)y). \tag{4.17}$$

Comparing (4.16) and (4.17) gives the result. □

Theorem 4.10. Let $\varpi_1 \in V_\pi$ be a cuspidal newform such that $A_{\varpi_1}(1,1) = 1$, and $p \nmid q$ a prime. The Hecke eigenvalues $\lambda(p^r, \pi)$ of π with respect to $T(p^r)$ are given by

$$\begin{aligned} \lambda(p,\pi) &= p^{3/2} A_{\varpi_1}(1,p), \\ \lambda(p^r,\pi) &= p^{3r/2}\left(A_{\varpi_1}(1,p^r) - p^{-1}A_{\varpi_1}(1,p^{r-2})\right), \quad r \ge 2. \end{aligned}$$

Proof. Plugging in $M = (1,1)$ to Proposition 4.9 gives

$$\lambda(p^r) A_{\varpi_1}(1,1) = \sum_{0 \le a,b \le r/2} \lambda^{(r)}_{a,b}(p) A_{\varpi_1}(1,1) = \sum_{0 \le c \le d \le r/2} \mathfrak{S}^{(r)}(c,d) p^{2c+d-\frac{3r}{2}} A_{\varpi_1}(p^{d-c}, p^{r-2d}). \tag{4.18}$$

We evaluate $\mathfrak{S}^{(r)}(c,d)$ explicitly. We set $A_a := \begin{pmatrix} p^c & a \\ & p^d \end{pmatrix}$, and partition the sum

$$\mathfrak{S}^{(r)}(c,d) = \sum_{0 \le a < p^d} \mathfrak{S}^{(r)}(c,d;a), \text{ where } \mathfrak{S}^{(r)}(c,d;a) := \sum_{\substack{\Gamma h_i \subseteq S(p^r) \\ A(h_i) = A_a}} \psi(\hat{x}_i),$$

and $A(h_i)$ denotes the top left 2×2 block of h_i. Using representatives in (4.13), we rewrite

$$\mathfrak{S}^{(r)}(c,d;a) = \sum_{\substack{\Gamma h_i \subseteq S(p^r) \\ A(h_i) = A_a}} \psi(\hat{x}_i) = \sum_{\substack{\Gamma h_i \subseteq S(p^r) \\ A(h_i) = A_a}} e\left(\frac{a}{p^c} + \frac{b_3}{p^{2d}}\right) = e\left(\frac{a}{p^c}\right) \sum_{\substack{\Gamma h_i \subseteq S(p^r) \\ A(h_i) = A_a}} e\left(\frac{b_3}{p^{2d}}\right).$$

The condition $BD \equiv 0 \pmod{p^r}$ in (4.13) says

$$p^{-r}BD = \begin{pmatrix} b_1 p^{-c} - ab_2 p^{-d-c} & b_2 p^{-d} \\ b_2 p^{-c} - ab_3 p^{-d-c} & b_3 p^{-d} \end{pmatrix} \in M_2(\mathbb{Z}). \tag{4.19}$$

Note that the summation over B depends only on $v_p(a)$. We partition the sum with respect to $v_p(a)$. For $v_p(a) \le c-2$, we have

$$\sum_{\substack{0 \le a < p^d \\ v_p(a) \le c-2}} \mathfrak{S}^{(r)}(c,d;a) = \sum_{0 \le v \le c-2} \sum_{\substack{0 < a' < p^{d-v} \\ (a',p)=1}} e\left(\frac{a'}{p^{c-v}}\right) \sum_{\substack{\Gamma h_i \subseteq S(p^r) \\ A(h_i) = A_{p^v}}} e\left(\frac{b_3}{p^{2d}}\right) = 0.$$

For $v_p(a) = c-1$, we have $d \ge c \ge 1$, and

$$\sum_{\substack{0 \le a < p^d \\ v_p(a) \le c-1}} \mathfrak{S}^{(r)}(c,d;a) = \sum_{\substack{0 < a' < p^{d-c+1} \\ (a',p)=1}} e\left(\frac{a'}{p}\right) \sum_{\substack{\Gamma h_i \subseteq S(p^r) \\ A(h_i) = A_{p^{c-1}}}} e\left(\frac{b_3}{p^{2d}}\right) = -p^{d-c} \sum_{\substack{\Gamma h_i \subseteq S(p^r) \\ A(h_i) = A_{p^{c-1}}}} e\left(\frac{b_3}{p^{2d}}\right).$$

The integrality conditions in (4.19) forces $p^{d+1} \mid b_3$, $p^d \mid b_2$, and $p^{d+1} \mid b_1 p^{d-c+1} + b_2$. Hence

$$\sum_{\substack{0\le a<p^d\\ v_p(a)\le c-1}} \mathfrak{S}^{(r)}(c,d;a) = -p^{d-c} \sum_{\substack{0\le b_1,b_2,b_3<p^r\\ p^{d+1}\mid b_3,\ p^d\mid b_2\\ p^{d+1}\mid b_1p^{d-c+1}+b_2}} e\left(\frac{b_3}{p^{2d}}\right) = \begin{cases} -p^{3r-c-2d-1} & \text{if } d=1,\\ 0 & \text{otherwise.}\end{cases}$$

For $v_p(a) \ge c$, the integrality condition in (4.19) forces $p^d \mid b_2, b_3$, and $p^c \mid b_1$. Hence

$$\sum_{\substack{0\le a<p^d\\ v_p(a)\ge c}} \mathfrak{S}^{(r)}(c,d;a) = p^{d-c} \sum_{\substack{0\le b_1,b_2,b_3<p^r\\ p^d\mid b_2,b_3\\ p^c\mid b_1}} e\left(\frac{b_3}{p^{2d}}\right) = \begin{cases} p^{3r-c-2d} & \text{if } d=0,\\ 0 & \text{otherwise.}\end{cases}$$

Hence we conclude

$$\mathfrak{S}^{(r)}(c,d) = \begin{cases} p^{3r} & \text{if } (c,d)=(0,0),\\ -p^{3r-4} & \text{if } (c,d)=(1,1),\\ 0 & \text{otherwise.}\end{cases}$$

Putting this back into (4.18) gives the statement. □

Hecke eigenvalues can also be expressed in terms of local Satake parameters α_p, β_p associated to π. Without loss of generality, assume $|\alpha_p| \ge |\beta_p| \ge 1$. Then up to some ordering we have $p^{\mu_\pi(p,1)} = \alpha_p$, $p^{\mu_\pi(p,2)} = \beta_p$, and $\sigma_\pi(p) = \mu_\pi(p,1)$. By an identity of Shimura [Shi63, Theorem 2], we have

$$\sum_{r=0}^{\infty} \lambda(p^r)x^r = (1-p^2x^2)(1-p^{3/2}\alpha_p x)^{-1}(1-p^{3/2}\alpha_p^{-1}x)^{-1}(1-p^{3/2}\beta_p x)^{-1}(1-p^{3/2}\beta_p^{-1}x)^{-1}. \quad (4.20)$$

For convenience, we define $\sigma_\pi^+(p) = \frac{3}{2} + \sigma_\pi(p)$.

Lemma 4.11. For a prime $p \nmid q$ and $r \ge 3$, the following inequality

$$\left|\lambda(p^{r-j})\right| \ge \frac{p^{(r-j)\sigma_\pi^+(p)}}{16}$$

holds for some $j \in \{0,1,2,3\}$.

Proof. We derive from (4.20) that

$$(1-p^{3/2}\alpha_p^{-1}x)(1-p^{3/2}\beta_p x)(1-p^{3/2}\beta_p^{-1}x)\sum_{r=0}^{\infty}\lambda(p^r)x^r = (1-p^2x^2)\sum_{r=0}^{\infty}\left(p^{3/2}\alpha_p\right)^r x^r.$$

Comparing coefficients gives

$$\begin{aligned}&\lambda(p^r) - \lambda(p^{r-1})p^{3/2}\left(\alpha_p^{-1}+\beta_p+\beta_p^{-1}\right) + \lambda(p^{r-2})p^3(\alpha_p^{-1}\beta_p + \alpha_p^{-1}\beta_p^{-1}+1) + \lambda(p^{r-3})p^{9/2}\alpha_p^{-1}\\ =&p^{3r/2}(\alpha_p^r - p^{-1}\alpha_p^{r-2}).\end{aligned}$$

Assume the contrary. Then the left hand side is bounded by

$$\frac{p^{r\sigma_\pi^+(p)}}{2} \le p^{r\sigma_\pi^+(p)} - p^{2+(r-2)\sigma_\pi^+(p)} \le p^{3r/2}\left|\alpha_p^r - p^{-1}\alpha_p^{r-2}\right|,$$

a contradiction. □

Lemma 4.12. Let $\varpi_1 \in V_\pi$ be a cuspidal newform such that $A_{\varpi_1}(1,1) = 1$, $p \nmid q$ a prime, and $r_0 \in \mathbb{N}_0$. Then the inequality

$$|A_{\varpi_1}(1,p^r)| \geq \frac{p^{r\sigma_\pi(p)}}{32}$$

holds for some $r_0 \leq r \leq r_0 + 5$.

Proof. By Lemma 4.11, we have

$$|\lambda(p^r)| \geq \frac{p^{r\sigma_\pi^+(p)}}{16}$$

for some $r_0 + 2 \leq r \leq r_0 + 5$. By Theorem 4.10, we have

$$p^{3r/2}\left(|A_{\varpi_1}(1,p^r)| + p^{-1}\left|A_{\varpi_1}(1,p^{r-2})\right|\right) \geq \frac{p^{r\sigma_\pi^+(p)}}{16},$$

and the statement follows. □

4.5 Sp(4) Kloosterman sums

Kloosterman sums for Sp(4) are defined in Section 3.1. They generalise in a natural way to the congruence subgroup $\Gamma_0(q)$. We consider the Bruhat decomposition

$$\mathrm{Sp}(4) = \coprod_{w \in W} G_w, \qquad G_w = UwTU_w.$$

Let $M, N, c \in \mathbb{N}^2$, and $w \in W$. Then, if

$$\psi_M(wc^*x(c^*)^{-1}w^{-1}) = \psi_N(x) \tag{4.21}$$

for all $x \in U_w(\mathbb{R})$, then the Kloosterman sum

$$\mathrm{Kl}_{q,w}(c,M,N) := \sum_{xwc^*x \in U(\mathbb{Z}))\backslash G_w(\mathbb{Q}) \cap \Gamma_0(q)/U(\mathbb{Z})} \psi_M(x)\psi_N(x') \tag{4.22}$$

is well-defined by an analogue of Proposition 3.19. If (4.21) does not hold, we set $\mathrm{Kl}_{q,w}(c,M,N) = 0$. From Section 3.5, the Kloosterman sum $\mathrm{Kl}_{q,w}(c,M,N)$ is nonzero only if $w = \mathrm{id}, s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta, w_0$.

Now suppose the entries of $M = (M_1, M_2)$ and $N = (N_1, N_2)$ are coprime to q. Considering the Bruhat decomposition of $\Gamma_0(q)$, we deduce that the Kloosterman sum $\mathrm{Kl}_{q,w}(c,M,N)$ is nonempty only if

$$q \mid c_1 \text{ for } w = s_\alpha s_\beta s_\alpha, \qquad q \mid c_1 \text{ and } q^2 \mid c_2 \text{ for } w = s_\beta s_\alpha s_\beta, w_0. \tag{4.23}$$

Meanwhile, the well-definedness condition (4.21) says that the Kloosterman sums are well-defined precisely if

$$N_2 = M_2\frac{c_1^2}{c_2^2} \text{ if } w = s_\alpha s_\beta s_\alpha, \quad N_1 = M_1\frac{c_2}{c_1^2} \text{ if } w = s_\beta s_\alpha s_\beta.$$

Hence the Kloosterman sums are well-defined only if

$$v_q(c_1) = v_q(c_2) \text{ if } w = s_\alpha s_\beta s_\alpha, \qquad v_q(c_2) = 2v_q(c_1) \text{ if } w = s_\beta s_\alpha s_\beta. \tag{4.24}$$

From the abstract definition in Section 3.1, the Kloosterman sums $\mathrm{Kl}_{q,w}(c,M,N)$ also enjoy certain multiplicativity in the moduli. We state one particular case. Let q be prime. For $c=(c_1,c_2)\in\mathbb{N}^2$, let $c_j'=q^{-v_q(c_j)}c_j$, $j=1,2$, and $c'=(c_1',c_2')$. Then we have

$$\mathrm{Kl}_{q,w}(c,M,N)=\mathrm{Kl}_{q,w}\left((q^{v_q(c_1)},q^{v_q(c_2)}),M',N'\right)\mathrm{Kl}_{1,w}(c',M'',N'') \tag{4.25}$$

for some $M',N',M'',N''\in\mathbb{N}^2$. Moreover, if the entries of M, N are coprime to q, then so are M', N'. From [DR98], we have a trivial bound

$$\mathrm{Kl}_{1,w}(c',M'',N'')\le |U(\mathbb{Z})\backslash G_w(\mathbb{Q})\cap \mathrm{Sp}(4,\mathbb{Z})/U_w(\mathbb{Z})|\le c_1'c_2'. \tag{4.26}$$

4.5.1 Evaluation of Kloosterman sums

For the proof of the theorems in Section 1.3 , we compute the following Kloosterman sums:

$$\begin{aligned}&\mathrm{Kl}_{q,s_\alpha s_\beta s_\alpha}\left((q,q),M,N\right), && \mathrm{Kl}_{q,s_\beta s_\alpha s_\beta}\left((q,q^2),M,N\right),\\ &\mathrm{Kl}_{q,w_0}\left((q,q^2),M,N\right), && \mathrm{Kl}_{q,w_0}\left((q,q^3),M,N\right).\end{aligned}$$

We refer to Section 2.2.4 for the Bruhat decomposition.

(i) Consider the Bruhat decomposition for summands in $\mathrm{Kl}_{q,s_\alpha s_\beta s_\alpha}\left((q,q),M,N\right)$:

$$\begin{aligned}\gamma&=\begin{pmatrix}1&\beta_1&\beta_2&\beta_3\\ &1&\beta_4&\beta_5\\ &&1&\\ &&-\beta_1&1\end{pmatrix}\begin{pmatrix}&&-q^{-1}&\\ &1&&\\ q&&&\\ &&&1\end{pmatrix}\begin{pmatrix}1&v_2/q&v_3/q&v_4/q\\ &1&v_4/q&\\ &&1&\\ &&-v_2/q&1\end{pmatrix}\\ &=\begin{pmatrix}\beta_2q&\beta_2v_2+\beta_1&\beta_2v_3-\beta_3v_2/q+\beta_1v_4/q-1/q&\beta_2v_4+\beta_3\\ \beta_4q&\beta_4v_2+1&\beta_4v_3-\beta_5v_2/q+v_4/q&\beta_4v_4+\beta_5\\ q&v_2&v_3&v_4\\ -\beta_1q&-\beta_1v_2&-\beta_1v_3-v_2/q&-\beta_1v_4+1\end{pmatrix}\in\Gamma_0(q),\end{aligned}$$

with $v_2,v_3,v_4 \pmod q$ chosen such that $(v_3,v_4,(q,v_2))=1$, and $\left((q,v_2)^2,qv_3+v_2v_4\right)=q$. As $\gamma\in\Gamma_0(q)$, by considering the lower left block, we deduce that $v_2=0$, and solve $\beta_1\equiv 0 \pmod 1$. The conditions on v_3,v_4 then simplify as $(q,v_3)=1$. Considering the second row, we solve

$$\beta_4\equiv-\frac{\overline{v_3}v_4}{q}\pmod 1,\quad \beta_5\equiv\frac{\overline{v_3}v_4^2}{q}\pmod 1.$$

So the Kloosterman sum is given by

$$\mathrm{Kl}_{q,s_\alpha s_\beta s_\alpha}\left((q,q),M,N\right)=\sum_{\substack{v_3 (\mathrm{mod}\ q)\\ (v_3,q)=1}}\ \sum_{v_4 (\mathrm{mod}\ q)} e\left(\frac{M_2\overline{v_3}v_4^2}{q}\right)=0.$$

(ii) Consider the Bruhat decomposition for summands in $\mathrm{Kl}_{q,s_\beta s_\alpha s_\beta}\left((q,q^2),M,N\right)$:

$$\begin{aligned}\gamma&=\begin{pmatrix}1&\beta_1&\beta_2&\beta_3\\ &1&\beta_4&\beta_5\\ &&1&\\ &&-\beta_1&1\end{pmatrix}\begin{pmatrix}&&&-q^{-1}\\ &&q^{-1}&\\ &q&&\\ -q&&&\end{pmatrix}\begin{pmatrix}1&&-v_{23}/q^2&v_{13}/q^2\\ &1&v_{13}/q^2&v_{14}/q^2\\ &&1&\\ &&&1\end{pmatrix}\\ &=\begin{pmatrix}-\beta_3q&\beta_2q&\beta_2v_{13}/q+\beta_1/q+\beta_3v_{23}/q&\beta_2v_{14}/q-\beta_3v_{13}/q-1/q\\ -\beta_5q&\beta_4q&\beta_4v_{13}/q+\beta_5v_{23}/q+1/q&\beta_4v_{14}/q-\beta_5v_{13}/q\\ 0&q&v_{13}/q&v_{14}/q\\ -q&-\beta_1q&-\beta_1v_{13}/q+v_{23}/q&-\beta_1v_{14}/q-v_{13}/q\end{pmatrix}\in\Gamma_0(q),\end{aligned}$$

with $v_{13}, v_{14}, v_{23} \pmod{q^2}$ chosen such that $(q^2, v_{13}, v_{14}) = q$, and $(q, v_{23}, v_{34}) = 1$, where $v_{34} = -\frac{v_{13}^2 + v_{14}v_{23}}{q^2}$. As $\gamma \in \Gamma_0(q)$, by considering the lower left block, we solve $\beta_1 \equiv 0 \pmod 1$. Then, $-\beta_1 v_{13}/q + v_{23}/q$ being an integer implies $q \mid v_{23}$, so $(q, v_{34}) = 1$. Write $v_{13} = qv'_{13}$, $v_{14} = qv'_{14}$, and $\beta_4 = \beta'_4/q$, $\beta_5 = \beta'_5/q$ for some $\beta'_4, \beta'_5 \in \mathbb{Z}$. By considering the second row, we deduce that

$$\beta'_4 v'_{13} + \beta'_5 v_{23}/q + 1, \beta'_4 v'_{14} - \beta'_5 v'_{13} \in q\mathbb{Z},$$

from which we deduce $\beta'_5 \equiv v'_{14}\overline{v_{34}} \pmod q$, and $\beta_5 \equiv \frac{v'_{14}\overline{v_{34}}}{q} \pmod 1$. Writing $v_{23} = qv'_{23}$, the Kloosterman sum is given by

$$\mathrm{Kl}_{q, s_\beta s_\alpha s_\beta}\left((q, q^2), M, N\right) = \sum_{\substack{v'_{13}, v'_{14}, v'_{23} \pmod q \\ (q, v'_{13}, v'_{14}) = 1 \\ (q, v_{34}) = 1}} e\left(\frac{M_2 v'_{14}\overline{v_{34}} + N_2 v'_{14}}{q}\right),$$

where $v_{34} = -({v'_{13}}^2 + v'_{14}v'_{23})$. We evaluate

$$\sum_{\substack{v'_{13}, v'_{23} \pmod q \\ (q, v'_{13}) = 1}} 1 + \sum_{\substack{v'_{13}, v'_{14}, v'_{23} \pmod q \\ (q, v'_{14}) = 1,\ (q, v_{34}) = 1}} e\left(\frac{M_2 v'_{14}\overline{v_{34}} + N_2 v'_{14}}{q}\right) = q(q-1) - \sum_{\substack{v'_{13}, v'_{14} \pmod q \\ (q, v'_{14}) = 1}} e\left(\frac{N_2 v'_{14}}{q}\right) = q^2.$$

(iii) Consider the Bruhat decomposition for summands in $\mathrm{Kl}_{q, w_0}\left((q, q^2), M, N\right)$:

$$\gamma = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} & & -q^{-1} & \\ & & & -q^{-1} \\ q & & & \\ & q & & \end{pmatrix} \begin{pmatrix} 1 & v_2/q & v_3/q & v_4/q \\ & 1 & v_{13}/q^2 & v_{14}/q^2 \\ & & 1 & \\ & & -v_2/q & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \beta_2 q & \beta_2 v_2 + \beta_3 q & \beta_2 v_3 + \beta_3 v_{13}/q + \beta_1 v_2/q^2 - 1/q & \beta_2 v_4 - \beta_1/q + \beta_3 v_{14}/q \\ \beta_4 q & \beta_4 v_2 + \beta_5 q & \beta_4 v_3 + \beta_5 v_{13}/q + v_2/q^2 & \beta_4 v_4 + \beta_5 v_{14}/q - 1/q \\ q & v_2 & v_3 & v_4 \\ -\beta_1 q & -\beta_1 v_2 + q & -\beta_1 v_3 + v_{13}/q & -\beta_1 v_4 + v_{14}/q \end{pmatrix} \in \Gamma_0(q),$$

with $v_2, v_3, v_4 \pmod q$ and $v_{13}, v_{14} \pmod{q^2}$ chosen such that $v_{13}q + v_2 v_{14} - v_4 q^2 = 0$, $(q, v_2, v_3, v_4) = 1$, and $(q^2, v_{13}, v_{14}, v_{23}, v_{34}) = 1$, where $v_{23} = \frac{v_2 v_{13} - v_3 q^2}{q}$ and $v_{34} = \frac{v_3 v_{14} - v_4 v_{13}}{q}$. As $\gamma \in \Gamma_0(q)$, by considering the lower left block, we deduce that $v_2 = 0$, and solve $\beta_1 \equiv 0 \pmod 1$. The last row being integers implies that $q \mid v_{13}, v_{14}$. Write $v_{13} = qv'_{13}$, $v_{14} = qv'_{14}$. The relation $v_{13}q + v_2 v_{14} - v_4 q^2 = 0$ says $v'_{13} = v_4$. We check that $q \mid v_{23}$ as well, so $(q, v_{34}) = 1$. Write $\beta_4 = \beta'_4/q$, $\beta_5 = \beta'_5/q$ for some $\beta'_4, \beta'_5 \in \mathbb{Z}$. By considering the second row, we deduce that

$$\beta'_4 v_3 + \beta'_5 v'_{13},\ \beta'_4 v_4 + \beta'_5 v'_{14} - 1 \in q\mathbb{Z},$$

from which we deduce $\beta'_5 \equiv v_3\overline{v_{34}} \pmod q$, and $\beta_5 = \frac{v_3\overline{v_{34}}}{q} \pmod 1$. The Kloosterman sum is given by

$$\mathrm{Kl}_{q, w_0}\left((q, q^2), M, N\right) = \sum_{\substack{v_3, v_4, v'_{14} \pmod q \\ (q, v_3, v_4) = 1 \\ (q, v_{34}) = 1}} e\left(\frac{M_2 v_3 \overline{v_{34}} + N_2 v'_{14}}{q}\right),$$

where $v_{34} = v_3 v'_{14} - v_4^2$.

Fix $v_4, v'_{14} \neq 0$. As $v_3 \neq 0$ varies, $\overline{v_3} v_{34} \equiv v'_{14} - v_4^2 \overline{v_3}$ runs through nonzero residues except v'_{14} modulo q; hence, as v_3 varies, $v_3 \overline{v_{34}}$ runs through all residues except $\overline{v'_{14}}$ modulo q. Hence

$$\sum_{\substack{v_3, v_4, v'_{14} (\bmod q) \\ (q, v_4)=1,\ (q, v'_{14})=1 \\ (q, v_{34})=1}} e\left(\frac{M_2 v_3 \overline{v_{34}} + N_2 v'_{14}}{q} \right) = - \sum_{\substack{v_4, v'_{14} (\bmod q) \\ (q, v_4)=1 \\ (q, v'_{14})=1}} e\left(\frac{M_2 \overline{v'_{14}} + N_2 v'_{14}}{q} \right) = -(q-1) S(M_2, N_2; q).$$

If $v_4 \neq 0$ and $v'_{14} = 0$, then $v_{34} = -v_4^2$. The corresponding part of the sum becomes

$$\sum_{\substack{v_3, v_4 (\bmod q) \\ (q, v_4)=1}} e\left(\frac{-M_2 v_3 \overline{v_4}^2}{q} \right) = 0.$$

Meanwhile, for $v_4 = 0$, we have $v'_{14} \neq 0$, and $v_{34} = v_3 v'_{14}$, so $v_3 \overline{v_{34}} = \overline{v'_{14}}$. Hence this part of the sum is

$$\sum_{\substack{v_3, v'_{14} (\bmod q) \\ (q, v_3)=1 \\ (q, v'_{14})=1}} e\left(\frac{M_2 \overline{v'_{14}} + N_2 v'_{14}}{q} \right) = (q-1) S(M_2, N_2; q).$$

Combining the parts above, we conclude that $\mathrm{Kl}_{q, w_0}\left((q, q^2), M, N \right) = 0$.

(iv) Consider the Bruhat decomposition for summands in $\mathrm{Kl}_{q, w_0}\left((q, q^3), M, N \right)$:

$$\gamma = \begin{pmatrix} 1 & \beta_1 & \beta_2 & \beta_3 \\ & 1 & \beta_4 & \beta_5 \\ & & 1 & \\ & & -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} & & -q^{-1} & \\ & & & -q^{-2} \\ q & & & \\ & q^2 & & \end{pmatrix} \begin{pmatrix} 1 & v_2/q & v_3/q & v_4/q \\ & 1 & v_{13}/q^3 & v_{14}/q^3 \\ & & 1 & \\ & & -v_2/q & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \beta_2 q & \beta_2 v_2 + \beta_3 q^2 & \beta_2 v_3 + \beta_3 v_{13}/q + \beta_1 v_2/q^3 - 1/q & \beta_2 v_4 - \beta_1/q^2 + \beta_3 v_{14}/q \\ \beta_4 q & \beta_4 v_2 + \beta_5 q^2 & \beta_4 v_3 + \beta_5 v_{13}/q + v_2/q^3 & \beta_4 v_4 + \beta_5 v_{14}/q - 1/q^2 \\ q & v_2 & v_3 & v_4 \\ -\beta_1 q & -\beta_1 v_2 + q^2 & -\beta_1 v_3 + v_{13}/q & -\beta_1 v_4 + v_{14}/q \end{pmatrix} \in \Gamma_0(q),$$

with $v_2, v_3, v_4 \pmod q$ and $v_{13}, v_{14} \pmod{q^2}$ chosen such that $v_{13} q + v_2 v_{14} - v_4 q^3 = 0$, $(q, v_2, v_3, v_4) = 1$, and $(q^2, v_{13}, v_{14}, v_{23}, v_{34}) = 1$, where $v_{23} = \frac{v_2 v_{13} - v_3 q^3}{q}$ and $v_{34} = \frac{v_3 v_{14} - v_4 v_{13}}{q}$. As $\gamma \in \Gamma_0(q)$, by considering the lower left block, we deduce that $v_2 = 0$, and solve $\beta_1 \equiv 0 \pmod 1$. The last row being integers implies that $q \mid v_{13}, v_{14}$. Write $v_{13} = q v'_{13}$, $v_{14} = q v'_{14}$. The relation $v_{13} q + v_2 v_{14} - v_4 q^3 = 0$ says $v'_{13} = v_4 q$. We check that $q \mid v_{23}$ as well, so $(q, v_{34}) = 1$. Write $\beta_4 = \beta'_4/q$, $\beta_5 = \beta'_5/q^2$ for some $\beta'_4, \beta'_5 \in \mathbb{Z}$. By considering the second row, we deduce that

$$\beta'_4 v_3 q + \beta'_5 v'_{13},\ \beta'_4 v_4 q + \beta'_5 v'_{14} - 1 \in q^2 \mathbb{Z},$$

from which we deduce $\beta'_5 \equiv v_3 \overline{v_{34}} \pmod{q^2}$, and $\beta_5 = \frac{v_3 \overline{v_{34}}}{q^2} \pmod 1$. The Kloosterman sum is given by

$$\mathrm{Kl}_{q, w_0}\left((q, q^3), M, N \right) = \sum_{\substack{v_3, v_4 (\bmod q),\ v'_{14} (\bmod q^2) \\ (q, v_3, v_4)=1,\ (q, v_{34})=1}} e\left(\frac{M_2 v_3 \overline{v_{34}} + N_2 v'_{14}}{q^2} \right),$$

where $v_{34} = v_3 v'_{14} - v_4^2 q$.

Fix $v_4 \neq 0$. Then from $(q, v_{34}) = 1$ we have $(q, v'_{14}) = 1$, and $v_3 \neq 0$. For a fixed v'_{14}, we see that as v_3 varies, $\overline{v_3} v_{34} \equiv v'_{14} - v_4^2 \overline{v_3} q$ runs through nonzero residues modulo q^2 that are congruent to $v'_{14} \pmod q$, except v'_{14}; hence, as v_3 varies, $v_3 \overline{v_{34}}$ runs through all residues modulo q^2 that are congruent to $v'_{14} \pmod q$, except $\overline{v'_{14}}$. Hence

$$\sum_{\substack{v_3, v_4 (\mathrm{mod}\ q) \\ v'_{14} (\mathrm{mod}\ q^2) \\ (q,v_3)=1,\ (q,v_4)=1 \\ (q,v'_{14})=1,\ (q,v_{34})=1}} e\left(\frac{M_2 v_3 \overline{v_{34}} + N_2 v'_{14}}{q^2}\right) = - \sum_{\substack{v_4 (\mathrm{mod}\ q) \\ v'_{14} (\mathrm{mod}\ q^2) \\ (q,v_4)=1,\ (q,v'_{14})=1}} e\left(\frac{M_2 \overline{v'_{14}} + N_2 v'_{14}}{q^2}\right) = -(q-1) S(M_2, N_2; q^2).$$

Meanwhile, for $v_4 = 0$, we have $(q, v'_{14}) = 1$, and $v_{34} = v_3 v'_{14}$, so $v_3 \overline{v_{34}} = \overline{v'_{14}}$. Hence this part of the sum is

$$\sum_{\substack{v_3 (\mathrm{mod}\ q) \\ v'_{14} (\mathrm{mod}\ q^2) \\ (q,v_3)=1,\ (q,v'_{14})=1}} e\left(\frac{M_2 \overline{v'_{14}} + N_2 v'_{14}}{q^2}\right) = (q-1) S(M_2, N_2; q).$$

Combining the parts above, we conclude that $\mathrm{Kl}_{q,w_0}\big((q, q^3), M, N\big) = 0$.

4.6 Poincaré series and the Kuznetsov formula

Let $E : \mathbb{R}_+^2 \to \mathbb{C}$ be a fixed function with compact support, and $X \in \mathbb{R}_+^2$ a "parameter". We define

$$E^{(X)}(y_1, y_2) = E(X_1 y_1, X_2 y_2),$$

and a right K-invariant function $F^{(X)} : \mathrm{Sp}(4, \mathbb{R}) \to \mathbb{C}$ by

$$F^{(X)}(xy) = \psi(x) E^{(X)}(\mathrm{y}(y)) \tag{4.27}$$

for $x \in U(\mathbb{R})$ and $y \in T(\mathbb{R}_+)$, where $\psi = \psi_{1,1}$ is as in (4.1). For $N \in \mathbb{N}^2$, we define the Poincaré series of level q to be

$$P_N^{(X)}(xy) = \sum_{\gamma \in P_0 \cap \Gamma_0(q) \backslash \Gamma_0(q)} F^{(X)}(\iota(N) \gamma xy).$$

Note that $F^{(X)}(\iota(N) xy) = \psi_N(x) E^{(X)}(N\, \mathrm{y}(y)) = \psi_N(x) E(XN\, \mathrm{y}(y))$. For $w \in W$, let $G_w = UwTU$, and $\Gamma_w := U(\mathbb{Z}) \cap w^{-1} U(\mathbb{Z})^\top w$. Let $R_w(q)$ be a complete system of coset representatives for $P_0 \cap \Gamma_0(q) \backslash \Gamma_0(q) \cap G_w / \Gamma_w$.

We compute the Fourier coefficients of the Poincaré series:

$$\begin{aligned}
&\int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} P_M^{(X)}(xy) \overline{\psi_N(x)} dx \\
&= \sum_{\gamma \in P_0 \cap \Gamma_0(q) \backslash \Gamma_0(q)} \int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} F^{(X)}(\iota(M) \gamma xy) \overline{\psi_N(x)} dx \\
&= \sum_{w \in W} \sum_{\gamma \in R_w(q)} \sum_{\ell \in \Gamma_w} \int_{U(\mathbb{Z}) \backslash U(\mathbb{R})} F^{(X)}(\iota(M) \gamma \ell xy) \overline{\psi_N(x)} dx \\
&= \sum_{w \in W} \sum_{c \in \mathbb{N}^2} \mathrm{Kl}_{q,w}(c, M, N) \int_{U_w(\mathbb{R})} F^{(X)}(\iota(M) c^* wxy) \overline{\psi_N(x)} dx.
\end{aligned}$$

For fixed y, it follows from Lemma 4.1 and E having compact support that the c-sum runs over a finite set, and $U_w(\mathbb{R})$ runs over a compact domain. In particular, the right hand side is absolutely convergent.

Now let ϖ be an automorphic form in the spectrum of $L^2(\Gamma_0(q)\backslash \mathrm{Sp}(4,\mathbb{R})/K)$, not necessarily cuspidal. By unfolding, (4.8) and a change of variables $\iota(N)y \mapsto y$, we obtain

$$\left\langle \varpi, P_N^{(X)} \right\rangle = \int_{T(\mathbb{R}_+)} \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varpi(xy)\psi_N(-x)\overline{E^{(X)}(N\cdot \mathrm{y}(y))}dxd^*y = N^\eta A_\varpi(N) \left\langle W_\mu, E^{(X)} \right\rangle .$$

By Parseval, we obtain

$$\left\langle P_M^{(X)}, P_N^{(X)} \right\rangle = M^\eta N^\eta \int_{(q)} \overline{A_\varpi(M)} A_\varpi(N) \left| \left\langle W_\mu, E^{(X)} \right\rangle \right|^2 d\varpi.$$

Meanwhile, unfolding the inner product directly gives

$$\begin{aligned}\left\langle P_M^{(X)}, P_N^{(X)} \right\rangle &= \int_{T(\mathbb{R}_+)} \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} P_M^{(X)}(xy)\psi_N(-x)\overline{E^{(X)}(N\cdot \mathrm{y}(y))}dxd^*y \\ &= \sum_{w\in W} \sum_{c\in\mathbb{N}^2} \mathrm{Kl}_{q,w}(c,M,N) \int_{T(\mathbb{R}^+)} \int_{U_w(\mathbb{R})} F^{(X)}(\iota(M)c^*wxy)\psi_N(-x)\overline{E(XN\cdot \mathrm{y}(y))}dxd^*y.\end{aligned}$$

Now define

$$A := \iota(XM)c^* w\iota(XN)^{-1}w^{-1} = \iota((XM)\cdot {}^w(XN))c^* \in T(\mathbb{R}_+). \tag{4.28}$$

Then $\mathrm{y}(A)^\eta c_1c_2 = ((XM)\cdot {}^w(XN))^\eta$. By change of variables $\iota(XN)y \mapsto y$, $\iota(XN)x\iota(XN)^{-1} \mapsto x$, we can express $\left\langle P_M^{(X)}, P_N^{(X)} \right\rangle$ as

$$\sum_{w\in W} \sum_{c\in\mathbb{N}^2} \mathrm{Kl}_{q,w}(c,M,N) \frac{(XM)^\eta (XN)^\eta}{c_1c_2\,\mathrm{y}(A)^\eta} \int_{T(\mathbb{R}_+)} \int_{U_w(\mathbb{R})} F^{(X)}(\iota(X)^{-1}Awxy)\psi_{X^{-1}}(-x)\overline{E(\mathrm{y}(y))}dxd^*y.$$

We then conclude a Kuznetsov-type trace formula.

Lemma 4.13. Let $M,N\in\mathbb{N}^2$, $X\in\mathbb{R}_+^2$, E a function on $\mathbb{R}_+^2$ with compact support, and define $F^{(X)}$ as in (4.27). Then

$$\begin{aligned}&\int_{(q)} \overline{A_\varpi(M)} A_\varpi(N) \left| \left\langle W_\mu, E^{(X)} \right\rangle \right|^2 d\varpi \qquad (4.29)\\ &= \sum_{w\in W} \sum_{c\in\mathbb{N}^2} \mathrm{Kl}_{q,w}(c,M,N) \frac{X^{2\eta}}{c_1c_2\,\mathrm{y}(A)^\eta} \int_{T(\mathbb{R}_+)} \int_{U_w(\mathbb{R})} F^{(X)}(\iota(X)^{-1}Awxy)\psi_{X^{-1}}(-x)\overline{E(\mathrm{y}(y))}dxd^*y,\end{aligned}$$

with A as in (4.28).

4.7 Proof of theorems

We establish the following proposition, from which the other theorems are proved.

Proposition 4.14. Keep the notations above. Let $m\in\mathbb{N}$ be coprime to q and $Z\ge 1$. Then

$$\int_{(q)} |A_\varpi(1,m)|^2 Z^{2\sigma_\pi(\infty)} \delta_{\lambda_\varpi\in I} d\varpi \ll_{I,\varepsilon} q^\varepsilon$$

uniformly in $mZ \ll q^2$ for a sufficiently small implied constant depending on I.

Proof. We take $X = (1, Z)$, $M = N = (1, m)$, and apply Lemma 4.13. By Lemma 4.8, there is a finite set of compactly supported functions E_j such that

$$Z^{2\eta_2+2\sigma_\pi(\infty)}\delta_{\lambda_\varpi\in I} \ll_I \sum_j \left|\left\langle W_\mu, E_j^{(X)}\right\rangle\right|^2. \tag{4.30}$$

Now we consider the arithmetic side of the Kuznetsov formula for a fixed $E^{(X)} = E_j^{(X)}$. It suffices to consider the Weyl elements $w \in W$ for which the Kloosterman sum $\mathrm{Kl}_{q,w}(c, M, N)$ does not vanish, namely, $w = \mathrm{id}, s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta, w_0$.

For $w = \mathrm{id}$, we have $c_1 = c_2 = 1$, and hence the contribution is $O(Z^{2\eta_2}) = \mathcal{O}(Z^3)$.

Now let $w \in \{s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta, w_0\}$. Apply Lemma 4.1 with $B = (XM) \cdot {}^w(XN)$. Concretely, we set

$$(B_1, B_2) = \begin{cases} (mZ, 1) & \text{if } w = s_\alpha s_\beta s_\alpha, \\ (1, (mZ)^2) & \text{if } w = s_\beta s_\alpha s_\beta, w_0. \end{cases}$$

Then we obtain

$$c_1 \ll B_1 B_2^{1/2} = mZ, \quad c_2 \ll B_1 B_2 = \begin{cases} mZ & \text{if } w = s_\alpha s_\beta s_\alpha, \\ (mZ)^2 & \text{if } w = s_\beta s_\alpha s_\beta, w_0. \end{cases}$$

We assume $mZ \ll q^2$ with a sufficiently small implied constant, such that

$$c_1, c_2 < q^2 \text{ for } w = s_\alpha s_\beta s_\alpha, \qquad \text{and } c_1 < q^2, c_2 < q^4 \text{ for } w = s_\beta s_\alpha s_\beta, w_0 \tag{4.31}$$

always hold. Now we consider the Kloosterman sums

$$\mathrm{Kl}_{q,w}(c, M, N) = \sum_{xc^*wx' \in U(\mathbb{Z})\backslash G_w(\mathbb{Q})\cap\Gamma_0(q)/U_w(\mathbb{Z})} \psi_M(x)\psi_N(x'),$$

where the entries of $M = (M_1, M_2)$ and $N = (N_1, N_2)$ are coprime to q. The Kloosterman sums are nonzero only when (4.23) and (4.24) are satisfied. By (4.25), (4.26) and (4.31), the problem reduces to computing the Kloosterman sums

$$\begin{aligned} &\mathrm{Kl}_{q,s_\alpha s_\beta s_\alpha}((q,q), M, N), \quad \mathrm{Kl}_{q,s_\beta s_\alpha s_\beta}\left((q,q^2), M, N\right), \\ &\mathrm{Kl}_{q,w_0}\left((q,q^2), M, N\right), \qquad \mathrm{Kl}_{q,w_0}\left((q,q^3), M, N\right). \end{aligned}$$

From Section 4.5.1, we see that only $\mathrm{Kl}_{q,s_\beta s_\alpha s_\beta}\left((q,q^2), M, N\right)$ does not vanish. So only $w = s_\beta s_\alpha s_\beta$ contributes.

The next step is to estimate for $w = s_\beta s_\alpha s_\beta$ the integral

$$\begin{aligned} &\left|\int_{T(\mathbb{R}_+)}\int_{U_w(\mathbb{R})} F^{(X)}(\iota(X)^{-1}Awxy)\psi_{X^{-1}}(-x)\overline{E(\mathrm{y}(y))}dxd^*y\right| \\ &\le \int_{T(\mathbb{R}_+)}\int_{U_w(\mathbb{R})} |E(\mathrm{y}(Awxy))E(\mathrm{y}(y))|\,. \end{aligned}$$

This integral is bounded by the size of the set of $x \in U_w(\mathbb{R})$ such that $\mathrm{y}(Awxy)$ lies in the support of E. Using Lemma 4.1 and Lemma 4.3, we deduce that

$$\begin{aligned} &\left|\int_{T(\mathbb{R}_+)}\int_{U_w(\mathbb{R})} F^{(X)}(\iota(X)^{-1}Awxy)\psi_{X^{-1}}(-x)\overline{E(\mathrm{y}(y))}dxd^*y\right| \\ &\ll_E \mathrm{vol}\left\{x \in U_w(\mathbb{R}) \;\middle|\; \Delta_1(wx) \ll_E \mathrm{y}(A)_1\,\mathrm{y}(A)_2^{1/2},\ \Delta_2(wx) \ll_E \mathrm{y}(A)_1\,\mathrm{y}(A)_2\right\} \ll_E \mathrm{y}(A)^{\eta(1+\varepsilon)}. \end{aligned}$$

So the contribution from $w = s_\beta s_\alpha s_\beta$ is given by

$$\sum_{c\in\mathbb{N}^2} \mathrm{Kl}_{q,w}(c,M,N)\frac{X^{2\eta}}{c_1c_2\,\mathrm{y}(A)^\eta}\int_{T(\mathbb{R}_+)}\int_{U_w(\mathbb{R})} F^{(X)}(\iota(X)^{-1}Awxy)\psi_{X^{-1}}(-x)\overline{E(\mathrm{y}(y))}dxd^*y$$
$$\ll_E \sum_{c_1'\ll mZ/q}\frac{Z^{2\eta_2}\,\mathrm{y}(A)^\varepsilon}{q} \ll Z^3q^\varepsilon.$$

Combining the estimates with (4.30), we obtain

$$\int_{(q)} |A_\varpi(1,m)|^2 Z^{3+2\sigma_\pi(\infty)}\delta_{\lambda_\varpi\in I}d\varpi \ll_I \int_{(q)} |A_\varpi(1,m)|^2 \left|\left\langle W_\mu, E^{(X)}\right\rangle\right|^2 d\varpi \ll_\varepsilon Z^3q^\varepsilon.$$

Dividing both sides by Z^3 yields the theorem. □

Proof of Theorem 1.4. It follows easily from Proposition 4.14, Theorem 4.10 and the estimate (4.15) that

$$\sum_{\pi\in\mathcal{F}_I(q)} \left|\lambda'(m,\pi)\right|^2 Z^{2\sigma_\pi(\infty)} \ll_\varepsilon q^{3+\varepsilon}\int_{(q)} |A_\varpi(1,m)|^2 Z^{2\sigma_\pi(\infty)}\delta_{\lambda_\varpi\in I} \ll_{I,\varepsilon} q^{3+\varepsilon}. \qquad □$$

Proof of Theorem 1.5. This is just a simple variation of the proofs above. Again we have

$$\sum_{\pi\in\mathcal{F}_I(q)} \Big|\sum_{\substack{m\le x\\(m,q)=1}} \alpha(m)\lambda'(m,\pi)\Big|^2 \ll_\varepsilon q^{3+\varepsilon}\int_{(q)}\Big|\sum_{\substack{m\le x\\(m,q)=1}}\alpha(m)A_\varpi(M)\Big|^2\delta_{\lambda_\varpi\in I}d\varpi$$

$$= q^{3+\varepsilon}\sum_{\substack{m_1,m_2\le x\\(m_1m_2,q)=1}}\alpha(m_1)\overline{\alpha(m_2)}\int_{(q)} A_\varpi(M_1)\overline{A_\varpi(M_2)}\delta_{\lambda_\varpi\in I}d\varpi,$$

where $M=(1,m)$, $M_1=(1,m_1)$, $M_2=(1,m_2)$. Now we apply Lemma 4.13 and evaluate the Kloosterman sums on the arithmetic side. For $w\neq \mathrm{id}$, apply Lemma 4.1 with $B = M_1\cdot {}^wM_2$. We get

$$c_1 \ll (m_1m_2)^{1/2}\le x,\quad c_2\ll m_1\le x \qquad \text{for } w = s_\alpha s_\beta s_\alpha,$$
$$c_1 \ll (m_1m_2)^{1/2}\le x,\quad c_2\ll m_1m_2\le x^2 \qquad \text{for } w = s_\beta s_\alpha s_\beta, w_0.$$

Note that when $x\ll q$ with a sufficiently small implied constant, the condition $(m,q)=1$ is void, and we deduce from (4.23) that the Kloosterman sums $\mathrm{Kl}_{q,w}(c,M,N)$ are empty for $w\neq\mathrm{id}$. Hence only the trivial Weyl element contributes, and we obtain the desired bound. □

Proof of Corollary 1.6. Note that the renormalisation $\lambda'(m,\pi) := m^{-3/2}\lambda(m,\pi)$ moves the critical strip to $0<\mathrm{Re}\, s<1$. Observe that for $\pi\in\mathcal{F}_I(q)$ an approximate functional equation has length $q^{1/2}$ (see [IK04, Section 5]). So, for all but $O(1)$ cuspidal representations $\pi\in\mathcal{F}_I(q)$ (and $\varepsilon<1/2$) we have

$$|L(1/2+it,\pi)|^2 \ll_{I,t,\varepsilon} q^\varepsilon \sum_{2^j=M\le q^{1/2+\varepsilon}}\frac{1}{M}\Big|\sum_{M\le m\le 2M}\lambda'_\pi(m)\Big|^2.$$

The statement then follows from Theorem 1.5. □

Proof of Theorem 1.3. We first assume $v = p \neq q$ is a finite place. We choose ν_0 maximal such that $p^{\nu_0} \ll q^2$ with an implied constant that is admissible to Proposition 4.14. Then by Lemma 4.12 and the estimate (4.15), there exists $\nu_0 - 5 \leq \nu_\pi \leq \nu_0$ such that

$$|A_\varpi(1, p^{\nu_\pi})|^2 \gg q^{-3-\varepsilon} p^{2\nu_\pi \sigma_\pi(p)}.$$

Note that $p^{\nu_\pi} \asymp q^2$. We apply Proposition 4.14 with $m = p^{\nu_\pi}$, $Z = 1$, and conclude that

$$N_p(\sigma, \mathcal{F}_I(q)) \leq \sum_{\pi \in \mathcal{F}_I(q)} \frac{p^{2\nu_\pi \sigma_\pi(p)}}{p^{2\nu_\pi \sigma}} \ll q^{3-4\sigma+\varepsilon} \int_{(q)} \sum_{\nu_0 - 5 \leq \nu \leq \nu_0} |A_\varpi(1, p^\nu)|^2 \, \delta_{\lambda_\varpi \in I} \ll_{I,\varepsilon} q^{3-4\sigma+\varepsilon}.$$

For $v = \infty$, we use the estimate (4.15), apply Proposition 4.14 with $m = 1$, $Z \ll q^2$, and conclude that

$$N_\infty(\sigma, \mathcal{F}_I(q)) \leq \sum_{\pi \in \mathcal{F}_I(q)} Z^{2\sigma_\pi(\infty) - 2\sigma} \ll q^{3-4\sigma+\varepsilon} \int_{(q)} |A_\varpi(1,1)|^2 \, Z^{2\sigma_\pi(\infty)} \ll_{I,\varepsilon} q^{3-4\sigma+\varepsilon}. \qquad \square$$

4.8 Appendix: Computation of Fourier coefficients

In this appendix, we outline an algorithm for computing arbitrary Fourier coefficients of a cuspidal newform $\varpi_1 \in V_\pi$ with $A_{\varpi_1}(1,1) = 1$. For this purpose, it suffices to compute the actions of $T(p)$ and $T_{0,1}^{(2)}(p)$, which generate the Hecke algebra. By Proposition 4.9, we compute

$$\lambda(p,\pi) A_{\varpi_1}(M_1, M_2) = p^{3/2}\big(A_{\varpi_1}(M_1, pM_2) \underbrace{+ A_{\varpi_1}(p^{-1}M_1, pM_2)}_{\text{if } p \mid M_1} \underbrace{+ A_{\varpi_1}(pM_1, p^{-1}M_2) + A_{\varpi_1}(M_1, p^{-1}M_2)}_{\text{if } p \mid M_2}\big), \tag{4.32}$$

and if $p \nmid M_2$,

$$\big(\lambda_{0,1}^{(2)}(p,\pi) + 1\big) A_{\varpi_1}(M_1, M_2) = p^2\big(A_{\varpi_1}(pM_1, M_2) \underbrace{+ A_{\varpi_1}(p^{-1}M_1, p^2M_2) + A_{\varpi_1}(p^{-1}M_1, M_2)}_{\text{if } p \mid M_1}\big). \tag{4.33}$$

We proceed to show how the Fourier coefficients $A_{\varpi_1}(p^{k_1}, p^{k_2})$ are obtained. Starting from $A_{\varpi_1}(1,1) = 1$, we apply (4.32) and (4.33) with $M = (1,1)$ and solve the coefficients

$$A_{\varpi_1}(p, 1) = p^{-2}\left(\lambda_{0,1}^{(2)}(p) + 1\right), \quad A_{\varpi_1}(1, p) = p^{-3/2}\lambda(p).$$

Inductively, suppose the Fourier coefficients $A_{\varpi_1}(p^{k_1}, p^{k_2})$ are known for all $k_1 + k_2 \leq r$. For $0 \leq k \leq r$, applying (4.32) with $M = (p^k, p^{r-k})$ yields the coefficient $A_{\varpi_1}(p^k, p^{r-k+1})$. Then, applying (4.33) with $M = (p^r, 1)$ yields the coefficient $A_{\varpi_1}(p^{k+1}, 1)$, since the coefficient $A_{\varpi_1}(p^{k-1}, p^2)$ has already been determined. This shows that the Fourier coefficients $A_{\varpi_1}(p^{k_1}, p^{k_2})$ with $k_1 + k_2 \leq r + 1$ can be expressed in terms of $\lambda(p)$ and $\lambda_{0,1}^{(2)}(p)$, finishing the induction.

Writing $X := p^{-3/2}\lambda(p,\pi)$ and $Y := p^{-2}\left(\lambda_{0,1}^{(2)}(p,\pi) + 1\right)$, the Fourier coefficients $A_{\varpi_1}(p^{k_1}, p^{k_2})$ for small k_i are computed in the following table:

(k_1, k_2)	$A_{\varpi_i}(p^{k_1}, p^{k_2})$
$(0,0)$	1
$(0,1)$	X
$(1,0)$	Y
$(0,2)$	$X^2 - Y - 1$
$(1,1)$	$XY - X$
$(2,0)$	$-X^2 + Y^2 + Y$
$(0,3)$	$X^3 - 2XY - X$
$(1,2)$	$X^2Y - X^2 - Y^2 - Y + 1$
$(2,1)$	$-X^3 + XY^2 + X$
$(3,0)$	$-2X^2Y + Y^3 + X^2 + 2Y^2 - 1$
$(0,4)$	$X^4 - 3X^2Y - X^2 + Y^2 + 2Y$
$(1,3)$	$X^3Y - X^3 - 2XY^2 + 2X$
$(2,2)$	$-X^4 + X^2Y^2 + X^2Y - Y^3 + 2X^2 - 2Y^2$
$(3,1)$	$-2X^3Y + XY^3 + 2X^3 + XY^2 - 2X$
$(4,0)$	$X^4 - 3X^2Y^2 + Y^4 + 3Y^3 - X^2 + Y^2 - 2Y$
$(0,5)$	$X^5 - 4X^3Y - X^3 + 3XY^2 + 4XY - X$
$(1,4)$	$X^4Y - X^4 - 3X^2Y^2 + X^2Y + Y^3 + 2X^2 + 2Y^2 - Y - 1$
$(2,3)$	$-X^5 + X^3Y^2 + 2X^3Y - 2XY^3 + 2X^3 - 2XY^2 - X$
$(3,2)$	$-2X^4Y + X^2Y^3 + 2X^4 + 3X^2Y^2 - Y^4 + X^2Y - 3Y^3 - 4X^2 - Y^2 + 2Y + 1$
$(4,1)$	$X^5 - 3X^3Y^2 + XY^4 + 2X^3Y + 2XY^3 - 3X^3 - 2XY + 2X$
$(5,0)$	$3X^4Y - 4X^2Y^3 + Y^5 - 2X^4 - 3X^2Y^2 + 4Y^4 + 3Y^3 + 3X^2 - 3Y^2 - 2Y$

From Theorem 4.10, we obtain $\lambda(p^2, \pi) = p^3(X^2 - Y - 1) - p^2$. Hence the Fourier coefficients can also be expressed in terms of eigenvalues $\lambda(p^r, \pi)$ of standard Hecke operators.

It is evident from the Proposition 4.9 that Fourier coefficients are multiplicative, that is,

$$A_{\varpi_1}(M_1N_1, M_2N_2) = A_{\varpi_1}(M_1, M_2)A_{\varpi_1}(N_1, N_2) \text{ if } (M_1M_2, N_1N_2) = 1. \tag{4.34}$$

Using (4.34), and (4.14) for negative coefficients, we are able to compute $A_{\varpi_1}(M)$ for every $M \in \mathbb{Z}^2$.

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