Symplectic non-squeezing theorem and Hamiltonian partial differential equations

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Dedicated to the loving memory of Stana Cicmilović

Abstract

The thesis is concerned with providing a first natural generalization of M. Gromov's non-squeezing symplectic result in infinite dimensional case and applying it within the context of Hamiltonian partial differential equations.

First part of the thesis covers a special case of the infinite dimensional generalization, adapting the approach suggested by A. Sukhov and A. Tumanov for the treatment of the finite dimensional case. We contend that this is an important generalization of Gromov's result and contribution to an open question whether the generalization holds in full generality in infinite dimensional case.

Second part covers an application of the aforementioned result to Hamiltonian equations. Namely, we recover known non-squeezing results for mass subcritical and critical nonlinear Schrödinger equation by R. Killip, M. Visan, X. Zhang and for Korteweg–De Vries equation by J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao. The first result for complex modified Korteweg–De Vries is obtained. All previous results relied heavily on well-posedness theory at symplectic regularity. We follow the same principle, however, first part of the thesis significantly simplifies proofs of known results. This is due to the fact that all previous approaches were based on reduction of initial equations of interest to the finite dimensional Hamiltonian flow, for which one would recall Gromov's result. Whilst the choice of the reduction to the finite dimensional case was usually an obvious one, infinite dimensional formulation is natural and more flexible as approximations are needed regardless of the equation.

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Contents

1 Introduction

The Euclidean space $(\mathbb{R}^{2d}, \|\cdot\|)$, which we identify with the complex space $(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$, has the standard symplectic structure defined on it by

$$\omega_{st}(\cdot,\cdot) = \operatorname{Im}\langle\cdot,\cdot\rangle.$$

Denote by $z = (z_1, \dots, z_d) \in \mathbb{R}^{2d}, z_i \in \mathbb{C}$, and by B_r a ball $B_r = \{z \in \mathbb{R}^{2d} : |z| \le r\}$ and Σ_R a cylinder such that $\Sigma_R = \{z \in \mathbb{R}^{2d} : |z_1| \le R\}$. A symplectomorphism $\varphi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is a C^1 diffeomorphism which preserves the symplectic structure, i.e. $\varphi^*\omega_{st} = \omega_{st}$. In [Gro85], Gromov has formulated and proved what is now called symplectic non-squeezing theorem in the finite dimensional set-up, and which shows that morphisms preserving symplectic structure imply rigidity of the following kind

Theorem 1.0.1 (Gromov). *There exists a symplectomorphism $\varphi : (B_r, \omega_{st}) \to (\mathbb{R}^{2d}, \omega_{st})$ such that $\varphi(B_r) \subset \Sigma_R$ if and only if $r \le R$.*

This paper has been seen as one of the most influental ones in symplectic topology and has inspired a lot of new research in this area. However, it is a long standing open question whether the non-squeezing theorem generalizes to infinite dimensional symplectic Hilbert spaces.

First results on this matter were due to Kuksin ([Kuk95a], [Kuk95b]), motivated by implications that the non-squeezing property has on the qualitative information of the flow. Firstly, a flow having the non-squeezing property does not allow the existence of stable critical points, and secondly, it prohibits uniform evacuation of a fixed frequency on a fixed ball of initial data. Kuksin's approach was based on proving that the flows of interest preserve symplectic capacities, loosely defined as limits of finite dimensional ones as one tries to approximate the PDE with a finite dimensional Hamiltonian ODE. Existence of symplectic capacities is equivalent to the symplectic non-squeezing theorem. Kuksin considered equations posed on the torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, whose flow is a compact perturbation of a linear one, namely the nonlinear string equation

$$\partial_{tt}u = \Delta u + p(u,t,x), \qquad x \in \mathbb{T},$$

where p is a smooth function that has at most polynomial growth as $|u| \to \infty$, the quadratic nonlinear wave equation

$$\partial_{tt}u = \Delta u + a(t,x)u + b(t,x)u^2, \qquad x \in \mathbb{T}^2,$$

the nonlinear membrane equation

$$\partial_{tt}u = -\Delta^2 u + p(u,t,x), \qquad x \in \mathbb{T}^2,$$

and the Schrödinger equation

$$-i\partial_t u = -\Delta u + V(x)u + \left[\frac{\partial}{\partial \overline{U}} G(U, \overline{U}, x)\right] * \xi, \qquad U = u * \xi, \qquad x \in \mathbb{T}^n$$

where G is a smooth function and ξ fixed real function, and $*$ denotes the convolution.

Bourgain ([Bou94]) was the first to treat a flow that is not compact perturbation of a linear one, namely defocusing cubic nonlinear Schrödinger equation (NLS) posed on the torus

$$i\partial_t u + \Delta u = |u|^2 u, \qquad x \in \mathbb{T}.$$

His approach was to approximate the flow by introducing a sharp frequency cutoff in the nonlinearity

$$i\partial_t u + \Delta u + P_N(u|u|^2) = 0,$$

where P_N is Dirichlet projection with respect to x, i.e. $P_N(f) = \sum_{|n|\leq N} \widehat{f}(n)e^{inx}$, hence obtaining a finite dimensional Hamiltonian flow on space of data ϕ such that $\phi = P_N\phi$, and use Gromov's result.

Motivated by their result [CKS$^+$03] on sharpness of well-posedness in the strong sense for Korteweg De-Vries equation (KdV)

$$\partial_t q = -q_{xxx} + 6qq_x,$$

Colliander, Keel, Staffilani, Takaoka and Tao proved non-squeezing for KdV on $H_0^{-\frac{1}{2}}(\mathbb{T})$ in [CKS$^+$05]. While following the ideas of localizing the nonlinearity as done by Bourgain, the cutoff that was used was a classical Littlewood-Paley one, which is in contrast to the sharp one in [Bou94]. This is due to delicate cancellative structure in the KdV equation which permits the decoupling of high and low frequencies. As a result of it, crude frequency projection leads to impossibility of uniform approximation of KdV by a truncated flow on balls of data, as shown in [CKS$^+$05]. Whilst the approximated flow with smooth frequency projection can easily be seen to be a finite dimensional Hamiltonian on the space of functions which have the uniform upper bound for frequency in Fourier series, the difficulty was proving directly that such approximation was a good one for KdV. Said authors overcome this difficulty by using Muira transform to pass to the modified KdV (mKdV) and prove its invertibility. Then the regularity of the inverse of Muira map, coupled with better smoothing properties of the mKdV and refining estimates for well-posedness in [CKS$^+$03], concludes the proof of approximation.

Up to that point, all known results were for subcritical flows. The first one, albeit a conditional one, for the critical flow and global-in-time non-squeezing was a result of Mendelson ([Men17]) for the Klein-Gordon equation on $\mathbb{T}^3$. Conditionality comes due to the absence of global well-posedness and any uniform control on the local time of existence for arbitrary data. Additionally, Mendelson proved non-squeezing for short time dynamics.

First results in the unbounded case are due to Killip, Visan and Zhang ([KVZ19], [KVZ21]) for the nonlinear Schrödinger equation in $L^2(\mathbb{R}^d)$. These results address both (de)focusing subcritical

$$i\partial_t u + \Delta u = \kappa |u|^{p-1} u,$$

for $1 \leq p < 1 + \frac{4}{d}, \kappa = \pm 1$, and defocusing critical nonlinearity

$$i\partial_t u + \Delta u = |u|^{\frac{4}{d}} u,$$

and follow the approach based on a finite dimensional approximation of NLS by introducing a smooth frequency cutoff in the nonlinearity

$$i\partial_t u + \Delta u = \mathcal{P}(F(\mathcal{P}(u))).$$

The subcritical equation is easier to handle, as expected. To overcome the issues in the critical-scaling case, said authors develop a general methodology for obtaining the uniform global space-time bounds for suitable Fourier truncations of dispersive PDE, which allow them to conclude the non-squeezing property of critical flow by the one of frequency truncated one. Inspired by Bahouri and Gérard's work [BG99] which showed how a nonlinear profile decomposition can be used to establish the well-posedness in the weak topology in the setting of energy-critical wave equation, Killip, Visan and Zhang prove and use profile decomposition for NLS to conclude the non-squeezing of NLS from the non-squeezing property of truncated systems. Moreover, the profile decomposition plays an important role for overcoming symmetries of the initial equation and allowing them to uniformly approximate truncated systems by finite dimensional Hamiltonian ODE.

Recently, non-squeezing property of KdV on the line was resolved by Ntekoume (see [Nte19]). Presented approach reproves the result for the torus obtained in [CKS+05] in an easier manner. The main ideas are based on well-posedness result of Killip and Visan for KdV on the line and the torus in H^{-1} ([KV19]). Even though a sharp result for well-posedness on the torus was previously obtained by Kappeler and Topalov in [KT06], Killip and Visan introduced flows that approximate KdV in norm topology, on bounded sets of data, uniformly on intervals of time, and thus obtained well-posedness both on the line and the circle, in contrast to [KT06]. Non-squeezing property of KdV then reduces to proving it for the approximate flows. Ntekoume proved that localizing in frequency approximate flows in a suitable way leads to finite dimensional approximation, from which one can conclude non-squeezing by invoking Gromov's result. We would like to point out that Ntekoume's result on the line is unaccessible by our approach; indeed, even though both the line and the circle case have the same Poisson structure, the former does not admit a non-degenerate almost complex structure, leading to the absence of a symplectic structure. Nevertheless, Ntekoume was able to prove non-squeezing behaviour in the line case as well.

All of the mentioned results depended on Gromov's finite dimensional result. A natural question was whether one can prove a genuine infinite dimensional analogue of it in contrast to finite dimensional approximation, or at least, find a weaker version of it, which can be used to conclude non-squeezing properties of infinite dimensional symplectomorphisms.

First results on infinite dimensional non-squeezing are due to Abbondandolo and Majer ([AM15]), for symplectomorphisms that map balls of data into convex sets. This was the first result whose techniques were purely of infinite dimensional nature. Convexity of the image allowed them to construct an infinite dimensional capacity, whose existence directly implies the non-squeezing property. However, since there is no reason to expect that nonlinear PDE take balls of initial data at any time to a convex set, aside from short time intervals, one cannot apply it to many equations of interest previously stated.

Second results are due to Sukhov and Tumanov in [ST16a],[ST16b]. Stated results are a continuation of their work [ST14], where said authors reproved Gromov's finite result [Gro85]. Motivated by Gromov's approach of constructing a pseudo-holomorphic disc with good boundary conditions and of desired area, instead of using topological tools as done by Gromov, Sukhov and Tumanov treat the construction of the disc as an analytic problem closely related to the Beltrami equation. They generalize the construction to the infinite dimensional case in [ST16b], for a class of symplectomorphisms that have uniform regularity behaviour with respect to Hilbert scales, allowing them to obtain compactness in infinite dimensional case. Assuming this behaviour, one can consider long-time flows - the scope of their result allows them to obtain non-squeezing for discrete Schrödinger equation. Alternatively, in [ST16a] they also prove that the non-squeezing holds under smallness assumption of the anti-holomorphic part of the symplectomorphism. Consequently, small C^2 perturbation of the identity and short-time flows are another class of non-squeezing flows. However, this is just as restrictive as Abbondandolo and Majer's result ([AM15]).

As expected and shown in [ST16a], many important bounds carry over from finite dimensional one, allowing the same fixed point argument to be used for construction of the pseudo-holomorphic disc. Recurring issue in infinite dimensional generalization (as in [ST16b]) is the compactness of the underlying function space in which all potential pseudo-holomorphic discs reside. The novelty of this thesis is the introduction of a class of weakly continuous symplectomorphisms that solve this issue. Moreover, these assumptions allow us to consider many important dispersive equations, regardless of short or long-time dynamics. We contend that our result is the first infinite dimensional one that is natural with respect to the applications to Hamiltonian PDE. In order to state it, let us introduce some notation.

Let $\mathbb{H}$ be a separable complex Hilbert space with a Hermitian inner product $\langle\cdot,\cdot\rangle$ and an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Every complex Hilbert space $\mathbb{H}$ has the standard symplectic structure given by

$$\omega(x,y) = \operatorname{Im}\langle x,y\rangle_{\mathbb{H}}.$$

Denote by $B(\mathbb{H})$ the space of bounded linear operators on $\mathbb{H}$ and by τ_{weak} the weak topology on $\mathbb{H}$.

Definition 1.0.2. *A symplectomorphism is a C^1 diffeomorphism $\varphi : \mathbb{H} \to \mathbb{H}$ such that $\varphi^*\omega = \omega$. A map $\varphi : \mathbb{H} \to \mathbb{H}$ is continuous with respect to weak topology if $\varphi : (\mathbb{H}, \tau_{weak}) \to (\mathbb{H}, \tau_{weak})$ is continuous in the topological sense.*

All equations of interest are maps that are continuous with respect to weak topology. Moreover, at least from current perspective, it appears that this is a natural way of looking at this equations (topologically wise) as the non-squeezing inequality is a weak type of statement. We denote the ball of radius r with $B_r := \{h \in \mathbb{H} : \|h\| \leq r\}$. Moreover, let

$$\Sigma_R := \{h \in \mathbb{H} : |\Pi_1 h| \leq R\}$$

be a cylinder of radius R, where $\Pi_1 : \mathbb{H} \to \mathbb{C}$ denotes the projection defined as $\Pi_1(h) := \langle h, e_1 \rangle$. The principal result of this thesis is the following

Theorem 1.0.3. *Let $\varphi : (\mathbb{H}, \omega) \to (\mathbb{H}, \omega)$ be a symplectomorphism such that φ and φ^{-1} are continuous with respect to the weak topology on $\mathbb{H}$ and such that the map*

$$D\varphi : (\mathbb{H}, \tau_{weak}) \to (B(\mathbb{H}), \|\cdot\|_{op})$$

is continuous. If $\varphi(\mathbb{B}_r) \subset \Sigma_R$, then $r \leq R$.

We would like to point out that even though the stated continuity of the map $D\varphi$ in Theorem 1.0.3 may seem like an independent assumption, in our applications the continuity of $D\varphi$ in operator norm will come off as a directly corollary of the weak continuity of $\varphi : (\mathbb{H}, \tau_{weak}) \to (\mathbb{H}, \tau_{weak})$. Assumption on the continuity of inverse φ^{-1} is stronger than needed - it suffices to assume that inverse maps bounded sets into bounded sets, from which the continuity with respect to weak topology follows. The result is also a generalization of Gromov's result since weak and strong topologies coincide in the finite dimensional case. However, with applications to equations in mind, one can restate non-squeezing inequality for arbitrary centered ball and cylinder in the following fashion

Corollary 1.0.4. *Let $\varphi : (\mathbb{H}, \omega) \to (\mathbb{H}, \omega)$ be a symplectomorphism such that φ and φ^{-1} are continuous with respect to the weak topology on $\mathbb{H}$ and such that the map*

$$D\varphi : (\mathbb{H}, \tau_{weak}) \to (B(\mathbb{H}), \|\cdot\|_{op})$$

is continuous. Let $h_0, l \in \mathbb{H}$ be such that $\|l\|_{\mathbb{H}} = 1$ and $0 < R < r < \infty$, $\alpha \in \mathbb{C}$. Then there exists

$$h_1 \in B_r(h_0) := \{h \in \mathbb{H} : \|h - h_0\|_{\mathbb{H}} < r\}$$

such that

$$|\langle \varphi(h_1), l \rangle - \alpha| > R.$$

Corollary follows directly by observation that

$$\langle \varphi(h_1) - \alpha l, l\rangle = \langle U^{-1}[\varphi(h_1) - \alpha l], U^{-1}l\rangle = \langle U^{-1}[\varphi(h_1) - \alpha l], e_1\rangle,$$

where U is a unitary operator on $\mathbb{H}$ such that $U(e_1) = l$, and the fact that unitary operators and translations are symplectomorphisms, hence reducing the Corollary 1.0.4 to Theorem 1.0.3.

We will be interested in reproving non-squeezing for the (sub)critical NLS on $\mathbb{R}^d$ and KdV on the torus. The result for mKdV on $\mathbb{R}$ is new and first to our knowledge. Symplectomorphism by definition has to be at least a C^1 map. Except mKdV, every flow ((sub)critical NLS, KdV) we consider is of at least of C^1 regularity with respect to norm induced topology ([Bou93a],[Bou93b],[CKS+03],[CKS+05],[CCT03][CKS+04]) in phase space and globally well-posed, meaning that symplectomoprhisms that we will be considering, i.e. flows for any fixed time, are well defined. The mKdV on the contrary is only continuous in L^2 - analytic continuity holds sharply at $H^{\frac{1}{4}}$ regularity ([KPV93], [KPV01], [CCT03]).

Both mass subcritical and critical NLS are continuous maps on L^2 with respect to weak topology. The issue with applying Theorem 1.0.3 is with the derivative continuity assumption. Translation symmetry of the equation prohibits this possibility. We deal with this issue by localizing the nonlinearity in space and use local smoothing to obtain a gain in regularity, and hence compactness. Local smoothing plays crucial role in establishing the non-squeezing property for truncated flows. The original case follows by arguments presented in following section.

Regarding KdV, just as in [Nte19], we shall reduce checking non-squeezing to approximate flows introduced in [KV19] and which were based on the integrable nature of the equation. The non-squeezing property of KdV will follow by the fact that the symplectic regularity is higher than one of endpoint of well-posedness, hence we will be able to obtain necessary equicontinuity of the set we want to approximate and use result of [KV19]. However, this regularity disparity will lead to compactness, which we shall exploit independently in order to prove that approximate flows satisfy assumptions of Theorem 1.0.3.

Lastly, regarding mKdV on the line, low regularity global well-posedness result by Harrop-Griffiths, Killip and Visan ([HGKV20]) plays a crucial role. Combining ideas of [KVZ18] and [KV19], authors introduced approximate flows which imply low regularity well-posedness theory for mKdV and NLS on the line. Reduction to proving non-squeezing for approximate flows rather than the original one is of the same spirit as the one for KdV.

1.1 Application of Theorem 1.0.3 to Hamiltonian PDE

We would like to present the ideas that inspired the assumptions of Theorem 1.0.3 and the overall approach one takes in order to conclude the non-squeezing property of flows of nonlinear equations. While the continuity with respect to weak topology holds for every equation of our interest, many notable equations, such as nonlinear Schrödinger one on $\mathbb{R}^d$, do not satisfy the continuity of derivative assumption of Theorem 1.0.3 - one needs to make some adjustments and take a look at an equation similar to the original one. For example, symmetries of said equation, translation one in particular, imply that in order to get compactness of solutions one has to localize in some sense. These adjustments can be interpreted as a family of one, having the property that if one fixes initial data, approximations converge to the solution of the initial equation. Moreover, these adjustments will crucially lead to a family of maps which will retain the same property of the initial map of interest - continuity with respect to the weak topology.

Families will be indexed by real numbers going to infinity, but we can reduce the index set to the interval $\mathcal{I} = [0,1]$, where 1 corresponds to the infinity. We define it as a metric space $(\mathcal{I}, d_1)$ via the norm $|\cdot|$ from $\mathbb{R}$. Moreover, for any other metric space (M, d_M), the product space $\mathcal{I} \times M$ is a metric space with the metric given by

$$d_2\Big((x,\tau_1),(y,\tau_2)\Big) = \Big(d_1(\tau_1,\tau_2)^2 + d_M(x,y)^2\Big)^{\frac{1}{2}}, \quad \forall x,y \in M, \quad \forall \tau_1,\tau_2 \in \mathcal{I},$$

and henceforth when talking about continuity of a map with product space for a domain, the topology in question will be the one induced by d_2. Naturally, ball $\mathbb{B}_r \subset \mathbb{H}$ with weak topology will be of particular interest, as it is metrizable with the metric we denote by d_w. In that case, X_r will denote said ball endowed with the topology induced by the metric d_w. This thesis follows the following general principle

Theorem 1.1.1. *Let $\mathbb{H}$ be a complex Hilbert space with the standard symplectic form and let*

$$\Phi_\tau := \Phi(\tau,\cdot) : \mathcal{I} \times \mathbb{H} \to \mathbb{H}$$

be a family of symplectomorphisms that exibits non-squeezing behaviour for a fixed functional $e_1 \in \mathbb{H}^*$ in the sense of Theorem 1.0.3 and for $\tau \in [0,1)$. Assume that for all $\tau \in \mathcal{I}$, $\Phi_\tau(\mathbb{B}_r) \subset \mathbb{B}_{cr}$ for every r, where the constant c does not depend on the index set $\mathcal{I}$, and that

$$\Phi_\tau := \Phi(\tau,\cdot) : \mathcal{I} \times X_r \to X_{cr}$$

is a continuous family as well. Then Φ_1 is uniformly approximated by Φ_τ in $C(X_r, X_{cr})$ and is consequently non-squeezing.

We shall distinguish different approximations. Some will be with respect to the norm of the phase space, for bounded sets of data, which provide a good approximation of the initial flow in an obvious way - the non-squeezing data mapped outside the cylinder for the approximate flow will be the same one for the original flow. However, as we are unable to approximate all flows of interest in this fashion, we shall be talking about approximations with respect to the weak topology on the space of initial data as well. As an example of the latter we state the nonlinear Schrödinger equation (NLS) on the line

$$i\partial_t u + \Delta u = |u|^2 u.$$

posed in $L^2(\mathbb{R})$. Even though the equation is continuous with respect to the weak topology on L^2, the derivative of the flow is not continuous in the sense of Theorem 1.0.3 due to translation symmetry of the equation. This can be readily seen by observing the sequence $u_n(t,x) = u(t,x+n) \rightharpoonup 0$. We overcome this difficulty by observing that the truncated flow (NLSR)

$$i\partial_t u_R + \Delta u_R = \chi_R |u_R|^2 u_R,$$

where χ_R is a characteristic function of the interval $[-R,R]$, fulfills the assumptions of Theorem 1.0.3. Even more so, this truncation is a good one, as we can approximate NLS uniformly for bounded initial data by NLSR. In particular, for fixed initial data $u(0)$, $u_R \rightharpoonup u$ as $R \to \infty$.

Motivated by previous, we generalize and denote by $\Phi_1 : \mathbb{H} \to \mathbb{H}$ the flow of interest and $\{\Phi_\tau\}_{\tau\in\mathcal{I}} : \mathbb{H} \to \mathbb{H}$ family of approximations indexed by set $\mathcal{I}$. The non-squeezing inequality is a weak type one, that is we want to obtain relevant qualitative information about the flow Φ_1 by evaluating it with a fixed functional $e_1 \in \mathbb{H}^*$ - we search for $u_0 \in \mathbb{B}_r$ satisfying $|e_1\big(\Phi_1(u_0)\big)| > R$. Hence we shall be looking at the approximations Φ_τ that converge weakly to Φ_1 pointwise, i.e. $\Phi_\tau(u_0) \rightharpoonup \Phi_1(u_0)$. Crucial for finding the witness of non-squeezing for Φ_1 from the ones of Φ_τ will be the uniform approximation of Φ_1 by Φ_τ in $C(X_r, X_{cr})$. This is the first point at which assumptions of Theorem 1.0.3 come into play - exploiting the weak continuity assumption for $\{\Phi_\tau\}$, we shall be looking at balls of fixed radius $\mathbb{B}_r \subset \mathbb{H}$ in the well-posedness space, which are compact when endowed with weak topology, and hence we are able to apply Arzela-Ascholi theorem in this setting. The radius cr will be the one given by the well-posedness theory - this comes as no surprise since all equations of interest are globally well-posed, hence the norm if uniformly bounded for all times.

Proof of Theorem 1.1.1. Firstly, let $u_\tau \in \mathbb{B}_r$ represent the initial data witnessing the non-squeezing in the sense of Theorem 1.0.3, that is $|e_1 \circ \Phi_\tau(u_\tau)| > R$. Define $u := \text{w-}\lim_\tau u_\tau \in X_r$. Secondly, since

$$\Phi_\tau := \Phi(\tau,\cdot) : \mathcal{I} \times X_r \to X_{cr}$$

is continuous and $\mathcal{I} \times X_r$ is compact, Φ is uniformly continuous, i.e.

$$(\forall \varepsilon > 0)(\exists \delta > 0) \;\; d_2\Big((x,\tau_1),(y,\tau_2)\Big) < \delta \implies d_w(\Phi_{\tau_1}(x), \Phi_{\tau_2}(y)) < \varepsilon,$$

$x, y \in X_r$, $\tau_1, \tau_2 \in \mathcal{I}$. Consequently, $\{\Phi_\tau\}_{\tau\in\mathcal{I}}$ is an uniformly equicontinuous family in $C(X_r, X_{cr})$. Arzela-Ascoli then implies that Φ_τ uniformly converges to Φ_1 in $C(X_r, X_{cr})$.

The non-squeezing property of Φ_1 follows from the ones of Φ_τ, $\tau \in [0,1)$, by

$$d_w(\Phi_\tau(u_\tau), \Phi(u)) \le d_w(\Phi_\tau(u_\tau), \Phi_\tau(u)) + d_w(\Phi_\tau(u), \Phi(u)).$$

□

2 Symplectic non-squeezing theorem

2.1 Symplectic geometry preliminaries

In contrast to the finite dimensional case, there are different notions of symplectic structure in infinite dimensional one. Namely,

Definition 2.1.1. *A strong (weak) symplectic form* ω *on a real Hilbert space* $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ *is a skew-symmetric continuous 2-form*

$$\omega : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$$

which is non-degenerate, in the sense that the associated linear mapping

$$\Omega : \mathbb{H} \to \mathbb{H}^*$$
$$\Omega : h \mapsto \omega(h, \cdot)$$

is an isomorphism (injective).

In this exposition we shall be dealing with strong symplectic forms exclusively, henceforth symplectic form will always be a strong one. Let $B(\mathbb{H})$ be the space of $\mathbb{R}$-linear bounded operators on $\mathbb{H}$ and denote by I the identity. Moreover, when talking about operators in this chapter, we assume that they are $\mathbb{R}$-linear, unless explicitly specified otherwise. We identify the tangent space at every point of $\mathbb{H}$ with $\mathbb{H}$.

Definition 2.1.2. *An almost complex structure on* $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ *is a continuous map*

$$J : (\mathbb{H}, \|\cdot\|) \to (B(\mathbb{H}), \|\cdot\|_{op})$$

such that $J^2(h) = -I$ *for every* $h \in \mathbb{H}$*. Additionally, the Hilbert product* $\langle\cdot,\cdot\rangle$ *and the induced norm* $\|\cdot\|$ *are said to be compatible with a symplectic structure* ω *on* $\mathbb{H}$ *if there exists an almost complex structure such that* $\langle J\cdot,\cdot\rangle = \omega(\cdot,\cdot)$*.*

Previous definition indicates that having two out of the three structures determines a third one. Moreover, for any symplectic form ω on $(\mathbb{H}, \langle\cdot,\cdot\rangle)$, there exists an equivalent inner product $\langle\cdot,\cdot\rangle_1$ such that the operator $J : \mathbb{H} \to \mathbb{H}$ defined by

$$\langle J\cdot,\cdot\rangle_1 = \omega(\cdot,\cdot),$$

is a complex structure on $\mathbb{H}$, i.e. $J^2 = -I$.

For any real Hilbert space $\mathbb{H}$ and an almost complex structure J on it, there exists an equivalent norm and an inner product on $\mathbb{H}$ making it a complex Hilbert space (Prop 2.2, [ST16a]). Lastly, all separable and infinite dimensional complex Hilbert spaces are isometrically isomorphic.

Previous comments allow us from now on to consider a separable complex Hilbert space $\mathbb{H}$ with an inner product $\langle\cdot,\cdot\rangle$. Standard symplectic structure is then given by

$$\omega_{st} = \operatorname{Im}\langle\cdot,\cdot\rangle,$$

and the standard almost complex structure J_{st} given as multiplication by $i : \mathbb{H} \to \mathbb{H}$, $i : h \mapsto ih$ for all $h \in \mathbb{H}$. Standard complex structure J_{st} is compatible with ω_{st}.

Let $\{e_n\}_{n\in\mathbb{N}}$ be the orthonormal basis of $\mathbb{H}$. We have coordinate series $h = \sum_{n=1}^{\infty} h_n e_n$, where $h_n = \langle h, e_n\rangle = x_n + iy_n \in \mathbb{C}$, for every $h \in \mathbb{H}$. Moreover, we have complex conjugation defined as $\bar{h} = \sum_{n=1}^{\infty} \bar{h}_n e_n$. Abusing the given notation, we shall also write the decomposition $h = h_1 + h_2$ for every $h \in \mathbb{H}$, where h_1 represents the projection of h onto subspace $\langle e_1\rangle$, that is $h_1 = \Pi_1(h)e_1$, where $\Pi_1(h) := \langle h, e_1\rangle$. It should be clear from the text which notation we are using.

Rewritten in complex coordinates, the standard symplectic structure is given as

$$\omega_{st} = \frac{i}{2}\sum_{k=1}^{\infty} dh_k \wedge d\bar{h}_k,$$

where $dh_k = dx_k + idy_k$ and $d\bar{h}_k = dx_k - idy_k$ are 1-forms coordinate-wise on $\mathbb{C} = \mathbb{R}^2$. In the rest of the paper, we will denote the standard symplectic structure by ω.

Definition 2.1.3. *A C^1 diffeomorphism $\varphi : (\mathbb{H}_1, \omega_1) \to (\mathbb{H}_2, \omega_2)$ is a* symplectomorphism *if $\varphi^*\omega_2 = \omega_1$, where the pull-back is defined as*

$$\varphi^*\omega_2[h](v_1, v_2) = \omega_2[\varphi(h)]\Big(D\varphi(v_1), D\varphi(v_2)\Big), \quad \forall h, v_1, v_2 \in \mathbb{H}_1.$$

Let J_1 be an almost complex structure on H_1 compatible with ω_1. Such diffeomorphism induces an almost complex structure on the image compatible with ω_2 which we call *induced almost complex structure* and which is given by

$$J_2 = \varphi_*(J_1) := D\varphi \circ J_1 \circ D\varphi^{-1}.$$

That J_2 is an almost complex structure can be computed directly.

One of the main objects in this section is a notion of a pseudo-holomorphic disc

Definition 2.1.4. *Denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ the unit disc. We call a C^1 smooth map $u : (\mathbb{D}, i) \to (\mathbb{H}, J)$ a J-*holomorphic disc *if*

$$J \circ du = du \circ i, \tag{2.1}$$

where $du(z) : T_z\mathbb{D} \to T_{u(z)}\mathbb{H}$ denotes the derivative.

We shall rewrite the J-holomorphic property (2.1) in terms of complex derivatives. Firstly, we write $z = x + iy \in \mathbb{C}$ and use the notation

$$dz = dx + idy$$

and

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2.2}$$

Moreover, the complex structure on $\mathbb{D}$ acts on its tangent space so that $\frac{\partial}{\partial y} = i\frac{\partial}{\partial x}$, where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the tangent vectors spanning the tangent space. Hence we also consider $\frac{\partial}{\partial \bar{z}}$, $\frac{\partial}{\partial z}$ as vectors spanning the same tangent space on $\mathbb{D}$. Denoting the vector by $\partial u = du(\frac{\partial}{\partial z})$, J-holomorphic condition (2.1) becomes

$$(J_{st} - J(u))\partial u = (J_{st} + J(u))\overline{\partial} u.$$

This follows readily by equalities

$$J(u(z)) \circ du\left(\frac{\partial}{\partial x}\right) = du\left(\frac{\partial}{\partial y}\right)$$
$$J(u(z)) \circ du\left(\frac{\partial}{\partial y}\right) = -du\left(\frac{\partial}{\partial x}\right)$$

and using previously introduced complex notation.

Since $u : \mathbb{D} \to \mathbb{H}$ is a map that is invariant with respect to the underlying almost complex structures i and J, and i is compatible with the standard symplectic form on $\mathbb{D}$, so is the case with J and ω on $\mathbb{H}$. Compatibility, and hence non-degeneracy of form ω implies that the operator $J_{st} + J(u(z))$ is invertible for every $z \in \mathbb{D}$ (see Prop. 2.8, [ST16a]), Indeed, the invertibility of the stated operator is equivalent to the one of $I - J_{st}J(u)$, which is equivalent to the form $\langle J_{st}J(u)h, h\rangle > 0$ being positive for all non-zero $h \in \mathbb{H}$. The last however holds because both $J_{st}, J(u)$ are compatible with ω in the sense of Definition 2.1.2. In particular the linear operator

$$L := (J_{st} + J(u))^{-1}(J_{st} - J(u))$$

is well defined. Moreover, L is antilinear with respect to $\mathbb{C}$, i.e., $J_{st}L = -LJ_{st}$. Hence, there exists a bounded linear with respect to $\mathbb{C}$ operator $A_J \in B(\mathbb{H})$ such that

$$Lh = A_J\bar{h}.$$

Indeed, every operator $S \in B(\mathbb{H})$ can be uniquely written in the form

$$Su = Pu + Q\overline{u},$$

where P and Q are linear w.r.t $\mathbb{C}$ operators in $B(\mathbb{H})$. Since L is antilinear w.r.t $\mathbb{C}$, that means that for every $u = u_1 + iu_2 \in \mathbb{H}$,

$$
\begin{aligned}
&Lu = Lu_1 + Liu_2 = Lu_1 - iLu_2 = (Pu_1 + Qu_1) - i(Pu_2 + Qu_2),\\
&Lu = (Pu_1 + Qu_1) + i(Pu_2 - Qu_2),
\end{aligned}
$$

that is, $Pu_2 = 0$ for every $u_2 \in \mathbb{H}$, hence $P = 0$, hence $Lu = Q\overline{u}$. We denote Q by A_J.

We call A_J the *complex representation* of J. Finally, the J-holomorphicity gives the following equation in complex coordinates which we call *Beltrami type equation*

$$
\bar{\partial}u(z) = A_J(u(z))\overline{\partial u}(z). \tag{2.3}
$$

Inspired by this, the centerpiece of this chapter is construction of a disc $u : \mathbb{D} \to \mathbb{H}$ that solves the equation

$$
\begin{aligned}
&\bar{\partial}u(z) = A(u(z))\overline{\partial u}(z) &(2.4)\\
&A : (\mathbb{H}, \tau_{weak}) \to (B(\mathbb{H}), \|\cdot\|_{op}),
\end{aligned}
$$

and that has desired properties, such as boundary conditions and area. Consequently, we shall be searching for a solution of said equation that has the integral form

$$
u = C\left(\bar{\partial}u\right) + \Phi, \tag{2.5}
$$

for suitably chosen operator C and a holomorphic function $\Phi : \mathbb{D} \to \mathbb{H}$.

Recalling the statement of Theorem 1.0.3 for a symplectomorphism $\varphi : \mathbb{H} \to \mathbb{H}$ and denoting by J the induced almost complex structure $\varphi_* J_{st}$, the proof of said theorem is based on the existence of a J-holomorphic disc $u : \mathbb{D} \to \mathbb{H}$ such that its boundary lies in $\partial\Sigma_R$, has an area equal to $R^2\pi$ and goes through the point $\varphi(0)$. Now, as the problem suggests, there are two separate issues emerging.

Firstly, regarding the boundary conditions and the area, the choice of C and Φ in the integral form (2.5) will implicitly give desired properties. Here C denotes modified classical Cauchy transform, introduced by Sukhov and Tumanov in [ST14] and used subsequently in [ST16a], [ST16b], in order to obtain stated properties of a disc which solves (2.4). These modifications reduce boundary conditions with respect to $\partial\Sigma_R$ to the cylinder with a triangle base, i.e. a linear boundary case, making it easier to construct explicitly the boundary behavior. Hence *Schwarz-Christoffel* mapping Φ comes into play as a modification of the standard Cauchy transform by a conformal map that achieves such transformation of the complex plane $\mathbb{C}$ into a desired polygon, namely triangle Δ_R, with corners $-R, R, iR$. This *Schwarz-Christoffel* map Φ fixes stated points. Consequently, by the nature of integral form (2.5), the solution u will fix stated points as well, making it unique with respect to Möbius symmetries of $\mathbb{D}$ (see Remark 2.1.8). Additionally, the degree of the map $\Pi_1 u : \partial\mathbb{D} = \mathbb{S}^1 \to \Pi_1(\partial\Sigma_R)$ equals 1, giving the desired area. The degree is meant in the topological sense, i.e. as the multiplying scalar $k \in \mathbb{Z}$ defining the homomorphism $[\Pi_1 u]_* : H_1(\mathbb{S}^1) \to H_1(\mathbb{S}^1)$. Section 2.1.1 covers details on this matter.

Secondly, one can establish existence of a non-trivial solution to (2.4) in an integral form (2.5) provided that there exists a constant $0 < a < 1$ such that $\|A(h)\|_{op} \leq a$ for all $h \in \mathbb{H}$. Validity of this statement will be postponed for later discussion. Nevertheless, existence of such uniform bound is a major obstacle we will face. A priori, φ need not have have uniformly bounded derivative, nor will this be the case for equations of interest which will be the topic of Chapter 3. Existence of a constant $M > 0$ such that $\|D\varphi(h)\|_{op} \leq M$ for all $h \in \mathbb{H}$ is a sufficient condition for existence of the constant $a < 1$. Consequently, we need to truncate the flow φ in some way, leading to A in (2.4), rather than A_J in Beltrami equation (2.3), hence why we distinguish equation (2.4) from (2.3). Nevertheless, we shall always refer to (2.3) when talking about the construction of the disc of equation (2.4). Even though the main difference stems from the a priori bound, the mentioned truncation will be such that it will preserve J on a desired set, implying that the solution to (2.4) will crucially still be a J-holomorphic disc on it.

To see that the uniform bound for the derivative is a sufficient condition for obtaining constant $a < 1$, recall firstly that $D\varphi(h)$ is a linear symplectomorphism for all $h \in \mathbb{H}$. As previously mentioned, every $\mathbb{R}$-linear operator admits an unique decomposition

$$(D\varphi)u = Pu + Q\overline{u},$$

by $\mathbb{C}$-linear bounded operators P and Q. In this particular case, $P = \partial_h \varphi$ and $Q = \partial_{\overline{h}} \varphi$. The following holds

Lemma 2.1.5 (Lemma 2.4, Prop 2.5, [ST16a]). *Operator P is invertible since φ is symplectic and*

$$\|Q\bar{P}^{-1}\| = \|Q\|(1 + \|Q\|^2)^{-1/2} < 1. \tag{2.6}$$

Moreover, the complex representation A_J is given by $A_J = Q\overline{P}^{-1}$.

Denoting by $\lambda = \|Q\|^2$, the function $\lambda \mapsto \lambda(1+\lambda)^{-1}$ is increasing for $\lambda > 0$, hence it follows directly that the uniform bound on $D\varphi$ implies the uniform bound on Q, so previous statements imply uniform bound $\|A_J(h)\|_{op} \leq a < 1$.

Returning to the Beltrami-type equation (2.3), solving it in an analytical fashion means we consider it as a $\bar{\partial}$ problem in infinite dimensional Hilbert space. Firstly, the $\bar{\partial}$ problem in the scalar case is the problem of solving the differential equation

$$\bar{\partial} f(z) = g(z),$$

for the function $f : \mathbb{C} \to \mathbb{C}$, and g known. Here $\bar{\partial}$ represents the complex derivative in the sense of (2.2). Classical Cauchy-Green transform

$$C(f)(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(z)}{z - \zeta} dz \wedge d\bar{z},$$

solves the $\overline{\partial}$-problem, that is $\overline{\partial}C(f) = f$ in the distributional sense. Moreover, it is bounded as an operator $C : L^p(\mathbb{C}) \to \dot{W}^{1,p}(\mathbb{C})$, for $p > 1$. Beurling transform is defined as

$$B(f) = \partial C(f)$$

in the distributional sense, or alternatively

$$B(f)(\zeta) := -\frac{1}{2\pi i} PV \int_{\mathbb{C}} \frac{f(z)}{(z-\zeta)^2} dz \wedge d\overline{z}.$$

Beurling transform is a bounded linear operator $B : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ for $p > 1$, and is an isometry for $p = 2$.

Since $\mathbb{H}$ is a Hilbert space, there exists a unique well-defined extension of any bounded operator $L \in B(L^p(\mathbb{D}))$ to an operator $L_{\mathbb{H}} \in B(L^p(\mathbb{D}, \mathbb{H}))$ such that for every $u \in L^p(\mathbb{D})$ and $h \in \mathbb{H}$, $L_{\mathbb{H}}(uh) = L(u)h$ holds. In the rest of the paper, we shall use the same notation for the Cauchy and Beurling operators and their extensions to Hilbert space valued functions. Moreover, we shall be using a modification of the Cauchy transform on $L^p(\mathbb{D}, \mathbb{H})$. Specifically, denoting by $i_1 : L^p(\mathbb{D}, \mathbb{H}) \to L^p(\mathbb{C}, \mathbb{H})$ the extension by zero to the entire complex plane and by $i_2 : L^p(\mathbb{C}, \mathbb{H}) \to L^p(\mathbb{D}, \mathbb{H})$ the restriction to the disc $\mathbb{D}$, Cauchy transform

$$\tilde{C} = i_2 \circ C \circ i_1,$$

also solves the $\bar{\partial}$ problem on the disc $\mathbb{D}$ and it is bounded as an operator $\tilde{C} : L^p(\mathbb{D}, \mathbb{H}) \to W^{1,p}(\mathbb{D}, \mathbb{H})$. Corresponding bounds hold for the Beurling transform $\tilde{B} := \partial \tilde{C}$. For clarity sake, we shall omit writing the tilde sign, and denote by C Cauchy transform on $\mathbb{D}$. Lastly, for $2 < p < \infty$, Morrey's embedding

$$W^{1,p}(\mathbb{D}) \hookrightarrow C^{0,1-2/p}(\mathbb{D}) \tag{2.7}$$

implies that $C : L^p(\mathbb{D}, \mathbb{H}) \to C^{0,1-2/p}(\mathbb{D}, \mathbb{H})$ is bounded as an operator as well.

We finish this discussion about Beltrami equation with the following lemma, which we will prove in similar form in Lemma 2.2.1. For the time being, the purpose of the following lemma is to provide a general model for proving the existence of Hilbert space valued Beltrami equation.

Lemma 2.1.6. *Let $A : (\mathbb{H}, \tau_{weak}) \to (B(\mathbb{H}), \|\cdot\|_{op})$ be continuous such that $\|A(h)\|_{op} \leq a < 1$, for all $h \in \mathbb{H}$. For any holomorphic function $f : \mathbb{D} \to \mathbb{H}$ there exists $u \in W^{1,p}(\mathbb{D}, \mathbb{H})$ for some $p > 2$, satisfying the integral identity*

$$u = C(\bar{\partial}u) + f$$

and which solves Beltrami equation (2.4).

Proof. Denote by $v = \bar{\partial} u$. For a solution of (2.4) in the form of $u = Cv + f$,

$$v = A(u)(\overline{Bv} + \overline{f'})$$

holds. Then

$$\Big(I - A(u) \circ \overline{B}\Big) v = A(u)\overline{f'}.$$

Since $\|A\| \leq a$ and $b_p := \|B\|_{B(L^p)} \searrow 1$ as $p \to 2$, for $p > 2$ close enough to 2, operator $I - A(u) \circ \overline{B}$ is invertible and

$$v = \Big[\Big(I - A(u) \circ \overline{B}\Big)^{-1} A(u)\Big]\overline{f'}, \tag{2.8}$$

hence we have the bound

$$\|v\|_{L^p} \leq \frac{a\|f'\|_{L^p}}{1 - ab_p}.$$

Then we have a priori bound for the solution u of (2.4) satisfying the integral identity

$$\|u\|_{W^{1,p}} \leq \frac{ac_p\|f'\|_{L^p}}{1 - ab_p} + \|f\|_{W^{1,p}} \lesssim_{p,f} 1. \tag{2.9}$$

In other words, fixing holomorphic f gives the a priori bound. Moreover, Morrey's embedding (2.7) implies a priori bound

$$\|u\|_{L^\infty(\mathbb{D},\mathbb{H})} \lesssim_{p,f} \|u\|_{C^{0,\alpha}(\mathbb{D},\mathbb{H})} \lesssim_{p,f} 1,$$

where $\alpha = \frac{p-2}{p}$. As a consequence, we only consider the space $C(\overline{\mathbb{D}}, X)$, where X is the ball $\mathbb{B}_M := \{h : \|h\| \leq M\}$ and M is a constant given by the last inequality. X denotes the set $\mathbb{B}_M$, but endowed with weak topology from $\mathbb{H}$, making it a compact space. More specifically,

$$\mathcal{B} := \Big\{u \in C(\overline{\mathbb{D}}, X) : \|u\|_{W^{1,p}} \leq M\Big\}$$

is the set, and the map is

$$\begin{aligned} &\mathcal{L} : C(\overline{\mathbb{D}}, X) \to C(\overline{\mathbb{D}}, X) \\ &\mathcal{L}(u) = C\Big[\Big(I - A(u) \circ \overline{B}\Big)^{-1} A(u)\overline{f'}\Big] + f. \end{aligned}$$

Then the $W^{1,p}$ regularity of the solutions and Morrey's inequality imply equicontinuity of all solutions to the equation (2.4) of the form $u = Cv + f$. Arzela-Ascoli implies the compactness of the set $\mathcal{L}(\mathcal{B})$. Existence follows by applying Theorem 2.1.7, stated below. □

Theorem 2.1.7 (Schauder's fixed point theorem, [Sch30])**.** *Let V be topological vector space and $F : V \to V$ continuous map. Let K be a nonempty convex closed set with compact image in itself. Then F has a fixed point.*

We remark that the uniform bound for A in the proof of Lemma 2.1.6 lead to a priori bound (2.9) for solutions of Beltrami equation (2.4). However, continuity of A with respect to weak topology plays crucial role for establishing the existence, allowing us to apply Arzela-Ascoli theorem.

Remark 2.1.8. *Notice as well that if u is a solution of (2.4), then for every conformal map $\psi : \mathbb{D} \to \mathbb{D}$, the map $u \circ \psi$ is a solution as well.*

2.1.1 Modified Cauchy transform

Let Q be a function in $\mathbb{D}$ and recall that C represents the classical Cauchy transform. We call Q a weight function. Introduce the operator

$$\begin{aligned} C_Q f(\zeta) &= Q(\zeta)\Big(C(f/Q)(\zeta) + \zeta^{-1}\overline{C(f/Q)(1/\bar{\zeta})}\Big) \\ &= Q(\zeta)\int_{\mathbb{D}}\left(\frac{f(z)}{Q(z)(z-\zeta)} + \frac{\overline{f(z)}}{\overline{Q(z)}(z-\zeta)}\right)\frac{dz\wedge d\bar{z}}{2\pi i}. \end{aligned} \tag{2.10}$$

As the boundary conditions are the crucial part of modifications, one observes that for $|\zeta| = 1$ one has

$$\begin{aligned} C_Q f(\zeta) &= Q(\zeta)\Big(C(f/Q)(\zeta) + \overline{\zeta C(f/Q)(\zeta)}\Big) \\ &= (Q(\zeta)/\sqrt{\zeta})\Big(\sqrt{\zeta}C(f/Q)(\zeta) + \overline{\sqrt{\zeta}C(f/Q)(\zeta)}\Big) \\ &= (Q(\zeta)/\sqrt{\zeta})\cdot\Big[\operatorname{Re}\sqrt{\zeta}C(f/Q)(\zeta)\Big]. \end{aligned} \tag{2.11}$$

In other words, for $|\zeta| = 1$

$$\operatorname{Im} C_Q f(\zeta) = Q(\zeta)/\sqrt{\zeta},$$

that is the imaginary part of the boundary values does not depend on the function f, but only on the choice of weight function Q.

Schwarz-Christoffel mapping f is a conformal transformation of the upper half-plane

$$\{\zeta \in \mathbb{C} : \operatorname{Im}\zeta > 0\}$$

onto the interior of a simple n-polygon in $\mathbb{C}$. Namely, if the polygon has interior angles $\alpha_1, \alpha_2, \ldots, \alpha_n$, then the mapping f is given by

$$f(\zeta) = \int^{\zeta} \frac{K}{(w-a_1)^{1-\frac{\alpha_1}{\pi}}(w-a_2)^{1-\frac{\alpha_2}{\pi}}\cdots(w-a_n)^{1-\frac{\alpha_n}{\pi}}}\,dw,$$

where K is a constant and $a_1 < a_2 < \ldots < a_n$ are the values, along the real axis of the half-plane, of points corresponding to the vertices of the polygon.

We now turn to stating explicit weights

$$R(\zeta) = e^{3\pi i/4}(\zeta-1)^{1/4}(\zeta+1)^{1/4}(\zeta-i)^{1/2}, \quad X(\zeta) = R(\zeta)/\sqrt{\zeta}, \quad Z(\zeta) = \zeta - 1.$$

The branch of R is chosen so that it is continuous in $\overline{\mathbb{D}}$, with $R(0) = e^{3\pi i/4}$, with *zeros being exactly the corners of the triangle* Δ_1, to which we want to reduce desired construction. Operator C_Z was firstly introduced by Vekua [Vek62], operators similar to C_R by Antoncev and Monakhov in [SNA67], [Mon83], whereas C_R was firstly introduced by Sukhov and Tumanov in [ST14].

With boundary condition in mind, we look at the function X only on the circle $\partial\mathbb{D}$ and choose the branch of $\sqrt{\zeta}$ continuous in $\mathbb{C}$ without positive real line. Then $\arg X$ is constant on each arc $\gamma_1 = \{e^{i\theta} : 0 < \theta < \pi/2\}$, $\gamma_2 = \{e^{i\theta} : \pi/2 < \theta < \pi\}$, $\gamma_3 = \{e^{i\theta} : \pi < \theta < 2\pi\}$, and equals to $3\pi/4$, $\pi/4$ and 0 respectively. Then

$$\begin{cases} \arg(X(\zeta)\cdot\sqrt{2}e^{i\frac{\pi}{4}}) = \frac{3\pi}{4} + \frac{\pi}{4} = \pi, & \zeta \in \gamma_1, \\ \arg(X(\zeta)\cdot\sqrt{2}e^{i\frac{7\pi}{4}}) = \frac{\pi}{4} + \frac{7\pi}{4} = 2\pi, & \zeta \in \gamma_2, \\ \arg(X(\zeta)\cdot e^{0} = 0 + 0 = 0, & \zeta \in \gamma_3, \end{cases} \tag{2.12}$$

Equivalently, the function X satisfies the boundary conditions

$$\begin{cases} \operatorname{Im}[(1+i)X(\zeta)] = 0, & \zeta \in \gamma_1, \\ \operatorname{Im}[(1-i)X(\zeta)] = 0, & \zeta \in \gamma_2, \\ \operatorname{Im}[X(\zeta)] = 0, & \zeta \in \gamma_3, \end{cases} \tag{2.13}$$

which represent the lines through 0 parallel to the sides of the triangle Δ_1. Moreover, for the weight R, boundary conditions observed in (2.11) for Cauchy transform C_R are given by equations (2.13).

We will need only the operators corresponding to two special weights, namely C_Z and C_R, which we shall denote by C_1, C_2 respectively. We also define formal derivatives $B_j f(\zeta) = \partial C_j f(\zeta), j = 1, 2$ as integrals in the sense of the Cauchy principal value.

Proposition 2.1.9 ([Vek62], [Mon83]; Proposition 4.1, [ST14])**.** *Following properties of operators $C_j, B_j, j = 1, 2$ hold*

(i) Each $B_j : L^p(\mathbb{D}) \to L^p(\mathbb{D})$, $j = 1, 2$, is a bounded linear operator for $p_1 < p < p_2$. Here for B_2 one has $p_1 = 1$ and $p_2 = \infty$ and for B_1 one has $p_1 = 4/3$ and $p_2 = 8/3$. For $2 < p < p_2$, one has $B_j f(\zeta) = (\partial/\partial\zeta)C_j f(\zeta)$ as Sobolev's derivatives.

(ii) Each $C_j : L^p(\mathbb{D}) \to W^{1,p}(\mathbb{D})$, $j = 1, 2$, is a bounded linear operator for $2 < p < p_2$. For $f \in L^p(\mathbb{D})$, $2 < p < p_2$, one has $(\partial/\partial\overline{\zeta})C_j f = f$ on $\mathbb{D}$ as Sobolev's derivative.

(iii) For every $f \in L^p(\mathbb{D})$, $p > 2$, the function $C_2 f$ satisfies $\operatorname{Re} C_2 f|_{\partial\mathbb{D}} = 0$ whereas $C_1 f$ satisfies the same boundary conditions (2.13) as X.

(iv) Each $B_j : L^2(\mathbb{D}) \to L^2(\mathbb{D})$, $j = 1, 2$, is an isometry.

(v) The function $p \mapsto \|B_j\|_{L^p}$ approaches $\|B_j\|_{L^2} = 1$ as $p \searrow 2$.

Notice that the boundedness of modified transforms does not hold for all $p > 1$ as they did for original operators, but rather for indices in the neighborhood of $p = 2$. However, for the purposes of this paper, this range of indices will suffice. For more details, as well as for the proofs, we refer to [ST14].

2.2 Proof of the non-squeezing Theorem 1.0.3

As shown by Gromov in [Gro85], the proof of Theorem 1.0.3 is based on the existence of a pseudoholomorphic disc $u : \mathbb{D} \to \mathbb{H}$ such that its boundary lies in $\partial\Sigma_R$, has an area equal to $R^2\pi$ and goes through the point $\varphi(0)$. Following lemma shows the existence of the disc with desired properties, with respect to weak continuity arising from assumptions of Theorem 1.0.3. We present the existence in the case of cylinder with a unit triangle basis $\Delta_1 \subset \mathbb{C}$, with corners $-1, 1, i$. The idea of reduction to triangle base was of Sukhov and Tumanov in [ST14]. General case of radius follows by obvious adding of constants in the proof. We shall denote the cylinder by $\tilde{\Delta}_1 \subset \mathbb{H}$.

Lemma 2.2.1. *Let $h_0 \in int(\tilde{\Delta}_1)$ be arbitrary. Moreover, let $A_J : (\mathbb{H}, \tau_{weak}) \to (B(\mathbb{H}), \|\cdot\|_{op})$ be continuous such that $\|A_J(h)\|_{op} \le a < 1$, for every $h \in \mathbb{H}$. Then there exists a solution u of the Beltrami-type equation (2.3), such that $u \in W^{1,p}(\mathbb{D}, \mathbb{H})$ for some $p > 2$, $area(u(\mathbb{D})) = 1$, $u(\partial\mathbb{D}) \subset \partial\tilde{\Delta}_1$ and $h_0 \in u(\mathbb{D})$. In particular, $\deg(\Pi_1 u(\partial\mathbb{D})) = 1$.*

Taking previous lemma into account, we prove Theorem 1.0.3

Proof of Theorem 1.0.3. Let $J = \varphi_* J_{st}$ be the induced almost complex structure on $\mathbb{H}$. Let $A_J : (\mathbb{H}, \tau_{weak}) \to (B(H), \|\cdot\|_{op})$ be the complex representation of J. As mentioned earlier, we would like to obtain a uniform bound for A_J. In order to do so, we shall construct two truncations. First one will serve to control the image of the disc we want to construct, namely to ensure that the disc cannot escape cylinder Σ_R. The second one is the one that implies uniform bound for A_J.

Firstly, let $\varepsilon > 0$ be arbitrary small and recall $\Pi_1(h) := \langle h, e_1 \rangle$. We define the cut-off $\widetilde{\eta_\varepsilon} : (\mathbb{H}, \tau_{weak}) \to [0, 1]$ in the following way

$$\widetilde{\eta_\varepsilon}(h) := \eta_\varepsilon(\Pi_1(h)), \text{ for every } h \in \mathbb{H},$$

where the $\eta_\varepsilon : \mathbb{C} \to [0, 1]$ is continuous cut-off such that

$$\eta_\varepsilon(z) = \begin{cases} 1 & , |z| \le \sup_{h \in B_{r-\varepsilon}} |\Pi_1(\varphi(h))| \\ 0 & , |z| \ge \sup_{h \in B_r} |\Pi_1(\varphi(h))| . \end{cases}$$

Secondly,

$$f(h) := g\left(\|d\varphi(\varphi^{-1}(h))\|_{op}\right)$$

where $g : \mathbb{R} \to [0,1]$ is continuous cut-off such that $g \equiv 1$ for $x \le M_\delta$, $g \equiv 0$ for $x \ge M$ (M_δ and M are specific constants to be specified shortly). We observe that $f : (\mathbb{H}, \tau_{weak}) \to [0,1]$ is continuous as a composition of continuous functions, starting with φ and $d\varphi$, then taking the operator norm as a function, and finally composing with g.

Let

$$M_\delta := \sup_{h \in B_{r-\delta}} \|d\varphi(h)\|_{op}$$

and

$$M := \sup_{h \in B_r} \|d\varphi(h)\|_{op},$$

for an arbitrary $\delta > 0$. Due to compactness of B_r, and continuity of $d\varphi$, M_1 and M_2 exist and are finite.

Finally, we define the complex representation

$$\widetilde{A}(h) := \widetilde{\eta_\varepsilon}(h) f(h) A_J(h).$$

With the abuse of notation, we shall denote $\widetilde{A_J}$ by A_J in the rest of the proof. We remark that these truncations still respect the weak topology, so the modified A_J will still retain the weak continuity condition. Moreover, this truncation implies the uniform bound for A_J. We distinguish two cases. First, if $f(h) = 0$, then $A_J(h) = 0$. Second, if $f(h) \neq 0$, then the definition of f implies that $\|d\varphi(\varphi^{-1}(h))\|_{op} \le M$ on the set $f^{-1}((0,1])$. Then by Lemma 2.1.5, there exists a uniform bound $a < 1$ on complex representation A_J on the set $f^{-1}((0,1])$. This is the bound for the A_J as well.

Note that the final truncation f does not affect the induced almost complex structure J on $\varphi(B_{r-\delta})$ since $f \equiv 1$ on a set whose subset is $\varphi(B_{r-\delta})$. This means that the disc attained by the Theorem 2.2.1 will be a J-holomorphic disc on $\varphi(B_{r-\delta})$.

Recall that $\omega = \frac{i}{2} \sum_{k=1}^{\infty} dh_k \wedge d\overline{h}_k$ and decompose every $h \in \mathbb{H}$ as $h = h_1 e_1 + \Pi_1^\perp(h)$. $\Pi_1^\perp$ represents the projection on the orthogonal space to e_1. A diffeomorphism m of $\mathbb{C}$ that preserves the volume, is symplectic with respect to $\omega_1 = dh_1 \wedge d\overline{h}_1$. Thus, the diffeomorphism $\varphi_1 : \mathbb{H} \to \mathbb{H}$ obtained as a tensor sum

$$\varphi_1(h) := m(h_1)e_1 + \Pi_1^\perp(h)$$

of m and of the identity map on the (symplectic) orthogonal of $\mathbb{C}e_1$ is a symplectomoprhism. Let m be a symplectomorphism such that it maps circle $B_R(0) \subset \mathbb{C}$ into $\sqrt{\pi}R\Delta_1$, where $\Delta_1 = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1 - |\operatorname{Re} z|\}$. Hence φ_1 transforms Σ_R to a triangle cylinder, denoted by $\tilde{\Delta}_R$, with base $\sqrt{\pi}R\Delta_1$. This reduction of the non-squeezing theorem to the case of a cylinder with triangle base was introduced by [ST14], in order to use modified Cauchy transforms previously introduced. Moreover, notice that in our case, with respect to Lemma 2.2.1, map φ_1 is weak-weak continuous. Hence, the continuity of A_J is preserved under such deformation of $\mathbb{H}$, i.e. we obtain our final almost complex representation $A_J \circ \varphi_1$.

Now from Lemma 2.2.1 follows that there exists a solution $u \in W^{1,p}(\mathbb{D}, \mathbb{H})$, for some $p > 2$, of equation (2.3) such that its area is equal to $R^2\pi$, $u(\partial\mathbb{D}) \subset \partial\tilde{\Delta}_R$ and that $\varphi(0) \in u(\mathbb{D})$.

We finish the proof in the same manner as Gromov ([Gro85]). Let D be the connected component of the preimage of $u^{-1}(u(\mathbb{D}) \cap \varphi(\mathbb{B}_{r-\varepsilon}))$. Then $\tilde{u} := \varphi^{-1}(u(D))$ is a closed J_{st}-holomorphic curve in $\mathbb{B}_{r-\varepsilon}$ with boundary contained in the boundary of the ball. Moreover, $0 \in \tilde{u}(D)$ and due to integration over smaller set and φ being a symplectomorphism, $area(\tilde{u}(D)) \leq area(u(D)) = R^2\pi$.

The approximation argument through finite dimensions for which Lelong's monotonicity formula holds gives us the inequality $r^2\pi \leq area(\tilde{u})$, which concludes the proof. □

2.3 Existence of the pseudoholomorphic disc

We paraphrase the approach of Sukhov and Tumanov introduced in [ST14], consequently adapted for infinite dimensional case in [ST16b]. Our contribution is showing that the fixed point argument still follows with respect to adaptations arising from assumptions of Theorem 1.0.3, namely that we can construct a compact function space of pseudo-holomorphic discs on which the fixed point map, introduced in [ST14], adjusted in [ST16b], is well-defined.

Proof of Lemma 2.2.1. We denote by $u : \mathbb{D} \to \mathbb{H}$ the solution of Beltrami equation (2.3) and write its decomposition $u = u_1 + u_2$, where $u_1 = \Pi_1(u)e_1$. Moreover, let $v = \overline{\partial}u$. Then it is obvious that $v = v_1 + v_2$ where $\overline{\partial}u_k = v_k$, for $k = 1, 2$. We look for a solution $u = u_1 + u_2 : \mathbb{D} \to \mathbb{H}$ of (2.3) of class $W^{1,p}(\mathbb{D}, \mathbb{H})$, $p > 2$, in the form introduced in [ST14]

$$\begin{cases} u_1 = C_1 v_1 + \Phi e_1, \\ u_2 = C_2 v_2 - C_2 v_2(\tau) + h_0 - \Pi_1(h_0)e_1, \end{cases} \tag{2.14}$$

for some $\tau \in \mathbb{D}$, and Schwarz-Christoffel mapping $\Phi : \mathbb{D} \to \Delta_1$ of the unit circle into the triangle, such that it preserves corners $\Phi(\pm 1) = \pm 1$ and $\Phi(i) = i$. Note that $\Phi \in W^{1,p}(\mathbb{D})$ for $2 \leq p < 4$, since

$$\Phi(z) := \int_0^z \frac{dw}{e^{3\pi i/4}(w-1)^{\frac{1}{4}}(w+1)^{\frac{1}{4}}(w-i)^{\frac{1}{2}}}.$$

Solutions of Beltrami equation (2.3) having integral form (2.14) have a priori estimate

$$\|u\|_{W^{1,p}(\mathbb{D},\mathbb{H})} \leq C(\Phi, a, h_0, p) < \infty$$

(see Proof of Theorem 3.2 in [ST16b] for explicit computations). This follows since the integral form (2.14) as a solution of (2.3) satisfies

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = A(u) \begin{pmatrix} \overline{B_1 v_1} + \overline{\Phi'} \\ \overline{B_2 v_2} \end{pmatrix}, \tag{2.15}$$

where we define $A : C(\overline{\mathbb{D}}, \mathbb{H}_w) \to (B(L^p(\mathbb{D}, \mathbb{H})), \|\cdot\|_{op})$, $\mathbb{H}_w$ denotes $(\mathbb{H}, \tau_{weak})$, as

$$A[u](v)(z) := A_J[u(z)](v(z)),$$

for $u \in C(\overline{\mathbb{D}}, \mathbb{H}_w)$ and $v \in L^p(\mathbb{D}, \mathbb{H})$. Observe that $\|A(u)\|_{B(L^p(\mathbb{D},\mathbb{H}))} \leq \sup\limits_{z \in \mathbb{D}} \|A_J(u(z))\|_{B(\mathbb{H})} \leq a < 1$, hence $A(u)$ is bounded for every $u \in C(\overline{\mathbb{D}}, \mathbb{H}_w)$.

Moreover, as $p > 2$, we have that because of Morrey's embedding (2.7) such solution u is actually in $C^{0,\alpha}(\mathbb{D}, \mathbb{H})$ ($\alpha = \frac{p-2}{p}$), where $\|u\|_{C^{0,\alpha}(\mathbb{D},\mathbb{H})} \lesssim \|u\|_{W^{1,p}(\mathbb{D},\mathbb{H})}$. Thus we obtain the radius M of the ball $\mathbb{B}^{\mathbb{H}}_M(0) = X$, depending on C.

This loop-sided argument for obtaining the a-priori bound for a solution of Beltrami equation (2.3) of the integral form (2.14), and hence the function space $C(\overline{\mathbb{D}}, X)$, can be thought of in three steps, each of which defines a continuous map denoted by $\Gamma, \mathcal{E}, \mathcal{C}$, and whose composition defines the fixed point map

$$\begin{aligned} \widetilde{\mathcal{F}}(u) &:= (\mathcal{C} \circ \mathcal{E} \circ \Gamma)(u) \\ \widetilde{\mathcal{F}} &: (C(\overline{\mathbb{D}}, X), \|\cdot\|_\infty) \to (C(\overline{\mathbb{D}}, X), \|\cdot\|_\infty). \end{aligned}$$

Here

$$\begin{aligned} \Gamma &: (C(\overline{\mathbb{D}}, X), \|\cdot\|_\infty) \to \left(B\left(L^p\left(\mathbb{D}, \mathbb{H}\right)\right), \|\cdot\|_{op}\right) \\ \Gamma(u) &:= \left(I - A(u)\overline{[B_2 \circ \Pi_1 e_1 + B_1 \circ (I - \Pi_1(\cdot) e_1)]}\right)^{-1} \circ A(u), \end{aligned}$$

represents obtaining the inverse of the Beltrami operator,

$$\begin{aligned} \mathcal{E} &: \left(B\left(L^p\left(\mathbb{D}, \mathbb{H}\right)\right) \|\cdot\|_{op}\right) \to \left(L^p\left(\mathbb{D}, \mathbb{H}\right), \|\cdot\|_{L^p}\right) \\ \mathcal{E}(T) &:= T(\Phi' e_1), \end{aligned}$$

represents obtaining $v = \bar{\partial} u$ from u by evaluation of a specific holomorphic function for operator $\Gamma(u)$, based on integral form (2.14), and

$$\begin{aligned} \mathcal{C} &: \left(L^p\left(\mathbb{D}, \mathbb{H}\right), \|\cdot\|_{L^p}\right) \to \left(W^{1,p}\left(\mathbb{D}, \mathbb{H}\right), \|\cdot\|_{W^{1,p}}\right) \\ \mathcal{C}(u) &:= C_1(\Pi_1(u) e_1) + C_2(u - \Pi_1(u) e_1) - [C_2(u - \Pi_1(u) e_1)](\tau) + \Phi e_1 + h_0 - \Pi_1(h_0) e_1, \end{aligned}$$

represents the integral form (2.14).

We justify continuity of $\widetilde{\mathcal{F}}$ by proving the continuity of each appearing term. Γ is well-defined since the invertibility of operator

$$I - A(u)\overline{[B_2 \circ \Pi_1 e_1 + B_1 \circ (I - \Pi_1(\cdot)e_1)]}$$

follows from the bound $\|A(u)\overline{[B_2 \circ \Pi_1 e_1 + B_1 \circ (I - \Pi_1(\cdot)e_1)]}\|_{B(L^p(\mathbb{D},\mathbb{H}))} \leq aC_p b_p < 1$. Continuity of Γ follows from the continuity of mapping

$$A : (C(\overline{\mathbb{D}}, X), \|\cdot\|_\infty) \to (B(L^p(\mathbb{D},\mathbb{H})), \|\cdot\|_{op}),$$

since $\Gamma(u)$ is obtained from $A(u)$ via operations in $B(L^p(\mathbb{D},\mathbb{H}))$, which are continuous with respect to operator norm. As the inequality

$$\|A(u_1) - A(u_2)\|_{(B(L^p(\mathbb{D},\mathbb{H}))} \leq \sup_{z \in \mathbb{D}} \|A_J(u_1(z)) - A_J(u_2(z))\|_{B(\mathbb{H})}$$

holds, the continuity of A follows from uniform norm on $C(\overline{\mathbb{D}}, X)$ and continuity of $A_J : (\mathbb{H}, \tau_{weak}) \to (B(\mathbb{H}), \|\cdot\|_{op})$. Map $\mathcal{E}$ is continuous as a bounded linear operator. Map $\mathcal{C}$ is continuous since it is obtained via operations on bounded linear operators. Hence, $\widetilde{\mathcal{F}}$ is continuous.

Moreover, note that the choice of function space $C(\overline{\mathbb{D}}, X)$ leads to observation of the embeddings

$$W^{1,p}(\mathbb{D},\mathbb{H}) \hookrightarrow C^{0,\alpha}(\mathbb{D},\mathbb{H}) \hookrightarrow C^{0,\alpha}(\overline{\mathbb{D}},\mathbb{H}_{weak}) \hookrightarrow C(\overline{\mathbb{D}},\mathbb{H}_{weak}).$$

Hence, as X is compact, Arzela-Ascoli theorem implies that the embedding $C^{0,\alpha}(\overline{\mathbb{D}}, X) \hookrightarrow C(\overline{\mathbb{D}}, X)$ is compact. We fix the convex set

$$S = \{u \in C(\overline{\mathbb{D}}, X) | u \in W^{1,p}(\mathbb{D},\mathbb{H}), \|u\|_{W^{1,p}} \leq C\}.$$

From construction of mapping $\widetilde{\mathcal{F}}$ and the a priori bounds from equation (2.3), one observes that $\widetilde{\mathcal{F}}(C(\overline{\mathbb{D}}, X)) \subset W^{1,p}(\mathbb{D},\mathbb{H})$, and more specifically,

$$\widetilde{\mathcal{F}}(S) \subset \{u \in C(\overline{\mathbb{D}}, X) | u \in W^{1,p}(\mathbb{D},\mathbb{H}), \|u\|_{W^{1,p}} \leq C\},$$

that is

$$\widetilde{\mathcal{F}}(S) \subset \{u \in C(\overline{\mathbb{D}}, X) | u \in C^{0,\alpha}(\overline{\mathbb{D}}, X), \|u\|_{W^{1,p}} \leq C\}.$$

Hence, $\widetilde{\mathcal{F}}(S) \subset S$, where $\widetilde{\mathcal{F}}(S)$ is compactly embedded in S.

Lastly, define a continuous map $\Psi : \mathbb{C} \to \overline{\mathbb{D}}$

$$\Psi(z) = \begin{cases} \Phi^{-1}(z), & z \in \bar{\Delta}, \\ \Phi^{-1}(\partial\Delta \cap [\langle h_0, e_1 \rangle, z]), & z \in \mathbb{C} \setminus \bar{\Delta}. \end{cases}$$

and a map

$$\widetilde{\Psi}(\tau) = \Psi\Big(\Pi_1(h_0) - C_1(\bar{\partial}\Pi_1(u)(\tau))\Big),$$

both of which were introduced in [ST14] as well. Finally, we define the map

$$\mathcal{F}(u,\tau) := \widetilde{\mathcal{F}}(u) \oplus \widetilde{\Psi}(\tau) : C(\overline{\mathbb{D}}, X) \times \mathbb{C} \to C(\overline{\mathbb{D}}, X) \times \mathbb{C}.$$

We fix the convex set $S_1 = S \times \overline{\mathbb{D}}$. As $\widetilde{\mathcal{F}}(S) \subset S$ and $\widetilde{\Psi}(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$, thus $\mathcal{F}(S_1) \subset S_1$. Schauder fixed point theorem implies the existance of the disc solving Beltrami equation (2.3) and having the integral form (2.14).

□

2.3.1 Properties of the disc

As stated previously, our approach was motivated by authors of [ST14], who reproved Gromov's finite result using Cauchy transforms that we have used in our proof as well. For that matter, the resulting properties of disc in our infinite setting and theirs in [ST16b] differ not, and the proofs from [ST16b] carry over completely but with one essential difference in the proof of Lemma 2.3.1 - addressing the holomorphicity of disc at points at which truncation is 0 and its implications on maximum principle implying that disc cannot leave the cylinder.

Lemma 2.3.1. *The solution map u of Beltrami equation (2.3) constructed in Lemma 2.2.1 satisfies $u_1(\overline{\mathbb{D}}) \subset \overline{\Delta}_1$, $u_1(\partial\mathbb{D}) \subset \partial\Delta_1$, and $\deg u_1 = 1$; here $\deg u_1$ denotes the degree of the map $u|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \partial\Delta_1$. In particular, u satisfies the required boundary conditions and $area(u(\mathbb{D})) = 1$. Lastly, for τ obtained as a fixed point of the second argument of the map $\mathcal{F}$ in the end of Lemma 2.2.1, $\tau \in int(\mathbb{D})$ and $u_1(\tau) = h_1$, hence $u(\tau) = h_0$.*

3 Application to Hamiltonian equations

3.1 Preliminaries and notation

Let $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ for dimension $d \geq 1$ and let $1 \leq p, q \leq \infty$. We denote by $\mathcal{S}(\mathbb{R}^d)$ the set of Schwartz functions on $\mathbb{R}^d$, or simply by $\mathcal{S}$ when the dimension is evident, and by $\mathcal{S}'$ the dual space, i.e. space of tempered distributions. We define mixed Lebesgue spaces as $L^{q,p}_{t,x}(\mathbb{R}\times\mathbb{R}^d)$ as

$$f \in L^{q,p}_{t,x}(\mathbb{R}\times\mathbb{R}^d) \Leftrightarrow \|f\|_{L^q_t L^p_x(\mathbb{R}\times\mathbb{R}^d)} := \Big\| \|f(t,\cdot)\|_{L^p_x(\mathbb{R}^d)} \Big\|_{L^q_t(\mathbb{R})} < \infty.$$

In general, for a Banach space $(X, \|\cdot\|_X)$, a space $L^p(\mathbb{R}, X)$ is defined as

$$f \in L^p(\mathbb{R}, X) \Leftrightarrow f : \mathbb{R} \to X \text{ is weakly measurable and } \|f\|_{L^p(\mathbb{R},X)} := \Big\| \|f(t,\cdot)\|_X \Big\|_{L^p(\mathbb{R})} < \infty.$$

We adopt the following convention for the Fourier transform, which we denote by $\mathcal{F}$,

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} f(x)dx, \ \ \xi \in \mathbb{R}^d,$$
$$\mathcal{F}(f)(k) = \hat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikx} f(x)dx, \ \ k \in \mathbb{Z}.$$

Concomitant to this, we define inhomogeneous and homogeneous Sobolev norms

$$\|f\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1+|\xi|^2)^s d\xi, \qquad \|f\|^2_{\dot{H}^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\xi|^{2s} d\xi$$
$$\|f\|^2_{H^s(\mathbb{T})} := \sum_{\xi\in 2\pi\mathbb{Z}} |\hat{f}(\xi)|^2 (1+\xi^2)^s d\xi, \qquad \|f\|^2_{\dot{H}^s(\mathbb{T})} := \sum_{\xi\in 2\pi\mathbb{Z}} |\hat{f}(\xi)|^2 \xi^{2s} d\xi$$

for $s \in \mathbb{R}$, and Sobolev spaces

$$H^s(\mathbb{R}^d) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \Big| \|f\|_{H^s} < \infty \Big\}, \qquad \dot{H}^s(\mathbb{R}^d) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \Big| \|f\|_{\dot{H}^s} < \infty \Big\},$$
$$H^s(\mathbb{T}) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \Big| \|f\|_{H^s} < \infty \Big\}, \qquad \dot{H}^s(\mathbb{T}) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \Big| \|f\|_{\dot{H}^s} < \infty \Big\}.$$

Let $\varphi : \mathbb{R}^d \to [0,1]$ be smooth, spherically symmetric function such that

$$\varphi(x) \equiv 1 \quad \text{for} \quad |x| \leq 1, \quad \text{and} \quad \varphi(x) \equiv 0 \quad \text{for} \quad |x| \geq 2.$$

Define projections onto low, respectively high, frequencies

$$\widehat{P_{\leq N} f}(\xi) := \varphi(\xi/N)\hat{f}(\xi), \qquad \widehat{P_{>N} f}(\xi) := (1-\varphi(\xi/N))\hat{f}(\xi).$$

Denoting by $S(t)$ the Schrödinger semigroup

$$S(t)f(x) = e^{it\Delta} f(x) := (4i\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} f(y)dy,$$

we state Strichartz estimates

Theorem 3.1.1 (Strichartz estimates, [KT98]). *Let (q,p) be an admissible pair, i.e.*

$$\frac{2}{q}+\frac{d}{p}=\frac{d}{2},\quad 2\le q,p\le\infty,\quad (q,p,d)\neq(2,\infty,2).$$

Then for any admissible exponent pairs $(q,p),(\tilde{q},\tilde{p})$, we have the following homogeneous Strichartz estimate (for the solution $S(t)u_0$ of the homogeneous problem $i\partial_t u+\Delta u=0$, $u|_{t=0}=u_0$)

$$\|S(t)u_0\|_{L_t^q L_x^p(\mathbb{R}\times\mathbb{R}^d)}\le C(q,p,d)\|u_0\|_{L_x^2(\mathbb{R}^d)},$$

and the inhomogeneous Strichartz estimate (for the solution $\int_0^t S(t-t')f(t')dt'$ of the inhomogeneous problem $i\partial_t u+\Delta u=f$, $u|_{t=0}=0$)

$$\left\|\int_0^t S(t-t')f(t')dt'\right\|_{L_t^q L_x^p(\mathbb{R}\times\mathbb{R}^d)}\le C(q,p,\tilde{q},\tilde{p})\|f\|_{L_t^{\tilde{q}'}L_x^{\tilde{p}'}(\mathbb{R}\times\mathbb{R}^d)},$$

where $\frac{1}{q}+\frac{1}{q'}=1$. In the above, the time definition domain $\mathbb{R}$ can be replaced by any time interval $[-T,T]$, $T>0$.

Regularity properties of the equation

$$i\partial_t u+\Delta u=f, \tag{3.1}$$

will be of interest. Firstly,

Definition 3.1.2. *Given an interval $0\in I\subset\mathbb{R}$, we say that $u:I\times\mathbb{R}^d\to\mathbb{C}$ is a solution to (3.1) with initial data u_0 at time $t=0$, if for any compact interval $J\subset I$, $u\in C_t^0L_x^2(J\times\mathbb{R}^d)\cap L_{t,x}^{\frac{2(d+2)}{d}}(J\times\mathbb{R}^d)$, and for all $t,t_0\in I$, Duhamel's formula*

$$u(t)=S(t-t_0)u(t_0)-i\int_{t_0}^t S(t-s)f(s)ds$$

holds.

Definition 3.1.3. *Strichartz norm, for $d\le 4$, is defined by*

$$\|u\|_S=\|u\|_{S_T}:=\|u\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R}^d)}+\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([-T,T]\times\mathbb{R}^d)}+\|u\|_{L_t^{\frac{d+4}{d}}L_x^{\frac{2(d+4)}{d}}([-T,T]\times\mathbb{R}^d)}.$$

In general case, $d\ge 5$, we omit the last term in the sum as it is not a Strichartz pair.

Denote by $B_R:=\{x\in\mathbb{R}^d:\|x\|\le R\}$.

Remark 3.1.4. *Solutions $u(t)$ of (3.1) with initial data $u_0 \in L^2$ have the following gain of regularity*

$$\|u(t)\|_{L_t^2 H_x^{1/2}([-T,T]\times B_R)} \lesssim R^{\frac{1}{2}}\Big\{\|u_0\|_{L_x^2} + \|f\|_{L_t^1 L_x^2([-T,T]\times\mathbb{R}^d)}\Big\},$$

which follows from local smoothing property of Schrödinger propagator ([CS88])

$$\|u\|_{L_{t,x}^2([-T,T]\times B_R)} \lesssim R^{\frac{1}{2}}\Big\{\Big\||\nabla|^{-\frac{1}{2}}u(0)\Big\|_{L_x^2} + \Big\||\nabla|^{-\frac{1}{2}}f\Big\|_{L_t^1 L_x^2([-T,T]\times\mathbb{R}^d)}\Big\}, \tag{3.2}$$

where $|\nabla|^s, s \in \mathbb{R}$, denotes a Fourier multiplier given by symbol $|\xi|^s$.

Lemma 3.1.5. *Fix $T > 0$ and let $u : [-2T, 2T] \times \mathbb{R}^d \to \mathbb{C}$ be a solution with initial data $u(0) \in L^2(\mathbb{R}^d)$ of the equation (3.1). Then*

$$\Big\|u(t+\tau, x+y) - u(t,x)\Big\|_{L_{t,x}^2([-T,T]\times B_R)} \lesssim_{T,R} (|y|^{\frac{1}{2}} + |\tau|^{\frac{1}{4}})\Big(\|u_0\|_{L^2} + \|f\|_{L_t^1 L_x^2([-2T,2T]\times\mathbb{R}^d)}\Big),$$

uniformly for $|\tau| \leq T$ and $y \in \mathbb{R}^d$.

Proof. Triangle inequality allows us to consider time and space regularity separately. Firstly, consider the linear Schrödinger equation

$$i\partial_t v + \Delta v = 0,$$

with initial data u_0, i.e.

$$v(t) = \begin{cases} 0 & , t < 0 \\ S(t)u_0 & , t > 0, \end{cases}$$

and for which we want to prove that

$$\|v(t+\tau, x+y) - v(t,x)\|_{L_{t,x}^2([-T,T]\times B_R)} \lesssim (|y|^{\frac{1}{2}} + |\tau|^{\frac{1}{4}})\|u_0\|_{L^2(\mathbb{R}^d)}.$$

Local smoothing, that is Remark 3.1.4 gives

$$\|v(t,x+y) - v(t,x)\|_{L^2([-T,T]\times B_R)} \leq |y|^{\frac{1}{2}}\|v(t,x)\|_{L_t^2 \dot{H}_x^{\frac{1}{2}}([-T,T]\times B_{R+|y|})} \lesssim_R |y|^{\frac{1}{2}}\|u_0\|_{L^2}.$$

Denoting by $v_N := P_{\leq N}u$ for some $N > 0$, we write

$$v(t+\tau) - v(t) = v(t+\tau) - v_N(t+\tau) + v_N(t+\tau) - v_N(t) + v_N(t) - v(t), \tag{3.3}$$

and estimate terms separately. Firstly, by (3.2),

$$\|v(t) - v_N(t)\|_{L^2([-T,T]\times B_R)} \lesssim_R \Big\||\nabla|^{-\frac{1}{2}}v_{>N}(0)\Big\|_{L_x^2} \lesssim_R N^{-\frac{1}{2}}\|u_0\|_{L_x^2}.$$

Since $i\partial_t v_N = -\Delta v_N$, and denoting by $\hat{w}_N = |\xi|^2 \hat{v}_N$, $i\partial_t w_N = -\Delta w_N$ with initial data $\hat{w}_N(0) = |\xi|^2 \hat{v}_N(0)$. Then by (3.2)

$$\begin{aligned}
\|v_N(t+\tau) - v_N(t)\|_{L^2([-T,T]\times B_R)} &\leq \|v_N(t+\tau)\|_{L^2([-\tau,0]\times B_R)} + \|v_N(t+\tau) - v_N(t)\|_{L^2([0,T]\times B_R)} \\
&\leq \tau^{\frac{1}{2}} \|u_0\|_{L^2} + \int_t^{t+\tau} \|\Delta v_N(s)\|_{L^2([0,T]\times B_R)} ds \\
&\lesssim_R \tau^{\frac{1}{2}} \|u_0\|_{L^2} + \int_t^{t+\tau} \left\| |\nabla|^{-\frac{1}{2}} w_N(0) \right\|_{L^2_x} ds \\
&\lesssim_{T,R} (\tau^{\frac{1}{2}} + N^{\frac{3}{2}} \tau) \|u_0\|_{L^2_x}.
\end{aligned}$$

Choosing N so that $\tau = N^{-2}$, and plugging previous inequalities in (3.3), one gets

$$\|v(t+\tau) - v(t)\|_{L^2([-T,T]\times B_R)} \lesssim \left(N^{-\frac{1}{2}} + \tau^{\frac{1}{2}} + N^{\frac{3}{2}}\tau\right) \|u_0\|_{L^2} \lesssim_{T,R} \tau^{\frac{1}{4}} \|u_0\|_{L^2}, \tag{3.4}$$

concluding the proof of the linear case.

In the nonlinear case, Lemma 3.1.4 implies the space regularity

$$\begin{aligned}
\left\| u(t,x+y) - u(t,x) \right\|_{L^2_{t,x}([-T,T]\times B_R)} &\leq |y|^{\frac{1}{2}} \left\| u(t,x) \right\|_{L^2_t \dot{H}^{\frac{1}{2}}_x([-T,T]\times B_R)} \\
&\lesssim_R |y|^{\frac{1}{2}} \left\{ \|u(0)\|_{L^2_x} + \|f\|_{L^1_t L^2_x([-2T,2T]\times \mathbb{R}^d)} \right\}.
\end{aligned}$$

For the time regularity we use Duhamel's formula

$$u(t) = S(t)u_0 + \int_0^t S(t-s) f(s) ds = v(t) + \int_0^t S(t-s) f(s) ds$$

and estimate each term separately. First term follows by (3.4). Regarding the second term, denote by

$$w(t,s) = \begin{cases} 0 & , t \leq s \text{ or } t < 0 \\ S(t-s)f(s) & , t > s\ . \end{cases}$$

Then using Minkowski and (3.4)

$$\begin{aligned}
&\left\| \int_0^{t+\tau} S(t+\tau-s) f(s) ds - \int_0^t S(t-s) f(s) ds \right\|_{L^2_{t,x}([-T,T]\times B_R)} \\
&\quad = \left\| \int_0^{2T} (w(t+\tau,s) - w(t,s)) ds \right\|_{L^2_{t,x}([-T,T]\times B_R)} \leq \int_0^{2T} \|w(t+\tau,s) - w(t,s)\|_{L^2_{t,x}([-T,T]\times B_R)} ds \\
&\quad \lesssim_{T,R} \int_0^{2T} \tau^{\frac{1}{4}} \|f(s)\|_{L^2_x} ds \lesssim_{T,R} \tau^{\frac{1}{4}} \int_0^{2T} \|f(s)\|_{L^2_x} ds \lesssim_{T,R} \tau^{\frac{1}{4}} \|f\|_{L^1_t L^2_x([-2T,2T]\times \mathbb{R}^d)}.
\end{aligned}$$

□

Remark 3.1.6. *In general case $d \geq 5$ we cannot bound the nonlinearity $f = |u|^{\frac{4}{d}} u$ in $L_t^1 L_x^2$ and we have to use dual Strichartz spaces instead. The changes in proof of Lemma 3.1.5 are standard, only the local smoothing estimate requires a different argument. It can be proven using U^2 and V^2 spaces (see [HHK09], [KT18], [KTV14], [KT05])*

$$\|u\|_{L^2 H^{1/2}((-T,T)\times B_R)} \leq \|u\|_{U_\Delta^2((-T,T);L^2)} \lesssim \|u_0\|_{L^2} + \|f\|_{DU_\Delta^2(-T,T);L^2)}$$

and

$$\|f\|_{DU_\Delta^2} \lesssim \|f\|_{L^{q'} L^{p'}}$$

where (q,p) is a Strichartz pair with $q > 2$.

3.2 Mass subcritical NLS

This section is concerned with proving the non-squeezing property of the flow of mass subcritical NLS in $L^2(\mathbb{R}^d)$

$$i\partial_t u + \Delta u = \kappa |u|^{p-1} u, \tag{3.5}$$

for $1 < p < 1 + \frac{4}{d}$, $\kappa = \pm 1$. Specifically,

Theorem 3.2.1. *Let $h_0, l \in L^2$ such that $\|l\|_{L^2} = 1$ and $0 < r_2 < r_1 < 0$, $\alpha \in \mathbb{C}$. Then for every time $T > 0$ there exists the initial data $u_0 \in \mathbb{B}_{r_1}(h_0)$ such that the solution u given by (3.5) satisfies*

$$|\langle u(T), l\rangle_{L^2} - \alpha| > r_2.$$

Equation (3.5) is Hamiltonian with energy functional

$$H(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\kappa}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx,$$

defined on $H : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$. The equation (3.5) also preserves the L^2 norm. Choosing

$$\omega(f,g) = \operatorname{Im}\int_{\mathbb{R}^d} f(x)\overline{g(x)} dx = \operatorname{Im}\langle f, g\rangle_{L^2}$$

for symplectic form and $J = i$ for an almost complex structure, equation (3.5) becomes a symplectic one. Note that ω is a strong symplectic form on L^2 and weak on $\mathcal{S}(\mathbb{R}^d)$ ([MNc$^+$20]).

We shall restrict ourselves to defocusing case, since the sign of κ plays no role in the following exposition. The goal is to approximate the said flow with its truncated version

$$i\partial_t u + \Delta u = \chi_R |u|^{p-1} u, \tag{3.6}$$

where χ_R represents a characteristic function of a ball $B(0,R) := \{x \in \mathbb{R}^d : \|x\| \leq R\}$, generated by Hamiltonian

$$H_R(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\kappa}{p+1}\int_{\mathbb{R}^d} \chi_R |u|^{p+1} dx,$$

and prove that (3.6) obeys the assumptions of Theorem 1.0.3, as it is symplectic in the same manner as (3.5). Truncation χ_R directly introduces a family of symplectomorphisms indexed by $R \geq 1$, or equivalently, by set $[0,1)$ via identity $\tau = 1 - R^{-1}$. Endpoint 1 of $\mathcal{I} = [0,1]$ corresponds to the initial NLS (3.5). To be more precise, denote the flow generated by (3.6) as Φ_R and (3.5) by Φ. We define the continuous family of symplectomorphisms

$$
\begin{aligned}
&\Psi : \mathcal{I} \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \\
&\Psi(1) := \Phi \\
&\Psi(\tau) := \Phi_R, \ \ \tau \in [0,1).
\end{aligned}
$$

Conservation of L^2 norm for all $\Psi(\tau)$, $\tau \in [0,1]$, directly reduces to consideration of the family

$$
\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1},
$$

where X_{r_1} denotes the ball B_{r_1} endowed with weak topology and centered at $0 \in L^2$. It suffices to consider stated family in the context of Theorem 1.0.3. The choice of family Ψ is motivated by the fact that symplectomorphisms $\Psi(\tau)$, $\tau \in [0,1)$, satisfy assumptions of Theorem 1.0.3 (equivalently Corollary 1.0.4), namely both weak continuity of the flow map $\Psi(\tau)$ and its derivative $D\Psi(\tau)$. The endpoint $\Psi(1)$ however does not have weakly continuous derivative, but it does satisfy the assumption of weak continuity of the flow $\Psi(1)$, allowing consideration of the continuity of family $\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1}$. Stated homotopy is indeed continuous, allowing non-squeezing property of $\Psi(1)$ to be deduced by uniform approximation argument (see Theorem 1.1.1) from flows $\Psi(\tau)$, $\tau \in [0,1)$, which are non-squeezing by Theorem 1.0.3.

Motivation behind the truncation χ_R of nonlinearity in (3.6) lies in local smoothing of Schrödinger operator. In other words, when localized in space, solutions to (3.5) and (3.6) gain a half of derivative in regularity as shown in Lemma 3.1.5, which coupled with well-posedness of said equations gives precompactness of the set of solutions in a suitable Lebesgue space. This allows us to uniformly control the nonlinearity $|u|^{p-1}u$ in a compact fashion. Namely, for initial data $u_{n,0} \rightharpoonup u_0$ and the corresponding solutions u_n, u of (3.6), nonlinearities $|u_n|^{p-1}u_n \rightharpoonup |u|^{p-1}u$ converge in $L^1_t L^2_x$, and even more, $\chi_R |u_n|^{p-1}u_n \to \chi_R |u|^{p-1}u$ in $L^1_t L^2_x$. While this observation holds true for NLS and approximate flows (3.6), localization χ_R of the nonlinearity is crucial for obtaining the weak continuity of the derivative of the flow (3.6), allowing us to apply Theorem 1.0.3.

Nevertheless, well-posedness of NLS and approximate flows (3.6) will allow uniform, independent of R, control of nonlinearities $|u|^{p-1}u$, leading to continuity with respect to weak topology in L^2 of the family Ψ.

We shall restrict ourselves to the case $d \leq 4$ in order to preserve clarity of the following exposition. General case $d \geq 5$ follows by Remark 3.1.6. We start by stating the following well-posedness theory

Lemma 3.2.2. *Let $u_0 \in L^2(\mathbb{R}^d)$ such that $\|u_0\|_{L^2} \lesssim 1$. Then there exists a unique global solution $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ to the approximate flow (3.6) (or NLS (3.5)) with initial data $u(0) = u_0$. Additionally, for any $T > 0$*

$$\|u\|_S \lesssim_T 1, \text{ and } \quad \left\| |u|^{p-1} u \right\|_{L^1_t L^2_x} \lesssim_T 1,$$

and flows are Lipschitz on bounded balls of initial data.

In other words, for bounded initial data and fixed time T, Strichartz norms for solutions of (3.5) and (3.6) are independent of space truncation χ_R. Same holds for nonlinearities $|u|^{p-1}u$ in L^1L^2. We briefly state the proof

Proof of Lemma 3.2.2. The proof follows classically by fixed point argument of the map

$$\Phi(u)(t) := S(t)u_0 - i \int_0^t S(t-s)|u(s)|^{p-1} u(s) ds.$$

Since for $u \in S$, $u \in L^1_t L^2_x([-T,T] \times \mathbb{R}^d)$ and $u^{1+4/d} \in L^1_t L^2_x([-T,T] \times \mathbb{R}^d)$, by interpolation

$$\|u\|_{L^p_t L^{2p}_x([-T,T]\times\mathbb{R}^d)} = \left\| |u|^{p-1} u \right\|_{L^1_t L^2_x([-T,T]\times\mathbb{R}^d)} \lesssim_{\|u\|_S} T^{\theta(p)}, \quad \theta(p) > 0,$$

follows, obtaining the bound for the nonlinear term. Contraction of maps Φ_R, Φ in Strichartz space S then follows by Strichartz bounds, bound for the nonlinearity in $L^1_t L^2_x$ and subcriticallity of exponent p, which imply smallness in time existence T.

Lastly, we prove the Lipschitz regularity of flows Φ_R, Φ . We only state it for Φ_R, as the bound will be relevant later on. Namely

$$\begin{aligned} \left\| \int_0^T S(t-s)\chi_R(|u|^{p-1}u - |v|^{p-1}v) ds \right\|_S &\lesssim \|\chi_R(|u|^{p-1}u - |v|^{p-1}v)\|_{L^1_t L^2_x} \\ &\lesssim \left(\|\chi_R |u|^{p-1}\|_{L_t^{\frac{p}{p-1}} L_x^{\frac{2p}{p-1}}} + \|\chi_R |v|^{p-1}\|_{L_t^{\frac{p}{p-1}} L_x^{\frac{2p}{p-1}}} \right) \|u - v\|_{L^p_t L^{2p}_x} \\ &\lesssim \|u - v\|_S^{c(p)}. \end{aligned} \tag{3.7}$$

□

Remark 3.2.3. *Lipschitz regularity of flow map Φ_R suggests the consideration of a linearized equation*

$$i\partial_t u + \Delta u = Vu, \quad V = \chi_R |v|^{p-1} \in L_t^{\frac{p}{p-1}} L_x^{\frac{2p}{p-1}}. \tag{3.8}$$

Equation (3.8) *is well-posed in the same sense as equations (3.5) and (3.6), as per* (3.7)*, and moreover, it is continuous as a map*

$$S \times L_t^{\frac{p}{p-1}} L_x^{\frac{2p}{p-1}} \to S : u_0 \times V \mapsto u. \tag{3.9}$$

This continuity extends to uniform one with respect to V on bounded balls of initial data in S.

Forthcoming discussion crucially lies on the precompactness of the set of solutions to (3.5) and (3.6), for bounded initial data. Observe the set

$$\begin{aligned}\mathcal{M}_{T,M} &= \left\{ \chi_{T,M}\Phi(u_0), \chi_{T,M}\Phi_R(u_0) \;\middle|\; \|u_0\|_{L^2} \lesssim 1,\;\; R \geq 1 \right\} \\ &:= \left\{ \chi_{T,M}\Psi(\tau, u_0) \;\middle|\; \|u_0\|_{L^2} \lesssim 1,\;\; \tau \in \mathcal{I} \right\},\end{aligned} \tag{3.10}$$

where $\chi_{T,M}$ is a smooth space-time localization on $[-T,T] \times B(0,M) \subset \mathbb{R} \times \mathbb{R}^d$. Following lemma, due to M. Riesz ([Rie33]), gives necessary and sufficient conditions for a set to be precompact in Lebesgue space

Lemma 3.2.4 ([Rie33])**.** *Fix $1 \leq p < \infty$. A family of functions $\mathcal{F} \subset L^p(\mathbb{R}^d)$ is precompact in this topology if and only if it obeys the following three conditions:*

(i) There exists $C > 0$ such that $\|f\|_{L^p} \leq C$ for all $f \in \mathcal{F}$.

(ii) For any $\varepsilon > 0$, there exists $\delta > 0$ so that $\int_{\mathbb{R}^d} |f(x+y) - f(x)|^p dx < \varepsilon$ for all $f \in \mathcal{F}$ and all $|y| < \delta$.

(iii) For any $\varepsilon > 0$ there exists R so that $\int_{|x| \geq R} |f(x)|^p dx < \varepsilon$ for all $f \in \mathcal{F}$.

Second and third condition from previous lemma are equicontinuity and tightness, respectively.

Corollary 3.2.5. *The set $\mathcal{M}_{T,M}$ is precompact in $L^r(\mathbb{R} \times \mathbb{R}^d)$, for $r \in [1, \frac{2(d+2)}{d})$.*

Proof. Precompactness in $L^2_{t,x}$ follows directly by invoking Lemma 3.2.4 . Indeed, uniform boundedness follows directly from independent of truncation χ_R Strichartz bounds given by Lemma 3.2.2. Tightness is directly implied by the space-time localization $\chi_{T,M}$. Equicontinuity follow from Lemma (3.1.5) by applying $\|\chi_{T,M}\|_{L^\infty} \leq 1$ and uniform Strichartz bounds (3.2.2).

Precompactness in L^r for general r follows by interpolation with respect to uniform, independent of truncation χ_R, Strichartz bounds $L^{\frac{2(d+2)}{d}}_{t,x}$ given by Lemma 3.2.2 and Hölder inequality. □

Remark 3.2.6. *Precompactness of $\mathcal{M}_{T,M}$ implies precompactness of the set of nonlinearities*

$$\mathcal{N}_{T,M} = \left\{ \chi_{T,M}|u|^{p-1}u \;\middle|\; \|u_0\|_{L^2} \lesssim 1,\;\; u \text{ solves (3.5),(3.6) for initial data } u_0 \right\}$$

in $L^1_t L^2_x$. Indeed, $\mathcal{N}_{T,M}$ is precompact in $L^1_{t,x}$ due to Hölder inequality ($a = \frac{2(d+2)}{d(p-1)} > \frac{2(d+2)}{d+4}$)

$$\left\| \chi_{T,M}|u|^{p-1}u - \chi_{T,M}|v|^{p-1}v \right\|_{L^1_{t,x}} \lesssim \left(\|u^{p-1}\|_{L^{\frac{2(d+2)}{d(p-1)}}_{t,x}} + \|v^{p-1}\|_{L^{\frac{2(d+2)}{d(p-1)}}_{t,x}} \right) \|\chi_{T,M}(u-v)\|_{L^{a'}_{t,x}}$$

Strichartz estimates and precompactness of $\mathcal{M}_{T,M}$. Precompactness in $L^1_t L^2_x$ follows from one in $L^1_{t,x}$ by interpolation, Strichartz estimates and the fact that $|u|^{4/d}u \in L^1_t L^2_x$.

Lemma 3.2.7. *Family $\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1}$ is continuous, where X_{r_1} represents the ball of initial data $\mathbb{B}_{r_1}(0) \subset L^2$ endowed with weak topology.*

Proof. We distinguish two cases coming from the continuity in the first variable in $\mathcal{I}$ - $\tau \to 1$ ($R \to \infty$) and $\tau_n \to \tau < 1$ ($R_n \to R \in \mathbb{R}$). We prove the continuity only in the former case, which is the harder case, coupled with a convergent sequence in the metric space (X_{r_1}, d_w). We omit the proof of the easier case, as it follows with minor adjustments of the following proof and hence is obvious. However, we would like to point out the the precompactness of the set $\mathcal{M}_{T,M}$ is what allows us to prove the continuity of the family uniformly on $\mathcal{I} \times X_{r_1}$.

Let $R_n > 0$ be a sequence such that $R_n \to \infty$ and $u_n(0), u(0) \in B_{r_1}$ be a sequence of initial data such $u_n(0) \rightharpoonup u(0)$. We want to prove that then

$$u_n := \Phi_{R_n}(u_n(0)) \rightharpoonup u := \Phi(u(0)).$$

Strichartz norms (Lemma 3.2.2) for all flows Φ_{t,R_n} are uniformly bounded by constants depending on r_1, T, p, h_0, and dimension d, and not by the truncation constant R, hence $\|u_n\|_S \lesssim_T 1$. Then the proof follows from the precompactness of set $\mathcal{M}_{T,M}$. Indeed, one has that there exists v such that, after passing to a subsequence,

$$\lim_{n\to\infty} \|u_n - v\|_{L^r([-T,T]\times B_M)} = 0,$$

for $1 \le r < \frac{2(d+2)}{d}$. Uniform Strichartz bounds for u_n imply that

$$\|v\|_{L^\infty L^2} + \|v\|_{L^{\frac{2(d+2)}{d}}} \lesssim 1.$$

We want to prove that $v = u_\infty$. Let us go back to the Duhamel's formula

$$u_n(t) = e^{it\Delta}u_n(0) - i\int_0^t e^{i(t-s)\Delta}\chi_{R_n}|u_n(s)|^{p-1}u_n(s)ds$$

and look at the weak limit of this equality. Then on the LHS we have v. As the Schrödinger semigroup is linear, first summand on RHS will be equal to $e^{it\Delta}u_\infty(0)$. It remains to prove that

$$\text{w-}\lim_{n\to\infty}\int_0^t e^{i(t-s)\Delta}\chi_{R_n}|u_n(s)|^{p-1}u_n(s)ds = \int_0^t e^{i(t-s)\Delta}|v(s)|^{p-1}v(s)ds.$$

Then the Strichartz bounds and well-posedness theory imply $v = u_\infty$.

Firstly, see that

$$\text{w-}\lim_{n\to\infty}\int_0^t e^{i(t-s)\Delta}\chi_{R_n}|u_n(s)|^{p-1}u_n(s)ds = \text{w-}\lim_{n\to\infty}\int_0^t e^{i(t-s)\Delta}|u_n(s)|^{p-1}u_n(s)ds.$$

Indeed, fixing arbitrary $\varphi \in L^2(\mathbb{R}^d)$, one has

$$
\begin{aligned}
\left|\left\langle \varphi, \int_0^t e^{i(t-s)\Delta}(1-\chi_{R_n})|u_n(s)|^{p-1}u_n(s)ds\right\rangle\right| &\le \left|\int_0^t \left\langle (1-\chi_{R_n})e^{-i(t-s)\Delta}\varphi, |u_n(s)|^{p-1}u_n(s)\right\rangle ds\right| \\
&\le \|(1-\chi_{R_n})e^{-i(t-s)\Delta}\varphi\|_{L^\infty L^2} T^\theta \|u_n\|_S^p \\
&\lesssim \|(1-\chi_{R_n})e^{-i(t-s)\Delta}\varphi\|_{L^\infty L^2} \to 0, \qquad (3.11)
\end{aligned}
$$

as $n \to \infty$. Hence we have reduced the problem to proving

$$
\text{w-}\lim_{n\to\infty} \int_0^t e^{i(t-s)\Delta}|u_n(s)|^{p-1}u_n(s)ds = \int_0^t e^{i(t-s)\Delta}|v(s)|^{p-1}v(s)ds
$$

in L^2. Similarly to the argument in inequality (3.11), that is exploiting that $e^{i(t-s)\Delta}$ creates a tight orbit $\{e^{i(t-s)\Delta}\varphi : t \in [0,T]\}$ and $\|u_n\|_S \lesssim 1$, we can reduce the last equality to the following one

$$
\text{w-}\lim_{n\to\infty} \int_0^t e^{i(t-s)\Delta}\chi_R|u_n(s)|^{p-1}u_n(s)ds = \int_0^t e^{i(t-s)\Delta}\chi_R|v(s)|^{p-1}v(s)ds,
$$

for some big enough $R > 0$.

Finally, the last equality follows from convergence of u_n to v locally in space and time

$$
\begin{aligned}
&\left|\left\langle \varphi, \int_0^t e^{i(t-s)\Delta}\chi_R(|u_n(s)|^{p-1}u_n(s) - |v(s)|^{p-1}v(s))ds\right\rangle\right| \\
&\quad\lesssim \left\|\int_0^t e^{i(t-s)\Delta}\chi_R(|u_n(s)|^{p-1}u_n(s) - |v(s)|^{p-1}v(s))ds\right\|_{L_t^\infty L_x^2} \\
&\quad\lesssim \left\|\chi_R(|u_n(s)|^{p-1}u_n(s) - |v(s)|^{p-1}v(s))\right\|_{L_t^1 L_x^2} \to 0,
\end{aligned}
$$

as $n \to \infty$ due to Remark 3.2.6. □

We now turn to proving that Φ_R is non-squeezing by proving the properties of the flow allowing us to invoke Theorem 1.0.3. Following lemma concerning continuity of Φ_R is a direct corollary of the continuity of family $\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1}$.

Corollary 3.2.8. *Approximate flow (3.6) maps weakly convergent sequence of initial data into a weakly convergent sequence, i.e. flow is continuous with respect to the weak topology on the space of initial data L^2.*

Corresponding derivative $D\Phi_R$ at point u_0 is given as an evolution map of the equation

$$
i\partial_t v + \Delta v = \chi_R\left[\frac{p-1}{2}|u|^{p-3}u^2\bar{v} + \frac{p+1}{2}|u|^{p-1}v\right], \qquad (3.12)
$$

where $v(t) := D\Phi_R[u_0](v_0)$. In order to make sense of the statement, let us discuss the well-posedness of (3.12). For that purpose, we can restrict ourselves to observing the equation

$$i\partial_t v + \Delta v = \chi_R |u|^{p-1} v, \tag{3.13}$$

as the right hand side of (3.12) can be bounded pointwise by $C|u|^{p-1}v$. We shall denote the time-dependent potential $|u|^{p-1}$ as V. We would like to point out that one should think of the derivative of flow (3.6) as a perturbation of the Laplacian by a rough potential V. The proof of the continuity of $D\Phi_R$ then reduces to proving that potentials converge in a suitable Lebesgue space.

Well-posedness of (3.12) follows in the same fashion as well-posedness of (3.13) by applying triangle inequalities accordingly in the proof that follows. Duhamel's formula for solution v can be written as

$$v(t) := D\Phi_R[u_0](v_0) = e^{it\Delta} v_0 - i \int_0^t e^{i(t-s)\Delta} \chi_R V(s) v(s) ds,$$

where we treat the potential V as a nonlinearity.

Lemma 3.2.9. *Cauchy problem for (3.13) is locally well-posed in $L^2(\mathbb{R}^d)$ in the following sense: There exists a positive time $T > 0$ depending on the norm of initial data $\|v_0\|_{L^2(\mathbb{R}^d)}, p, d$ and a unique solution $v = v(t,x)$ defined on the time interval $[-T, T]$ such that*

$$v \in S_T := \left\{ v \in C([-T,T], L^2) \;\middle|\; v \in L^{\frac{2(d+2)}{d}}([-T,T] \times \mathbb{R}^d) \right\}.$$

Moreover, the evolution map $D\Phi_R[u_0] : L^2 \longrightarrow L^2$ is Lipschitz on balls of initial data with a constant that depends on $\|v_0\|_{L^2(\mathbb{R}^d)}, p, d$.

Proof. This was previously discussed in Remark 3.2.3. We remark that T depends on the size of data, and a priori time t at which we are looking at the flow of (3.6) may be greater. However, for the purpose of proving the next theorem, we can restrict ourselves to ball of initial data of small scale, in order to obtain $t < T$. $\square$

Remark 3.2.10. *Space of potentials*

$$\mathcal{V}_{T,M} = \left\{ \chi_{T,M} |u|^{p-1} \;\middle|\; \|u_0\|_{L^2} \lesssim 1, \;\; u \text{ solves (3.5),(3.6) for initial data } u_0 \right\}$$

is precompact in $L_t^r L_x^{2r}(\mathbb{R} \times \mathbb{R}^d)$, $1 \le r < \frac{d+4}{d(p-1)}$. This follows in the same manner as Remark 3.2.6.

Having made sense of $D\Phi_R(u_0)$, we claim that

Lemma 3.2.11. *Derivative $D\Phi_R : (L^2(\mathbb{R}^d), \tau_{weak}) \longrightarrow (B(L^2(\mathbb{R}^d)), \|\cdot\|_{op})$ is continuous.*

Proof. Continuity follows by continuity argument in Remark 3.2.3 and Remark 3.2.10, since $\frac{p}{p-1} < \frac{d+4}{d(p-1)}$, which is equivalent to subcriticality of exponent p. Strichartz bounds in Lemma 3.2.2, resp. Lemma 3.2.9, allow uniform treatment of terms $|u|^{p-1}$, resp. v, in (3.13).

More specifically, let $u_n(0) \rightharpoonup u(0)$ be a weakly convergent sequence and u_n, u the corresponding Cauchy solutions to (3.6), similarly with potentials V_n, V. Lemma 3.2.8 implies that $u_n \rightharpoonup u$ uniquely. Let

$$\begin{aligned} i\partial_t w_n + \Delta w_n &= \chi_R V_n w_n, \\ i\partial_t w + \Delta w &= \chi_R V w, \end{aligned}$$

be solutions of (3.13) for same initial data $w(0) = w_n(0) = w_0$. Denoting $v = w - w_n$, we get

$$i\partial_t v + \Delta v = \chi_R V v - w_n \chi_R (V_n - V),$$

for initial data $v(0) = 0$. We want to prove that solutions $v := [D\Phi_{T,R}(u(0)) - D\Phi_{T,R}(u_n(0))]w_0$ converge to 0 uniformly on a ball of initial data that has a small radius. We note that the Strichartz norms of w_n are uniformly bounded by a constant depending on the size of initial data, which we have fixed. Well-posedness of (3.13) gives the inequality

$$\begin{aligned} \|v\|_X &\lesssim \left\| \int_0^T S(t-s) w_n \chi_R (V_n - V) ds \right\|_S \\ &\lesssim \|w_n \chi_R (V_n - V)\|_{L_t^1 L_x^2} \leq \|w_n\|_{L_t^p L_x^{2p}} \|\chi_R (V_n - V)\|_{L_t^{\frac{p}{p-1}} L_x^{\frac{2p}{p-1}}} \\ &\lesssim \|\chi_R (V_n - V)\|_{L_t^{\frac{p}{p-1}} L_x^{\frac{2p}{p-1}}}. \end{aligned}$$

□

Remark 3.2.12. *Lemmata 3.2.8 and 3.2.11 imply that the flow (3.6) up to time $T := T(r_1 + \|h_0\|)$ dependent on the size of data in $\mathbb{B}_{r_1}(h_0)$, obeys assumptions of Corollary 1.0.4. However, one can extend this result to the flow at any time t by iteration. Indeed, rewriting $\Phi_{t,R} = \Phi_{t-NT,R} \circ (\Phi_{T,R})^N$, the chain rule gives $D\Phi_{t,R} = D\Phi_{t-NT,R} \circ (D\Phi_{T,R})^N$. Then that $D\Phi_{t,R} : (L^2, \tau_{weak}) \to B(L^2)$ is continuous follows from Lemma 3.2.11, and the fact that $\Phi_{T,R} : (L^2, \tau_{weak}) \to (L^2, \tau_{weak})$ is continuous.*

3.3 Mass critical NLS

Now we turn our attention to proving the non-squeezing property of mass critical defocusing NLS in $L^2(\mathbb{R}^d)$

$$i\partial_t u + \Delta u = |u|^{\frac{4}{d}} u, \tag{3.14}$$

that is,

Theorem 3.3.1. *Let $h_0, l \in L^2$ such that $\|l\|_{L^2} = 1$ and $0 < r_2 < r_1 < 0$, $\alpha \in \mathbb{C}$. Then for every time $T > 0$ there exists the initial data $u_0 \in B_{r_1}(h_0)$ such that the solution u given by mass critical NLS (3.14) satisfies*

$$|\langle u(T), l\rangle - \alpha| > r_2.$$

Equation is invariant with respect to scaling, namely if $u(t,x)$ solves (3.14) on $[0,T]$ with initial data $u(0,x) = u_0(x)$, then

$$\lambda^{\frac{d}{2}} u(\lambda^2 t, \lambda x) \tag{3.15}$$

solves (3.14) on $[0, \frac{T}{\lambda^2}]$ with initial data $\lambda^{\frac{d}{2}} u_0(\lambda x)$. Additionally, $L^2(\mathbb{R}^d)$ and $L^{\frac{2(d+2)}{d}}(\mathbb{R} \times \mathbb{R}^d)$ norms are invariant, hence Strichartz norm $\|\cdot\|_S$ as well. Equation is Hamiltonian with energy functional

$$H(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{d}{2(d+2)}\int_{\mathbb{R}^d} |u|^{\frac{2d+4}{d}} dx.$$

defined on $H : \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$. Symplectic property of the flow (3.14) follows in the same fashion as the one for subcritical one presented in Section 3.2.

The goal is to approximate the said flow with the truncated version introduced in [KVZ21], following ideas of [CKS+05], and choosing a smooth frequency projector. Namely, authors of [KVZ21] introduced the flow $\Phi_{\mathcal{P}_D}$

$$i\partial_t u + \Delta u = \mathcal{P}_D(F(\mathcal{P}_D(u))), \tag{3.16}$$

where $F(u) = |u|^{\frac{4}{d}} u$ and $\mathcal{P}_D$ represents a Fourier multiplier $m_D(\xi)$ defined as follows: Let $\varphi : \mathbb{R}^d \to [0,1]$ be the bump function used in definition of Littlewood-Paley projections. For $1 \le D \in 2^{\mathbb{Z}}$ define

$$\begin{aligned} m_D(\xi) &:= \frac{1}{\log_2(2D)} \sum_{N \ge 1}^{D} \varphi(\xi/N) \\ &= \varphi(\xi) + \sum_{N \ge 2}^{D} \left[\frac{\log_2(2D) - \log_2(N)}{\log_2(2D)}\right][\varphi(\xi/N) - \varphi(2\xi/N)] \end{aligned} \tag{3.17}$$

Symbol $m_D : \mathbb{R}^d \to [0,1]$ is a Mikhlin multiplier uniformly for $D \ge 1$ and it is compactly supported. Namely, $m_D(\xi) \equiv 1$ if $|\xi| \le \frac{1}{2}$ and $m_D(\xi) \equiv 0$ if $|\xi| > 2D$.

Global existence theory proven in [KVZ21] relies on the critical one, which is resolved by Dodson ([Dod16a],[Dod16b],[Dod12]), and observing solutions of (3.16) as regularization of a solution of (3.14) or a perturbation of a linear solution. The following result is obtained

Theorem 3.3.2 ([KVZ21]). *Given $M > 0$, there are constants $C(M)$ and $D_0(M)$ so that the following holds: For any $u_0 \in L^2(\mathbb{R}^d)$ with $\|u_0\|_{L^2} \leq M$ and any $D \geq D_0(M)$, there exists a unique global solution u of the frequency truncated flow (3.16); moreover,*

$$\|u\|_{S(\mathbb{R})} \leq C(M). \tag{3.18}$$

In particular, there exist $u_\pm \in L^2(\mathbb{R}^d)$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{L^2} = 0.$$

Since we only care about bounded data for concluding non-squeezing behaviour, previous theorem directly gives the lower bound for D for which we need to consider flows (3.16). Hence in the forthcoming discussion, D will be implicitly known and fixed. Pursuing clarity, we shall omit D in $\mathcal{P}_D$, and simply write $\mathcal{P}$. Moreover, we shall denote by $\mathcal{P}_n$ a Fourier multiplier with symbol $m_D(\xi/n)$, $n \in \mathbb{R}$, $n \geq 1$.

Finally, we want to approximate NLS (3.14) with

$$i\partial_t u + \Delta u = \mathcal{P}_n(F(\mathcal{P}_n(u))), \tag{3.19}$$

on a bounded ball of initial data. The reasoning behind the choice of the flow (3.19) is that it approximates (3.14) in weak topology as $n \to \infty$, uniformly on balls of initial data. Then the non-squeezing property of (3.14) follows directly from one of (3.19). Uniform weak approximation was already proven in [KVZ21], relying on (3.18) and using profile decomposition. Specially, consideration of solutions to (3.19) with respect to n reduces to the one of (3.16) by scaling $\lambda(n) = \frac{1}{n}$. Moreover, denoting by Φ the mass-critical flow (3.14) and by $\Phi_{\mathcal{P}_n}$ the flow of (3.19), scaling invariance of Strichartz norm and (3.18) imply that

$$\max\left\{\|\Phi(u_0)\|_S, \|\Phi_{\mathcal{P}_n}(u_0)\|_S\right\} \lesssim C(\|u_0\|_{L^2}), \forall n \in \mathbb{R}. \tag{3.20}$$

These observations, just like in the mass subcritical case, lead to the consideration of the family of symplectomorphisms

$$\begin{aligned}
&\Psi : \mathcal{I} \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)\\
&\Psi(1) := \Phi\\
&\Psi(\tau) := \Phi_{\mathcal{P}_n},\ \ \tau \in [0,1),
\end{aligned}$$

where the correspondence between $n \geq 1$ and $\tau \in [0,1)$ is obvious. It is natural to ask how is the family defined for all data in L^2 given the global existence Theorem 3.3.2 and the existence of global solutions up to the size of initial data dependent on the truncation $\mathcal{P}$. However, frequency truncated flow (3.19) is a subcritical flow, as shall be discussed more in details later on. This observation, coupled with the conservation of L^2 norm for flows

(3.14) and (3.19) implies the the family is well defined for all data and all times $T > 0$, and we consider the continuous family

$$\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1}.$$

Nevertheless, uniform Strichartz bounds of Theorem 3.3.2 are crucial for allowing the family to be considered in the context of Theorem 1.1.1, i.e. to discuss continuity of the family with respect to weak topology for a fixed ball B_{r_1}.

We shall go a step further and introduce additional truncation following the idea of the mass-subcritical case, namely space localization of the nonlinearity. More specifically, we consider the flow $\Phi_{\mathcal{P},R}$

$$i\partial_t u + \Delta u = \mathcal{P}(\chi_R F(\mathcal{P}(u))). \tag{3.21}$$

As in (3.14), equation (3.21) conserves the L^2 norm of the solution and the energy given by

$$H_{\mathcal{P},R}(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{d}{2d+4}\int_{\mathbb{R}^d} \chi_R |\mathcal{P}(u)|^{\frac{2d+4}{d}} dx.$$

Equation (3.21) is Hamiltonian generated by $H_{\mathcal{P},R}$. It is symplectic in the same manner as mass-subcritical NLS (3.5).

The proof of non-squeezing property of $\Phi_{\mathcal{P},R}$ (3.21) by invoking Theorem 1.0.3 shall be the same as in Section 3.2, since this flow is subcritical (as is (3.19)), and the local existence theory will allow to prove the continuity of the derivative in an easier way. Consequently, we shall introduce the family of symplectomorphisms in the same spirit as the one in Section 3.2

$$\begin{aligned}
&\widetilde{\Psi}_{\mathcal{P}} : \mathcal{I} \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)\\
&\widetilde{\Psi}_{\mathcal{P}}(1) := \Phi_{\mathcal{P}}\\
&\widetilde{\Psi}_{\mathcal{P}}(\tau) := \Phi_{\mathcal{P},R}, \ \ \tau \in [0,1),
\end{aligned}$$

where $R \geq 1$ and $\tau \in [0,1)$ correspond in an obvious way, which will imply the non-squeezing property of $\Phi_{\mathcal{P}}$ (3.19).

We remark that the global theory of frequency truncated flow (3.19) developed in [KVZ21] is necessary in order to obtain uniform bounds for the truncated flows and reduce non-squeezing of NLS (3.14) to the one of (3.19) via the family Ψ, but for the sake of proving that $\Phi_{\mathcal{P}_n}$ (3.19) is non-squeezing, local existence theory shall suffice for establishing it via family $\widetilde{\Psi}$, as expected due to Section 3.2.

Firstly, let us prove the compactness of all solutions to NLS (3.14) and frequency truncated flow (3.19) with bounded initial data in a suitable mixed-norm space. We consinder initial data u_0 such that $\|u_0\|_{L^2} \lesssim 1$ and subsequent $\mathcal{P} = \mathcal{P}_D$, where D is given by Theorem 3.3.2. We define the set

$$\mathcal{M}_{T,M} = \Big\{\chi_{T,M}\Phi(u_0), \chi_{T,M}\Phi_{\mathcal{P}_n}(u_0) \ \Big| \ \|u_0\|_{L^2} \lesssim 1\Big\}, \tag{3.22}$$

for all $n \in \mathbb{R}$ and any finite smooth space-time localization $\chi_{T,M}$ on $[-T,T] \times B(0,M) \subset \mathbb{R} \times \mathbb{R}^d$. Similarly to sub-critical case, we shall restrict ourselves to the case $d \le 4$, for the sake of clarity of exposition.

Lemma 3.3.3. *The set $\mathcal{M}_{T,M}$ is precompact in $L^r(\mathbb{R} \times \mathbb{R}^d)$, for $r \in [1, \frac{2(d+2)}{d})$.*

Proof. Precompactness in $L^2_{t,x}$ follows directly by applying Lemma 3.2.4 . Indeed, uniform boundedness follows directly from uniform Strichartz bounds (3.20). Equicontinuity follows from Lemma 3.1.5 and uniform Strichartz bounds (3.20). Tightness is implicit in space-time localization $\chi_{T,M}$.

Precompactness in L^r for general r follows by interpolation with respect to uniform Strichartz bounds (3.20) and Hölder inequality. □

Remark 3.3.4. *Precompactness of $\mathcal{M}_{T,M}$ implies precompactness of the set*

$$\mathcal{N}_{T,M} = \left\{ \chi_{T,M} |u|^{p-1} u \;\middle|\; \|u_0\|_{L^2} \lesssim 1, \;\; u \text{ solves (3.14),(3.19) for initial data } u_0 \right\}$$

in $L^r_{t,x}$, $1 \le r < \frac{2(d+2)}{d+4}$, just like in the subcritical case, and Strichartz bounds (3.20).

These observations imply the following

Lemma 3.3.5. *Family $\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1}$ is continuous.*

Proof. Just like in Section 3.2, we distinguish two cases coming from the continuity in the first variable in $\mathcal{I}$ - $\tau \to 1$ ($n \to \infty$) and $\tau_n \to \tau < 1$ ($m_n \to m \in \mathbb{R}$). We prove the continuity only in the former case, which is the harder case, coupled with a convergent sequence in the metric space (X_{r_1}, d_w). We omit the proof of the easier case, as it follows with minor adjustments of the following proof and hence is obvious. The only difference is the fact that $m_n \to m \in \mathbb{R}$ corresponds to scaling, and hence the convergence of Fourier multipliers $m_D(\xi/m_n) \to m_D(\xi/m)$ ($\mathcal{P}_{m_n} \to \mathcal{P}_m$ in strong operator topology).

Let $n \to \infty$ and $u_n(0), u(0) \in B_{r_1}$ be a sequence of initial data such $u_n(0) \rightharpoonup u(0)$ in L^2. We want to prove that then

$$u_n := \Phi_{\mathcal{P}_n}(u_n(0)) \rightharpoonup u := \Phi(u(0)).$$

Since initial data $u_n(0)$ is bounded, the sequence u_n converges due to precompactness of $\mathcal{M}_{T,M}$, hence there exists v such that, after passing to a subsequence,

$$\lim_{n\to\infty} \|u_n - v\|_{L^r([0,T]\times B_M)} = 0,$$

for $1 \le r < \frac{2(d+2)}{d}$, fixed $T > 0$ and any $M > 0$. Moreover, this implies that $u_n(t) \rightharpoonup v(t)$ in L^2 for almost every $t \in [-T,T]$. Uniform Strichartz bounds for u_n imply that

$$\|v\|_{L^\infty L^2} + \|v\|_{L^{\frac{2(d+2)}{d}}} \lesssim 1,$$

hence, per Remark 3.3.4,

$$\lim_{n\to\infty} \left\|\chi_{T,M}|u_n|^{p-1}u_n - \chi_{T,M}|v|^{p-1}v\right\|_{L^r_{t,x}} = 0, \quad 1 \le r < \frac{2(d+2)}{d+4},$$

and

$$|u_n|^{p-1}u_n \rightharpoonup |v|^{p-1}v \quad \text{in } L^r_{t,x}, \quad 1 \le r \le \frac{2(d+2)}{d+4}. \tag{3.23}$$

We observe Duhamel's formula

$$u_n(t) = e^{it\Delta}u_n(0) - i\int_0^t e^{i(t-s)\Delta}\mathcal{P}_n(F(\mathcal{P}_n(u_n)))ds$$

and look at the weak limit of this equality. On the LHS we have v and the first summand on RHS will be equal to $e^{it\Delta}u(0)$. It remains to prove that

$$\text{w-}\lim_{n\to\infty}\int_0^t e^{i(t-s)\Delta}\mathcal{P}_n(F(\mathcal{P}_n(u_n)))ds = \int_0^t e^{i(t-s)\Delta}F(v)ds.$$

Then, as RHS of above Duhamel's formula is continuous with respect to time, after changing v at a set of measure zero to make it continuous with respect to time in L^2, Strichartz bounds and well-posedness theory imply $v = u$.

Firstly, see that

$$\text{w-}\lim_{n\to\infty}\int_0^t e^{i(t-s)\Delta}\mathcal{P}_n(F(\mathcal{P}_n(u_n)))ds = \text{w-}\lim_{n\to\infty}\int_0^t e^{i(t-s)\Delta}F(\mathcal{P}(u_n))ds.$$

Indeed, fixing arbitrary $\varphi \in L^2(\mathbb{R}^d)$, and without loss of generality assuming that $\varphi \in \mathcal{S}$, one has

$$\begin{aligned}\left|\left\langle \varphi, \int_0^t e^{i(t-s)\Delta}(1-\mathcal{P}_n)F(\mathcal{P}_n(u_n))ds\right\rangle\right| &\le \left|\int_0^t \left\langle (1-\mathcal{P}_n)e^{-i(t-s)\Delta}\varphi, F(\mathcal{P}_n(u_n))\right\rangle ds\right| \\ &\le \|(1-\mathcal{P}_n)e^{-i(t-s)\Delta}\varphi\|_{L^{\frac{2(d+2)}{d}}}\|F(\mathcal{P}_n(u_n))\|_{L^{\frac{2(d+2)}{d+4}}} \\ &\lesssim \|(1-\mathcal{P}_n)e^{-i(t-s)\Delta}\varphi\|_{L^{\frac{2(d+2)}{d}}} \to 0, \end{aligned} \tag{3.24}$$

as $n \to \infty$. Hence we have reduced the problem to proving

$$\text{w-}\lim_{n\to\infty}\int_0^t e^{i(t-s)\Delta}F(\mathcal{P}_n(u_n))ds = \int_0^t e^{i(t-s)\Delta}F(v)ds$$

in L^2. Since $\|\mathcal{P}_n v\|_{L^{\frac{2(d+2)}{d}}} \lesssim \|v\|_{L^2_x} \lesssim 1$ and $\|F(\mathcal{P}_n v) - F(v)\|_{L^{\frac{2(d+2)}{d+4}}} \lesssim \|(1-\mathcal{P}_n)v\|_{L^{\frac{2(d+2)}{d}}} \to 0$, as $n \to \infty$, the proof reduces to

$$\text{w-}\lim_{n\to\infty}\int_0^t e^{i(t-s)\Delta}F(\mathcal{P}_n u_n)ds = \int_0^t e^{i(t-s)\Delta}F(\mathcal{P}_n v)ds.$$

Similarly to the argument in inequality (3.24), that is exploiting that $e^{i(t-s)\Delta}$ creates a tight equicontinuous orbit $\{e^{i(t-s)\Delta}\varphi : t \in [0,T]\}$ and $\|u_n\|_S \lesssim 1$, we can reduce the last equality to the following one

$$\text{w-}\lim_{n\to\infty} \int_0^t e^{i(t-s)\Delta} \chi_R F(\mathcal{P}_n u_n) ds = \int_0^t e^{i(t-s)\Delta} \chi_R F(\mathcal{P}_n v) ds.$$

for some big enough $R > 0$. However,

$$\left\| \chi_R \Big(\mathcal{P}_n u_n - \mathcal{P}_n v \Big) \right\|_{L^2_{t,x}} \lesssim \|\chi_{2R}(u_n - v)\|_{L^2_{t,x}} + (nR)^{-1} \Big\{ \|u_n\|_{L^2_{t,x}} + \|v\|_{L^2_{t,x}} \Big\},$$

and $\|\mathcal{P}_n u_n\|_{L^{\frac{2(d+2)}{d}}} \lesssim \|u_n\|_{L^2_x} \lesssim 1$, thus $\lim\limits_{n\to\infty} \left\| \chi_{T,R}(\mathcal{P}_n u_n - \mathcal{P}_n v) \right\|_{L^r_{t,x}} = 0$, $1 \le r < \frac{2(d+2)}{d}$. Then similarly to (3.23)

$$\lim_{n\to\infty} \left\| \chi_{T,R} F(\mathcal{P}_n u_n) - \chi_{T,R} F(\mathcal{P}_n v) \right\|_{L^r_{t,x}} = 0, \quad 1 \le r < \frac{2(d+2)}{d+4},$$

and

$$F(\mathcal{P}_n u_n) \rightharpoonup F(\mathcal{P}_n v) \quad \text{in } L^r_{t,x}, \quad 1 \le r \le \frac{2(d+2)}{d+4}, \tag{3.25}$$

which finishes the proof since $\varphi \in \mathcal{S}$. □

We now turn to proving that $\Phi_{\mathcal{P}_n}$ (3.19) has non-squeezing property. We omit n in $\mathcal{P}_n$ in the following, as it plays no role. In contrast to previous exposition, where we relied heavily on uniform Strichartz bounds with respect to the frequency truncation (3.20), and which hold globally in time, the proof of non-squeezing of flow (3.19) will rely heavily on local well-posedness, as we will treat it as what it is - a subcritical flow. Firstly,

Lemma 3.3.6. *Let $u_0 \in L^2(\mathbb{R}^d)$ such that $\|u_0\|_{L^2} \lesssim 1$. Then there exists a unique global solution $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ to the approximate flows (3.19) and (3.21) with initial data $u(0) = u_0$. Additionally, for any $T > 0$*

$$\|u\|_S \lesssim_T 1, \textit{ and } \quad \left\| |u|^{p-1} u \right\|_{L^1_t L^2_x} \lesssim_T 1.$$

Proof. Subcriticality of (3.19) and (3.21) follows from the observation that the following inequalities hold

$$\|\mathcal{P}(u)\|_{L^\infty} \le_{\mathcal{P}} \|u\|_{L^2},$$
$$\|\mathcal{P}(u)\|_{L^2} \le \|u\|_{L^2},$$

and interpolating these bounds. Then, observing the Strichartz norms for the nonlinearity

$$\begin{aligned}\Big\| \int_0^T S(t-s)\mathcal{P}(\chi_R F(\mathcal{P}(u)))ds\Big\|_S &\le \|\mathcal{P}(\chi_R F(\mathcal{P}(u)))\|_{L^1L^2} \le \|F(\mathcal{P}(u))\|_{L^1L^2}\\ &\le T\|F(\mathcal{P}(u))\|_{L^\infty L^2} = T\|\mathcal{P}(u)\|^{\frac{d+4}{d}}_{L^\infty L^{\frac{2(d+4)}{d}}}\\ &\lesssim_{\mathcal{P}} T\|u\|^{\frac{d+4}{d}}_{L^\infty L^2} \lesssim_{\mathcal{P}} T\|u\|^{\frac{d+4}{d}}_S,\end{aligned}$$

local well-posedness follows directly, and global well-posedness from the conservation of L^2 norm. □

The approach we will take then will be essentially the same as the one in previous Section 3.2, which comes as no surprise due to subcritical nature. Denote by

$$\mathcal{M}^{\mathcal{P}}_{T,M} = \Big\{\chi_{T,M}\Phi_{\mathcal{P}}(u_0), \chi_{T,M}\Phi_{\mathcal{P},R}(u_0) \;\Big|\; \|u_0\|_{L^2} \lesssim 1\Big\},$$

for any finite smooth space-time localization $\chi_{T,M}$ on $[0,T]\times B(0,M) \subset \mathbb{R}\times\mathbb{R}^d$. We omit the proof of the following lemma, as it follows by the same arguments as the proof of Lemma 3.2.5.

Lemma 3.3.7. *The set $\mathcal{M}^{\mathcal{P}}_{T,M}$ is precompact in $L^r(\mathbb{R}\times\mathbb{R}^d)$, for $r \in [1, \frac{2(d+2)}{d})$.*

As it will be of use in forthcoming proofs due to recurring terms of type $\mathcal{P}u$, the set

Lemma 3.3.8.

$$\widetilde{\mathcal{M}}^{\mathcal{P}}_{T,M} = \Big\{\chi_{T,M}\mathcal{P}\Phi_{\mathcal{P}}(u_0), \chi_{T,M}\mathcal{P}\Phi_{\mathcal{P},R}(u_0) \;\Big|\; \|u_0\|_{L^2} \lesssim 1\Big\}$$

is precompact in $L^r_{t,x}(\mathbb{R}\times\mathbb{R}^d)$ for $1 \le r < \infty$.

Proof. For clarity sake, we shall only pay attention to solutions of (3.21), denoting them by u. Since $\|u\|_S \lesssim_{\mathcal{P}} 1$, then $\mathcal{P}u \in L^r_{t,x}([0,T]\times\mathbb{R}^d)$ for $2 \le r \le \infty$. Tightness follows directly via space-time localization $\chi_{T,M}$. Equicontinuity once again follows directly from Lemma (3.1.5) and Lemma 3.3.6.

Then lemma 3.2.4 ensures precompactness in stated Lebesgue space. Range $1 \le r \le 2$ follows from previous bounds and Hölder inequality. □

Remark 3.3.9. *As a consequence, if u_n converges u due to precompactness of $\mathcal{M}^{\mathcal{P}}_{T,M}$, then it also converges weakly in $L^2_{t,x}$, which combined with the fact that $\mathcal{P}$ is bounded linear operator, means that $\mathcal{P}u_n \rightharpoonup \mathcal{P}u$ in $L^2_{t,x}$. However, precompactness of $\widetilde{\mathcal{M}}^{\mathcal{P}}_{T,M}$ then implies that $\chi_M\mathcal{P}u_n \to \chi_M\mathcal{P}u$ in $L^r_{t,x}$, $1 \le r < \infty$.*

Previous observations imply

Corollary 3.3.10. *Precompactness of $\widetilde{\mathcal{M}}^{\mathcal{P}}_{T,M}$ implies that*

$$\mathcal{N}^{\mathcal{P}}_{T,M} = \left\{ \chi_{T,M} |\mathcal{P}u|^{p-1}\mathcal{P}u \;\middle|\; \|u_0\|_{L^2} \lesssim 1,\ \ u \text{ solves (3.19),(3.21) for initial data } u_0 \right\}$$

is precompact in $L^1_t L^2_x$.

Corollary 3.3.10 is the analogue of precompactness of nonlinearities in the subcritical case covered in Remark 3.2.6, adjusted to the nonlinearity of the flow $\Phi_{\mathcal{P}}$ (3.19), and as result implies in the same manner that

Lemma 3.3.11. *Family $\widetilde{\Psi} : \mathcal{I} \times X_{r_1} \to X_{r_1}$ is continuous.*

Proof. The proof is exactly the same as the one for Lemma 3.2.7 with obvious adjustments for frequency projection in the nonlinearity (rewriting $\mathcal{P}u$ instead of u), whilst relying on the compactness of space localized solutions of equations (3.21) and (3.19), i.e. the set $\mathcal{M}^{\mathcal{P}}_{T.M}$, and nonlinearities $\mathcal{N}^{\mathcal{P}}_{T,M}$. □

Finally, let us prove that flow given by (3.21) fulfills assumptions of Corollary 1.0.4. Following lemma is a direct corollary of the continuity of family $\widetilde{\Psi} : \mathcal{I} \times X_{r_1} \to X_{r_1}$.

Lemma 3.3.12. *Space-frequency truncated flow $\Phi_{\mathcal{P},R}$ (3.21) maps weakly convergent sequence of initial data into a weakly convergent sequence, i.e. flow is continuous with respect to weak topology on space of initial data L^2.*

Similarly to Section 3.2, derivative of the flow (3.21)

$$u(t) := \Phi_{\mathcal{P},R}(u_0) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-s)\Delta}\mathcal{P}(\chi_R|\mathcal{P}u(s)|^{\frac{4}{d}}\mathcal{P}u(s))ds.$$

at point u_0 is given by

$$i\partial_t v + \Delta v = \mathcal{P}\chi_R\left[\frac{p-1}{2}|\mathcal{P}u|^{p-3}(\mathcal{P}u)^2\mathcal{P}\bar{v} + \frac{p+1}{2}|\mathcal{P}u|^{p-1}\mathcal{P}v\right], \tag{3.26}$$

where $p = \frac{4}{d} + 1$ is the critical exponent. Well-posedness theory of (3.26) and reduction to simply observing

$$i\partial_t v + \Delta v = \mathcal{P}\chi_R\left[|\mathcal{P}u|^{p-1}\mathcal{P}v\right], \tag{3.27}$$

holds in the same manner as in the previous section. Likewise, same holds for iterating the flow up to a desired time, by applying the time obtained by well-posedness theory dependent on the size of ball of data. Denote by $V := |\mathcal{P}u|^{p-1}$.

Lemma 3.3.13. *Derivative $D\Phi_{\mathcal{P},R} : (L^2, \tau_{weak}) \to (B(L^2), \|\cdot\|_{op})$ is continuous.*

Proof. Let $u_n(0) \rightharpoonup u(0)$ be a weakly convergent sequence and u_n, u the corresponding Cauchy solutions to (3.21), similarly with potentials V_n, V. Let

$$\begin{aligned} i\partial_t w_n + \Delta w_n &= \mathcal{P}\chi_R[V_n \mathcal{P} w_n], \\ i\partial_t w + \Delta w &= \mathcal{P}\chi_R[V \mathcal{P} w], \end{aligned}$$

be solutions of (3.26) for same initial data $w(0) = w_n(0) = w_0$. Denoting $v = w - w_n$, for initial data $v(0) = 0$, we get

$$i\partial_t v + \Delta v = \mathcal{P}\chi_R[V\mathcal{P}v] + \mathcal{P}\chi_R[(V_n - V)\mathcal{P}w_n],$$

We want to prove that solutions $v := [D\Phi_{T,R}(u(0)) - D\Phi_{T,R}(u_n(0))]w_0$ converge to 0 uniformly on a ball of initial data that has a small radius. We note that the Strichartz norms of w_n are uniformly bounded by a constant depending on the size of initial data, which we have fixed. Well-posedness of (3.27) and Lemma 3.3.8 gives the inequality

$$\begin{aligned} \|v\|_X &\lesssim \left\| \int_0^T S(t-s)\mathcal{P}\chi_R[(V_n - V)\mathcal{P}w_n] ds \right\|_S \\ &\lesssim \|\mathcal{P}\chi_R[(V_n - V)\mathcal{P}w_n]\|_{L_t^1 L_x^2} = \|\chi_R[(V_n - V)\mathcal{P}w_n]\|_{L_t^1 L_x^2} \\ &\leq \|\mathcal{P}w_n\|_{L_{t,x}^\infty} \|\chi_R(V_n - V)\|_{L_t^1 L_x^2} \lesssim_{\mathcal{P}} \|w_n\|_{L^\infty L^2} \|\chi_R(V_n - V)\|_{L_t^1 L_x^2}, \end{aligned}$$

completing the proof. $\square$

3.4 KdV on the torus

We turn to proving the non-squeezing property of the KdV flow on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$

$$\partial_t q = -q_{xxx} + 6qq_x \tag{3.28}$$

in $H_0^{-\frac{1}{2}}(\mathbb{T}) = \{q \in H^{-\frac{1}{2}}(\mathbb{T}) : \hat{q}(0) := \int_{\mathbb{T}} q\,dx = 0\}$, namely

Theorem 3.4.1. *Let $T > 0$, $0 < R < r < \infty$, $\alpha \in \mathbb{C}$ and $h_0 \in H^{-\frac{1}{2}}(\mathbb{T}), l \in H^{\frac{1}{2}}(\mathbb{T})$ such that $\|l\|_{H^{\frac{1}{2}}} = 1$. Then there exists $u_0 \in B_r(h_0) := \{h \in H_0^{-\frac{1}{2}} : \|h - h_0\|_{H^{-\frac{1}{2}}} \leq r\}$ such that the corresponding solution $u(T)$ of KdV (3.28) satisfies*

$$|\langle u(T), l\rangle - \alpha| > R.$$

KdV is Hamiltonian with respect to the Poisson structure defined by

$$\{F, G\} := \int \frac{\delta F}{\delta q} \frac{\partial}{\partial x}\left(\frac{\delta G}{\delta q}\right) dx.$$

for $F, G : C^\infty \to \mathbb{R}$, and functional

$$H_{KdV}(q) = \int_{\mathbb{T}} \frac{1}{2} q_x^2 + q^3 dx.$$

For future reference, another important Hamiltonian is momentum

$$P(q) := \frac{1}{2} \int |q|^2 dx,$$

which generates translations. Moreover, P and H_{KdV} Poisson commute $\{P, H_{KdV}\} = 0$, i.e. one quantity is preserved under the flow generated by another one, and as a consequence the flows generated by them commute, at least for data in C^∞.

Given Poisson structure is degenerate due to existence of Casimir function $q \mapsto \int q$. Poisson structure is defined via almost complex structure $J = \partial_x$, which is degenerate as well, and L^2 product. Motivated by this, choice of space of initial data $H_0^{-\frac{1}{2}}$ leads to restricting the flow to symplectic leaf on which J becomes non-degenerate, meaning we obtain the symplectic form $\omega_{-\frac{1}{2}} : H_0^{-\frac{1}{2}} \times H_0^{-\frac{1}{2}} \to \mathbb{C}$ given by

$$\omega_{-\frac{1}{2}}(u, v) := \int_{\mathbb{T}} u(x) \partial_x^{-1} v(x) dx, \tag{3.29}$$

and where $\partial_x^{-1} : H_0^{-\frac{1}{2}}(\mathbb{T}) \to H_0^{\frac{1}{2}}(\mathbb{T})$ is the inverse to the differential operator ∂_x defined via Fourier transform by

$$\widehat{\partial_x^{-1} f}(k) := \frac{1}{ik} \hat{f}(k), \quad k \neq 0.$$

Operator ∂_x^{-1} is well-defined since $\hat{f}(0) = 0$ for all $f \in H_0^{-\frac{1}{2}}(\mathbb{T})$. Moreover, for a real function u, $\overline{\hat{u}(k)} = \hat{u}(-k)$, $k \in \mathbb{N}$. Plancherel theorem then allows us to rewrite (3.29) as

$$\begin{aligned} \omega_{-\frac{1}{2}}(u, v) &= \sum_{k=-\infty, k \neq 0}^{\infty} \hat{u}(-k) \frac{1}{ik} \hat{v}(k) \\ &= \sum_{k=1}^{\infty} \frac{1}{ik} (\hat{u}(-k)\hat{v}(k) - \hat{u}(k)\hat{v}(-k)) \\ &= \sum_{k=1}^{\infty} \frac{2}{k} (\operatorname{Im}(\hat{v}(k) \overline{\hat{u}(k)})) \end{aligned} \tag{3.30}$$

In contrast to previously discussed NLS, the underlying symplectic space (and the form itself) is not the standard one. However, having statement of non-squeezing Theorem 1.0.3 in mind and motivated by (3.30), this can be readily solved by observing a bounded linear symplectomorphism

$$S : H_0^{-\frac{1}{2}}(\mathbb{T}) \longrightarrow l^2(\mathbb{Z}^*),$$

given by

$$S(e^{ikx}) = \frac{1}{\sqrt{k}} e_k, \quad k \neq 0,$$

where $\mathbb{H} := l^2(\mathbb{Z}^*)$ has orthonormal basis $\{e_k\}_{k\in\mathbb{Z}^*}$. That is, S is rescaling the corresponding basis vectors. Conjugation of the flow Φ^{KdV} with S

$$\Phi := S \circ \Phi^{KdV} \circ S^{-1} : \mathbb{H} \longrightarrow \mathbb{H},$$

gives a symplectomorphism that preserves the standard symplectic form on $\mathbb{H}$, and due to linearity of S, analyzing properties of the flow Φ^{KdV} is equivalent to the one of Φ. Hence, it suffices to prove the weak continuity properties of Φ^{KdV} in order to conclude its symplectic non-squeezing behaviour.

Well-posedness in H^{-1} was initially obtained in [KT06]. However, key strategy in our proof is based on an approach of Killip and Visan [KV19] on well-posedness in H^{-1}. Well-posedness at that regularity is proven by observing an approximation of KdV flow in H^{-1} norm, uniformly on precompact sets of initial data. Namely, the authors of [KV19] introduce family of Hamiltonians H_κ, which Poisson commute with H_{KdV}

$$\{H_{KdV}, H_\kappa\} = \{P, H_\kappa\} = \{H_{KdV}, P\} = 0$$

and for which the asymptotic expansion holds

$$H_\kappa(q) = H_{KdV}(q) + O(\kappa^{-2}), \quad \text{for} \quad q \in C^\infty.$$

The choice of Hamiltonian H_κ is motivated by integrable nature of KdV and associated Lax pair

$$L(t) := -\partial_x^2 + q(t,x) \quad \text{and} \quad P(t) := -4\partial_x^3 + 3(\partial_x q(t,x) + q(t,x)\partial_x).$$

Choosing $\kappa \geq 1$ and defining resolvent

$$R_0(\kappa) = (-\partial_x^2 + \kappa^2)^{-1},$$

they prove that quantity

$$\alpha(\kappa, q) := \sum_{l=2}^{\infty} \frac{(-1)^l}{l} \operatorname{Tr}\left\{\left(\sqrt{R_0} q \sqrt{R_0}\right)^l\right\},$$

gives important information about high frequency part of H^{-1} norm, proving to be essential for obtaining well-posedness of KdV at that regularity. The series converges provided that $\kappa^{-\frac{1}{2}} \|q\|_{H^{-1}} \leq \delta$ for some $\delta > 0$. We adopt the notation from [KV19]

$$B_{\delta,\kappa} := \{q \in H^{-\frac{1}{2}}(\mathbb{T}) : \kappa^{-\frac{1}{2}} \|q\|_{H^{-1}} \leq \delta\}.$$

More specifically, they prove that the analytic flow (eq. (3.11) in Proposition 3.2, [KV19]), denoted by Φ_κ^{KdV},

$$\partial_t q = 16\kappa^5 g'(x,\kappa,q) + 4\kappa^2 q', \tag{3.31}$$

generated by Hamiltonian

$$H_\kappa(q) := -16\kappa^5 \alpha(\kappa,q) + 4\kappa^2 P(q), \tag{3.32}$$

approximates KdV flow on H^{-1}, uniformly on precompact set of initial data, for $\kappa \geq 1$ large. Here $g(x,\kappa,q)$ denotes the diagonal part of continuous integral kernel $G(x,y,\kappa,q)$ associated with the resolvent $R := (L+\kappa^2)^{-1}$, which is a Hilbert-Schmidt operator. Moreover, [KV19] shows that $g : B_{\delta,\kappa} \to H^1$ is a diffeomorphism. Functional derivative is defined using the L^2 pairing, namely

$$\left.\frac{d}{d\theta}\right|_{\theta=0} F(f_1 + \theta f_2) = \left\langle \frac{\delta F}{\delta q}(f_1), f_2 \right\rangle_{L^2}.$$

In order to make notation simpler, in the following we denote $\Phi^{KdV}, \Phi_\kappa^{KdV}$ by Φ, Φ_κ. Crucially, [KV19] shows that for a precompact set of data $Q \subset H^{-1} \cap C^\infty$

$$\lim_{\kappa\to\infty} \sup_{u_0 \in Q} \|\Phi(u_0) - \Phi_\kappa(u_0)\|_{H^{-1}} = 0.$$

Approximation suggests that non-squeezing property of (3.28) could follow from one of (3.31). Both flows are analytic ([CKS+03], [KV19] resp.). As we shall see in the following, dealing with weak topology (which seems to be omnipresent in non-squeezing approach) will not be an issue due to compactness of embedding $H^{-\frac{1}{2}}(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$, and being able to push the initial data down to the sharp regularity for which we have the well-posedness.

We remark that in the rest of the section we shall be discussing flows (3.28) and (3.31) for mean-zero functions, and will be using non-homogeneous Sobolev norms. However, both are not an issue due to the fact that both flows are mean-preserving and equivalence of homogeneous and non-homogeneous Sobolev norm on $H_0^{-\frac{1}{2}}(\mathbb{T})$.

Choosing $\kappa \geq 1$ big enough allows the following exposition to be well-defined in accordance with $g : B_{\delta,\kappa} \subset H^{-1} \to H^1$ being defined for suitable data. For the sake of Theorem 3.4.1, we can assume without loss of generality that the ball is centered around 0 as translations are symplectomorphisms, and observe that then we are only interested in initial data in a ball of fixed radius r, which directly imposes bounds on δ. Indeed, restricting to set of data with $H^{-\frac{1}{2}}$ norm at most r, one only needs to consider κ such that $\sqrt{\kappa} \geq \frac{r}{\delta}$. However, the problem with concluding non-squeezing property of the flow Φ_κ (3.31) by trying to prove assumptions of Theorem 1.0.3 is that the flow itself is not defined for all data in $H^{-\frac{1}{2}}$, but rather for data in $B_{\delta,\kappa}$. We overcome this difficulty by defining a smooth symplectic flow $\widetilde{\Phi}_\kappa$ on $H^{-\frac{1}{2}}$ whose restriction to a ball $B_{\delta,\kappa}$ is the flow Φ_κ (3.31). Specifically, without going into rigorous details for the time being, the flow $\widetilde{\Phi}_\kappa$ will be generated by a Hamiltonian

$$\widetilde{H}_\kappa = \eta\alpha + 4\kappa^2 P,$$

where η is such a function that ensures that $\widetilde{H}_\kappa$ is well defined for all data in $H^{-\frac{1}{2}}$.

Just like in the NLS case, κ introduces a family of symplectomorphisms indexed by $\sqrt{\kappa} \geq \frac{r}{\delta}$, or equivalently, by set $[0,1)$. Endpoint 1 of $\mathcal{I} = [0,1]$ corresponds to the KdV flow (3.28). We define the family of symplectomorphisms

$$\begin{aligned}
&\Psi : \mathcal{I} \times H_0^{-\frac{1}{2}}(\mathbb{T}) \to H_0^{-\frac{1}{2}}(\mathbb{T}) \\
&\Psi(1) = \Phi^{KdV} \\
&\Psi(\tau) = \widetilde{\Phi}_\kappa, \ \ \tau \in [0,1).
\end{aligned}$$

Global well-posedness of KdV and Φ_κ (3.31) obtained in [KV19] leads to consideration of family

$$\Psi : \mathcal{I} \times X_r \to X_{cr},$$

where X_r denotes the ball B_r from Theorem 3.4.1, endowed with weak topology, and the constant c is given by well-posedness result [KV19] (and is independent of κ). We will justify the choice of this family soon via explicit construction of $\widetilde{\Phi}_\kappa$, which will show that $\widetilde{\Phi}_\kappa$ is indeed an extension of Φ_κ for all times on ball of initial data B_r. All these observations reduce the proof of non-squeezing of KdV in the context of Theorem 1.1.1, for which we will have to prove that the constructed family Ψ satisfies its assumptions.

Now we turn our attention to the extension of Φ_κ (3.31), which we denoted by $\widetilde{\Phi}_\kappa$. Define Hamiltonian

$$\widetilde{H}_\kappa(q) = \widetilde{H}(\kappa,q) := -16\kappa^5\eta(\kappa^{-1}\|q\|^2_{H^{-1}})\alpha(\kappa,q) + 4\kappa^2 P(q), \tag{3.33}$$

where $\eta : [0,\infty) \to [0,1]$ is a smooth cut-off such that $\eta \equiv 1$ on $[0,\frac{\delta}{2}]$ and $\eta \equiv 0$ on $[\delta,\infty)$. Note that functional derivative of $f(q) := \eta(\kappa^{-1}\|q\|^2_{H^{-1}}) : H^{-\frac{1}{2}} \to \mathbb{R}$ is given by

$$\frac{\delta f}{\delta q} := 2\kappa^{-1}\eta'(\kappa^{-1}\|q\|^2_{H^{-1}})B_2(q),$$

where $B_2(f) := \mathcal{F}^{-1}[(1+\xi^2)^{-1}\widehat{f}(\xi)]$. Moreover, B_2 is a bounded linear map that gives two derivative of regularity to initial data, i.e. $B_2 : H^s \to H^{s+2}$ for every $s \in \mathbb{R}$. Equation generated by Hamiltonian $\widetilde{H}_\kappa$ (3.33) is then given by

$$\partial_t q = 16\kappa^4\Big[\kappa\eta(\kappa^{-1}\|q\|^2_{H^{-1}})g(x,\kappa,q) + 2\eta'(\kappa^{-1}\|q\|^2_{H^{-1}})\alpha(\kappa,q)B_2(q)\Big]' + 4\kappa^2 q'. \tag{3.34}$$

Denote by $F : [1,\infty) \times H^{-1} \to H^1$ the smooth map

$$F(\kappa, q) := 16\kappa^5 \eta(\kappa^{-1}\|q\|^2_{H^{-1}}) g(x,\kappa,q) + 32\kappa^4 \eta'(\kappa^{-1}\|q\|^2_{H^{-1}}) \alpha(\kappa,q) B_2(q).$$

F is indeed smooth, since $g, \alpha : [1,\infty) \times B_{\kappa,\delta} \subset [1,\infty) \times H^{-1} \to H^1$ are analytic and $B_2 : H^s \to H^{s+2}$ is a bounded linear operator for every $s \in \mathbb{R}$. In particular, for any $\kappa \geq 1$, $F_\kappa := F(\kappa, \cdot) : H^{-1} \to H^1$. Denoting by $\iota : H^{-\frac{1}{2}} \to H^{-1}$ the compact embedding, one rewrites flow $\widetilde{\Phi}_\kappa$ (3.34) as

$$\partial_t q = [F_\kappa \circ \iota(u)]' + 4\kappa^2 q'.$$

Term $4\kappa^2 q'$ coming from momentum $P(q)$ generates translations, hence it is an unitary linear operator on $H^{-\frac{1}{2}}$, which we shall denote by $L_\kappa(t)(q(x)) := q(x + 4\kappa^2 t)$. Equivalently, solution to (3.34) is then given by Duhamel's formula

$$\widetilde{\Phi}_\kappa(q(0)) := q(t,x) = L_\kappa(t) q(0) + \int_0^t L_\kappa(t-s)[F_\kappa \circ \iota(q)]' ds \tag{3.35}$$

Notice that similarly to the case of Lemma **??**, we do not have to care about which time t we are fixing, since composition of flows that are defined up to a time dependent on the size of the data preserves all analytic and symplectic properties we need. Firstly, we shall start with well-posedness theory

Lemma 3.4.2. *Approximate flow $\widetilde{\Phi}_\kappa$ (3.34) is globally well-posed in $H^{-\frac{1}{2}}$.*

Proof. Local well-posedness, both in H^{-1} and $H^{-\frac{1}{2}}$, follow from the fact that $\widetilde{H}_\kappa$ is smooth with respect to both norms, hence the functional derivative $\frac{\partial \widetilde{H}_\kappa}{\partial q}$ is Lipschitz on bounded sets of data. Existence follows directly by applying fixed point argument.

Global well-posedness in $H^{-\frac{1}{2}}$ follows from one in H^{-1}. Namely, local well-posedness in H^{-1} gives time of existence depending on the size of initial data in H^{-1}. Having the definition of flow $\widetilde{\Phi}_\kappa$ in mind, this creates a dichotomy. In the case of initial data u_0 which is above threshold $\|u_0\|_{H^{-1}} > \sqrt{\kappa}\delta$, dynamics of $\widetilde{\Phi}_\kappa$ is governed by momentum P, which generates translations L_κ. This operator is unitary, hence solution with such data is global. Otherise, when $\|u_0\|_{H^{-1}} \leq \sqrt{\kappa}\delta$, local well-posedness of $\widetilde{\Phi}_\kappa$ gives iteration in time depending only on κ, δ. This allows for infinite, if needed, iteration in time, leading to global well-posedness as well. $\square$

Lemma 3.4.3. *For a ball of initial data $B_r \subset H_0^{-\frac{1}{2}}$, there exists κ_0 depending on r such that for all $\kappa \geq \kappa_0$, $\widetilde{\Phi}_\kappa$ is an extension of Φ_κ on ball of data B_r, for all times $T > 0$, and*

$$\left\|\widetilde{\Phi}_{\kappa,T}(q_0)\right\|_{H^{-\frac{1}{2}}} \lesssim_r 1, \ \ \forall q_0 \in B_r. \tag{3.36}$$

Consequently, $\widetilde{\Phi}_\kappa$ approximates KdV as well, that is for $Q = B_r \cap C^\infty$ one has

$$\lim_{\kappa \to \infty} \sup_{u \in Q} \sup_{t \leq T} \|\Phi_t(u) - \widetilde{\Phi}_{\kappa,t}(u)\|_{H^{-1}} = 0. \tag{3.37}$$

Proof. Flows preserving $\alpha(\kappa,\cdot)$, for all κ greater than some fixed value κ_0, have a priori bound based on initial data (see Theorem 2.4, [KVZ18]). Namely, authors of [KVZ18] prove that for initial data $q(0)$ and a global, in the sense for all time, trajectory $q(t) \in H^{-1}$, along which $\frac{d}{dt}\alpha(\kappa, q(t)) = 0$ for all $\kappa \geq \kappa_0 := 1 + 90\|q(0)\|_{H^{-1}}$, the bound

$$\|q(t)\|_{H^{-1}} \leq 40\kappa_0^2 \|q(0)\|_{H^{-1}}$$

holds. Even more so, same holds for higher regularity

$$\|q(t)\|_{H^s} \lesssim (1 + 90\|q(0)\|_{H^{-1}})^{|s|}\|q(0)\|_{H^s} \lesssim_{\|q(0)\|_{H^s}} 1$$

($-1 \leq s < 0$), as shown in the same theorem. As shown in [KV19], Φ_κ on $B_{\kappa,\delta}$ preserves $\alpha(\kappa_1)$ for all κ_1 greater than some fixed value depending on κ and δ.

In our particular case, as we are only interested in data such that $\|q(0)\|_{H^{-\frac{1}{2}}} \leq r$, we can pick κ_0 so that $(40\kappa_0^2)^{-1}\kappa^{-\frac{1}{2}}\|\Phi_\kappa(q(0))\|_{H^{-1}} \leq \kappa^{-\frac{1}{2}} r \leq \delta$, for all $\kappa \geq \kappa_0$. This ensures that for $q(0) \in B_r$ and for all time $T > 0$, $\eta(\kappa^{-1}\|\tilde{\Phi}(q(0))\|^2_{H^{-1}}) \equiv 1$. Hence $\widetilde{\Phi}_\kappa\big|_{B_r} = \Phi_\kappa$ for all $\kappa \geq \kappa_0$ and (3.36) holds.

Embedding $Q := B_r \cap C^\infty \subset H^{-\frac{1}{2}}$ into H^{-1} creates an equicontinuous set with respect to H^{-1} norm topology. The limit

$$\lim_{\kappa\to\infty} \sup_{u\in Q} \sup_{t\leq T} \|[\Phi_t(\Phi_{\kappa,t})^{-1}]u - u\|_{H^{-1}} = 0,$$

holds, as proven in [KV19] for any bounded equicontinuous set Q of Schwartz data in H^{-1}. Since $\widetilde{\Phi}_\kappa|_Q \equiv \Phi_\kappa$ for big enough κ, approximation (3.37) holds. □

Henceforth, when talking about κ, we shall always assume that it is greater that κ_0 given by previous lemma. This directly implies how the family Ψ is defined, but as previously mentioned, this is fine with application to Theorem 1.1.1 in mind and the constants appearing in it.

Lemma 3.4.4. *Family $\Psi : \mathcal{I} \times X_r \to X_{cr}$ is continuous.*

Proof. The constant c is given by global bounds (3.36) of Lemma 3.4.3. For continuity discussion we shall fix arbitrary $l \in H^{\frac{1}{2}}$ such that $\|l\|_{H^{\frac{1}{2}}} \leq 1$. Density argument allows to assume without the loss of generality that $l \in C^\infty$, since for any $\varepsilon > 0$ there exists $\tilde{l} \in C^\infty$ such that $\|l - \tilde{l}\|_{H^{\frac{1}{2}}} < \varepsilon$. Similarly, for any initial data $q_0 \in X_r$ we may assume that $q_0 \in C^\infty$ since the well-posedness of KdV (3.28) and $\widetilde{\Phi}_\kappa$ (3.34) imply that for any $\varepsilon > 0$ and every $\tau \in \mathcal{I}$ exists $\tilde{q}_0 \in H_0^{-\frac{1}{2}}$ such that $\|\Psi_\tau(q_0) - \Psi_\tau(\tilde{q}_0)\|_{H_0^{-\frac{1}{2}}} < \varepsilon$.

We distinguish two cases of continuity in the $\mathcal{I}$ variable - first one is the endpoint $\tau_n \to 1$ ($\kappa_n \to \infty$), which corresponds to $\widetilde{\Phi}_\kappa$ convergence to KdV, and the second one when

$\tau_n \to \tau < 1$ ($\kappa_n \to \kappa < \infty$). Regardless of this dichotomy, we shall always have a sequence $q_n(0,x) \to q(0,x)$ in X_r, that is $q_n(0,x) \rightharpoonup q(0,x)$ in $H_0^{-\frac{1}{2}}$. Then $q_n(0,x) \to q(0,x)$ in H^{-1}, and hence the sequence is a precompact set in H^{-1}.

Regarding the first case, let $\kappa_n \to \infty$. Since

$$\begin{aligned}
\left|\left\langle \Psi(1,q(0)) - \Psi(\tau_n, q_n(0)), l\right\rangle\right| &= \left|\left\langle \Phi(q(0)) - \widetilde{\Phi}_{\kappa_n}(q_n(0)), l\right\rangle\right| \\
&\leq \|\Phi(q(0)) - \widetilde{\Phi}_{\kappa_n}(q_n(0))\|_{H^{-1}}\|l\|_{H^1} \\
&\lesssim \left\|\Phi(q(0)) - \Phi(q_n(0))\right\|_{H^{-1}} + \left\|\Phi(q_n(0)) - \widetilde{\Phi}_{\kappa_n}(q_n(0))\right\|_{H^{-1}},
\end{aligned}$$

well-posedness of KdV and (3.37) imply that $\widetilde{\Phi}_{\kappa_n}(q_n(0)) \to \Phi(q(0))$ in X_{cr}.

Regarding the second case, $\kappa_n \to \kappa$, we can equivalently observe the closed interval $\tau \in I \subset \mathcal{I}\backslash\{1\}$ (corresponding finite closed interval $\tilde{I}$ such that $\kappa \in \tilde{I}$) and the compact set $I \times X_r$. Recall that for $\tau \in I$

$$\Psi(\tau, q_0) = \widetilde{\Phi}_\kappa(q_0) := q(\kappa, t) = L_\kappa(t)q_0 + \int_0^t L_\kappa(t-s)[F(\kappa, \iota(q))]' ds.$$

That the time t in the formula of Ψ makes sense and that it is continuous as a map $\Psi : I \times H^{-1} \to H^{-1}$ follows from the boundedness of $\tilde{I}$, smoothness of the Hamiltonian $\widetilde{H}(\kappa, q) : \tilde{I} \times H^{-1} \to \mathbb{R}$ and the fact that the gradient $\nabla\widetilde{H}(\kappa, q)$ is Lipschitz on bounded balls of data in H^{-1}, with Lipschitz constant independent of κ. Since $q_n(0) \to q(0)$ in H^{-1}, $\Psi(\tau_n, q_n(0)) \to \Psi(\tau, q(0))$ in H^{-1} and

$$\left|\left\langle \Psi(\tau_n, q_n(0)) - \Psi(\tau, q(0)), l\right\rangle\right| \leq \|\Psi(\tau_n, q_n(0)) - \Psi(\tau, q(0))\|_{H^{-1}}\|l\|_{H^1}$$

conclude the proof.

□

We now turn to proving that $\widetilde{\Phi}_\kappa$ (3.34) is non-squeezing in the sense of Theorem 1.0.3.

Lemma 3.4.5. *Nonlinear flow $\widetilde{\Phi}_\kappa$ (3.34) is continuous with respect to weak topology in $H_0^{-\frac{1}{2}}$.*

Proof. Let $q_n(0,x) \rightharpoonup q(0,x)$ in $H_0^{-\frac{1}{2}}$. Then $q_n(0,x) \to q(0,x)$ in H^{-1}. Local well-posedness of (3.34) in H^{-1} implies that $q_n(t) \to q(t)$ uniformly in time on $[0,T]$ in H^{-1} norm topology. That $q_n(t) \rightharpoonup q(t)$ in the weak sense $H_0^{-\frac{1}{2}}$ then follows from strong convergence in H^{-1} and density argument for functionals being represented by functions of C^∞ regularity. □

Derivative of $\widetilde{\Phi}_\kappa$ (3.34) at point q_0 and solution q is given by the equation

$$\partial_t v = [DF_\kappa \circ \iota(q)v]' + 4\kappa^2 v', \tag{3.38}$$

or, equivalently, by Duhamel's formula

$$v(t) := D\widetilde{\Phi}_\kappa[q_0]v(0) = L_\kappa(t)v_0 + \int_0^t L_\kappa(t-s)\Big[[DF_\kappa \circ \iota(q)]v\Big]' ds. \tag{3.39}$$

In order to make sense of the derivative flow, one has to prove that it is well-posed in $H^{-\frac{1}{2}}$. This readily follows from

$$\begin{aligned}
\|v\|_{H^{-\frac{1}{2}}} &\leq \|v(0)\|_{H^{-\frac{1}{2}}} + \int_0^t \|\Big[[DF_\kappa \circ \iota(q(s))]v(s)\Big]'\|_{H^{-\frac{1}{2}}} \\
&\leq \|v(0)\|_{H^{-\frac{1}{2}}} + \int_0^t \|[DF_\kappa \circ \iota(q(s))]v(s)\|_{H^1} \\
&\leq \|v(0)\|_{H^{-\frac{1}{2}}} + t\|DF_\kappa \circ \iota(q)\|_{B(H^{-\frac{1}{2}},H^1)}\|v\|_{H^{-\frac{1}{2}}},
\end{aligned}$$

and the fact that F_κ is smooth. These inequalities give existence up to a time depending on the size of data $v(0)$, but we can obviously iterate up to any finite time T by applying the well-defined flow, as the data blows up at most exponentially.

We can now prove the weak continuity of the derivative of flow (3.34) and conclude non-squeezing property of $\widetilde{\Phi}_\kappa$

Lemma 3.4.6. *Derivative* $D\widetilde{\Phi}_\kappa : (H^{-\frac{1}{2}}, \tau_{weak}) \longrightarrow (B(H^{-\frac{1}{2}}), \|\cdot\|_{op})$ *is continuous.*

Proof. Let $q_n(0) \rightharpoonup q(0)$ in $H^{-\frac{1}{2}}$, and let q_n, q be corresponding Cauchy solutions of (3.34). Well-posedness of (3.34) in H^{-1} gives uniform convergence of $\|q_n - q\|_{H^{-1}}$ on $[0,T]$. Let w_n and w be solutions of (3.38) with the same initial data $w(0)$.

$$\begin{aligned}
\partial_t w_n &= \Big[[DF_\kappa \circ \iota(q_n)]w_n\Big]' + 4\kappa^2 w_n', \\
\partial_t w &= \Big[[DF_\kappa \circ \iota(q)]w\Big]' + 4\kappa^2 w'.
\end{aligned}$$

We want to prove that as $n \to \infty$, $v_n := w_n - w$ goes to 0. Previous equations give

$$\partial_t v_n = \Big[(DF_\kappa \circ \iota(q_n) - DF_\kappa \circ \iota(q))w_n\Big]' + \Big[DF_\kappa \circ \iota(q)v_n\Big]' + 4\kappa^2 v_n'.$$

Writing down Duhamel's formula we get

$$v_n(t) = \int_0^T \Big[DF_\kappa \circ \iota(q)v_n\Big]' ds + \int_0^T \Big[(DF_\kappa \circ \iota(q_n) - DF_\kappa \circ \iota(q))w_n\Big]' ds.$$

Once again, scaling with respect to q and correspondingly rescaling the data w_0 up to a fixed constant allows us to move the first summand on RHS to LHS and enables us to directly obtain the bound on norm of difference of derivatives

$$\begin{aligned}\|v(t)\|_{H^{-\frac{1}{2}}} &\lesssim \left\| \int_0^T \left[(DF_\kappa \circ \iota(q_n) - DF_\kappa \circ \iota(q)) w_n \right]' ds \right\|_{H^{-\frac{1}{2}}} \\ &\lesssim_T \sup_{t\in[0,T]} \left\| \left[(DF_\kappa \circ \iota(q_n) - DF_\kappa \circ \iota(q)) w_n \right]' \right\|_{H^{-\frac{1}{2}}} \\ &\lesssim_T \sup_{t\in[0,T]} \left\| (DF_\kappa \circ \iota(q_n) - DF_\kappa \circ \iota(q)) w_n \right\|_{H^1} \\ &\lesssim_T \sup_{t\in[0,T]} \|(DF_\kappa \circ \iota(q_n) - DF_\kappa \circ \iota(q))\|_{B(H^{-\frac{1}{2}},H^1)} \|w_n\|_{H^{-\frac{1}{2}}}.\end{aligned}$$

Then continuity in operator norm follows from compactness of ι and smoothness of F_κ. □

3.5 AKNS/mKdV

We now consider nonlinear Schrödinger equation

$$iu_t + u_{xx} = \pm 2|u|^2 u \tag{3.40}$$

and complex modified Korteweg-de Vries equation

$$u_t + u_{xxx} = \pm 6|u|^2 u_x, \tag{3.41}$$

in $L^2(\mathbb{R})$. Principal result of this section is as follows

Theorem 3.5.1. *Let $h_0, l \in L^2(\mathbb{R})$ such that $\|l\|_{L^2} = 1$ and $0 < r_2 < r_1 < 0$, $\alpha \in \mathbb{C}$. Then for every time $T > 0$ there exists the initial data $u_0 \in B_{r_1}(h_0)$ such that the solution u given by mKdV (3.41) satisfies*

$$|\langle u(T), l\rangle - \alpha| > r_2.$$

Equations are invariant with respect to the scaling

$$u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$$

and

$$u(x,t) \to \lambda u(\lambda x, \lambda^3 t)$$

respectively, while initial data scales in the same manner for both $u_0(x) \to \lambda u_0(\lambda x)$. Both equations are Hamiltonian with respect to the same symplectic structure

$$\omega(\cdot,\cdot) = \operatorname{Im}\langle \cdot,\cdot\rangle_{L^2}.$$

Poisson structure is given by

$$\{F,G\} := -i \int \frac{\delta F}{\delta u}\frac{\delta G}{\delta v} - \frac{\delta F}{\delta v}\frac{\delta G}{\delta u} dx,$$

for $F, G : \mathcal{S} \to \mathbb{R}$, where $v := \pm\bar{u}$. Hamiltonians generating the flows are

$$H_{NLS}(u) := \int |u'|^2 + |u|^4 dx \quad \text{and} \quad H_{mKdV}(u) := -i \int u'\bar{u}'' + 3u^2\bar{u}\bar{u}' dx.$$

Two other important Hamiltonians are mass and momentum

$$M(u) := \int u\bar{u} dx \quad \text{and} \quad P(u) := -i \int u\bar{u}' dx,$$

which generate phase shifts and translations, respectively. All four stated Hamiltonians Poisson commute, i.e. for $* = NLS, mKdV$

$$\{H_{NLS}, H_{mKdV}\} = \{H_*, M\} = \{H_*, P\} = \{M, P\} = 0,$$

indicating intricate, yet important fact - NLS and mKdV are completely integrable.

The line case for NLS was already covered in one of the previous sections, but we shall present another proof that is more in line with the integrable nature of the equation, and which is closely tied to the one of mKdV. Regarding mKdV on the line, as mentioned previously, the flow itself is not uniformly continuous on $L^2(\mathbb{R})$, hence it cannot be smooth nor symplectic. Nonetheless we can conclude non-squeezing property of mKdV by approximation arguments based on flows that are actually smooth and symplectic. Similarly to the previous discussion on KdV, integrability plays important role. Both equations admit Lax pairs, while a part of it is the same for both equations

$$L(\kappa) = L_0(\kappa) + \begin{bmatrix} 0 & u \\ -\bar{u} & 0 \end{bmatrix} \text{ where } L_0(\kappa) := \begin{bmatrix} \kappa - \partial & 0 \\ 0 & \kappa + \partial \end{bmatrix}. \tag{3.42}$$

and $\kappa \in \mathbb{R}$ with $|\kappa| \geq 1$. For sufficiently small $\delta > 0$, $L(\kappa)$ is invertible as an operator on $L^2(\mathbb{R})$ for all $u \in B^s_\delta$

$$B^s_\delta := \{q \in H^s : \|q\|_{H^s} \leq \delta\},$$

$-\frac{1}{2} < s \leq 0$ and for all κ. We denote $B_\delta := B^0_\delta$. The inverse $R(\kappa) = L(\kappa)^{-1}$ admits an integral kernel $G(x, y, \kappa)$ whose entries prove to be crucial for treatment of well-posedness and via which we define the following quantities

$$\begin{aligned}
\gamma(x,\kappa) &:= \operatorname{sign}(\kappa)[G_{11}(x,x,\kappa) + G_{22}(x,x,\kappa)] - 1, \\
g_{12}(x,\kappa) &:= \operatorname{sign}(\kappa) G_{12}(x,x,\kappa), \\
g_{21}(x,\kappa) &:= \operatorname{sign}(\kappa) G_{21}(x,x,\kappa).
\end{aligned}$$

Important quantity is logarithmic perturbation determinant given by

$$A(\kappa, u) := \operatorname{sign}(\kappa) \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l} \operatorname{Tr}\left\{ \left(\sqrt{R_0}(L - L_0)\sqrt{R_0} \right)^l \right\},$$

which as expected plays an important role in integrability theory since for any $|\kappa_i| \geq 1$, $i = 1, 2$

$$\{A(\kappa_1), A(\kappa_2)\} = \{A(\kappa_1), H_{NLS}\} = \{A(\kappa_1), H_{mKdV}\} = 0.$$

This observation suggests that both equations can be treated in a similar way for well-posedness results, which turns out to be true, as can be seen in [HGKV20], in which authors obtain well-posedness on $\mathbb{R}$ at low regularity. Inspired by ideas of [KVZ18], [KV19], authors of [HGKV20] make use of classical asymptotic expansion

$$A(\kappa) = \frac{1}{2\kappa}M + \frac{-i}{(2\kappa)^2}P + \frac{(-i)^2}{(2\kappa)^3}H_{NLS} + \frac{(-i)^3}{(2\kappa)^4}H_{mKdV} + O(\kappa^{-5})$$

for data in $B_\delta \cap \mathcal{S}$, which suggests that flows generated by Hamiltonians

$$H^\kappa_{NLS} := -8\kappa^3 \operatorname{Re} A(\kappa) + 4\kappa^2 M \quad \text{and} \quad H^\kappa_{mKdV} := 16\kappa^4 \operatorname{Im} A(\kappa) + 4\kappa^2 P,$$

should prove useful for establishing well-posedness of NLS and mKdV at low regularity. In particular and crucially, Hamiltonians $H^\kappa_{NLS}, H^\kappa_{mKdV}, H_{NLS}, H_{mKdV}$ all Poisson commute for data in B^s_δ. Inner product on L^2 leads to the notion of functional derivatives defined by

$$\frac{d}{d\theta}\bigg|_{\theta=0} F(f + \theta g) = \left\langle \overline{g}, \frac{\delta F}{\delta u} \right\rangle + \left\langle g, \frac{\delta F}{\delta v} \right\rangle,$$

for $F : \mathcal{S} \to \mathbb{R}$ of at least C^1 regularity. Then flows generated by Hamiltonians $H^\kappa_{NLS}, H^\kappa_{mKdV}$ are given by equations

$$i u_t = 4\kappa^3 (g_{12}(\kappa, u) - g_{12}(-\kappa, u)) + 4\kappa^2 u \tag{3.43}$$

and

$$u_t = 8\kappa^4 (g_{12}(\kappa, u) + g_{12}(-\kappa, u)) + 4\kappa^2 u_x, \tag{3.44}$$

respectively and we denote them by $\Phi^{NLS}_\kappa, \Phi^{mKdV}_\kappa$. Here g_{12} denotes the upper right entry of integral kernel associated to matrix valued $R - R_0$ (we refer to previous section on KdV for notation). Also, $\frac{\delta A}{\delta u} = g_{12}$ holds. Since the map $g_{12} : B^s_\delta \to H^{s+1}$, $g_{12} : u \mapsto g_{12}(\kappa, u)$ is a diffeomorphism for every $-\frac{1}{2} < s \leq 0$ and $\kappa \geq 1$, these equations are well-posed and analytic for data in B^s_δ.

Lastly, $* = \{NLS, mKdV\}$, [HGKV20] shows that for a precompact set $Q \subset B^s_\delta \cap \mathcal{S}$ of data

$$\lim_{\kappa \to \infty} \sup_{u_0 \in Q} \|\Phi^*(u_0) - \Phi^*_\kappa(u_0)\|_{H^s} = 0, \tag{3.45}$$

for fixed $-\frac{1}{2} < s < 0$. Scaling of NLS (3.40) and mKdV (3.41) coupled with previous approximation establishes well-posedness at that regularity for all data in H^s. This observation suggests that flows (3.43),(3.44) serve as a good approximation for concluding that (3.40),(3.41) are non-squeezing. However, while scaling of the equations (3.40),(3.41) allows reduction to consideration of the small data case and defining flows (3.43),(3.44) on ball B^s_δ, proving to be crucial for treatment of well-posedness, (3.43),(3.44) are not defined on the entire space of initial data. This is a problem with having application of Theorem 1.0.3 in mind, as symplectomoprhisms should be defined for all data in L^2. We remedy this issue by observing that non-squeezing statement is also scale invariant, which we shall write down explicitly soon, and constructing symplectomorphisms whose restriction on small ball of data B^s_δ coincides with (3.43),(3.44). More precisely, but omitting forthcoming details, the symplectomorphisms approximating mKdV on B_δ will be generated by Hamiltonians

$$\widetilde{H}_\kappa = \eta \cdot \operatorname{Im} A(\kappa, \chi_{R(\kappa)} \cdot) + 4\kappa^2 P,$$

for suitably chosen cut-off η and space truncation $\chi_{R(\kappa)}$ ($R(\kappa)$ denoting dependence on κ), which ensures that Hamiltonians are well-defined for all data in L^2. We shall denote these flows by $\tilde{\Phi}_\kappa$.

Just like in the KdV case, $\kappa \geq 1$ introduces a family of symplectomorphisms equivalently indexed by $\tau \in [0,1)$. Endpoint 1 of $\mathcal{I} = [0,1]$ corresponds to the mKdV flow (3.41). We define the family of symplectomorphisms, reducing the proof of non-squeezing of mKdV in the context of Theorem 1.1.1,

$$\begin{aligned}
&\Psi : \mathcal{I} \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) \\
&\Psi(1) = \Phi^{mKdV} \\
&\Psi(\tau) = \tilde{\Phi}_\kappa, \ \ \tau \in [0,1).
\end{aligned}$$

Conservation of L^2 norm for all $\Psi(\tau)$, $\tau \in [0,1]$, implies the consideration of the family

$$\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1},$$

where X_{r_1} denotes the ball B_{r_1} from Theorem 3.5.1, endowed with weak topology and centered at $0 \in L^2$. Once again, centering at 0 does not make us lose on generality, as translations are symplectomorphisms. However, as we wish to use the approximation (3.45) in order to obtain the continuity of $\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1}$ at endpoint case $\tau \to 1$, arbitrary r_1 will not suffice since (3.45) holds for small data case. Following arguments show how scaling of mKdV and of non-squeezing statement allows us to reduce everything to small data setting, i.e. to assume that $r_1 \ll 1$.

Remark 3.5.2. *Firstly, using the terms appearing in Theorem 3.5.1, introduce the following notation*

$$\begin{aligned}
&r_{2,\lambda} := \sqrt{\lambda} r_2, \ \ r_{1,\lambda} := \sqrt{\lambda} r_1, \ \ \alpha_\lambda = \sqrt{\lambda}\alpha, \ \ T_\lambda = \lambda^{-3} T, \\
&h_\lambda(x) = \lambda h(\lambda x), \ \ \|h_\lambda\|_{L^2} = \sqrt{\lambda}\|h\|_{L^2} \\
&l_\lambda(x) = \sqrt{\lambda} l(\lambda x), \ \ \|l_\lambda\|_{L^2} = \|l\|_{L^2} = 1,
\end{aligned}$$

and recall the scaling of mKdV

$$u(t,x) \to \lambda^{-1} u(\lambda^{-3} t, \lambda^{-1} x), \quad u_0(x) \to \lambda^{-1} u_0(\lambda^{-1} x).$$

Taking $\lambda \ll 1$, the inequalities

$$\begin{aligned} \|u(0)-h\|_{L^2} &= \sqrt{\lambda}^{-1} \|u_\lambda(0)-h_\lambda\|_{L^2} < \sqrt{\lambda}^{-1} r_{1,\lambda} = r_1, \\ \left|\langle u(T), l\rangle - \alpha\right| &= \sqrt{\lambda}^{-1} \left|\langle u_\lambda(T_\lambda), l_\lambda\rangle - \alpha_\lambda\right| > \sqrt{\lambda}^{-1} r_{2,\lambda} = r_2 \end{aligned}$$

show that non-squeezing for flow at time $T > 0$ and general size of ball of initial data r_1 follows from one of flow at rescaled time $\lambda^{-3}T$ and for arbitrarily small data of size $\sqrt{\lambda} r_1$.

Similarly to KdV case, we shall exploit the fact that symplectic regularity is higher that one accessible by well-posedness theory, and pushing ball of initial data to lower regularity to obtain equicontinuity. The issue of tightness on $\mathbb{R}$ is resolved by the fact that when evaluating bounded set of data with respect to fixed functional, it suffices to localize in space set of testing data. We remark that this is a significant contrast to treatment of KdV in [KV19], where equicontinuity was enough to prove that approximations are justified, independently of geometry of $\mathbb{R}$ and $\mathbb{T}$. Tightness, and hence local smoothing, plays central role in obtaining well-posedness at low regularity in [HGKV20]. Lastly, motivated by [KV19], authors of [HGKV20] prove that equicontinuity and tightness of set of data is an invariant property preserved by mKdV and Φ_κ^{mKdV} flows, and by diffeomorphism g_{12}.

Regarding the case of NLS and mKdV posed on torus, [HGKV20] shows that g_{12} is diffeomorphism in the line case. However, one readily sees that the same is true in the torus case for suitably chosen κ based on the size of data, as shown in [KVZ18] for the Lax structure for ANKS/mKdV hierarchy, which holds in both cases. Hence flows (3.43),(3.44) are well-defined for suitably small data in the torus case, and for which we shall prove that non-squeezing property holds. Those flows are analytic, and in essence, behave in the exact same manner as Φ_κ^{KdV} flows, meaning that one exploits the gain of regularity of nonlinearity and the fact that well-posedness theory of flows holds below the regularity of space at which the flow is symplectic. This approach may lead to obtaining Bourgain's result for NLS in [Bou94]. On the other hand, mKdV on $\mathbb{T}$ is not well-posed at L^2 regularity, since $H^{\frac{1}{2}}$ is the sharp threshold for continuous dependence of the initial data, as shown by Chapouto [Cha21].

Proofs for NLS and mKdV are same, so for clarity sake we will only present the one for mKdV and denote mKdV and Φ_κ^{mKdV} flows by Φ, Φ_κ. Only difference is that the linear propagators are different, but both are uniformly bounded by 1 for all $\kappa \geq 1$. However, we will distinguish the line and the torus case, since the latter can be directly proven to fulfill assumptions of Corollary 1.0.4, whilst former needs adjustments that are, in a way, of the same nature as ones for NLS discussion in Section 3.2.

We finish this preliminary part with two lemmata that will be of relevance later on. First one also serves as a complimentary result to the approximation (3.45) for precompact sets.

Lemma 3.5.3. *Fix any* $-\frac{1}{2} < s < 0$. *For any compactly supported smooth function* $\varphi : \mathbb{R} \to \mathbb{R}$ *and a bounded equicontinuous set* $Q \subset H^s$

$$\lim_{\kappa\to\infty} \sup_{u\in Q} \|\varphi(\Phi(u) - \Phi_\kappa(u))\|_{L^\infty H^s} = 0.$$

Proof. Let $\operatorname{supp}\varphi \subset [-R, R]$. Proposition 7.1 from [HGKV20] assures that for smooth normalized cut-off ψ, denoting by $\phi_h := \psi(\cdot - h)$, and for a set of bounded equicontinuous data Q, one has uniform convergence

$$\psi_h g_{12}(\Phi\Phi_\kappa^{-1} u_0) \to \psi_h g_{12}(u_0)$$

in $C([0,T], H^{1+s})$ as $\kappa \to \infty$ and for all $h \in \mathbb{R}$. Here the inverse Φ_κ^{-1} denotes the time reversed flow. Denote by $\bar{\Phi}_\kappa := \Phi\Phi_\kappa^{-1}$. Then the following uniform convergence with respect to κ holds since

$$\begin{aligned}
&\sup_{u_0\in Q} \left\|\varphi\Big[g_{12}(\Phi(u_0)) - g_{12}(\Phi_\kappa(u_0))\Big]\right\|_{L^\infty H^{1+s}} \\
&= \sup_{u_0\in Q} \left\|\varphi\Big[g_{12}(\Phi[\Phi_\kappa]^{-1}\Phi_\kappa(u_0)) - g_{12}(\Phi_\kappa(u_0))\Big]\right\|_{L^\infty H^{1+s}} \\
&\lesssim_R \sup_{u\in Q^*} \left\|\psi_h[g_{12}(\bar{\Phi}_\kappa(u)) - g_{12}(u)]\right\|_{L^\infty H^{1+s}},
\end{aligned}$$

where $Q^* := \{\Phi_\kappa u_0 : u_0 \in Q\}$ is an equicontinuous set in H^s. Equicontinuity is an invariance of flows Φ, Φ_κ, proven in [HGKV20] (Prop. 4.5). Then, previous inequality and the following identity

$$\varphi g_{12}(f) = -(2\kappa - \partial)^{-1}\Big[\varphi' g_{12}(f) + \varphi f(\gamma(f) + 1)\Big],$$

alongside with invertibility of $L = 2\kappa - \partial : H^{1+s} \to H^s$, imply uniform convergence with respect to κ of the following

$$\lim_{\kappa\to\infty} \sup_{u\in Q^*} \left\|\varphi'\Big[g_{12}(\bar{\Phi}_\kappa u) - g_{12}(u)\Big] + \varphi\Big((\bar{\Phi}_\kappa u)(\gamma(\bar{\Phi}_\kappa u) + 1) - u(\gamma(u) + 1)\Big)\right\|_{L^\infty H^s} = 0.$$

Just like previously,

$$\begin{aligned}
\lim_{\kappa\to\infty} \sup_{u\in Q^*} \left\|\varphi'\Big[g_{12}(\bar{\Phi}_\kappa u) - g_{12}(u)\Big]\right\|_{L^\infty H^s} &\leq \lim_{\kappa\to\infty} \sup_{u\in Q^*} \left\|\varphi'\Big[g_{12}(\bar{\Phi}_\kappa u) - g_{12}(u)\Big]\right\|_{L^\infty H^{1+s}} \\
&\lesssim_R \lim_{\kappa\to\infty} \sup_{u\in Q^*} \left\|\psi_h\Big[g_{12}(\bar{\Phi}_\kappa u) - g_{12}(u)\Big]\right\|_{L^\infty H^{1+s}} = 0,
\end{aligned}$$

which in turn reduces to

$$
\begin{aligned}
0 &= \lim_{\kappa\to\infty} \sup_{u\in Q^*} \Big\| \varphi\Big((\bar\Phi_\kappa u)(\gamma(\bar\Phi_\kappa u)+1) - u(\gamma(u)+1) \Big) \Big\|_{L^\infty H^s} \\
&= \lim_{\kappa\to\infty} \sup_{u\in Q^*} \Big\| \varphi(\bar\Phi_\kappa u - u) + \varphi\Big((\bar\Phi_\kappa u)\gamma(\bar\Phi_\kappa u) - u\gamma(u) \Big) \Big\|_{L^\infty H^s}.
\end{aligned}
$$

Since

$$
\|\varphi(\bar\Phi_\kappa u)\gamma(\bar\Phi_\kappa u)\|_{L^\infty H^s} \le \kappa^{-(s+\frac{1}{2})} \|\bar\Phi_\kappa u\|_{H^s} \|\gamma(\bar\Phi_\kappa u)\|_{H^{1+s}} \le \kappa^{-2s-1} \|\bar\Phi_\kappa u\|_{H^s}^3 \lesssim_\delta \kappa^{-2s-1},
$$

we obtain

$$
0 = \lim_{\kappa\to\infty} \sup_{u\in Q^*} \Big\| \varphi(\bar\Phi_\kappa u - u) \Big\|_{L^\infty H^s} = \lim_{\kappa\to\infty} \sup_{u_0\in Q} \Big\| \varphi(\Phi(u_0) - \Phi_\kappa(u_0)) \Big\|_{L^\infty H^s}.
$$

□

Lemma 3.5.4. *mKdV (3.41) is continuous with respect to the weak topology in L^2.*

We delay the proof of Lemma 3.5.4 for the end of the following Subsection 3.5.1, as we shall need to introduce additional structure in order to prove it.

3.5.1 Line case

As in the case for NLS, translation symmetry for $\Phi_\kappa^{NLS}, \Phi_\kappa^{mKdV}$ suggests that we have to localize the nonlinearity in some way. Additionally, we need flows to be defined for all data in L^2. Motivated by these observations, define Hamiltonians

$$
H^{NLS}(\kappa, R, \cdot) = H_{\kappa,R}^{NLS}(\cdot) := -8\kappa^3 \eta(\|\chi_R \cdot\|_{H^s}^2) \operatorname{Re} A(\kappa, \chi_R \cdot) + 4\kappa^2 M(\cdot) \tag{3.46}
$$

and

$$
H^{mKdV}(\kappa, R, \cdot) = H_{\kappa,R}^{mKdV}(\cdot) := 16\kappa^4 \eta(\|\chi_R \cdot\|_{H^s}^2) \operatorname{Im} A(\kappa, \chi_R \cdot) + 4\kappa^2 P(\cdot), \tag{3.47}
$$

where $\eta : [0,\infty) \to [0,1]$ is a smooth cut-off such that $\eta \equiv 1$ on $[0, 0.49\delta]$ and $\eta \equiv 0$ on $[0.5\delta, \infty)$ and $-\frac{1}{2} < s < 0$ is arbitrary. We shall denote them by $\Phi_{\kappa,R}^{NLS}$ and $\Phi_{\kappa,R}^{mKdV}$, respectively. Here χ_R presents a smooth cut-off such that $\chi_R \equiv 1$ on $[-R, R]$ and $\chi_R \equiv 0$ for $|x| \ge R+1$. Note that functional derivative of $f(u) := \eta(\|\chi_R u\|_{H^s}^2) : L^2 \to \mathbb{R}$ is given by

$$
\frac{\delta f}{\delta v} := \eta'(\|\chi_R u\|_{H^s}^2) \chi_R L_s(\chi_R u),
$$

where $L_s(f) := \mathcal{F}^{-1}[(1+\xi^2)^s \widehat{f}]$. Moreover, L_s is a bounded linear map that gives $-2s$ regularity to initial data, i.e. $L_s : H^t \to H^{t-2s}$ for every $t \in \mathbb{R}$.

Then Hamiltonians (3.46) and (3.47) generate flows

$$\begin{aligned} iu_t =&4\kappa^3\chi_R\eta(\|\chi_R u\|_{H^s}^2)(g_{12}(\kappa,\chi_R u)-g_{12}(-\kappa,\chi_R u)) \\ &+8\kappa^3\chi_R L_s(\chi_R u)\eta'(\|\chi_R u\|_{H^s}^2)\operatorname{Re}A(\kappa,\chi_R u)+4\kappa^2 u \end{aligned} \tag{3.48}$$

and

$$\begin{aligned} u_t =&8\kappa^4\chi_R\eta(\|\chi_R u\|_{H^s}^2)(g_{12}(\kappa,\chi_R u)+g_{12}(-\kappa,\chi_R u)) \\ &+16\kappa^4\chi_R L_s(\chi_R u)\eta'(\|\chi_R u\|_{H^s}^2)\operatorname{Im}A(\kappa,\chi_R u)+4\kappa^2 u_x, \end{aligned} \tag{3.49}$$

Linear propagators are given by $L_\kappa^{NLS}(t)(u) := e^{i4\kappa^3 t}u$ and $L_\kappa^{mKdV}(t)(u) := e^{i4\kappa^4\partial_x}u$ respectively. We shall only present approach one takes for mKdV - the one for NLS follows in the exact same fashion with obvious, but purely technical, adaptations. Note that

$$\|L_\kappa^{NLS}(t)\|_{op} = \|L_\kappa^{mKdV}(t)\|_{op} = 1.$$

Equations make sense definition wise since $A(\kappa)$ and $g_{12}(\kappa)$ are defined on B_δ. We shall extensively be using the fact that $g_{12} : B_\delta^s \to H^{1+s}$ is diffeomorphism, and for that purpose we shall fix $-\frac{1}{2} < s < 0$. Moreover, denote by

$$F(\kappa,u) := 8\kappa^4\chi_R\eta(\|u\|_{H^s}^2)(g_{12}(\kappa,u)+g_{12}(-\kappa,u))+16\kappa^4\chi_R L_s(u)\eta'(\|u\|_{H^s}^2)\operatorname{Im}A(\kappa,u)$$

a smooth map $F : [1,\infty)\times H^s \to L^2$. Smoothness of F follows from the fact that

$$\begin{aligned} &\eta(\|\cdot\|_{H^s}^2) : H^s \to \mathbb{R},\ \ A(\kappa,\cdot) : B_\delta^s \to \mathbb{C}, \\ &g_{12}(\kappa,\cdot) : B_\delta^s \to H^{s+1},\ \ \chi_R L_s : L^2 \to L^2, \end{aligned}$$

are smooth. Additionally, denote by $\chi_R\cdot : L^2 \to H^s$ a compact linear operator and by $F_\kappa(\cdot) := F(\kappa,\cdot) : H^s \to L^2$. Equation (3.49) than stands

$$u_t = F_\kappa(\chi_R u) + 4\kappa^2 u_x.$$

Corresponding Duhamel's formula is

$$u(t) = L_\kappa(t)u_0 + \int_0^t L_\kappa(t-s)F_\kappa(\chi_R u(s))ds,$$

and we denote the flow by $\Phi_{\kappa,R}$. Now that we have introduced the notation, let us address the well-posedness theory.

Lemma 3.5.5. *Fix $\kappa \geq 1$. For every $R > 0$, the flow $\Phi_{\kappa,R}$ given by equation (3.49) is globally well-posed in $L^2(\mathbb{R})$ and locally well-posed in H^s. The flow is Lipschitz on bounded sets, with constant depending only on δ and κ, and not on truncation constant R. Lastly, $\Phi_{\kappa,R}$ preserves the L^2 norm.*

Proof. Since

$$\|F(\chi_R u) - F(\chi_R u)\|_{L^2} \le \|DF\|\|\chi_R(u-v)\|_{H^s} \lesssim \|u-v\|_{H^s} \lesssim \|u-v\|_{L^2},$$

local well-posedness in L^2 and H^s follow directly by fixed point argument. Global well-posedness follows from the fact that L^2 norm is preserved along the flow. Indeed, computing Poisson bracket

$$\begin{aligned}\{H^{mKdV}_{\kappa,R}, M\} &= \{16\kappa^4\eta(\|\chi_R\cdot\|^2_{H^s})\mathrm{Im}\,A(\kappa,\chi_R\cdot) + 4\kappa^2 P, M\} \\ &= \eta(\|\chi_R\cdot\|^2_{H^s})\{A(\kappa,\chi_R\cdot), M\} + \{\eta(\|\chi_R\cdot\|^2_{H^s}), M\}A(\kappa,\chi_R\cdot) + \{P, M\} = 0.\end{aligned}$$

□

Remark 3.5.6. *Moreover, notice that the conservation of L^2 norm implies that for small initial data data $\|u_0\|_{L^2} < 0.25\delta$ flow of $\Phi_{\kappa,R}$ is given by equation*

$$iu_t = 4\kappa^3\chi_R(g_{12}(\kappa,\chi_R u) - g_{12}(-\kappa,\chi_R u)) + 4\kappa^2 u,$$

since $\|\Phi_{\kappa,R}(u_0)\|_{H^s} \le \|\Phi_{\kappa,R}(u_0)\|_{L^2} = \|u_0\|_{L^2} < 0.25\delta$, for all times $T > 0$,, and hence $\eta \equiv 1$ along the trajectory. Then the choice of localized Hamiltonians is warranted $\Phi_{\kappa,R}$, aside from one being able to prove that flows fulfill assumptions of Theorem 1.0.3, by the fact that they approximate Φ_κ in uniform fashion for bounded sets of Schwartz initial data $Q := B_{0.5\delta} \cap \mathcal{S} \subset L^2$, for κ large, reminiscent of uniform approximation of Φ by Φ_κ on bounded intervals obtained in Lemma 3.5.3. More specifically,

Denote by $\varphi_R := \varphi(\frac{x}{R})$ a rescaled version of a smooth cut-off such that $\varphi \equiv 1$ on $[-1,1]$ and $\varphi(x) \equiv 0$ for $|x| \ge 2$.

Lemma 3.5.7. *Let $Q := B_{0.25\delta} \cap \mathcal{S} \subset L^2$. There exists some $-\frac{1}{2} < s_0 < 0$, so that for $r = r(\kappa) := 2\kappa^3$ and every $-\frac{1}{2} < s \le s_0$*

$$\lim_{\kappa\to\infty} \sup_{u_0\in Q} \|\varphi_{\kappa^3}(\Phi_\kappa u_0 - \Phi_{\kappa,r} u_0)\|_{L^\infty H^s} = 0.$$

Proof. Let $R := \kappa^3$. For every $\kappa \ge 1$ and $r = r(\kappa) = 2\kappa^3$, $\chi_r \equiv 1$ on $\operatorname{supp}\varphi_R$.

Uniform convergence of the difference of solutions of flow Φ_κ and $\Phi_{\kappa,r}$ for data in $B_{0.25\delta}$ can be equivalently stated by observing equation generated by the difference of Hamiltonians $H^{mKdV}_\kappa - H^{mKdV}_{\kappa,r}$

$$u_t = 8\kappa^4(g_{12}(\kappa,u) + g_{12}(-\kappa,u)) - 8\kappa^4\chi_r(g_{12}(\kappa,\chi_r u) + g_{12}(-\kappa,\chi_r u)).$$

Solution is given by Duhamel's formula

$$u(t) = 8\kappa^4\int_0^t [(g_{12}(\kappa,u) + g_{12}(-\kappa,u)) - \chi_r(g_{12}(\kappa,\chi_r u) + g_{12}(-\kappa,\chi_r u))]ds,$$

and one can see that this flow is well-posed in H^s, $-\frac{1}{2} < s < 0$, and is actually a map into H^{1+s}. With the statement of lemma in mind, we have interest in

$$\varphi_R u(t) = 8\kappa^4 \int_0^t \varphi_R[(g_{12}(\kappa,u) + g_{12}(-\kappa,u)) - (g_{12}(\kappa,\chi_r u) + g_{12}(-\kappa,\chi_r u))]ds,$$

for initial data in $B_{0.25\delta}$. Lemma 3.5.5 also gives global well-posedness of solutions to this equation for data in $B_{0.25\delta}$. Independence of the sign of κ for all functions in question allows us to reduce last line to control of the norm

$$\kappa^4 \|\varphi_R(g_{12}(u) - g_{12}(\chi_r u))\|_{L^\infty H^s} \le \kappa^3 \|\varphi_R(g_{12}(u) - g_{12}(\chi_r u))\|_{L^\infty H^{1+s}}.$$

We make use again of identities introduced in [HGKV20], namely

$$\begin{aligned}
&g_{12}'(f) = 2\kappa g_{12}(f) + f(\gamma(f)+1),\\
&\varphi_R g_{12}(f) = -(2\kappa-\partial)^{-1}\varphi_R' g_{12}(f) - (2\kappa-\partial)^{-1}\varphi_R f(\gamma(f)+1),
\end{aligned}$$

denoting by $L := -(2\kappa-\partial)^{-1}$. Then

$$\begin{aligned}
\varphi_R(g_{12}(u) - g_{12}(\chi_r u)) &= L[\varphi_R'(g_{12}(u) - g_{12}(\chi_r u))] + L[\varphi_R u(\gamma(u)+1) - \varphi_R \chi_r u(\gamma(\chi_r u)+1)]\\
&= L[\varphi_R'(g_{12}(u) - g_{12}(\chi_r u))] + L[\varphi_R u(\gamma(u) - \gamma(\chi_r u))].
\end{aligned}$$

We bound summands individually in the following way

$$\begin{aligned}
\|L[\varphi_R'(g_{12}(u) - g_{12}(\chi_r u))]\|_{L^\infty H^{1+s}} &\lesssim (|\kappa|R)^{-1}\|u\|_{H^s_\kappa} \lesssim_\delta (|\kappa|R)^{-1} \lesssim_\delta (|\kappa|)^{-4},\\
\|L[\varphi_R u(\gamma(u) - \gamma(\chi_r u))]\|_{L^\infty H^{1+s}} &\lesssim \kappa^{-1}\|\varphi_R u\gamma(u)\|_{H^s_\kappa} \lesssim \kappa^{-1-(s+\frac{1}{2})}\|u\|_{H^s_\kappa}\|\gamma(u)\|_{H^{1+s}_\kappa}\\
&\lesssim \kappa^{-2-2s}\|u\|^3_{H^s_\kappa} \lesssim \kappa^{-2-5s}\|u\|^3_{L^2} \lesssim_\delta \kappa^{-2-5s}.
\end{aligned}$$

Hence choosing s_0 so that $-2-5s_0 < -3$, we get that

$$\|\varphi_R u\|_{L^\infty H^{s_0}} \le \kappa^3 \|\varphi_R(g_{12}(u) - g_{12}(\chi_r u))\|_{L^\infty H^{1+s_0}} \lesssim_\delta \kappa^{1-5s_0},$$

concluding the proof. □

Lemma 3.5.7 finalizes the definition of family of Ψ. Fix $s_0 \in (-\frac{1}{2}, 0)$ and denote by $r(\kappa) = 2\kappa^3$, both given by Lemma 3.5.7. Denote by $\widetilde{\Phi}_\kappa := \Phi_{\kappa,r(\kappa)}$ the flow generated by a smooth Hamiltonian $H(\kappa, 2\kappa^3) : H^{s_0} \to \mathbb{R}$ (3.47). Recall the definition of (3.47) and the arbitrary choice of s. In what follows we can choose any $-\frac{1}{2} < s \le s_0$, but for clarity sake, we shall fix $s = s_0$. Since $\kappa \ge 1$, note that flows $\Psi(\tau)$, $\tau \in [0,1)$ are generated by a smooth Hamiltonian

$$\begin{aligned}
&\widetilde{H} : [0,1) \times H^{s_0} \to \mathbb{R}\\
&\widetilde{H}(\tau, u) := H(\kappa, 2\kappa^3, u).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\Psi : \mathcal{I} \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) \\
&\Psi(1) = \Phi^{mKdV} \\
&\Psi(\tau) = \widetilde{\Phi}_\kappa, \;\; \tau \in [0,1).
\end{aligned}$$

Correspondence between τ and κ is obvious. As proven, all flows in the family Ψ preserve L^2 norm, and with Remark 3.5.2 in mind, we can consider for $r_1 \le \frac{\delta}{4}$ the family

$$\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1}.$$

Lemma 3.5.8. *Family $\Psi : \mathcal{I} \times X_{r_1} \to X_{r_1}$ is continuous.*

Proof. For continuity discussion we shall fix arbitrary $l \in L^2$ such that $\|l\|_{L^2} \le 1$. Density argument allows to assume without the loss of generality that $l \in \mathcal{S}$ where $\operatorname{supp} l \subset [-M,M]$ for some $M > 0$, since for any $\varepsilon > 0$ there exists $\tilde{l} \in C_c^\infty$ such that $\|l - \tilde{l}\|_{L^2} < \varepsilon$. Similarly, for any initial data $u_0 \in X_{r_1}$ we may assume that $u_0 \in \mathcal{S}$ since the well-posedness of mKdV (3.41) and $\widetilde{\Phi}_\kappa$ (3.49) imply that for any $\varepsilon > 0$ and every $\tau \in \mathcal{I}$ exists $\tilde{u}_0 \in L^2$ such that $\|\Psi_\tau(u_0) - \Psi_\tau(\tilde{u}_0)\|_{L^2} < \varepsilon$.

We distinguish two cases of continuity in the $\mathcal{I}$ variable - first one is the endpoint $\tau_n \to 1$ ($\kappa_n \to \infty$), which corresponds to $\tilde{\Phi}_\kappa$ convergence to mKdV, and the second one when $\tau_n \to \tau < 1$ ($\kappa_n \to \kappa < \infty$). Regardless of this dichotomy, we shall always have a Schwartz sequence $u_n(0,x) \to u(0,x)$ in X_{r_1}, that is $u_n(0,x) \rightharpoonup u(0,x)$ in L^2. Hence the sequence is a bounded equicontinuous set in H^{s_0}, which we denote by $Q := \{u_n(0), u(0)\}$.

Regarding the first case, let $\kappa_n \to \infty$. Then

$$\begin{aligned}
\left|\left\langle \Psi(1,u(0)) - \Psi(\tau_n, u_n(0)), l\right\rangle\right| &= \left|\left\langle \Phi(u(0)) - \widetilde{\Phi}_{\kappa_n}(u_n(0)), l\right\rangle\right| \\
&= \left|\left\langle \varphi_M\Big(\Phi(u(0)) - \widetilde{\Phi}_{\kappa_n}(u_n(0))\Big), l\right\rangle\right| \\
&\le \|\varphi_M\Big(\Phi(u(0)) - \widetilde{\Phi}_{\kappa_n}(u_n(0))\Big)\|_{H^{s_0}} \|l\|_{H^{-s_0}} \\
&\lesssim \left\|\varphi_M\Big(\Phi(u(0)) - \Phi(u_n(0))\Big)\right\|_{H^{s_0}} + \left\|\varphi_M\Big(\Phi(u_n(0)) - \Phi_{\kappa_n}(u_n(0))\Big)\right\|_{H^{s_0}} \\
&\quad + \left\|\varphi_M\Big(\Phi_{\kappa_n}(u_n(0)) - \widetilde{\Phi}_{\kappa_n}(u_n(0))\Big)\right\|_{H^{s_0}}.
\end{aligned}$$

We estimate the convergence of each term separately. First one follows from Lemma 3.5.4 for regularity L^2 and compact embedding $\varphi_M \cdot : L^2 \to H^{s_0}$. The second one from equicontinuity of set Q and Lemma 3.5.3. Convergence of the third term follows from inequality

$$\left\|\varphi_M\Big(\Phi_{\kappa_n}(u_n(0)) - \widetilde{\Phi}_{\kappa_n}(u_n(0))\Big)\right\|_{H^{s_0}} \le \left\|\varphi_{\kappa_n^3}\Big(\Phi_{\kappa_n}(u_n(0)) - \widetilde{\Phi}_{\kappa_n}(u_n(0))\Big)\right\|_{H^{s_0}}$$

which holds for all κ_n such that $\kappa_n^3 \ge M$, equicontinuity of set Q in H^{s_0} and Lemma 3.5.7.

Regarding the second case, $\kappa_n \to \kappa$, we can equivalently observe the closed interval $\tau \in I \subset \mathcal{I}\backslash\{1\}$ (corresponding a finite closed neighborhood $\tilde{I}$ of κ) and the compact set $I \times X_r$. Recall that for $\tau \in I$

$$\Psi(\tau, u(0)) = \widetilde{\Phi}_\kappa(q_0) := u(t) = L_\kappa(t)u(0) + \int_0^t L_\kappa(t-s)F_\kappa(\chi_{2\kappa^3}u(s))ds.$$

That the time t in the formula of Ψ makes sense and that it is continuous as a map $\Psi : I \times L^2 \to L^2$ follows from the boundedness of $\tilde{I}$, smoothness of the Hamiltonian $\widetilde{H}(\kappa, u) : \tilde{I} \times L^2 \to \mathbb{R}$ and the fact that the gradient $\nabla\widetilde{H}(\kappa, u)$ is Lipschitz on bounded balls of data in L^2, with Lipschitz constant independent of κ due to uniformity from boundedness of $\tilde{I}$. Denot by $\tilde{\kappa}$ the right endpoint of $\tilde{I}$.

We want to prove that $u_n := \widetilde{\Phi}_{\kappa_n}(u_n(0)) \rightharpoonup u := \widetilde{\Phi}_\kappa(u(0))$. Since $\|u_n\|_{L_t^\infty L_x^2} \lesssim 1$, we define

$$v := \text{w-}\lim_{n\to\infty} u_n,$$

where the weak limit is taken in L^2, and hence $\|v\|_{L_t^\infty L_x^2} \lesssim 1$. Then $\|\chi_R(u_n - v)\|_{H^{s_0}} \to 0$, for every space localization χ_R ($R > 0$). We take weak limits in L^2 of

$$u_n(t) = L_{\kappa_n}(t)u_n(0) + \int_0^t L_{\kappa_n}(t-s)F_{\kappa_n}(\chi_{2\kappa_n^3}u_n(s))ds.$$

LHS equals v. Also,

$$\begin{aligned}
\left|\langle L_{\kappa_n}u_n(0) - L_\kappa u(0), l\rangle\right| &= \left|\langle L_{\kappa_n}u_n(0) - L_\kappa u_n(0), l\rangle\right| + \left|\langle L_\kappa u_n(0) - L_\kappa u(0), l\rangle\right| \\
&= \left|\langle u_n(0), (L_{\kappa_n} - L_\kappa)l\rangle\right| + \left|\langle u_n(0) - u(0), L_\kappa l\rangle\right| \\
&\le \|u_n(0)\|_{L^2}\|(L_{\kappa_n} - L_\kappa)l\|_{L^2} + \left|\langle u_n(0) - u(0), L_\kappa l\rangle\right|,
\end{aligned}$$

shows that the limit for the first term on RHS is $L_\kappa u(0)$, due to continuity of $L_{(\cdot)} : \tilde{I} \to B(L^2)$ with respect to strong operator topology and weak convergence $u_n(0) \rightharpoonup u(0)$. Lastly, we want to show that

$$\text{w-}\lim_{n\to\infty} \int_0^t L_{\kappa_n}(t-s)F_{\kappa_n}(\chi_{2\kappa_n^3}u_n(s))ds = \int_0^t L_\kappa(t-s)F_\kappa(\chi_{2\kappa^3}v(s))ds,$$

which would coupled with well-posedness imply that $v = u$, concluding the proof. However, the last limit holds in a stronger fashion, i.e. with respect to L^2 norm. Firstly, rewrite

$$\begin{aligned}
&\int_0^t L_{\kappa_n}(t-s)F_{\kappa_n}(\chi_{2\kappa_n^3}u_n(s))ds - \int_0^t L_\kappa(t-s)F_\kappa(\chi_{2\kappa^3}v(s))ds \\
&= \int_0^t \Big[L_{\kappa_n}(t-s)F_{\kappa_n}(\chi_{2\kappa_n^3}u_n(s))ds - L_\kappa(t-s)F_\kappa(\chi_{2\kappa^3}v(s)) \pm L_{\kappa_n}(t-s)F_{\kappa_n}(\chi_{2\kappa_n^3}v(s))\Big]ds.
\end{aligned}$$

Firstly,

$$
\begin{aligned}
&\Big\| \int_0^t L_{\kappa_n}(t-s) F_{\kappa_n}(\chi_{2\kappa_n^3} u_n(s)) ds - \int_0^t L_{\kappa_n}(t-s) F_{\kappa_n}(\chi_{2\kappa_n^3} v(s)) \Big\|_{L^2} \\
&\quad \le \left\| F_{\kappa_n}(\chi_{2\kappa_n^3} u_n(s)) - F_{\kappa_n}(\chi_{2\kappa_n^3} v(s)) \right\|_{L^\infty L^2} \\
&\quad \le \left\| \chi_{2\kappa_n^3}\big(u_n(s) - v(s)\big) \right\|_{L^\infty H^{s_0}} \le \left\| \chi_{2\tilde{\kappa}^3}\big(u_n(s) - v(s)\big) \right\|_{L^\infty H^{s_0}} \to 0.
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
&\Big\| \int_0^t L_{\kappa}(t-s) F_{\kappa}(\chi_{2\kappa^3} v(s)) ds - \int_0^t L_{\kappa_n}(t-s) F_{\kappa_n}(\chi_{2\kappa_n^3} v(s)) \Big\|_{L^2} \\
&\le \left\| F_{\kappa}(\chi_{2\kappa^3} v(s)) - F_{\kappa_n}(\chi_{2\kappa_n^3} v(s)) \right\|_{L^\infty L^2} + \left\| \big(L_\kappa(t-s) - L_{\kappa_n}(t-s)\big) F_\kappa(\chi_{2\kappa^3} v(s)) \right\|_{L^\infty L^2}
\end{aligned}
$$

which converges to 0 as well, due to smoothness of $F : \tilde{I} \times H^{s_0} \to L^2$ and continuity of $L_{(\cdot)} : \tilde{I} \to B(L^2)$ with respect to strong operator topology. □

Remark 3.5.9. *We would like to comment about the regularity s_0. The sole purpose of it is to ensure that we have approximation in the sense of Lemma 3.5.7. Hence, for any definition of $\Phi_{\kappa,R}$ (3.49) (and hence choice of s), the entire construction of family Ψ holds, continuity properties as well, as long as $s \le s_0$. That the existence of s_0 does not depend on the definition of $\Phi_{\kappa,R}$ (i.e. s), one can readily see it from Remark 3.5.6, which translates directly into the proof of Lemma 3.5.7, as cut-off η from the definition of H_κ 3.47 becomes obsolete for small data case. This observation is crucial for obtaining weak continuity of mKdV in Lemma 3.5.4.*

Now we turn to proving non-squeezing property of $\Phi_{\kappa,R}$, and we fix arbitrary regularity $-\frac{1}{2} < \tilde{s} < s_0 < 0$ and use it as regularity in the definition of $\Phi_{\kappa,R}$ (3.49). As a special case, $\widetilde{\Phi}_\kappa$ will have non-squeezing property as well. Note that we will be independently proving the weak continuity of $\widetilde{\Phi}_\kappa$, since it does not follow from Lemma 3.5.8, where we considered only small data case.

Lemma 3.5.10. *Flow $\Phi_{\kappa,R}$ (3.49) is continuous with respect to weak topology in H^s, for every $s > \tilde{s}$.*

Proof. We prove the weak continuity for arbitrary regularity $\tilde{s} < \bar{s} < 0$. That continuity holds for higher regularities $t > \bar{s}$ follows directly by the weak continuity for $\bar{s}$, embedding $H^t \to H^{\bar{s}}$ and density argument for functionals.

Let $u_n(0,x) \rightharpoonup u(0,x)$ in $H^{\bar{s}}$. Let u_n be corresponding Cauchy solutions of (3.49), and define

$$
v(t) := \text{w-}\lim_{n\to\infty} u_n(t)
$$

for each t, where weak limit is taken in $H^{\bar{s}}$. Hence $v \in L^\infty([0,T], H^{\bar{s}})$. We want to prove that v is then the solution of (3.49) for initial data $u(0)$. Sequence $\chi_R u_n$ converges strongly and uniformly on time interval $[0,T]$ to $\chi_R v$ in $H^{\tilde{s}}$. Taking weak limits in Duhamel's formula

$$u_n(t) = L_\kappa(t)u_n(0) + \int_0^t L_\kappa(t-s)F_\kappa(\chi_R u_n(s))ds$$

one obtains

$$v(t) = L_\kappa(t)u(0) + \text{w-}\lim_{n\to\infty}\int_0^t L_\kappa(t-s)F_\kappa(\chi_R u_n(s))ds.$$

Proving that

$$\text{w-}\lim_{n\to\infty}\int_0^t L_\kappa(t-s)F_\kappa(\chi_R u_n(s))ds = \int_0^t L_\kappa(t-s)F_\kappa(\chi_R v(s))ds$$

would conclude the proof by invoking local well-posedness. However, last equalities hold in stronger fashion, that is taking limits with respect to norm in L^2

$$\begin{aligned}\left\|\int_0^t L_\kappa(t-s)\Big[F_\kappa(\chi_R u_n(s)) - F_\kappa(\chi_R v(s))\Big]ds\right\|_{L^2} &\lesssim \sup_{[0,t]}\Big\|F_\kappa(\chi_R u_n) - F_\kappa(\chi_R v)\Big\|_{L^2}\\ &\lesssim \sup_{[0,t]}\Big\|\chi_R(u_n - v)\Big\|_{H^{\tilde{s}}}.\end{aligned}$$

□

Derivative of flow given by (3.49) at point u_0 is given by the equation

$$\partial_t v = D[F_\kappa \circ (\chi_R \cdot)](u)v + 4\kappa^2 v', \tag{3.50}$$

or equivalently, by Duhamel's formula

$$v(t) = L_\kappa(t)v_0 + \int_0^t L_\kappa(t-s)\Big[DF_\kappa(\chi_R u(s))\chi_R v(s)\Big]ds, \tag{3.51}$$

where $v(t) := D\Phi_{\kappa,R}[u_0]v(0)$. In order to make sense of the derivative flow, one has to prove that it is well-posed in L^2. This readily follows from

$$\begin{aligned}\|v\|_{L^2} &\lesssim \|v(0)\|_{L^2} + \int_0^t \|\Big[DF_\kappa(\chi_R u(s))v(s)\Big]\|_{L^2}\\ &\lesssim \|v(0)\|_{L^2} + t\|DF_\kappa(\chi_R u)\|_{B(H^{\bar{s}},L^2)}\|v\|_{H^{\bar{s}}}\\ &\lesssim \|v(0)\|_{L^2} + t\|DF_\kappa(\chi_R u)\|_{B(H^{\bar{s}},L^2)}\|v\|_{L^2},\end{aligned}$$

and the fact that F_κ is smooth on $H^{\tilde{s}}$, hence $\|DF_\kappa(\chi_R u)\|_{B(H^{\bar{s}},L^2)} \lesssim \|u_0\|_{L^2}$. These inequalities give existence up to a time depending on the size of data $v(0)$, but we can obviously iterate up to any finite time T by applying the well-defined flow, as the data blows up at most exponentially.

Proving the weak continuity of the derivative of flow $\Phi_{\kappa,R}$ concludes the proof of its non-squeezing property

Lemma 3.5.11. *Derivative $D\Phi_{\kappa,R} : (L^2, \tau_{weak}) \longrightarrow (B(L^2), \|\cdot\|_{op})$ is continuous.*

Proof. Let $u_n(0) \rightharpoonup u(0)$, and let u_n, u be corresponding Cauchy solutions of (3.49). Let w_n and w be solutions with the same initial data $w(0)$, $\|w_0\|_{L^2} \le \delta_1$ for some constant δ_1.

$$\partial_t w_n = DF_\kappa(\chi_R u_n)\chi_R w_n + 4\kappa^2 w_n',$$
$$\partial_t w = DF_\kappa(\chi_R u)\chi_R w + 4\kappa^2 w'.$$

We want to prove that as $n \to \infty$, $v := w_n - w$ goes to 0. Previous equations give

$$\partial_t v = [DF_\kappa(\chi_R u_n) - DF_\kappa(\chi_R u)]\chi_R w_n + DF_\kappa(\chi_R u)\chi_R v + 4\kappa^2 v'.$$

Writing down Duhamel's formula we get

$$\begin{aligned} v(t) = &\int_0^t L_\kappa(t-s)[DF_\kappa(\chi_R u(s))\chi_R v(s)]ds \\ &+ \int_0^t L_\kappa(t-s)\Big[[DF_\kappa(\chi_R u_n(s)) - DF_\kappa(\chi_R u(s))]\chi_R w_n(s)\Big]ds. \end{aligned}$$

Once again, scaling with respect to u and correspondingly rescaling the data w_0 up to a fixed constant allows us to move the first summand on RHS to LHS and enables us to directly obtain the bound on norm of difference of derivatives

$$\begin{aligned} \|v(t)\|_{L^2} &\lesssim_T \sup_{t\in[0,T]} \Big\|\Big[[DF_\kappa(\chi_R u_n(s)) - DF_\kappa(\chi_R u(s))]\chi_R w_n(s)\Big]\Big\|_{L^2} \\ &\lesssim_T \sup_{t\in[0,T]} \|DF_\kappa(\chi_R u_n) - DF_\kappa(\chi_R u)\|_{B(H^{\bar{s}},L^2)}\|w_n\|_{H^{\bar{s}}} \\ &\lesssim_{T,\delta} \sup_{t\in[0,T]} \|DF_\kappa(\chi_R u_n) - DF_\kappa(\chi_R u)\|_{B(H^{\bar{s}},L^2)}. \end{aligned}$$

Since $\chi_R u_n$ converges strongly to $\chi_R u$ in $H^{\bar{s}}$ due to Lemma 3.5.10, smoothness of F implies that the last term converges to 0 as $n \to \infty$. These bounds serve up to a time dependent on the size of the data v, but iterating the flow for arbitrary finite time gives the desired result for flow at any fixed time T. □

Proof of Lemma 3.5.4. Scaling of mKdV (3.41) reduces the proof to small data case. Hence, we want to prove that $\Phi : X_{r_1} \to X_{r_1}$ is continuous. Here X_{r_1} denotes the ball $B_{r_1} \subset L^2$ endowed with weak topology, and r_1 is chosen with an upper bound as in Lemma 3.5.8.

Without a loss of generality, fix a Schwartz function $l \in B_1 \subset H^1$ which is compactly supported on $[-M, M]$.

Let $u_n(0) \rightharpoonup u(0)$ in B_{r_1}. Well-posedness of mKdV implies that we can assume without a loss of generality that initial data is Schwartz. Hence set $Q := \{u(0), u_n(0) : n \in \mathbb{N}\} \subset \mathcal{S}$ is bounded and equicontinuous in $H^{\tilde{s}}$. Then

$$\begin{aligned}\left|\langle \Phi(u_n) - \Phi(u), l\rangle\right| &= \left|\langle \varphi_M\big(\Phi(u_n) - \Phi(u)\big), l\rangle\right| \le \left\|\varphi_M\big(\Phi(u_n) - \Phi(u)\big)\right\|_{H^{\tilde{s}}} \\ &\le \left\|\varphi_M\big(\Phi(u) - \Phi_\kappa(u)\big)\right\|_{H^{\tilde{s}}} + \left\|\varphi_M\big(\Phi(u_n) - \Phi_\kappa(u_n)\big)\right\|_{H^{\tilde{s}}} \\ &+ \left\|\varphi_M\big(\Phi_\kappa(u_n) - \Phi_\kappa(u)\big)\right\|_{H^{\tilde{s}}} = I_1 + I_2 + I_3.\end{aligned}$$

where I_1, I_2, I_3 denote the three terms correspondingly. We estimate first two terms jointly

$$I_1 + I_2 \le 2 \sup_{q \in Q} \left\|\varphi_M\big(\Phi(q) - \Phi_\kappa(q)\big)\right\|_{H^{\tilde{s}}}, \tag{3.52}$$

and the third one

$$\begin{aligned}I_3 &\le \left\|\varphi_M\big(\widetilde{\Phi}_\kappa(u) - \Phi_\kappa(u)\big)\right\|_{H^{\tilde{s}}} + \left\|\varphi_M\big(\widetilde{\Phi}_\kappa(u_n) - \Phi_\kappa(u_n)\big)\right\|_{H^{\tilde{s}}} \\ &+ \left\|\varphi_M\big(\widetilde{\Phi}_\kappa(u_n) - \widetilde{\Phi}_\kappa(u)\big)\right\|_{H^{\tilde{s}}} = J_1 + J_2 + J_3,\end{aligned}$$

where J_1, J_2, J_3 denote the three terms correspondingly. Flow $\widetilde{\Phi}_\kappa$ (3.49) is defined for regularity $\tilde{s}$. Now

$$J_1 + J_2 \le 2 \sup_{q \in Q} \left\|\varphi_{\kappa^3}\big(\widetilde{\Phi}_\kappa(q) - \Phi_\kappa(q)\big)\right\|_{H^{\tilde{s}}}, \tag{3.53}$$

for $\kappa \ge M$. For every $\varepsilon > 0$, there exist κ_1, κ_2, so that right hand sides of (3.52), (3.53) are strictly bounded by ε, which follows from Lemma 3.5.3 and Lemma 3.5.7 respectively. Define $\kappa_3 := \max\{M, \kappa_1, \kappa_2\}$. Then

$$(\exists n_0)\ n \ge n_0 \implies \left\|\varphi_M\big(\widetilde{\Phi}_{\kappa_3}(u_n) - \widetilde{\Phi}_{\kappa_3}(u)\big)\right\|_{H^{\tilde{s}}} < \varepsilon,$$

which follows from Lemma 3.5.10 and compact embedding $\varphi_M : L^2 \to H^{\tilde{s}}$, concluding the proof. □

3.5.2 Torus case

Denote by

$$B^s_{\delta,\kappa} := \{q \in H^s(\mathbb{T}) : \kappa^{-\frac{1}{2}} \|q\|_{H^s} \le \delta\}.$$

For proving well-posedness, once again, we shall be using the fact that for $|\kappa| \ge 1$ large, map $g_{12} : B^s_{\delta,\kappa} \to H^{1+s}$ is a diffeomorphism, and for that purpose we shall fix $-\frac{1}{2} < s < 0$. Having Theorem 1.0.3 in mind and the choice of ball of initial data B_r in the said theorem, we will consider only κ such that $\kappa \geq \kappa_0 := \left(\frac{r}{\delta}\right)^2$, which ensures that Φ_κ^* (3.43),(3.44) are defined for data $B_r \subset L^2$. Similarly to the line case, we introduce Hamiltonians

$$\widetilde{H}_\kappa^{NLS} := -8\kappa^3 \eta(\kappa^{-1}\|\cdot\|_{H^s}^2)\mathrm{Re}\, A(\kappa,\cdot) + 4\kappa^2 M \tag{3.54}$$

and

$$\widetilde{H}_\kappa^{mKdV} := 16\kappa^4 \eta(\kappa^{-1}\|\cdot\|_{H^s}^2)\mathrm{Im}\, A(\kappa,\cdot) + 4\kappa^2 P, \tag{3.55}$$

where $\eta : [0,\infty) \to [0,1]$ is a smooth cut-off such that $\eta \equiv 1$ on $[0, \frac{\delta^2}{2}]$ and $\eta \equiv 0$ on $[\delta^2, \infty)$. Choice of cut-off function implies that these Hamiltonians are well defined on entire L^2, and more, on H^s as well. Hamiltonians (3.54) and (3.55) generate flows, which we denote by $\widetilde{\Phi}_\kappa^{NLS}$ and $\widetilde{\Phi}_\kappa^{mKdV}$ respectably, given by equations

$$\begin{aligned} iu_t =& 4\kappa^3 \eta(\kappa^{-1}\|u\|_{H^s}^2)(g_{12}(\kappa,u) - g_{12}(-\kappa,u)) \\ &+ 8\kappa^2 \chi_R L_s(u)\eta'(\kappa^{-1}\|u\|_{H^s}^2)\mathrm{Re}\, A(\kappa,u) + 4\kappa^2 u \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} u_t =& 8\kappa^4 \eta(\kappa^{-1}\|u\|_{H^s}^2)(g_{12}(\kappa,u) + g_{12}(-\kappa,u)) \\ &+ 16\kappa^3 L_s(u)\eta'(\kappa^{-1}\|u\|_{H^s}^2)\mathrm{Im}\, A(\kappa,u) + 4\kappa^2 u_x. \end{aligned} \tag{3.57}$$

Here $L_s(f) := \mathcal{F}^{-1}[(1+\xi^2)^s \widehat{f}]$ is the same bounded linear map that gives $-2s$ regularity to initial data, i.e. $L_s : H^t(\mathbb{T}) \to H^{t-2s}(\mathbb{T})$ for every $t \in \mathbb{R}$. We shall only present the proof for mKdV, and will denote $\widetilde{\Phi}_\kappa^{mKdV}$ by $\widetilde{\Phi}_\kappa$ and $\widetilde{H}_\kappa^{mKdV}$ by $\widetilde{H}_\kappa$. Proofs for NLS follow in the same fashion.

Denote by

$$F(\kappa,u) := 8\kappa^4 \eta(\kappa^{-1}\|u\|_{H^s}^2)(g_{12}(\kappa,u) + g_{12}(-\kappa,u)) + 16\kappa^3 L_s(u)\eta'(\kappa^{-1}\|u\|_{H^s}^2)\mathrm{Im}\, A(\kappa,u)$$

a smooth map $F : [\kappa_0,\infty) \times H^s \to L^2$. Smoothness follows directly from regularity of the maps

$$\eta(\|\cdot\|_{H^s}^2) : H^s \to \mathbb{R},\ \ A(\kappa,\cdot) : B_{\delta,\kappa}^s \to \mathbb{C},\ \ g_{12}(\kappa,\cdot) : B_{\delta,\kappa}^s \to H^{s+1},\ \ L_s : H^s \to H^{-s}.$$

That $\widetilde{H}_\kappa : H^s \to \mathbb{R}$ is smooth follows by the same arguments. Denoting by $F_\kappa(\cdot) := F(\kappa,\cdot) : H^s \to L^2$ and $\iota : L^2 \to H^s$ the compact embedding, equation (3.57) can then be rewritten as

$$u_t = F_\kappa(\iota(u)) + 4\kappa^2 u_x,$$

while Duhamel's formula is given by

$$u(t) = L_\kappa(t)(u(0)) + \int_0^t L_\kappa(t-s)F_\kappa(\iota(u(s)))ds.$$

Due to conservation of L^2 norm, one readily sees from the definition of Hamiltonians that they are proper extension of Φ_κ from B_r to entire L^2, for every time $T > 0$.

Lemma 3.5.12. *Fix $\kappa \geq \kappa_0$. Flow $\widetilde{\Phi}_\kappa$ (3.57) is globally well-posed in H^s and locally well-posed in L^2. The flow is Lipschitz on bounded sets, with constant depending only on δ and κ. Lastly, it preserves the L^2 norm.*

Proof. Local well-posedness, both in H^s and L^2, follow from the fact that $\widetilde{H}_\kappa$ is smooth with respect to both norms, hence the functional derivative $\frac{\partial \widetilde{H}_\kappa}{\partial q}$ is Lipschitz on bounded sets of data. Existence follows directly by applying fixed point argument.

Global existence in H^s follows by the same arguments of Lemma 3.4.2. Preservation of L^2 follows by the same computation for the line case (Lemma 3.5.5). □

Lemma 3.5.13. *Flow $\widetilde{\Phi}_\kappa$ (3.57) is continuous with respect to weak topology in L^2.*

Proof. Let $u_n(0,x) \rightharpoonup u(0,x)$ in L^2. Then $u_n(0,x) \to u(0,x)$ in H^s. Local well-posedness of (3.57) in H^s implies that $u_n(t) \to u(t)$ uniformly in time on $[0,T]$ in H^s norm topology. That $u_n(t) \rightharpoonup u(t)$ in the weak sense L^2 then follows from strong convergence in H^s and density argument for functionals being represented by functions of C^∞ regularity. □

Derivative of (3.57) at point u_0 is given by the equation

$$\partial_t v = D[F_\kappa \circ \iota](u)v + 4\kappa^2 v'. \tag{3.58}$$

Since $F_\kappa : H^s \to L^2$ is smooth, $DF_\kappa(u) : H^s \to L^2$ is a bounded linear operator for every $u \in H^s$. Equivalently, by Duhamel's formula

$$v(t) = L_\kappa(t)v_0 + \int_0^t L_\kappa(t-s)\Big[D[F_\kappa \circ \iota](u(s))v(s)\Big]ds, \tag{3.59}$$

where $v(t) := D\widetilde{\Phi}_\kappa[u_0]v(0)$. Derivative $D\widetilde{\Phi}_\kappa$, as a flow, is well-defined in this manner as well-posedness of (3.58) follows from

$$\begin{aligned}
\|v\|_{L^2} &\lesssim_\kappa \|v(0)\|_{L^2} + \int_0^t \|\Big[D[F_\kappa \circ \iota](u(s))\Big]v(s)\|_{L^2} \\
&\lesssim_\kappa \|v(0)\|_{L^2} + t\|D[F_\kappa \circ \iota](u)\|_{B(H^s,L^2)}\|v\|_{H^s} \\
&\lesssim_\kappa \|v(0)\|_{L^2} + t\|D[F_\kappa \circ \iota](u)\|_{B(H^s,L^2)}\|v\|_{L^2},
\end{aligned}$$

and the fact that F_κ is smooth on H^s, hence $\|DF_\kappa(\iota(u))\|_{B(H^s,L^2)} \lesssim \|u_0\|_{L^2}$. These inequalities give existence up to a time depending on the size of data $v(0)$, but we can obviously iterate up to any finite time T by applying the well-defined flow, as the data blows up at most exponentially. Proving the weak continuity of the derivative of flow $\widetilde{\Phi}_\kappa$ concludes the proof of its non-squeezing property by invoking Theorem 1.0.3

Lemma 3.5.14. *The derivative $D\widetilde{\Phi}_\kappa : (L^2, \tau_{weak}) \longrightarrow (B(L^2), \|\cdot\|_{op})$ is continuous.*

Proof. Let $u_n(0) \rightharpoonup u(0)$, and let u_n, u be corresponding Cauchy solutions of (3.57). Let w_n and w be solutions with the same initial data $w(0)$, $\|w_0\|_{L^2} \le \delta_1$ for some constant δ_1.

$$\begin{aligned}\partial_t w_n &= D[F_\kappa \circ \iota](u_n)w_n + 4\kappa^2 w_n',\\ \partial_t w &= D[F_\kappa \circ \iota](u)w + 4\kappa^2 w'.\end{aligned}$$

We want to prove that as $n \to \infty$, $v := w_n - w$ goes to 0. Previous equations give

$$\partial_t v = \Big[D[F_\kappa \circ \iota](u_n) - D[F_\kappa \circ \iota](u)\Big]w_n + D[F_\kappa \circ \iota](u)v + 4\kappa^2 v'.$$

Writing down Duhamel's formula we get

$$v(t) = \int_0^t L_\kappa(t-s)\Big[D[F_\kappa \circ \iota](u_n(s)) - D[F_\kappa \circ \iota](u(s))\Big]w_n(s)ds + \int_0^t L_\kappa(t-s)D[F_\kappa \circ \iota](u)v ds.$$

Scaling with respect to time and correspondingly rescaling the data w_0 up to a fixed constant allows us to move the first summand on RHS to LHS and enables us to directly obtain the bound on norm of difference of derivatives

$$\begin{aligned}\|v(t)\|_{L^2} &\lesssim_T \sup_{t\in[0,T]} \left\|\Big[D[F_\kappa \circ \iota](u_n(s)) - D[F_\kappa \circ \iota](u(s))\Big]w_n(s)\right\|_{L^2}\\ &\lesssim_T \sup_{t\in[0,T]} \|D[F_\kappa \circ \iota](u_n) - D[F_\kappa \circ \iota](u)\|_{B(H^s,L^2)}\|w_n\|_{H^s}\\ &\lesssim_T \sup_{t\in[0,T]} \|D[F_\kappa \circ \iota](u_n) - D[F_\kappa \circ \iota](u)\|_{B(H^s,L^2)}.\end{aligned}$$

Since u_n converges weakly to u in L^2, it converges strongly H^s due to compact embedding $L^2 \hookrightarrow H^s$, and smoothness of F_κ implies that the last term converges to 0 as $n \to \infty$. These bounds serve up to a time dependent on the size of the data v, but iterating the flow for arbitrary finite time gives the desired result for flow at any fixed time T. $\square$

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