Supersymmetric methods in random matrix theory

Dissertation

zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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aus
Bonn

Bonn, Juni 2021

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

Erscheinungsjahr 2021

Tag der mündlichen Prüfung: 06. Oktober 2021

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Summary

Randomness and chaos are key ingredients in the description of nature and are therefore central elements in mathematics and physics. A conducting metal becomes an insulator if there are enough random defects in its structure. This phase transition generated by randomness (also called Anderson transition [And58]) is a central point of study. It is an unproven conjecture that in dimension $d \geqslant 3$ there is such a phase transition between diffusive and isolated states while in $d = 1$ there are proofs that only localization occurs.

This doctoral thesis provides insights into supersymmetric methods relevant for the study of two prominent random matrix models describing disordered materials: random Schrödinger operators and random band matrices.

The main idea in the following is - using the supersymmetric approach - to establish dual representations for the quantity of interest, which in turn can be studied via analytic tools, inspired by statistical mechanics.

Chapter 1 provides an introduction to supersymmetry, summarizing main definitions and results. We present basic properties of the two random matrix models mentioned above. In particular, we introduce the density of states as the main object of study.

Chapter 2 concerns the averaged density of states for a two-dimensional random band matrix ensemble with fixed but large band width W. We rigorously prove smoothness and convergence to Wigner's semicircle law with a precision $W^{-2+\delta}$ in the infinite volume limit. This extends the result of Disertori et al. [DPS02] from three to two dimensions. The proof uses the supersymmetric approach and a cluster expansion. This part is published in [DL17].

The supersymmetric representation introduced in Chapter 2 requires a certain regularity of the probability distribution. Chapter 3 gives a new supersymmetric dual representation by introducing polar coordinates. This can be applied to a large class of random matrix models, where only integrability of the random distribution is required. As an application of this new representation, we consider the linear correlated Lloyd model - a random Schrödinger model with Cauchy distributed random variables. In the case of non-negative correlations, we recover the well-known exact formula for the density of states proved by Lloyd and Simon [Llo69, Sim83] in the first place. Moreover, we examine a toy model with a single small negative correlation and show that the density of states is well-approximated by the exact formula above. This part is published in [DL20].

Acknowledgements

First of all, I would like to express my deep gratitude to my advisor Margherita Disertori for her constant support and guidance during the past years. I am thankful to her for all the time she spent, sharing her great intuition as well as her remarkable accuracy in technical details. Moreover, I thank her for encouraging me to visit various conferences and summer schools to both present our results and get to know new research fields as well as the large community in the field of mathematical physics.

Second, I want to thank Patrik Ferrari for agreeing to review this thesis, as well as the other members of the thesis committee, Martin Rumpf and Bastian Kubis.

I would like to thank the Deutsche Forschungsgemeinschaft (DFG) for the funding under the Collaborative Research Center 1060 "The mathematics of emergent effects" and the support by the Bonn International Graduate School for Mathematics (BIGS).

I am very grateful to Susanne Hilger and Jonas Jansen for many stimulating discussions and to Anna Kraut and Sarah Hertrampf for editorial feedback and proof reading on parts of this thesis.

Last but not least, I thank my family for their continuous support and love; especially Angelika, Hildegard and Wolfgang for taking care of Nora, Sebastian for motivation and encouragement, and finally, Jan and Nora for their love.

Contents

1. Overview

1.1. Introduction

The purpose of physics is the accurate and yet easy description of observations in nature. With increasing complexity of the problem, chaos and randomness come into play.

For example, it can be observed that a conducting metal becomes an insulator if the defects in its atomic structure exceed a certain threshold [And58]. To model this phenomenon, mathematical physicists introduced certain random matrix models: random Schrödinger operators and random band matrices. The phase transition between conductance and localization in dimension $d = 3$ is still an unproven conjecture for these models (except in special lattice models [Kle98]). Localization in the whole spectrum was proven in $d = 1$ and is conjectured to hold also in $d = 2$.

In this thesis, we use supersymmetric methods to study the random matrix models mentioned above. The supersymmetric approach was introduced by Berezin [Ber66]. It combines ordinary and anticommuting variables to provide a dual representation for spectral quantities such as the averaged Green's function (or the average of products of Green's functions). This representation can be seen as a statistical mechanical model (in a loose sense) with supermatrix-valued spins and is a convenient starting point to study spectral properties of our models.

1.1.1. Quantum Mechanics

We start with a short introduction to quantum mechanics (cf. [RS72]). It describes the physical properties at the scale of atoms and subatomic particles (e.g. electrons). Here a particle-wave duality can be observed. A quantum mechanical particle in $\mathbb{R}^d$ or $\mathbb{Z}^d$ is characterized by a wave function $\psi \in \mathcal{H}$, where $\mathcal{H} = L^2(\mathbb{R}^d)$ or $l^2(\mathbb{Z}^d)$ is a Hilbert space. The Schrödinger operator[1] $H : \mathcal{H} \to \mathcal{H}$ is self adjoint and describes the time evolution of the state ψ. It is defined by

$$H = H_0 + V, \tag{1.1.1}$$

where H_0 describes the kinetic energy and the external potential V is a multiplication operator. Solving the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = H\psi$$

[1] Note that H is generally unbounded. In this case H is only defined on a dense linear subset $D(\mathcal{H}) \subset \mathcal{H}$.

we obtain

$$\psi(t) = \exp(-itH)\psi_0,$$

where ψ_0 is the state at $t = 0$ and $\exp(-itH)$ is well-defined via the spectral theorem. The dynamical properties of ψ are closely related to the spectrum of H, e.g. eigenvalues of H are energy levels of the system.

1.1.2. Random Matrices

To be able to study heavy nuclei, Wigner replaced rather complicated potentials by random ones [Wig55] and thus laid the foundation to random matrix models. Today there are many fields in both, mathematics and physics, where random matrices come into place (cf. [Meh04, AW15]).

In this thesis we study two prominent examples modelling conducting properties of disordered materials: random Schrödinger operators and Gaussian random band matrices. We consider only discrete models on $\mathbb{Z}^d$ or on finite sets $\Lambda \subset \mathbb{Z}^d$.

Random Schrödinger operators

Replacing the potential in the Schrödinger operator (1.1.1) by a random one, we obtain a random Schrödinger operator $H(\omega) : l^2(\Lambda) \to l^2(\Lambda)$, $\omega \in \Omega$, given by

$$H(\omega) = -\Delta + \lambda V(\omega),$$

where $-\Delta$ is now the discrete Laplacian on Λ, $\lambda > 0$ is a parameter modelling the disorder and $V(\omega) = \{V_j(\omega)\}_{j \in \Lambda}$ is a multiplication operator with V_j random variables. For V_j independent and identically distributed this is called the Anderson model (introduced by Anderson [And58]) and is broadly studied. Nevertheless there are many open questions. For example the phase transition in $d = 3$, mentioned above, is unproven in general.

Gaussian random band matrices

Another interesting model are Gaussian random band matrices. Again we have a hermitian random operator $H(\omega) : l^2(\Lambda) \to l^2(\Lambda)$, $\omega \in \Omega$. This time each entry is random, but entries far from the diagonal are negligible small. Precisely,

$$H_{ii} \sim \mathcal{N}_{\mathbb{R}}(0, J_{ii}), \qquad H_{ij} \sim \mathcal{N}_{\mathbb{C}}(0, J_{ij}) \quad \text{for } i < j,$$

with $J_{ij} \ll 1$ for $|i - j| > W$, where W is the fixed bandwidth.

Note that both models behave similarly with the relation $\lambda \sim W^{-1}$. For example in $d = 3$, we expect localization for small W and large λ, while for large W and small λ we expect extended states.

1.1.3. Localization and Delocalization

In both cases we study a random hermitian operator $H(\omega)$ on $l^2(\mathbb{Z}^d)$, $\omega \in \Omega$. Its spectral properties are related to the physical behaviour of localization and diffusion. First of all the spectrum of H is deterministic and so are the pure point, singular and absolutely continuous parts if H is ergodic and (essentially) self-adjoint (cf. [Pas80, KS80, KM82] for random Schrödinger and [PF92, Chapter 2.B] for more general cases).

One distinguishes spectral and dynamical localization [Sto11, Chapter 3]. Spectral localization in $I \subset \mathbb{R}$ means that the spectrum in I is only pure point, i.e. $\Sigma(H) \cap I = \Sigma_{pp}(H) \cap I$ almost surely. On the other hand dynamical localization in $I \subset \mathbb{R}$ means there exist constants $C < \infty$ and $c > 0$ such that for all $x, y <\in \mathbb{Z}^d$

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}} |\langle \delta_y, \mathrm{e}^{-itH_\omega}\chi_I(H_\omega)\delta_x\rangle|\right] \leqslant C\,\mathrm{e}^{-c|x-y|},$$

i.e. solutions of time dependent Schrödinger equation are staying localized in space, uniformly in time.

One can prove that dynamical localization implies spectral localization and indeed dynamical localization is the more interesting notion since it implies also absence of quantum transport. Precisely, assuming dynamical localization, one can prove that for any $\psi \in l^2(\mathbb{Z}^d)$ with compact support we have

$$\sup_t \left\| |X|^p\,\mathrm{e}^{itH_\omega}\chi_I(H_\omega)\psi \right\| < \infty$$

for all $p \geqslant 0$ and for a.e. $\omega \in \Omega$. This means all moments of the position operator are finite in time. Hence, we have bounded states.

In the case of delocalization and unbounded states we would observe a presence of quantum transport meaning

$$\left\| |X|^p\,\mathrm{e}^{itH_\omega}\chi_I(H_\omega)\psi \right\| \to \infty \text{ for } t \to \infty.$$

1.1.4. Results and Conjectures

There are several results for localization, in particular for random Schrödinger operators. Diffusion is an open problem, except in special cases like the Bethe lattice. In the following we give a non-exhaustive overview of existing results.

Random Schrödinger operators

In $d = 1$ there is localization in the whole spectrum [GMP77, Car82]. In $d = 2$ this is conjectured to be the same. For $d \geqslant 2$ localization is proved only at large disorder or at the edge oft the spectrum [AM93, FS83]. It is an open problem to prove that there is a phase transition for $d \geqslant 3$.

On tree graphs as the Bethe lattice there are proofs for both localization and delocalization [AM93, Kle98, EHS07, AW15].

Random band matrices

For random band matrices there are even less results [Bou18]. There are a few results on the density of states [CFGK87, CCGI93, DPS02, DLS21, YYY21], but no localization results except in $d = 1$ [Shc14, SS17]. Nevertheless one expects that the random band matrix model behave similar to random Schrödinger with $\lambda \sim W^{-1}$.

1.1.5. Methods

There are various methods to study random matrices. For example models with rotation invariant measures $\mathrm{e}^{\operatorname{Tr} V(H)}\,\mathrm{d}H$, like the Gaussian ensembles, can be solved by orthogonal polynomial tools [Meh04].

Random Schrödinger operators can be studied via fractional moment methods [AM93] and multiscale analysis [FS83].

Another approach is supersymmetry. This technique was pioneered by Berezin [Ber87] and Efetov [Efe99]. For other introductions to the subject we recommend [Weg16, Var04]. The idea is to introduce anticommuting variables χ_j, i.e.

$$\chi_j \chi_k = -\chi_k \chi_j .$$

Note that this implies $\chi_j^2 = 0$. One can define integration over these variables. See Section 1.2 for details. This enables us to rewrite e.g. the Green's function as a supersymmetric integral, i.e. an integral over a supervector $\Phi = (z, \chi)$, where we can evaluate the average more easily. There are various applications of supersymmetry in random matrix theory (cf. [Dis04, Mir00, FM91, LSZ08]).

1.1.6. Structure of this thesis

The remaining thesis is structured as follows. The present introduction proceeds with four further sections. First, we give a basic introduction to the two fields "supermathematics" and "random matrix theory". Then we devote one section each to summarize and sketch the research results of the publications [DL17] and [DL20].

The introduction is followed by two chapters giving the two publications in full detail.

In the following we start with defining Grassmann variables and the supersymmetric formalism in Section 1.2. We also prove some important results for supersymmetric change of variables and Gaussian integrals.

In Section 1.3 we present in more detail two models for conductance in disordered materials: the random Schrödinger operator and random band matrices. In particular, we introduce the density of states for these operators, which is an important spectral quantity.

Section 1.4 deals with random band matrices on a two dimensional lattice and summarizes results on the averaged density of states both in finite and infinite volume. This is the topic of the first paper [DL17].

The second paper [DL20] is summarized in Section 1.5 and deals with an alternative supersymmetric representation, where validity extends to far less regular distributions, such as the Cauchy distribution. It is based on a supersymmetric version of polar coordinates. We also give some applications to the Lloyd model, a random Schrödinger model.

1.2. Supersymmetry

In this section we introduce the concept of supersymmetry. We consider Grassmann variables, i.e. objects which anti-commute. Precisely, two Grassmann variables v and w fulfil

$$v \wedge w = -w \wedge v,$$

where $\wedge$ is the anticommutative wedge product. Note that

$$v^2 = v \wedge v = -v \wedge v = 0.$$

These objects were introduced by Hermann Günther Grassmann in 1844 [Gra44]. They are well adapted to describe fermionic systems but proved also useful in a very different context, e.g. two dimensional lattice models like Ising and dimer model.

Supermathematics or supersymmetry deals with combining Grassmann variables with ordinary real or complex variables. In 1966 Felix A. Berezin developed the concept of supermathematics by introducing a notion of integration for Grassmann variables and the Berezinian, which generalizes the Jacobian [Ber87, Ber66]. Supermathematics also apply in statistical physics by providing dual representations for partition functions and correlations, where saddle point methods can be applied. A special case is the field of random matrices which are studied in this thesis.

We give a short introduction to Grassmann variables and supermathematics. Detailed descriptions on this formalism can be found in [Efe99, Var04, Weg16].

1.2.1. Grassmann calculus

Basic definition

We define the Grassmann algebra and related basic concepts. Moreover, we state basic algebraic properties. For our purpose we consider only finite dimensional vector spaces over real or complex numbers, i.e. $K = \mathbb{R}, \mathbb{C}$ and $V = K^N$.

Definition 1.2.1 (Grassmann algebra)**.** Let V be a finite dimensional vector space over a field K and denote the antisymmetric tensor product by

$$\begin{aligned} \wedge : V \times V &\to V \otimes_{as} V, \\ (v, w) &\mapsto v \wedge w = vw = -wv = -w \wedge v. \end{aligned}$$

The corresponding *Grassmann algebra* $\mathcal{G}$ is defined by

$$\mathcal{G} := \bigoplus_{k \geqslant 0} V^k,$$

where $V^0 = K, V^1 = V$ and $V^k = V^{k-1} \otimes_{as} V$ for $k \geqslant 2$.

Definition 1.2.2 (Even and odd elements)**.** We distinguish the sets of even and odd elements $\mathcal{G} = \mathcal{G}^0 \bigoplus \mathcal{G}^1$, where

$$\mathcal{G}^0 = \bigoplus_{k \geqslant 0} V^{2k}, \qquad \mathcal{G}^1 = \bigoplus_{k \geqslant 0} V^{2k+1}.$$

Elements in $\mathcal{G}^0$ are called even or *Bosonic variables*, elements in $\mathcal{G}^1$ odd, *Grassmann* or *Fermionic variables.*

Proposition 1.2.3. *The following properties hold.*

1. *Let V be N-dimensional, $N \in \mathbb{N}$. Then $V^k = \varnothing$ for $k > N$.*
2. *$\mathcal{G}$ is an associative algebra with unit.*
3. *$\mathcal{G}^0$ is an associative algebra with unit. All $v \in \mathcal{G}^0$ commute with all elements $w \in \mathcal{G}$: $vw = wv$.*
4. *$\mathcal{G}^1$ is not an algebra. All $v, w \in \mathcal{G}^1$ anticommute: $vw = -wv$. Moreover, $v^2 = 0$ for all $v \in \mathcal{G}^1$.*

Proposition 1.2.4. *A Grassmann algebra is a $\mathbb{Z}_2$-*graded algebra*, i.e. each element $v \in \mathcal{G}$ can be written as*

$$v = v^{(0)} + v^{(1)} \qquad \text{with } v^{(0)} \in \mathcal{G}^0, v^{(1)} \in \mathcal{G}^1.$$

Sums of elements in $\mathcal{G}^\sigma$ belong to $\mathcal{G}^\sigma$ while products of elements in $\mathcal{G}^\sigma$ and $\mathcal{G}^{\sigma'}$ belong to $\mathcal{G}^{\sigma+\sigma'}$, where $\sigma, \sigma', \sigma + \sigma' \in \mathbb{Z}_2$.

Generators

To do hands-on calculations with Grassmann elements, it is useful to have a concrete representation. Let $(\chi_1, \ldots, \chi_N)$ be a basis of the vector space V. By definition each element $a \in \mathcal{G}$ can be written as

$$a = \sum_{I \in \mathcal{P}(N)} a_I \chi^I, \tag{1.2.1}$$

where $\mathcal{P}(N)$ is the power set of $\{1, \ldots, N\}$, $a_I \in K$ for all $I \in \mathcal{P}(N)$ and $\chi^I = \prod_{j \in I} \chi_j$ is the *ordered* product for $I \in \mathcal{P}(N)$. Generalizing this representation we define the notion of generators.

Definition 1.2.5 (Generators)**.** A set $(\chi_1, \ldots, \chi_N)$ is called *a family of generators* of $\mathcal{G}$ if all χ_i are odd and for each $a \in \mathcal{G}$ there exists a unique representation of the form (1.2.1).

Remark. A priori, one can use generators which are not necessary odd. Since we want to preserve parity (e.g. later in the supersymmetric applications), we postulate that generators have to be odd.

Example. Each basis of V is a family of generators, but a family of generators is not necessary a basis of V: For $N = 3$ and (e_1, e_2, e_3) basis of V, the following is a family of generators of $\mathcal{G}$

$$\chi_1 = e_1, \quad \chi_2 = e_2, \quad \chi_3 = e_3 + e_1 e_2 e_3.$$

Indeed we can represent the e_j, and hence all $a \in \mathcal{G}$, by the χ_j via

$$e_1 = \chi_1, \quad e_2 = \chi_2, \quad e_3 = \chi_3 - \chi_1 \chi_2 \chi_3.$$

In the following we use $\mathcal{G} = \mathcal{G}[\chi] = \mathcal{G}[\chi_1, \ldots, \chi_N]$ and $a = a(\chi) = a(\chi_1, \ldots, \chi_N)$ to refer to a special set of generators. If we want to emphasize the underlying field K, we write $\mathcal{G} = \mathcal{G}_K = \mathcal{G}_K[\chi]$.

Some more definitions

We give some more definitions.

Definition 1.2.6 (Parity)**.** For *homogenous* elements (i.e. purely even or purely odd variables), we define the *parity operator* π via

$$\pi(a) = \sigma \quad \text{if } a \in \mathcal{G}^\sigma, \quad \sigma \in \{0, 1\}.$$

Definition 1.2.7 (Body and soul)**.** Each element $a \in \mathcal{G}$ can be decomposed in a unique way into

$$a = \operatorname{ord}(a) + \operatorname{nil}(a)$$

with $\operatorname{ord}(a) \in K$ and $\operatorname{nil}(a) \in \bigoplus_{k>0} V^k$. The element $\operatorname{ord}(a) \in K$ is called the ordinary part, domain or *body* of a and $\operatorname{nil}(a)$ the nilpotent part or *soul*.

Note that the nilpotent part is indeed nilpotent: $(\operatorname{nil}(a))^{N+1} = 0$.

Differentiation

Although Grassmann variables, i.e. odd elements in the Grassmann algebra, have neither a domain (indeed $\operatorname{ord} a = 0$) nor a notion of distance, there are notions of differentiating and integrating over Grassmann variables. To define differentiation we use the fact, that elements $a \in \mathcal{G}$ are always linear in a single Grassmann variable and take the corresponding coefficient. Note that we have to distinguish between right- and left-derivatives.

An element $a(\chi) \in \mathcal{G}[\chi]$ of the form (1.2.1) is linear in each generator χ_j

$$a(\chi_j) = a(0) + \chi_j a_l = a(0) + a_r \chi_j, \tag{1.2.2}$$

where

$$\begin{aligned}
a(0) &= \sum_{I\in\mathcal{P}(N):j\notin I} a_I\chi^I \\
a_l &= \sum_{I\in\mathcal{P}(N):j\in I} a_I\sigma_l(I,\{j\})\chi^{I\setminus\{j\}} \qquad a_r = \sum_{I\in\mathcal{P}(N):j\in I} a_I\sigma_r(I,\{j\})\chi^{I\setminus\{j\}},
\end{aligned}$$

with signs $\sigma_l(I,I')$ and $\sigma_r(I,I')$ for $I'\subset I$ defined by

$$\chi^I = \sigma_l(I,I')\chi^{I'}\chi^{I\setminus I'} \quad \chi^I = \sigma_r(I,I')\chi^{I\setminus I'}\chi^{I'}.$$

Definition 1.2.8. The *left resp. right derivative* of a in χ_j is the coefficient right resp. left to χ_j, i.e. we define

$$\frac{\overrightarrow{\partial}}{\partial\chi_j}a(\chi) := a_l \qquad a(\chi)\frac{\overleftarrow{\partial}}{\partial\chi_j} := a_r.$$

Proposition 1.2.9. *The following properties hold.*

1. *For odd elements, right and left derivative coincide and we can write*

$$\frac{\overrightarrow{\partial}}{\partial\chi_j}a = a\frac{\overleftarrow{\partial}}{\partial\chi_j} = \frac{\partial a}{\partial\chi_j}.$$

2. *In general, right and left derivative are different. Decomposing $a = a^{(0)} + a^{(1)}$ into even and odd part, we have*

$$\frac{\overrightarrow{\partial}}{\partial\chi_j}a = -a^{(0)}\frac{\overleftarrow{\partial}}{\partial\chi_j} + a^{(1)}\frac{\overleftarrow{\partial}}{\partial\chi_j}.$$

3. *The product rule for homogeneous a, b with grade $\pi(a), \pi(b) \in \mathbb{Z}_2$ reads*

$$\begin{aligned}
\frac{\overrightarrow{\partial}}{\partial\chi_j}(ab) &= \left(\frac{\overrightarrow{\partial}}{\partial\chi_j}a\right)b + (-1)^{\pi(a)}a\left(\frac{\overrightarrow{\partial}}{\partial\chi_j}b\right), \\
(ab)\frac{\overleftarrow{\partial}}{\partial\chi_j} &= a\left(b\frac{\overleftarrow{\partial}}{\partial\chi_j}\right) + (-1)^{\pi(b)}\left(a\frac{\overleftarrow{\partial}}{\partial\chi_j}\right)b.
\end{aligned}$$

Note that the first line holds also for a homogeneous and b arbitrary and the second for b homogeneous and a arbitrary.

4. *For two families of generators χ and $\eta = \eta(\chi)$, the chain rule reads*

$$\begin{aligned}
\frac{\overrightarrow{\partial}}{\partial\chi_j}a(\eta(\chi)) &= \sum_k \left(\frac{\overrightarrow{\partial}}{\partial\chi_j}\eta_k(\chi)\right)\left[\frac{\overrightarrow{\partial}}{\partial\eta_k}a(\eta)\right](\chi), \\
a(\eta(\chi))\frac{\overleftarrow{\partial}}{\partial\chi_j} &= \sum_k \left[a(\eta)\frac{\overleftarrow{\partial}}{\partial\eta_k}\right](\chi)\left(\eta_k(\chi)\frac{\overleftarrow{\partial}}{\partial\chi_j}\right).
\end{aligned} \tag{1.2.3}$$

5. *Note that multiple derivatives anticommute*

$$\frac{\overrightarrow{\partial}}{\partial\chi_j}\frac{\overrightarrow{\partial}}{\partial\chi_k}a = -\frac{\overrightarrow{\partial}}{\partial\chi_k}\frac{\overrightarrow{\partial}}{\partial\chi_j}a \qquad a\frac{\overleftarrow{\partial}}{\partial\chi_j}\frac{\overleftarrow{\partial}}{\partial\chi_k} = -a\frac{\overleftarrow{\partial}}{\partial\chi_k}\frac{\overleftarrow{\partial}}{\partial\chi_j}.$$

Proof. We proof the product and chain rule. The remaining properties follow directly.

3. We prove the first identity. The second one works analogously. Writing $a = a_0 + \chi_j a_1$ and $b = b_0 + \chi_j b_1$, we obtain for the right-hand side

$$\tfrac{\overrightarrow{\partial}}{\partial\chi_j}(ab) = \tfrac{\overrightarrow{\partial}}{\partial\chi_j}(a_0 + \chi_j a_1)(b_0 + \chi_j b_1) = a_1 b_0 + (-1)^{\pi(a_0)} a_0 b_1.$$

While the left-hand side gives

$$\begin{aligned}(LHS) &= a_1(b_0 + \chi_j b_1) + (-1)^{\pi(a)}(a_0 + \chi_j a_1)b_1 \\ &= a_1 b_0 + (-1)^{\pi(a)} a_0 b_1 + \chi_j[(-1)^{\pi(a_1)} a_1 b_1 + (-1)^{\pi(a)} a_1 b_1],\end{aligned}$$

which coincides with the right-hand side since $\pi(a) = \pi(a_0) = \pi(a_1) + 1$.

4. For the chain rule we prove again the first identity. Note that generators are odd by definition. We fix the index j and N generators χ and $\eta(\chi)$ and write $\eta_k(\chi) = a_k + \chi_j b_k$, where a_k is odd, b_k is even and both depend on $\{\chi_i\}_{i\neq j}$. Since a is a polynomial in η and the chain rule is clearly linear, it remains to prove it for monomials $a(\eta) = \prod_{k=1}^n \eta_k$. We use induction. For $n = 1$ we have

$$\tfrac{\overrightarrow{\partial}}{\partial\chi_j}\eta_1(\chi) = \tfrac{\overrightarrow{\partial}}{\partial\chi_j}(a_1 + \chi_j b_1) = b_1 = \sum_k b_k\delta_{k1} = \sum_k \left(\tfrac{\overrightarrow{\partial}}{\partial\chi_j}\eta_k\right)\left[\tfrac{\overrightarrow{\partial}}{\partial\eta_k}\eta_1\right](\chi).$$

Let our identity hold for n. Then by product rule

$$\begin{aligned}\tfrac{\overrightarrow{\partial}}{\partial\chi_j}a(\eta(\chi)) &= \tfrac{\overrightarrow{\partial}}{\partial\chi_j}\prod_{k=1}^{n+1}\eta_k(\chi) \\ &= \tfrac{\overrightarrow{\partial}}{\partial\chi_j}\eta_1(\chi) \times \prod_{k=2}^{n+1}\eta_k(\chi) - \eta_1(\chi)\tfrac{\overrightarrow{\partial}}{\partial\chi_j}\prod_{k=2}^{n+1}\eta_k(\chi) \\ &= b_1\prod_{k=2}^{n+1}\eta_k(\chi) - \eta_1(\chi)\sum_{l=1}^{N}\tfrac{\overrightarrow{\partial}}{\partial\chi_j}\eta_l(\chi)\tfrac{\overrightarrow{\partial}}{\partial\eta_l}\prod_{k=2}^{n+1}\eta_k(\chi) \\ &= b_1\prod_{k=2}^{n+1}\eta_k(\chi) - \eta_1(\chi)\sum_{l=2}^{n+1} b_l(-1)^l \prod_{k=2,k\neq l}^{n+1}\eta_k(\chi) \\ &= \sum_{l=1}^{n+1} b_l(-1)^{l+1}\prod_{k=1,k\neq l}^{n+1}\eta_k(\chi) \\ &= \sum_k\left(\tfrac{\overrightarrow{\partial}}{\partial\chi_j}\eta_k\right)\tfrac{\overrightarrow{\partial}}{\partial\eta_k}\prod_{k=1}^{n+1}\eta_k(\chi) = \sum_k\left(\tfrac{\overrightarrow{\partial}}{\partial\chi_j}\eta_k\right)\tfrac{\overrightarrow{\partial}}{\partial\eta_k}a(\eta(\chi)).\end{aligned}$$

$\square$

Integration

To motivate the introduction of a notion of integration over a Grassmann variable, we require three conditions for an integral $\int \mathrm{d}\chi_i$:

1. Linearity: $\int \mathrm{d}\chi_j(a(\chi)+\lambda b(\chi)) = \int \mathrm{d}\chi_j a(\chi) + \lambda \int \mathrm{d}\chi_j g$ for all $\lambda \in K, a(\chi), b(\chi) \in \mathcal{G}$.
2. The result is independent of the integration variable: $\frac{\overrightarrow{\partial}}{\partial\chi_j}\left[\int \mathrm{d}\chi_j a(\chi)\right] = 0$ for all $a(\chi) \in \mathcal{G}$.
3. Integrating the (left) derivative yields zero: $\int \mathrm{d}\chi_j[\frac{\overrightarrow{\partial}}{\partial\chi_j}a(\chi)] = 0$ for all $a(\chi) \in \mathcal{G}$.

Using the decomposition in Eq. (1.2.2), the last conditions reads

$$0 = \int \mathrm{d}\chi_j \frac{\overrightarrow{\partial}}{\partial\chi_j} a(\chi_j) = \int \mathrm{d}\chi_j a_l = \left[\int \mathrm{d}\chi_j \ 1\right] a_l,$$

hence $\int \mathrm{d}\chi_j 1 = 0$. By linearity and the fact that both $a(0)$ and a_l are independent of χ_j, we can write

$$\int \mathrm{d}\chi_j a(\chi_j) = \left[\int \mathrm{d}\chi_j 1\right] a(0) + \left[\int \mathrm{d}\chi_j\chi_j\right] a_l = 0 + \left[\int \mathrm{d}\chi_j\chi_j\right] a_l,$$

It remains to define $\int \mathrm{d}\chi_i \ \chi_i$ independent of χ_i as some arbitrary (non zero) constant. We choose

$$\int \mathrm{d}\chi_j \ \chi_j = 1.$$

Note that sometimes it is set to $\frac{1}{\sqrt{2\pi}}$. Consequently, we have

$$\int \mathrm{d}\chi_j a(\chi_j) = \int \mathrm{d}\chi_j a(0) + \int \mathrm{d}\chi_j \chi_j a_l = a_l = \frac{\overrightarrow{\partial}}{\partial\chi_j} a(\chi_j).$$

Hence, integration is equivalent to differentiation - sometimes up to a constant. We define integration over multiple Grassmann variables as the ordered product of the left derivatives.

Definition 1.2.10. Let $a(\chi) \in \mathcal{G}[\chi]$. The integration over a subset of generators $\chi_j, j \in I$ is defined by

$$\int \mathrm{d}\chi^I a(\chi) := \left(\frac{\overrightarrow{\partial}}{\partial\chi}\right)^I a(\chi) = \sum_{J\in\mathcal{P}(N):I\subset J} a_J \ \sigma_l(J,I) \ \chi^{J\setminus I},$$

where $\mathrm{d}\chi^I = \prod_{j\in I} \mathrm{d}\chi_j$ is again an ordered product.

Since $\int \mathrm{d}\chi_i = \frac{\overrightarrow{\partial}}{\partial\chi_i}$, the $\mathrm{d}\chi_i$ are anticommutative objects, too and e.g.

$$\int \mathrm{d}\chi_i \mathrm{d}\chi_j \ \chi_i\chi_j = -\int \mathrm{d}\chi_i \left[\int \mathrm{d}\chi_j\chi_j\right] \chi_i = \int \mathrm{d}\chi_i \ \chi_i = -1.$$

Functions

If $f : \mathbb{R} \to \mathbb{R}$ is smooth, we can define an extension of f to $\mathcal{G}^0$, which we denote also by $f : \mathcal{G}^0 \to \mathcal{G}^0$ via its Taylor expansion

$$f(a) := \sum_{j \geqslant 0} f^{(j)}(\mathrm{ord}\,(a)) \frac{\mathrm{nil}\,(a)^j}{j!},$$

where $f^{(j)}$ is the j-th derivative of $f : \mathbb{R} \to \mathbb{R}$. Note that the sum is finite since $\mathrm{nil}\,(a)$ is nilpotent. In particular, we can extend the exponential function to $\exp : \mathcal{G}^0 \to \mathcal{G}^0$. This new exponential satisfies $\exp(a + b) = \exp(a)\exp(b)$ for $a, b \in \mathcal{G}^0$ as the ordinary one.

Gaussian integrals

We give the real and complex versions of Gaussian integrals and then the Fermionic analogue.

Let $A \in \mathbb{R}^{N \times N}$ be symmetric and positive definite. Then (see e.g. [AS09])

$$\int_{\mathbb{R}^N} \prod_k \mathrm{d}x_k \, \mathrm{e}^{-\frac{1}{2}\sum_{ij} x_i A_{ij} x_j} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}.$$

The Grassmann analogue gives the Pfaffian (see e.g. [Weg16, Chapter 5.1]). More interesting for us is the analogue of the following complex Gaussian integral. Let $\mathrm{d}\bar{z}\,\mathrm{d}z = 2\,\mathrm{d}(\mathrm{Re}\,z)\,\mathrm{d}(\mathrm{Im}\,z)$ and $A \in \mathbb{C}^{N \times N}$ with positive definite Hermitian part $(A + A^*)/2$. Then (see e.g. [AS09])

$$\int_{\mathbb{C}^n} \prod_k \mathrm{d}\bar{z}_k \, \mathrm{d}z_k \, \mathrm{e}^{-\sum_{ij} \bar{z}_i A_{ij} z_j} = \frac{(2\pi)^N}{\det A}.$$

To find the analogue of this second identity, we consider a Grassmann algebra with $2N$ generators $(\chi_1, \ldots, \chi_N, \bar{\chi}_1, \ldots, \bar{\chi}_N)$. Note that the bar notation has no deeper reason than doubling the number of generators in order to mimic the complex case.

Proposition 1.2.11. *Let $A \in \mathbb{C}^{N \times N}$ be arbitrary (we need no positivity). Then*

$$\int \prod_k \mathrm{d}\bar{\chi}_k \, \mathrm{d}\chi_k \, \mathrm{e}^{-\sum_{ij} \bar{\chi}_i A_{ij} \chi_j} = \det A,$$

where the exponential is defined by its Taylor expansion.

Proof. We expand the exponential

$$\mathrm{e}^{-\sum_{ij} \bar{\chi}_i A_{ij} \chi_j} = \sum_{n \geqslant 0} \frac{1}{n!} \left(-\sum_{ij} \bar{\chi}_i A_{ij} \chi_j \right)^n = \sum_{n=0}^{N} \frac{1}{n!} \left(-\sum_{ij} \bar{\chi}_i A_{ij} \chi_j \right)^n$$

and note that the sum is finite, precisely $n \leqslant N$. We integrate over exactly $2N$ variables. Hence, only terms with $2N$ generators contribute to the integral, i.e. only the summand for $n = N$. This follows since there are $2N$ integration variables and $\chi_i A_{ij} \chi_j$ has degree 2.

Let $\mathfrak{P}(N)$ be the set of permutations of $\{1, \ldots, N\}$ and $\sigma(p)$ the sign of p. Then we can reorganize this term as follows

$$\begin{aligned}
\frac{1}{N!}\left(-\sum_{ij}\bar{\chi}_i A_{ij}\chi_j\right)^N =&\frac{(-1)^N}{N!}\sum_{\substack{i_1,\ldots,i_N\\ j_1,\ldots,j_N}}\prod_{k=1}^N \bar{\chi}_{i_k} A_{i_k j_k}\chi_{j_k}\\
=&(-1)^N \sum_{p,q\in\mathfrak{P}(N)}\prod_{k=1}^N \bar{\chi}_{q(k)} A_{q(k)(p(k))}\chi_{p(k)}\\
=&(-1)^N \sum_{p\in\mathfrak{P}(N)}\prod_{k=1}^N \bar{\chi}_k A_{kp(k)}\chi_{p(k)}\\
=&(-1)^N \prod_{k=1}^N \bar{\chi}_k\chi_k \sum_{p\in\mathfrak{P}(N)}\sigma(p)\prod_{k=1}^N A_{kp(k)}\\
=&(-1)^N \prod_{k=1}^N \bar{\chi}_k\chi_k \det A,
\end{aligned}$$

where we used that only terms with disjoint i_k and j_k give a non-zero contribution and hence both $(i_1, \ldots, i_N)$ and $(j_1, \ldots, j_N)$ are some permutations of $\{1, \ldots, N\}$. Therefore, we rewrite the sum over the i's and j's as sum over all permutations of N indices. The sum over the i's cancels $1/N!$. Reordering the Grassmann variables give the sign of the permutation and we end up with the determinant. The sign $(-1)^N$ vanishes by integrating over the Grassmann variables

$$\int\prod_k \mathrm{d}\bar{\chi}_k\,\mathrm{d}\chi_k(-1)^N\prod_j \bar{\chi}_k\chi_k = 1.$$

□

Corollary 1.2.12. *Let $A \in \mathbb{C}^{N\times N}$. Then*

$$\int\prod_k \mathrm{d}\bar{\chi}_k\,\mathrm{d}\chi_k\,\mathrm{e}^{-\sum_{ij}\bar{\chi}_i A_{ij}\chi_j}\chi_l\bar{\chi}_m = (-1)^{m+l}A^{(ml)} = A^{-1}_{lm}\det A,$$

where $A^{(ml)}$ is the minor of A, i.e. the determinant of the matrix with line m and row l cancelled. The second identity holds if A is invertible.

Proof. Expanding the exponential, we note that

$$\chi_l\bar{\chi}_m\exp(-\sum_{ij}\bar{\chi}_i A_{ij}\chi_j) = \chi_l\bar{\chi}_m\exp(-\sum_{ij,i\neq m,j\neq l}\bar{\chi}_i A_{ij}\chi_j).$$

Then

$$
\begin{aligned}
&\int \prod_k \mathrm{d}\bar{\chi}_k \,\mathrm{d}\chi_k \,\mathrm{e}^{-\sum_{ij,i\neq k,j\neq l} \bar{\chi}_i A_{ij} \chi_j} \chi_l \bar{\chi}_m \\
&= -\int \mathrm{d}\bar{\chi}_m \,\mathrm{d}\chi_l \;\chi_l \bar{\chi}_m \int \prod_{k\neq l,m} \mathrm{d}\bar{\chi}_k \,\mathrm{d}\chi_k \,\mathrm{d}\bar{\chi}_l \,\mathrm{d}\chi_m \,\mathrm{e}^{-\sum_{ij,i\neq m,j\neq l} \bar{\chi}_i A_{ij} \chi_j}.
\end{aligned}
$$

Now we bring the integration variables into the right order. Wlog $l < m$ and

$$
\begin{aligned}
&-\int \prod_{k\neq l,m} \mathrm{d}\bar{\chi}_k \,\mathrm{d}\chi_k \,\mathrm{d}\bar{\chi}_l \,\mathrm{d}\chi_m \\
=&(-1)^{m+l} \int \mathrm{d}\bar{\chi}_1 \,\mathrm{d}\chi_1 \cdots \mathrm{d}\bar{\chi}_{l-1} \,\mathrm{d}\chi_{l-1} \;\; \mathrm{d}\bar{\chi}_l \,\mathrm{d}\chi_{l+1} \cdots \mathrm{d}\bar{\chi}_{m-1} \,\mathrm{d}\chi_m \\
&\qquad \mathrm{d}\bar{\chi}_{m+1} \,\mathrm{d}\chi_{m+1} \cdots \mathrm{d}\bar{\chi}_N \,\mathrm{d}\chi_N.
\end{aligned}
$$

The result now follows from Proposition 1.2.11. The second identity comes from linear algebra. □

Change of variables

Similar to the results for Gaussian formulas, the analogue of the Jacobian in a change of variables is the *inverse* determinant of the derivatives.

Theorem 1.2.13. *Let $(\chi_1, \ldots, \chi_N)$ and $(\eta_1, \ldots, \eta_N)$ be two sets of generators with $\eta = \eta(\chi)$. For all $a \in \mathcal{G}[\eta]$ it holds that*

$$
\int \mathrm{d}\eta \; a(\eta) = \int \mathrm{d}\chi \; a(\eta(\chi)) (\det J)^{-1}, \quad J = \left(\frac{\partial \eta_i}{\partial \chi_j} \right)_{ij}.
$$

Note that $\eta_j = \eta_j(\chi)$ are odd thus left and right derivatives coincide and $\frac{\partial \eta_i}{\partial \chi_j} \in \mathcal{G}^0$ such that $\det J$ is well-defined. Moreover $\det J \neq 0$ since both sets $(\eta_1, \ldots, \eta_N)$ and $(\chi_1, \ldots, \chi_N)$ are generators.

Proof. We prove the result for a linear transformation $\eta_j = \sum_k J_{jk} \chi_k$. For the general result cf. [Weg16, Chapter 5.2]. We start on the right-hand side and insert the definition

$$
\begin{aligned}
I = \int \mathrm{d}\chi \; a(\eta(\chi)) (\det J)^{-1} &= \left[\prod_j \overrightarrow{\frac{\partial}{\partial \chi_j}} \right] a(\eta(\chi)) (\det J)^{-1} \\
&= \sum_{k_1,\ldots,k_N} \left[\prod_j \frac{\partial \eta_{k_j}}{\partial \chi_j} \overrightarrow{\frac{\partial}{\partial \eta_{k_j}}} \right] a(\eta(\chi)) (\det J)^{-1},
\end{aligned}
$$

where we used the chain rule (1.2.3). Note that $\partial\eta_k/\partial\chi_j$ is even, commutes and depends only on χ. Moreover, the k_j are disjoint since $(\overrightarrow{\frac{\partial}{\partial\eta_k}})^2 = 0$. Then

$$
\begin{aligned}
I &= \sum_{k_1,\ldots,k_N} \left[\prod_j \overrightarrow{\frac{\partial}{\partial\eta_{k_j}}}\right]\left[\prod_l \frac{\partial\eta_{k_l}}{\partial\chi_l}\right] a(\eta(\chi))(\det J)^{-1} \\
&= \sum_{p\in\mathfrak{P}(N)} \prod_j \overrightarrow{\frac{\partial}{\partial\eta_{p(j)}}} \left[\prod_l \frac{\partial\eta_{p(l)}}{\partial\chi_l}\right] a(\eta(\chi))(\det J)^{-1} \\
&= \prod_j \overrightarrow{\frac{\partial}{\partial\eta_j}} \left[\sum_{p\in\mathfrak{P}(N)} \operatorname{sgn}(p) \prod_l \frac{\partial\eta_{p(l)}}{\partial\chi_l}\right] a(\eta(\chi))(\det J)^{-1} \\
&= \prod_j \overrightarrow{\frac{\partial}{\partial\eta_j}}\, a(\eta(\chi)) = \int \mathrm{d}\eta\; a(\eta).
\end{aligned}
$$

Note that ordering the derivates in η_j gives the sign of the permutation p. Hence, we obtain the Leibniz formula of the determinant. $\square$

1.2.2. Supermathematics

Supervectors and supermatrices

We want to combine and mix even and odd variables in the following. The prefix "super" stands always for objects containing even and odd elements. We start with vectors and matrices.

Definition 1.2.14 (Supervectors and Supermatrices)**.** Let $\mathcal{G}$ be a Grassmann algebra. Let $p, q \in \mathbb{N}$. A *supervector* Φ is a collection of p Bosonic variables $x = (x_i)_{i=1}^p \in (\mathcal{G}^0)^p$ and q Fermionic variables $\chi = (\chi_j)_{j=1}^q \in (\mathcal{G}^1)^q$

$$\Phi = \begin{pmatrix} x \\ \chi \end{pmatrix}. \tag{1.2.4}$$

A *supermatrix* M is a linear transformation between supervectors, i.e.

$$\Psi = M\Phi, \qquad M = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}, \tag{1.2.5}$$

where a, b are $p \times p$ and $q \times q$ matrices in $\mathcal{G}^0$ and α, β are $p \times q$ and $q \times p$ matrices in $\mathcal{G}^1$.

Superdeterminant and supertrace

We want to introduce a superdeterminant and a supertrace having similar properties as their ordinary counterparts. Therefore we postulate:

1. The supertrace $\operatorname{Str} M$ depends only on the diagonal blocks a and b.

2. $\operatorname{Sdet}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \dfrac{\det a}{\det b}$,

3. $\operatorname{Sdet}(MM') = (\operatorname{Sdet} M)(\operatorname{Sdet} M')$,

4. $\ln \operatorname{Sdet} M = \operatorname{Str} \ln M$, where
$$\ln M := \begin{pmatrix} \ln a & 0 \\ 0 & \ln b \end{pmatrix} - \sum_{n \geqslant 1} \frac{(-1)^n}{n} \begin{pmatrix} 0 & a^{-1}\alpha \\ b^{-1}\beta & 0 \end{pmatrix}^n,$$
which is well-defined since the sum is finite.

The first postulate is expectable by the properties of the ordinary trace. The second one is motivated from the fact that the superdeterminant should replace the Jacobian in a change of variables formula. Remember that we found the inverse of the determinant for the fermionic transformation (Theorem 1.2.13). The other two are basic properties.

Lemma 1.2.15. *If 1.-4. hold, then*
$$\operatorname{Str}\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} = \operatorname{Tr} a - \operatorname{Tr} b.$$
Moreover if additionally $\operatorname{ord}(a)$ *and* $\operatorname{ord}(b)$ *are invertible,*
$$\operatorname{Sdet}\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} = \frac{\det(a - \alpha b^{-1}\beta)}{\det b} = \frac{\det a}{\det(b - \beta a^{-1}\alpha)}.$$

Proof. For the supertrace, we use the properties 1., 2. and 4.
$$\begin{aligned} \operatorname{Str}\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} &\overset{1.}{=} \operatorname{Str}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \overset{4.}{=} \ln \operatorname{Sdet} \exp\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \ln \operatorname{Sdet}\begin{pmatrix} \mathrm{e}^a & 0 \\ 0 & \mathrm{e}^b \end{pmatrix} \\ &\overset{2.}{=} \ln \frac{\det \exp(a)}{\det \exp(b)} = \ln \det \exp(a) - \ln \det \exp(b) = \operatorname{Tr} a - \operatorname{Tr} b. \end{aligned}$$
For the superdeterminant we write M as
$$M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}(1 + X), \quad \text{where } X = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^{-1}\alpha \\ b^{-1}\beta & 0 \end{pmatrix}, \tag{1.2.6}$$
and compute with 4.
$$\begin{aligned} \ln \operatorname{Sdet}(1 + X) = \operatorname{Str} \ln(1 + X) &= -\sum_{k \geqslant 1} \frac{(-1)^k}{k} \operatorname{Str} X^k \\ &= -\sum_{k \geqslant 1} \frac{(-1)^{2k}}{2k} [\operatorname{Tr}(AB)^k - \operatorname{Tr}(BA)^k] \\ &= \begin{cases} -\sum_{k \geqslant 1} \frac{1}{k} \operatorname{Tr}(AB)^k & = \ln \det(1 - AB) \\ \sum_{k \geqslant 1} \frac{1}{k} \operatorname{Tr}(BA)^k & = \ln \det(1 - BA) \end{cases} \end{aligned}$$
where we used that the sum is finite, $\operatorname{Tr}(\beta\alpha) = -\operatorname{Tr}(\alpha\beta)$ for odd α and β and
$$X^{2k} = \begin{pmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{pmatrix}, \quad X^{2k+1} = \begin{pmatrix} 0 & (AB)^k A \\ (BA)^k B & 0 \end{pmatrix}.$$
By multiplicity and 2. the formulas for Sdet follow. $\square$

Inverse of a supermatrix

The inverse of a supermatrix is similar to the inverse of a block matrix with some modifications.

Proposition 1.2.16. *A supermatrix M is invertible, if* $\operatorname{ord}(a)$ *and* $\operatorname{ord}(b)$ *are invertible and is given by*

$$M^{-1} = \begin{pmatrix} (a-\alpha b^{-1}\beta)^{-1} & -(a-\alpha b^{-1}\beta)^{-1}\alpha b^{-1} \\ -b^{-1}\beta(a-\alpha b^{-1}\beta)^{-1} & b^{-1}+b^{-1}\beta(a-\alpha b^{-1}\beta)^{-1}\alpha b^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} a^{-1}+a^{-1}\alpha(b-\beta a^{-1}\alpha)^{-1}\beta a^{-1} & -a^{-1}\alpha(b-\beta a^{-1}\alpha)^{-1} \\ -(b-\beta a^{-1}\alpha)^{-1}\beta a^{-1} & (b-\beta a^{-1}\alpha)^{-1} \end{pmatrix}$$

Proof. We use the decomposition (1.2.6) and compute $(1+X)^{-1}$ via the Taylor expansion $(1-x)^{-1} = \sum_{n\geqslant 0} x^n$:

$$(1+X)^{-1} = \sum_{n\geqslant 0}(-X)^n = \sum_{n\geqslant 0}\begin{pmatrix} (AB)^n & 0 \\ 0 & (BA)^n \end{pmatrix} - \sum_{n\geqslant 0}\begin{pmatrix} 0 & (AB)^n A \\ (BA)^n B & 0 \end{pmatrix}$$
$$= \begin{pmatrix} (1-AB)^{-1} & -(1-AB)^{-1}A \\ -(1-BA)^{-1}B & (1-BA)^{-1} \end{pmatrix}.$$

Multiplication with the inverse diagonal yields the result. □

Grassmann valued functions

Another way to combine even and odd elements are $\mathcal{G}$-valued functions on a domain $U \subset \mathbb{R}^p$. These objects can be integrated by a "superintegral".

Definition 1.2.17. Let $U \subset \mathbb{R}^p$ open, $\mathcal{G} = \mathcal{G}(\chi_1, \ldots, \chi_q)$ a Grassmann algebra. The *algebra of smooth $\mathcal{G}$-valued functions* on U is defined by

$$\mathcal{A}_{p,q}(U,\chi) := \left\{ f = f(x,\chi) = \sum_{I\in\mathcal{P}(q)} f_I(x)\chi^I : f_I \in C^\infty(U) \right\}.$$

We call $y_i(x,\chi)$, $\eta_j(x,\chi)$, for $i = 1,\ldots p$, $j = 1,\ldots,q$ *generators* of $\mathcal{A}_{p,q}(U,\chi)$ if $\pi(y_i) = 0$, $\pi(\eta_j) = 1$ and

1. $\{(\operatorname{ord}(y_1(x,0)), \ldots, \operatorname{ord}(y_p(x,0))), x \in U\}$ is a domain in $\mathbb{R}^p$,
2. we can write all $f \in \mathcal{A}_{p,q}(U,\chi)$ as $f = \sum_I f_I(y)\eta^I$.

Note that (x,χ) are generators for $\mathcal{A}_{p,q}(U,\chi)$.

Change of variables and Berezinian

A change of variables in a superintegral is a parity preserving transformation between systems of generators of $\mathcal{A}_{p,q}(U,\chi)$. The generalized Jacobian for such a coordinate transformation is called Berezinian. It is the superdeterminant of the partial derivatives. Note that the following formula holds only for functions with compact support, i.e. functions $f \in \mathcal{A}_{p,q}(U,\chi)$ such that $f_I \in C_c^\infty(U)$ for all $I \in \mathcal{P}(q)$.

Theorem 1.2.18. *Let $U \subset \mathbb{R}^p$ open, x,χ and $y(x,\chi)$, $\eta(x,\chi)$ two sets of generators of $\mathcal{A}_{p,q}(U,\chi)$. Denote the isomorphism between the generators by*

$$\psi : (x,\chi) \mapsto (y(x,\chi),\eta(x,\chi))$$

and $V = \operatorname{ord}(\psi(U)) = \{(\operatorname{ord}(y_1(x,0)),\ldots,\operatorname{ord}(y_p(x,0))), x \in U\} \subset \mathbb{R}^p$. Then for all $f \in \mathcal{A}_{p,q}(V,\eta)$ with compact support, we have

$$\int_V \mathrm{d}y\,\mathrm{d}\eta\ f(y,\eta) = \int_U \mathrm{d}x\,\mathrm{d}\chi\ f\circ\psi(x,\chi)\ \operatorname{Sdet}(J\psi), \tag{1.2.7}$$

where $\operatorname{Sdet}(J\psi)$ *is called the* Berezinian *defined by*

$$J\psi = \begin{pmatrix} \frac{\partial y}{\partial x} & y\frac{\overleftarrow{\partial}}{\partial\chi} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial\chi}\end{pmatrix}, \qquad \operatorname{Sdet}\begin{pmatrix} a & \alpha \\ \beta & b\end{pmatrix} = \det(a-\alpha b^{-1}\beta)\det b^{-1}.$$

Integration over even elements x and y means integration over the body $\operatorname{ord}(x)$ *and* $\operatorname{ord}(y)$ *in the corresponding regions U and V.*

The above result holds also for $U = \mathbb{R}^p$ and $f \in C(\mathbb{R}^p)$ with sufficient decay at ∞.

Proof. The following proof is based on [Dis11, Chapter 6.4.2]. We parametrize the transformation of generators with a variable $t \in [0,T]$ such that $(z(t),\gamma(t))$ is a set of generators for all $t \in [0,T]$ and

$$\begin{aligned} &z(0) = x && \gamma(0) = \chi && U(0) = V \\ &z(T) = y(x,\chi) && \gamma(T) = \eta(x,\chi) && U(T) = U. \end{aligned}$$

Then we study

$$g(t) = \int_{U(t)} \mathrm{d}x\,\mathrm{d}\chi\ \operatorname{Sdet} J(t,0)\ f(z(t)(x,\chi),\gamma(t)(x,\chi)),$$

where

$$J(t_2,t_1) = \begin{pmatrix} \frac{\partial z(t_2)}{\partial z(t_1)} & z(t_2)\frac{\overleftarrow{\partial}}{\partial\gamma(t_1)} \\ \frac{\partial\gamma(t_2)}{\partial z(t_1)} & \frac{\partial\gamma(t_2)}{\partial\gamma(t_1)}\end{pmatrix}.$$

Note that $g(0)$ gives the left-hand side and $g(T)$ the right-hand side of Eq. (1.2.7). To show that g is constant, we calculate the derivative

$$
\begin{aligned}
g'(t) =& \frac{\mathrm{d}}{\mathrm{d}s} g(t+s)|_{s=0} \\
=& \int_{U(t)} \mathrm{d}x\,\mathrm{d}\chi \ \mathrm{Sdet}\, J(t,0) \frac{\mathrm{d}}{\mathrm{d}s} \left[\mathrm{Sdet}\, J(t+s,t) f(z(t+s), \gamma(t+s)) \right]_{s=0},
\end{aligned}
$$

where we used

$$
J(t+s,0) = J(t+s,t)J(t,0).
$$

When the derivative falls on the domain, we obtain a boundary integral which vanishes since f has compact support. Differentiating the superdeterminant, we note that the same formula as in the ordinary case applies and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} \left[\mathrm{Sdet}\, J(t+s,t) \right]_{s=0} =& \mathrm{Sdet}\, J(t,t) \left[\mathrm{Str}\left(J(t+s,t)^{-1} \frac{\mathrm{d}}{\mathrm{d}s} J(t+s,t) \right) \right]_{s=0} \\
=& \mathrm{Str} \left[\frac{\mathrm{d}}{\mathrm{d}s} J(t+s,t)) \right]_{s=0} \\
=& \sum_j \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial}{\partial z_j(t)} z_j(t+s)|_{s=0} - \frac{\mathrm{d}}{\mathrm{d}s} \overrightarrow{\frac{\partial}{\partial \gamma_j(t)}} \gamma_j(t+s)|_{s=0} \\
=& \sum_j \frac{\partial}{\partial z_j(t)} z_j'(t) - \overrightarrow{\frac{\partial}{\partial \gamma_j(t)}} \gamma_j'(t).
\end{aligned}
$$

Then

$$
\begin{aligned}
& \tfrac{\mathrm{d}}{\mathrm{d}s} \left[\mathrm{Sdet}\, J(t+s,t) f(z(t+s), \gamma(t+s)) \right]_{s=0} \\
=& \tfrac{\mathrm{d}}{\mathrm{d}s} \left[\mathrm{Sdet}\, J(t+s,t) \right]_{s=0} f(z(t),\gamma(t)) + \mathrm{Sdet}\, J(t,t) \tfrac{\mathrm{d}}{\mathrm{d}s} \left[f(z(t+s), \gamma(t+s)) \right]_{s=0} \\
=& \sum_j \tfrac{\partial}{\partial z_j(t)} [z_j'(t) f(z(t),\gamma(t))] - \overrightarrow{\tfrac{\partial}{\partial \gamma_j(t)}} [\gamma_j'(t) f(z(t),\gamma(t))],
\end{aligned}
$$

where we used that γ_j' is homogenous. Now set $t = 0$

$$
g'(0) = \int_V \mathrm{d}x\,\mathrm{d}\chi \sum_j \frac{\partial}{\partial x_j} [z_j'(0) f(x,\chi)] - \overrightarrow{\frac{\partial}{\partial \chi_j}} [\gamma_j'(0) f(x,\chi)] = 0.
$$

The first term vanishes since it is a boundary integral and f has compact support. In the second term there are two derivatives in χ_j.

Now we remark that $z(t), \gamma(t)$ is a set of generators, hence we can write $z_j'(t), \gamma_j'(t)$ in terms of $z(t)$ and $\gamma(t)$ and notice that

$$
g'(t) = \int_{U(t)} \mathrm{d}x\,\mathrm{d}\chi \ \mathrm{Sdet}\, J(t,0) h(z(t)(x,\chi), \gamma(t)(x,\chi)).
$$

Hence, $g'(t)$ has the same form as $g(t)$ with f replaced by h. Repeating the argument above, we find that $g'(t) = 0$ for all $t \geqslant 0$ and hence $g(0) = g(T)$. □

Supersymmetric Gaussian Integral

The supersymmetric version of the Gaussian integral formula yields indeed an inverse superdeterminant.

Theorem 1.2.19. *Let M be a supermatrix as in Eq.* (1.2.5). *We consider a Grassmann algebra $\mathcal{G}[\chi_1, \dots, \chi_q, \bar{\chi}_1, \dots \bar{\chi}_q]$ with $2q$ generators, a supervector $\Phi = \begin{pmatrix} z \\ \chi \end{pmatrix}$ with p complex variables $z_1, \dots, z_p$ and q odd variables $\chi_1, \dots \chi_q$ and its transposed $\Phi^* = (\bar{z}, \bar{\chi})$. We define $\mathrm{d}\Phi^* \, \mathrm{d}\Phi = \mathrm{d}\bar{z} \, \mathrm{d}z \, \mathrm{d}\bar{\chi} \, \mathrm{d}\chi$ and $\Phi^* M \Phi = (\bar{z}, az) + (\bar{\chi}, \beta z) + (\bar{z}, \alpha\chi) + (\bar{\chi}, b\chi)$ with $(\bar{z}, az) = \sum_{ij} \bar{z}_i a_{ij} z_j$. Then*

$$\int \mathrm{d}\Phi^* \, \mathrm{d}\Phi \, \mathrm{e}^{-\Phi^* M \Phi} = (2\pi)^p (\operatorname{Sdet} M)^{-1}.$$

Proof. We apply the change of variables formula above. We transform even and odd variables by

$$w = z + a^{-1}\alpha\chi, \qquad \bar{w} = \bar{z}, \qquad \eta = \chi + b^{-1}\beta z, \qquad \bar{\eta} = \bar{\chi}.$$

Indeed

$$\Phi^* M \Phi = \Phi^* \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & a^{-1}\alpha \\ b^{-1}\beta & 1 \end{pmatrix} \Phi = (\bar{w}, aw) + (\bar{\eta}, b\eta).$$

Note that $\operatorname{ord} \bar{w} = \overline{\operatorname{ord} w}$ since the transformation adds only a nilpotent component to z. We calculate the Berezinian

$$\operatorname{Sdet} J^{-1} = \operatorname{Sdet} \begin{pmatrix} 1 & 0 & a^{-1}\alpha & 0 \\ 0 & 1 & 0 & 0 \\ b^{-1}\beta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \det(1 - a^{-1}\alpha b^{-1}\beta)^{-1}.$$

Applying the change of variables formula (Theorem 1.2.18) and evaluating the Gaussian integrals (Proposition 1.2.11), we get

$$\begin{aligned} \int \mathrm{d}\Phi^* \, \mathrm{d}\Phi \, \mathrm{e}^{-\Phi^* M \Phi} &= \det(1 - a^{-1}\alpha b^{-1}\beta)^{-1} \int \mathrm{d}\bar{w} \, \mathrm{d}w \, \mathrm{d}\bar{\eta} \, \mathrm{d}\eta \, \mathrm{e}^{-(\bar{w}, aw) - (\bar{\eta}, b\eta)} \\ &= \det(1 - a^{-1}\alpha b^{-1}\beta)^{-1} \frac{(2\pi)^p}{\det a} \det b = (2\pi)^p (\operatorname{Sdet} M)^{-1}. \end{aligned}$$

□

1.3. Random Matrices

A random matrix is a family of random variables $A = (a_{ij})_{i,j\in I}$ with given probability laws for some index set I. One is interested in properties of eigenvalues and eigenvectors. For $I \subset \mathbb{N}$ finite, A is an ordinary matrix. We are studying the case $I = \mathbb{Z}^d$ which defines a random operator on the Hilbert space $l^2(\mathbb{Z}^d)$ but for simplicity we also speak of random matrices.

Wigner introduced random matrices to model resonance of heavy nuclei in the 1950s [Wig55]. Today random matrix models have various applications in mathematics and physics. For surveys we recommend [Meh04, Sto01, AW15].

In this thesis, we focus on two models that characterize conductivity in disordered materials: random Schrödinger operators, which are also known as the Anderson model, and random band matrices. In both models, spectral properties of the random operators describe the dynamical behaviour of free electrons in disordered materials such as metals, alloys or crystals with defects. With growing disorder the conductivity of certain materials decreases and localization effects occur. This phase transition is still an open conjecture in mathematics except in the case of a Bethe lattice [Kle98]. It should be observed mathematically by a change in the nature of the spectrum of the underlying random operator.

The Anderson model is well-studied although there are still open conjectures (cf. [KK08, AW15, CL90]). In comparison, there are only few rigorous results on random band matrices [Bou18].

In the following we define both models. To study spectral properties we introduce the density of states that measures the number of eigenvalues per unit volume. This quantity is one of the easiest to study, but also the starting point to more advanced questions. In our papers the density of states is the central quantity we study.

1.3.1. General setting

Random operators on $l^2(\mathbb{Z}^d)$

Let $(a_{ij})_{i,j\in\mathbb{Z}^d}$ be a collection of complex-valued random variables on a probability space Ω such that $a_{ij} = \bar{a}_{ji}$ for all $i,j \in \mathbb{Z}^d$ and $\sum_{i\in\mathbb{Z}^d} |a_{ij}|^2 < \infty$ for all $j \in \mathbb{Z}^d$ with probability 1. Since $i,j \in \mathbb{Z}^d$, a realisation $(a_{ij}(\omega))_{ij}$ with $\omega \in \Omega$ is not an ordinary matrix (indexed in $I \subset \mathbb{N}$) but a multidimensional matrix operator. Consider the Hilbert space

$$\mathcal{H} = l^2(\mathbb{Z}^d) = \{\varphi : \mathbb{Z}^d \to \mathbb{C} : \sum_{i\in\mathbb{Z}^d} |\varphi_i|^2 < \infty\}$$

and the countable dense subset

$$D = \{\varphi \in l^2(\mathbb{Z}^d) : \text{only finitely many } \varphi_i \neq 0\}. \tag{1.3.1}$$

Let $\mathcal{L}(D, \mathcal{H})$ be all linear maps from D to $\mathcal{H}$. Then $H : \Omega \to \mathcal{L}(D, \mathcal{H})$

$$(H(\omega)\varphi)_i = \sum_{j\in\mathbb{Z}^d} a_{ij}(\omega)\varphi_j$$

is a symmetric random operator not necessarily bounded (cf. [PF92, 1.4(c)]).

We will often need the finite volume version of H. Let $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ be a cube in $\mathbb{Z}^d$,

$$\mathcal{H}_L = l^2(\Lambda_L) = \{\varphi : \Lambda_L \to \mathbb{C}\} = \mathbb{C}^{\Lambda_L}$$

the finite volume Hilbert space and $H_L = H|_{\Lambda_L}$ the restriction of H to Λ_L.

Ergodicity

Ergodicity generalizes the concept of a family of independent, identically distributed random variables $(X_i)_{i\in I}$ to "almost independent" random variables if the "distance" between i and j is large. It becomes useful in the definition of the density of states. We give the definition of an ergodic operator and state the ergodic theorem, which extends the strong law of large numbers to ergodic processes. We restrict the definitions below to the case relevant for our purpose, i.e. $I = \mathbb{Z}^d$ and $\mathcal{H} = l^2(\mathbb{Z}^d)$. This summary is taken from [KK08, Chapter 4].

Definition 1.3.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A family $\{T_i : \Omega \to \Omega\}_{i\in\mathbb{Z}^d}$ of measure preserving transformations (i.e. $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$) is called *ergodic*, if any event $A \in \mathcal{F}$ that is invariant under $\{T_i\}_{i\in\mathbb{Z}^d}$ has probability zero or one.

Definition 1.3.2. A stochastic process $(X_i)_{i\in\mathbb{Z}^d}$ is called *ergodic* if there exists an ergodic family of measure preserving transformations $\{T_i\}_{i\in\mathbb{Z}^d}$ such that $X_i(T_j\omega) = X_{i-j}(\omega)$.

Proposition 1.3.3. *Let $\{T_i\}_{i\in\mathbb{Z}^d}$ be an ergodic family of measure preserving transformations. Let Y be a random variable invariant under T_i (i.e. $Y(T_i\omega) = Y(\omega)$ for all i, ω). Then Y is almost surely constant, i.e. there exists $C \in \mathbb{C}$ such that $\mathcal{P}(Y = C) = 1$.*

Theorem 1.3.4 (Birkhoff). *If $(X_i)_{i\in\mathbb{Z}^d}$ is an ergodic process with $\mathbb{E}[|X_0|] < \infty$, then for $\mathbb{P}$-almost all ω*

$$\lim_{L\to\infty} \frac{1}{|\Lambda_L|} \sum_{i\in\Lambda_L} X_i = \mathbb{E}[X_0].$$

Definition 1.3.5. A random operator H on $l^2(\mathbb{Z}^d)$ is called *ergodic* if there exists a homomorphism between an ergodic family of measure preserving transformations $\{T_i\}_{i\in\mathbb{Z}^d}$ and a group of unitary operators $\{U_i : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)\}_{i\in\mathbb{Z}^d}$ such that for all $i \in \mathbb{Z}^d$

$$H(T_i\omega) = U_i H(\omega) U_i^*.$$

1.3.2. Random Schrödinger operator

Random Schrödinger operators are a well-studied class of random matrices. The following definitions and properties are taken from [AW15] and [KK08, Chapter 3]. The discrete random Schrödinger operator $H : D \to l^2(\mathbb{Z}^d)$ (D as in Eq. (1.3.1)) is given by

$$H = -\Delta + \lambda V,$$

where $-\Delta$ is the discrete Laplace operator

$$(-\Delta\varphi)_j = \sum_{k\in\mathbb{Z}^d:|j-k|=1} (\varphi_j - \varphi_k),$$

$\lambda > 0$ is the parameter of disorder and the potential $(V\varphi)_j(\omega) = V_j(\omega)\varphi_j$ is a diagonal multiplication operator. The V_j are real random variables. If V_j are independent and identically distributed, the model is called *Anderson model*.

The discrete Laplacian

As a quadratic form, we can write

$$\langle\psi, -\Delta\varphi\rangle = \frac{1}{2}\sum_{i\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d:|i-j|=1} \overline{(\psi_i - \psi_j)}(\varphi_i - \varphi_j).$$

Now it is easy to see that $-\Delta : l^2(\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ is symmetric. Moreover, it is bounded and hence self-adjoint. By Fourier transformation one obtains that the spectrum is purely absolutely continuous and equals $\Sigma(-\Delta) = [0, 4d]$. The kernel of the Laplacian is

$$a_{ij} = \begin{cases} -1 & |i-j| = 1, \\ 2d & i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.2}$$

Hence, $-\Delta$ is a special case of the multidimensional matrix operators defined above.

Random potential

Let the V_j be independent, identically distributed random variables with common distribution $\mathbb{P}_0$ and denote by

$$\operatorname{supp}\mathbb{P}_0 = \{x \in \mathbb{R} : \mathbb{P}_0((x-\varepsilon, x+\varepsilon)) > 0 \text{ for all } \varepsilon > 0\}$$

the support of $\mathbb{P}_0$. If $\mathbb{P}_0$ is compact, the operator $V(\omega)$ is bounded and hence also $H(\omega) = -\Delta + \lambda V(\omega)$. Therefore $H(\omega)$ is defined on $l^2(\mathbb{Z}^d)$ and selfadjoint. If $\operatorname{supp}\mathbb{P}_0$ is not compact, we work with $H : D \to \mathcal{H}$. Then H is well-defined but not bounded. Moreover, V and therefore H are essentially selfadjoint on D.

Ergodicity

If the family V_j is an ergodic process, e.g. V_j independent and identically distributed, then H is an ergodic operator. Indeed, assume there exists an ergodic family $\{T_j\}_{j\in\mathbb{Z}^d}$ of measure preserving transformations such that

$$V_i(T_j\omega) = V_{i-j}(\omega).$$

We define $\{U_i\}_{i\in\mathbb{Z}^d}$, a family of translations on $\mathcal{H}$ by $(U_i\varphi)_j = \varphi_{j-i}$ for $\varphi \in \mathcal{H}$. Note that the Laplacian commutes with the U_i. Hence, $H = -\Delta + \lambda V$ is ergodic

$$H(T_i\omega) = -\Delta + \lambda V(T_i\omega) = -\Delta + \lambda V_{\cdot-i}(\omega) = U_iH(\omega)U_i^*.$$

Finite volume and boundary conditions

We can define a random Schrödinger operator H_L in finite volume Λ_L by providing certain boundary conditions for the discrete Laplacian $-\Delta$. For the purpose of this introduction we take free boundary conditions, i.e. $H_L = H|_{\Lambda_L}$. One can also choose e.g. periodic, Dirichlet or Neumann boundary conditions. For details we refer to the literature [KK08, Chapter 5.2].

1.3.3. Random band matrices

A random band matrix is a multidimensional matrix operator $\{a_{ij}\}_{i,j\in\mathbb{Z}^d}$ with independent (up to the symmetry condition $a_{ij} = \bar{a}_{ji}$) *not* identically distributed entries. The band structure is established by the condition that outside a band width W the entries are zero or negligible (with large probability), i.e. $|a_{ij}| \ll 1$ for $|i-j| > W$. There are several options to model this band structure. Here we will consider the case of the Gaussian ensemble with the following Gaussian entries

$$a_{ii} \sim \mathcal{N}_{\mathbb{R}}(0, J_{ii}), \qquad a_{ij} \sim \mathcal{N}_{\mathbb{C}}(0, J_{ij}) \quad \text{for } i < j,$$

where $<$ is an order relation on $\mathbb{Z}^d$. The covariance J_{ij} decays to zero for $|i-j| \gg W$. A natural choice is $J_{ij} = \frac{1}{W}\mathbb{1}_{|i-j|\leqslant W}$. We choose the smoother variant

$$J = (-W^2\Delta + \mathbb{1})^{-1},$$

where $-\Delta$ is (the kernel of) the discrete Laplace operator on $l^2(\mathbb{Z}^d)$ (cf. (1.3.2)) and $\mathbb{1}$ is the unit matrix in $\mathbb{R}^{\mathbb{Z}^d\times\mathbb{Z}^d}$.

This fulfils the two conditions $a_{ij} = \bar{a}_{ji}$ and $\mathbb{E}[\sum_{i\in\mathbb{Z}^d}|a_{ij}|^2] < \infty$ for all $j \in \mathbb{Z}^d$, hence $\sum_{i\in\mathbb{Z}^d}|a_{ij}|^2 < \infty$ almost surely for all $j \in \mathbb{Z}^d$. Therefore, there is a symmetric random operator $H : \Omega \to \mathcal{L}(D, \mathcal{H})$ defined by

$$(H(\omega)\varphi)_i = \sum_{j\in\mathbb{Z}^d} a_{ij}(\omega)\varphi_j.$$

In particular, H is essentially self-adjoint.

Ergodicity

Since the entries of the covariance $J_{ij} = f(|i-j|)$ depend only on the distance of the indices, the a_{ij} are ergodic with $\{T_k\}_{k\in\mathbb{Z}^d}$ the ergodic family of measure preserving transformations defined via

$$a_{ij}(T_k\omega) = a_{i-k,j-k}(\omega).$$

We define as above $\{U_i\}_{i\in\mathbb{Z}^d}$ the family of translations on $\mathcal{H}$ by $(U_i\varphi)_j = \varphi_{j-i}$ for $\varphi \in \mathcal{H}$. Hence, H is ergodic

$$H(T_i\omega) = U_iH(\omega)U_i^*.$$

Extreme cases and GUE

Varying the band width W, the band matrix model interpolates between two extreme cases: For $W = 0$, the covariance gives the identity and we have only non zero entries on the diagonal $a_{ii} \sim \mathcal{N}_{\mathbb{R}}(0,1)$ and H becomes a multiplication operator $(H(\omega)\varphi)_i = a_{ii}(\omega)\varphi_i$.

For W large, the entries in the band become (almost) identically distributed and $W \to \infty$ approximates the Gaussian unitary ensemble (GUE), where all entries (up to symmetry) are independent and identically distributed Gaussian random variables.

1.3.4. Density of states

An important spectral quantity is the density of states. It gives the number of eigenvalues per unit volume in an interval. There are several quantities that can easily be mixed up. In the following, we introduce the different observables, we are interested in, both in finite and infinite volume. In particular, we will consider the density of state measure ν, the integrated density of states N (the distribution function of ν) and the density of states ρ (the Radon-Nikodym derivative of ν, if ν is absolutely continuous with respect to the Lebesgue measure).

Density of states in finite volume

Let $\Lambda_L = [-L, L]^d \cap \mathbb{Z}^d$ be a finite cube and $H_L : \Omega \to \mathcal{L}(l^2(\Lambda_L), l^2(\Lambda_L))$ a symmetric random operator in $l^2(\Lambda_L)$. Note that $l^2(\Lambda_L)$ is finite dimensional and, if H_L is symmetric, it is selfadjoint and the spectrum $\Sigma(H_L(\omega)) \subset \mathbb{R}$. For a fixed realisation $H(\omega)$, the *empirical density of states measure* in finite volume is the point measure

$$\nu_{L,\omega} = \frac{1}{|\Lambda_L|} \sum_{\lambda_n \in \Sigma(H_L(\omega))} \delta_{\lambda_n}. \tag{1.3.3}$$

To obtain a non random quantity, we introduce the *averaged density of states measure* in finite volume $\bar{\nu}_L = \mathbb{E}[\nu_L]$ via Riesz representation theorem as

$$\int f(x)\, \mathrm{d}\bar{\nu}_L(x) = \mathbb{E}\left[\int f(x)\, \mathrm{d}\nu_{L,\omega}(x)\right] \tag{1.3.4}$$

for all $f \in C_b(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous and bounded}\}$. Note that in infinite volume the related quantities will coincide since they are non random by ergodicity (cf. the next paragraph).

To do hands-on calculations, we want to write $\bar{\nu}_L$ in terms of the resolvent $G^+_{L,\omega}$, also called *Green's function*

$$G^+_{L,\omega}(x + i\varepsilon) = ((x + i\varepsilon)\mathbb{1} - H_L(\omega))^{-1},$$

where $x \in \mathbb{R}$ and $\varepsilon > 0$. We note that

$$g_{L,\omega}(x + i\varepsilon) := -\frac{1}{\pi|\Lambda_L|} \operatorname{Im} \operatorname{Tr} G^+_{L,\omega}(x + i\varepsilon) = \frac{1}{\pi|\Lambda_L|} \sum_{\lambda_n \in \Sigma(H_L(\omega))} \frac{\varepsilon}{(\lambda_n - x)^2 + \varepsilon^2} \tag{1.3.5}$$

approximates the point measure $\nu_{L,\omega}$ for $\varepsilon \downarrow 0$. Hence, for all $f \in C_b(\mathbb{R})$

$$\int f(x)\,\mathrm{d}\bar{\nu}_L(x) = \mathbb{E}\left[\int f(x)\,\mathrm{d}\nu_{L,\omega}(x)\right] = \mathbb{E}\left[\lim_{\varepsilon\downarrow 0}\int f(x) g_{L,\omega}(x+i\varepsilon)\,\mathrm{d}x\right].$$

Lemma 1.3.6. *Let $f \in C_b(\mathbb{R})$. Assume there exists $C > 0$ such that $\mathbb{E}[g_{L,\omega}(x+i\varepsilon)] \leqslant C$ for all $x \in \mathbb{R}, \varepsilon > 0$. Then*

$$\mathbb{E}\left[\lim_{\varepsilon\downarrow 0}\int f(x) g_{L,\omega}(x+i\varepsilon)\,\mathrm{d}x\right] = \int f(x) \lim_{\varepsilon\downarrow 0}\mathbb{E}[g_{L,\omega}(x+i\varepsilon)]\,\mathrm{d}x.$$

Proof. Note that $\int g_{L,\omega}(x+i\varepsilon)\,\mathrm{d}x = \frac{1}{|\Lambda|}\sum_{\lambda_j}\int \frac{\varepsilon}{(x-\lambda_j)^2+\varepsilon^2}\,\mathrm{d}x = K < \infty$ is bounded independent of ε. Hence, we can bound $|\int f(x) g_{L,\omega}(x+i\varepsilon)\,\mathrm{d}x| \leqslant \|f\|_{L^\infty} K$ independent of ε. Therefore, we can apply dominated convergence to interchange the average and the limit $\varepsilon \downarrow 0$

$$\mathbb{E}\left[\lim_{\varepsilon\downarrow 0}\int f(x) g_{L,\omega}(x+i\varepsilon)\,\mathrm{d}x\right] = \lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\int f(x) g_{L,\omega}(x+i\varepsilon)\,\mathrm{d}x\right].$$

By Fubini we can bring the average inside the integral and obtain

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\int f(x) g_{L,\omega}(x+i\varepsilon)\,\mathrm{d}x\right] = \lim_{\varepsilon\downarrow 0}\int f(x)\mathbb{E}[g_{L,\omega}(x+i\varepsilon)]\,\mathrm{d}x.$$

We can pull the ε-limit back inside the integral by dominated convergence because $\mathbb{E}[g_{L,\omega}(x+i\varepsilon)]$ is bounded uniformly in ε and x:

$$\lim_{\varepsilon\downarrow 0}\int f(x)\mathbb{E}[g_{L,\omega}(x+i\varepsilon)]\,\mathrm{d}x = \int f(x)\lim_{\varepsilon\downarrow 0}\mathbb{E}[g_{L,\omega}(x+i\varepsilon)]\,\mathrm{d}x.$$

□

We will see that $\mathbb{E}[g_{L,\omega}(x+i\varepsilon)]$ satisfies indeed the above bound for the models we consider. Assuming the condition above we can define the *finite volume averaged density of states* $\bar{\rho}_L : \mathbb{R} \to \mathbb{R}$ as

$$\bar{\rho}_L(x) = -\frac{1}{\pi|\Lambda_L|}\lim_{\varepsilon\downarrow 0}\mathbb{E}[\operatorname{Im}\operatorname{Tr} G^+_{L,\omega}(x+i\varepsilon)]. \tag{1.3.6}$$

Note that, in general, we cannot bring the limit inside the integral without the average since $g_{L,\omega}$ converges pointwise to 0 a.e. as $\varepsilon \downarrow 0$:

$$\begin{aligned}\int f(x)\,\mathrm{d}\nu_{L,\omega}(x) &= \lim_{\varepsilon\downarrow 0}\int f(x) g_{L,\omega}(x+i\varepsilon)\,\mathrm{d}x = \sum_{\lambda_n} f(\lambda_n)\\ &\neq \int f(x)\lim_{\varepsilon\downarrow 0} g_{L,\omega}(x+i\varepsilon)\,\mathrm{d}x = 0.\end{aligned}$$

Density of states in infinite volume

The following construction is taken from [KK08, Chapter 5.1] and [AW15, Chapter 3.4] and can be applied to random Schrödinger models. For random band matrices, we refer to the literature [PF92, Theorem 4.11].

Let H be the corresponding ergodic symmetric random operator and $H_L = H|_{\Lambda_L}$ the restriction of H to the finite cube. We start from the functional $\nu_{L,\omega} : L^\infty(\mathbb{R}) \to \mathbb{R}$ for bounded measurable functions (see Eq. (1.3.3))

$$\nu_{L,\omega}(f) = \frac{1}{|\Lambda_L|} \operatorname{Tr} f(H_L(\omega)) = \int f(x)\, \mathrm{d}\nu_{L,\omega}(x). \tag{1.3.7}$$

By ergodicity this converges for fixed $f \in L^\infty(\mathbb{R})$ to $\mathbb{E}[\langle \delta_0, f(H)\delta_0 \rangle]$ for almost all $\omega \in \Omega$ as $L \to \infty$. This defines a positive, non-random measure ν again via Riesz representation theorem (cf. Eq. (1.3.4)) by

$$\int f(x)\, \mathrm{d}\nu(x) := \mathbb{E}\left[\langle \delta_0, f(H)\delta_0 \rangle\right].$$

Indeed, ν is a probability measure since we get $\nu(\mathbb{R}) = \mathbb{E}(\langle \delta_0, \delta_0 \rangle) = 1$ by taking $f = 1$ constant.

It remains to show that this is indeed the weak limit of the measure $\nu_{L,\omega}$ above almost surely, i.e. there exists a set Ω_0 of probability one such that

$$\lim_{L\to\infty} \int f(x)\, \mathrm{d}\nu_{L,\omega}(x) = \int f(x)\, \mathrm{d}\nu(x) \tag{1.3.8}$$

for all $f \in C_b(\mathbb{R})$ and all $\omega \in \Omega_0$. We refer to [KK08, Theorem 5.5] and [AW15, Thereom 3.14].

Note that if we have other than free boundary conditions for the Laplacian in the random Schrödinger case, we need to modify the procedure above since H_L is no longer just the restriction of H to the finite cube. If the difference between H_L and $H|_{\Lambda_L}$ remains "trace class", we obtain the same result (cf. [AW15, Theorem 3.15]).

Definition 1.3.7. Finally, we can define the *density of states measure* ν as the probability measure

$$\nu(A) = \mathbb{E}[\langle \delta_0, \chi_A(H)\delta_0 \rangle],$$

the *integrated density of states* $N(E)$ as the distribution function of ν

$$N(E) = \nu((-\infty, E))$$

and the *density of states* ρ as the Radon-Nikodym derivative of the absolutely continuous part ν_{ac}

$$\rho(x) = \frac{\mathrm{d}\nu_{ac}(x)}{\mathrm{d}x}. \tag{1.3.9}$$

Connection to Green's function

We want to build a connection to the Green's function representation of the averaged density of states in finite volume (Eq. (1.3.6)).

We give a short excursion on Herglotz-Pick functions. The following definitions and statements are taken from [AW15, Appendix B]. We consider the Borel-Stieltjes transformation for any finite Borel measure μ given by $F : \mathbb{C}^+ \to \mathbb{C}^+$

$$F(z) = \int_{\mathbb{R}} \frac{1}{u-z}\, \mathrm{d}\mu(u). \tag{1.3.10}$$

This is a Herglotz-Pick function. Its boundary value $\lim_{\varepsilon\downarrow 0} F(x+i\varepsilon)$ for $x \in \mathbb{R}$ exists and is finite almost everywhere. Moreover, it determines the function uniquely and gives information on the absolutely continuous and singular part of the corresponding measure. We obtain e.g. the Radon-Nikodym derivative of the absolutely continuous measure via the underlying theory by

$$\frac{\mathrm{d}\mu_{ac}(x)}{\mathrm{d}x} = \lim_{\varepsilon\to 0} \frac{1}{\pi} \operatorname{Im} F(x+i\varepsilon) \tag{1.3.11}$$

for a.e. $x \in \mathbb{R}$.

We consider again our density of states measure ν which is non random. Hence, $\int f(x)\,\mathrm{d}\nu(x) = \mathbb{E}\left[\int f(x)\,\mathrm{d}\nu(x)\right]$. We insert Eq. (1.3.8) and apply dominated convergence to change the average and the limit $L \to \infty$ since $|f(x)\,\mathrm{d}\nu_{L,\omega}(x)| \leqslant \|f\|_\infty$ is bounded independent of L and ω:

$$\begin{aligned}\int f(x)\,\mathrm{d}\nu(x) &= \mathbb{E}\left[\int f(x)\,\mathrm{d}\nu(x)\right] = \mathbb{E}\left[\lim_{L\to\infty}\int f(x)\,\mathrm{d}\nu_{L,\omega}(x)\right]\\ &= \lim_{L\to\infty}\mathbb{E}\left[\int f(x)\,\mathrm{d}\nu_{L,\omega}(x)\right] = \lim_{L\to\infty}\int f(x)\,\mathrm{d}\bar{\nu}_L(x).\end{aligned} \tag{1.3.12}$$

Let F_ν be the Borel-Stieltjes transform of ν. Then by Eq. (1.3.10) and (1.3.7)

$$F_\nu(z) = \lim_{L\to\infty}\int \frac{1}{x-z}\,\mathrm{d}\bar{\nu}_L(x) = \lim_{L\to\infty}\frac{1}{|\Lambda_L|}\mathbb{E}\left[\operatorname{Tr}\left(H_L(\omega)-z\right)^{-1}\right].$$

In particular, $\lim_{\varepsilon\downarrow 0} F(x+i\varepsilon)$ exists and is almost surely finite by Herglotz-Pick theory. Hence the density of states ρ is by (1.3.11)

$$\rho(x) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\operatorname{Im} F_\nu(x+i\varepsilon) = \lim_{\varepsilon\downarrow 0}\lim_{L\to\infty}\mathbb{E}[g_{L,\omega}(x+i\varepsilon)]\text{for a.e. } x, \tag{1.3.13}$$

where $g_{L,\omega}$ is defined in Eq. (1.3.5).

It depends on the region in the spectrum, whether this is the infinite volume limit of the finite volume averaged density of states (i.e. whether we can interchange the limits $\varepsilon \to 0$ and $L \to \infty$).

Proposition 1.3.8. *Consider an interval $[a,b] \subset \mathbb{R}$. Assume that there are $C_1, C_2 < \infty$ independent of ω and L such that*

$$\sup_{x\in[a,b]} |\mathbb{E}[g_{L,\omega}(x+i\varepsilon)]| < C_1, \qquad \sup_{x\in[a,b]} |\lim_{\varepsilon\downarrow 0} \mathbb{E}[g_{L,\omega}(x+i\varepsilon)]| < C_2.$$

Then $\nu = \nu_{ac}$ in $[a,b]$ and

$$\rho(x) = \frac{\mathrm{d}\nu_{ac}(x)}{\mathrm{d}x} = \lim_{\varepsilon\downarrow 0}\lim_{L\to\infty} \mathbb{E}[g_L(x+i\varepsilon)] = \lim_{L\to\infty}\lim_{\varepsilon\downarrow 0} \mathbb{E}[g_L(x+i\varepsilon)].$$

We remark that $\nu = \nu_{ac}$ does *not* imply that the spectrum of H is absolutely continuous since ν is not the spectral measure of H.

Proof. Let $f \in C_b((a,b))$. We insert Eq. (1.3.12) and apply Lemma 1.3.6 since $\mathbb{E}[g_{L,\omega}(x+i\varepsilon)]$ is bounded independent of ε in $[a,b]$:

$$\int_a^b f(x)\,\mathrm{d}\nu(x) = \lim_{L\to\infty}\int_a^b f(x)\,\mathrm{d}\bar{\nu}_L(x) = \lim_{L\to\infty}\int_a^b f(x)\lim_{\varepsilon\downarrow 0}\mathbb{E}[g_L(x+i\varepsilon)]\,\mathrm{d}x$$

To bring the limit of $L\to\infty$ inside the integral we use the second bound and dominated convergence:

$$\lim_{L\to\infty}\int_a^b f(x)\lim_{\varepsilon\downarrow 0}\mathbb{E}[g_L(x+i\varepsilon)]\,\mathrm{d}x = \int_a^b f(x)\lim_{L\to\infty}\lim_{\varepsilon\downarrow 0}\mathbb{E}[g_L(x+i\varepsilon)]\,\mathrm{d}x \tag{1.3.14}$$

and hence combining (1.3.13) and (1.3.14)

$$\rho(x) = \frac{\mathrm{d}\nu_{ac}(x)}{\mathrm{d}x} = \lim_{\varepsilon\downarrow 0}\lim_{L\to\infty} \mathbb{E}[g_L(x+i\varepsilon)] = \lim_{L\to\infty}\lim_{\varepsilon\downarrow 0} \mathbb{E}[g_L(x+i\varepsilon)].$$

□

1.4. Density of States for random band matrix in two dimensions

In the following section we summarize the results and ideas of [DL17]. We start with a finite volume version of the random band matrix operator introduced in Section 1.3.3 on the two-dimensional lattice $\mathbb{Z}^2$. For the three-dimensional lattice, [DPS02] proved that the density of states of this model equals Wigner's semicircle law up to an error depending on the band width W. The main result of [DL17] extents this result from the three- to the two-dimensional case.

We remind here the model and the density of states and give the main result and the idea of the proof. Moreover, we compare our result and techniques with the ones of [DPS02]. There are two main differences: on the one hand, $d = 2$ is a limit case and makes the estimates more involved. The main idea is to extend the volume of the cubes used in the cluster expansion. On the other hand, we organize the cluster expansion in a simpler way. In contrast to [DPS02], we also perform a preliminary step of integration by parts, which leads to more reduced formulas.

1.4.1. Model and result

Model

In the following we set $d = 2$ and consider a finite cube $\Lambda \subset \mathbb{Z}^2$. Let $H \in \mathbb{C}^{\Lambda\times\Lambda}$ be a hermitian random matrix with independent entries distributed as

$$H_{ii} \sim \mathcal{N}_{\mathbb{R}}(0, J_{ii}), \qquad H_{ij} \sim \mathcal{N}_{\mathbb{C}}(0, J_{ij}), \quad \text{for } i < j,$$

where $<$ is the order relation on $\mathbb{Z}^2$. The covariance J is defined by

$$J_{ij} := (-W^2\Delta + \mathbb{1})^{-1}_{ij} \lesssim \mathrm{e}^{-|i-j|/W} \quad \text{for } |i-j| > W,$$

where $-\Delta \in \mathbb{R}^{\Lambda\times\Lambda}$ is the discrete Laplacian on Λ with periodic boundary conditions and $W \gg 1$ is the band width. Periodic boundary conditions in this context mean that we identify opposite sites of the cube Λ as nearest neighbours:

$$-\Delta_{ij} = \begin{cases} 2 & i = j, \\ -1 & i, j \text{ nearest neighbours in the torus,} \\ 0 & \text{otherwise.} \end{cases}$$

The difference to the free Laplacian is trace like and hence the infinite volume limit is the same [AW15, Theorem 3.15].

Density of States

In Section 1.3.4 we have seen that it is useful to study the finite volume averaged density of states given by

$$\bar{\rho}_\Lambda(E) = \frac{1}{|\Lambda|}\mathbb{E}\left[\sum_j \delta_{\lambda_j}(E)\right] = -\frac{1}{\pi|\Lambda|}\lim_{\varepsilon\downarrow 0}\mathbb{E}[\operatorname{Im}\operatorname{Tr} G^+(E+i\varepsilon)], \tag{1.4.1}$$

where $E \in \mathbb{R}$, λ_j are the random eigenvalues of H and G^+ is the resolvent

$$G^+(E+i\varepsilon) := ((E+i\varepsilon)\cdot\mathbb{1} - H)^{-1}.$$

Note that the second equality in Eq. (1.4.1) is meant in distributional sense. Nevertheless, we show that the limit $\varepsilon \downarrow 0$ exists pointwise.

Result

In the following we bound $E \mapsto \bar{\rho}_\Lambda(E)$ uniformly in Λ in some region $E \in \mathcal{I}$. We expect the spectrum to be $\Sigma = [-2, 2]$. We consider the interval $\mathcal{I} = \{E : \eta < |E| \leqslant 1.8, \eta > 0\}$ to be well inside the spectrum and to avoid 0 for technical reasons.

We give our theorem in $d = 2$ and the one in $d = 3$ of [DPS02].

Theorem 1.4.1 (Disertori, Lager 2017)**.** *For $d = 2$ and each fixed $\alpha \in (0,1)$, there exists a value $W_0(\alpha)$ such that for all $W \geqslant W_0(\alpha)$ and $E \in \mathcal{I}$*

$$|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \leqslant W^{-2}\, \mathrm{e}^{K(\ln W)^\alpha},$$
$$|\partial_E^n \bar{\rho}_\Lambda(E)| \leqslant C_n W^{-1} (\ln W)^{n(\alpha+1)}\, \mathrm{e}^{K(\ln W)^\alpha} \qquad \forall n \geqslant 0,$$

where ρ_{SC} is Wigner's semicircle law defined as

$$\rho_{SC}(E) = \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{E^2}{4}} & \text{if } |E| \leqslant 2, \\ 0 & \text{if } |E| > 2. \end{cases} \tag{1.4.2}$$

The constants C_n and K are independent of Λ and W. Both estimates hold uniformly in Λ and hence also in the infinite volume limit $\Lambda \to \mathbb{Z}^2$.

Theorem 1.4.2 (Disertori, Pinson, Spencer 2002)**.** *For $d = 3$, there exists a value W_0 such that for all $W \geqslant W_0$ and $E \in \mathcal{I}$*

$$|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \leqslant W^{-2},$$
$$|\partial_E^n \bar{\rho}_\Lambda(E)| \leqslant C_n \qquad \forall n \geqslant 0,$$

where ρ_{SC} is Wigner's semicircle law defined in (1.4.2)*. The constants C_n are independent of Λ and W. Both estimates hold uniformly in Λ and hence also in the infinite volume limit $\Lambda \to \mathbb{Z}^3$.*

Note that we insert an α dependence to deal with the problems arising in $d = 2$. Moreover, our bounds are slightly weaker than the ones in $d = 3$. We specify this in Section 1.4.3. First we give the idea of the proof.

1.4.2. Idea of the proof

The proof follows the ideas of [DPS02]. Starting from a supersymmetric representation of G^+, we perform a saddle point analysis and prove estimates in a finite cube of volume $W^2 \ln W^\alpha$. This differs from the natural choice of W^d, which works in $d = 3$. We need a larger volume to suppress contributions from the non-dominant saddle point. Then we perform a cluster expansion to extend the estimates to infinite volume.

Dual representation

We start from the trace of G^+ and represent each entry G^+_{kk} as a complex Gaussian integral. Then we replace the normalization factor by a fermionic Gaussian integral and end up with

$$\begin{aligned}\tfrac{1}{|\Lambda|}\sum_{k\in\Lambda} G^+_{kk}(E + i\varepsilon) &= \tfrac{-i}{|\Lambda|} \det\left[-i\tfrac{E+i\varepsilon-H}{2\pi}\right] \int \mathrm{d}\bar{z}\, \mathrm{d}z\, \mathrm{e}^{i(\bar{z},(E+i\varepsilon-H)z)} \sum_{k\in\Lambda} z_k \bar{z}_k \\ &= \tfrac{-i}{|\Lambda|} \int \mathrm{d}\Phi^*\, \mathrm{d}\Phi\, \mathrm{e}^{i\sum_{i,j\in\Lambda} \Phi_i^*(\delta_{ij}(E+i\varepsilon)-H_{ij})\Phi_j} \sum_{k\in\Lambda} z_k \bar{z}_k,\end{aligned}$$

where $\mathrm{d}\Phi^* \mathrm{d}\Phi = \mathrm{d}\bar{z}\,\mathrm{d}z\,\mathrm{d}\bar{\chi}\,\mathrm{d}\chi$ and Φ is a supervector cf. Eq. (1.2.4). We drop the argument $E + i\varepsilon = E_\varepsilon$ of G^+ in the following. Note that we use here the normalization of the fermionic integral $\int \mathrm{d}\chi\, \chi = \frac{1}{\sqrt{2\pi}}$ to eliminate the 2π factor arising from the complex Gaussian integral and hence by Proposition 1.2.11

$$\int \mathrm{d}\bar{\chi}\, \mathrm{d}\chi\, \mathrm{e}^{-(\bar{\chi}, M\chi)} = \det \tfrac{M}{2\pi}.$$

The probability measure is Gaussian, hence all moments are bounded. Therefore, we can shift the average inside the integral and evaluate it explicitly:

$$\mathbb{E}\left[\mathrm{e}^{-i\sum_{i,j\in\Lambda} \Phi_i^* H_{ij} \Phi_j}\right] = \mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij} (\Phi_i^*\Phi_j)(\Phi_j^*\Phi_i)}.$$

Then we apply the so-called Hubbard-Stratonovich transformation to introduce a new superintegral. We write

$$\mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij} (\Phi_i^*\Phi_j)(\Phi_j^*\Phi_i)} = \int \prod_{j\in\Lambda} \mathrm{d}M_j\, \mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij}^{-1} \mathrm{Str}\,[M_i M_j]}\, \mathrm{e}^{-i\sum_{j\in\Lambda} \Phi_j^* M_j \Phi_j},$$

where M is a collection of $|\Lambda|$ many 2×2 supermatrices

$$M_j = \begin{pmatrix} a_j & \bar{\rho}_j \\ \rho_j & ib_j \end{pmatrix},$$

where $a_j, b_j \in \mathbb{R}$ and $\bar{\rho}_j, \rho_j$ are Grassmann variables. By Fubini we can exchange the integrals over $\mathrm{d}M$ and $\mathrm{d}\Phi^*\,\mathrm{d}\Phi$.

$$\mathbb{E}[\mathrm{Tr}\, G^+] = \int \mathrm{d}M\, \mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij}^{-1} \mathrm{Str}\,[M_i M_j]} \int \mathrm{d}\Phi^*\, \mathrm{d}\Phi\, \mathrm{e}^{i\sum_{j\in\Lambda} \Phi_j^* (E_\varepsilon - M_j)\Phi_j} \sum_k |z_k|^2 \tag{1.4.3}$$

By a preliminary step of integration by parts, we eliminate the factor $\sum_k |z_k|^2$ as follows:

$$\begin{aligned} -i\int \mathrm{d}a\, \mathrm{e}^{-\frac{1}{2}a^t J^{-1} a}\, \mathrm{e}^{-i\sum_{j\in\Lambda} a_j |z_j|^2} \sum_{k\in\Lambda} |z_k|^2 &= \int \mathrm{d}a\, \mathrm{e}^{-\frac{1}{2}a^t J^{-1} a} \sum_{k\in\Lambda} \partial_{a_k}\big[\mathrm{e}^{-i\sum_{j\in\Lambda} a_j |z_j|^2}\big] \\ = -\int \mathrm{d}a \sum_{k\in\Lambda} \partial_{a_k}\big[\mathrm{e}^{-\frac{1}{2}a^t J^{-1} a}\big]\, \mathrm{e}^{-i\sum_{j\in\Lambda} a_j |z_j|^2} &= \int \mathrm{d}a\, \mathrm{e}^{-\frac{1}{2}a^t J^{-1} a}\, \mathrm{e}^{-i\sum_{j\in\Lambda} a_j |z_j|^2} \sum_{k\in\Lambda} a_k, \end{aligned}$$

where we used $(1, J^{-1}a) = (1, (-W^2\Delta + 1)a) = \sum_k a_k$. Note that there are no boundary terms. Because of periodic boundary conditions on Λ, the integral is translation invariant and we can replace $\frac{1}{|\Lambda|}\sum_k a_k$ with a_0. Using Theorem 1.2.19, we can integrate over the variables Φ^* and Φ in Eq.(1.4.3) where the observable is now a_0. We end up with

$$\begin{aligned} \tfrac{1}{|\Lambda|}\mathbb{E}[\mathrm{Tr}\, G^+] &= \int \mathrm{d}M\, \mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij}^{-1} \mathrm{Str}\,[M_i M_j]} \prod_{j\in\Lambda} \mathrm{Sdet}\,[E_\varepsilon - M_j]^{-1} a_0 \\ &= \int \mathrm{d}a\, \mathrm{d}b\, \mathrm{e}^{-\frac{1}{2}((a, J^{-1}a) + (b, J^{-1}b))} \prod_{j\in\Lambda} \tfrac{E_\varepsilon - ib_j}{E_\varepsilon - a_j}\ \det\left[\tfrac{J^{-1} - F}{2\pi}\right]\ a_0, \end{aligned} \tag{1.4.4}$$

where

$$F(a,b)_{ij} = \delta_{ij} \tfrac{1}{(E_\varepsilon - a_j)(E_\varepsilon - ib_j)}. \tag{1.4.5}$$

Because of the integration by parts, we have the simple observable a_0 in contrast to $\frac{1}{E_\varepsilon - a_0} \times$ (contribution in the determinant) ([DPS02, Eq. (3.1)]).

Saddle point analysis

The leading term in Eq. (1.4.4) is

$$\mathrm{e}^{-\frac{1}{2}((a,J^{-1}a)+(b,J^{-1}b))} \prod_{j\in\Lambda} \tfrac{E_\varepsilon - ib_j}{E_\varepsilon - a_j}.$$

Observing that $(a, J^{-1}a) = \sum_j a_j^2 + W^2 \sum_{i\sim j}(a_i - a_j)^2$ is small only if the $a_j \approx a$ are approximately constant, we obtain critical points if we set

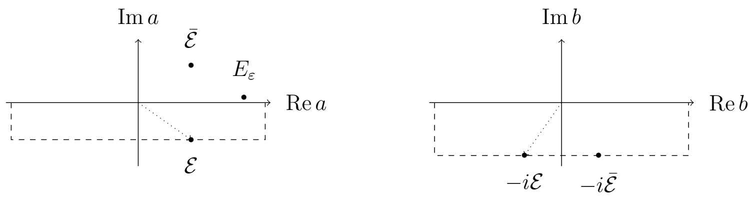
$$\mathrm{e}^{-\frac{|\Lambda|}{2}(a^2+b^2)} \left(\tfrac{E-ib}{E-a}\right)^{|\Lambda|} = \mathrm{e}^{-|\Lambda|\left(\frac{a^2}{2}+\ln(E-a)+\frac{b^2}{2}-\ln(E-ib)\right)} \sim 1.$$

Then the critical points are $a_s^\pm = \mathcal{E}_r \pm i\mathcal{E}_i$ and $b_s^\pm = -i\mathcal{E}_r \pm \mathcal{E}_i$, where

$$\mathcal{E} = \mathcal{E}_r - i\mathcal{E}_i = \tfrac{E}{2} - i\sqrt{1 - \tfrac{E^2}{4}}. \tag{1.4.6}$$

We want to apply a complex contour deformation to integrate through the saddle points. To avoid crossing the singularity $E + i\varepsilon$, we choose a_s^+ and we will see later that b_s^+ is the dominant saddle (cf. Figure 1.1).

Figure 1.1.: Complex saddle points $a_s^\pm$ and $b_s^\pm$ in the complex plane.



Finding the semicircle law

Applying the complex contour deformation mentioned above, we can take the limit $\lim_{\varepsilon\to 0}$ and obtain the semicircle law plus an error term. It remains to show that the error term is small. Precisely we obtain the following result:

Lemma 1.4.3. *After performing the complex deformation $a_j \mapsto a_j + a_s^+$ and $b_j \mapsto b_j + b_s^+$ we obtain*

$$\lim_{\varepsilon\to 0} \frac{1}{|\Lambda|} \mathbb{E}[\operatorname{Tr} G^+] = a_s^+ + I_{rem} \quad \textit{with} \quad I_{rem} = \int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b)a_0, \tag{1.4.7}$$

where

- $\mathrm{d}\mu_B(a,b)$ *is the Gaussian measure with complex covariance* B *given by* $B := (-W^2\Delta + (1-\mathcal{E}^2))^{-1}$ *and* $\mathcal{E}$ *defined in Eq.* (1.4.6)*,*
- $\mathcal{R}(a,b) := \det[1+DB]\ \mathrm{e}^{\mathcal{V}(a,b)}$*, with*
  - $D_{ij} = D_{ij}(a,b) = \delta_{ij} D_j(a,b)$ *is diagonal,*
  - $D_j(a,b) = \mathcal{E}^2 - F(a+a_s^+, b+b_s^+)_{jj}$ *with* F *defined in Eq.* (1.4.5)*,*
  - $\mathcal{V}(a,b) = \sum_{j\in\Lambda} V(a_j) - V(ib_j)$*, with* $V(x) = \int_0^1 \frac{x^3(1-t)^2}{(\mathcal{E}-tx)^3}\,\mathrm{d}t$.

Note that $-\frac{1}{\pi}\mathrm{Im}\, a_s^+ = \rho_{SC}(E)$. The difficult part now is to show that

$$\left| -\frac{1}{\pi}\mathrm{Im}\,\lim_{\varepsilon\to 0}\frac{1}{|\Lambda|}\mathbb{E}[\mathrm{Tr}\, G^+] - \rho_{SC}\right| = \frac{1}{\pi}|\mathrm{Im}\, I_{rem}| \leqslant \frac{1}{\pi}|I_{rem}| \leqslant \mathcal{O}(W^{-\delta})$$

for some $\delta > 0$.

Up to this point, the procedure is independent of the dimension. From now on it becomes crucial since the dimension has an impact on the decay of the covariance B and hence on the estimates of the remaining integral I_{rem} in (1.4.7). We discuss this in the next section.

1.4.3. The difficulties of d=2

To estimate I_{rem}, we partition the domain of integration into five sets: two small field regions, in which all a, b are near the saddle points and three large field regions, where at least one variable is far from the saddle point (cf. Eq. (2.3.7)).

The large field regions work similar in $d = 2$ and $d = 3$ and give exponentially small contribution in W. This decay is used to control various W factors arising from the cluster expansion later.

In the two small field region I_1 and I_2, all variables a, b are centred around one saddle point, respectively. To estimate these contributions we use the decay of the covariance $|B_{ij}|$. Here the dimension comes into place. In $d = 3$ we obtain

$$|B_{ij}| \leqslant \frac{K}{W^2(1+|i-j|)}\,\mathrm{e}^{-m_r|i-j|/W}$$

while in $d = 2$ we have

$$|B_{ij}| \leqslant \begin{cases} \frac{1}{W^{3/2}|i-j|^{1/2}}\,\mathrm{e}^{-m_r|i-j|/W} & |i-j| > \frac{W}{m_r}, \\ \frac{1}{W^2}\ln\frac{W}{1+|i-j|} & |i-j| \leqslant \frac{W}{m_r}, \end{cases}$$

where $m_r = \mathrm{Re}\,(1-\mathcal{E})$.

This weaker decay for small distances is the main problem in $d = 2$. Indeed estimating the integral in the small field regions in $d = 3$ we get the following estimates (cf. [DPS02, Eq. (5.32) and (5.44)]) which allow the natural choice $|\Lambda| = W^d = W^3$.

$$\left|\int d\mu_B(a,b)\chi_{I_1}\mathcal{R}(a,b)a_0\right| \leqslant K\, e^{K|\Lambda|W^{-3}} \sim K,$$
$$\left|\int d\mu_B(a,b)\chi_{I_2}\mathcal{R}(a,b)a_0\right| \leqslant K\, e^{K|\Lambda|W^{-2}}\, e^{-c|\Lambda|W^{-2}\frac{1}{8}} \sim e^{-cW},$$

where we denote by $K < \infty$ different constants independent of W and Λ and $c > 0$ is fixed. On the contrary in $d = 2$ the same estimates are (cf. Lemma 2.3.6)

$$\left|\int d\mu_B(a,b)\chi_{I_1}\mathcal{R}(a,b)a_0\right| \leqslant K\, e^{K|\Lambda|W^{-2}} \left(\frac{\ln W}{W^2}\right)^{1/2} e^{K|\Lambda|W^{-3}(\ln W)^{3/2}}, \tag{1.4.8}$$
$$\left|\int d\mu_B(a,b)\chi_{I_2}\mathcal{R}(a,b)a_0\right| \leqslant K\, e^{K|\Lambda|W^{-2}}\, e^{-c|\Lambda|W^{-2}\ln W}. \tag{1.4.9}$$

Taking the natural choice $|\Lambda| = W^d = W^2$, the second saddle gives a contribution $\sim W^{-c}$ which is not enough to suppress the various W factors that will arise from the cluster expansion. This weak bound comes from the ln-behaviour of B_{ij} for small distances. The solution is to extend the volume slightly to $|\Lambda| = W^2(\ln W)^\alpha$ for some $\alpha \in (0,1)$. Then the second saddle is suppressed nearly exponentially in W by $e^{K(\ln W)^\alpha - c(\ln W)^{1+\alpha}} = e^{-c\ln W[(\ln W)^\alpha - \frac{K}{c}]} \ll 1$ for W large enough.

The price to pay is that we get a prefactor of order $e^{K|\Lambda|W^{-2}} \sim e^{K(\ln W)^\alpha}$ at the dominant saddle instead of e^K in the case $d = 3$. It can be compensated by the observable which is of order W^{-2}. This equilibrium between conflicting effects is possible because $d = 2$ is a limit case.

1.4.4. Supersymmetric cluster expansion and integration by parts

Cluster expansion

Both [DPS02] and [DL17] use an inductive cluster expansion (cf. [Riv91, Chapter III.1]). More modern versions as the Brydges-Kennedy-Taylor forest formula [AR95] or the Erice-cluster expansion [Bry86] may be used as well but would complicate the procedure.

Let us start with an easy example. Consider a normalized Gaussian integral with positive definite covariance $C \in \mathbb{R}^{\Lambda\times\Lambda}$ and some diagonal observable $\mathcal{O}(x) = \prod_j \mathcal{O}_j(x_j)$:

$$F = \int d\mu_C(x)\mathcal{O}(x).$$

We divide Λ into equally sized, disjoint cubes $\Lambda = \bigcup_j \triangle_j$. Let $\triangle_0$ be some given cube. Then we manipulate C with an interpolation parameter $s \in [0,1]$ as follows

$$C(s)_{ij} = \begin{cases} sC_{ij} & \text{if } i \in \triangle_0, j \notin \triangle_0 \text{ or vice versa,} \\ C_{ij} & \text{otherwise,} \end{cases}$$

to decouple $\triangle_0$ from the remaining volume. Note that this is equivalent to

$$C(s) = sC + (1-s)[C_{\triangle_0\triangle_0} + C_{\triangle_0^c\triangle_0^c}]$$

(where $C_{\triangle\triangle}$ is C restricted to $\triangle$). In particular, $C(0) = C_{\triangle_0\triangle_0} + C_{\triangle_0^c\triangle_0^c}$ is block-diagonal and $C(s)$ is positive definite since it is a convex combination of positive definite matrices. Setting $F(s) = \int \mathrm{d}\mu_{C(s)}(x)\mathcal{O}(x)$ we obtain by the fundamental theorem of calculus

$$F = [F(s)]_{s=1} = [F(s)]_{s=0} + \int_0^1 \mathrm{d}s\ \partial_s F(s),$$

where $F(0) = \int_{\triangle_0} \mathrm{d}\mu_{C_{\triangle_0\triangle_0}}(x)\mathcal{O}_{\triangle_0} \times \int_{\triangle_0^c} \mathrm{d}\mu_{C_{\triangle_0^c\triangle_0^c}}(x)\mathcal{O}_{\triangle_0^c}$ partitions the domain of integration into $\triangle_0$ and its complement. Using integration by parts, one can move the derivative of the covariance to the observable and get, using $\partial_s C(s) = C_{\Delta_0^c\Delta_0} + C_{\Delta_0\Delta_0^c}$

$$\partial_s F(s) = \sum_{i\in\triangle_0, j\notin\triangle_0} C_{ij} \int \mathrm{d}\mu_{C(s)}(x)\partial_{x_i}\partial_{x_j}\mathcal{O}(x) = \sum_{i\in\triangle_0, j\notin\triangle_0} C_{ij}F_{(i,j)}(s).$$

We have extracted a new cube $\triangle_1$ containing j. This procedure can be continued inductively. In the second step one fixes the points i and j and extracts a new cube $\triangle_2$ containing neither of them. This produces a tree structure on the extracted cubes and one ends up with a sum over polymers P (i.e. unions of disjoint cubes)

$$F = \sum_{\substack{\mathcal{P}=(P_1,\dots P_n)\\ \text{polymers}}} \prod_{k=1}^{n} \left[\sum_{\substack{T_k\\ \text{tree on } P_k}} \sum_{\substack{V\\ \text{vertex of } T_k}} \prod_{l\in T} C_{V_l} \int_{[0,1]^{|T|}} \prod_l \mathrm{d}s_l\ M_T(s)\ F_V(s) \right],$$

where we sum over all disjoint polymers $\mathcal{P} = (P_1, \dots P_n)$. On each polymer we sum over all possible tree structures and then, for each tree edge connecting $\triangle$ and $\triangle'$, we sum over the endpoints $V = \{i,j\}$ of the covariance C_{ij} for $i \in \triangle$ and $j \in \triangle'$. Finally $M_T(s)$ is a product of s factors. This expansion should allow easier estimates than the whole integral before.

To perform the cluster expansion in our case, we start from the supersymmetric integral

$$I_{rem} = \int \mathrm{d}\mu_B(M)\,\mathrm{e}^{\mathcal{V}(M)} a_0,$$

where $\mathcal{V}(M) = \sum_j \mathcal{V}(M_j)$, $\mathcal{V}(M_j) = \mathcal{V}_j(a,b) + \bar{\rho}_j\rho_j D_j$ and $\mathrm{d}\mu_B(M)$ is the Gaussian measure for $M = \begin{pmatrix} a & \bar{\rho} \\ \rho & ib \end{pmatrix}$, i.e. $\mathrm{d}\mu_B(M) = \mathrm{d}M\,\mathrm{e}^{-\frac{1}{2}\mathrm{Str}\,(M,B^{-1}M)}$. The procedure is very similar to the sketch above with two main differences.

First of all, we have a complex covariance $B = (C^{-1} + im_i^2)^{-1}$ but we manipulate the real covariance C instead of B and set $B(s) = (C(s)^{-1} + im_i^2)^{-1}$. This extracts a multilink consisting of one, two or three cubes at a time and leads to a more complicate propagator $G(s)CG(s)$ instead of C (cf. Lemma 2.4.4). Note that this propagator depends on the coupling parameters s. This is why we use this (a bit old fashioned) inductive cluster expansion instead of other versions. Nevertheless interpolating instead B directly would lead to expressions like$(\mathrm{Re}\, B(s)^{-1})^{-1}$ which are very difficult to control.

Secondly, we have a supersymmetric integral. This improves the procedure since we have no normalization factor and the integral outside the polymer containing 0 yields 1 (cf. Lemma 2.4.3).

We obtain the following

$$I_{rem} = \sum_{\substack{P \text{ polymer:} \\ 0 \in P}} \sum_{\substack{T \text{ tree} \\ \text{on } P}} \sum_{\substack{V \\ \text{multilink}}} \int_{[0,1]^{|T|}} \prod_q \mathrm{d}s_q \prod_q [G(s)CG(s)]_{V_q} M_T(s)\; F_V(s),$$

where

$$F_V(s) = \int \mathrm{d}\mu_{B(s)}(M) \prod_q \mathrm{Str}\,(\partial_{M_{i_q}} \partial_{M_{j_q}}) \left[a_0\, \mathrm{e}^{\mathcal{V}(M)}\right].$$

To bound I_{rem} we first integrate the fermionic part of $F_V(s)$ and estimate the remaining part. Therefore, we partition the region of integration in each cube again in small and large field regions and use the finite volume estimates (cf. Eq. (1.4.8) and (1.4.9)). At the dominant saddle we gain from each derivative a factor $W^{-1}\sqrt{\ln W}$. This can be extracted in the other regions as well since we have proven (almost) exponential decay in W.

The decay of $G(s)CG(s)$ enables us to sum over both: the vertex position inside each cube and the cube position. To sum also over the tree structure, we demand that the remaining factor $g = K^{(\ln W)^\alpha} W^{-1/3+\varepsilon} \ll 1$ for W large enough. Then

$$\sum_{r \geqslant 1} \sum_{T \text{ tree with } r \text{ vertices}} g^{r-1} < \infty.$$

For details we refer to Chapter 2. As a result, we obtain

$$|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \leqslant \frac{1}{W}\, \mathrm{e}^{(\ln W)^\alpha}.$$

To improve the bound to W^{-2}, we need some additional step of integration by parts.

Remark. [DPS02] use the same approach as we do, but without the supersymmetric matrix M which simplifies and clarifies the formulas. Moreover, in [DPS02] the observable has a more involved expression. Finally our bounds are more tricky because the covariance has a weaker decay in $d = 2$ than in $d = 3$.

Integration by parts

Before applying the cluster expansion we again use integration by parts (cf. Eq. (2.4.5)) and write

$$I_{rem} = \int \mathrm{d}\mu_B(M)\, \mathrm{e}^{\mathcal{V}(M)} a_0 = -\sum_{l_0} B_{0l_0} \int \mathrm{d}M \partial_{a_{l_0}} \left[\mathrm{e}^{-\frac{1}{2}\mathrm{Str}\,(M, B^{-1}M)} \right] \mathrm{e}^{\mathcal{V}(M)}$$
$$= \sum_{l_0} B_{0l_0} \int \mathrm{d}\mu_B(M) \partial_{a_{l_0}} \mathrm{e}^{\mathcal{V}(M)}.$$

Hence, we transform the observable a_0 into a partial derivative of the potential. This simplifies the procedure since now there is only one term in the integral F_V, namely $\mathrm{e}^{\mathcal{V}(M)}$, which collects derivatives - from the observable and from the cluster expansion. The additional derivative gives a factor W^{-2} which improves the estimates at the dominant saddle and we end up with the stated correction

$$|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \leqslant W^{-2}\, \mathrm{e}^{K(\ln W)^\alpha}.$$

The decay of B enables the summation over $l_0 \in \Lambda$. Note that this step also simplifies the analysis of the derivatives of G^+ a lot.

1.4.5. Conclusion

We have seen that $d = 2$ is a limit case because of weaker estimates for the covariance. Therefore, we expect the correction to the semicircle law to be larger than in $d = 3$. By extending the underlying volume from the natural choice W^2 to $W^2(\ln W)^\alpha$ for some $\alpha \in (0, 1)$ we obtain control over the non-dominant saddle and can apply a cluster expansion.

To optimize α one would need to track the constants c and K more carefully. To sum over the cluster expansion in the end, we choose $W_0(\alpha)$ such that a certain term $g = K^{(\ln W)^\alpha} W^{-1/3+\varepsilon} \ll 1$ for $W \geqslant W_0(\alpha)$ large enough. Another constraint for $W_0(\alpha)$ and therefore also for α is that the decay at the second saddle $\mathrm{e}^{-c(\ln W)^{1+\alpha}}$ needs to bound factors W^n as described above. Note that we collect not too many W factors, precisely not more than three per extracted tree line.

In addition we obtain more reduced and simplified formulas by a more compact notation for the supersymmetric approach and some apriori steps of integration by parts.

1.5. Supersymmetric polar coordinates with applications to the Lloyd model

We have seen that spectral properties of random operators are encoded in the average of the Green's function. For probability distributions with enough finite moments, the standard supersymmetric approach offers a useful dual representation as we have seen in the case of Gaussian band matrices. In this section we summarize the results of [DL20], where we enlarge the applicability of the supersymmetric approach. We introduce an alternative dual representation that remains valid for a very large class of probability distributions, precisely for all these that have an integrable random distribution. In particular, the moments can be infinite. This representation is based on supersymmetric polar coordinates.

As an application, we study the density of states for the Lloyd model, which is a random Schrödinger model with Cauchy distributed random variables. We study three cases: the classical Lloyd model with independent random variables, the case with positive linearly correlated random variables and a third model with some localized negative perturbations to the classical case. For the first two cases we recover known results of [Llo69, Sim83]. The third case provides a new application as far as we know.

1.5.1. Motivation and Setting

Consider a random Schrödinger operator $H : l^2(\Lambda) \to l^2(\Lambda)$ on a finite volume $\Lambda \subset \mathbb{Z}^d$ given by

$$H = -\Delta + \lambda V, \tag{1.5.1}$$

where $-\Delta$ is the discrete Laplacian on Λ with certain boundary conditions and V a multiplication operator $(V\varphi)_j = V_j\varphi_j$, where $\{V_j\}_{j\in\Lambda}$ are random variables with joint distribution μ.

Let us assume first the V_j to be i.i.d. We have seen in Section 1.3.4 that $\mathbb{E}[\mathrm{Tr}\,(z-H)^{-1}]$ leads to the density of states. The supersymmetric approach enables to rewrite this expression as a supersymmetric integral:

$$(z-H)^{-1}_{jj} = \int [\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\,\mathrm{e}^{i\Phi^*(z+\Delta-\lambda V)\Phi}|z_j|^2,$$

where $\Phi = (z,\chi)$ is a supervector consisting of $|\Lambda|$ complex variables z_j and $|\Lambda|$ Grassmann variables χ_j (cf. Section 1.2). Here we use the convention $\int \mathrm{d}\chi\ \chi = 1$ and set $[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi] := (2\pi)^{-N}\prod_{j=1}^N \mathrm{d}\bar{z}_j\,\mathrm{d}z_j\,\mathrm{d}\bar{\chi}_j\,\mathrm{d}\chi_j$ with a now included factor $(2\pi)^N$. If we are able to move the average inside the integral, we have to evaluate only

$$\mathbb{E}[\,\mathrm{e}^{-i\lambda\Phi^*V\Phi}] = \prod_j \mathbb{E}[\,\mathrm{e}^{-i\lambda\Phi_j^*\Phi_jV_j}] = \prod_j \mathbb{E}[\,\mathrm{e}^{-i\lambda\bar{z}_jz_jV_j}(1-i\lambda\bar{\chi}_j\chi_jV_j)]$$

instead of $\mathbb{E}[\mathrm{Tr}\,(z-H)^{-1}]$, where we used that $(\bar{\chi}_j\chi_j)^2 = 0$.

Interchanging average and integral by Fubini is only possible if we have integrable functions. Let now μ_0 be the probability density of a single V_j and assume $\mathbb{E}[|V_j|] < \infty$. Then

$$\mathbb{E}[\,\mathrm{e}^{-i\lambda\Phi_j^*\Phi_j V_j}] = \int \mathrm{e}^{-i\lambda\Phi_j^*\Phi_j x}\,\mathrm{d}\mu_0(x) = \hat{\mu}_0(\lambda\Phi_j^*\Phi_j),$$

i.e. we can represent the average as a product of Fourier transformations of μ_0 at $\Phi_j^*\Phi_j$, an even element in the Grassmann algebra. The function $\hat{\mu}_0(\lambda\Phi_j^*\Phi_j)$ is well-defined if $\mu_0 \in C^1$. If we consider a general joint density, we obtain

$$\mathbb{E}[\exp(-i\lambda\Phi^* V\Phi)] = \int \mathrm{e}^{-i\lambda\Phi^*\Phi x}\,\mathrm{d}\mu(x) = \hat{\mu}(\lambda\Phi^*\Phi).$$

This formula holds if $\hat{\mu}$ admits enough derivatives.

For Cauchy distributed variables we have no finite moment. If $\mathrm{d}\mu = \frac{1}{\pi}\frac{1}{1+x^2}\,\mathrm{d}x$ then $\mathbb{E}[|V|] = \infty$ for $V \sim \mu$. Random Schrödinger operators with Cauchy distributed V (also called Lloyd model) have been studied with other tools by Lloyd [Llo69] and Simon [Sim83]. One obtains an explicit representation for the density of states:

$$\begin{aligned}\bar{\rho}_\Lambda^{(H)}(E) &= \lim_{\varepsilon\to 0}\mathbb{E}[\operatorname{Tr} G_\Lambda^{(H)}(E+i\varepsilon)]\\ &= \operatorname{Tr}\left((E+i\lambda)\mathbb{1}_\Lambda - H_0\right)^{-1} = \operatorname{Tr} G_\Lambda^{(H_0)}(E+i\lambda),\end{aligned}$$

where $H_0 = -\Delta_0$ is the free Laplacian and λ is the parameter of disorder.

Our goal is to construct a new supersymmetric representation that can be applied also to less regular distributions such as the Cauchy distribution and more generally for integrable V. With this we can reprove the above results [Llo69, Sim83]. Moreover, we study a toy model with single negative correlations.

We expect that our formula can also help in other cases as a starting point for standard methods such as saddle point analysis, cluster expansions or renormalization (cf. [Fre20]).

1.5.2. Supersymmetric change of variables

A supersymmetric change of variables is a priori only known for functions with compact support (cf. [Efe99, Chapter 2.5] and [Weg16, Chapter 10.3] or Theorem 1.2.18). If we consider functions with non-compact support, we obtain additional boundary terms that become very complicated for an arbitrary transformation.

In the following we consider supersymmetric polar coordinates mapping $U = \mathbb{C}\setminus\{0\}$ to $\mathbb{R}^+ \times (0, 2\pi)$. Hence, 0 is a boundary term of U and a compact supported function f needs to fulfil $f(0) = 0$. For a general f we have derived a compact formula with simple explicit expressions for boundary terms.

In the following we use the notation from Section 1.2 above. Consider first a Grassmann algebra with two generators $\mathcal{G}[\bar{\chi}, \chi]$. The idea of supersymmetric polar coordinates is to transform between $(\bar{z}, z, \bar{\chi}, \chi)$ with $z \in \mathbb{C}$ and $(r, \theta, \bar{\rho}, \rho)$ with $r \in \mathbb{R}^+$ and $\theta \in (0, 2\pi)$

such that $\bar{z}z + \bar{\chi}\chi = r^2$. If we take

$$z = \mathrm{e}^{i\theta}(r - \tfrac{1}{2}\bar{\rho}\rho) \qquad \bar{z} = \mathrm{e}^{-i\theta}(r - \tfrac{1}{2}\bar{\rho}\rho)$$
$$\chi = \sqrt{r}\rho \qquad \bar{\chi} = \sqrt{r}\bar{\rho}$$

we have indeed $\bar{z}z + \bar{\chi}\chi = (r - \frac{1}{2}\bar{\rho}\rho)^2 + r\bar{\rho}\rho = r^2$.

Note that 0 is a boundary point for this transformation since $U = \mathbb{C}\backslash\{0\}$ is mapped to $\mathbb{R}^+ \times (0, 2\pi)$. If f has compact support in U (i.e. especially $f(0) = 0$) we can apply Theorem 1.2.18, the standard version of the supersymmetric change of coordinates formula, where the Jacobian is replaced by the Berezinian. On the contrary, for functions with $f(0) \neq 0$, we get additional boundary terms.

In the following we prove a one-dimensional version of our result. In that case we obtain the desired integral in polar coordinates with a constant Berezinian and a single boundary term at 0.

Theorem 1.5.1. *Let $f \in \mathcal{G}_{2,2}(\mathbb{C})$ be integrable, i.e. all $f_I : \mathbb{C} \to \mathbb{C}$ are integrable, we have*

$$\begin{aligned} I(f) &:= \int_{\mathbb{C}} \mathrm{d}\bar{z}\,\mathrm{d}z\,\mathrm{d}\bar{\chi}\,\mathrm{d}\chi\; f(\bar{z}, z, \bar{\chi}, \chi) \\ &= 2\int_{\mathbb{R}^+\times(0,2\pi)} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho\; f \circ \Psi(r, \theta, \bar{\rho}, \rho) + 2\pi\; f \circ \Psi(0), \end{aligned}$$

where $\Psi(r, \theta, \bar{\rho}, \rho) = (\,\mathrm{e}^{-i\theta}(r - \frac{1}{2}\bar{\rho}\rho),\, \mathrm{e}^{i\theta}(r - \frac{1}{2}\bar{\rho}\rho), \sqrt{r}\bar{\rho}, \sqrt{r}\rho)$.

Proof. We split our transformation Ψ into three steps. The first one is the change to standard complex polar coordinates, the second a rescaling of the Grassmann variables and the third a translation of $|z|$ in the "Grassmann plane":

$$\begin{array}{ccccccc} z & \overset{\Psi_1}{\to} & re^{i\theta} & \overset{\Psi_2}{\to} & re^{i\theta} & \overset{\Psi_3}{\to} & \left(r - \frac{1}{2}\bar{\rho}\rho\right) e^{i\theta}, \\ \chi & \overset{\Psi_1}{\to} & \chi & \overset{\Psi_2}{\to} & \sqrt{r}\rho & \overset{\Psi_3}{\to} & \sqrt{r - \frac{1}{2}\bar{\rho}\rho}\;\rho = \sqrt{r}\rho. \end{array}$$

The first transformation $\Psi_1(r, \theta, \chi, \bar{\chi}) = (r\,\mathrm{e}^{i\theta}, r\,\mathrm{e}^{-i\theta, \chi, \bar{\chi}})$ is an ordinary transformation with Jacobian $2r$. The second one $\Psi_2(r, \theta, \rho, \bar{\rho}) = (r, \theta, \sqrt{r}\rho, \sqrt{r}\bar{\rho})$ ia a linear transformation in the purely fermionic variables, where we can apply Theorem 1.2.13. Here, we obtain a factor r^{-1} which cancels the Jacobian from above up to a constant. The third transformation $\Psi_3(r, \theta, \rho, \bar{\rho}) = (r - \frac{1}{2}\bar{\rho}\rho, \theta, \rho\bar{\rho})$ mixes bosonic and fermionic variables and produces the boundary term. We calculate

$$\begin{aligned} I(f) &= \int_{\mathbb{C}} \mathrm{d}\bar{z}\,\mathrm{d}z\,\mathrm{d}\bar{\chi}\,\mathrm{d}\chi\; f(\bar{z}, z, \bar{\chi}, \chi) \\ &= \int_{\mathbb{R}^+\times(0,2\pi)} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\chi}\,\mathrm{d}\chi\; 2r\; f \circ \Psi_1(r, \theta, \bar{\chi}, \chi) \\ &= 2\int_{\mathbb{R}^+\times(0,2\pi)} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho\; f \circ \Psi_1 \circ \Psi_2(r, \theta, \bar{\rho}, \rho). \end{aligned}$$

For the third step, we expand $\tilde{f} = f \circ \Psi_1 \circ \Psi_2 \circ \Psi_3$ as follows

$$f \circ \Psi_1 \circ \Psi_2(r, \theta, \bar{\rho}, \rho) = \tilde{f}(r + \tfrac{\bar{\rho}\rho}{2}, \theta, \bar{\rho}, \rho) = \tilde{f}(r, \theta, \bar{\rho}, \rho) + \tfrac{\bar{\rho}\rho}{2} \partial_r \tilde{f}(r, \theta, \bar{\rho}, \rho).$$

Inserting this we have

$$I(f) = 2 \int_{\mathbb{R}^+ \times (0,2\pi)} \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\bar{\rho} \, \mathrm{d}\rho \; \tilde{f}(r, \theta, \bar{\rho}, \rho) - \int_{\mathbb{R}^+ \times (0,2\pi)} \mathrm{d}r \, \mathrm{d}\theta \; \partial_r \tilde{f}(r, \theta, 0, 0),$$

where we integrated over the Grassmann variables in the second term. By integration by parts

$$\int_{\mathbb{R}^+ \times (0,2\pi)} \mathrm{d}r \, \mathrm{d}\theta \; \partial_r \tilde{f}(r, \theta, 0, 0) = - \int_{(0,2\pi)} \mathrm{d}\theta \; \tilde{f}(0, \theta, 0, 0) = -2\pi \tilde{f}(0),$$

where we used that $\lim_{r \to \infty} \tilde{f}(r, \theta, 0, 0) = 0$ since f (and hence $\tilde{f}$) is integrable and $\tilde{f}(0, \theta, 0, 0)$ is independent of θ. $\square$

If we consider now $f \in \mathcal{G}_{2N,2N}(\mathbb{C}^N)$, we obtain for each index two contributions, a boundary term and the full integral, hence we get in total 2^N terms. Note that in our notation $[\, \mathrm{d}\Phi^* \, \mathrm{d}\Phi]$ a factor $(2\pi)^N$ is included.

Theorem 1.5.2. *Let $f \in \mathcal{G}_{2N,2N}(\mathbb{C}^N)$ be integrable, i.e. all $f_I : \mathbb{C}^N \to \mathbb{C}$ are integrable, we have*

$$I(f) = \int_{\mathbb{C}^N} [\, \mathrm{d}\Phi^* \, \mathrm{d}\Phi] \; f(\Phi^*, \Phi) = \sum_{\alpha \in \{0,1\}^N} I_\alpha(f)$$

with multiindex α and

$$I_\alpha(f) = \pi^{-|1-\alpha|} \int_{(\mathbb{R}^+ \times (0,2\pi))^{1-\alpha}} (\, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\bar{\rho} \, \mathrm{d}\rho)^{1-\alpha} \; f \circ \Psi_\alpha(r, \theta, \bar{\rho}, \rho),$$

where $(\, \mathrm{d}r)^{1-\alpha} = \prod_{j:\alpha_j = 0} \mathrm{d}r_j$ (and the same for $\theta, \bar{\rho}$ and ρ) and Ψ_α is given by $\Psi_\alpha : (r, \theta, \bar{\rho}, \rho) \mapsto (z, \bar{z}, \chi, \bar{\chi})$ with

$$\begin{cases} z_j(r_j, \theta_j, \bar{\rho}_j, \rho_j) & = \delta_{\alpha_j 0} \, \mathrm{e}^{i\theta_j} (r_j - \frac{1}{2}\bar{\rho}_j \rho_j), \\ \bar{z}_j(r_j, \theta_j, \bar{\rho}_j, \rho_j) & = \delta_{\alpha_j 0} \, \mathrm{e}^{-i\theta_j} (r_j - \frac{1}{2}\bar{\rho}_j \rho_j), \\ \chi_j(r_j, \theta_j, \bar{\rho}_j, \rho_j) & = \delta_{\alpha_j 0} \, \sqrt{r_j} \rho_j, \\ \bar{\chi}_j(r_j, \theta_j, \bar{\rho}_j, \rho_j) & = \delta_{\alpha_j 0} \, \sqrt{r_j} \bar{\rho}_j. \end{cases}$$

Proof. Generalize Theorem 1.5.1 to N sets of variables. For details see Section 3.3. $\square$

1.5.3. Dual representation for random Schrödinger operators

The above formula provides an alternative supersymmetric dual representation. We apply it in the following to the averaged density of states for random Schrödinger operators.

We consider a random Schrödinger operator $H = -\Delta + \lambda V$ on Λ defined in Eq. (1.5.1), where $\{V_j\}_{j\in\Lambda}$ are real random variables with integrable joint density μ.

Theorem 1.5.3. *The supersymmetric polar coordinate representation for the average of the trace of the Green's function $G^+(E+i\varepsilon) = (E+i\varepsilon-H)^{-1}$ for a random Schrödinger operator H reads:*

$$\mathbb{E}[\operatorname{Tr} G^+(E+i\varepsilon)] = \sum_{\alpha\in\{0,1\}^\Lambda} \int_{(\mathbb{R}^+\times(0,2\pi))^{1-\alpha}} \Big(\tfrac{\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar\rho\,\mathrm{d}\rho}{\pi}\Big)^{1-\alpha} \hat\mu(\{\lambda r_j^2\}_{j\in\Lambda})|_{r^\alpha=0}$$
$$\times g\circ\Psi_\alpha(r,\theta,\bar\rho,\rho) \sum_{\substack{j\in\Lambda:\\ \alpha_j=0}} (r_j^2 - r_j\bar\rho_j\rho_j),$$

where $g(\Phi^*,\Phi) = \mathrm{e}^{i\Phi^*(E+i\varepsilon+\Delta)\Phi}$ and $\hat\mu(\{\lambda r_j^2\}_{j\in\Lambda})$ is the $|\Lambda|$-dimensional, joint Fourier transform of μ. If in addition the V_j's are independent and identically μ_0-distributed then

$$\hat\mu(\{\lambda r_j^2\}_{j\in\Lambda})|_{r^\alpha=0} = \prod_{j\in\Lambda:\alpha_j=0} \hat\mu_0(\lambda r_j).$$

Proof. We start from the classical supersymmetric approach also used in Section 1.4.2. The entry of an inverse matrix has a complex Gaussian integral representation. The corresponding normalization factor can be replaced by a fermionic Gaussian integral. We obtain

$$\begin{aligned}\operatorname{Tr}(E+i\varepsilon-H)^{-1} &= -i\det\big[-i\tfrac{E+i\varepsilon-H}{2\pi}\big]\int \mathrm{d}\bar z\,\mathrm{d}z\,\mathrm{e}^{i(\bar z,(E+i\varepsilon-H)z)}\sum_{k\in\Lambda} z_k\bar z_k\\ &= -i\int[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\,\mathrm{e}^{i\sum_{i,j\in\Lambda}\Phi_i^*(\delta_{ij}(E+i\varepsilon+\lambda V_j)+\Delta_{ij})\Phi_j}\sum_{k\in\Lambda} z_k\bar z_k.\end{aligned}$$

Note that we stick to the normalization of the fermionic integral $\int \mathrm{d}\chi\,\chi = 1$ as introduced in Section 1.2. The factor 2π from the complex Gaussian integral is hidden in $[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi] = (2\pi)^{-N}\,\mathrm{d}\bar z\,\mathrm{d}z\,\mathrm{d}\bar\chi\,\mathrm{d}\chi$. Now we apply Theorem 1.5.2. As a result, randomness appears only in $\exp(i\lambda\sum_{j:\alpha_j=0} r_j^2 V_j)$ which is a bounded function. Hence, we can insert the average inside the integral and obtain the above formula since

$$\mathbb{E}\Bigg[\exp(i\lambda\sum_{j:\alpha_j=0} r_j^2 V_j)\Bigg] = \hat\mu(\lambda r^2)|_{r^\alpha=0}.$$

$\square$

Remark. We can apply Theorem 1.5.2 to other important quantities in this context, e.g. to the generating function $\mathcal{G}$ given by

$$\mathcal{G}_\varepsilon(E,\tilde{E}) = \mathbb{E}\left[\frac{\det((E+i\varepsilon)\mathbb{1}_\Lambda - H_\Lambda)}{\det((\tilde{E}+i\varepsilon)\mathbb{1}_\Lambda - H_\Lambda)}\right].$$

and the Green's function squared $\mathbb{E}[|G_\Lambda(E+i\varepsilon)_{jk}|^2]$. In this last case we need two sets of Grassmann variables (see Theorem 3.2.2).

1.5.4. Application to Lloyd model

We apply the above representation to derive the density of states for the Lloyd model, a random Schrödinger model with Cauchy distributed random variables V_j. More precisely we consider linear correlated random potentials, i.e. $V_j = \sum_k T_{jk} W_k$, where $W_k \sim \text{Cauchy}(0,1)$ are i.i.d. random variables, $T_{jk} = T_{kj} \in \mathbb{R}$ and $\sum_j T_{jk} > 0$.

In the following we discuss two cases: first we study the (positive) correlated Lloyd model with $T_{jk} \geqslant 0$ and $\sum_j T_{jk} > 0$. Note that the classical Lloyd model is a special case of this with $T_{jk} = \delta_{jk}$. Our representation provides a new proof for the results of [Llo69] and [Sim83].

As a new application we consider in a second case a toy model with a single negative correlation, i.e. $T_{jj} = 1$ and $T_{21} = T_{12} = -\delta^2$ with $0 < \delta < 1$ and $T_{jk} = 0$ otherwise. The indices 1 and 2 denote two fixed, nearest neighbour points $i_1, i_2 \in \Lambda$ with $|i_1 - i_2| = 1$.

Theorem 1.5.4 (Positive correlated Lloyd model)**.** *Let $T_{jk} \geqslant 0$ and $\sum_j T_{jk} > 0$. Then we have*

$$\lim_{\varepsilon\to 0} \mathbb{E}[\operatorname{Tr} G_\Lambda(E+i\varepsilon)] = \operatorname{Tr}(E\mathbb{1}_\Lambda + i\lambda\hat{T} - H_0)^{-1},$$

where $H_0 = -\Delta$ and $\lambda\hat{T}$ is a diagonal matrix with $\hat{T}_{ij} = \delta_{ij}\sum_k T_{jk}$.

In particular both, the classical and the (positive) correlated Lloyd model, have the same (averaged) density of states as the free Laplacian $H_0 = -\Delta$ with constant imaginary mass λ and variable mass $\lambda\hat{T}$, respectively.

Idea of the proof. Apply Theorem 1.5.3 and evaluate $\hat{\mu}_0$. Then we transform back from polar coordinates to the classical supersymmetric representation. For details see Section 3.4. □

In the case of a single negative correlation (the toymodel described above) we obtain the following result.

Theorem 1.5.5 (Toy model)**.** *Let $T_{jj} = 1$, $T_{21} = T_{12} = -\delta^2$ with $0 < \delta < 1$ and $T_{jk} = 0$ otherwise, $\lambda > 0$ and $0 < \delta \ll (1+\lambda^{-1})^{-1}$. Then*

$$\lim_{\varepsilon\to 0} \mathbb{E}[\operatorname{Tr} G_\Lambda(E+i\varepsilon)] = \operatorname{Tr}(E\mathbb{1}_\Lambda + i\lambda\hat{T} - H_0)^{-1}\left[1 + \mathcal{O}\Big((\delta(1+\lambda^{-1}))^2\Big) + \mathcal{O}(|\Lambda|^{-1})\right].$$

Idea of the proof. Follows from Theorem 1.5.3 by integrating first over the uncorrelated variables in Λ and estimating the remaining integral. For details see Section 3.4. □

1.5.5. Conclusion

We have introduced a new supersymmetric dual representation which can be applied to a broader set of models, more precisely to those with integrable distributions. As an application we reproved results of [Llo69] and [Sim83] for the Lloyd model with positive correlated potential and enlarged these to a toy model with a single negative correlation. In general this new dual representation can serve as a starting point to standard tools as saddle point analysis, cluster expansions or renormalization.

2. Density of States for Random Band Matrix in two dimensions

2.1. Introduction and main result

General setting. It is a well known fact that conducting properties of disordered materials can be related, in the context of quantum mechanics, to the statistics of eigenvalues and eigenvectors of certain random matrix ensembles [And58]. The most famous example are random Schrödinger operators, whose lattice version is characterized by the random matrix $H_\Lambda : \Lambda \times \Lambda \to \mathbb{R}$, on a subset Λ of $\mathbb{Z}^d$, defined by $H_\Lambda = -\Delta + \lambda V$, where $-\Delta$ is the discrete Laplacian, V is a diagonal matrix with random diagonal entries and $\lambda > 0$ is a parameter encoding the strength of the disorder. The entries V_j are generally assumed to be independent identically distributed (for instance Gaussian). As $\Lambda \uparrow \mathbb{Z}^d$, this model exhibits a localized phase for all λ in $d = 1$ and at large disorder $\lambda \gg 1$ in $d \geqslant 2$. The localized phase is conjectured to hold also at weak disorder $\lambda \ll 1$ in $d = 2$, while a phase transition is conjectured in $d \geqslant 3$. Though the localized phase is well understood, the weak disorder regime in $d \geqslant 2$ remains an open problem. For a review of definitions and results see for instance [KK08].

Another relevant model in this context is the random band matrix (RBM) ensemble, characterized by a self-adjoint matrix $H_\Lambda : \Lambda \times \Lambda \to \mathbb{K}$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$, whose entries are *all* independent (up to self-adjointness) random variables *not* identically distributed with negligible entries outside a band of width $W \geqslant 0$, i.e. $|H_{ij}| \ll 1$, with large probability, when $|i - j| > W$. As in the case of random Schrödinger, when $\Lambda \uparrow \mathbb{Z}^d$, band matrices are believed to exhibit a phase transition in $d \geqslant 3$ between a localized phase at small W and an extended phase at large W, while the localized phase is conjectured to hold for all W in $d = 1, 2$.

Two important examples of RBM are the 'smooth Gaussian' and the N-orbital model. In the first case, the matrix elements are Gaussian:

$$H_{ii} \sim \mathcal{N}_\mathbb{R}(0, J_{ii}), \qquad H_{ij} \sim \mathcal{N}_\mathbb{C}(0, J_{ij}), \quad \text{for } i < j,$$

where $<$ denotes an order relation on $\mathbb{Z}^d$, and the band structure is encoded in the covariance $J_{ij} = J_{ji} = f(|i - j|)$ decaying to zero when $|i - j| \gg W$. In the second case the covariance J_{ij} is short range, for instance $J = \mathrm{Id} + a\Delta$ for some $a > 0$, but each matrix element H_{ij} is itself a $N \times N$ matrix with i.i.d. entries $(H_{ij})_{\alpha\beta} \sim \mathcal{N}_\mathbb{C}(\frac{1}{N} J_{ij})$ $\forall i < j$, or $i = j$ and $\alpha < \beta$ and $(H_{ii})_{\alpha\alpha} \sim \mathcal{N}_\mathbb{R}(\frac{1}{N} J_{ii})$. The band width in this case is $W = 2N$. These models are difficult to analyse with standard random matrix tools, since the probability distribution is not invariant under unitary rotations. At the moment, most results available deal with the one dimensional case (cf. [CMI90, CCGI93, Sch09, Sod10, BGP14, Shc14, BE16, Pch15]). Recently localization at strong disorder (i.e. small band width) in any dimension was proved for a large class of $N-$orbital models by Peled, Schenker, Shamis and Sodin [PSSS17].

In this paper we consider the density of states $\rho_\Lambda(E) := \frac{1}{|\Lambda|}\sum_j \delta_{\lambda_j}(E)$, where $E \in \mathbb{R}$ is the energy and λ_j are the (random) eigenvalues of H. Since the probability distribution is translation invariant and the bandwidth W is fixed, by standard ergodicity arguments (see [PF92]) this measure is non random in the thermodynamic limit. Similar results hold in $d = 1$ also for the (non ergodic) case when W diverges together with the matrix size [MPK92]. We will therefore concentrate on the *averaged density of states* (DOS) defined by

$$\bar{\rho}_\Lambda(E) := \mathbb{E}[\rho_\Lambda(E)] := \frac{1}{|\Lambda|}\mathbb{E}\left[\sum_j \delta_{\lambda_j}(E)\right] = -\frac{1}{\pi|\Lambda|}\lim_{\varepsilon\to 0}\mathbb{E}[\operatorname{Im}\operatorname{Tr} G_\Lambda^+(E_\varepsilon)], \tag{2.1.1}$$

where $E_\varepsilon := E + i\varepsilon$, with $\varepsilon > 0$, $\mathbb{E}$ denotes the average with respect to the probability distribution of H and, for any $z \in \mathbb{C}$, the Green's function (or resolvent) is defined by

$$G_\Lambda^+(z) := (z \cdot \mathbb{1} - H)^{-1} = (z - H)^{-1}.$$

By standard analyticity arguments, the limit $\varepsilon \to 0$ in (2.1.1) above exists and is finite for Lebesgue a.e. $E \in \mathbb{R}$ (see for ex. [AW15, App. B]).

In dimension larger than one rigorous results on the density of states for RBM were obtained by [DPS02, CFGK87], and more recently by [PSSS17], based on seminal work by Wegner [Weg79]. In the case of the classical GUE ensemble, corresponding to $d = 1$, $\Lambda = 1, \ldots, N$ and $J_{ij} = 1/N\ \forall i, j$, the density of states, in the limit $N \uparrow \infty$ is given a.s. by Wigner's famous semicircle law

$$\rho_{SC}(E) = \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{E^2}{4}} & \text{if } |E| \leqslant 2, \\ 0 & \text{if } |E| > 2. \end{cases} \tag{2.1.2}$$

The model. We consider a Gaussian complex RBM ensemble defined on a *two dimensional* discrete cube $\Lambda \subset \mathbb{Z}^2$, centered at the origin, with covariance

$$J_{ij} := (-W^2\Delta + \mathbb{1})^{-1}_{ij}, \tag{2.1.3}$$

where $-\Delta \in \mathbb{R}^{\Lambda\times\Lambda}$ is the discrete Laplace operator on Λ with periodic boundary conditions, i.e. $(x, -\Delta x) = \sum_{i\sim j}(x_i - x_j)^2$ for any $x \in \mathbb{R}^\Lambda$ and we write $i \sim j$ when i and j are nearest neighbors in the torus $\mathbb{Z}^2/(\Lambda\mathbb{Z}^2)$. The parameter $W \in \mathbb{R}$ is large but fixed. Note that J_{ij} is exponentially small for distances $|i - j| \geqslant W$. Hence all matrix elements outside a (two-dimensional) band of width W centered around the diagonal are small with high probability and this model describes a "smoothed-out" version of a band matrix ensemble with band width W.

For this model, the averaged density of states (2.1.1) exists and takes a finite value for all $E \in \mathbb{R}$. In the *three dimensional* case, Disertori, Pinson and Spencer [DPS02] derived an explicit representation of the function $\bar{\rho}_\Lambda(E)$, in the bulk of the spectrum, in terms of a convergent sum of certain integrals. Using this representation they obtained detailed information on the function $E \mapsto \bar{\rho}_\Lambda(E)$ and its derivatives, in the limit $\Lambda \to \mathbb{Z}^3$ for fixed but large band width W (weak disorder regime). In particular they proved that the limit expression coincides with Wigner's semicircle law with a precision $1/W^2$. Similar results were obtained for the $N-$orbital model in [CFGK87] in the case of dominant diagonal disorder. In this paper we construct an extension of the representation derived in [DPS02] to the two dimensional case, and use it to derive precise information (such as smoothness) on the function $E \mapsto \bar{\rho}_\Lambda(E)$ for energies in the bulk of the spectrum, in the limit $\Lambda \to \mathbb{Z}^2$.

The proof in [DPS02] used the so-called *supersymmetric approach* (SUSY), pioneered by K. Efetov [Efe83] based on seminal work by Schäfer and Wegner [SW80, Weg79] and further developed (among others) by Y. Fyodorov, A. Mirlin and M. Zirnbauer [Mir00, FM91, LSZ08]. A good introduction to random matrix theory and SUSY can be found in [HG19]. This is a duality transformation that allows one to write averages in H as new integrals where a saddle approximation may be justified: $\mathbb{E}[f(H)] = \int \tilde{f}(M)\, \mathrm{e}^{-F(M)} \prod_{j\in\Lambda} \mathrm{d}M_j$, where M_j is a small matrix containing both complex and Grassmann (odd) elements, F can be seen as the free energy functional in some statistical mechanical model, and $\tilde{f}$ is the new observable. The Grassmann variables can be always integrated out exactly (though the combinatorics involved may be quite difficult) and the resulting measure is complex but *normalized*. The measure $\exp(-F(M))\,\mathrm{d}M$ depends on the probability measure P but also on the observable f, and has internal (odd) symmetries, *inherited by the observable only*. Different rigorous versions of the SUSY approach have been tested on the standard matrix ensembles GUE and GOE (where other techniques also apply) [Dis04, Sha13]. When $\Lambda \uparrow \mathbb{Z}^d$, the integral is expected to concentrate near the saddle manifold determined by $\partial_M F(M) = 0$. This reduces to *isolated points* in the case of the DOS, and the main difficulty is to obtain estimates on the fluctuations that are uniform in the volume. In [DPS02] the dual representation is studied via a complex translation coupled with a cluster expansion, which effectively factorizes the integral over regions of volume W^3. A key feature of the dual representation, in three dimensions, is the presence of a *double well* structure, with one well suppressed by an exponential factor $\exp(-W)$, inside each region. In the two dimensional case, the second well is only weakly suppressed, and the arguments used in [DPS02] do not apply.

Main result. This article is devoted to prove the following result which extends [DPS02, Theorem 1] to the two dimensional case.

Remember that in the case of GUE the spectrum of H, in the thermodynamic limit, is concentrated on $\{|E| \leqslant 2\}$ (see (2.1.2)). In the case of a band matrix, non rigorous arguments suggest that most of the spectrum remains in the same interval. Here, we restrict to energies E in the bulk and avoid $E = 0$ for technical reasons. Precisely, for $\eta > 0$ small but fixed, we consider the interval

$$\mathcal{I} = \{E : \eta < |E| \leqslant 1.8\}. \tag{2.1.4}$$

Theorem 2.1.1. *For $d = 2$ and each fixed $\alpha \in (0, 1)$, there exists a value $W_0(\alpha)$ such that for all $W \geqslant W_0(\alpha)$ and $E \in \mathcal{I}$*

$$|\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| \leqslant W^{-2}\, \mathrm{e}^{K(\ln W)^\alpha}, \tag{2.1.5}$$

$$|\partial_E^n \bar{\rho}_\Lambda(E)| \leqslant C_n W^{-1} (\ln W)^{n(\alpha+1)}\, \mathrm{e}^{K(\ln W)^\alpha} \qquad \forall n \geqslant 0, \tag{2.1.6}$$

where ρ_{SC} is Wigner's semicircle law defined in (2.1.2)*. The constants C_n and K are independent of Λ and W. Both estimates hold uniformly in Λ and hence also in the infinite volume limit $\Lambda \to \mathbb{Z}^2$.*

Remark. Note that we obtain the semicircle law with a precision

$$\bar{\rho}_\Lambda(E) = \rho_{SC}(E) + \mathcal{O}\left(W^{-2+\delta}\right)$$

for small $\delta > 0$ depending on $W_0(\alpha)$, while in $d = 3$ one obtains $\mathcal{O}(W^{-2})$ [DPS02, eq. (2.7)]. Moreover, (2.1.6) implies a W-independent estimate on the derivatives up to a certain order $n_0(W)$

$$|\partial_E^n \bar{\rho}_\Lambda(E)| \leqslant C_n \quad \forall n \leqslant n_0(W),$$

with $\lim_{W\to\infty} n_0(W) = \infty$.

Strategy. The strategy is similar to the one in [DPS02]. We establish a dual representation for the averaged DOS via the supersymmetric approach and apply a complex translation into the saddle points. To overcome the second well problem, we modify the factorization procedure, using slightly larger blocks (of size $W^2(\ln W)^\alpha$ instead of the natural W^2) for our cluster expansion. This yields better estimates for configurations near the second saddle, but creates new problems for the 'good' configurations, near the main saddle. An equilibrium between these two conflicting effects is possible because $d = 2$ is a 'limit' case. Finally, as in [DPS02], we apply a non standard cluster expansion, extracting at each step a 'multi-link' consisting of three instead of one connection. Here, in contrast to [DPS02], we use the (super-) symmetric structure of the dual representation to reformulate the cluster expansion in a more compact and transparent way. We also use the symmetry to simplify the dual representation and a number of other equalities.

Remark. Note that Wegner-type estimates on the integrated density of states can be obtained, in any dimension, by softer methods (see [PSSS17]). Here, the dual representation plus cluster expansion give an explicit representation of the function $\frac{1}{|\Lambda|}\mathbb{E}[\mathrm{Tr}\,(z - H)^{-1}]$ where the limit $\mathrm{Im}\, z \to 0$ can be taken explicitely. This representation remains valid in the thermodynamic limit and allows to study detailed properties (such as smoothness and main contributions) of the limit function.

Organization of the paper. In Section 2.2, the dual representation and the complex contour deformation are introduced. This allows to reformulate the problem as Theorem 2.2.3. Finally a sketch of the proof's strategy is given. In Section 2.3, we summarize some properties and preliminary estimates that will be needed in the proof. We also prove the main result in a finite volume. These results build a foundation for the infinite volume case. In Section 2.4, the cluster expansion is introduced and the limit $\Lambda \uparrow \mathbb{Z}^2$ is analyzed. A short introduction to the supersymmetric formalism is given in App. 2.A, and a proof of the dual representation is given in App. 2.B. Finally, App. 2.C collects some results on the discrete Laplace operator in $d = 2$ and some matrix inequalities, together with their proof. A list of symbols can be found at the end.

Notation. Since we apply many estimates in the paper, we denote by K any large positive constant independent of W and Λ.

Acknowledgements We are grateful to Tom Spencer for encouraging us to pursue the $d = 2$ case and for numerous discussions. We thank Sasha Sodin for useful discussions on representations of the discrete Laplacian. We are also grateful to Martin Lohmann, Susanne Hilger, Richard Höfer and Anna Kraut for helpful discussions and suggestions related to this paper. Very special thanks go to David Brydges for sharing his many insights on the model and inspiring discussions on cluster expansions. Finally, we acknowledge the Deutsche Forschungsgemeinschaft for support through CRC 1060 "The Mathematics of Emergent Effects" and the Hausdorff Center for Mathematics.

2.2. Reformulating the problem

We perform a duality transformation and a complex contour deformation to rewrite the average of the Green's function as a Gaussian integral with some remainder (cf. Lemma 2.2.2). This reduces the proof of Theorem 2.1.1 to bound this functional integral appropriately (cf. Theorem 2.2.3). In the end we give ideas for the proof.

2.2.1. Duality transformation

The first step in the proof is to represent, via the supersymmetric formalism (cf. App. 2.A), the trace in (2.1.1) as a functional integral where a saddle point analysis can be justified. Recall the definition of J given by (2.1.3). A normalized Gaussian measure with covariance J is defined by

$$\mathrm{d}\mu_J(a,b) := \det\left[\tfrac{J^{-1}}{2\pi}\right] \mathrm{e}^{-\frac{1}{2}((a,J^{-1}a)+(b,J^{-1}b))} \prod_{j\in\Lambda} \mathrm{d}a_j\, \mathrm{d}b_j, \tag{2.2.1}$$

with $a, b \in \mathbb{R}^\Lambda$, $\prod_{j\in\Lambda} \mathrm{d}a_j\, \mathrm{d}b_j$ is the product measure and $(a, J^{-1}a) = \sum_{i,j\in\Lambda} a_i J^{-1}_{ij} a_j$. With this definition we can state the following lemma which is a variant of [DPS02, Lemma 1].

Lemma 2.2.1. *For any space dimension $d \geqslant 1$, the following identities hold:*

$$\frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G_\Lambda^+(E_\varepsilon)] = \int \mathrm{d}\mu_J(a,b) \prod_{j\in\Lambda} \tfrac{E_\varepsilon - ib_j}{E_\varepsilon - a_j} \ \det[1-FJ] \ a_0, \tag{2.2.2}$$

$$\partial_E^n \frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G_\Lambda^+(E_\varepsilon)] = \int \mathrm{d}\mu_J(a,b) \prod_{j\in\Lambda} \tfrac{E_\varepsilon - ib_j}{E_\varepsilon - a_j} \ \det[1-FJ] \ a_0 \left[\sum_{j\in\Lambda} a_j - ib_j\right]^n, \tag{2.2.3}$$

where $\mathrm{d}\mu_J(a,b)$ is the normalized Gaussian measure defined above and $F = F(a,b)$ is a diagonal matrix with entries

$$F(a,b)_{ij} = \delta_{ij} \tfrac{1}{(E_\varepsilon - a_j)(E_\varepsilon - ib_j)}.$$

Proof. The idea of the proof is to write $(E_\varepsilon - H)^{-1}_{ii}$ as a complex Gaussian integral and represent the normalization as a Fermionic Gaussian integral. Then the average over H can be computed easily and one can integrate the Fermionic variables again. This is analog to the procedure in [DPS02], but we apply an additional step of integration by parts to simplify our result. The second identity is proven similarly. For convenience of the reader, a sketch of the proof is given in App. 2.B. □

The integrals above are well-defined only for $\varepsilon > 0$ since for each a_j there is a pole at $a_j = E_\varepsilon$. Note that there are no singularities in b_j.

Remark. In this dual representation, the only contribution from the observable is the term a_0. By the same techniques, we obtain

$$1 = \mathbb{E}[1] = \int \mathrm{d}\mu_J(a,b) \prod_{j\in\Lambda} \tfrac{E_\varepsilon - ib_j}{E_\varepsilon - a_j} \ \det[1 - JF(a,b)]\,. \tag{2.2.4}$$

2.2.2. Contour deformation

By the same saddle analysis performed in [DPS02, Section 4], we expect the complex normalized measure

$$\mathrm{d}\mu_J(a,b) \prod_{j\in\Lambda} \tfrac{E_\varepsilon - ib_j}{E_\varepsilon - a_j} \ \det[1 - F(a,b)J] \tag{2.2.5}$$

to be concentrated near the constant configurations given by $a_j = a_s^\pm$, $b_j = b_s^\pm$ for all $j \in \Lambda$, where $a_s^\pm, b_s^\pm$ are the saddle points $a_s^\pm = \mathcal{E}_r \mp i\mathcal{E}_i$, $b_s^\pm = -i\mathcal{E}_r \mp \mathcal{E}_i$, and

$$\mathcal{E} = \mathcal{E}_r - i\mathcal{E}_i = \tfrac{E}{2} - i\sqrt{1 - \tfrac{E^2}{4}}$$

has the useful properties $E - \mathcal{E} = \bar{\mathcal{E}}$ and $\mathcal{E}\bar{\mathcal{E}} = 1$ for all $|E| < 2$.

We perform a translation of the real axis in the complex plane in order to pass through a saddle point. For the variables a, we translate to the saddle $a_s^+ = \mathcal{E}$ to avoid crossing the pole in $a = E_\varepsilon$. The variables b have no pole and both saddle points have the same imaginary part. Hence a complex translation allows to pass through both saddles. We will prove later that $b_s^+ = -i\mathcal{E}$ is the dominant one. In the next lemma we show that, after the deformation, we can take the limit $\varepsilon \to 0$, and the translated measure can be reorganized as

$$\mathrm{d}\mu_J(a+\mathcal{E}, b-i\mathcal{E}) \prod_{j\in\Lambda} \tfrac{E_\varepsilon - i(b_j - i\mathcal{E})}{E_\varepsilon - (a_j+\mathcal{E})} \; \det\left[1 - F(a+\mathcal{E}, b-i\mathcal{E})J\right] = \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b),$$

where the Gaussian measure $\mathrm{d}\mu_B(a,b)$ has now a complex covariance.

Lemma 2.2.2. *By a complex deformation the functional integrals* (2.2.2) *and* (2.2.3), *in the limit* $\varepsilon \to 0$, *can be written as*

$$\lim_{\varepsilon\to 0} \frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G_\Lambda^+(E_\varepsilon)] = a_s^+ + \int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b)a_0, \tag{2.2.6}$$

$$\lim_{\varepsilon\to 0} \partial_E^n \frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G_\Lambda^+(E_\varepsilon)] = \int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b)a_0 \left(\sum_{j\in\Lambda} a_j - ib_j\right)^n \tag{2.2.7}$$

$$= \sum_{j_1,\dots,j_n} \int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b)a_0 \prod_{k=1}^n (a_{j_k} - ib_{j_k}), \tag{2.2.8}$$

where $a, b \in \mathbb{R}^\Lambda$, *and* $\mathrm{d}\mu_B(a,b)$ *is the normalized Gaussian measure as defined in* (2.2.1) *with complex covariance*

$$B := (-W^2\Delta + (1-\mathcal{E}^2))^{-1}. \tag{2.2.9}$$

The remainder $\mathcal{R}(a,b)$ *is defined by*

$$\mathcal{R}(a,b) := \det[1 + DB] \; \mathrm{e}^{\mathcal{V}(a,b)}, \tag{2.2.10}$$

where $D_{ij} = D_{ij}(a,b) = \delta_{ij}D_j(a,b)$ *is a diagonal matrix, and we defined*

$$\begin{aligned} D_j(a,b) &= \mathcal{E}^2 - F(a+a_s^+, b+b_s^+)_{jj} = \mathcal{E}^2 - \frac{1}{(\bar{\mathcal{E}} - a_j)(\bar{\mathcal{E}} - ib_j)} \\ &= -\int_0^1 \left(\frac{a_j}{(\bar{\mathcal{E}} - ta_j)^2(\bar{\mathcal{E}} - itb_j)} + \frac{ib_j}{(\bar{\mathcal{E}} - ta_j)(\bar{\mathcal{E}} - itb_j)^2} \right) \mathrm{d}t, \end{aligned} \tag{2.2.11}$$

$$\mathcal{V}(a,b) = \sum_{j\in\Lambda} \mathcal{V}_j(a,b) = \sum_{j\in\Lambda} V(a_j) - V(ib_j), \quad V(x) = \int_0^1 \frac{x^3(1-t)^2}{(\bar{\mathcal{E}} - tx)^3}\,\mathrm{d}t. \tag{2.2.12}$$

Proof. By Cauchy's theorem, we can perform the translations $a_j \mapsto a_j + a_s^+$ and $b_j \mapsto b_j + b_s^+$ for all $j \in \Lambda$ and take the limit $\varepsilon \to 0$ inside the functional integral (2.2.2). Note that translating to a_s^+ ensures that there is no additional contribution from the pole $E + i\varepsilon$. Using (2.2.4), the integral with constant a_s^+ gives 1. The measure (2.2.5) after the translation is reorganized as follows. Expanding around $a = b = 0$ we can write

$$\mathrm{d}\mu_J(a+\mathcal{E}, b-i\mathcal{E}) \prod_{j\in\Lambda} \tfrac{E_\varepsilon - i(b_j - i\mathcal{E})}{E_\varepsilon - (a_j+\mathcal{E})} = \mathrm{d}\mu_B(a,b) \tfrac{\det B}{\det J} e^{\mathcal{V}(a,b)}.$$

where $\mathcal{V}(a,b) = O(|a|^3 + |b^3|)$ since linear contributions vanish (we are expanding around the saddle) and constant terms cancel. Finally

$$\det[1 - F(a+\mathcal{E}, b - i\mathcal{E})J]\tfrac{\det B}{\det J} = \det[1 + DB].$$

To obtain the second identity, note that $(a_j + a_s^+) - i(b_j + b_s^+) = a_j - ib_j$, and the integral with constant a_s^+ vanishes since it corresponds to the derivative of a constant. □

Remark. Note that now there is no pole in a_j if $|E| < 2$ since $|\bar{\mathcal{E}} - a_j| \geqslant |\mathcal{E}_i| > 0$ for all $a_j \in \mathbb{R}$. For b_j, a singularity appears from the determinant for the special case $E = 0$. As the same factor appears outside the determinant, this is a removable singularity. Nevertheless we avoid $E = 0$ in the definition of the interval $\mathcal{I}$ (2.1.4).

With these representations, the proof of Theorem 2.1.1 is reduced to prove the following theorem since $\mathrm{Im}\,(a_s^+) = \sqrt{1 - E^2/4}$ yields the semicircle law (2.1.2).

Theorem 2.2.3. *Under the same assumptions as in Theorem 2.1.1, we have*

$$\left| \int \mathrm{d}\mu_B(a,b) \mathcal{R}(a,b) a_0 \right| \leqslant W^{-2} K^{(\ln W)^\alpha} \tag{2.2.13}$$

$$\left| \sum_{j_1,\dots,j_n \in \Lambda} \int \mathrm{d}\mu_B(a,b) \mathcal{R}(a,b) a_0 \prod_{k=1}^n (a_{j_k} - ib_{j_k}) \right| \leqslant C_n W^{-1} (\ln W)^{n(\alpha+1)}\, \mathrm{e}^{K(\ln W)^\alpha}. \tag{2.2.14}$$

2.2.3. Strategy of the proof: finite and infinite volume

To prove the results above, we will need to estimate integrals of the following form

$$\int \mathrm{d}\mu_B(a,b) \mathcal{R}(a,b) \mathcal{O}(a,b), \tag{2.2.15}$$

where $\mathcal{O}(a,b) := \prod_k a_k \prod_l b_l$ is a *local* observable, i.e. a product of finitely many field factors a_k and b_l. We will show that, inserting absolute values inside (2.2.15) leads to the following estimate

$$\left| \int \mathrm{d}\mu_B(a,b) \mathcal{R}(a,b) \mathcal{O}(a,b) \right| \leqslant \mathrm{e}^{K \frac{|\Lambda|}{W^2}} \int \mathrm{d}\mu_C(a,b) \left| \mathrm{e}^{\mathrm{Tr}\, DB} \right| \, \left| \mathrm{e}^{\mathcal{V}(a,b)} \right| \, |\mathcal{O}(a,b)| \,,$$

where C is a real covariance (defined in (2.3.1) below), and D, B and $\mathcal{V}$ were defined in Lemma 2.2.2. Guided by the saddle point approach, we will partition the domain of integration into different regions, respectively near to and far from the saddle points, and estimate the integral in each region separately (cf. Lemma 2.3.6). To obtain the finite volume estimate of (2.2.13), an additional preliminary step of integration by parts is needed to improve the error estimates. All this is done in Section 2.3.

These arguments work only in finite volume, since the factor $\exp(K|\Lambda|W^{-2})$ diverges as $|\Lambda| \to \infty$. To deal with this problem, we will partition Λ into cubes (of finite, but large volume). Applying a suitable cluster expansion, we can write (2.2.15) as a sum of the form

$$\sum_Y c_Y F_Y,$$

where Y are polymers, i.e. unions of cubes, and the constant c_Y is an exponentially small factor controlling the sum. Finally, F_Y is a functional integral depending only on the fields inside Y, and can be estimated by the same tools as in the finite volume case. The precise definitions and details are given in Section 2.4.

2.3. Preliminary results

In this section, we start by collecting in Section 2.3.1 and 2.3.2 some results and bounds we will need later. Finally in Section 2.3.3 we prove an estimate for the absolute value of the integral (2.2.15) in a large but finite volume. The proof uses a partition of the integration domain into regions, selecting values of (a, b) in the vicinity or far from the saddles.

2.3.1. Properties of the covariance

The Hessian $B^{-1} = -W^2\Delta + (1 - \mathcal{E}^2)$ has a complex mass term

$$1 - \mathcal{E}^2 = 2\left(1 - \tfrac{E^2}{4}\right) + iE\sqrt{1 - \tfrac{E^2}{4}} =: m_r^2 + i\sigma_E m_i^2, \quad \sigma_E := \operatorname{sgn}(E).$$

For $|E| < 2$, $m_r^2 > 0$, hence the integrals (2.2.6) and (2.2.7) are finite. We introduce the real covariance C defined by

$$C := [\operatorname{Re}(B^{-1})]^{-1} = (-W^2\Delta + m_r^2)^{-1}. \tag{2.3.1}$$

Note that $B^{-1} = C^{-1} + i\sigma_E m_i^2$ and $C > 0$ both as a quadratic form and pointwise. The decay of C_{ij} depends on the space dimension d. For $d = 2$, we have

$$0 < C_{ij} \leqslant \begin{cases} \frac{K}{W^2} \ln\left(\frac{W}{m_r(1+|i-j|)}\right) & \text{if } |i-j| \leqslant \frac{W}{m_r}, \\ \frac{K}{|i-j|^{1/2} W^{3/2}} \, \mathrm{e}^{-\frac{m_r}{W}|i-j|} & \text{if } |i-j| > \frac{W}{m_r}. \end{cases} \tag{2.3.2}$$

Morover $|B_{ij}|$ has the same decay as C_{ij}. A proof is given in App. 2.C.

Remark. For $d = 3$, the decay is easier:

$$C_{ij} \leqslant \frac{K}{W^2(1+|i-j|)} \, \mathrm{e}^{-\frac{m_r}{W}|i-j|} \quad \forall i, j \in \Lambda. \tag{2.3.3}$$

Because of the log-behavior for small distances, estimating the error terms (2.1.5)-(2.1.6) in $d = 2$ is more difficult than in $d = 3$ (cf. [DPS02, eq. (2.6)-(2.7)]).

2.3.2. Some useful estimates

We frequently use the following statement to estimate determinants.

Lemma 2.3.1. *For any complex matrix A with* $\operatorname{Tr} A^*A < \infty$*, we have*

$$|\det[1+A]| \leqslant \left|\mathrm{e}^{\operatorname{Tr} A}\right| \mathrm{e}^{\frac{1}{2}\operatorname{Tr} A^*A}. \tag{2.3.4}$$

Proof. Consider the matrix $M = A + A^* + A^*A$, which is self-adjoint and diagonalizable with real eigenvalues λ_i. Then

$$|\det[1+A]|^2 = \det[1+M] = \prod_i (1+\lambda_i) \leqslant \mathrm{e}^{\sum_i \lambda_i} = \mathrm{e}^{\operatorname{Tr} M} \leqslant \left|\mathrm{e}^{2\operatorname{Tr} A}\right| \mathrm{e}^{\operatorname{Tr} A^*A},$$

where we apply $1+\lambda_i \leqslant \mathrm{e}^{\lambda_i}$ for all $\lambda_i \in \mathbb{R}$. □

In the finite volume estimates, we will insert quadratic terms in a and b into the measure and change the covariance from C to $C_f := (C^{-1} - fm_r^2)^{-1}$. We estimate the change of the normalization factor $\det[C^{-1}/C_f^{-1}]$ as follows.

Lemma 2.3.2. *For $d = 2, W \gg 1$ and $0 < f < 1/2$, there exist some constants $K > 0$ (independent of W and f) such that*

$$\det\left[\frac{C^{-1}}{C_f^{-1}}\right] \leqslant \frac{1}{1-f} \exp\left[\frac{Kf|\Lambda|}{W^2} \ln\left(\frac{W^2}{1-f}\right)\right].$$

Proof. We use the explicite eigenvalues of C^{-1} and C_f^{-1} to write

$$\begin{aligned}\det\left[\frac{C^{-1}}{C_f^{-1}}\right] &= \prod_k \frac{2\sum_{l=1}^d (1-\cos k_l)W^2 + m_r^2}{2\sum_{l=1}^d (1-\cos k_l)W^2 + m_r^2(1-f)} \\ &\leqslant \frac{1}{1-f} \exp\left[m_r^2 f \sum_{k\neq 0} \left(2\sum_{l=1}^d (1-\cos k_l)W^2 + m_r^2(1-f)\right)^{-1}\right],\end{aligned}$$

where we extract the zero's mode and apply $1+\lambda_i \leqslant \mathrm{e}^{\lambda_i}$. Approximating the sum in the exponential yields the above result. □

Finally, we give the Brascamp-Lieb inequality [BL76], which is used in the estimates near the dominant saddle point.

Theorem 2.3.3 (Brascamp-Lieb inequality)**.** *Let $\mathcal{H}(x)$ be a positive Hamiltonian, symmetric under $x \mapsto -x$ and let $\mathrm{d}\mu_{\mathcal{H}}(x)$ be a Gibbs measure given by*

$$\mathrm{d}\mu_{\mathcal{H}}(x) := \mathrm{d}x_1 \cdots \mathrm{d}x_N \frac{1}{Z(\mathcal{H})} \mathrm{e}^{-\frac{1}{2}\mathcal{H}(x)},$$

where $Z(\mathcal{H}) := \int \mathrm{d}x_1 \cdots \mathrm{d}x_N \exp\left(-\mathcal{H}(x)/2\right)$ *is the partition function. If* $\mathcal{H}'' \geqslant C^{-1} > 0$*, the following inequalities hold:*

$$\int \mathrm{d}\mu_{\mathcal{H}}(x)|x_i|^n \leqslant \int \mathrm{d}\mu_C(x)|x_i|^n, \quad \forall n > 0 \text{ and} \tag{2.3.5}$$
$$\int \mathrm{d}\mu_{\mathcal{H}}(x)\, \mathrm{e}^{(v,x)} \leqslant \int \mathrm{d}\mu_C(x)\, \mathrm{e}^{(v,x)},$$

where $\mathrm{d}\mu_C(x)$ *is the free Gaussian measure and* $v \in \mathbb{R}^N$ *and* $(v,x) = \sum_{i=1}^N v_i x_i$.

Remark. A direct consequence of Brascamp-Lieb inequality is the following estimate, which holds under the same assumptions as above:

$$\begin{aligned}\int \mathrm{d}\mu_{\mathcal{H}}(x) \prod_i |x_i|^{n_i}\, \mathrm{e}^{(v,x)} &= \int \mathrm{d}\mu_{\mathcal{H}_v}(x) \prod_i |x_i|^{n_i} \int \mathrm{d}\mu_{\mathcal{H}}(x)\, \mathrm{e}^{(v,x)} \\ &\leqslant \prod_i \left[\int \mathrm{d}\mu_{\mathcal{H}_v}(x) \prod_i |x_i|^n\right]^{n_i/n} \int \mathrm{d}\mu_{\mathcal{H}}(x)\, \mathrm{e}^{(v,x)} \\ &\leqslant \sqrt{(2n-1)!!} \prod_i C_{ii}^{n_i/2}\, \mathrm{e}^{\frac{1}{2}(v,Cv)}, \quad \forall n_i \geqslant 0,\end{aligned} \tag{2.3.6}$$

where $n = \sum_i n_i$ and we changed in the first line the measure to $\mathcal{H}_v = \mathcal{H} - (v,\cdot)$ with $\mathcal{H}_v'' = \mathcal{H}''$. In the second line we applied a generalized Hölder estimate. In the last line, the Gaussian integrals are computed exactly after applying Brascamp Lieb and Cauchy Schwarz.

2.3.3. Finite volume estimates

In the following we prove Theorem 2.2.3 in finite volume by partitioning the domain of integration and estimating the functional integrals in each region separately.

Inserting absolute values

To control the infinite volume limit, we will need to estimate integrals of the form (2.2.15) with $\mathcal{O}(a,b) = \mathcal{O}_{m,n}(a,b) = \prod_{k=1}^p |a_{j_k}|^{m_k} \prod_{l=1}^q |b_{j_l}|^{n_l}$, with $p,q \in \mathbb{N}$, $m_k, n_l \in \mathbb{N}$ and $j_k, j_l \in \Lambda$ for all $k \leqslant p, l \leqslant q$ and $m := \sum_{k=1}^p m_k$ and $n := \sum_{l=1}^q n_l$. Following [DPS02], we put the absolute values inside the integral (2.2.15) and replace the complex covariance B (2.2.9) by the real one C (2.3.1). The next two lemmas are the analogs in $d = 2$ of [DPS02, Lemma 3 and 4].

Lemma 2.3.4. *The absolute value of the complex measure* $\mathrm{d}\mu_B$ *is bounded by*

$$|\,\mathrm{d}\mu_B(a,b)| \leqslant \mathrm{e}^{K\frac{|\Lambda|}{W^2}}\, \mathrm{d}\mu_C(a,b).$$

Proof. The measure $\mathrm{d}\mu_B(a,b)$ can be written as

$$|\,\mathrm{d}\mu_B(a,b)| = \left|\frac{\det B^{-1}}{\det C^{-1}}\right| \mathrm{d}\mu_C(a,b) = |\det[1 + i\sigma_E m_i^2 C]| d\mu_C(a,b).$$

Applying (2.3.4) with $A = i\sigma_E m_i^2 C$, $\mathrm{Tr}\, A$ is purely imaginary and, using (2.3.2),

$$\mathrm{Tr}\, A^* A = m_i^4 \mathrm{Tr}\, C^* C \leqslant \sum_{i,j\in\Lambda} \tfrac{K}{W^3|i-j|}\, \mathrm{e}^{-\frac{m_r}{W}|i-j|} + \sum_{|i-j|\leqslant \frac{W}{m_r}} \tfrac{K}{W^4} \ln^2\left(\tfrac{W}{m_r(1+|i-j|)}\right) \leqslant \sum_{i\in\Lambda} \tfrac{K}{W^2}.$$

□

Lemma 2.3.5. *The determinant in the remainder* (2.2.10) *can be bounded by*

$$|\det[1 + DB]| \leqslant \mathrm{e}^{K\frac{|\Lambda|}{W^2}} \left|\mathrm{e}^{\mathrm{Tr}\, DB}\right|.$$

Proof. Applying again (2.3.4), we need to bound $\mathrm{Tr}\,(DB)^*(DB) = \sum_{i,j\in\Lambda} |D_j|^2 |B_{ij}|^2$. We estimate D by its supremum norm, $\sup_{j\in\Lambda} \sup_{a_j,b_j\in\mathbb{R}} |D_j(a_j, b_j)| \leqslant K$. Finally we bound $\mathrm{Tr}\, B^*B$ by $K|\Lambda|W^{-2}$ as we did above for C. □

Applying the two lemmas in (2.2.15), we have

$$\left|\int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b)\mathcal{O}_{m,n}(a,b)\right| \leqslant \mathrm{e}^{K\frac{|\Lambda|}{W^2}} F^{m,n},$$

where $F^{m,n} := \int \mathrm{d}\mu_C(a,b) \left|\mathrm{e}^{\mathrm{Tr}\, DB}\right| \left|\mathrm{e}^{\mathcal{V}(a,b)}\right| |\mathcal{O}_{m,n}(a,b)|$.

Partition of the integration domain

Guided by the saddle point picture, we partition, as in [DPS02], the domain of integration into regions near and far from the saddle points: $1 = \sum_{k=1}^5 \chi[I^k]$ with

$$\begin{aligned}
I^1 &:= \{a, b : |a_j|, |b_j - b_{j'}| \leqslant \delta\, \forall j, j' \in \Lambda && \text{and } |b_0| \leqslant 2\delta\},\\
I^2 &:= \{a, b : |a_j|, |b_j - b_{j'}| \leqslant \delta\, \forall j, j' \in \Lambda && \text{and } |b_0 - 2\mathcal{E}_i| \leqslant 2\delta\},\\
I^3 &:= \{a, b : b_j \in \mathbb{R}\, \forall j \in \Lambda && \text{and } \exists j_0 \in \Lambda : |a_{j_0}| > \delta\},\\
I^4 &:= \{a, b : |a_j| \leqslant \delta\, \forall j \in \Lambda && \text{and } \exists j_0, j_0' \in \Lambda : |b_{j_0} - b_{j_0'}| > \delta\},\\
I^5 &:= \{a, b : |a_j|, |b_j - b_{j'}| \leqslant \delta\, \forall j, j' \in \Lambda && \text{and } |b_0|, |b_0 - 2\mathcal{E}_i| > 2\delta\},
\end{aligned} \tag{2.3.7}$$

for $\delta = \delta(W) > 0$ small to be fixed later. Hence, we can write $F^{m,n} = \sum_{s=1}^5 F_s^{m,n}$, where

$$F_s^{m,n} := \int \mathrm{d}\mu_C(a,b)\ \chi[I^s]\ \mathrm{e}^{\mathrm{Re}\,\mathrm{Tr}\, DB}\ \left|\mathrm{e}^{\mathcal{V}(a,b)}\right|\ |\mathcal{O}_{m,n}(a,b)|.$$

In the "small field" regions I^1 and I^2, all a variables are near the saddle, and the b variables are all near the first saddle at 0 in I^1, or near the second one at $2\mathcal{E}_i$ in I^2. The main contribution to $F^{m,n}$ comes from the region I^1, while I^2 is suppressed by a small factor from the determinant. In the "large field" regions I^s, $s = 3, 4, 5$, at least one variable is far away from the saddle points. Their contribution is exponentially suppressed by the corresponding probabilities $\int \mathrm{d}\mu_C\ \chi[I^s]$.

The following lemma gives the precise estimates on $F_s^{m,n}$. Since we proceed analog to [DPS02, Section 5], only the main ideas and the crucial steps are given in the proof.

Lemma 2.3.6. *Let $\delta = W^{-\nu}$, for some $0 < \nu < 1$ and $W \gg 1$. Then for any $|\Lambda|$ we have*

$$\begin{aligned} F_1^{m,n} &\leqslant K^{m+n+1} \left(\tfrac{\ln W}{W^2}\right)^{(m+n)/2} \sqrt{(2m)!!(2n)!!}\ \mathrm{e}^{K|\Lambda|W^{-3}(\ln W)^{3/2}}, \\ F_2^{m,n} &\leqslant K^{n+m+1}\, \mathrm{e}^{-c|\Lambda|W^{-2}\ln W}, \end{aligned} \tag{2.3.8}$$

where in the second line $c > 0$ is independent of W and $|\Lambda|$. Moreover, there exists $W_0(\nu) \gg 1$ such that for any $W \geqslant W_0(\nu)$ and $W^{3\nu} \leqslant |\Lambda| \leqslant (C_{jj})^{-2}\delta^2 \leqslant KW^{4-2\nu}(\ln W)^{-2}$, we have

$$\begin{aligned} F_s^{m,n} &\leqslant K^{n+m+1}W^n \prod_{k=1}^{p} \sqrt{m_k!} \prod_{l=1}^{q} \sqrt{n_l!}\ \mathrm{e}^{-K\delta^2 W^2(\ln W)^{-1}} \qquad \text{for } s = 3, 4, \\ F_5^{m,n} &\leqslant K^{n+m+1}W^n \prod_{k=1}^{p} \sqrt{m_k!} \prod_{l=1}^{q} \sqrt{n_l!}\, \mathrm{e}^{-K\delta^2|\Lambda|}, \end{aligned}$$

Remark. In the following, we want to fix the volume of our cube Λ to an appropriate finite size. The natural choice would be W^2. This would ensure the global prefactor $\mathrm{e}^{K|\Lambda|W^{-2}}$ from Lemma 2.3.4 is bounded by a constant independent of W. On the other hand the contribution of the second saddle would be suppressed only by some W^{-c} (cf. (2.3.8)) which is not enough to compensate various W factors arising in the cluster expansion. Extending the volume to $W^2(\ln W)^\alpha$ for fixed $\alpha \in (0,1)$ reinforces the decay to $\mathrm{e}^{-c(\ln W)^{1+\alpha}}$ which bounds an arbitrary factor W^n for $\alpha > 0$. The price to pay is a worse estimate on the global prefactor $\mathrm{e}^{K|\Lambda|W^{-2}} \leqslant \mathrm{e}^{K(\ln W)^\alpha}$. For $\alpha < 1$ this can be compensated by the observable, which is of order $O(W^{-2})$ after extracting the leading contribution (cf. (2.1.6)).

Proof. Following [DPS02], we first perform some (region dependent) estimates on the exponential terms $\mathrm{Re}\,\mathrm{Tr}\, DB + \mathrm{Re}\,\mathcal{V}(a,b)$ and insert the results in the measure. In region I^1 the resulting measure is no longer Gaussian, hence we apply a Brascamp-Lieb inequality. In the other regions the measure remains Gaussian. The decay comes from $\mathrm{Re}\,\mathrm{Tr}\, DB$ in I^2 and from a small probability argument in the large field regions. New features of $d = 2$ appear in the choice of the volume of the cube $|\Lambda|$ and in the bounds of B and C, for example we have $|B_{jj}| \leqslant KW^{-2}\ln W$.

Region $\mathbf{I^1}$ In the first region, all variables a_j and b_j are small and we bound

$$\mathrm{Re}\, V(x) \leqslant K|x|^3 \quad \text{and} \quad |D_j(a,b)| \leqslant K(|a_j| + |b_j|) \quad \text{for } |x|, |a_j|, |b_j| < \delta. \tag{2.3.9}$$

Then $\mathrm{Re}\,\mathrm{Tr}\, DB \leqslant \sum_{j\in\Lambda} |D_j||B_{jj}| \leqslant \sum_{j\in\Lambda} K(|a_j| + |b_j|)W^{-2}\ln W$ and

$$F_1^{m,n} \leqslant \int \mathrm{d}\mu_C(a,b)\chi[I^1]\, \mathrm{e}^{K\sum_{j\in\Lambda}(|a_j|+|b_j|)W^{-2}\ln W + |a_j|^3 + |b_j|^3} \prod_{k=1}^{p} |a_{j_k}|^{m_k} \prod_{l=1}^{q} |b_{j_l}|^{n_l}.$$

We define the Hamiltonian of the Gibbs measure by

$$\mathcal{H}(x) := x^t C^{-1} x - K\sum_{j\in\Lambda} |x_j|W^{-2}\ln W + |x_j|^3$$

and $Z(\mathcal{H}) := \int\prod_{j\in\Lambda} \mathrm{d}x_j \exp(-\mathcal{H}(x)/2)\chi[I^1]$. Then we can write

$$F_1^{m,n} \leqslant \left(\tfrac{Z(\mathcal{H})}{Z_0}\right)^2 \int \mathrm{d}\mu_{\mathcal{H}}(a,b) \prod_{k=1}^{p} |a_{j_k}|^{m_k} \prod_{l=1}^{q} |b_{j_l}|^{n_l},$$

where $Z_0 = \det[C^{-1}/2\pi]^{1/2}$. Repeating the proof of [DPS02, Lemma 5] in $d = 2$,

$$Z(\mathcal{H}) \leqslant \mathrm{e}^{\sum_j (C_f)_{jj}^{3/2} + (C_f)_{jj}^{1/2} W^{-2}} Z_0 \leqslant \mathrm{e}^{K|\Lambda|W^{-3}(\ln W)^{3/2}} Z_0. \tag{2.3.10}$$

where $C_f^{-1} := C^{-1} - f m_r^2 \leqslant \mathcal{H}''$ for $f = \mathcal{O}(\delta)$, and we used $\delta < 1$. When $m > 0$,

$$\prod_{k=1}^{p} |a_{j_k}|^{m_k} \leqslant \frac{1}{m} \sum_{k=1}^{p} m_k |a_{j_k}|^m.$$

The same holds for b. Applying the Brascamp-Lieb inequality (2.3.5) and a Cauchy-Schwarz estimate, we obtain a factor $\sqrt{(2m)!!}(C_f)_{j_k j_k}^{m/2} \leqslant \sqrt{(2m)!!}(KW^{-2}\ln W)^{m/2}$ for each $|a_{j_k}|^m$ and an analog factor for $|b_{j_k}|^n$.

Region $\mathbf{I^2}$ As in [DPS02], we can bound the factors $a_j^{m_j}$ and $b_j^{n_j}$ by constants and the potential by

$$\operatorname{Re} V(a_j) \leqslant \tfrac{m_r^2}{2} f_a a_j^2, \qquad \operatorname{Re} V(ib_j) \leqslant \tfrac{m_r^2}{2} f_b b_j^2 + (1-f_b)2\mathcal{E}_i m_r^2 (b_j - \mathcal{E}_i), \tag{2.3.11}$$

with $f_a = f_b = \mathcal{O}(\delta)$. Analog to [DPS02, Lemma 6] the trace can be estimated as

$$\operatorname{Re} D_j B_{jj} \leqslant -2cW^{-2}\ln W, \tag{2.3.12}$$

where $c > 0$ is independent of W and Λ. Combining these estimates and using Lemma 2.3.2 in the second step, we obtain

$$\begin{aligned} F_2^{m,n} &\leqslant K^{m+n}\,\mathrm{e}^{-2c|\Lambda|W^{-2}\ln W} \int \mathrm{d}\mu_C(a,b)\,\mathrm{e}^{\frac{m_r^2}{2}\sum_{j\in\Lambda}\left(f_a a_j^2 + f_b b_j^2\right)}\,\mathrm{e}^{(1-f_b)2\mathcal{E}_i m_r^2 (b_j - \mathcal{E}_i)} \\ &\leqslant K^{m+n}\,\mathrm{e}^{-2(c-K(f_a+f_b))|\Lambda|W^{-2}\ln W} \frac{K}{\sqrt{(1-f_a)(1-f_b)}} \int \mathrm{d}\mu_{C_{f_b}}(b)\,\mathrm{e}^{(1-f_b)2\mathcal{E}_i m_r^2 (b_j - \mathcal{E}_i)} \\ &\leqslant K^{m+n+1}\,\mathrm{e}^{-2(c-K\delta)|\Lambda|W^{-2}\ln W} \leqslant K^{m+n+1}\,\mathrm{e}^{-c|\Lambda|W^{-2}\ln W}, \end{aligned}$$

where the remaining integral in the second line is 1 and we used $\delta < c/2K$ for W large enough.

Regions $\mathbf{I^3}$ and $\mathbf{I^4}$ As in [DPS02], for arbitrary a_j and b_j in $\mathbb{R}$ we can bound

$$\operatorname{Re} V(a_j) \leqslant \tfrac{m_r^2}{2} f_a a_j^2, \quad \operatorname{Re} V(ib_j) \leqslant \tfrac{m_r^2}{2} f_b b_j^2 + \mathcal{O}(1-f_b), \quad \operatorname{Re} D_j B_{jj} \leqslant \tfrac{K}{W^2}\ln W, \tag{2.3.13}$$

where $f_a, f_b \in (1/2, 1)$. Inserting the quadratic terms into the measure, and using a small fraction of the remaining mass, we bound

$$|a_j|^{m_j} \leqslant \left(\frac{K}{\sqrt{\varepsilon(1-f_a)}}\right)^{m_j} \sqrt{m_j!}\ \mathrm{e}^{\frac{1}{2}\varepsilon m_r^2(1-f_a)a_j^2}$$

and the same for b_j, where $0 < \varepsilon \ll 1$ is small but fixed and independent of W. Using Lemma 2.3.2, and $\ln W^2(1-f)^{-1} \leqslant K_m \ln W$ for all $f \in [0, 1-W^{-m}], m \in \mathbb{N}$, we obtain

$$\begin{aligned} F_s^{m,n} &\leqslant \frac{\mathcal{K}_{m,n}K^{m+n}}{(1-f_a)^{m/2}(1-f_b)^{n/2}}\, \mathrm{e}^{K|\Lambda|(W^{-2}\ln W+(1-f_b))} \int \mathrm{d}\mu_C(a,b)\, \mathrm{e}^{\frac{m_r^2}{2}\sum_{j\in\Lambda}(\tilde{f}_a a_j^2+\tilde{f}_b b_j^2)}\chi[I^s] \\ &\leqslant \mathcal{K}_{m,n}K^{m+n}W^{n+1}\, \mathrm{e}^{K|\Lambda|W^{-2}\ln W} \int \mathrm{d}\mu_{C_{\tilde{f}_a}}(a)\, \mathrm{d}\mu_{C_{\tilde{f}_b}}(b)\ \chi[I^s], \end{aligned}$$

where $\mathcal{K}_{m,n} = \prod_{k=1}^p \sqrt{m_k!}\prod_{l=1}^q \sqrt{n_l!}$, and $\tilde{f}_a = f_a + \varepsilon(1-f_a)$ (same for $\tilde{f}_b$). In the second line we take $f_a \in (1/2, 3/4)$ and $f_b = 1 - W^{-2}$, to ensure that all error terms in the exponent are not larger than the first one, i.e. $|\Lambda|W^{-2}\ln W$. Applying [DPS02, Lemma 8], we bound the remaining integral by:

$$\begin{aligned} \int \mathrm{d}\mu_{C_{\tilde{f}_a}}(a)\chi[I^3] &\leqslant \mathrm{e}^{-x\delta}\sum_{j\in\Lambda} \mathrm{e}^{\frac{1}{2}x^2(C_{\tilde{f}_a})_{jj}} \leqslant |\Lambda|\,\mathrm{e}^{-K\delta^2W^2(\ln W)^{-1}}, \\ \int \mathrm{d}\mu_{C_{\tilde{f}_b}}(b)\chi[I^4] &\leqslant \mathrm{e}^{-x\delta}\sum_{j,j'\in\Lambda} \mathrm{e}^{\frac{1}{2}x^2[(C_{\tilde{f}_b})_{jj}+(C_{\tilde{f}_b})_{j'j'}-2(C_{\tilde{f}_b})_{jj'}]} \leqslant |\Lambda|^2\,\mathrm{e}^{-K\delta^2W^2(\ln W)^{-1}}, \end{aligned}$$

where we set $x = K\delta W^{-2}\ln W$ and in the first line we used $(C_{\tilde{f}_a})_{jj} \simeq W^{-2}\ln W$. In the second line, $(C_{\tilde{f}_b})_{jj} \simeq W^{-2}\ln W + W^2|\Lambda|^{-1}$, since $(1-\tilde{f}_b) = O(W^{-2})$. The additional term is canceled by the sum $(C_{\tilde{f}_b})_{jj} + (C_{\tilde{f}_b})_{j'j'} - 2(C_{\tilde{f}_b})_{jj'}$. Now, inserting the constraints we assumed on $|\Lambda|$ and δ we obtain the result.

Region $\mathbf{I^5}$ The proof in region I^5 is similar to the one in the other large field region, with the difference that the exponential decay comes from the bound of the potential in b that can be improved to

$$\operatorname{Re} V(ib_j) \leqslant \tfrac{m_r^2}{2} f_b b_j^2 + \mathcal{O}(1-f_b) - c\delta^2 \qquad \text{with } f_b = 1 - W^{-2}. \tag{2.3.14}$$

By the same arguments as above, we obtain a factor $\exp(-c|\Lambda|\delta^2)$ from the last term which gives the main behaviour of the integral since $\delta > W^{-1}(\ln W)^{1/2}$ for W large enough. □

Improved estimates

Let us now fix $|\Lambda| = W^2(\ln W)^\alpha$, with $\alpha \in (0,1)$ as discussed is the remark below Lemma 2.3.6. We want to apply Lemma 2.3.6 to (2.2.6) and (2.2.8) to prove (2.1.5) and (2.1.6). For the correction to the semicircle law in (2.1.5) we obtain

$$\begin{aligned} \left|\int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b)a_0\right| &\leqslant \mathrm{e}^{K(\ln W)^\alpha}\left[\frac{(\ln W)^{1/2}}{W} + \mathrm{e}^{-c(\ln W)^{1+\alpha}} + \mathrm{e}^{-K\delta^2W^2((\ln W)^{-1}+(\ln W)^\alpha)}\right] \\ &\leqslant \mathrm{e}^{K(\ln W)^\alpha}\frac{(\ln W)^{1/2}}{W} \end{aligned}$$

which is not the desired estimate. To estimate the derivatives in (2.2.8), we need to extract enough W factors to control the sum over the indices j_k. If we apply Lemma 2.3.6 naively, we obtain

$$|(2.2.8)| \leqslant K_n \sum_{j_1,\dots,j_n} \mathrm{e}^{K(\ln W)^\alpha} \left(\frac{(\ln W)^{1/2}}{W}\right)^{n+1} = K_n\, \mathrm{e}^{K(\ln W)^\alpha} \frac{(\ln W)^{1/2}}{W} \left(W(\ln W)^{\alpha+1/2}\right)^n,$$

which grows in W algebraically for $n > 0$. To improve these bounds similar to [DPS02], we apply a few preliminary steps of integration by parts. This is done in the next lemma.

Lemma 2.3.7. *For general $\Lambda \subset \mathbb{Z}^d$, the integrals* (2.2.6) *and* (2.2.8) *can be written as*

$$\lim_{\varepsilon\to 0} \frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G_\Lambda^+(E_\varepsilon)] - a_s^+ = \sum_{l_0\in\Lambda} B_{0l_0} \int \mathrm{d}\mu_B(a,b)\partial_{a_{l_0}}\mathcal{R}(a,b),$$

$$\lim_{\varepsilon\to 0} \partial_E \frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G_\Lambda^+(E_\varepsilon)] = \sum_{j_1\in\Lambda}\sum_{l_0,l_1\in\Lambda} B_{0l_0}B_{j_1l_1} \int \mathrm{d}\mu_B(a,b)\partial_{x_{l_1}}\partial_{a_{l_0}}\mathcal{R}(a,b) + \delta_{j_1,l_1}$$

$$\lim_{\varepsilon\to 0} \partial_E^n \frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G_\Lambda^+(E_\varepsilon)] = \sum_{\substack{j_1,\dots,j_n\in\Lambda\\ l_0,\dots,l_n\in\Lambda}} B_{0l_0}\prod_{m=1}^n B_{j_ml_m} \int \mathrm{d}\mu_B(a,b)\prod_{m=1}^n \partial_{x_{l_m}}\partial_{a_{l_0}}\mathcal{R}(a,b),$$

with $\partial_{x_l} = \partial_{a_l} + i\partial_{b_l}$.

Proof. We use integration by parts. For the first equation we only need to apply one step of integration by parts. For the derivatives, the case $n = 1$ is special. We calculate

$$\begin{aligned}
&\sum_{j\in\Lambda}\int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b)a_0(a_j - ib_j)\\
=&\sum_{j\in\Lambda}\int \mathrm{d}\mu_B(a,b)\sum_{l_0\in\Lambda} B_{0l_0}\Big((a_j-ib_j)\partial_{a_{l_0}}\mathcal{R}(a,b) - \delta_{jl_0}\mathcal{R}(a,b)\Big)\\
=&\sum_{j\in\Lambda}\sum_{l_0,l_1\in\Lambda} B_{0l_0}B_{jl_1}\int \mathrm{d}\mu_B(a,b)\partial_{x_{l_1}}\partial_{a_{l_0}}\mathcal{R}(a,b) - \delta_{jl_0},
\end{aligned}$$

where we used in the last step that $\int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b) = 1$.

For $n \geqslant 2$, we apply several steps of integration by parts. Writing $x_j = a_j - ib_j$,

$$\begin{aligned}
&\sum_{j_1,\dots,j_n\in\Lambda}\int \mathrm{d}\mu_B(a,b)\mathcal{R}(a,b)a_0\prod_{m=1}^n x_{j_m}\\
=&\sum_{j_1,\dots,j_n\in\Lambda}\int \mathrm{d}\mu_B(a,b)\sum_{l_0\in\Lambda} B_{0l_0}\left(\prod_{m=1}^n x_{j_m}\partial_{a_{l_0}}\mathcal{R}(a,b) + \mathcal{R}(a,b)\partial_{a_{l_0}}\prod_{m=1}^n x_{j_m}\right)\\
=&\sum_{j_1,\dots,j_n\in\Lambda}\sum_{l_0,\dots,l_n\in\Lambda} B_{0l_0}\prod_{m=1}^n B_{j_ml_m}\int \mathrm{d}\mu_B(a,b)\prod_{m=1}^n\partial_{x_{l_m}}\partial_{a_{l_0}}\mathcal{R}(a,b)
\end{aligned}$$

where the last term in the second line corresponds to the derivative of a constant, hence equals zero. In the third line we used $\partial_{x_i}x_j = 0$ for all i, j. ∎

These representations give the stated decay in finite volume $|\Lambda| = W^2(\ln W)^\alpha$:

Lemma 2.3.8. *For fixed* $|\Lambda| = W^2(\ln W)^\alpha$ *we have*

$$|\lim_{\varepsilon\to 0} \frac{1}{|\Lambda|}\mathbb{E}[\mathrm{Tr}\, G^+_\Lambda(E_\varepsilon)] - a^+_s| \leqslant \mathrm{e}^{K(\ln W)^\alpha} W^{-2}\ln W,$$
$$|\lim_{\varepsilon\to 0} \partial^n_E \frac{1}{|\Lambda|}\mathbb{E}[\mathrm{Tr}\, G^+_\Lambda(E_\varepsilon)]| \leqslant C_n.$$

Proof. We apply Lemma 2.3.6 on the representations of the previous lemma. Deriving the remainder $\mathcal{R}(a,b)$, we obtain

$$\partial_{a_l}\left(\mathcal{R}(a,b)\right) = \left(\det[1+DB]\partial_{a_l}V(a_l) + \det B\,\det_{\{l\},\{l\}}[B^{-1}+D]\,\partial_{a_l}D_l\right)\,\mathrm{e}^{\mathcal{V}(a,b)}.$$

In the first summand, we can bound $|\partial_{a_l}V(a_l)| \leqslant K|a_l|^2$. In the second one, we bound $|\partial_{a_l}D_l| \leqslant K$. In region I^1 the matrix $B^{-1}+D$ is invertible and

$$\left|\det B\ \det_{\{l\},\{l\}}[B^{-1}+D]\right| = \left|\det[1+DB](B^{-1}+D)^{-1}_{ll}\right| \leqslant |\det[1+DB]|\tfrac{\ln W}{W^2}.$$

In the other regions, it suffices to write the expression above as $|\det(1+M)|$ (for a certain matrix M) and bound it similar to Lemma 2.3.5. Using Lemma 2.3.6, the integral is bounded by

$$\sum_{l\in\Lambda}|B_{0l}|\,\mathrm{e}^{K(\ln W)^\alpha}W^{-2}\ln W \leqslant \mathrm{e}^{K(\ln W)^\alpha}W^{-2}\ln W$$

since $\sum_{l\in\Lambda}|B_{0l}| \leqslant K$. This proves the first part.

For $n=1$ the first integral yields a factor $(W^{-1}(\ln W)^{1/2})^3$ which controls the sum over j easily. In the second term, the sum over j disappears. In both cases the sum over l is performed by B_{0l} and is bounded by a constant.

For $n \geqslant 2$, note that each factor B_{ij} controls a sum over $i\in\Lambda$ or $j\in\Lambda$. The largest contribution appears when all l_m are different. In this case the expression above is bounded by $W^{-2}(\ln W)^{1+(2+\alpha)n} \ll 1$ for all $n \leqslant n_0(W)$. When $l_m = l_{m'}$ the factor $W^{-2}\ln W$ comes from $B_{j_{m'}l_{m'}}$, since no sum over $l_{m'}$ is needed. □

Remark. Note that the representation for the first derivative is special, but the additional term is easy to handle, since it directly give control over the sums over Λ, hence we neglect it in the following.

Large volume

We can easily extend the above result from one cube of volume $W^2(\ln W)^\alpha$ to a finite union of cubes of this volume. The procedure is independent of the space dimension and follows [DPS02, Corollary 1]. The idea is to decouple the cubes by replacing the periodic Laplacian in the covariance by one with Neumann boundary conditions.

Theorem 2.3.9. *For $d = 2$ and each fixed $\alpha \in (0,1)$, there exists a value $W_0(\alpha)$ such that for $W \geqslant W_0(\alpha)$ and Λ a union of N cubes of volume $W^2(\ln W)^\alpha$ we have for all $E \in \mathcal{I}$*

$$\begin{aligned} |\bar{\rho}_\Lambda(E) - \rho_{SC}(E)| &\leqslant W^{-2}\, \mathrm{e}^{NK(\ln W)^\alpha} \\ |\partial_E^n \bar{\rho}_\Lambda(E)| &\leqslant C_n N^n\, \mathrm{e}^{NK(\ln W)^\alpha}, \end{aligned}$$

where ρ_{SC} is Wigner's semicircle law (2.1.2) *and C_n depends only on n.*

Remark. Note that here n can take any value independent of W.

Proof. We consider again the dual representations (2.2.6) and (2.2.7), apply the steps of integration by parts described above and pull the sums in front of the integral. The measure is again bounded by Lemma 2.3.4, and the determinant $\det 1 + DB$ by Lemma 2.3.5. When derivatives fall on the determinant, this is replaced by terms of the form $\det B \det_{\mathcal{J}\mathcal{J}}(B^{-1} + D)$ for an index set $\mathcal{J}$, which can be bounded in the same way. Collecting a factor $\exp(K|\Lambda|W^{-2}) = \exp(NK(\ln W)^\alpha)$, we need to estimate an integral of the form (2.2.15). Note that all terms in the integral factorize over the cubes except the measure $\mathrm{d}\mu_C$. Before applying Lemma 2.3.6, we insert the partition of the integration domain in each cube separately: $1 = \prod_\triangle \sum_{s_\triangle=1}^5 \chi[I_\triangle^{s_\triangle}]$. We estimate the terms $\operatorname{Re}\operatorname{Tr} DB$ and $\operatorname{Re}\mathcal{V}$ in each cube depending on the region (as in (2.3.9), (2.3.11), (2.3.12), (2.3.13), (2.3.14)). Inserting the quadratic contributions into the measure and extracting the normalizing factor, we obtain

$$\sum_{\{s_\triangle\}_{\triangle\in\Lambda}} \sqrt{\det \tfrac{C^{-1}}{C_{f_a}^{-1}} \det \tfrac{C^{-1}}{C_{f_b}^{-1}}} \int \mathrm{d}\mu_{C_{f_a}}(a)\, \mathrm{d}\mu_{C_{f_b}}(b) |\mathcal{O}(a,b)|\, \mathrm{e}^{\sum_\triangle h_{s_\triangle}(a,b)} \prod_\triangle \chi[I_\triangle^{s_\triangle}],$$

where we collect all non-quadratic (cubic, linear and constant) terms in $h_{s_\triangle}(a,b)$, and $C_f^{-1} = C^{-1} - \hat{f}m_r^2$, and $\hat{f} = \sum_{\triangle, s_\triangle>1} f_\triangle \mathbf{1}_\triangle$ is a block diagonal matrix. Note that, for the moment, only the mass in regions with $s_\triangle > 1$ has been modified. Now we can bound the normalization factor in each cube as usual since

$$\tfrac{\det C^{-1}}{\det C_f^{-1}} = \det[1 + \hat{f}m_r^2 C_f] \leqslant \det[1 + \hat{f}m_r^2 C_f^N] = \prod_{\triangle: s_\triangle > 1} \det[1_\triangle + f_\triangle m_r^2 C_{f_\triangle}^{\triangle,N}] \quad (2.3.15)$$

where $C_f^N = (-W^2\Delta_N(1-\hat{f})m_r^2)^{-1}$, $-\Delta_N$ is the Laplacian with Neumann boundary conditions on the cube boundaries, and $C_{f_\triangle}^{\triangle,N}$ is this covariance restricted to $\triangle$. To prove the inequality above, we use $C_f \leqslant C_f^N$ and $\hat{f} \geqslant 0$ as quadratic forms, and the minmax-principle to compare the corresponding eigenvalues.

As in [DPS02, Lemma 8], we estimate the characteristic function $\chi[I_\triangle^3]$ by $K\,\mathrm{e}^{\pm\sum_{j\in\triangle} xa_j}$. A similar bound holds for $\chi[I_\triangle^4]$. Now, apart from the cubic contributions in the first region, all terms depending on a or b in the integral are of the form $|a|^n$ or $\exp((a,v))$, for some vector v. The same holds for b. We are then reduced to estimate an integral of the following form

$$\int \mathrm{d}\mu_{C_{f_a}}(a)\, \mathrm{d}\mu_{C_{f_b}}(b) \prod_{\triangle: s_\triangle=1} \mathrm{e}^{\mathcal{F}_\triangle(a)+\mathcal{F}_\triangle(b)} \prod_{j\in\triangle} |a_j|^{m_j} |b_j|^{n_j} \prod_{\triangle: s_\triangle>1} \mathrm{e}^{(a,v_\triangle)+(b,w_\triangle)},$$

where $\mathcal{F}_\triangle(a) = K\sum_{j\in\triangle} |a_j|^3 + |a_j|W^{-2}$, $K > 0$ is some constant, $n_j, m_j \geqslant 0$, $v_\triangle, w_\triangle$ are some vectors. Defining $\mathcal{H}(a) = (a, C_{f_a}^{-1}a)/2 - \sum_{\triangle:s_\triangle=1}\mathcal{F}_\triangle(a)$ (same for b), we can apply Brascamp-Lieb (2.3.6) and (2.3.10). As a result the integral above is bounded by

$$K^N\,\mathrm{e}^{\frac{1}{2}(v,C_{f_a}v)+\frac{1}{2}(w,C_{f_b}w)}\prod_{\triangle:s_\triangle=1}\prod_{j\in\triangle}\sqrt{(2n_j-1)!!}\sqrt{(2m_j-1)!!}\;(C_{f_a})_{jj}^{m_j/2}(C_{f_b})_{jj}^{n_j/2} \tag{2.3.16}$$

where now $\hat{f} = \sum_\triangle f_\triangle \mathbf{1}_\triangle$, and $f_\triangle > 0$ for all cubes. Note that, to avoid heavy notations, we write C_f in this case, too. Now we replace C_f by C_f^N in the exponent and hence obtain factorized estimates over each cube. Since C_f^N decays in the same way as C_f, the bounds now work as before.

Finally, when estimating n derivatives in E, we collect a factor N^n from the sums over the j_k's. □

This result is not sufficient to deal with the case of very large (or infinite) volume. To handle this case, we introduce in the next section a cluster expansion.

2.4. Proof of Theorem 2.2.3

Following [DPS02] we will apply a cluster expansion which is a variation of the rooted Brydges-Kennedy Taylor forest formula (cf. [Bry86],[AR95]) to decouple an appropriate finite region containing the observable from the remaining volume. In contrast to [DPS02], we perform first the steps of integration by parts described in Section 2.3.3. This preliminary procedure simplifies the extraction of the correct decay later. The cluster expansion and the preliminary steps of integrations by parts are more easily implemented by going back to the original representation of the integrals (2.2.2),(2.2.3) in terms of Bosonic and Fermionic variables, as in App. 2.B. This is done in Section 2.4.1 below. In Section 2.4.2 a cluster expansion is applied to the supersymmetric representation obtained in Section 2.4.1. The following Sections 2.4.3-2.4.5 bound the different terms in the cluster expansion. More precisely, in Section 2.4.3 we give an estimate on the propagators and in Section 2.4.4 we bound the functional integral on a finite set of cubes. Section 2.4.5 is devoted to combining all bounds and performing the sum over vertex and cube positions as well as the tree structure. Finally in Section 2.4.6, we sketch the procedure for the derivatives.

2.4.1. Supersymmetric representation

To modify the dual representation introduced in Lemma 2.2.1 we introduce a family $(\bar\rho_j, \rho_j)_{j\in\Lambda}$ of Grassmann variables (cf. App. 2.A). For each $j \in \Lambda$ we denote by M_j the supermatrix $M_j = \begin{pmatrix} a_j & \bar\rho_j \\ \rho_j & ib_j \end{pmatrix}$. Note that the trace is replaced by $\operatorname{Str} M_j = a_j - ib_j$ (cf. (2.A.5)). With these notations we can state the new representaion.

Lemma 2.4.1. *The integrals in* (2.2.6) *and* (2.2.8) *can be reorganized to yield*

$$\lim_{\varepsilon\to 0}\frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G^+_\Lambda(E_\varepsilon)] - a_s^+ = \int \mathrm{d}\mu_B(M)\, \mathrm{e}^{\mathcal{V}(M)} a_0, \tag{2.4.1}$$

$$\lim_{\varepsilon\to 0}\partial_E^n\frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G^+_\Lambda(E_\varepsilon)] = \int \mathrm{d}\mu_B(M)\, \mathrm{e}^{\mathcal{V}(M)} a_0 \prod_{k=0}^{n} \operatorname{Str} M_{j_k}, \tag{2.4.2}$$

where the supersymmetric gaussian measure is defined by

$$\mathrm{d}\mu_B(M) := \mathrm{d}M\, \mathrm{e}^{-\frac{1}{2}\operatorname{Str}(M,B^{-1}M)} = \mathrm{d}\mu_B(a,b)\,\mathrm{d}\mu_B(\bar\rho,\rho) \tag{2.4.3}$$

with product measure $\mathrm{d}M = \prod_{j\in\Lambda}\mathrm{d}M_j = \prod_{j\in\Lambda}\mathrm{d}a_j\,\mathrm{d}b_j\,\mathrm{d}\bar\rho_j\,\mathrm{d}\rho_j$,

$$\mathrm{d}\mu_B(\bar\rho,\rho) := \prod_{j\in\Lambda}\mathrm{d}\bar\rho_j\,\mathrm{d}\rho_j \det\left[\tfrac{2\pi}{B^{-1}}\right] \mathrm{e}^{-(\bar\rho,B^{-1}\rho)},$$

and $\operatorname{Str}(M,B^{-1}M) = \sum_{i,j\in\Lambda} B^{-1}_{ij}\operatorname{Str}(M_iM_j)$*, Finally, all non Gaussian terms in the integral are collected in the exponent* $\mathcal{V}(M) = \sum_{j\in\Lambda}\mathcal{V}(M_j)$*, defined by*

$$\begin{aligned}\mathcal{V}(M_j) &:= -\ln\operatorname{Sdet}[\bar{\mathcal{E}} - M_j] - \mathcal{E}\operatorname{Str} M_j - \tfrac{\mathcal{E}^2}{2}\operatorname{Str} M_j^2\\ &= \int_0^1 (1-t)^2 \operatorname{Str}\frac{M^3}{(\bar{\mathcal{E}}-tM)^3}\,\mathrm{d}t = \mathcal{V}_j(a,b) + \bar\rho_j\rho_j D_j,\end{aligned} \tag{2.4.4}$$

where $\mathcal{V}_j(a,b)$ *is the potential introduced in* (2.2.12)*. Here we abuse notation by using the same letter for the potential* $\mathcal{V}(M_j)$ *and* $\mathcal{V}(a,b)$*, since the two expressions are closely related.*

Remark. This representation simplifies the cluster expansion since the covariance appears only in the Gaussian measure. Note that the normalization constants for the real and Fermionic variables above cancel each other.

Proof. We replace the determinant in $\mathcal{R}(a,b)$ by a Fermionic integral using (2.A.2) and collect all remaining terms into the exponent $\mathcal{V}(M)$. □

The following result is the analog of Lemma 2.3.7 in this new formalism.

Lemma 2.4.2. *The expressions* (2.4.1) *and* (2.4.2) *can be reorganized as follows*

$$\lim_{\varepsilon\to 0}\frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G^+_\Lambda(E_\varepsilon)] - a_s^+ = \sum_{l_0\in\Lambda} B_{0l_0}\int \mathrm{d}\mu_B(M)\partial_{a_{l_0}}\mathrm{e}^{\mathcal{V}(M)} =: \sum_{l_0\in\Lambda} B_{0l_0}F_\Lambda^{(l_0)}, \tag{2.4.5}$$

$$\begin{aligned}\lim_{\varepsilon\to 0}\partial_E^n\frac{1}{|\Lambda|}\mathbb{E}[\operatorname{Tr} G^+_\Lambda(E_\varepsilon)] &= \sum_{l_0,\dots,l_n} B_{0l_0}\prod_{m=1}^{n} B_{j_ml_m}\int \mathrm{d}\mu_B(M)\prod_{m=1}^{n}\operatorname{Str}\partial_{M_{l_m}}\partial_{a_{l_0}}\mathrm{e}^{\mathcal{V}(M)}\\ &=: \sum_{l_0,\dots,l_n} B_{0l_0}\prod_{m=1}^{n} B_{j_ml_m}F_\Lambda^{(l_0,\dots,l_n)},\end{aligned} \tag{2.4.6}$$

where ∂_{M_j} *is defined in* (2.B.5).

Proof. We apply integration by parts as in Lemma 2.3.7. and use the following relations: $(\mathrm{Str}\,\partial_{M_i})\mathrm{Str}\,M_j^n = n\delta_{ij}\mathrm{Str}\,M_j^{n-1}$ and $[\mathrm{Str}\,\partial_{M_i}, \mathrm{Str}\,\partial_{M_j}] = 0$. □

Note that $\partial_{a_{l_0}}$ moves the local observable a_0 in 0 to the local observable $\partial_{a_{l_0}}\mathcal{V}(M_{l_0})$ at position l_0. Moreover, the B-factors enable summation over $j_1, \ldots, j_n$ and l_0.

Remark. In the remaining we will show that applying a cluster expansion to $F_\Lambda^{(l_0)}$ and $F_\Lambda^{(l_0,\ldots,l_n)}$ defined in (2.4.5) and (2.4.6) yields the stated estimate for the semicircle law (2.2.13), but not for the derivatives (2.2.14), since in this last case one may not be able to extract enough fine structure to sum over the indices $l_1, \ldots, l_n$. This happens when two or more of the l_k coincide and we obtain linear terms in M from the derivatives $\prod_{m=1}^n \mathrm{Str}\,\partial_{M_{l_m}}\partial_{a_{l_0}}\exp(\mathcal{V}(M))$. In this case, we need to apply again integration by parts on the resulting field factors before performing the cluster expansion. Nevertheless, for clarity, we first prove the cluster expansion only for (2.4.6). It is easy to see that the same approach works for the (more involved) expression we obtain after some steps of integration by parts. A detailed description of the procedure can be found in Section 2.4.6 below.

The following lemma will simplify the cluster expansion, since the integrals over regions without observable contributions turn out to be trivial.

Lemma 2.4.3. *If we restrict the functional integrals $F_\Lambda^{(l_0)}$ and $F_\Lambda^{(l_0,\ldots,l_n)}$ defined in* (2.4.5) *and* (2.4.6) *to a set $Y^C = \Lambda\backslash Y$ not containing l_0, we have for $m > 0$ and indices $l_j \in Y^C$ for $j = 1, \ldots, m$ that*

$$F_{Y^C} = \int \mathrm{d}\mu_{B_{Y^C}}(M)\ \mathrm{e}^{\sum_{j\in Y^C}\mathcal{V}(M_j)} = 1,$$

$$F_{Y^C}^{(l_1,\ldots,l_m)} = \int \mathrm{d}\mu_{B_{Y^C}}(M)\prod_{j=1}^m \mathrm{Str}\,\partial_{M_{l_j}}\ \mathrm{e}^{\sum_{j\in Y^C}\mathcal{V}(M_j)} = 0,$$

where B_{Y^C} is the covariance restricted to the volume Y^C.

Proof. Using the definition of $\mathcal{V}(M)$, we can write

$$F_{Y^C} = \int \mathrm{d}M\,\mathrm{e}^{-\frac{1}{2}\mathrm{Str}\,(M,\tilde{J}^{-1}M)}\ \mathrm{e}^{(\mathcal{E},\mathrm{Str}\,M)}\int \mathrm{d}\bar{\Phi}\,\mathrm{d}\Phi\,\mathrm{e}^{i(\bar{\Phi},(\bar{\mathcal{E}}-M)\Phi)},$$

$$F_{Y^C}^{(l_1,\ldots,l_m)} = \int \mathrm{d}M\,\mathrm{e}^{-\frac{1}{2}\mathrm{Str}\,(M,\tilde{J}^{-1}M)}\ \mathrm{e}^{(\mathcal{E},\mathrm{Str}\,M)}$$
$$\times\left(\sum_{P_1,P_2}\prod_{j_1\in P_1}(-\mathcal{E}^2\mathrm{Str}\,M_{j_1})\prod_{j_2\in P_2}\mathrm{Str}\,\partial_{M_{j_2}}\right)\int \mathrm{d}\bar{\Phi}\,\mathrm{d}\Phi\,\mathrm{e}^{i(\bar{\Phi},(\bar{\mathcal{E}}-M)\Phi)},$$

where $\tilde{J}^{-1} = B_{Y^C}^{-1} + \mathcal{E}^2$ and we insert for the superdeterminant a superintegral with measure $\mathrm{d}\bar{\Phi}\,\mathrm{d}\Phi = \prod_{j\in\Lambda}\mathrm{d}\bar{z}_j\,\mathrm{d}z_j\,\mathrm{d}\bar{\chi}_j\,\mathrm{d}\chi_j$. In the second line we sum over all partitions $P_1 \cup P_2 = \{1, \ldots, m\}$ with $P_1 \cap P_2 = \varnothing$. Note that we can rewrite both $\mathrm{Str}\,\partial_{M_j}$ and $\mathrm{Str}\,M_j$ using integration by parts (in Φ for the first, in M and then in Φ for the second) as $\partial_{\bar{\mathcal{E}}}$ and $\sum_k \mathcal{E}^2\tilde{J}_{jk}^{-1}\partial_{\bar{\mathcal{E}}}$. For a general $\Lambda' \subset \Lambda$, the restriction of B^{-1} to Λ' does not have the form $-W^2\Delta + (1-\mathcal{E}^2)$, but $\operatorname{Re}\tilde{J}^{-1} \geqslant 1$ still holds (cf. Lemma 2.C.2). Hence we can interchange the measures and perform integration over M by completing the square. Inserting (2.B.2), we obtain

$$F_{YC} = \mathbb{E}_{\tilde{J}}\left[\int \mathrm{d}\bar{\Phi}\,\mathrm{d}\Phi\, \mathrm{e}^{i(\bar{\Phi},(\mathcal{E}+A-H)\Phi)}\right] = \mathbb{E}_{\tilde{J}}[1] = 1,$$

$$F_{YC}^{(l_1,\ldots,l_m)} = \left(\sum_{P_1,P_2}\prod_{j_1\in P_1}\sum_{k_{j_1}}\mathcal{E}^2\tilde{J}^{-1}_{j_1 k_{j_1}}\right)\partial^m_{\mathcal{E}}\mathbb{E}_{\tilde{J}}[1] = 0$$

where A is a diagonal matrix with $A_j = \mathcal{E}\sum_k \tilde{J}_{jk}$. Note that $\operatorname{Im}(\mathcal{E}+A-H) > 0$, since $\operatorname{Re}\mathcal{E}$ and $\operatorname{Im}\tilde{J}$ have the same sign. Hence the integrals above are well-defined. □

2.4.2. Cluster expansion

In the following, we prove a cluster expansion for the integrals $F_\Lambda^{(l_0)}$ and $F_\Lambda^{(l_0,\ldots,l_n)}$ defined in (2.4.5) and (2.4.6). We partition a large but finite volume Λ into disjoint cubes $\triangle$ of fixed volume $W^2(\ln W)^\alpha$. By interpolating the covariance, the functional integral over Λ can be rewritten as a sum of local integrals over unions of these cubes called polymers. Here, we use a non-standard cluster expansion interpolating in the real covariance C instead of B and setting $B(s) = (C(s)^{-1} + i\sigma_E m_i^2)^{-1}$ (cf. also the remark below). This is done in an inductive procedure. Because of the interpolation in C, we extract a "multi-link" consisting of three edges instead of a single edge in each step.

Before stating the result, we give a few notations: The volume Λ is divided into cubes $\triangle$ of size $W^2(\ln W)^\alpha$. Denote by $\triangle_0$ the root cube containing l_0. In each step we extract a *generalized cube* $\tilde{\triangle} = (\triangle, \triangle', \triangle'')$ connected via a multi-link (i,k,k',j), where $k' \in \triangle, i \in \triangle', j \in \triangle''$ and k is in the volume already extracted. The links (i,k) and (k',j) are "weak" while (k,k') is "strong" in the sense that it prescribes the tree structure. The collection of $\triangle_0$ and the extracted generalized cube is called the *generalized polymer* $\tilde{Y} = (\triangle_0, \tilde{\triangle}_1, \ldots, \tilde{\triangle}_r)$.

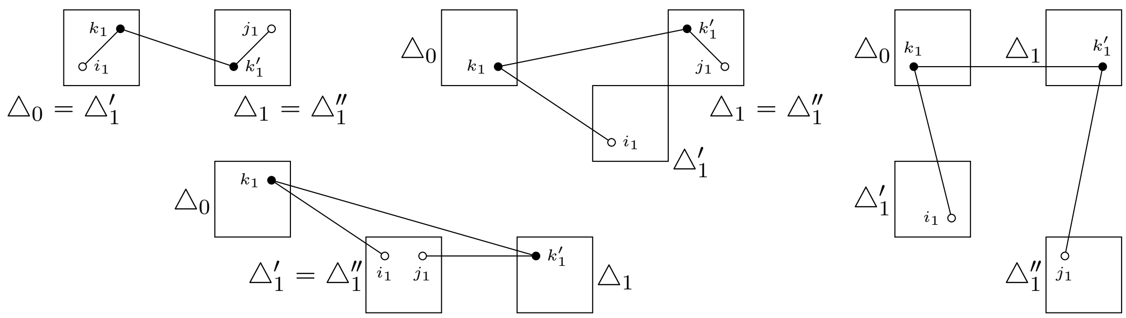


Figure 2.1.: Some examples for the first generalized cube $\tilde{\triangle}_1 = (\triangle_1, \triangle_1', \triangle_1'')$ extracted by the first link $l_1 = (i_1, k_1, k_1', j_1)$ with $k_1 \in \triangle_0$, $k_1' \in \triangle_1$, $i_1 \in \triangle_1'$ and $j_1 \in \triangle_1''$. The cubes may coincide with the unique constraint $\triangle_0 \neq \triangle_1$.

Lemma 2.4.4. *For* $(*)$ *equals to the set of fixed indices* (l_0) *and* $(l_0,\ldots,l_n)$*, respectively, we can write* $F_\Lambda^{(l_0)}$ *and* $F_\Lambda^{(l_0,\ldots,l_n)}$ *defined in* (2.4.5) *and* (2.4.6) *as*

$$
\begin{aligned}
F_\Lambda^{(*)} = & \sum_{\substack{\tilde{Y}: \\ (*)\in\tilde{Y}}} \sum_{\substack{T \text{ on } \tilde{Y}, \\ |T|=r}} \int_{[0,1]^r} \mathrm{d}s_r \cdots \mathrm{d}s_1 M_T(s) \\
& \times \sum_{\substack{(i_q,j_q) \\ (k_q,k'_q)}} \prod_{q=1}^{r} G_q(s)_{i_q k_q} C_{k_q k'_q} G_q(s)_{k'_q j_q} \; F_T^{(*)}[s](\{i_q, j_q\}_{q=1}^r),
\end{aligned}
$$

where $\tilde{Y} = (\triangle_0, \tilde{\triangle}_1, \ldots, \tilde{\triangle}_r)$ *is a generalized polymer consisting of the root cube* $\triangle_0$ *(containing* l_0*) and* r *ordered generalized cubes* $\tilde{\triangle}_q$*,* $q = 1,\ldots,r$*. Each* $\tilde{\triangle}_q = (\triangle_q, \triangle'_q, \triangle''_q)$ *is a collection of three cubes not necessarily disjoint with the unique constraint* $\triangle_q \cap (\triangle_0 \cup \bigcup_{p=1}^{q-1} \tilde{\triangle}_p) = \emptyset$*. For* $(*) = (l_0,\ldots,l_n)$*, the generalized polymer* $\tilde{Y}$ *needs to contain all the indices* $l_0,\ldots,l_n$*. We sum over all ordered trees* T *on the generalized polymer, such that the* q*-th tree link connects* $\tilde{\triangle}_q$ *with* $\triangle_0 \cup \bigcup_{p=1}^{q-1} \tilde{\triangle}_p$*.*

Each tree link consists of three lines (i_q,k_q)*,* (k_q,k'_q) *and* (k'_q,j_q)*, where the* $k_q - k'_q$ *connection forms the tree structure. Precisely,* $k_q \in \triangle_0 \cup \bigcup_{p=1}^{q-1} \tilde{\triangle}_p$ *is in the generalized polymer up to index* $q-1$*,* $k'_q \in \triangle_q$*,* $i_q \in \triangle'_q$ *and* $j_q \in \triangle''_q$*. Note that the position of* $\triangle'$ *and* $\triangle''$ *is arbitrary and they can coincide with each other or an already extracted cube. For each link, we have an interpolation parameter* $0 \leqslant s_q \leqslant 1$*. The functional integrals* $F_T^{(l_0)}$ *and* $F_T^{(l_0,\ldots,l_n)}$ *are defined by*

$$
\begin{aligned}
F_T^{(l_0)}[s](\{i_q,j_q\}) &:= \int \mathrm{d}\mu_{B(s)}(M) \prod_{q=1}^{r} \operatorname{Str}\left(\partial_{M_{i_q}} \partial_{M_{j_q}}\right) \left[\partial_{a_{l_0}} \mathrm{e}^{\mathcal{V}(M)}\right], \qquad (2.4.7) \\
F_T^{(l_0,\ldots,l_n)}[s](\{i_q,j_q\}) &:= \int \mathrm{d}\mu_{B(s)}(M) \prod_{q=1}^{r} \operatorname{Str}\left(\partial_{M_{i_q}} \partial_{M_{j_q}}\right) \left[\prod_{m=1}^{n} \operatorname{Str} \partial_{M_{l_m}} \partial_{a_{l_0}} \mathrm{e}^{\mathcal{V}(M)}\right],
\end{aligned}
$$

where $B(s) := (C(s)^{-1} + i\sigma_E m_i^2)^{-1}$ *and* $C(s)_{ij} := s_{ij} C_{ij}$ *with*

$$
s_{ij} := \begin{cases} 1 & \text{if } \exists\ q : i,j \in \tilde{\triangle}_q, \\ \prod_{p=q'}^{q-1} s_p & \text{if } \exists\ q' < q : i \in \tilde{\triangle}_q \text{ and } j \in \tilde{\triangle}_{q'} \text{ or vice versa}, \\ 0 & \text{otherwise}, \end{cases}
$$

$M_T(s)$ *is a product of* s *factors extracted by the derivative* $\partial_{s_q} B(s)$*. The propagator* $G_q(s)^{-1} := (1 + i\sigma_E m_i^2 C(s))_{|s_p=1 \forall p>q}$ *depends only on the first* q *interpolation parameters* $s_1,\ldots,s_q$*.*

Remark. The most standard way to do a cluster expansion would be to interpolate directly in the total complex covariance B. The propagator $G(s)CG(s)$ would then be replaced by B. Nevertheless, in order to bound our expressions, we need $(\operatorname{Re} B(s)^{-1})^{-1}$ to behave similar to C, and it is not easy to compare these two operators.

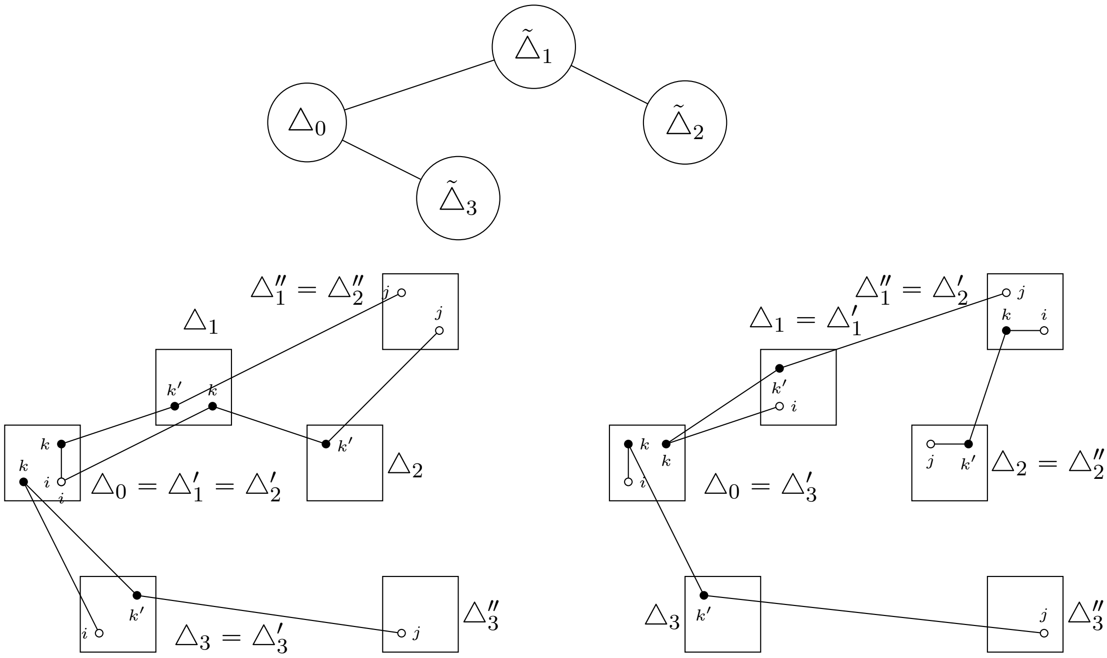


Figure 2.2.: Two examples of links and underlying generalized polymere for a fixed tree structure on $\tilde{Y}$. Note that cubes and even the notes can coincide as long as the conditions $k_q \in \bigcup_{p=1}^{q-1} \tilde{\triangle}_p \cup \triangle_0$ and $k_q' \in \tilde{\triangle}_q$ are fulfilled.

One may also use a standard cluster expansion (e.g. a Brydges-Kennedy Taylor forest formula or Erice type cluster expansion [AR95, Bry86]) in the real covariance C. Since the propagator $G(s)CG(s)$ has an s-dependence, derivatives in s could also fall on it, which complicates the algebra involved in factorizing the contributions from different connected components.

Therefore, we use the same "inductive" interpolation scheme as in [DPS02] (analog to older versions of cluster expansions, cf. [Riv91, Chapter III.1]).

Proof. We construct the cluster expansion by an inductive argument. The large volume is divided into cubes of size $W^2(\ln W)^\alpha$. In the following, we want to extract the set of cubes interacting with the observable. Therefore, we test if there exists a connection between the root cube $\triangle_0$ and some other cube $\triangle \subset \Lambda$.

We introduce an interpolating covariance $B(s_1)$ with $0 \leqslant s_1 \leqslant 1$, which satisfies $B(1) = B$ while $B(0)$ decouples the root cube $\triangle_0$ from the rest of the volume. We define $B(s_1)^{-1} := C(s_1)^{-1} + i\sigma_E m_i^2$, where

$$C(s_1)_{ij} := \begin{cases} s_1 C_{ij} & \text{if } i \in \triangle_0 \text{ and } j \in \triangle \neq \triangle_0 \text{ or vice versa,} \\ C_{ij} & \text{otherwise.} \end{cases}$$

This is equivalent to $C(s_1) = s_1 C + (1-s_1)(C_{\triangle_0\triangle_0} + C_{\triangle_0^C\triangle_0^C})$, where $C_{\triangle\triangle}$ is the covariance C restricted to the set $\triangle$. By this definition, $C(s_1)$ is still a positive operator because it is a convex combination of positive operators. Define

$$F_\Lambda^{(l_0)}[s_1] := \int \mathrm{d}\mu_{B(s_1)}(M)\partial_{a_{l_0}}\, \mathrm{e}^{\mathcal{V}(M)}$$

and $F_\Lambda^{(l_0,\dots,l_n)}[s_1]$ similarly. Note that for $s_1 = 1$ we have $F_\Lambda^{(*)}[s_1]_{|s_1=1} = F_\Lambda^{(*)}$. By the fundamental theorem of calculus

$$F_\Lambda^{(*)}[s_1]_{|s_1=1} = F_\Lambda^{(*)}[s_1]_{|s_1=0} + \int_0^1 \mathrm{d}s_1\ \partial_{s_1} F_\Lambda^{(*)}[s_1].$$

$F_\Lambda^{(*)}[s_1]_{|s_1=0}$ corresponds to decoupling $\triangle_0$ from the remaining volume. By Lemma 2.4.3, the integral over $\triangle_0^C$ yields one in the case $* = l_0$ or if all indices $l_0, \dots, l_n$ are in $\triangle_0$, and zero otherwise. The derivative is written by integration by parts as

$$\int \partial_{s_1} \mathrm{d}\mu_{B(s_1)}(M)[\cdot] = \int \mathrm{d}\mu_{B(s_1)}(M) \sum_{i_1 j_1} \partial_{s_1} B(s_1)_{i_1 j_1} \tfrac{1}{2}\mathrm{Str}\, \partial_{M_{i_1}} \partial_{M_{j_1}} [\cdot],$$

Moreover, the propagator $\partial_{s_1} B(s_1)_{i_1 j_1}$ gives three connections

$$\partial_{s_1} B(s_1)_{i_1 j_1} = \sum_{\triangle_1 \neq \triangle_0} \sum_{\substack{k_1 \in \triangle_0 \\ k_1' \in \triangle_1}} G(s_1)_{i_1 k_1} C_{k_1 k_1'} G(s_1)_{k_1' j_1} + G(s_1)_{i_1 k_1'} C_{k_1' k_1} G(s_1)_{k_1 j_1},$$

where $G(s_1) = (1 + i\sigma_E m_i^2 C(s_1))^{-1}$. Since the matrices C and $G(s_1)$ are symmetric, one can rewrite the second summand as $G(s_1)_{j_1 k_1} C_{k_1 k_1'} G(s_1)_{k_1' i_1}$. To sum the two terms, note that the supertrace is invariant under changing i_1 and j_1. Therefore we obtain

$$\partial_{s_1} F_\Lambda^{(*)}[s_1] = \sum_{\substack{(i_1,j_1) \\ (k_1,k_1')}} \left(G(s_1)_{i_1 k_1} C_{k_1 k_1'} G(s_1)_{k_1' j_1} \right) F_\Lambda^{(*)}[s_1]((i_1, j_1)),$$

where

$$F_\Lambda^{(l_0)}[s_1]((i_1, j_1)) = \int \mathrm{d}\mu_{B(s_1)}(M) \mathrm{Str}\, \partial_{M_{i_1}} \partial_{M_{j_1}} \partial_{a_{l_0}}\, \mathrm{e}^{\mathcal{V}(M)}$$

and $F_\Lambda^{(l_0,\dots,l_n)}[s_1]((i_1, j_1))$ is defined similarly.

For each pair (k_1, k_1') with $k_1 \in \triangle_0$ and $k_1' \in \triangle_1$, there is a strong connection between $\triangle_0$ and $\triangle_1$, but there is no corresponding derivative in the functional integral as for i_1 and j_1. If i_1 or j_1 belong to some cube $\triangle \nsubseteq \triangle_0 \cup \triangle_1$, they give some additional connections. Therefore, the first step of induction extracts a link consisting of three connection between the four points i_1, j_1, k_1 and k_1'. This link connects $\triangle_0$ to a set of one, two or three new cubes, which we call the generalized cube $\tilde{\triangle}_1$ (cf. Figure 2.1).

Now, we fix $(i_1, j_1), (k_1, k_1')$ corresponding to a connection between $\triangle_0$ and $\tilde{\triangle}_1$. We test if there is a connection between $\tilde{\triangle}_{0,1} = \triangle_0 \cup \tilde{\triangle}_1$ and any other cube $\triangle'$. For this, we define for $0 \leqslant s_2 \leqslant 1$ the real interpolating covariance as

$$C(s_1, s_2)_{ij} = \begin{cases} s_2 C(s_1)_{ij} & \text{if } i \in \tilde{\triangle}_{0,1} \text{ and } j \notin \tilde{\triangle}_{0,1} \text{ or vice versa,} \\ C(s_1)_{ij} & \text{otherwise.} \end{cases}$$

We can write $C(s_1, s_2)$ again as a convex combination of positive operators

$$C(s_1, s_2) = s_2 C(s_1) + (1 - s_2)(C_{\tilde{\triangle}_{0,1}\tilde{\triangle}_{0,1}}(s_1) + C_{\tilde{\triangle}^C_{0,1}\tilde{\triangle}^C_{0,1}}(s_1)),$$

thus $C(s_1, s_2)$ is still positive. Now, $F^{(*)}_\Lambda[s_1]((i_1, j_1)) = F^{(*)}_\Lambda[s_1, s_2]((i_1, j_1))_{|s_2=1}$. By the fundamental theorem of calculus

$$F^{(*)}_\Lambda[s_1, s_2]((i_1, j_1))_{|s_2=1} = F^{(*)}_\Lambda[s_1, s_2]((i_1, j_1))_{|s_2=0} + \int_0^1 \mathrm{d}s_2\ \partial_{s_2} F^{(*)}_\Lambda[s_1, s_2]((i_1, j_1)).$$

As before $F^{(*)}_\Lambda[s_1, s_2]((i_1, j_1))_{|s_2=0}$ corresponds to the functional integral restricted to $\tilde{\triangle}_{0,1}$ (if all indices $l_0, \dots, l_n$ are in $\tilde{\triangle}_{0,1}$, otherwise it is zero). The derivative in s_2 of $F^{(*)}_\Lambda[s_1, s_2]((i_1, j_1))$ gives

$$\sum_{\substack{(i_2,j_2),(k_2,k'_2)\\ k_2\in\tilde{\triangle}_{0,1},k'_2\notin\tilde{\triangle}_{0,1}}} \left[G(s_1, s_2)_{i_2k_2} C(s_1)_{k_2k'_2} G(s_1, s_2)_{k'_2j_2}\right] F^{(*)}_\Lambda[s_1, s_2]((i_1, j_1), (i_2, j_2)).$$

Note that i_2 and j_2 are arbitrary but k_2 needs to be in $\tilde{\triangle}_{0,1}$ and k'_2 in a new cube.

We repeat this argument until we construct all possible connected components containing the root cube. Note that in the second case, only generalized polymers containing all indices $l_0, \dots, l_n$ give a non-zero contribution. This is a finite sum for Λ fixed. The $k_r - k'_r$ connections build a tree structure on the generalized cubes, while the positions of i_r and j_r are arbitrary (cf. Figure 2.2). □

2.4.3. Decay of $\mathbf{G_q}(\mathrm{s})$ and $\mathrm{B}(\mathrm{s})$

First we determine the decay of the propagator $G_q(s)$ and the interpolated complex covariance $B(s)$. Note that, for $C(s)$, we can simply use that $C(s)_{ij} \leqslant C_{ij}$.

Lemma 2.4.5. *The decays of $B(s)$ and $G_q(s)$, respectively, are given by*

$$|B(s)_{ij}| \leqslant |C_{ij}| + \tfrac{K}{W^2}\, \mathrm{e}^{-f\frac{m_r}{W}|i-j|}$$
$$|G_p(s)_{ij}| \leqslant \delta_{ij} + |C_{ij}| + \tfrac{K}{W^2}\, \mathrm{e}^{-f\frac{m_r}{W}|i-j|}$$

where $f = \inf[1/2, g]$ with a constant $g < 1$ independent of W.

Remark. Note that the decay of $B(s)$ and $G(s)$ is bounded by

$$|C_{ij}| + \tfrac{K}{W^2}\, \mathrm{e}^{-f\frac{m_r}{W}|i-j|} \leqslant \begin{cases} \frac{K}{W^2} \ln\left(\frac{W}{m_r(1+|i-j|)}\right) & \text{if } |i-j| \leqslant \frac{W}{m_r}, \\ \frac{K}{W^2}\, \mathrm{e}^{-f\frac{m_r}{W}|i-j|} & \text{if } |i-j| > \frac{W}{m_r}. \end{cases}$$

Proof. The proof works exactly like the one in [DPS02, Lemma 15], replacing the three dimensional decay (2.3.3) with the two dimensional decay given in (2.3.2), the key relation being $\sum_{k\in\Lambda}(C(s)_{jk}\exp(\mu|k-j|))^2 \leqslant KW^{-2}$ for $\mu < m_r/(2W)$. □

2.4.4. Bounding the functional integrals

To estimate (2.4.7), we fix a generalized polymer $\tilde{Y}$ and indices $\{i_q, j_q\}$, and we define $\mathcal{J} = \{i_q, j_q : q = 1, \ldots, r\} \subseteq Y$ as the set of all derived indices. Then the corresponding integrand can be written as

$$\prod_{p=1}^{r} \operatorname{Str} \partial_{M_{i_p}} \partial_{M_{j_p}} \left[\partial_{a_{l_0}} e^{\mathcal{V}(M)} \right] = \partial_{a_{l_0}} \prod_{j \in Y/\mathcal{J}} e^{\mathcal{V}(M_j)} \sum_{d \in \mathcal{D}} \prod_{j \in \mathcal{J}} \partial_{M_j}^{d_j} e^{\mathcal{V}(M_j)}$$

where $\partial_{M_j}^{d_j} := \partial_{a_j}^{d_j(a)} \partial_{b_j}^{d_j(b)} \partial_{\bar{\rho}_j}^{d_j(\bar{\rho})} \partial_{\rho_j}^{d_j(\rho)}$, and $\mathcal{D} = \{d = \{d_j\}_{j \in \mathcal{J}}\}$ is a set of multi-indices with $d_j = (d_j(a), d_j(b), d_j(\bar{\rho}), d_j(\rho))$. Note that $d_j(\bar{\rho}), d_j(\rho) \in \{0, 1\}$ and $|d_j| := d_j(a) + d_j(b) + d_j(\bar{\rho}) + d_j(\rho)$ equals the multiplicity of j in $\mathcal{J}$. For the case $j = l_0$, we have an additional derivative in a_{l_0} which needs to be treated separately. Computing the derivatives for each $j \in \mathcal{J} \cup \{l_0\}$ and each multi-index d_j,

$$\partial_{a_{l_0}}^{\delta_{j l_0}} \partial_{M_j}^{d_j} e^{\mathcal{V}(M_j)} = \sum_{r_j} M_j^{r_j} C_{d_j, r_j}(a_j, b_j) \, e^{\mathcal{V}_j(a,b)} \, e^{-\bar{\rho}_j \rho_j D_j(a,b)[1 - d_j(\bar{\rho}) d_j(\rho)]},$$

where $M_j^{r_j} := a_j^{r_j(a)} b_j^{r_j(b)} \bar{\rho}_j^{r_j(\bar{\rho})} \rho_j^{r_j(\rho)}$, $r_j = (r_j(a), r_j(b), r_j(\bar{\rho}), r_j(\rho))$ are the remaining powers of the variables in M, and $C_{d_j, r_j}(a_j, b_j)$ is a bounded function remaining after derivatives have been taken. Note that we use the notation $d_j = 0$ for $j \notin \mathcal{J}$, and the same for r_j for $j \notin \mathcal{J} \cup \{l_0\}$. Using the definitions (2.4.4), (2.2.11), (2.2.12), and the relation $\partial_a^d \exp(V(a)) = \sum_{k=1}^{d} \binom{d}{k} \partial_a^{d-k} \left[(\partial_a V(a))^k \right] \exp(V(a))$, one can see that

$$|C_{d_j, r_j}(a_j, b_j)| \leqslant K^{d_j(a) + d_j(b)} d_j(a)! d_j(b)!$$

independent of r_j for all (a_j, b_j) configurations. Note that $n_j := |r_j| + |d_j| \geqslant 3$ and $|r_j| \leqslant 3|d_j|$ for all $j \in \mathcal{J} \backslash \{l_0\}$. If $l_0 \notin \mathcal{J}$, we have $|r_{l_0}| = 2$ and if $l_0 \in \mathcal{J}$, we have at least $n_{l_0} \geqslant 2$.

Lemma 2.4.6. *The functional integral* (2.4.7) *is bounded by*

$$|F_T^{(l_0)}[s](\{i_q, j_q\})| \leqslant K_1^{|Y| (\ln W)^\alpha} \sum_{d \in \mathcal{D}} \sum_{\{r_j\}_{j \in \mathcal{J}}} \prod_{\triangle \in Y} \left[K_2^{n_\triangle} r_\triangle! \prod_{j \in \mathcal{J} \cap \triangle} d_j! \left(\tfrac{\ln W}{W^2} \right)^{\frac{r_j}{2}} \right], \tag{2.4.8}$$

where K_1 and K_2 are constants, and $|Y|$ denotes the number of cubes in Y.

Proof. We first compute the Fermionic integral and estimate the resulting determinant (see the two lemmas below). We obtain

$$\begin{aligned} &\left| \int d\mu_{B(s)}(\bar{\rho}, \rho) \, e^{-\sum_{j \in J_4} \bar{\rho}_j \rho_j D_j} \left(\prod_{j \in J_1} \rho_j \bar{\rho}_j \right) \left(\prod_{j \in J_2} \rho_j \right) \left(\prod_{j \in J_3} \bar{\rho}_j \right) \right| \\ &\leqslant \prod_{\triangle \in Y} r_\triangle(\rho)^{r_\triangle(\rho)} r_\triangle(\bar{\rho})^{r_\triangle(\bar{\rho})} \left(\frac{\sqrt{\ln W}}{W} \right)^{r_\triangle(\rho) + r_\triangle(\bar{\rho})} e^{\operatorname{Re} \operatorname{Tr} (B(s) D)_{J_3 \cup J_4, J_3 \cup J_4}} \, e^{K(|Y| (\ln W)^\alpha + |\mathcal{J}|)}, \end{aligned} \tag{2.4.9}$$

where $Y = \bigcup_{j=1}^{5} J_j$ is a partition given by

$$\begin{aligned}
J_1 &= \{j \in Y : r_j(\rho) = r_j(\bar\rho) = 1\}, \\
J_2 &= \{j \in Y : r_j(\rho) = 1, r_j(\bar\rho) = 0\}, \\
J_3 &= \{j \in Y : r_j(\rho) = 0, r_j(\bar\rho) = 1\}, \\
J_4 &= \{j \in Y : r_j(\rho) = r_j(\bar\rho) = 0, d_j(\rho) = d_j(\bar\rho) = 0\}, \\
J_5 &= \{j \in Y : r_j(\rho) = r_j(\bar\rho) = 0, d_j(\rho) = d_j(\bar\rho) = 1\}.
\end{aligned}$$

The remaining Bosonic functional integral is bounded by generalizing the results of the finite volume case. We first insert the partition into the different domains of integration for each cube separately. In I^1 we need the factors $|a_j|^{r_j(a)}|b_j|^{r_j(b)}$ of $M_j^{r_j}$ since they give additional small factors of order $W^{-1}(\ln W)^{1/2}$. In the other regions these factors can be bounded by $\exp(\mathcal{V}_j(a,b))$ and we include them into $C(a_j)C(b_j)$.

Summarizing the above procedure, we estimate $|F_T^{(l_0)}[s](\{i_q, j_q\})|$ by

$$\begin{aligned}
&\mathrm{e}^{K(|Y|(\ln W)^\alpha + |\mathcal{J}|)} \int \big|\,\mathrm{d}\mu_{B(s)}(a,b)\big|\ \mathrm{e}^{\mathrm{Re}\,\mathrm{Tr}\,(B(s)D)_{J_3J_3} + \mathrm{Re}\,\mathrm{Tr}\,(B(s)D)_{J_4J_4}} \prod_{j\in Y} \big|\,\mathrm{e}^{\mathcal{V}_j(a,b)}\big| \\
&\times \sum_{d\in\mathcal{D}} \sum_{\{r_j\}_{j\in\mathcal{J}}} \prod_{\triangle\in Y} \frac{r_\triangle(\rho)^{r_\triangle(\rho)} r_\triangle(\bar\rho)^{r_\triangle(\bar\rho)}}{W^{r_\triangle(\rho)+r_\triangle(\bar\rho)}} \\
&\times \prod_{j\in\mathcal{J}\cap\triangle} \big(\chi[I_\triangle^1]|a_j|^{r_j(a)}|b_j|^{r_j(b)} + \chi[(I_\triangle^1)^C]\big)\, K^{d_j(a)+d_j(b)} d_j(a)!d_j(b)!
\end{aligned}$$

We first apply Lemma 2.3.4 which also holds for $B(s)$ and $C(s)$. Proceeding as in the proof of Theorem 2.3.9 we insert the bounds of Lemma 2.3.6. As a result we obtain (2.3.15) and (2.3.16) with C replaced by $C(s)$ and C_f replaced by $C_f(s) = (C(s)^{-1} - fm_r^2)^{-1} > 0$. Now we have $C(s) < C_N = (-W^2\Delta_N + m_r^2)^{-1}$ since $C(s)$ can be represented as a quadratic form of block diagonal pieces of C and each of these is smaller (as a quadratic form) than C_N by the arguments of Lemma 2.C.2. This decouples the different cubes. In I^1 we obtain for each $j \in \mathcal{J}$ a factor

$$r_j(a)!r_j(b)!\left(\tfrac{K(\ln W)^{1/2}}{W}\right)^{r_j(a)+r_j(b)}.$$

In the other regions we extract this factor from the exponential decay. The factorials in $d_j(a)$ and $d_j(b)$ are bounded by $d_j!$ and the factors in $r_\triangle$ are bounded by $K^{r_\triangle} r_\triangle!$. Finally, we end up with (2.4.8). Note that we obtain an additional factor $W^{-2}\ln W$ in the case, where $l_0 \notin \mathcal{J}$. In the other case, we extract the precision later. □

Lemma 2.4.7. *The Fermionic integral (2.4.9) can be written as*

$$\prod_{\triangle\in Y} r_\triangle(\rho)^{r_\triangle(\rho)} r_\triangle(\bar\rho)^{r_\triangle(\bar\rho)} \left(\tfrac{\sqrt{\ln W}}{W}\right)^{r_\triangle(\rho)+r_\triangle(\bar\rho)} \sigma \det M, \tag{2.4.10}$$

where σ is a sign and the matrix M is given by $M = (M_{J_iJ_j})_{i,j=1}^5$ with blocks

$$
\begin{aligned}
(M_{J_iJ_j})_{\alpha\beta} &= \left(\tfrac{W^2}{r_\triangle(\rho_\alpha)r_\triangle(\bar\rho_\beta)\ln W}B(s)_{\alpha\beta}\right)_{\alpha\in J_i,\beta\in J_{j'}} && \textit{for } i,j \in \{1,2\},\\
(M_{J_iJ_j})_{\alpha\beta} &= \left(\tfrac{W}{r_\triangle(\rho_\alpha)\sqrt{\ln W}}(DB(s))_{\alpha\beta}\right)_{\alpha\in J_i,\beta\in J_j} && \textit{for } i \in \{1,2\} \textit{ and } j \in \{3,4\},\\
(M_{J_iJ_j})_{\alpha\beta} &= \left(\tfrac{W}{r_\triangle(\bar\rho_\beta)\sqrt{\ln W}}B(s)_{\alpha\beta}\right)_{\alpha\in J_i,\beta\in J_{j'}} && \textit{for } i \in \{3,4,5\} \textit{ and } j \in \{1,2\},\\
(M_{J_iJ_j})_{\alpha\beta} &= ((1+DB(s))_{\alpha\beta})_{\alpha\in J_i,\beta\in J_j} && \textit{for } i \in \{3,4,5\}, j \in \{3,4\},\\
(M_{J_iJ_j})_{\alpha\beta} &= 0 && \textit{for } i \in \{1,2,3,4\}, j=5,\\
(M_{J_iJ_j})_{\alpha\beta} &= \delta_{\alpha\beta} && \textit{for } i=j=5,
\end{aligned}
$$

where $j' = 1$ for $j = 1$ and $j' = 3$ for $j = 2$.

Proof. Note that the integral is zero unless $|J_2| = |J_3|$ because of the symmetry of the Fermionic Gaussian integral. Computing the Fermionic integral (2.4.9) by (2.A.3), it is equal to

$$\sigma \det B(s) \det_{J_1\cup J_3, J_1\cup J_2}(B(s)^{-1} + \tilde D),$$

where σ is a sign, $\tilde D$ is a diagonal matrix with $\tilde D_j = D_j[1 - d_j(\bar\rho)d_j(\rho)]$ and $\det_{IJ} A$ is the determinant of the minor of A, where the rows with indices in I and the columns with indices in J are crossed out. Since we estimate the absolute value in the next step, we do not need the precise sign. We assume without loss of generality that the indices $j \in \Lambda$ are ordered such that $B(s)$ is a block matrix of the form $B(s) = (B(s)_{J_iJ_j})_{i,j=1}^5$ and $B(s)_{J_i,J_j} = (B(s)_{\alpha\beta})_{\alpha\in J_i,\beta\in J_j}$.

To simplify this expression the minor is extended to a $|\Lambda| \times |\Lambda|$ matrix without changing the determinant up to a sign in the following way:

$$
M_1 = \begin{bmatrix}
A & * & * & * & * \\
0 & 0 & (B(s)^{-1})_{J_2J_3} & (B(s)^{-1})_{J_2J_4} & (B(s)^{-1})_{J_2J_5} \\
0 & A' & * & * & * \\
0 & 0 & (B(s)^{-1})_{J_4J_3} & (B(s)^{-1} + D)_{J_4J_4} & (B(s)^{-1})_{J_4J_5} \\
0 & 0 & (B(s)^{-1})_{J_5J_3} & (B(s)^{-1})_{J_5J_4} & (B(s)^{-1})_{J_5J_5}
\end{bmatrix},
$$

where the blocks A and A' have determinant one. The blocks $*$ can be chosen arbitrarily. We choose A and A' as the identity, $(M_1)_{J_1J_2} = 0$ and the other freely selectable blocks $(M_1)_{J_iJ_j}$ as $(B(s)^{-1} + D)_{J_iJ_j}$. By multiplying with $B(s)$ from the left, we obtain

$$
\tilde M = B(s)M_1 = \begin{bmatrix}
B(s)_{J_1J_1} & B(s)_{J_1J_3} & (B(s)D)_{J_1J_3} & (B(s)D)_{J_1J_4} & 0 \\
B(s)_{J_2J_1} & B(s)_{J_2J_3} & (B(s)D)_{J_2J_3} & (B(s)D)_{J_2J_4} & 0 \\
B(s)_{J_3J_1} & B(s)_{J_3J_3} & (1 + B(s)D)_{J_3J_3} & (B(s)D)_{J_3J_4} & 0 \\
B(s)_{J_4J_1} & B(s)_{J_4J_3} & (B(s)D)_{J_4J_3} & (1 + B(s)D)_{J_4J_4} & 0 \\
B(s)_{J_5J_1} & B(s)_{J_5J_3} & (B(s)D)_{J_5J_3} & (B(s)D)_{J_5J_4} & (1)_{J_5J_5}
\end{bmatrix}
$$

Extracting a factor $r_{\triangle_j}(\rho)\sqrt{\ln W}/W$ for each $j \in J_1 \cup J_2$ from lines J_1 and J_2 and a factor $r_{\triangle_j}(\bar\rho)\sqrt{\ln W}/W$ for each $j \in J_1 \cup J_3$ from columns J_1 and J_2, we obtain (2.4.10). Note that columns of $B(s)$ with indices in J_3 become columns with indices in J_2 in $\tilde M$ such that we need to extract the factors from column J_2. ☐

Lemma 2.4.8. *The determinant of the matrix M can be bounded by*

$$|\det M| \leqslant K\, \mathrm{e}^{\operatorname{Re}\operatorname{Tr}(B(s)D)_{J_3\cup J_4, J_3\cup J_4}}\, \mathrm{e}^{K(|Y|(\ln W)^\alpha + |\mathcal{J}|)}.$$

Proof. We use the usual bound for determinants (2.3.4) with $A = M - 1$. Since

$$\operatorname{Tr} A^*A = \sum_{i,j=1}^{5} \operatorname{Tr}(A_{J_iJ_j})^* A_{J_iJ_j} \text{ and } \operatorname{Tr}(A_{J_iJ_j})^* A_{J_iJ_j} = \sum_{\alpha\in J_i,\beta\in J_j} \bar A_{\alpha\beta} A_{\alpha\beta},$$

each block can be bounded separately. For $j = 5$ and all i the trace above is zero. For $i = 3, 4, 5$ and $j = 3, 4$, we have $A_{J_iJ_j} = (DB(s))_{J_iJ_j}$ and we estimate

$$\operatorname{Tr}(A_{J_iJ_j})^* A_{J_iJ_j} = \sum_{\alpha\in J_i,\beta\in J_j} |B(s)_{\alpha\beta}|^2 |D_\beta|^2 \leqslant K \sum_{\alpha\in Y} \frac{1}{W^2} \leqslant K|Y|(\ln W)^\alpha,$$

where we used $|D_\beta| \leqslant K$ and the decay of $B(s)$ (Lemma 2.4.5). For $i = 1, 2$ and $j = 3, 4$, we only have off-diagonal terms and $A_{J_iJ_j} = \left(\frac{W}{r_\triangle(\rho)\sqrt{\ln W}} B(s)D\right)_{J_iJ_j}$. Using $r_\triangle \geqslant 1$ and $|D_\beta| \leqslant K$, we estimate

$$\operatorname{Tr}(A_{J_iJ_j})^* A_{J_iJ_j} \leqslant K \tfrac{W^2}{\ln W} \sum_{\alpha\in J_i,\beta\in J_j} |B(s)_{\alpha\beta}|^2 \leqslant K \tfrac{W^2}{\ln W} \sum_{\alpha\in J_i} \tfrac{1}{W^2} \leqslant K \tfrac{|\mathcal{J}|}{\ln W},$$

where the sum over $\beta \in J_j$ is extended to $\beta \in \Lambda$. We bound the trace similarly for the case $i = 3, 4, 5$ and $j = 1, 2$, where $A_{J_iJ_j} = \left(\frac{W}{r_\triangle(\bar\rho)\sqrt{\ln W}} B(s)\right)_{J_iJ'_j}$. Extending the sum over $\alpha \in J_i$ to $\alpha \in \Lambda$, we end up with

$$\operatorname{Tr}(A_{J_iJ_j})^* A_{J_iJ_j} \leqslant \tfrac{W^2}{\ln W} \sum_{\alpha\in J_i,\beta\in J_j} |B(s)_{\alpha\beta}|^2 \leqslant K \tfrac{W^2}{\ln W} \sum_{\beta J_j} \tfrac{1}{W^2} \leqslant K \tfrac{|\mathcal{J}|}{\ln W},$$

For $i, j = 1, 2$, we have $A_{J_iJ_j} = \left(\frac{W^2}{r_\triangle(\rho) r_\triangle(\bar\rho)\ln W} B(s) - 1\right)_{J_iJ'_j}$. Summing the trace of the quadratic term and the corresponding block of $\operatorname{Re}\operatorname{Tr} A$, terms linear in M cancel and we end up with

$$\tfrac{1}{2}\operatorname{Tr}((M_{J_iJ_j})^* - 1)(M_{J_iJ_j} - 1) + \operatorname{Re}\operatorname{Tr}(M_{J_iJ_j} - 1) \leqslant \tfrac{1}{2}\operatorname{Tr}(M_{J_iJ_j})^* M_{J_iJ_j}.$$

Now, the term $\operatorname{Tr}(M_{J_iJ_j})^* M_{J_iJ_j}$ is bounded by using the factor $r_\triangle$ explicitly. Rewriting the sum over $\alpha \in J_i$ and $\beta \in J_j$ into a sum over cubes, we obtain

$$\begin{aligned} \operatorname{Tr}(M_{J_iJ_j})^* M_{J_iJ_j} &\leqslant \sum_{\alpha\in J_i,\beta\in J_j} \frac{W^4}{r_{\triangle_\alpha}(\rho)^2 r_{\triangle_\beta}(\bar\rho)^2 (\ln W)^2} |B(s)_{\alpha\beta}|^2 \\ &\leqslant \sum_{\substack{\triangle:\triangle\cap J_i\neq\varnothing \\ \triangle':\triangle'\cap J_j\neq\varnothing}} \sum_{\substack{\alpha\in\triangle\cap J_i \\ \beta\in\triangle'\cap J_j}} \frac{W^4}{r_\triangle(\rho)^2 r_{\triangle'}(\bar\rho)^2 (\ln W)^2} |B(s)_{\alpha\beta}|^2. \end{aligned}$$

The number of summands in the second sum is bounded by $r_\triangle(\rho) r_{\triangle'}(\bar\rho)$. Applying the estimate $|\alpha - \beta| \geqslant W(\operatorname{dist}(\triangle_\alpha, \triangle_\beta) - 1)$ and the decay of $B(s)$, we end up with

$$\begin{aligned}\operatorname{Tr}(M_{J_iJ_j})^* M_{J_iJ_j} \leqslant \sum_{\substack{\triangle:\triangle\cap J_i\neq\varnothing\\ \triangle':\triangle'\cap J_j\neq\varnothing}} & \tfrac{K}{r_\triangle(\rho) r_{\triangle'}(\bar\rho)(\ln W)^2} \Big(\delta_{\operatorname{dist}(\triangle,\triangle')-1<\frac{1}{m_r}} \ln^2 \Big[\tfrac{W}{m_r} \tfrac{1}{W(\operatorname{dist}(\triangle,\triangle')-1)+1}\Big] \\ & + \delta_{\operatorname{dist}(\triangle,\triangle')-1\geqslant\frac{1}{m_r}} \, \mathrm{e}^{-m_r(\operatorname{dist}(\triangle,\triangle')-1)}\Big),\end{aligned}$$

where the factor W^4 cancels. The sum over $\triangle'$ is bounded by a constant independent of W because of the exponential decay. Therefore,

$$\operatorname{Tr}(M_{J_iJ_j})^* M_{J_iJ_j} \leqslant K \sum_{\triangle:\triangle\cap J_i\neq\varnothing} \frac{1}{r_\triangle(\rho)} \leqslant K|\mathcal{J}|.$$

Combining these estimates, we end up with the result. □

2.4.5. Summing up

In this section we will put together the estimates above to complete the proof. The large factorials and combinatoric factors arising from the bound of the functional integral and the sum over the cube positions will be controlled by fractions of the exponential decay of $G_q(s)CG_q(s)$, while the non-exponential part will allow to sum over the vertex positions i, j, k, k' inside each fixed cube. Finally the sum over the tree structure will be achieved by a standard argument.

Reorganizing W factors. Before performing the estimates, we extract additional W factors from G as follows:

$$\begin{aligned}&\prod_{q=1}^{r} |G_q(s)_{i_qk_q}| \; |C_{k_qk'_q}| \; |G_q(s)_{k'_qj_q}| \prod_{\triangle\in Y} \Big(\tfrac{(\ln W)^{1/2}}{W}\Big)^{r_\triangle} \qquad (2.4.11)\\ &= \prod_{q=1}^{r} \Big|\tfrac{W^2}{\ln W} G_q(s)_{i_qk_q}\Big| \; \Big|\tfrac{\ln W}{W^2} C_{k_qk'_q}\Big| \; \Big|\tfrac{W^2}{\ln W} G_q(s)_{k'_qj_q}\Big| \prod_{j\in\mathcal{J}\cup\{l_0\}} \Big(\tfrac{(\ln W)^{1/2}}{W}\Big)^{n_j},\end{aligned}$$

where we remember that $d_{l_0} = 0$ and $r_{l_0} = 2$ if $l_0 \notin \mathcal{J}$.

Factorials. We extract a small fraction of the exponential to control finite powers of factorials $d_\triangle!^p$. The number $d_\triangle$ counts the number of multi-link starting points i_q and endpoints j_q inside $\triangle$. Denoting by q_0 the first (smallest) index in this family, one can see that for all $q > q_0$, the cubes containing the vertex k'_q are pairwise disjoint and different from $\triangle$. For $d_\triangle$ large, more than half of these cubes have distance of order $W(\ln W)^{\alpha/2} d_\triangle^{1/2}$ from $\triangle$ since we are in a finite dimensional space. Therefore we gain a factor of order $\exp(-(\ln W)^{\alpha/2} d_\triangle^{3/2})$ from the exponential decay of GCG. A small fraction from this beats finite powers of factorials in $d_\triangle$ and $n_\triangle$:

$$\prod_{q=1}^{r} \mathrm{e}^{-\varepsilon|i_q-k_q|/W}\, \mathrm{e}^{-\varepsilon|k_q-k'_q|/W}\, \mathrm{e}^{-\varepsilon|k'_q-j_q|/W} \leqslant \prod_{\triangle} \frac{K}{d_\triangle!^p} \leqslant \prod_{\triangle} \frac{K^{n_\triangle}}{n_\triangle!^{p/4}}, \tag{2.4.12}$$

where we used $n_\triangle \leqslant 4d_\triangle$ in the last step.

Applying Lemma 2.4.6, the bound of the factorials (2.4.12) and the reorganization of the W factors in (2.4.11), we have

$$\begin{aligned}
&\prod_{q=1}^{r} \left(|G_q(s)_{i_qk_q}||C_{k_qk'_q}||G_q(s)_{k'_qj_q}|\right) |F_T[s](\{i_q,j_q\})| \qquad (2.4.13)\\
&\leqslant \frac{\ln W}{W^2} K^{(\ln W)^\alpha} \left[\prod_{q=1}^{r} \mathrm{e}^{-f'd(\triangle'_q,\triangle_q)/W}\, \mathrm{e}^{-f'd(\triangle_q,\triangle_{\mathcal{A}(q)})/W}\, \mathrm{e}^{-f'd(\triangle_{\mathcal{A}(q)},\triangle''_q)/W}\right]\\
&\times \sum_{d\in\mathcal{D}} \sum_{\{r_j\}_{j\in\mathcal{J}}} \left(\frac{K^{(\ln W)^\alpha}(\ln W)^{1/2}}{W}\right)^{n_{l_0}-2} \prod_{j\in\mathcal{J}\setminus\{l_0\}} \left(\frac{K^{(\ln W)^\alpha}(\ln W)^{1/2}}{W}\right)^{n_j} \prod_{q=1}^{r} \tilde{G}_{i_qk_q}\tilde{C}_{k_qk'_q}\tilde{G}_{k'_qj_q}
\end{aligned}$$

where $f' = fm_r - \varepsilon$ is the remaining mass, $d(\triangle,\triangle')$ is the distance between the centers of the cubes $\triangle$ and $\triangle'$, and $\tilde{G}$ and $\tilde{C}$ are the prefactors of the exponential decay of G and C given by

$$\begin{aligned}
\tilde{C}_{ij} &= \delta_{|i-j|\leqslant \frac{W}{m_r}} \frac{K\ln W}{W^4} \ln\left(\frac{W}{m_r}\frac{1}{|i-j|+1}\right) + \delta_{|i-j|> \frac{W}{m_r}} \frac{K\ln W}{W^{7/2}|i-j|^{1/2}}\\
\tilde{G}_{ij} &= \delta_{ij}\frac{W^2}{\ln W} + \delta_{|i-j|\leqslant \frac{W}{m_r}} \frac{K}{\ln W} \ln\left(\frac{W}{m_r}\frac{1}{|i-j|+1}\right) + \delta_{|i-j|> \frac{W}{m_r}} \frac{1}{\ln W}.
\end{aligned}$$

Sum over the vertex position inside each cube. Remember that $\tilde{\triangle}_q = (\triangle_q, \triangle'_q, \triangle''_q)$, $q = 0,\dots r$. For each $\tilde{\triangle}_q$ we call $\tilde{\triangle}_{\mathcal{A}(q)}$ the ancestor of $\tilde{\triangle}_q$ in the tree, and $\triangle_{\mathcal{A}(q)}$ the cube in $\tilde{\triangle}_{\mathcal{A}(q)}$ containing k_q. Let us now fix the tree structure T, the position of the above cubes, and the multiplicities $d \in \mathcal{D}$.

Lemma 2.4.9. *The sum over the vertex positions i_q, j_q, k_q, k'_q compatible with the above constraints is bounded by*

$$\sum_{\substack{i_q\in\triangle'_q,\, j_q\in\triangle''_q\\ k_q\in\triangle_q,\, k'_q\in\triangle_{\mathcal{A}(q)}}}^{(T,d)} \prod_{q=1}^{r} \tilde{G}_{i_qk_q}\tilde{C}_{k_qk'_q}\tilde{G}_{k'_qj_q} \leqslant K^{d_{l_0}}(\ln W)^{\alpha d_{l_0}} \prod_{j\in\mathcal{J}\setminus\{l_0\}} K^{d_j} W^2 (\ln W)^{5d_j/2} \tag{2.4.14}$$

Proof. Each multi-link consists of four vertices i_q, j_q, k_q, k'_q, where k_q and k'_q must belong to different cubes while i_q and j_q are arbitrary. For $j = i_q$ or $j = j_q$, we say

- j is new in step q if the qth multi-link extracts j and j was never extracted before.

- j is old in step q if the qth multi-link extracts j and j was already extracted.

Since the multi-indices d and the tree structure are fixed, the fact that j is old or new is preserved when summing over its position inside the cubes. We consider the different cases. Note that we only sum over i_q and j_q if they are new. If both i_q and j_q are of the same type (old or new), we distribute the resulting factor to both indices. If one is old and one is new, the resulting factor counts only for the new index.

a) $i_q \neq j_q$ and both i_q and j_q are new. We sum over i_q and j_q:

$$\sum_{k_q\in\triangle_{k_q}} \sum_{i_q\in\triangle_{i_q}} \tilde{G}_{i_qk_q} \sum_{k'_q\in\triangle_{k'_q}} \tilde{C}_{k_qk'_q} \sum_{j_q\in\triangle_{j_q}} \tilde{G}_{k'_qj_q} \leqslant KW^4(\ln W)^{9\alpha/2-1}$$

Therefore, we pay a factor $W^2(\ln W)^{9\alpha/4-1/2}$ for i_q and the same factor for j_q.

b) $i_q \neq j_q$ and i_q is new and j_q is old. The same estimate holds for i_q old and j_q new. Then we sum only over i_q.

$$\sum_{k'_q\in\triangle_{k'_q}} \tilde{G}_{k'_qj_q} \sum_{k_q\in\triangle_{k_q}} \tilde{C}_{k_qk'_q} \sum_{i_q\in\triangle_{i_q}} \tilde{G}_{i_qk_q} \leqslant KW^2(\ln W)^{7\alpha/2-1}.$$

Hence, we need to bound a factor $W^2(\ln W)^{7\alpha/2-1}$ for i_q and no factor for j_q.

c) $i_q \neq j_q$ and i_q and j_q are old. Then, i_q and j_q are both fixed and

$$\sum_{k_q\in\triangle_{k_q}} \tilde{G}_{i_qk_q} W^{-4}\ln^2 W \sum_{k'_q\in\triangle_{k'_q}} \tilde{G}_{k'_qj_q} \leqslant K(\ln W)^{2\alpha},$$

where we bound $\tilde{C}_{k_qk'_q} \leqslant W^{-4}\ln^2 W$. For both i_q and j_q we collect a factor $K(\ln W)^{\alpha}$.

d) $i_q = j_q$ and i_q is new. Then, we sum over i_q

$$\sum_{k_q\in\triangle_{k_q}} \sum_{k'_q\in\triangle_{k'_q}} \tilde{C}_{k_qk'_q} \sum_{i_q\in\triangle_{i_q}} \tilde{G}_{i_qk_q}\tilde{G}_{k'_qi_q} \leqslant KW^2(\ln W)^{5\alpha/2}$$

and obtain a factor $W^2(\ln W)^{5\alpha/2}$ for i_q and no factor for j_q.

e) $i_q = j_q$ and i_q is old. Then, i_q is fixed and

$$\sum_{k_q\in\triangle_{k_q}} \tilde{G}_{i_qk_q} W^{-4}\ln^2 W \sum_{k'_q\in\triangle_{k'_q}} \tilde{G}_{k'_qj_q} \leqslant K(\ln W)^{2\alpha}.$$

We gain a factor $(\ln W)^{\alpha}$ for i_q and j_q. Note that for $l_0 \in \mathcal{J}$, l_0 is always old. □

Combining the products over $j \in \mathcal{J}$ in (2.4.13) and (2.4.14) we obtain

$$\prod_{j\in\mathcal{J}\setminus\{l_0\}} \left(\frac{K^{(\ln W)^{\alpha}}(\ln W)^{1/2}}{W}\right)^{n_j} K^{d_j}W^2(\ln W)^{5d_j/2} \leqslant \prod_{j\in\mathcal{J}\setminus\{l_0\}} \left(\frac{K^{(\ln W)^{\alpha}}(\ln W)^{3}}{W^{1/3}}\right)^{n_j}, \tag{2.4.15}$$

where we used $n_j \geqslant 3$ for all $j \in \mathcal{J}, j \neq l_0$. The point $j = l_0$ is special since $n_{l_0} \geqslant 2$. But since the position l_0 is fixed, l_0 is always "old" and we obtain

$$\left(K^{(\ln W)^\alpha} \tfrac{(\ln W)^{1/2}}{W}\right)^{n_{l_0}-2} (K(\ln W)^\alpha)^{d_{l_0}}. \tag{2.4.16}$$

Finally, we perform the sum over the multi-indices d and r, compatibles with the fixed tree structure. The sum over $r_j = (r_j(a), r_j(b), r_j(\bar{\rho}), r_j(\rho))$ gives a factor K^r since $|r_j| \leqslant 3|d_j|$ and $\sum_{j\in\mathcal{J}} |d_j| = 2r$. The sum over $d_j = (d_j(a), d_j(b), d_j(\bar{\rho}), d_j(\rho))$ can be estimated by an integral over a simplex of length r, giving an additional factor K^r.

Combining these factors with (2.4.15) and (2.4.16) we obtain the bound g^{r-1}, where $g = K^{(\ln W)^\alpha} W^{-1/3+\varepsilon}$ with $0 < \varepsilon \ll 1/3$, hence $g \ll 1$ for W large.

Sum over the cube position and the tree structure. For a fixed tree structure we use the remaining exponential decay of GCG to sum over the positions of the all cubes inside $\tilde{\triangle}_q$ for all $1 \leqslant q \leqslant r$, starting from the leafs (i.e. vertices with degree 1) and going towards the root $\triangle_0$. For each multi-link connecting a generalized cube $\tilde{\triangle}_q$ to its ancestor $\tilde{\triangle}_{\mathcal{A}(q)}$ the position of $\triangle_q, \triangle'_q$ and $\triangle''_q$ is summed over using the exponential decay of GCG. This costs only a constant factor for each cube. Finally we pay a factor 3 to choose the position of the ancestor in $\tilde{\triangle}_{\mathcal{A}(q)}$. We end up with

$$\left|F_\Lambda^{(l_0)}\right| = \tfrac{\ln W}{W^2}\, \mathrm{e}^{K(\ln W)^\alpha} \left[1 + \sum_{r\geqslant 1} \sum_T \sum_{\text{unordered orders}} \int_{[0,1]^r} \prod_{q=1}^r \mathrm{d}s_q |M_T(s)|\; g^{r-1}\right].$$

Integrating over the interpolating factors s cancels the last sum over the orders of the trees (cf. [Riv91, Lemma III.1.1]): $\sum_{\text{orders}} \int_{[0,1]^r} \prod_{q=1}^r \mathrm{d}s_q |M_T(s)| = 1$. The remaining sum is written as

$$\sum_{r\geqslant 1}\sum_T g^{r-1} \leqslant 1 + \sum_{r\geqslant 2}\sum_T \sqrt{g^r} = \sum_{\deg_{\tilde{\triangle}_0}\geqslant 1} \prod_{i_0=1}^{\deg_{\tilde{\triangle}_0}} \sqrt{g} \left[\sum_{\deg_{\tilde{\triangle}_{i_0}}\geqslant 1} \prod_{i_1=1}^{\deg_{\tilde{\triangle}_{i_0}}-1} \sqrt{g} \sum \cdots \right],$$

where $\deg_{\tilde{\triangle}}$ denotes the degree of the generalized cube $\tilde{\triangle}$ in the tree T. Since $g \ll 1$, we can sum from the leaves towards the root using a standard procedure (cf. [DPS02, Section 6.3.4]) and bound the sum above by a constant. It suffices to assume $\sqrt{g} < 1/4$ to make this procedure work. Hence $W_0(\alpha)$ need to be chosen large enough that $K^{(\ln W)^\alpha} W^{-1/3+\varepsilon} < 1/16$ for all $W \geqslant W_0(\alpha)$. Finally we estimate the sum over l_0 using the exponential decay of $|B_{0l_0}|$. As a result

$$\left|\int \mathrm{d}\mu_B(M)\, \mathrm{e}^{\mathcal{V}(M)} a_0\right| \leqslant \sum_{l_0\in\Lambda} |B_{0l_0}| \left|F_\Lambda^{(l_0)}\right| \leqslant \tfrac{\ln W}{W^2}\, \mathrm{e}^{K(\ln W)^\alpha}.$$

This proves the first part of Theorem 2.2.3.

2.4.6. Derivatives

Bounding the derivative is similar to the procedure above. Our starting point is

$$\sum_{j_1,\dots,j_n} \sum_{l_0,\dots,l_n} B_{0l_0} \prod_{m=1}^{n} B_{j_m l_m} F_\Lambda^{(l_0,\dots,l_n)}.$$

Since the B factors control the sums over l_0 and over the j_m's, we have only n remaining sums of $l_1,\dots,l_n$ over the volume Λ. We observe that a cluster expansion of $F_\Lambda^{(l_0,\dots,l_n)}$ extracts only trees such that all indices $l_0,\dots,l_n$ are in the connected cluster. We can extract a fraction of the exponential decay of GCG to sum over the 'coarse' position of the $l_0,\dots,l_n$, i.e. the position of the cubes containing the indices. Finally, to sum over the index position inside each cube, we need to extract at least a factor $(W^2(\ln W)^\alpha)^{-1}$ for each $l_1,\dots,l_n$.

As mentioned above, applying the cluster expansion directly to $F_\Lambda^{(l_0,\dots,l_n)}$ is not enough to extract this fine structure. Problems arise when two or more of the l_j's coincide and we have contributions from $\prod_{m=1}^n \operatorname{Str} \partial_{M_{l_m}} \partial_{a_{l_0}} \exp(\mathcal{V}(M))$ of the form $(\operatorname{Str} \partial_{M_l})^n \mathcal{V}(M_l)$ with $n \geqslant 2$. Since the lowest order contribution of $\mathcal{V}(M)$ is cubic, we obtain linear or constant terms in M. Note that constant terms vanish, since $(\operatorname{Str} \partial_{M_i})^n \operatorname{Str} M_j^n = \delta_{ij} n! \operatorname{Str} 1 = 0$. Linear terms may display a problem if a derivative of the cluster expansion falls on them. In this case, we have no field factor left and we gain only only $W^{-1}(\ln W)^{1/2}$ from the derivative (cf. eq. (2.4.11)) of the cluster expansion, which is not enough for the fine structure estimates.

In the special case l_0, problems arise for terms of the form $(\operatorname{Str} \partial_{M_{l_0}})^n \partial_{a_{l_0}} \mathcal{V}(M_{l_0})$, with $n \geqslant 1$, since we obtain again linear or constant terms in a_{l_0}. For linear terms we have the same problem as above. Note that also in this case the constant term vanishes since the whole integral corresponds to the derivative of a constant (cf. proof of Lemma 2.2.2 and Lemma 2.4.3) except in the special case when $n \geqslant 2$ and all l_k coincide. Indeed, in this case the integral coincides with (2.2.4) and hence yields one, but this is no problem since we can sum over the remaining indices using the B factors.

To solve these problems, we apply integration by parts on the linear contributions of the form $\operatorname{Str} M_{l_j}$ with $l_j \neq l_0$ as in (2.B.4) *before* performing the cluster expansion. Each new B factor that we obtain ensures summation over at least one old index, while a new index to be summed, coupled with an $\operatorname{Str} \partial_M$, appears. Again, the derivative may fall on the exponential $\exp(\mathcal{V}(M))$ (extracting a new term $\operatorname{Str} M^2$ at lowest order) or a prefactor $\operatorname{Str} M^n$ for $n \geqslant 1$. If $n = 1$ the integral vanishes by the same arguments as above. For $n = 2$ we obtain a new linear term, where we need to perform integration by parts. In all other cases we obtain enough fine structure. Note that the procedure ends after at most $2n$ steps.

Again a derivative falling on another linear contribution vanishes since by the same arguments as above. Therefore we end up with functional integrals of the form

$$\sum_{k\in\Lambda \forall k \in \mathcal{K}} \int \mathrm{d}\mu_B(M) a_{k_0}^{m_{k_0}} \prod_{k\in\mathcal{K}} \operatorname{Str} M_k^{m_k} \,\mathrm{e}^{\mathcal{V}(M)}$$

where $m_{k_0} \geqslant 1$, $m_k \geqslant 2$ and $|\mathcal{K}| \leqslant n$. Note that again the index k_0 is special and a constant term i.e. $m_{k_0} = 0$ means the integral corresponds to the derivative of a constant.

Applying here the cluster expansion yields a connected tree containing all indices $k_0, \ldots, k_n$. We obtain a functional integral of the form

$$F_T^{(k_0,\ldots,k_n)}[s](\{i_q, j_q\}) = \int \mathrm{d}\mu_{B(s)}(M) \prod_{q=1}^{r} \operatorname{Str}\left(\partial_{M_{i_q}} \partial_{M_{j_q}}\right) \left[a_{k_0}^{m_{k_0}} \prod_{k \in \mathcal{K}} \operatorname{Str} M_k^{m_k} \, \mathrm{e}^{\mathcal{V}(M)} \right]$$

and bound it similar to Lemma 2.4.6. Note that the indices $k_0, \ldots, k_n$ need to be treated separately as l_0 before. We obtain $n_{k_0} \geqslant 1$ and $n_k \geqslant 2$ for $k \in \mathcal{K}$. Since we sum later over these indices, they are 'old' and hence we obtain at least a total factor $W^{-(2n+1)}(\ln W)^n$. Collecting all W contributions we get

$$W^{2n}(\ln W)^{n\alpha} W^{-(2n+1)}(\ln W)^n \, \mathrm{e}^{K(\ln W)^\alpha} = W^{-1}(\ln W)^{n(\alpha+1)} \, \mathrm{e}^{K(\ln W)^\alpha} \leqslant 1$$

for W large enough (depending on n). Note that the first factor comes from the sum over the index position inside each cube, and the last from the contribution of the root cube (see end of Section 2.4.5).

2.A. Supersymmetric Formalism

We will summarize the main ideas of the supersymmetric formalism (see [Efe99] for an easy-to-read introduction and [Ber87] for a detailed description).

Definition 2.A.1 (Grassmann algebra)**.** Let $N \in \mathbb{N}$ and let V be a vector space over a field $\mathbb{K}$ with basis $(\alpha_1, \ldots, \alpha_N)$ and denote the antisymmetric tensor product by

$$\begin{aligned} \wedge : V \times V &\to V \otimes_{as} V, \\ (v, w) &\mapsto v \wedge w = vw = -wv. \end{aligned}$$

The corresponding *Grassmann algebra* is defined by

$$\mathcal{A} := \bigoplus_{k \geqslant 0} V^k,$$

where $V^0 = \mathbb{K}$, $V^1 = V$ and $V^k = V^{k-1} \otimes_{as} V$ for $k \geqslant 2$. This is an associative algebra with unit. We distinguish between the subsets of even elements $\mathcal{A}^0 := \bigoplus_{k \geqslant 0} V^{2k}$ and odd elements $\mathcal{A}^1 := \bigoplus_{k \geqslant 0} V^{2k+1}$. While the even elements form an algebra again, this is not true for $\mathcal{A}^1$. Even elements commute with all elements in the Grassmann algebra and are called *Bosonic variables.* On the other hand, two odd elements anticommute and are called *Fermionic (or Grassmann) variables.*

The generators $(\alpha_1, \ldots, \alpha_N)$ of $\mathcal{A}$ are Grassmann variables, and are provided with the anticommutation property $\alpha_i \alpha_j = -\alpha_j \alpha_i$ for all $i, j = 1, \ldots, N$. Note that this directly implies $\alpha_i^2 = 0$. Hence, any element in $\mathcal{A}$ is a finite polynomial of the form

$$f(\alpha_1, ..., \alpha_N) = f_0 + \sum_{k=1}^{N} \sum_{i_1 < ... < i_k} f_{i_1,...,i_k} \alpha_{i_1} \cdots \alpha_{i_k},$$

where $f_0, f_{i_1,...,i_k} \in \mathbb{K}$ and f_0 is called *spectrum* of $f(\alpha_1, ..., \alpha_N)$.

Definition 2.A.2 (Grassmann integration)**.** As a formal symbol, we define the integral over a Grassmann variable as $\int \mathrm{d}\alpha_i \; 1 = 0$ and $\int \mathrm{d}\alpha_i \; \alpha_i = \frac{1}{\sqrt{2\pi}}$. To define integration of multiple variables, we assume Fubini's theorem applies, but the differentials anticommute.

Notation. To keep the notation as short as possible, we write for any two families $(\zeta_i)_{i \in I}$ and $(\xi_i)_{i \in I}$ of Bosonic and/or Fermionic variables, the sum over the corresponding index set I as $(\zeta, \xi) = \sum_{i \in I} \zeta_i \xi_i$.

Gaussian integral. We will often use the following Gaussian integral formulas. Let $x \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$. For $M \in \mathbb{C}^{n \times n}$ with positive definite Hermitian part,

$$\begin{aligned} \int \mathrm{d}x \, \mathrm{e}^{-\frac{1}{2}(x, Mx)} \, \mathrm{e}^{(x,y)} &= \tfrac{(2\pi)^{n/2}}{\sqrt{\det M}} \, \mathrm{e}^{\frac{1}{2}(y, M^{-1} y)}, \\ \int \mathrm{d}\bar{z} \, \mathrm{d}z \, \mathrm{e}^{-(\bar{z}, Mz)} \, \mathrm{e}^{(\bar{v}, z) + (\bar{z}, w)} &= \tfrac{(2\pi)^n}{\det M} \, \mathrm{e}^{(\bar{v}, M^{-1} w)}, \end{aligned}$$

where $y, v, w \in \mathbb{C}^n$ and the measures are usual Lebesgue product measures, i.e. $\mathrm{d}x = \prod_{i=1}^n \mathrm{d}x_i$, $\mathrm{d}\bar{z} \, \mathrm{d}z = \prod_{i=1}^n \mathrm{d}\bar{z}_i \, \mathrm{d}z_i$ and $\mathrm{d}\bar{z}_i \, \mathrm{d}z_i = 2 \, \mathrm{dRe} \, z_i \, \mathrm{dIm} \, z_i$. Note that the formulas remain valid if we replace y_i, v_i and w_i by even elements of $\mathcal{A}$. A direct consequence of the latter are the following identities

$$\int \mathrm{d}\bar{z} \, \mathrm{d}z \;\; \mathrm{e}^{(\bar{z}, Mz)} = \tfrac{(2\pi)^n}{\det M} \quad \text{and} \quad \int \mathrm{d}\bar{z} \, \mathrm{d}z \;\; z_k \; \bar{z}_l \;\; \mathrm{e}^{-(\bar{z}, Mz)} = M_{kl}^{-1} \tfrac{(2\pi)^n}{\det M}. \tag{2.A.1}$$

Using Definition 2.A.2 above, we obtain similar Fermionic formulas. Let $(\chi_i)_{i=1}^n$ and $(\bar{\chi}_i)_{i=1}^n \subset \mathcal{A}_1$ be two families of the Grassmann variables, where the $\bar{\chi}_i$'s are independent of the χ_i's. For an arbitrary $M \in \mathbb{C}^{n \times n}$, we have

$$\int \mathrm{d}\bar{\chi} \, \mathrm{d}\chi \, \mathrm{e}^{-(\bar{\chi}, M\chi)} = (2\pi)^{-n} \det M, \tag{2.A.2}$$

$$\int \mathrm{d}\bar{\chi} \, \mathrm{d}\chi \, \mathrm{e}^{-(\bar{\chi}, M\chi)} \, \mathrm{e}^{(\bar{\rho}, \chi) + (\bar{\chi}, \rho)} = (2\pi)^{-n} \det M \, \mathrm{e}^{(\bar{\rho}, M^{-1} \rho)},$$

$$\int \mathrm{d}\bar{\chi} \, \mathrm{d}\chi \, \mathrm{e}^{-(\bar{\chi}, M\chi)} \prod_{i \in I} \chi_i \prod_{j \in J} \bar{\chi}_j = \sigma_{IJ} \delta_{|I|=|J|} (2\pi)^{-n} \mathrm{det}_{JI} M, \tag{2.A.3}$$

where $\mathrm{d}\bar{\chi} \, \mathrm{d}\chi = \prod_{i=1}^n \mathrm{d}\bar{\chi}_i \, \mathrm{d}\chi_i$ and $(\rho_i)_{i=1}^n$ and $(\bar{\rho}_i)_{i=1}^n \subset \mathcal{A}_1$ are two families of Grassmann variables. Moreover, σ_{IJ} is a sign, $I, J \subset \{1, \ldots, n\}$ are two index sets and $\det_{JI} M$ is the determinant of the minor of M where the rows with indices in J and the columns with indices of I are crossed out.

Supervectors and Supermatrices To combine real or complex variables with Grassmann ones, we introduce the notation of a *supervector* Φ consisting of p Bosonic variable $X = (X_i)_{i=1}^p \in (\mathcal{A}^0)^p$ and q Fermionic variable $\alpha = (\alpha_j)_{j=1}^q \in (\mathcal{A}^1)^q$ by

$$\Phi = \begin{pmatrix} X \\ \alpha \end{pmatrix}.$$

A *supermatrix* is a linear transformation between supervectors, i.e.

$$\Phi' = \mathbf{M}\Phi, \quad \mathbf{M} = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix}, \tag{2.A.4}$$

where a, b are $p \times p$ and $q \times q$ matrices in $\mathcal{A}_0$ and σ, ρ are $p \times q$ and $q \times p$ matrices in $\mathcal{A}_1$. We denote supermatrices by bold face capital letters. For the supermatrix $\mathbf{M}$, we define the notation of a *supertrace* and a *superdeterminant* as

$$\operatorname{Str} \mathbf{M} := \operatorname{Tr} a - \operatorname{Tr} b \quad \text{and} \quad \operatorname{Sdet} \mathbf{M} := \det[a - \sigma b^{-1}\rho] \det[b^{-1}]. \tag{2.A.5}$$

Finally, the inverse of the supermatrix $\mathbf{M}$ is given by

$$\mathbf{M}^{-1} = \begin{pmatrix} (a - \sigma b^{-1}\rho)^{-1} & -(a - \sigma b^{-1}\rho)^{-1}\sigma b^{-1} \\ -b^{-1}\rho(a - \sigma b^{-1}\rho)^{-1} & b^{-1} + b^{-1}\rho(a - \sigma b^{-1}\rho)^{-1}\sigma b^{-1} \end{pmatrix}.$$

Let $\mathbf{M}$ be a supermatrix of the form (2.A.4) and Φ a supervector and Φ^* its adjoint

$$\Phi = \begin{pmatrix} z \\ \chi \end{pmatrix} \quad \text{and} \quad \Phi^* = (\bar{z}, \bar{\chi}), \tag{2.A.6}$$

where $z \in \mathbb{C}^p$, $\chi = (\chi_j)_{j=1}^q$ and $\bar{\chi} = (\bar{\chi}_j)_{j=1}^q$ are again independent families of Grassmann variables. We can write the superdeterminant as a Gaussian integral

$$\int \mathrm{d}\Phi^* \, \mathrm{d}\Phi \, \mathrm{e}^{-(\bar{\Phi}, \mathbf{M}\Phi)} = \operatorname{Sdet} \mathbf{M^{-1}}, \tag{2.A.7}$$

where $\mathrm{d}\Phi^* \, \mathrm{d}\Phi = \mathrm{d}\bar{\chi} \, \mathrm{d}\chi \, \mathrm{d}\bar{z} \, \mathrm{d}z$. Below, we consider only the special case $p = q = 1$.

2.B. Proof of Lemma 2.2.1

We combine (2.A.1) and (2.A.2) to rewrite the Green's function as a Gaussian integral. Let $\chi = (\chi_i)_{i \in \Lambda}$ and $\bar{\chi} = (\bar{\chi}_i)_{i \in \Lambda}$ be two families of Grassmann variables, $z = (z_j)_{j \in \Lambda} \in \mathbb{C}^\Lambda$ and $\Phi = (\Phi_j)_{j \in \Lambda}$ and $\Phi^* = (\Phi_j^*)_{j \in \Lambda}$ two sets of supervectors defined as in (2.A.6). Using the fact that $(-i(E_\varepsilon - H))^{-1}$ has positive definite Hermitian part, we write

$$\begin{aligned} \sum_{k \in \Lambda} G_\Lambda^+(E_\varepsilon)_{kk} &= -i(2\pi)^{-|\Lambda|} \det[-i(E_\varepsilon - H)] \int \mathrm{d}\bar{z} \, \mathrm{d}z \, \mathrm{e}^{i(\bar{z}, (E_\varepsilon - H)z)} \sum_{k \in \Lambda} z_k \bar{z}_k \\ &= -i \int \mathrm{d}\Phi^* \, \mathrm{d}\Phi \, \mathrm{e}^{i \sum_{i,j \in \Lambda} (\Phi_i, (\delta_{ij} \mathbf{E}_\varepsilon - \mathbf{H}_{ij})\Phi_j)} \sum_{k \in \Lambda} z_k \bar{z}_k, \end{aligned} \tag{2.B.1}$$

where the product measure is defined as in (2.A.7). Note that the bold face printed $\mathbf{E}_\varepsilon$ and $\mathbf{H}_{ij}$ are 2×2 supermatrices with diagonal entries E_ε and H_{ij}, respectively, and vanishing off-diagonal entries. Since the contribution of the random matrix H appears only in the exponential, using a Hubbard-Stratonovitch transformation as in [DPS02, Lemma 1], we can rewrite the average over H as

$$\mathbb{E}\left[\mathrm{e}^{-i\sum_{i,j\in\Lambda}(\bar{\Phi}_i,\mathbf{H}_{ij}\Phi_j)}\right] = \mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\Lambda}J_{ij}(\Phi_i^*\Phi_j)(\Phi_j^*\Phi_i)} = \mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\Lambda}J_{ij}\mathrm{Str}\,(\Phi_i\Phi_i^*)(\Phi_j\Phi_j^*)} \tag{2.B.2}$$

$$= \int\prod_{j\in\Lambda}\mathrm{d}\mathbf{M}_j\,\mathrm{e}^{-\frac{1}{2}\sum_{i,j\in\Lambda}J_{ij}^{-1}\mathrm{Str}\,[\mathbf{M}_i\mathbf{M}_j]}\,\mathrm{e}^{-i\sum_{j\in\Lambda}(\bar{\Phi}_j,\mathbf{M}_j\Phi_j)}, \tag{2.B.3}$$

where

$$\mathbf{M}_j = \begin{pmatrix} a_j & \bar{\rho}_j \\ \rho_j & ib_j \end{pmatrix} \quad \text{and} \quad \mathrm{d}\mathbf{M}_j := \mathrm{d}a_j\,\mathrm{d}b_j\,\mathrm{d}\bar{\rho}_j\,\mathrm{d}\rho_j,$$

$(a_j)_{j\in\Lambda}$ and $(b_j)_{j\in\Lambda}$ are families of real variables and $(\rho_j)_{j\in\Lambda}$ and $(\bar{\rho}_j)_{j\in\Lambda}$ are two families of Grassmann variables.

The expression for the observable $-i\sum_{k\in\Lambda}\bar{z}_k z_k$ can be written as a derivative $\sum_{k\in\Lambda}\partial_{a_k}$ of the second exponential in (2.B.3). Hence, applying integration by parts in the variables a_j, the derivative falls on the first exponential in (2.B.3) which yields $\sum_{k\in\Lambda}\sum_{l\in\Lambda}J_{kl}^{-1}a_l = \sum_{l\in\Lambda}a_l$. Since the integral expression is still translation invariant in Λ, by relabeling the indices we can now substitute the sum $|\Lambda|^{-1}\sum_{l\in\Lambda}a_l$ by a_0. This step simplifies the integral compared to [DPS02, eq.(3.1)].

By (2.A.7), the integral over the supervector yields

$$\frac{1}{|\Lambda|}\sum_{k\in\Lambda}\mathbb{E}[G_\Lambda^+(E_\varepsilon)_{kk}] = \int\prod_{j\in\Lambda}\mathrm{d}\mathbf{M}_j\,\mathrm{e}^{-\sum_{i,j\in\Lambda}J_{ij}^{-1}\mathrm{Str}\,[\mathbf{M}_i\mathbf{M}_j]}\prod_{j\in\Lambda}\mathrm{Sdet}\,[\mathbf{E}_\varepsilon-\mathbf{M}_j]^{-1}a_0.$$

Finally, we insert the expressions for the supermatrix $\mathbf{M}$ and perform the integration over the Grassmann variables applying (2.A.2). This proves (2.2.2).

For (2.2.3), note that each E-derivative of (2.B.1) results into $i\sum_{k\in\Lambda}\bar{z}_k z_k+\bar{\chi}_k\chi_k$, which can be replaced by $\sum_{k\in\Lambda}\mathrm{Str}\,\mathbf{M}_k$. Now, using the definition (2.4.3) with B replaced by J, integration by parts in $\mathrm{Str}\,\mathbf{M}_k$ yields

$$\int\mathrm{d}\mu_J(\mathbf{M})\mathrm{Str}\,\mathbf{M}_j\mathcal{F}(\mathbf{M}) = \sum_k J_{kj}\int\mathrm{d}\mu_J(\mathbf{M})\mathrm{Str}\,\partial_{\mathbf{M}_j}\mathcal{F}(\mathbf{M}), \tag{2.B.4}$$

where

$$\partial_{\mathbf{M}_j} = \begin{pmatrix} \partial_{a_j} & -\partial_{\rho_j} \\ \partial_{\bar{\rho}_j} & i\partial_{b_j} \end{pmatrix} \tag{2.B.5}$$

and $\mathcal{F}(\mathbf{M})$ is any smooth function such that the integral above exists.

2.C. Estimates of the covariance

Let $d = 2$, $\Lambda \subset \mathbb{Z}^2$ a finite cube, $-\Delta_\Lambda^P$ the discrete Laplacian on Λ with periodic boundary conditions and $-\Delta$ the discrete Laplacian on $\mathbb{Z}^2$. We consider the two covariances $C_m^\Lambda := (-\Delta_\Lambda^P + m^2)^{-1}$ for the finite cube Λ and $C_m^\infty = (-\Delta + m^2)^{-1}$ for $\mathbb{Z}^2$, with $m > 0$. We will prove the following result.

Lemma 2.C.1. *The finite volume covariance C_m^Λ satisfies*

$$0 < (C_m^\Lambda)_{ij} \leqslant \begin{cases} K \ln\left(\frac{1}{m(1+|i-j|)}\right) & \text{if } |i-j| \leqslant \frac{1}{m} \\ \frac{K}{(|i-j|m)^{1/2}} \,\mathrm{e}^{-m|i-j|} & \text{if } |i-j| > \frac{1}{m}, \end{cases} \tag{2.C.1}$$

provided the mass is small $0 < m \ll 1$ and $m|\Lambda|^{1/2} > 1$. Moreover for all $m < 1$ the diagonal part satisfies

$$(C_m^\Lambda)_{jj} \geqslant (C_m^\infty)_{jj} \geqslant K_1 \ln(m^{-1}) + K_2$$

for some constants $K_1, K_2 > 0$ uniformly in Λ.

Remark. The decay for J and C in (2.3.2) follow directly from this result. The same holds for the complex covariance B since $|B_{ij}| \leqslant C_{ij}$ (see (2.C.6) below).

Proof. First we establish a series expansion and write C_m^Λ as a series in C_m^∞. Using the ideas of Salmhofer [Sal99] in the continuous case, we prove the desired decay for C_m^∞. Finally we conclude that C_m^Λ has the same decay.

Step 1: Series expansion To compare C_m^Λ and C_m^∞, we can write the two Laplacians as $-\Delta_\Lambda^P = 4\mathbb{1}_\Lambda - N_\Lambda^P$, and $-\Delta = 4\mathbb{1}_{\mathbb{Z}^2} - N_{\mathbb{Z}^2}$, where $\mathbb{1}$ is the identity matrix on Λ and $\mathbb{Z}^2$, respectively, and N is the matrix with entries $N_{ij} = 1$ if $|i-j| = 1$ and $N_{ij} = 0$ otherwise. Note that one uses the periodic distance $|\cdot|_P$ in the torus Λ in the case of periodic boundary conditions. The covariances can then be written as a series

$$C_m^\Lambda = (D - N_\Lambda^P)^{-1} = \sum_{k=0}^\infty D^{-1}(N_\Lambda^P D^{-1})^k, \quad C_m^\infty = \sum_{k=0}^\infty D_{\mathbb{Z}^2}^{-1}(N_{\mathbb{Z}^2} D_{\mathbb{Z}^2}^{-1})^k \tag{2.C.2}$$

where $D = (4+m^2)\mathbb{1}_\Lambda$ and $D_{\mathbb{Z}^2} = (4+m^2)\mathbb{1}_{\mathbb{Z}^2}$ are diagonal matrices. This representation is obtained by iterating the identity

$$(A+B)^{-1} - A^{-1} = -A^{-1}B(A+B)^{-1},$$

for matrices A and B with A and $A+B$ invertible. To prove convergence, we use the structure of N_Λ^P and rewrite the sum as a sum over paths

$$\sum_{k=0}^\infty \left(D^{-1}(N_\Lambda^P D^{-1})^k\right)_{ij} = \sum_{k=0}^\infty \sum_{\gamma \in \Gamma_{ij}^\Lambda, |\gamma| = k} \lambda^{k+1} \leqslant \sum_{k=0}^\infty 4^k \lambda^{k+1} = \frac{\lambda}{1-4\lambda} = \frac{1}{m^2} < \infty,$$

where $\lambda = (4+m^2)^{-1}$ and Γ^Λ_{ij} is the set of all paths from i to j in the torus Λ. In the second step, we bound the number of paths from i to j of length k by the number of all paths of length k starting in i, i.e. 4^k. Therefore, the sum converges and is well-defined. By the same arguments, we can write

$$(C^\infty_m)_{ij} = \sum\nolimits_{\gamma \in \Gamma^{\mathbb{Z}^2}_{ij}} \lambda^{|\gamma|+1} < \infty,$$

where $\Gamma^{\mathbb{Z}^2}_{ij}$ is the set of all paths from i to j in $\mathbb{Z}^2$. The formulas above imply directly that $(C^\Lambda_m), (C^\infty_m)_{ij} > 0$. Now, we can identify each point $x \in \mathbb{Z}^2$ with a point $\tilde{x} = x + n|\Lambda|^{1/2}$ in Λ, where $n \in \mathbb{Z}^2$. By identifying each path $\gamma \in \Gamma^{\mathbb{Z}^2}_{ij}$ in $\mathbb{Z}^2$ with the corresponding path in $\tilde{\gamma} \in \Gamma^\Lambda_{ij}$ in the torus, we easily obtain $(C^\infty_m)_{ij} \leqslant (C^\Lambda_m)_{ij}$, hence the first part of Lemma 2.C.1. In order to get the inverse relation we need to characterize the paths in Γ^Λ_{ij}, which cannot be identified with paths in $\Gamma^{\mathbb{Z}^2}_{ij}$. These are exactly the paths in Γ^Λ_{ij} such that their corresponding paths in $\mathbb{Z}^2$ do not end in j but rather in $j_n = j + n|\Lambda|^{1/2}$, with $n \in \mathbb{Z}^2 \backslash \{0\}$. These paths cross the boundary of the cube Λ in such a way that for at least one of the 2 space dimensions the differences n_1 and/or n_2 of the number of crossings in positive and negative direction, respectively, is non-vanishing. We write

$$(C^\Lambda_m)_{ij} = \sum_{n \in \mathbb{Z}^2} \sum\nolimits_{\tilde{\gamma} \in \Gamma^{\mathbb{Z}^2}_{ij_n}} \lambda^{|\gamma|+1} = \sum_{n \in \mathbb{Z}^2} (C^\infty_m)_{ij_n}, \tag{2.C.3}$$

where $j_n = j + n|\Lambda|^{1/2}$ as before.

Step 2: Decay of $\mathbf{C^\infty_m}$ We prove that C^∞_m has the desired decay (2.C.1) The proof is similar to the proof in the continuous case in [Sal99, Lemma 1.10], but the expressions become more complicated in the discrete case. We give a sketch of the main steps. By its Fourier representations, the covariance for $\mathbb{Z}^2$ can be written as

$$(C^\infty_m)_{ij} = \int_{[-\pi,\pi]^2} \frac{\mathrm{e}^{i(k,i-j)}}{2\sum_{l=1}^2 (1-\cos k_l) + m^2} \, \mathrm{d}^2 k, \tag{2.C.4}$$

where $k = (2\pi)/|\Lambda|^{1/d} n$. First, by rescaling $k \to mk$, we obtain

$$(C^\infty_m)_{ij} = \int_{[-\pi/m,\pi/m]^2} \frac{\mathrm{e}^{im(k,i-j)}}{2m^{-2}\sum_{l=1}^2 (1-\cos(mk_l)) + 1} \, \mathrm{d}^2 k.$$

We can assume that $i_1 - j_1 \geqslant |i_2 - j_2| > 0$. Considering the integrand as a function in k_1, there are two poles

$$k_1^\pm = \pm i \tfrac{1}{m} \operatorname{arcosh}\left(\tfrac{m^2}{2} + 2 - \cos(mk_2)\right) = \pm i r(k_2)$$

of order one in the complex plain. Closing the integration contour for k_1 to the rectangle with vertices $-m^{-1}\pi, m^{-1}\pi, m^{-1}\pi + iy$ and $-m^{-1}\pi + iy$ such that the sign $\operatorname{sgn} y(i_1 - j_1) = 1$ and send $|y| \to \infty$, we can apply the residue theorem and the following integral in k_2 remains

$$(C^\infty_m)_{ij} = 2\pi \int_{-\pi/m}^{\pi/m} \frac{\mathrm{e}^{imk_2(i_2-j_2)} \, \mathrm{e}^{-m(i_1-j_1)r(k_2)}}{\frac{2}{m} \sinh \operatorname{arcosh}\left(\frac{m^2}{2} + 2 - \cos(mk_2)\right)} \, \mathrm{d}k_2. \tag{2.C.5}$$

Using $\sinh \operatorname{arcosh} z = \sqrt{z^2-1}$ for $z \geqslant 1$, the absolute value of the integral can be bounded by

$$|(C_m^\infty)_{ij}| \leqslant 2\pi \int_0^{\pi/m} \frac{\mathrm{e}^{-r(k_2)t}}{\frac{1}{m}\sqrt{\left(\frac{m^2}{2}+2-\cos(mk_2)\right)^2-1}}\,\mathrm{d}k_2,$$

where $t = m|i-j|/2$. Note that we showed above that $(C_m^\infty)_{ij} \geqslant 0$. Let us assume $t \geqslant 1$ first. The residual $r(k_2)$ is monotone increasing for $k_2 \in [0, m^{-1}\pi]$ and bounded by

$$r(k_2) \geqslant \begin{cases} r(0) + ck_2^2 = 1 + O(m) + ck_2^2 & \text{if } k_2 \leqslant 1, \\ r(0) + ck_2 = 1 + O(m) + ck_2 & \text{if } k_2 > 1, \end{cases}$$

where c is independent of m and k_2. One can bound the square root in the denominator for all $k_2 \in [0, m^{-1}\pi]$ by

$$\sqrt{\left(\frac{m^2}{2}+2-\cos(mk_2)\right)^2-1} \geqslant m.$$

Therefore, the integral is bounded by

$$\begin{aligned}
(C_m^\infty)_{ij} \leqslant& 2\pi\,\mathrm{e}^{-t}\left(\int_0^1 \mathrm{e}^{-tck_2^2}\,\mathrm{d}k_2 + \int_1^{\pi/m} \mathrm{e}^{-tck_2}\,\mathrm{d}k_2\right) \\
\leqslant& 2\pi\,\mathrm{e}^{-t}\left(\frac{1}{\sqrt{t}}\int_0^{\sqrt{t}} \mathrm{e}^{-ck_2^2}\,\mathrm{d}k_2 + \frac{1}{t}\int_t^{\pi t/m} \mathrm{e}^{-ck_2}\,\mathrm{d}k_2\right) \\
\leqslant& \frac{2\pi}{\sqrt{t}}\,\mathrm{e}^{-t}\left(\int_0^\infty \mathrm{e}^{-ck_2^2}\,\mathrm{d}k_2 + \int_1^\infty \mathrm{e}^{-ck_2}\,\mathrm{d}k_2\right) \leqslant \frac{K}{\sqrt{t}}\,\mathrm{e}^{-t},
\end{aligned}$$

where in the last line we used $t \geqslant 1$. This proves the second part of (2.C.1). In the case $0 < t \leqslant 1$, we perform first in (2.C.5) the change of variables

$$s = r(k_2)t \quad \equiv \quad k_2(s) = \frac{1}{m}\arccos\left(2 + \frac{m^2}{2} - \cosh\left(\frac{ms}{t}\right)\right).$$

Inserting the Jacobian

$$\frac{\mathrm{d}k_2}{\mathrm{d}s} = \frac{\sinh\left(\frac{ms}{t}\right)}{t\sqrt{1-\left(2+\frac{m^2}{2}-\cosh\left(\frac{ms}{t}\right)\right)^2}},$$

and repeating the arguments after (2.C.5), we obtain

$$(C_m^\infty)_{ij} \leqslant K\int_{s_0}^{s_1} \frac{\mathrm{e}^{-s}}{\frac{t}{m}\sqrt{1-\left(2+\frac{m^2}{2}-\cosh\left(\frac{ms}{t}\right)\right)^2}}\,\mathrm{d}s \sim K\int_t^\infty \frac{\mathrm{e}^{-s}}{\sqrt{s^2-t^2}}\,\mathrm{d}s \sim K\ln t^{-1},$$

where $s_0 = r(0)t$ and $s_1 = r(\pi/m)t$, and we used again $m \ll 1$. It remains to consider the case $i = j$. Using the Fourier integral representation one can see that $(C_m^\infty)_{jj} \leqslant K\ln m^{-1}$, hence we can change the bound for small distances to

$$(C_m^\infty)_{ij} \leqslant K\ln\left(\frac{1}{m(1+|i-j|)}\right) \qquad \text{if } |i-j| \leqslant \frac{1}{m}$$

Step 3: Conclusion In order to estimate (2.C.3), we divide the sum into two pieces:

$$(C_m^\Lambda)_{ij} = \sum_{n\in\mathbb{Z}^2} (C_m^\infty)_{ij_n} = \sum_{n\in\mathbb{Z}^2:|n|<2} (C_m^\infty)_{ij_n} + \sum_{n\in\mathbb{Z}^2:|n|\geqslant 2} (C_m^\infty)_{ij_n}.$$

For the first sum, note that $(C_m^\Lambda)_{ij}$ depends only on the distance $|i-j|_P$ and we can assume that the periodic distance is reached inside the cube, i.e. $|i-j|_P = |i-j|$. Then we can estimate $|i-j_n| \geqslant |i-j|$ and therefore each covariance $(C_m^\infty)_{ij_n} \leqslant (C_m^\infty)_{ij}$. Since the sum contains finitely terms, the first sum decays as C_m^∞ with a modified constant K in front. To control the second sum note that $|i-j_n| \geqslant \max_k (n_k-1)|\Lambda|^{1/2} \geqslant |\Lambda|^{1/2} \geqslant m^{-1}$ for $|n| \geqslant 2$. Extracting the desired decay from each $(C_m^\infty)_{ij_n}$, a fraction of the exponential decay (2.C.1) remains in the sum that allows to perform the sum and yields a constant.

Finally, to prove the second part of Lemma 2.C.1, we partition the integration region of (2.C.4) into $\|k\| \leqslant 1$, and $\|k\| > 1$. The integral over the second region is bounded below by a constant, while the the integral over the first region generates the $\ln m^{-1}$ contribution. □

Remark. For the case of a complex mass as in B, note that we can apply the same series expansion as in (2.C.2) and estimate the absolute value by

$$|B_{ij}| \leqslant W^{-2} \sum_{k=0}^{\infty} |(\tilde{D}^{-1}(N_\Lambda^P \tilde{D}^{-1})^k)_{ij}| \leqslant W^{-2} \sum_{k=0}^{\infty} ((D^{-1}(N_\Lambda^P D^{-1})^k)_{ij} = C_{ij}, \qquad (2.C.6)$$

where $\tilde{D}$ is a diagonal matrix with entries $4 + (m_r^2 + im_i^2)/W^2$ and $\operatorname{Re}\tilde{D} = D$.

Lemma 2.C.2. *Let $C \in \mathbb{R}^{N\times N}$ be a real symmetric matrix such that $C^{-1} \geqslant c\,\mathrm{Id}$ as a quadratic form, for some $c > 0$. Let $B = (C^{-1} + im\,\mathrm{Id})^{-1}$, with $m \in \mathbb{R}$. Then, the restriction of B to any subset $Y \subset \{1,\ldots,N\}$, satisfies $\operatorname{Re}(B_Y)^{-1} \geqslant c\,\mathrm{Id}_Y$ for any choice of m.*

Proof. Using Schur's complement, we can write

$$\operatorname{Re} B_Y^{-1} = C_{YY}^{-1} - C_{YY^C}^{-1} C_{Y^CY^C}^{-1}((C_{Y^CY^C}^{-1})^2 + m^2)^{-1} C_{Y^CY}^{-1}.$$

By assumption we have for all $v \in \mathbb{R}^Y$ and $w \in \mathbb{R}^{Y^C}$

$$(v, C_{YY}^{-1}v) + (v, C_{YY^C}^{-1}w) + (w, C_{Y^CY}^{-1}v) + (w, C_{Y^CY^C}^{-1}w) \geqslant c((v,v) + (w,w)).$$

Choosing $w = -C_{Y^CY^C}^{-1}((C_{Y^CY^C}^{-1})^2 + m^2)^{-1} C_{Y^CY}^{-1} v$, we obtain an even better bound than the desired result for $\operatorname{Re}(B_Y)^{-1}$. □

List of symbols

Λ	$\subset \mathbb{Z}^2$, discrete cube.
H	$: \Lambda \times \Lambda \to \mathbb{C}$ random band matrix.
W	band width.
$\bar\rho_\Lambda(E)$	averaged density of states in finite volume Λ.
$G^+_\Lambda(z)$	Green's function, $z \in \mathbb{C}$.
$\rho_{SC}(E)$	Wigner's semicircle law.
E_ε	$= E + i\varepsilon$ energy with imaginary part.
J	initial covariance.
$\mathcal{I}$	energy interval.
α	$\in (0,1)$, parameter entering in the definition of the reference volume in the cluster expansion.
a, b	$\in \mathbb{R}^\Lambda$ integration variables.
$a^\pm_s, b^\pm_s$	saddle points.
$\mathcal{E}$	$= \mathcal{E}_r - i\mathcal{E}_i = \frac{E}{2} - i\sqrt{1 - \frac{E^2}{4}}$, value of saddle point a^+_s.
B	new complex covariance, obtained after contour deformation.
C	new real covariance.
$\mathrm{d}\mu_J(a,b)$	Gaussian measure with covariance J.
$\mathcal{R}(a,b)$	remainder in the functional integral after contour deformation.
D	diagonal matrix depending on a, b.
$\mathcal{V}(a,b)$	effective potential after contour deformation.
$V(x)$	cubic Taylor remainder.
$\mathcal{O}(a,b)$	local observable, later $\mathcal{O}_{m,n}(a,b)$.
m_r, m_i	real and imaginary part of complex mass term $1 - \mathcal{E}^2$ of C.
I^s	$\subset \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$, partition of integration domain, $s = 1, ...5$.
$F^{m,n}_s$	functional integral with local observable $\mathcal{O}_{m.m}$ restricted to I^s.
M	$= (M_j)_{j\in\Lambda}$ set of 2×2 supermatrices.
$\bar\rho_j, \rho_j)_{j\in\Lambda}$	set of Grassmann variables.
$\mathrm{d}\mu_B(M)$	Gaussian measure in both complex and Grassmann variables.
$\mathcal{V}(M)$	effective potential depending on the supermatrix M.
s_p	inductively introduced interpolation parameters.
$C(s)$	interpolated real covariance $C(s)_{ij} = s_{ij} C_{ij}$.
$B(s)$	interpolated complex covariance $(C(s)^{-1} + i\sigma_E m_i^2)^{-1}$.
$G_q(s)$	propagator depending only on $s_1, \ldots, s_q$.
$\tilde Y$	$(\triangle_0, \tilde\triangle_1, \ldots, \tilde\triangle_r)$ generalized polymer.
T	ordered tree on generalized polymer $\tilde Y$.
$\triangle$	cube in $\mathbb{Z}^2$ of size $W^2 (\ln W)^\alpha$.
$\triangle_0$	root cube containing 0.
$\tilde\triangle$	$= (\triangle, \triangle', \triangle'')$ generalized cube.
i, k, k', j	$\in \mathbb{Z}^2$ indices summed over $i \in \triangle', j \in \triangle'', k' \in \triangle, k \in$ "old" cubes.

3. Supersymmetric Polar Coordinates with applications to the Lloyd model

3.1. Introduction

A major open problem in mathematical physics is the existence of an Anderson transition in dimension three and higher for random Schrödinger operators. These operators model transport in disordered media, a classical example being electrical conductivity in metals with impurities. In this paper, we consider the quantum mechanical problem of an electron moving on a lattice $\mathbb{Z}^d$ and interacting with a random potential. The corresponding mathematical model is the so-called discrete Random Schrödinger operator, or Anderson's tight binding model [And58], acting on the Hilbert space $l^2(\mathbb{Z}^d)$ and defined by

$$H := -\Delta_{\mathbb{Z}^d} + \lambda V,$$

where $\Delta_{\mathbb{Z}^d}$ is the lattice Laplacian $(\Delta\psi)(j) = \sum_{k:|j-k|=1}(\psi(j) - \psi(k))$, and V is a multiplication operator $(V\psi)(j) = V_j\psi(j)$. Here, $\{V_j\}_{j\in\mathbb{Z}^d}$ is a collection of random variables (independent or correlated) and $\lambda > 0$ is a parameter expressing the strength of disorder. Physical information are encoded in the spectral properties of H. For a large class of random potentials V localization of the eigenfunctions has been proved in $d = 1$ for arbitrary disorder and in $d \geqslant 2$ for large disorder or at the band edge. A localization - delocalization transition has been proved on tree graphs, and is conjectured to hold on $\mathbb{Z}^d$, for $d \geqslant 3$. A detailed up-to-date review on the model, known results and tools can be found in the book by Aizenman and Warzel [AW15].

Finite volume criteria allow to reconstruct properties of H from the Green's function (or resolvent) of a finite volume approximation H_Λ, by taking the thermodynamic limit $\Lambda \uparrow \mathbb{Z}^d$. More precisely, let $\Lambda \subset \mathbb{Z}^d$ be a finite cube centered around the origin with volume $|\Lambda| = N$. We define the Random Schrödinger operator $H_\Lambda \in l^2(\Lambda)$ on Λ as

$$H_\Lambda = -\Delta + \lambda V, \tag{3.1.1}$$

where $\Delta = \Delta_\Lambda$ is the discrete Laplacian on Λ

$$(\Delta\psi)(j) = \sum_{k\in\Lambda:|j-k|=1} (\psi(k) - \psi(j)) + \text{ eventual boundary terms.}$$

The relevant quantities are expressions of the form

$$\mathbb{E}[G_\Lambda(z_1)_{j_1,k_1} \dots G_\Lambda(z_n)_{j_n,k_n}], \tag{3.1.2}$$

where $G_\Lambda(z) := (z\mathbb{1}_\Lambda - H_\Lambda)^{-1}$, $z \in \mathbb{C}\backslash\sigma(H)$, and $\mathbb{E}$ denotes the average with respect to the random vector V.

In particular the (averaged) density of states $\bar{\rho}_\lambda(E)$ satisfies the relation,

$$\int_{\mathbb{R}} \frac{1}{z-E}\ \bar{\rho}_\lambda(E)\,\mathrm{d}E = \frac{1}{\pi|\Lambda|}\mathbb{E}[\operatorname{Tr} G_\Lambda(z)],$$

hence (see for example [AW15, Section 4 and Appendix B])

$$\bar{\rho}_\Lambda(E) := -\frac{1}{\pi|\Lambda|}\lim_{\varepsilon\to 0^+}\mathbb{E}[\operatorname{Im}\operatorname{Tr} G_\Lambda(E+i\varepsilon)],$$

where $E \in \mathbb{R}$. Regularity properties of $\bar{\rho}_\Lambda(E)$ and its derivatives can be inferred from the generating function

$$\mathcal{G}_\varepsilon(E,\tilde{E}) = \mathbb{E}\left[\frac{\det((E+i\varepsilon)\mathbb{1}_\Lambda - H_\Lambda)}{\det((\tilde{E}+i\varepsilon)\mathbb{1}_\Lambda - H_\Lambda)}\right]. \tag{3.1.3}$$

For example

$$\operatorname{Tr} G_\Lambda(E+i\varepsilon) = -\partial_{\tilde{E}}\mathcal{G}_\varepsilon(E,\tilde{E})|_{\tilde{E}=E} = \partial_E\mathcal{G}_\varepsilon(E,\tilde{E})|_{\tilde{E}=E}. \tag{3.1.4}$$

Information on the nature of the spectrum can be deduced from the thermodynamic limit of

$$\mathbb{E}[|G_\Lambda(E+i\varepsilon)_{jk}|^2], \qquad \text{or} \qquad \rho_2(E,E+\omega) := \mathbb{E}[\rho_\Lambda(E)\rho_\Lambda(E+\omega)]$$

where the spectral parameter ε and the energy difference ω must be taken of order $|\Lambda|^{-1}$.

A possible tool to analyse these objects is the so-called supersymmetric (SUSY) approach. It allows to rewrite averages of the form (3.1.2) as an integral involving only the Fourier transform of the probability distribution, at the cost of introducing Grassmann variables in the intermediate steps. A short introduction on Grassmann variables and their application in our context is given in Appendix 3.A. For more details see for example the following monographs: [Var04, Ber87, Weg16, DeW92]. This formalism proved to be especially useful in the case of random operators arising from quantum diffusion problems [Efe99]. The supersymmetric approach was applied with success to study Anderson localization as well as phase transitions on tree-graphs [Wan01, Bov90, CK86, KMP86]. All these applications are based on variations of the following key fact.

Theorem 3.1.1. *Let H_Λ be as in Eq.* (3.1.1) *and assume the V_j are independent random variables with probability measure μ_j such that $\int v_j^2 d\mu_j(v_j) < \infty$ $\forall j$, i.e., its Fourier transform $\hat{\mu}_j(t) := \int e^{-itv_j} d\mu_j(v_j)$ is twice differentiable with bounded first and second derivatives.*

Let $\mathcal{A} = \mathcal{A}[\{\chi_j, \bar{\chi}_j\}_{j\in\Lambda}\}]$ be a Grassmann algebra, $z \in \mathbb{C}^\Lambda$ a family of complex variables and set $\Phi_j := (z_j, \chi_j)^t$, $\Phi_j^* := (\bar{z}_j, \bar{\chi}_j)$ such that $\Phi_j^*\Phi_k = \bar{z}_j z_k + \bar{\chi}_j\chi_k$ is an even element in $\mathcal{A}$ for all $j, k \in \Lambda$. For any matrix $A \in \mathbb{C}^{\Lambda\times\Lambda}$, we define

$$\Phi^* A\Phi := \Phi^{\mathrm{diag}}(A,A)\Phi = \sum_{j,k\in\Lambda} A_{jk}\Phi_j^*\Phi_k,$$

where $\operatorname{diag}(A,A)$ *is a* $2|\Lambda|\times 2|\Lambda|$ *block diagonal matrix. In particular* $\Phi^*\Phi=\sum_{j\in\Lambda}\Phi_j^*\Phi_j$. *Finally, for any even element* $a=b_a+n_a$ *in* $\mathcal{A}^0$ *with* $n_a^3=0$ *we define (cf. Eq. (3.A.2))*

$$\hat{\mu}_j(a)=\mathbb{E}[e^{iaV_j}]:=\hat{\mu}_j(b_a)+\hat{\mu}_j'(b_a)n_a+\tfrac{1}{2}\hat{\mu}_j''(b_a)n_a^2. \tag{3.1.5}$$

Then the generating function (3.1.3) *can be written as*

$$\mathcal{G}_\varepsilon(E,\tilde{E})=\int[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\ \mathrm{e}^{i\Phi^*(\mathbf{E}+i\varepsilon+\Delta)\Phi}\prod_{j\in\Lambda}\hat{\mu}_j(\lambda\Phi_j^*\Phi_j), \tag{3.1.6}$$

where we defined $[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]:=\prod_{j\in\Lambda}(2\pi)^{-1}\,\mathrm{d}\bar{z}_j\,\mathrm{d}z_j\,\mathrm{d}\bar{\chi}_j\,\mathrm{d}\chi_j$, $\Phi^*\varepsilon\Phi=\varepsilon\Phi^*\Phi$ *and* $\mathbf{E}=\operatorname{diag}(\tilde{E}\mathbb{1}_{|\Lambda|},E\mathbb{1}_{|\Lambda|})$ *is a diagonal matrix. Moreover*

$$\begin{aligned}\mathbb{E}[|G_\Lambda(E+i\epsilon)_{jk}|^2]=\int&[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\,[\,\mathrm{d}\tilde{\Phi}^*\,\mathrm{d}\tilde{\Phi}]\ \mathrm{e}^{i\Phi^*(\mathbf{E}+i\varepsilon+\Delta)\Phi-i\tilde{\Phi}^*(\mathbf{E}-i\varepsilon+\Delta)\tilde{\Phi}}\\&\times z_j\bar{z}_k\tilde{z}_k\bar{\tilde{z}}_j\prod_{j\in\Lambda}\hat{\mu}_j(\lambda(\Phi_j^*\Phi_j-\tilde{\Phi}_j^*\tilde{\Phi}_j)).\end{aligned} \tag{3.1.7}$$

A similar representation holds for the two-point function $\rho_2(E,\tilde{E})$.

Remark. In the formulas above both $\hat{\mu}_j(\lambda(\Phi_j^*\Phi_j))$ and $\hat{\mu}_j(\lambda(\Phi_j^*\Phi_j-\tilde{\Phi}_j^*\tilde{\Phi}_j))$ are well defined. Indeed, the even elements $a_1:=\Phi_j^*\Phi_j$ and $a_2:=\Phi_j^*\Phi_j-\tilde{\Phi}_j^*\tilde{\Phi}_j$, have nilpotent part $n_{a_1}=\bar{\chi}_j\chi_j$ and $n_{a_2}=\bar{\chi}_j\chi_j-\bar{\tilde{\chi}}_j\tilde{\chi}_j$, respectively. The result then follows from $n_{a_1}^2=0=n_{a_2}^3$, together with Eq. (3.1.5).

Note that we have taken independent variables above only to simplify notations. In the general case, the product of one-dimensional Fourier transforms is replaced by a joint Fourier transform. The generalized formula will hold as long as the Fourier transform admits enough derivatives.

Proof. We write $\mathcal{G}_\varepsilon(E,\tilde{E})$ and $\mathbb{E}[|G_\Lambda(E+i\varepsilon)_{jk}|^2]$ as a supersymmetric integral (cf. Theorem 3.A.1)

$$\begin{aligned}&\mathcal{G}(E,\tilde{E})=\mathbb{E}\left[\int[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\ \mathrm{e}^{i\Phi^*(\mathbf{E}+i\varepsilon+\Delta-\lambda V)\Phi}\right]\\&\mathbb{E}[|G_\Lambda(E+i\varepsilon)_{jk}|^2]=\\&\qquad\mathbb{E}\left[\int[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\ [\,\mathrm{d}\tilde{\Phi}^*\,\mathrm{d}\tilde{\Phi}]\ \mathrm{e}^{i\Phi^*(\mathbf{E}+i\varepsilon+\Delta-\lambda V)\Phi-i\tilde{\Phi}^*(\mathbf{E}-i\varepsilon+\Delta-\lambda V)\tilde{\Phi}}z_j\bar{z}_k\tilde{z}_k\bar{\tilde{z}}_j\right]\end{aligned}$$

This step holds for any choice of $V\in\mathbb{R}^\Lambda$. Note that we need two copies of SUSY variables to represent $\mathbb{E}[|G_\Lambda(E+i\varepsilon)_{jk}|^2]$. When $\mathrm{d}\mu_j$ admits two finite moments, we can move the average inside. The result follows. $\square$

The aim of this paper is to extend this representation to probability distributions with less regularity. To this purpose we introduce a supersymmetric version of polar coordinates which allows to reexpress $e^{i\lambda V_j\Phi_j^*\Phi_j}$ as $e^{i\lambda V_j x_i}$, where $x_j\in\mathbb{R}$ is a real variable.

As a result, the formula can be extended to any probability distribution on $N = |\Lambda|$ real variables. In contrast to the ordinary ones, supersymmetric polar coordinates introduce correction terms due to the boundary of the integration domain. The simple formula above will then be replaced by a sum of integrals.

As a concrete example, we consider the so-called Lloyd model, with V defined as $V_j := \sum_{k\in\Lambda} T_{jk} W_k$, where $\{W_k\}_{k\in\Lambda}$ is a family of i.i.d. random variables with Cauchy distribution $\mathrm{d}\mu(x) = \pi^{-1}(1+x^2)^{-1}\,\mathrm{d}x$. The standard (uncorrelated) Lloyd model corresponds to $T_{jk} = \delta_{jk}$. In this case the variables $\{V_j\}_{j\in\Lambda}$ are independent and Cauchy distributed. Note that $\mathrm{d}\mu(x)$ has no finite moments. For this model, the averaged Green's function (and hence the density of states) can be computed exactly whenever $T_{jk} \geqslant 0\ \forall j,k$ (non-negative correlation) [Llo69, Sim83].

Using supersymmetric polar coordinates, we show here that for the non-negative linearly correlated Lloyd model Eq. (3.1.6) and (3.1.7) remain valid, with an appropriate redefinition of $\hat{\mu}(b_a + n_a)$. In this case, one can easily recover the exact formula for the averaged Green's function. The formula remains valid also in the case of linear negative correlation, at the price of adding additional correction terms, due to boundary effects.

We expect the supersymmetric representation will help to study problems not yet accessible via other tools, such as negative correlations or the two point function at weak disorder. As a first test, we considered a simplified model with small negative correlations localized on one site. For this toymodel we used the supersymmetric representation to prove that the density of states remains in the vicinity of the exact formula. Our result holds in any dimension and arbitrary volume.

Overview of this article. In Section 3.2 we state the main results of the paper, and give some ideas about the proofs. More precisely, Section 3.2.1 introduces supersymmetric polar coordinates (Theorem 3.2.1), with a general integrated function f, not necessarily compactly supported. Applications to $\mathcal{G}_\varepsilon(E,\tilde{E})$ and $\mathbb{E}[|G_\Lambda(E+i\varepsilon)_{jk}|^2]$ are given in Theorem 3.2.2. The detailed proofs of both theorems can be found in Section 3.3. In Subsection 3.2.2 we consider the Lloyd model and give an application of the formula for a simple toymodel. The corresponding proofs are in Section 3.4.

3.2. Main results

3.2.1. Supersymmetric polar coordinates

For an introduction to the supersymmetric formalism see Appendix 3.A.

Consider first $\mathcal{A}[\bar{\chi},\chi]$ a Grassmann algebra with two generators. The idea of supersymmetric polar coordinates is to transform between generators $(\bar{z}, z, \bar{\chi}, \chi)$ of $\mathcal{A}_{2,2}(\mathbb{C})$

and $(r, \theta, \bar{\rho}, \rho)$ of $\mathcal{A}_{2,2}(\mathbb{R}^+ \times (0, 2\pi))$ [1] such that $\bar{z}z + \bar{\chi}\chi = r^2$. A reasonable change is

$$\Psi(r, \theta, \bar{\rho}, \rho) = \begin{pmatrix} z(r, \theta, \bar{\rho}, \rho) \\ \bar{z}(r, \theta, \bar{\rho}, \rho) \\ \chi(r, \theta, \bar{\rho}, \rho) \\ \bar{\chi}(r, \theta, \bar{\rho}, \rho) \end{pmatrix} := \begin{pmatrix} \mathrm{e}^{i\theta}(r - \frac{1}{2}\bar{\rho}\rho) \\ \mathrm{e}^{-i\theta}(r - \frac{1}{2}\bar{\rho}\rho) \\ \sqrt{r}\rho \\ \sqrt{r}\bar{\rho} \end{pmatrix} \tag{3.2.1}$$

Indeed, we have $\bar{z}z + \bar{\chi}\chi = (r - \frac{1}{2}\bar{\rho}\rho)^2 + r\bar{\rho}\rho = r^2$.

Note that 0 is a boundary point for polar coordinates since it maps $\mathbb{R}^+ \times (0, 2\pi)$ to $\mathbb{C}\backslash\{0\}$. For functions with compact support in $U = \mathbb{C}\backslash\{0\}$ a SUSY version of the standard coordinate change formula applies, where the Jacobian is replaced by a Berezinian, c.f. Theorem 3.A.3. On the contrary, functions with $f(0) \neq 0$ have no compact support in the domain $U = \mathbb{C}\backslash\{0\}$ and we collect additional boundary terms as the following theorem shows.

Theorem 3.2.1 (Supersymmetric polar coordinates). *Let $N \in \mathbb{N}$, $\mathcal{A}_{2N}$ the complex Grassmann algebra generated by $\{\bar{\chi}_j, \chi_j\}_{j=1}^N$ and $\{\Phi_j^*, \Phi_j\}_{j=1}^N$ a set of supervectors defined as in Theorem 3.1.1. Let $f \in \mathcal{A}_{2N,2N}(\mathbb{C}^N)$ be integrable, i.e., all $f_I : \mathbb{C}^N \to \mathbb{C}$ are integrable. Then*

$$I(f) = \int_{\mathbb{C}^N} [\,\mathrm{d}\Phi^* \,\mathrm{d}\Phi]\; f(\Phi^*, \Phi) = \sum_{\alpha \in \{0,1\}^N} I_\alpha(f) \tag{3.2.2}$$

with multiindex α and

$$I_\alpha(f) = \pi^{-|1-\alpha|} \int_{(\mathbb{R}^+ \times (0,2\pi))^{1-\alpha}} (\,\mathrm{d}r \,\mathrm{d}\theta \,\mathrm{d}\bar{\rho} \,\mathrm{d}\rho)^{1-\alpha}\; f \circ \Psi_\alpha(r, \theta, \bar{\rho}, \rho), \tag{3.2.3}$$

where $(\,\mathrm{d}r)^{1-\alpha} = \prod_{j:\alpha_j=0} \mathrm{d}r_j$ and Ψ_α is given by $\Psi_\alpha : (r, \theta, \bar{\rho}, \rho) \mapsto (z, \bar{z}, \chi, \bar{\chi})$ with

$$\begin{cases} z_j(r_j, \theta_j, \bar{\rho}_j, \rho_j) & = \delta_{\alpha_j 0}\; \mathrm{e}^{i\theta_j}(r_j - \frac{1}{2}\bar{\rho}_j\rho_j), \\ \bar{z}_j(r_j, \theta_j, \bar{\rho}_j, \rho_j) & = \delta_{\alpha_j 0}\; \mathrm{e}^{-i\theta_j}(r_j - \frac{1}{2}\bar{\rho}_j\rho_j), \\ \chi_j(r_j, \theta_j, \bar{\rho}_j, \rho_j) & = \delta_{\alpha_j 0}\, \sqrt{r_j}\rho_j, \\ \bar{\chi}_j(r_j, \theta_j, \bar{\rho}_j, \rho_j) & = \delta_{\alpha_j 0}\, \sqrt{r_j}\bar{\rho}_j. \end{cases}$$

Proof. See Section 3.3. □

Remark. For f compactly supported on $\mathbb{C}\backslash\{0\}$ (this means in particular $f(0) = 0$), we recover the result of Theorem 3.A.3. Namely for $\alpha = 0$, we obtain the right-hand side of Theorem 3.A.3 while all contributions from $\alpha \neq 0$ vanish.

Example. To illustrate the idea behind the above result, consider the following simple example. Let φ be the smooth compactly supported function $\varphi : \mathbb{R} \to \mathbb{R}$, given by

$$\varphi(x) = \begin{cases} \mathrm{e}^{-(1-2|x|^2)^{-1}} & \text{if } |x| < \frac{1}{\sqrt{2}} \\ 0 & \text{otherwise.} \end{cases}$$

[1] cf. Definition 3.A.2. Note that $\bar{\rho}, \rho \in \mathcal{A}^1[\bar{\chi}, \chi]$.

Note that $\varphi(0) = \mathrm{e}^{-1} \neq 0$, hence $f(\bar{z}, z, \bar{\chi}, \chi) = \varphi(\bar{z}z + \bar{\chi}\chi)$ is a smooth function without compact support in $\mathbb{C}\backslash\{0\}$. By a straightforward computation, we have

$$\begin{aligned} I(f) &= \int_{|z|<\frac{1}{\sqrt{2}}} [\,\mathrm{d}\Phi^* \,\mathrm{d}\Phi]\ \mathrm{e}^{-(1-2\bar{z}z)^{-1}}(1 - 2(1-2\bar{z}z)^{-2}\bar{\chi}\chi) \\ &= \frac{1}{2\pi}\int_0^{\frac{1}{\sqrt{2}}} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\ 4r\,\mathrm{e}^{-(1-2r^2)^{-1}}(1-2r^2)^{-2} = e^{-1}, \end{aligned}$$

where we expand the expression in the Grassmann variables and change to ordinary polar coordinates after integrating over the Grassmann variables. Applying formulas (3.2.2) and (3.2.3), we obtain directly

$$I(f) = \pi^{-1}\int_{\mathbb{R}^+\times(0,2\pi)} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho\ f\circ\Psi(r,\theta,\bar{\rho},\rho) +\ f\circ\Psi(0) = e^{-1},$$

where the first integral vanishes, since $f\circ\Psi$ is independent of $\bar{\rho}$ and ρ.

Now consider the generating function (3.1.3). In the case of an integrable density without other regularity conditions, we obtain the following result.

Theorem 3.2.2. *Let $\Lambda \subset \mathbb{Z}^d$ be a finite volume and $H_\Lambda = -\Delta + \lambda V$ be the Schrödinger operator introduced in Eq.* (3.1.1)*, where $\{V_j\}_{j\in\Lambda}$ is a family of real random variables with integrable joint density μ. Then the generating function* (3.1.3) *can be written as*

$$\mathcal{G}_\varepsilon(E,\tilde{E}) = \sum_{\alpha\in\{0,1\}^\Lambda} \int_{(\mathbb{R}^+\times(0,2\pi))^{1-\alpha}} \left(\tfrac{\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho}{\pi}\right)^{1-\alpha} \hat{\mu}(\{\lambda r_j^2\}_{j\in\Lambda})|_{r^\alpha=0}\, g\circ\Psi_\alpha(r,\theta,\bar{\rho},\rho) \tag{3.2.4}$$

where $g(\Phi^*,\Phi) = \mathrm{e}^{i\Phi^*(\mathbf{E}+i\varepsilon+\Delta)\Phi}$, $\mathbf{E} = \mathrm{diag}\,(\tilde{E}\mathbb{1}_{|\Lambda|}, E\mathbb{1}_{|\Lambda|})$ and $\hat{\mu}(\{\lambda r_j^2\}_{j\in\Lambda})$ is the $|\Lambda|$-dimensional, joint Fourier transform of μ. Similarly

$$\begin{aligned} &\mathbb{E}[|G_\Lambda(E+i\varepsilon)_{jk}|^2] \\ &= \sum_{\substack{\alpha\in\{0,1\}^\Lambda \\ \tilde{\alpha}\in\{0,1\}^\Lambda}} \pi^{-|1-\alpha|-|1-\tilde{\alpha}|} \int_{\substack{(\mathbb{R}^+\times(0,2\pi))^{1-\alpha} \\ \times(\mathbb{R}^+\times(0,2\pi))^{1-\tilde{\alpha}}}} (\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho)^{1-\alpha}(\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\theta}\,\mathrm{d}\bar{\tilde{\rho}}\,\mathrm{d}\tilde{\rho})^{1-\tilde{\alpha}} \\ &\qquad \hat{\mu}(\{\lambda(r_j^2-\tilde{r}_j^2)\}_{j\in\Lambda})_{r^\alpha=0=\tilde{r}^{\tilde{\alpha}}}\, g^+\circ\Psi_\alpha(r,\theta,\bar{\rho},\rho)\ g^-\circ\Psi_{\tilde{\alpha}}(r,\theta,\bar{\rho},\rho), \end{aligned}$$

where $g^+(\Phi^*,\Phi) = \bar{z}_k z_j\,\mathrm{e}^{i\Phi^*(\mathbf{E}+i\varepsilon+\Delta)\Phi}$ and $g^-(\tilde{\Phi}^*,\tilde{\Phi}) = \overline{\tilde{z}}_j\tilde{z}_k\,\mathrm{e}^{-i\tilde{\Phi}^*(\mathbf{E}-i\varepsilon+\Delta)\tilde{\Phi}}$.

Idea of the proof. Again we write $\mathcal{G}_\varepsilon(E,\tilde{E})$ and $|G_\Lambda(E+i\varepsilon)_{jk}|^2$ as a supersymmetric integral (Theorem 3.A.1). Note that we need two copies of SUSY variables to represent $|G_\Lambda(E+i\varepsilon)_{jk}|^2$. Taking the average inside at this point would cause problems. Hence we apply first our polar-coordinate formula Theorem 3.2.1. Since r is now real, the expression $\mathbb{E}[\,\mathrm{e}^{i\lambda\sum_j V_j r_j^2}]$ is the standard Fourier transform $\hat{\mu}(\{\lambda r_j^2\}_{j\in\Lambda})$. Details can be found in Section 3.3. □

3.2.2. Applications to the Lloyd model

As a concrete example, we consider the Lloyd model with linear correlated random potentials, i.e. $V_j = \sum_k T_{jk} W_k$, where $W_k \sim \text{Cauchy}(0,1)$ are i.i.d. random variables, $T_{jk} = T_{kj} \in \mathbb{R}$ and $\sum_j T_{jk} > 0$.

We discuss three cases:

1. the classical Lloyd model, where $T_{jk} = \delta_{jk}$, hence $V_j \sim \text{Cauchy}(0,1)$ are i.i.d.
2. the (positive) correlated Lloyd model, where $T_{jk} \geqslant 0$ with $\sum_j T_{jk} > 0$.
3. a toymodel with single negative correlation, i.e. $T_{jj} = 1$ and $T_{21} = T_{12} = -\delta^2$ with $0 < \delta < 1$ and $T_{jk} = 0$ otherwise. The indices 1 and 2 denote two fixed, nearest neighbour points $i_1, i_2 \in \Lambda$ with $|i_1 - i_2| = 1$.

Proposition 3.2.3. *When $T_{jk} \geqslant 0$ for all j,k (Case 1. and 2. above) we have*

$$\mathcal{G}_\varepsilon(E,\tilde{E}) = \int [\,\mathrm{d}\Phi^* \,\mathrm{d}\Phi]\; g(\Phi^*,\Phi)\, \mathrm{e}^{-\sum_k \lambda \sum_j T_{jk} \Phi_j^* \Phi_j}.$$

where $g(\Phi^*,\Phi) := \mathrm{e}^{i\Phi^*(\mathbf{E}-i\varepsilon+\Delta)\Phi}$. For the toymodel (Case 3. above) a similar formula holds with additional correction terms. Precisely

$$\mathcal{G}_\varepsilon(E,\tilde{E}) = \sum_{\beta\in\{++,+-,-+\}} \int_{\mathcal{I}_\beta} [\,\mathrm{d}\Phi^* \,\mathrm{d}\Phi]\; h(\Phi^*,\Phi)\, \mathrm{e}^{-\lambda \sum_{j=1}^2 T_j^\beta \Phi_j^* \Phi_j} + R(h)$$

where $h(\Phi^*,\Phi) = g(\Phi^*,\Phi)\,\mathrm{e}^{-\lambda\sum_{j\neq 1,2}\Phi_j^*\Phi_j}$, we defined $T^{++} = (1-\delta^2)(1,1)$, $T^{+-} = (1+\delta^2)(1,-1)$ and $T^{-+} = (1+\delta^2)(-1,1)$ and

$$\begin{aligned} \mathcal{I}_{++} &= \{z \in \mathbb{C}^N : \delta|z_2| < |z_1| < |z_2|/\delta\},\\ \mathcal{I}_{+-} &= \{z \in \mathbb{C}^N : |z_1| > |z_2|/\delta\},\\ \mathcal{I}_{-+} &= \{z \in \mathbb{C}^N : |z_1| < \delta|z_2|\}. \end{aligned} \tag{3.2.5}$$

Moreover, the additional boundary term is given by

$$\begin{aligned} R(h) = -\frac{1}{\pi^2} \int_{\mathbb{R}^+\times(0,2\pi)^2} & \mathrm{d}r_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 [\,\mathrm{d}\hat{\Phi}^*\,\mathrm{d}\hat{\Phi}]\; h\circ\Psi_{12}(r_2,\theta_1,\theta_2,\hat{\Phi}^*,\hat{\Phi})\\ &\times \lambda r_2 \delta^2 \left[\mathrm{e}^{-\lambda(1-\delta^4)r_2^2} + \delta^2\,\mathrm{e}^{-\lambda(1-\delta^4)\delta^2 r_2^2}\right], \end{aligned} \tag{3.2.6}$$

where $\hat{\Phi} = (\Phi_j)_{j\in\Lambda\setminus\{1,2\}}$ and

$$\Psi_{12}(r_2,\theta_1,\theta_2) = (\Phi_1,\Phi_2) = \begin{pmatrix} \mathrm{e}^{i\theta_1}\delta r_2 & \mathrm{e}^{i\theta_2} r_2 \\ 0 & 0 \end{pmatrix}$$

The same formulas hold for $\mathbb{E}[\operatorname{Tr} G_\Lambda(E+i\varepsilon)]$ with g replaced by $g_1(\Phi^*,\Phi) = \sum_j |z_j|^2\,\mathrm{e}^{i\Phi^*(E-i\varepsilon+\Delta)\Phi}$.

Idea of the proof. We use the representation from Theorem 3.2.2 and insert the Fourier transform of the given density. For non-negative correlations we can then undo the coordinate change. When negative correlations are present this operation generates additional correction terms. For details see Section 3.4. □

In the case of non-negative correlations we recover exact formulas, as follows.

Theorem 3.2.4. *Let $T_{jk} = \delta_{jk}$ (classical Lloyd model). We have*

$$\lim_{\varepsilon \to 0} \mathcal{G}_\varepsilon(E, \tilde{E}) = \frac{\det((E + i\lambda)\mathbb{1}_\Lambda - H_0)}{\det((\tilde{E} + i\lambda)\mathbb{1}_\Lambda - H_0)}, \tag{3.2.7}$$

where $H_0 = -\Delta$. In particular

$$\lim_{\varepsilon \to 0} \mathbb{E}[\operatorname{Tr} G_\Lambda(E + i\varepsilon)] = \operatorname{Tr}\left((E + i\lambda)\mathbb{1}_\Lambda - H_0\right)^{-1}. \tag{3.2.8}$$

For $T_{jk} \geqslant 0$ (non-negative correlation) Eq. (3.2.7) and (3.2.8) still hold, with $\lambda \mathbb{1}_\Lambda$ replaced by the diagonal matrix $\lambda \hat{T}$, where $\hat{T}_{ij} = \delta_{ij} \sum_k T_{jk}$.

In particular both, the classical and the (positive) correlated Lloyd model have the same (averaged) density of states as the free Laplacian $H_0 = -\Delta$ with imaginary mass λ and $\lambda \hat{T}$, respectively.

Idea of the proof. Follows from Proposition 3.2.3. For details see Section 3.4. □

Note that the results on the density of states above can be derived also by other methods (cf. [Llo69] and [Sim83]).

In the case of localized negative correlation (the toymodel in Case 3. above) we obtain the following result.

Theorem 3.2.5 (Toymodel)**.** *Consider T_{jk} be as in Case 3. above, $\lambda > 0$ and $0 < \delta \ll (1 + \lambda^{-1})^{-1}$. Then*

$$\begin{aligned} &\lim_{\varepsilon \to 0} \mathbb{E}[\operatorname{Tr} G_\Lambda(E + i\varepsilon)] = \\ &\qquad \operatorname{Tr}\left(E\mathbb{1}_\Lambda + i\lambda\hat{T} - H_0\right)^{-1} \left[1 + \mathcal{O}\Big((\delta(1 + \lambda^{-1}))^2\Big) + \mathcal{O}(|\Lambda|^{-1})\right]. \end{aligned}$$

Idea of the proof. Follows from Proposition 3.2.3 by integrating first over the uncorrelated variables in Λ and estimating the remaining integral. For details see Section 3.4. □

3.3. Supersymmetric polar coordinates

3.3.1. Proof of Theorem 3.2.1

Proof of Theorem 3.2.1. The idea is to apply the coordinate change Ψ from Eq. (3.2.1) for each $j \in \{0, \ldots, N\}$. To simplify the procedure, we divide it into $\Psi_1 \circ \Psi_2 \circ \Psi_3$, where

Ψ_1 is a change from ordinary polar coordinates into complex variables, Ψ_2 rescales the odd variables and Ψ_3 translates the radii into super space. Note that only the last step mixes ordinary and Grassmann variables and produces boundary terms.

We first change the complex variables $z_j, \bar{z}_j$ for all j into polar coordinates

$$\begin{aligned}\psi_1 : (0,\infty) \times [0, 2\pi) &\to \mathbb{C}\backslash\{0\}\\ (r,\theta) &\mapsto z(r,\theta), \quad z_j(r_j,\theta_j) = r_j\, \mathrm{e}^{i\theta_j} \quad \forall j.\end{aligned}$$

The Jacobian is $\prod_{j=1}^N 2r_j$ and by an ordinary change of variables

$$I(f) = \frac{1}{(2\pi)^N} \int_{(\mathbb{R}^+ \times (0,2\pi))^N} \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\bar{\chi}\, \mathrm{d}\chi \prod_{j=1}^N 2r_j\ f \circ \Psi_1(r,\theta,\bar{\chi},\chi),$$

where $\Psi_1 = \psi_1 \times \mathbb{1}$. Note that no boundary terms arise. Now we rescale the odd variables by

$$\psi_2(\bar{\rho},\rho) := (\bar{\chi}(\bar{\rho},\rho), \chi(\bar{\rho},\rho)) \quad \begin{cases} \bar{\chi}_j(\bar{\rho}_j,\rho_j) := \sqrt{r_j}\bar{\rho}_j \\ \chi_j(\bar{\rho}_j,\rho_j) := \sqrt{r_j}\rho_j \end{cases} \quad \forall j.$$

There are again no boundary terms since we have a purely odd transformation. The Berezinian is given by $\prod_{j=1}^N r_j^{-1}$. Since ψ_2 is a linear transformation, this can also be computed directly. This cancels with the Jacobian from Ψ_1 up to a constant. Hence

$$I(f) = \frac{1}{\pi^N} \int_{(\mathbb{R}^+ \times (0,2\pi))^N} \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\bar{\rho}\, \mathrm{d}\rho\ f \circ \Psi_1 \circ \Psi_2(r,\theta,\bar{\rho},\rho),$$

where $\Psi_2 = \mathbb{1} \times \psi_2$. After these transformations, we have $\bar{z}_j z_j + \bar{\chi}_j\chi_j = r_j^2 + r_j\bar{\rho}_j\rho_j = (r_j + \frac{1}{2}\bar{\rho}_j\rho_j)^2$. We set $\Psi_3(r,\theta,\bar{\rho},\rho) = (r - \frac{1}{2}\bar{\rho}\rho, \theta, \bar{\rho}, \rho)$. Hence $\Psi = \Psi_1 \circ \Psi_2 \circ \Psi_3$ is the Ψ from Eq. (3.2.1):

$$\begin{array}{ccccccc} z_j & \overset{\Psi_1}{\mapsto} & r_j e^{i\theta_j} & \overset{\Psi_2}{\mapsto} & r_j e^{i\theta_j} & \overset{\Psi_3}{\mapsto} & \left(r_j - \frac{1}{2}\bar{\rho}_j\rho_j\right) e^{i\theta_j}, \\ \chi_j & \overset{\Psi_1}{\mapsto} & \chi_j & \overset{\Psi_2}{\mapsto} & \sqrt{r_j}\rho_j & \overset{\Psi_3}{\mapsto} & \sqrt{r_j - \frac{1}{2}\bar{\rho}_j\rho_j}\ \rho_j = \sqrt{r_j}\rho_j. \end{array}$$

We expand $\tilde{f} = f \circ \Psi_1 \circ \Psi_2 \circ \Psi_3$ as follows

$$f \circ \Psi_1 \circ \Psi_2(r,\theta,\bar{\rho},\rho) = \tilde{f}(r + \tfrac{\bar{\rho}\rho}{2}, \theta, \bar{\rho}, \rho) = \sum_{\alpha \in \{0,1\}^N} \left(\tfrac{\bar{\rho}\rho}{2}\right)^\alpha \partial_r^\alpha \tilde{f}(r,\theta,\bar{\rho},\rho). \tag{3.3.1}$$

Note that we can set $\rho_j = 0$ and $\bar{\rho}_j = 0$ for $\alpha_j = 1$ in $\partial_r^\alpha \tilde{f}$. We use the short-hand notation $\partial_r^\alpha \tilde{f}(r,\theta,\bar{\rho},\rho)|_{\bar{\rho}^\alpha=\rho^\alpha=0}$. Inserting this into the integral I and applying integration by parts in r^α, we obtain

$$\begin{aligned} I(f) =& \tfrac{1}{\pi^N} \int_{(\mathbb{R}^+ \times (0,2\pi))^N} \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\bar{\rho}\, \mathrm{d}\rho \sum_{\alpha \in \{0,1\}^N} 2^{-|\alpha|} \left(\bar{\rho}\rho\right)^\alpha \partial_r^\alpha \tilde{f}(r,\theta,\bar{\rho},\rho) \qquad (3.3.2)\\ =& \sum_{\alpha \in \{0,1\}^N} \tfrac{1}{2^{|\alpha|}\pi^N} \int_{(\mathbb{R}^+)^{1-\alpha} \times (0,2\pi)^N} (\,\mathrm{d}r)^{1-\alpha}\, \mathrm{d}\theta (\,\mathrm{d}\bar{\rho}\, \mathrm{d}\rho)^{1-\alpha} \tilde{f}(r,\theta,\bar{\rho},\rho)|_{r^\alpha=\bar{\rho}^\alpha=\rho^\alpha=0}, \end{aligned}$$

where we applied $\int_{(\mathbb{R}^+)^\alpha}(dr)^\alpha \partial_r^\alpha \tilde{f} = (-1)^\alpha \tilde{f}_{|r^\alpha=0}$ and $\int(\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho)^\alpha(\bar{\rho}\rho)^\alpha = (-1)^\alpha$. Note that $\tilde{f}(r,\theta,\bar{\rho},\rho)|_{r^\alpha=\bar{\rho}^\alpha=\rho^\alpha=0} = f\circ\Psi_\alpha$ is independent of θ_j for $\alpha_j = 1$ and we can integrate $\int(\,\mathrm{d}\theta)^\alpha = (2\pi)^{|\alpha|}$. This proves the theorem. $\square$

3.3.2. Proof of Theorem 3.2.2

Proof of Theorem 3.2.2. Applying Theorem 3.A.1 to $\mathcal{G}_\varepsilon(E,\tilde{E})$ yields

$$\mathcal{G}_\varepsilon(E,\tilde{E}) = \mathbb{E}\left[\int [\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\,\mathrm{e}^{i\Phi^*(\mathbf{E}+i\varepsilon-\lambda V+\Delta)\Phi}\right].$$

Note that we cannot interchange the average with the integral, since the average of the supersymmetric expression $\mathrm{e}^{i\lambda\Phi^* V\Phi}$ may be ill-defined if infinite moments are present. But after applying Theorem 3.2.1 we get

$$\mathcal{G}_\varepsilon(E,\tilde{E}) = \sum_{\alpha\in\{0,1\}^\Lambda} \pi^{-|1-\alpha|}\mathbb{E}\left[\int_{(\mathbb{R}^+\times(0,2\pi))^{1-\alpha}} (\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho)^{1-\alpha}\,\mathrm{e}^{-i\lambda\sum_j V_j r_j^2} g\circ\Psi_\alpha(r,\theta,\bar{\rho},\rho)\right],$$

where $g(\Phi^*,\Phi) = \mathrm{e}^{i\Phi^*(\mathbf{E}+i\varepsilon+\Delta)\Phi}$. Now we can take the average inside the integral. The same arguments hold for $\mathbb{E}[|G_\Lambda(E+i\varepsilon)_{jk}|^2]$. $\square$

3.4. Applications to the Lloyd model

3.4.1. Proof of Proposition 3.2.3

We will need the following well-known result for the proof of the proposition.

Lemma 3.4.1. *Let $A \sim$ Cauchy$(0,1)$ and $t\in\mathbb{R}$. Then* $\mathbb{E}[\,\mathrm{e}^{itA}] = \mathrm{e}^{-|t|}$.

Proof. Let $t \geqslant 0$. We take the principal value and apply the residue theorem.

$$\lim_{R\to\infty}\int_{[-R,R]}\frac{\mathrm{e}^{itx}}{\pi(1+x^2)}\,\mathrm{d}x = \lim_{R\to\infty}\left[2\pi i\mathrm{Res}_{\,i}\frac{\mathrm{e}^{itx}}{\pi(1+x^2)} - \int_\gamma \frac{\mathrm{e}^{itx}\,\mathrm{d}x}{\pi(1+x^2)}\right] = \mathrm{e}^{-t},$$

where $\gamma(s) = R\,\mathrm{e}^{is}$ for $s\in[0,\pi]$. The case $t<0$ follows analogously by closing the contour from below. $\square$

Proof of Proposition 3.2.3. Starting from the representation (3.2.4) of Theorem 3.2.2, we use Lemma 3.4.1 to determine the Fourier transform

$$\hat{\mu}(\{\lambda r_j^2\}_{j\in\Lambda}) = \mathbb{E}[\,\mathrm{e}^{i\lambda\sum_{j,k}T_{jk}W_k r_j^2}] = \mathrm{e}^{-\sum_k\lambda|\sum_j T_{jk}r_j^2|}.$$

As long as $r_j\in\mathbb{R}$, this is well-defined and the integral remains finite for arbitrary correlation T. When $T_{jk}\geqslant 0$ for all j,k, we can drop the absolute value and obtain

$$\hat{\mu}(\{\lambda r_j^2\}_{j\in\Lambda}) = \mathrm{e}^{-\sum_k\lambda\sum_j T_{jk}r_j^2} = \tilde{\mu}\circ\Psi_\alpha,$$

where $\tilde{\mu}(\Phi^*,\Phi) = \exp[-\sum_k \lambda \sum_j T_{jk}\Phi_j^*\Phi_j]$ is a smooth, integrable function in $\mathcal{A}_{2N,2N}(\mathbb{C}^N)$, which can be transformed back to ordinary supersymmetric coordinates by Theorem 3.2.1.

In the case of the toymodel, our function is continuous but only piecewise smooth. We partition the integration domain into regions, where our function is smooth. In polar coordinates the regions (3.2.5) become

$$\begin{aligned}
\mathcal{I}_{++} &= \{0 < \delta r_2 < r_1 < \tfrac{r_2}{\delta}\} \times (0,\infty)^{\Lambda\setminus\{1,2\}} &&= \{r \in (0,\infty)^\Lambda : \delta r_2 < r_1 < \tfrac{r_2}{\delta}\},\\
\mathcal{I}_{+-} &= \{0 < \tfrac{r_2}{\delta} < r_1\} \times (0,\infty)^{\Lambda\setminus\{1,2\}} &&= \{r \in (0,\infty)^\Lambda : r_1 > \tfrac{r_2}{\delta}\},\\
\mathcal{I}_{-+} &= \{0 < r_1 < \delta r_2\} \times (0,\infty)^{\Lambda\setminus\{1,2\}} &&= \{r \in (0,\infty)^\Lambda : r_1 < \delta r_2\}.
\end{aligned}$$

Hence $(0,\infty)^\Lambda$ can be written as the disjoint union $\mathcal{I}_{++} \cup \mathcal{I}_{+-} \cup \mathcal{I}_{-+} \cup \mathcal{N}$, where $\mathcal{N}$ is a set of measure 0. Using T^β defined above, we can write

$$\begin{aligned}
I_1 &= \sum_{\alpha\in\{0,1\}^\Lambda} \pi^{-|1-\alpha|} \int_{(\mathbb{R}^+\times(0,2\pi))^{1-\alpha}} (\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho)^{1-\alpha} \hat{\mu}(\{\lambda r_j^2\}_{j\in\Lambda})|_{r^\alpha=0}\; g\circ\Psi_\alpha(r,\theta,\bar{\rho},\rho)\\
&= \sum_\beta \sum_{\alpha\in\{0,1\}^\Lambda} \pi^{-|1-\alpha|} \int_{(\mathbb{R}^+\times(0,2\pi))^{1-\alpha}} (\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho)^{1-\alpha} \chi(\mathcal{I}_\beta)|_{r^\alpha=0}\\
&\qquad\qquad \times\; \mathrm{e}^{-\lambda(\delta_{\alpha_1 0} r_1^2 T_1^\beta + \delta_{\alpha_2 0} r_2^2 T_2^\beta)} h\circ\Psi_\alpha(r,\theta,\bar{\rho},\rho),
\end{aligned}$$

where $\beta \in \{++,+-,-+\}$ and $h(\Phi^*,\Phi) = g(\Phi^*,\Phi)\,\mathrm{e}^{-\lambda\sum_{j\neq 1,2}\Phi_j^*\Phi_j}$ is independent of β. Finally, $\chi(\mathcal{I}_\beta)$ is the characteristic function of $\mathcal{I}_\beta$ and $r^\alpha = 0$ means $r_j = 0$ for $\alpha_j = 1$.

To transform back we need to repeat the proof of Theorem 3.2.1 on the different domains. Consider the integral

$$I_2 = \sum_\beta \int_{\mathcal{I}_\beta} [\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\; h(\Phi^*,\Phi)\,\mathrm{e}^{-\lambda\sum_{j=1}^2 T_j^\beta \Phi_j^*\Phi_j},$$

where $\mathcal{I}_\beta$ are the corresponding subsets of $\mathbb{C}^\Lambda$ (cf. Eq. (3.2.5)). We will show that inserting polar coordinates in I_2, we recover I_1 plus some correction terms. In each region, the integrated function is smooth and we can apply the first two transformations Ψ_1 and Ψ_2 from the proof of Theorem 3.2.1 and obtain

$$I_2 = \frac{1}{\pi^{|\Lambda|}} \sum_\beta \int_{\mathcal{I}_\beta\times(0,2\pi)^{|\Lambda|}} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho\;\; \mathrm{e}^{-\lambda\sum_{j=1}^2 T_j^\beta (r_j + \frac{1}{2}\bar{\rho}_j\rho_j)^2} h\circ\Psi_1\circ\Psi_2(r,\theta,\bar{\rho},\rho).$$

Replacing as in Eq. (3.3.1) the integrand by the Taylor-expansion of $\tilde{f}_\beta = \mathrm{e}^{-\lambda\sum_{j=1}^2 (T_\beta)_j r_j^2}\tilde{h}$, with $\tilde{h} = h\circ\Psi_1\circ\Psi_2\circ\Psi_3$, we obtain

$$\begin{aligned}
I_2 &= \sum_{\alpha\in\{0,1\}^\Lambda} I_\alpha, \text{ where}\\
I_\alpha &= \frac{1}{\pi^{|\Lambda|}2^{|\alpha|}} \sum_\beta \int_{\mathcal{I}_\beta\times(0,2\pi)^{|\Lambda|}} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho\;(\bar{\rho}\rho)^\alpha\; \partial_r^\alpha \tilde{f}_\beta(r,\theta,\bar{\rho},\rho).
\end{aligned}$$

Applying now integration by parts as in Eq. (3.3.2) generates additional boundary terms. More precisely, when $\alpha_1 = \alpha_2 = 0$, no derivatives in r_1 and r_2 appear and $\mathcal{I}_\beta = \tilde{\mathcal{I}}_\beta \times (0,\infty)^{\Lambda\setminus\{1,2\}}$. Hence no additional terms arise and

$$I_\alpha = \pi^{-|1-\alpha|} \sum_\beta \int_{(\mathbb{R}^+\times(0,2\pi))^{1-\alpha}} (\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar\rho\,\mathrm{d}\rho)^{1-\alpha}\,\mathrm{e}^{-\lambda\sum_{j=1}^2 (T_\beta)_j r_j^2}\chi(\tilde{\mathcal{I}}_\beta)\; h\circ\Psi_\alpha(r,\theta,\bar\rho,\rho).$$

For $\alpha_1 = 1$ and $\alpha_2 = 0$ (or vice versa), additional boundary terms do appear but cancel since the function is continuous:

$$\begin{aligned}
I_\alpha =& \frac{1}{\pi^{|\Lambda|}2^{|\alpha|}} \sum_\beta \int_{\mathcal{I}_\beta\times(0,2\pi)^{|\Lambda|}} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar\rho\,\mathrm{d}\rho\;(\bar\rho\rho)^\alpha\;\partial_{r_1}\left[h^{(\alpha)}(r,\theta,\bar\rho,\rho)\,\mathrm{e}^{-\lambda\sum_{j=1}^2 T_j^\beta r_j^2}\right]\\
=& \frac{1}{\pi^{|\Lambda|}2^{|\alpha|}} \int_{(\mathbb{R}^+)^{|\Lambda|-1}\times(0,2\pi)^{|\Lambda|}} \mathrm{d}\hat r\,\mathrm{d}\theta\,\mathrm{d}\bar\rho\,\mathrm{d}\rho\;(\bar\rho\rho)^\alpha\left[h^{(\alpha)}\,\mathrm{e}^{-\lambda\sum_{j=1}^2 T_j^{-+}r_j^2}\right]_{r_1=0}^{r_1=\delta r_2}\\
&+\left[h^{(\alpha)}\,\mathrm{e}^{-\lambda\sum_{j=1}^2 T_j^{++}r_j^2}\right]_{r_1=\delta r_2}^{r_1=r_2/\delta} + \left[h^{(\alpha)}\,\mathrm{e}^{-\lambda\sum_{j=1}^2 T_j^{+-}r_j^2}\right]_{r_1=r_2/\delta}^{r_1=\infty}\\
=& -\frac{1}{\pi^{|\Lambda|}2^{|\alpha|}} \int_{(\mathbb{R}^+)^{|\Lambda|-1}\times(0,2\pi)^{|\Lambda|}} \mathrm{d}\hat r\,\mathrm{d}\theta\,\mathrm{d}\bar\rho\,\mathrm{d}\rho\;(\bar\rho\rho)^\alpha h^{(\alpha)}|_{r_1=0}\,\mathrm{e}^{-\lambda T_2^{-+}r_2^2},
\end{aligned}$$

where $\mathrm{d}\hat r = \prod_{j\neq 1}\mathrm{d}r_j$ and $h^{(\alpha)} = \partial_r^{\hat\alpha}\tilde h$ and $\hat\alpha_j = \alpha_j$ for all $j \neq 1,2$, $\hat\alpha_1 = \hat\alpha_2 = 0$. Note that in the second step all terms except the first one cancel because of continuity: $\sum_{j=1}^2 T_j^{-+}r_j^2|_{r_1=\delta r_2} = \sum_{j=1}^2 T_j^{++}r_j^2|_{r_1=\delta r_2}$ and $\sum_{j=1}^2 T_j^{++}r_j^2|_{r_1=r_2/\delta} = \sum_{j=1}^2 T_j^{+-}r_j^2|_{r_1=r_2/\delta}$. We can apply now integration by parts for $r^{\hat\alpha}$ as before. Note that for $r_1 = 0$ the sets $\mathcal{I}_{++} = \mathcal{I}_{+-} = \varnothing$ and we obtain only contributions from the set $\mathcal{I}_{-+} = \{r_2 \in \mathbb{R}^+\}$ which is the same as writing $\sum_\beta \chi(\mathcal{I}_\beta)|_{r_1=0}$.
When $\alpha_1 = \alpha_2 = 1$, we obtain additional boundary terms which do not cancel. Applying integration by parts in r_1, we need to evaluate

$$\partial_{r_2}[h^{(\alpha)}\,\mathrm{e}^{-\lambda\sum_{j=1}^2 (T_\beta)_j r_j^2}] = (\partial_{r_2}h^{(\alpha)} - 2\lambda T_2^\beta r_2 h^{(\alpha)})\,\mathrm{e}^{-\lambda\sum_{j=1}^2 (T_\beta)_j r_j^2}$$

on the different boundaries. The contributions of $\partial_{r_2}h^{(\alpha)}\,\mathrm{e}^{-\lambda\sum_{j=1}^2 (T_\beta)_j r_j^2}$ cancel as above by continuity except for the term at $r_1 = 0$. The contributions from the second summand remain:

$$\begin{aligned}
I_\alpha =& \frac{1}{\pi^{|\Lambda|}2^{|\alpha|}} \sum_\beta \int_{\mathcal{I}_\beta\times(0,2\pi)^{|\Lambda|}} \mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\bar\rho\,\mathrm{d}\rho\;(\bar\rho\rho)^\alpha\partial_{r_1}\partial_{r_2}\left[h^{(\alpha)}\,\mathrm{e}^{-\lambda\sum_{j=1}^2 (T_\beta)_j r_j^2}\right]\\
=& \frac{1}{\pi^{|\Lambda|}2^{|\alpha|}} \int_{(\mathbb{R}^+)^{|\Lambda|-1}\times(0,2\pi)^{|\Lambda|}} \mathrm{d}\hat r\,\mathrm{d}\theta\,\mathrm{d}\bar\rho\,\mathrm{d}\rho\;(\bar\rho\rho)^\alpha\partial_{r_2}\left[-h^{(\alpha)}\,\mathrm{e}^{-\lambda T_2^{-+}r_2^2}\right]_{r_1=0} + R_\alpha(h),
\end{aligned}$$

where $R_\alpha(h)$ is the remaining part defined below in Eq. (3.4.1). In the first integral, we can apply integration by parts in r_2 and $r^{\hat\alpha}$ as before and the result is independent of β. It remains to consider

$$\begin{aligned}
R_\alpha(h) =& \frac{1}{\pi^{|\Lambda|}2^{|\alpha|}} \int_{(\mathbb{R}^+)^{|\Lambda|-1}\times(0,2\pi)^{|\Lambda|}} \mathrm{d}\hat r\,\mathrm{d}\theta\,\mathrm{d}\bar\rho\,\mathrm{d}\rho\;(\bar\rho\rho)^\alpha\;2\lambda r_2 \qquad (3.4.1)\\
&\times\left[h^{(\alpha)}|_{r_1=\delta r_2}(T_2^{++}-T_2^{-+})\,\mathrm{e}^{-\lambda(1-\delta^4)r_2^2} + h^{(\alpha)}|_{r_1=\frac{r_2}{\delta}}(T_2^{+-}-T_2^{++})\,\mathrm{e}^{-\lambda(1-\delta^4)\delta^{-2}r_2^2}\right].
\end{aligned}$$

Here, we can integrate over $r^{\hat{\alpha}}$, but the integral over r_2 remains:

$$
\begin{aligned}
R_\alpha(h) = -\pi^{-|1-\hat{\alpha}|} &\int_{(\mathbb{R}^+\times(0,2\pi))^{1-\hat{\alpha}}\times\mathbb{R}^+\times(0,2\pi)^2} (\,\mathrm{d}r\,\mathrm{d}\theta)^{1-\hat{\alpha}}(\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho)^{1-\alpha}\,\mathrm{d}r_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\ \lambda r_2 \\
\times &\left[\tilde{h}|_{r^{\hat{\alpha}}=\bar{\rho}^\alpha=\rho^\alpha=0,r_1=\delta r_2}\delta^2\,\mathrm{e}^{-\lambda(1-\delta^4)r_2^2} + \ \tilde{h}|_{r^{\hat{\alpha}}=\bar{\rho}^\alpha=\rho^\alpha=0,r_1=r_2/\delta}\,\mathrm{e}^{-\lambda(1-\delta^4)\delta^{-2}r_2^2}\right].
\end{aligned}
$$

By rescaling the second term $r_2 \mapsto \delta^2 r_2$, we obtain

$$
\begin{aligned}
I_2 - I_1 = -\sum_{\hat{\alpha}}\pi^{-|1-\hat{\alpha}|} &\int_{(\mathbb{R}^+\times(0,2\pi))^{1-\hat{\alpha}}\times\mathbb{R}^+\times(0,2\pi)^2} (\,\mathrm{d}r\,\mathrm{d}\theta)^{1-\hat{\alpha}}(\,\mathrm{d}\bar{\rho}\,\mathrm{d}\rho)^{1-\alpha}\,\mathrm{d}r_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \\
&\times \lambda r_2\delta^2\ \tilde{h}|_{r^{\hat{\alpha}}=\bar{\rho}^\alpha=\rho^\alpha=0,r_1=\delta r_2}\left[\mathrm{e}^{-\lambda(1-\delta^4)r_2^2} + \delta^2\,\mathrm{e}^{-\lambda(1-\delta^4)\delta^2 r_2^2}\right].
\end{aligned}
$$

Note that we can transform the variables of $\Lambda\backslash\{1,2\}$ back to flat coordinates by Theorem 3.2.1 and obtain $I_2 - I_1 = R(h)$ that finishes the proof. $\square$

3.4.2. Proof of Theorem 3.2.4

Proof of Theorem 3.2.4. We start from the result of Propostion 3.2.3.

In both models, the classical and the positive correlated one, we have $T_{jk} \geqslant 0$ and $\sum_k T_{jk} > 0$, hence the body of $\lambda\sum_j T_{jk}\Phi_j^*\Phi_j$ is strictly positive except on a set of measure 0. We end up with

$$
\mathcal{G}_\varepsilon(E,\tilde{E}) = \int[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\,\mathrm{e}^{i\Phi^*(\hat{E}+i\varepsilon+i\lambda\hat{T}+\Delta)\Phi},
$$

where we can take the average $\varepsilon \to 0$ and go back to the original representation. $\square$

3.4.3. Proof of Theorem 3.2.5

Proof of Theorem 3.2.5. Using Eq. (3.1.4) and the result of Proposition 3.2.3, we get

$$
\begin{aligned}
\mathbb{E}[\operatorname{Tr} G_\Lambda(E+i\varepsilon)] &= \mathbb{E}\Big[\int[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\,\mathrm{e}^{i\Phi^*(E+i\varepsilon-\lambda V+\Delta)\Phi}\sum_{j\in\Lambda}|z_j|^2\Big] \\
&= I_{++} + I_{+-} + I_{-+} + R(h)
\end{aligned}
$$

where for $\beta = (++), (+-)$ or $(-+)$ we have

$$
I_\beta = \int_{\mathcal{I}_\beta}[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\,\mathrm{e}^{i\Phi^*(E+i\varepsilon+\Delta)\Phi}\sum_{j\in\Lambda}|z_j|^2\,\mathrm{e}^{-\lambda(T_1^\beta\Phi_1^*\Phi_1+T_2^\beta\Phi_2^*\Phi_2+\sum_{k\neq1,2}\Phi_k^*\Phi_k)},
$$

and for $h(\Phi^*,\Phi) = \sum_j|z_j|^2\,\mathrm{e}^{-\lambda\sum_{j\neq1,2}\Phi_j^*\Phi_j}\,\mathrm{e}^{i\Phi^*(E-i\varepsilon+\Delta)\Phi}$ the remainder $R(h)$ is defined in Eq. (3.2.6).

We will show that the main contribution comes from $\mathcal{I}_{++}$ and indeed

$$
\text{body }(T_1^{++}\Phi_1^*\Phi_1 + T_2^{++}\Phi_2^*\Phi_2) = (1-\delta^2)[|z_1|^2+|z_2|^2] > 0 \quad \forall(z_1,z_2)\neq(0,0).
$$

In the following we show that I_{+-} and I_{-+}, as well as $R(h)$ are small in terms of δ.

Analysis of the I_β terms. Integrating out the Grassmann variables, we obtain for all β

$$I_\beta = \int_{\mathcal{I}_\beta} \mathrm{d}\bar{z}\,\mathrm{d}z \sum_{j\in\Lambda} |z_j|^2 \det\left[\tfrac{C_\beta+\varepsilon}{2\pi}\right] \mathrm{e}^{-\bar{z}(C_\beta+\varepsilon)z},$$

where C_β has the block structure

$$\begin{aligned} C_\beta &= \begin{pmatrix} A_\beta & -iD \\ -iD^T & B \end{pmatrix}, \quad A_\beta := A_0 + \lambda \mathrm{diag}\, T^\beta, \quad A_0 := -i(E+\Delta)|_{\{1,2\}} \\ B &:= (\lambda - i(E+\Delta))_{|\Lambda\setminus\{1,2\}}, \quad D^T := (d_1, d_2), \end{aligned} \tag{3.4.2}$$

and we defined the vectors $d_1, d_2 \in \mathbb{R}^{\Lambda\setminus\{1,2\}}$ as $d_1(j) = \delta_{|i_1-j|,1}$, $d_2(j) = \delta_{|i_2-j|,1}$, where i_1, i_2 are the positions of $1,2$. Note that the blocks B and D are independent of β and $\mathrm{Re}\, B > 0$. On the contrary $\mathrm{Re}\, C_\beta > 0$ holds only for $\beta = (+,+)$. We set then $\varepsilon = 0$ in our formulas and reorganize $I_{++} + I_{+-} + I_{-+}$ as follows

$$\begin{aligned} [I_{++} + I_{+-} + I_{-+}]_{|\varepsilon=0} &= \int \mathrm{d}\bar{z}\,\mathrm{d}z \sum_{j\in\Lambda} |z_j|^2 \det\left[\tfrac{C_{++}}{2\pi}\right] \mathrm{e}^{-\bar{z}C_{++}z} \\ &+ \int_{\mathcal{I}_{+-}} \mathrm{d}\bar{z}\,\mathrm{d}z \sum_{j\in\Lambda} |z_j|^2 \left(\det\left[\tfrac{C_{+-}}{2\pi}\right] \mathrm{e}^{-\bar{z}C_{+-}z} - \det\left[\tfrac{C_{++}}{2\pi}\right] \mathrm{e}^{-\bar{z}C_{++}z}\right) \\ &+ \int_{\mathcal{I}_{-+}} \mathrm{d}\bar{z}\,\mathrm{d}z \sum_{j\in\Lambda} |z_j|^2 \left(\det\left[\tfrac{C_{-+}}{2\pi}\right] \mathrm{e}^{-\bar{z}C_{-+}z} - \det\left[\tfrac{C_{++}}{2\pi}\right] \mathrm{e}^{-\bar{z}C_{++}z}\right) \\ &= \mathrm{Tr}\, C_{++}^{-1} + \int_{\mathcal{I}_{+-}} (\cdots) + \int_{\mathcal{I}_{-+}} (\cdots) = \mathrm{Tr}\, C_{++}^{-1} (1 + \mathcal{E}_{+-} + \mathcal{E}_{-+}) \end{aligned}$$

To estimate $\mathcal{E}_{+-}$ and $\mathcal{E}_{-+}$, we integrate over the variables $w = (z_j)_{j\in\Lambda, j\neq 1,2}$ exactly. We define $z = (\hat{z}, w)$, $\hat{z} = (z_1, z_2)$. Then

$$\begin{aligned} \bar{z}C_\beta z &= \bar{w}Bw - i\bar{w}D^t\hat{z} - i\bar{\hat{z}}Dw + \bar{\hat{z}}A_\beta\hat{z}, \\ \sum_{j\in\Lambda} |z_j|^2 &= \bar{\hat{z}}\hat{z} + \sum_{l\in\Lambda\setminus\{1,2\}} |w_l|^2. \end{aligned}$$

Integrating over w we get

$$\begin{aligned} &\int \mathrm{d}\bar{w}\,\mathrm{d}w \det\left[\tfrac{B}{2\pi}\right] \mathrm{e}^{-\bar{w}Bw}\, \mathrm{e}^{-i\bar{w}D^t\hat{z} - i\bar{\hat{z}}Dw} (\bar{w}w + \bar{\hat{z}}\hat{z}) \\ &= \mathrm{e}^{-\bar{\hat{z}}DB^{-1}D^t\hat{z}} (\mathrm{Tr}\, B^{-1} - \bar{\hat{z}}DB^{-2}D^t\hat{z} + \bar{\hat{z}}\hat{z}) = \mathrm{e}^{-\bar{\hat{z}}DB^{-1}D^t\hat{z}} (\mathrm{Tr}\, B^{-1} + \bar{\hat{z}}M\hat{z}), \end{aligned}$$

where we defined $M := 1 - DB^{-2}D^T$. Then for $\beta = (+-), (-+)$ and $\beta' = \beta$ or $\beta' = (++)$ we have

$$\begin{aligned} &\int_{\mathcal{I}_\beta} \mathrm{d}\bar{z}\,\mathrm{d}z \det\left[\tfrac{C_{\beta'}}{2\pi}\right] \mathrm{e}^{-\bar{z}C_{\beta'}z} \sum_{j\in\Lambda} |z_j|^2 \\ &\quad = \int_{\mathcal{I}_\beta} \mathrm{d}\bar{\hat{z}}\,\mathrm{d}\hat{z} \det\left[\tfrac{S_{\beta'}}{2\pi}\right] \mathrm{e}^{-\bar{\hat{z}}S_{\beta'}\hat{z}} \left(\mathrm{Tr}\, B^{-1} + \bar{\hat{z}}M\hat{z}\right), \end{aligned}$$

where $S_{\beta'} = A_{\beta'} + DB^{-1}D^T$ is the Schur complement of the 2×2 block of $C_{\beta'}$ corresponding to $1, 2$. We also used $\det C_{\beta'} = \det B \det S_{\beta'}$. We consider now the error term $\mathcal{E}_{-+}$. The error term $\mathcal{E}_{+-}$ works analogously. From the results above we get

$$\begin{aligned}
\mathcal{E}_{-+} &= \frac{1}{\operatorname{Tr} C_{++}^{-1}} \int_{\mathcal{I}_{-+}} \mathrm{d}\bar{z}\,\mathrm{d}z \sum_{j\in\Lambda} |z_j|^2 \left(\det\left[\tfrac{C_{-+}}{2\pi}\right] \mathrm{e}^{-\bar{z}C_{-+}z} - \det\left[\tfrac{C_{++}}{2\pi}\right] \mathrm{e}^{-\bar{z}C_{++}z} \right) \\
&= \int_{|z_1|<\delta|z_2|} \mathrm{d}\bar{\hat{z}}\,\mathrm{d}\hat{z} \det\left[\tfrac{S_{++}}{2\pi}\right] \mathrm{e}^{-\bar{\hat{z}}S_{++}\hat{z}} \,\frac{\operatorname{Tr} B^{-1} + \bar{\hat{z}} M \hat{z}}{\operatorname{Tr} C_{++}^{-1}} \left(\mathrm{e}^{-\bar{\hat{z}}X\hat{z}} \det(1 + S_{++}^{-1}X) - 1 \right),
\end{aligned}$$

where we used S_{++}^{-1} invertible and we defined

$$X := A_{-+} - A_{++} = 2\lambda \begin{pmatrix} -1 & 0 \\ 0 & \delta^2 \end{pmatrix}, \text{ hence } \bar{\hat{z}}X\hat{z} = 2\lambda(\delta^2|z_2|^2 - |z_1|^2) > 0.$$

Now we change the coordinate z_1 to $v = z_1 z_2^{-1}\delta^{-1}$. As a short-hand notation write $S = S_{++}$. We have

$$\bar{\hat{z}}S\hat{z} = |z_2|^2(\mathbf{v}^* S \mathbf{v}), \qquad \bar{\hat{z}}M\hat{z} = |z_2|^2 \mathbf{v}^* M \mathbf{v}, \qquad \bar{\hat{z}}X\hat{z} = |z_2|^2(\mathbf{v}^* X \mathbf{v}),$$

where $\mathbf{v} = (\delta v, 1)^t$ and $\mathbf{v}^* = (\delta\bar{v}, 1)$. Note that $\operatorname{Re} S > 0$ and

$$(\mathbf{v}^* X \mathbf{v}) = 2\lambda\delta^2(1 - |v|^2) \geqslant 0, \tag{3.4.3}$$

therefore we can perform the integral over z_2 exactly

$$\begin{aligned}
\mathcal{E}_{-+} =& \det\left[\tfrac{S}{2\pi}\right] \int_{|v|<1} \mathrm{d}\bar{z}_2\,\mathrm{d}z_2\,\mathrm{d}\bar{v}\,\mathrm{d}v\ \delta^2|z_2|^2\, \mathrm{e}^{-|z_2|^2\mathbf{v}^*S\mathbf{v}}\, \frac{\operatorname{Tr} B^{-1} + |z_2|^2\mathbf{v}^*M\mathbf{v}}{\operatorname{Tr} C_{++}^{-1}} \\
&\times \left(\mathrm{e}^{-|z_2|^2\mathbf{v}^*X\mathbf{v}} \det(1 + S^{-1}X) - 1 \right) \\
=& \delta^2 \int_{|v|<1} \frac{\mathrm{d}\bar{v}\,\mathrm{d}v}{2\pi} \left[\frac{\operatorname{Tr} B^{-1}}{\operatorname{Tr} C_{++}^{-1}} \left(\frac{\det(S+X)}{(\mathbf{v}^*(S+X)\mathbf{v})^2} - \frac{\det S}{(\mathbf{v}^*S\mathbf{v})^2} \right) \right. \\
&\left. + \frac{2\mathbf{v}^*M\mathbf{v}}{\operatorname{Tr} C_{++}^{-1}} \left(\frac{\det(S+X)}{(\mathbf{v}^*(S+X)\mathbf{v})^3} - \frac{\det S}{(\mathbf{v}^*S\mathbf{v})^3} \right) \right] \\
=& \delta^2 \int_{|v|<1} \frac{\mathrm{d}\bar{v}\,\mathrm{d}v}{2\pi} \left(\frac{\det(S+X)}{(\mathbf{v}^*(S+X)\mathbf{v})^2} - \frac{\det S}{(\mathbf{v}^*S\mathbf{v})^2} \right) + O(|\Lambda|^{-1}),
\end{aligned}$$

where we applied Lemma 3.4.2 below and

$$\frac{\operatorname{Tr} B^{-1}}{\operatorname{Tr} C_{++}^{-1}} = 1 - \frac{\operatorname{Tr} S_{++}^{-1} M}{\operatorname{Tr} C_{++}^{-1}} = 1 + O(|\Lambda|^{-1}). \tag{3.4.4}$$

Applying Lemma 3.4.2 again, together with Eq. (3.4.3) we get

$$\begin{aligned}
&\frac{\det(S+X)}{(\mathbf{v}^*(S+X)\mathbf{v})^2} - \frac{\det S}{(\mathbf{v}^*S\mathbf{v})^2} = -\frac{(\mathbf{v}^*X\mathbf{v})\det S}{(\mathbf{v}^*(S+X)\mathbf{v})^2(\mathbf{v}^*S\mathbf{v})} \left[2 + \frac{(\mathbf{v}^*X\mathbf{v})}{(\mathbf{v}^*S\mathbf{v})} \right] \\
&+ \frac{X_{11}S_{22} + X_{22}S_{11} + X_{11}X_{22}}{(\mathbf{v}^*(S+X)\mathbf{v})^2} = \mathcal{O}\Big((1 + \frac{1}{\lambda^2}) \left[1 + \delta^2(1 + \frac{1}{\lambda^2}) \right] \Big).
\end{aligned}$$

Analysis of $R(h)$. Note that we can set $\varepsilon = 0$ in $R(h)$. By the notations in Eq. (3.4.2), we can write

$$R(h) = -\tfrac{1}{\pi^2}\int_{\mathbb{R}^+\times(0,2\pi)^2} \mathrm{d}r_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2[\,\mathrm{d}\hat{\Phi}^*\,\mathrm{d}\hat{\Phi}]\lambda r_2\delta^2\left[\mathrm{e}^{-\lambda(1-\delta^4)r_2^2} + \delta^2\,\mathrm{e}^{-\lambda(1-\delta^4)\delta^2 r_2^2}\right]$$
$$\times\left((1+\delta^2)r_2^2 + \sum_j |w_j|^2\right)\,\mathrm{e}^{-\hat{\Phi}^* B\hat{\Phi}}\,\mathrm{e}^{ir_2(\bar{w}D^T v_\theta + \bar{v}_\theta D w)}\,\mathrm{e}^{-r_2^2\bar{v}_\theta A_0 v_\theta},$$

where $v_\theta^t = (\,\mathrm{e}^{i\theta_1}\delta,\,\mathrm{e}^{i\theta_2})$. Integrating over the Grassmann variables, we obtain

$$R(h) = -\tfrac{1}{\pi^2}\int_{\mathbb{R}^+\times(0,2\pi)^2} \mathrm{d}r_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\,\mathrm{d}\bar{w}\,\mathrm{d}w\lambda r_2\delta^2\left[\mathrm{e}^{-\lambda(1-\delta^4)r_2^2} + \delta^2\,\mathrm{e}^{-\lambda(1-\delta^4)\delta^2 r_2^2}\right]$$
$$\times\left((1+\delta^2)r_2^2 + \sum_j |w_j|^2\right)\det\left[\tfrac{B}{2\pi}\right]\,\mathrm{e}^{-\bar{w}Bw}\,\mathrm{e}^{ir_2(\bar{w}D^T v_\theta + \bar{v}_\theta D w)}\,\mathrm{e}^{-r_2^2\bar{v}_\theta A_0 v_\theta},$$

Define $S_0 = A_0 + DB^{-1}D^T$. Integrating over w and r_2, we obtain

$$\begin{aligned}
\frac{R(h)}{\operatorname{Tr} C_{++}^{-1}} &= -\tfrac{1}{\pi^2}\tfrac{1}{\operatorname{Tr} C_{++}^{-1}}\int_{\mathbb{R}^+\times(0,2\pi)^2} \mathrm{d}r_2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\lambda r_2\delta^2\left[\mathrm{e}^{-\lambda(1-\delta^4)r_2^2} + \delta^2\,\mathrm{e}^{-\lambda(1-\delta^4)\delta^2 r_2^2}\right]\\
&\quad\times\left(\bar{v}_\theta M v_\theta r_2^2 + \operatorname{Tr} B^{-1}\right)\,\mathrm{e}^{-r_2^2\bar{v}_\theta S_0 v_\theta}\\
&= -\tfrac{1}{\pi^2}\tfrac{\operatorname{Tr} B^{-1}}{\operatorname{Tr} C_{++}^{-1}}\int_{(0,2\pi)^2}\mathrm{d}\theta_1\,\mathrm{d}\theta_2\tfrac{\lambda\delta^2}{2}\left[\tfrac{1}{\lambda(1-\delta^4)+\bar{v}_\theta S_0 v_\theta} + \tfrac{\delta^2}{\lambda\delta^2(1-\delta^4)+\bar{v}_\theta S_0 v_\theta}\right]\\
&\quad-\tfrac{1}{\pi^2}\int_{(0,2\pi)^2}\mathrm{d}\theta_1\,\mathrm{d}\theta_2\tfrac{\lambda\delta^2}{2}\tfrac{\bar{v}_\theta M v_\theta}{\operatorname{Tr} C_{++}^{-1}}\left[\tfrac{1}{(\lambda(1-\delta^4)+\bar{v}_\theta S_0 v_\theta)^2} + \tfrac{\delta^2}{(\lambda\delta^2(1-\delta^4)+\bar{v}_\theta S_0 v_\theta)^2}\right].
\end{aligned}$$

Similar to the estimates above, we insert absolute values and use Lemma 3.4.2 and Eq. (3.4.4) to bound the first term by $\delta^2(1+\mathcal{O}(|\Lambda|^{-1}))$ and the second one by $\delta^2\mathcal{O}(\lambda^{-1}|\Lambda|^{-1})$. □

Lemma 3.4.2. *Let $\eta > 0$ and $\mu_\lambda = \frac{\lambda\eta}{\lambda+4d\,\mathrm{e}^\eta}$. Let B, M, C_{++} and S_{++} be the matrices in the proof above. Set $0 < \delta \leqslant \frac{1}{2}$. Then*

1. $|B_{ij}^{-1}| \leqslant \frac{2}{\lambda}\,\mathrm{e}^{-\mu_\lambda|i-j|}$ *and* $\operatorname{Re}(f^*B^{-1}f) \geqslant \frac{\lambda}{\lambda^2+(4d)^2}\|f\|^2\ \forall f \in \mathbb{C}^{\Lambda\setminus\{1,2\}}$
2. $|\operatorname{Tr} C_{++}^{-1}| \geqslant \frac{|\Lambda|\lambda}{K(\lambda+1)^2}$.
3. $\operatorname{Re}(f^*S_{++}f) \geqslant \frac{\lambda}{2}\|f\|^2\ \forall f \in \mathbb{C}^{\Lambda\setminus\{1,2\}}$. *Moreover* $|(S_{++})_{jk}| \leqslant K(\lambda + \frac{1}{\lambda})$ *for all* $j,k = 1,2$.

Proof. (*i*) We have $B = i(-\Delta_{|\Lambda\setminus\{1,2\}} - (E+i\lambda))$. The upper bound follows by Combes-Thomas [AW15][Sect 10.3]. For the lower bound note that

$$f^*\operatorname{Re} B^{-1}f = \lambda\|B^{-1}f\|^2$$

Moreover $\|Bg\|^2 = \lambda^2\|g\|^2 + g^*(E + \Delta_{|\Lambda\setminus\{1,2\}})g \leqslant (\lambda^2 + (4d)^2)\|g\|^2$. The result follows setting $g = B^{-1}f$.
(ii) As in (i) above $f^*\mathrm{Re}\, C_{++}^{-1}f \geqslant \lambda(1-\delta^2)\|C_{++}^{-1}f\|^2$. We can write $C_{++} = \lambda - \lambda\delta^2\mathbb{1}_{1,2} - i(E+\Delta)$, where $\mathbb{1}_{1,2}$ is the diagonal matrix $(\mathbb{1}_{1,2})_{ij} = \delta_{ij}[\delta_{ji_1} + \delta_{ji_2}]$. Hence

$$C_{++}^*C_{++} = (\lambda - \lambda\delta^2\mathbb{1}_{1,2})^2 + (E+\Delta)^2 + i\lambda\delta^2[\mathbb{1}_{1,2}, \Delta].$$

The result follows by inserting this decomposition in $\|C_{++}g\|^2$ for $g = C_{++}^{-1}f$.
(iii) Using (i) we have

$$\mathrm{Re}\, f^*Sf = \lambda(1-\delta^2)\|f\|^2 + \mathrm{Re}\, f^*DB^{-1}D^tf \geqslant \lambda(1-\delta^2)\|f\|^2.$$

The upper bound follows from (i) too. $\square$

3.A. Super analysis

We collect here only a minimal set of definitions for our purpose. For details, see [Ber87, Var04, Weg16, DeW92].

3.A.1. Basic definitions

Let $q \in \mathbb{N}$. Let $\mathcal{A} = \mathcal{A}_q = \mathcal{A}[\chi_1, \ldots, \chi_q]$ be the Grassmann algebra over $\mathbb{C}$ generated by $\chi_1, \ldots, \chi_q$, i.e.

$$\mathcal{A} = \oplus_{i=0}^q V^i,$$

where V is the complex vector space with basis $(\chi_1, \ldots, \chi_q)$, $V^0 = \mathbb{C}$ and $V^j = V^{j-1} \wedge V$ for $j \geqslant 2$ with the anticommutative product $\wedge$

$$\chi_i \wedge \chi_j = -\chi_j \wedge \chi_i.$$

As a short hand notation, we write in the following $\chi_i\chi_j = \chi_i \wedge \chi_j$ and for $I \subset \{1, \ldots, q\}$ denote $\chi^I = \prod_{j\in I}\chi_j$ the *ordered* product of the χ_j with $j \in I$. Then each $a \in \mathcal{A}$ has the form

$$a = \sum_{I\in\mathcal{P}(q)} a_I\chi^I, \tag{3.A.1}$$

where $\mathcal{P}(q)$ is the power set of $\{1, \ldots, q\}$ and $a_I \in \mathbb{C}$ for all $I \in \mathcal{P}(q)$. We distinguish even and odd elements $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$, where

$$\mathcal{A}^0 = \oplus_{i=0}^{\lfloor q/2\rfloor}V^{2i}, \quad \mathcal{A}^1 = \oplus_{i=0}^{\lfloor q/2\rfloor}V^{2i+1}.$$

The parity operator p for homogeneous (i.e. purely even, resp. purely odd) elements is defined by

$$p(a) = \begin{cases} 0 & \text{if } a \in \mathcal{A}^0, \\ 1 & \text{if } a \in \mathcal{A}^1. \end{cases}$$

Note that even elements commute with all elements in $\mathcal{A}$ and two odd elements anticommute. For an even element $a \in \mathcal{A}^0$, we write $a = b_a + n_a$, where n_a is the nilpotent part and $b_a = a_\varnothing \in \mathbb{C}$ is called the body of a.

Let $U \subset \mathbb{R}$ open. For any function $f \in C^\infty(U)$, we define

$$\begin{aligned} f : \mathcal{A}^0 &\to \mathcal{A}^0 \\ a &\mapsto f(a) = f(b_a + n_a) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b_a) n_a^k \end{aligned} \tag{3.A.2}$$

via its Taylor expansion. Note that the sum above is always finite.

3.A.2. Differentiation

Let $I' \subset I$. We define the signs $\sigma_l(I, I')$ and $\sigma_r(I, I')$ via

$$\chi^I = \sigma_l(I, I')\chi^{I'}\chi^{I\setminus I'} \quad \chi^I = \sigma_r(I, I')\chi^{I\setminus I'}\chi^{I'}.$$

Then the left- resp. right-derivative of an element a of the form (3.A.1) is defined as

$$\begin{aligned} \frac{\overrightarrow{\partial}}{\partial \chi_j} a &:= \sum_{I \in \mathcal{P}(q): j \in I} a_I \ \sigma_l(I, \{j\}) \ \chi^{I\setminus\{j\}}, \\ a \frac{\overleftarrow{\partial}}{\partial \chi_j} &:= \sum_{I \in \mathcal{P}(q): j \in I} a_I \ \sigma_r(I, \{j\}) \ \chi^{I\setminus\{j\}}. \end{aligned}$$

3.A.3. Integration

The integration over a subset of (odd) generators $\chi_j, j \in I$ is defined by

$$\int \mathrm{d}\chi^I a := \left(\frac{\overrightarrow{\partial}}{\partial \chi} \right)^I a = \sum_{J \in \mathcal{P}(q): I \subset J} a_J \ \sigma_l(J, I) \ \chi^{J\setminus I},$$

where a has the form (3.A.1) and $\mathrm{d}\chi^I = \prod_{j \in I} \mathrm{d}\chi_j$ is again a ordered product. Note that the one forms $\mathrm{d}\chi_i$ are anticommutative objects and e.g. $\int \mathrm{d}\chi_i \, \mathrm{d}\chi_j \chi_i \chi_j = -\int \mathrm{d}\chi_i \, \mathrm{d}\chi_j \chi_j \chi_i = -1$.

Gaussian integral. There is a useful Gaussian integral formula for Grassmann variables. We rename our basis as $(\chi_1, \ldots, \chi_q, \bar{\chi}_1, \ldots \bar{\chi}_q)$. Then for $M \in \mathbb{C}^{q \times q}$

$$\int \mathrm{d}\bar{\chi} \, \mathrm{d}\chi \, \mathrm{e}^{-\sum_{i,j} \bar{\chi}_i M_{ij} \chi_j} = \det M, \tag{3.A.3}$$

where $\mathrm{d}\bar{\chi} \, \mathrm{d}\chi = \prod_{j=1}^q \mathrm{d}\bar{\chi}_j \, \mathrm{d}\chi_j$. Combining this with complex Gaussian integral formulas, we obtain the following result.

Theorem 3.A.1 (Supersymmetric integral representation). *Let $A_1, A_2 \in \mathbb{C}^{n\times n}$ with $\operatorname{Re} A_1 > 0$. Let $\Phi = (z,\chi)^t \in \mathbb{C}^n \times V^n$ be a supervector and $\Phi^* = (\bar{z}, \bar{\chi}) \in \mathbb{C}^n \times V^n$ its transpose. With the notations $[\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi] = (2\pi)^{-n}\,\mathrm{d}\bar{z}\,\mathrm{d}z\,\mathrm{d}\bar{\chi}\,\mathrm{d}\chi$ and $\Phi^* A\Phi = \sum_{j,k=1}^n \bar{z}_j (A_1)_{jk} z_k + \bar{\chi}_j (A_2)_{jk}\chi_k$ for a block matrix $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ (a supermatrix with odd parts 0) we can write*

$$\frac{\det A_2}{\det A_1} = \int [\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\,\mathrm{e}^{-\Phi^* A\Phi}$$

and

$$(A_1^{-1})_{jk} = \int [\,\mathrm{d}\Phi^*\,\mathrm{d}\Phi]\ \bar{z}_k z_j\,\mathrm{e}^{-\Phi^*\hat{A}_1\Phi},$$

where $\hat{A}_1 = \begin{pmatrix} A_1 & 0 \\ 0 & A_1 \end{pmatrix}$.

Proof. Combine Eq. (3.A.3) with the complex Gaussian integral formulas

$$\det A_1 = \frac{1}{(2\pi)^n}\int \mathrm{d}\bar{z}\,\mathrm{d}z\,\mathrm{e}^{-\bar{z}A_1 z}, \quad (A_1^{-1})_{jk} = \frac{\det A_1}{(2\pi)^n}\int \mathrm{d}\bar{z}\,\mathrm{d}z\ \bar{z}_k z_j\,\mathrm{e}^{-\bar{z}A_1 z}.$$

Note that while Eq. (3.A.3) holds for all matrices $A \in \mathbb{C}^{n\times n}$, we need the additional condition $\operatorname{Re} A > 0$ for the complex ones to ensure that the complex integral is finite. □

3.A.4. Grassmann algebra functions and change of variables

In this section, we denote the body of an even element a by $b(a)$ instead of b_a.

Definition 3.A.2. Let $U \subset \mathbb{R}^p$ open. The *algebra of smooth $\mathcal{A}[\chi_1,\ldots,\chi_q]$-valued functions* on a domain U is defined by

$$\mathcal{A}_{p,q}(U) := \left\{ f = f(x,\chi) = \sum_{I\in\mathcal{P}(q)} f_I(x)\chi^I : f_I \in C^\infty(U) \right\}.$$

We call $y_i(x,\chi)$, $\eta_j(x,\chi)$, for $i = 1,\ldots p$, $j = 1,\ldots,q$ *generators* of $\mathcal{A}_{p,q}(U)$ if $p(y_i) = 0$, $p(\eta_j) = 1$ and

1. $\{(b(y_1(x,0)),\ldots,b(y_p(x,0))), x \in U\}$ is a domain in $\mathbb{R}^p$,
2. we can write all $f \in \mathcal{A}_{p,q}(U)$ as $f = \sum_I f_I(y)\eta^I$.

Note that (x,χ) are generators for $\mathcal{A}_{p,q}(U)$.

A change of variables is then a parity preserving transformation between systems of generators of $\mathcal{A}_{p,q}(U)$. A practical change of variable formula for super integrals is currently only known for functions with compact support, i.e. functions $f \in \mathcal{A}_{p,q}(U)$ such that $f_I \in C_c^\infty(U)$ for all $I \in \mathcal{P}(q)$.

Theorem 3.A.3. *Let $U \subset \mathbb{R}^p$ open, x, χ and $y(x,\chi)$, $\eta(x,\chi)$ two sets of generators of $\mathcal{A}_{p,q}(U)$. Denote the isomorphism between the generators by*

$$\psi : (x,\chi) \mapsto (y(x,\chi), \eta(x,\chi))$$

and $V = b(\psi(U)) = \{(b(y_1(x,0)), \ldots, b(y_p(x,0))), x \in U\}$. Then for all $f \in \mathcal{A}_{p,q}(V)$ with compact support, we have

$$\int_V \mathrm{d}y\, \mathrm{d}\eta f(y,\eta) = \int_U \mathrm{d}x\, \mathrm{d}\chi f \circ \psi(x,\chi) \mathrm{Sdet}\,(J\psi),$$

where $\mathrm{Sdet}\,(J\psi)$ *is called the Berezinian defined by*

$$J\psi = \begin{pmatrix} \frac{\partial y}{\partial x} & y\frac{\overleftarrow{\partial}}{\partial \chi} \\ \frac{\partial \eta}{\partial x} & \eta\frac{\overleftarrow{\partial}}{\partial \chi} \end{pmatrix}, \qquad \mathrm{Sdet}\begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} = \det(a - \sigma b^{-1}\rho)\det b^{-1}.$$

Integration over even elements x and y means integration over the body $b(x)$ and $b(y)$ in the corresponding regions U and V.

Proof. See [Ber87, Theorem 2.1] or [Var04, Theorem 4.6.1]. □

Remark. Applying an isomorphism ψ that changes only the odd elements, Theorem 3.A.3 holds also for smooth, integrable functions that are not necessarily compactly supported. Changing also the even elements for a non compactly supported function, boundary integrals can occur.

Bibliography

[AM93] M. Aizenman and S. Molchanov. Localization at large disorder and at extreme engeries - an elementary derivation. *Communications in mathematical physics*, 1993.

[And58] P. W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109:1492–1505, Mar 1958.

[AR95] A. Abdesselam and V. Rivasseau. Trees, forests and jungles: a botanical garden for cluster expansions. In *Constructive physics (Palaiseau, 1994)*, volume 446 of *Lecture Notes in Phys.*, pages 7–36. Springer, Berlin, 1995.

[AS09] A. Altland and B. D. Simons. *Condensed Matter Field Theory.* Cambridge University Press, 2009.

[AW15] M. Aizenman and S. Warzel. *Random operators*, volume 168 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015. Disorder effects on quantum spectra and dynamics.

[BE16] Z. Bao and L. Erdős. Delocalization for a class of random block band matrices. *Probability Theory and Related Fields*, 167(3-4):673–776, January 2016.

[Ber66] F. A. Berezin. *The method of second quantization.* Pure and applied physics; 24. Acad. Press, New York u.a., 1966.

[Ber87] F. A. Berezin. *Introduction to superanalysis.* Mathematical physics and applied mathematics; 9. Reidel, Dordrecht u.a., 1987.

[BGP14] F. Benaych-Georges and S. Péché. Largest eigenvalues and eigenvectors of band or sparse random matrices. *Electron. Commun. Probab.*, 19:no. 4, 1–9, 2014.

[BL76] H. J. Brascamp and E. H. Lieb. On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation. *J. Functional Analysis*, 22(4):366–389, 1976.

[Bou18] Paul Bourgade. Random band matrices. arXiv:1807.03031, 2018.

[Bov90] A. Bovier. The density of states in the Anderson model at weak disorder: A renormalization group analysis of the hierarchical model. *J. Stat. Phys.*, 59(3-4):745–779, 1990.

[Bry86] D. C. Brydges. A short course on cluster expansions. Les Houche, Session XLIII, 1986.

[Car82] R. Carmona. Exponential localization in one dimensional disordered systems. *Duke Mathematical Journal*, 49(1), March 1982.

[CCGI93] G. Casati, B. V. Chirikov, I. Guarneri, and F. M. Izrailev. Band-random-matrix model for quantum localization in conservative systems. *Phys. Rev. E*, 48:R1613–R1616, Sep 1993.

[CFGK87] F. Constantinescu, G. Felder, K. Gawedzki, and A. Kupiainen. Analyticity of density of states in a gauge-invariant model for disordered electronic systems. *Journal of Statistical Physics*, 48(3-4):365–391, 1987.

[CK86] M. Campanino and A. Klein. A supersymmetric transfer matrix and differentiability of the density of states in the one-dimensional anderson model. *Communications in Mathematical Physics*, 104(2):227–241, 1986.

[CL90] R. Carmona and J. Lacroix. *Spectral theory of random Schrödinger operators*. Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1990.

[CMI90] G. Casati, L. Molinari, and F. Izrailev. Scaling properties of band random matrices. *Phys. Rev. Lett.*, 64:1851–1854, Apr 1990.

[DeW92] B. S. DeWitt. *Supermanifolds*. Cambridge monographs on mathematical physics. Cambridge Univ. Press, Cambridge u.a., 2. ed. edition, 1992.

[Dis04] M. Disertori. Density of states for GUE through supersymmetric approach. *Rev. Math. Phys.*, 16(09):1191–1225, 2004.

[Dis11] M. Disertori. Modèles de matrices aléatoires et supersymétrie. *Habilitation à diriger des recherches*, 2011.

[DL17] M. Disertori and M. Lager. Density of States for Random Band Matrices in Two Dimensions. *Annales Henri Poincaré*, 18(7):2367–2413, Jul 2017.

[DL20] M. Disertori and M. Lager. Supersymmetric polar coordinates with applications to the lloyd model. *Mathematical Physics, Analysis and Geometry*, 23(1), January 2020.

[DLS21] M. Disertori, M. Lohmann, and S. Sodin. The density of states of 1d random band matrices via a supersymmetric transfer operator. *Journal of spectral theory*, 2021.

[DPS02] M. Disertori, H. Pinson, and T. Spencer. Density of states for random band matrices. *Comm. Math. Phys.*, 232(1):83–124, 2002.

[Efe83] K. Efetov. Supersymmetry and theory of disordered metals. *Adv. in Phys.*, 32(1):53–127, 1983.

[Efe99] K Efetov. *Supersymmetry in disorder and chaos.* Cambridge Univ. Press, New York, NY u.a., 1. paperback ed. edition, 1999.

[EHS07] P Exner, M. Helm, and P. Stollmann. Localization on a qunatum graph with a random potential on the edges. *Reviews in Mathematical Physics*, 19(09):923–939, October 2007.

[FM91] Y. V. Fyodorov and A. D. Mirlin. Scaling properties of localization in random band matrices: a σ-model approach. *Phys. Rev. Lett.*, 67(18):2405–2409, 1991.

[Fre20] L. Fresta. *Supersymmetry and Renormalization inthe Theory of Random Schrödinger Operators.* PhD thesis, Universität Zürich, 2020.

[FS83] J. Fröhlich and T. Spencer. Absence of diffusion in the anderson tight binding model for large disorder or low energy. *Communications in mathematical physics*, 1983.

[GMP77] I. Ya. Gol'dshtein, S. A. Molchanov, and L. A. Pastur. A pure point spectrum of the stochastic one-dimensional schrödinger operator. *Functional Analysis and Its Applications*, 11(1):1–8, 1977.

[Gra44] H. G. Grassmann. Lineare ausdehnungslehre. *Wigand, Leipzig*, 1844.

[HG19] F. Haake and S. Gnutzmann. *Quantum Signatures of Chaos.* Springer complexity. Springer International Publishing AG, Cham, 2019.

[KK08] W. Kirsch and F. Klopp. An invitation to Random Schrödinger operators. *Panorama Synthèse*, pages 1–119, 2008.

[Kle98] A. Klein. Extended states in the anderson model on the bethe lattice. *Advances in mathematics (New York. 1965)*, 1998.

[KM82] W. Kirsch and F. Martinelli. On the spectrum of schrödinger operators with a random potential. *Communications in mathematical physics*, 1982.

[KMP86] A. Klein, F. Martinelli, and J. F. Perez. A rigorous replica trick approach to anderson localization in one dimension. *Communications in Mathematical Physics*, 106(4):623–633, 1986.

[KS80] H. Kunz and B. Souillard. Sur le spectre des opérateurs aux différences finies aléatoires. *Communications in mathematical physics*, 1980.

[Llo69] P. Lloyd. Exactly solvable model of electronic states in a three-dimensional disordered Hamiltonian: non-existence of localized states. *Journal of Physics C: Solid State Physics*, 1969.

[LSZ08] P. Littelmann, H.-J. Sommers, and M. R. Zirnbauer. Superbosonization of invariant random matrix ensembles. *Comm. Math. Phys.*, 283(2):343–395, 2008.

[Meh04] M. L. Mehta. *Random matrices.* Pure and applied mathematics ; v. 142. Elsevier/Academic Press, Amsterdam ;San Diego, CA, 3rd ed edition, 2004.

[Mir00] A. D. Mirlin. Statistics of energy levels and eigenfunctions in disordered systems. *Phys. Rep.*, 326(5-6):259–382, 2000.

[MPK92] S. A. Molchanov, L. A. Pastur, and A. M. Khorunzhii. Limiting eigenvalue distribution for band random matrices. *Theoretical and Mathematical Physics*, 90(2):108–118, 1992.

[Pas80] L. A Pastur. Spectral properties of disordered systems in the one-body approximation. *Communications in mathematical physics*, 1980.

[Pch15] V. Pchelin. Poisson statistics for random deformed band matrices with power law band width, 2015.

[PF92] L. A Pastur and A. L Figotin. *Spectra of Random and almost periodic operators.* Die Grundlehren der mathematischen Wissenschaften ; 297. Springer, Berlin u.a., 1992.

[PSSS17] R. Peled, J. Schenker, M. Shamis, and S. Sodin. On the wegner orbital model. *International Mathematics Research Notices*, 2019(4):1030–1058, Jul 2017.

[Riv91] V. Rivasseau. *From perturbative to constructive renormalization.* Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1991.

[RS72] M. Reed and B. Simon. *Methods of modern mathematical physics. 1, Functional analysis.* Acad. Press, New York u.a., 1972.

[Sal99] M. Salmhofer. *Renormalization.* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1999. An introduction.

[Sch09] J. Schenker. Eigenvector Localization for Random Band Matrices with Power Law Band Width. *Communications in Mathematical Physics*, 290(3):1065–1097, 2009.

[Sha13] M. Shamis. Density of states for Gaussian unitary ensemble, Gaussian orthogonal ensemble, and interpolating ensembles through supersymmetric approach. *J. Math. Phys.*, 54(11), 2013.

[Shc14] T. Shcherbina. On the Second Mixed Moment of the Characteristic Polynomials of 1D Band Matrices. *Comm. Math. Phys.*, 328(1):45–82, 2014.

[Sim83] B. Simon. Equality of the density of states in a wide class of tight-binding Lorentzian random models. *Physical Review B*, 1983.

[Sod10] S. Sodin. The spectral edge of some random band matrices. *Ann. of Math. (2)*, 172(3):2223–2251, 2010.

[SS17] M. Shcherbina and T. Shcherbina. Characteristic polynomials for 1D random band matrices from the localization side. *Comm. Math. Phys.*, 351(3):1009–1044, 2017.

[Sto01] P. Stollmann. *Caught by disorder: bound states in random media.* Progress in mathematical physics ; 20. Birkhäuser, Boston u.a., 2001.

[Sto11] G. Stolz. An introduction to the mathematics of anderson localization. arXiv:1104.2317, 2011.

[SW80] L. Schäfer and F. Wegner. Disordered system with n orbitals per site: Lagrange formulation, hyperbolic symmetry, and Goldstone modes. *Z. Phys. B*, 38:113–126, 1980.

[Var04] V. S. Varadarajan. *Supersymmetry for mathematicians: an introduction*, volume 11 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2004.

[Wan01] W. Wang. Localization and universality of poisson statistics for the multidimensional anderson model at weak disorder. *Inventiones mathematicae*, 146(2):365–398, Nov 2001.

[Weg79] F. Wegner. The mobility edge problem: continuous symmetry and a conjecture. *Z. Phys. B*, 35:207–210, 1979.

[Weg16] F Wegner. *Supermathematics and its applications in statistical physics: Grassmann variables and the method of supersymmetry.* Lecture notes in physics ; 920 ; volume 920. Springer, Heidelberg ; New York ; Dordrecht ; London, 2016.

[Wig55] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *The Annals of Mathematics*, 62(3):548, November 1955.

[YYY21] F. Yang, H. Yau, and J. Yin. Delocalization and quantum diffusion of random band matrices in high dimensions i: Self-energy renormalization. arXiv:2104.12048, 2021.