Black Hole Microstate Counting in Four-Dimensional $\mathcal{N} = 4$ String Compactifications

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Summary

In this thesis we present research on the spectrum of supersymmetric Bogomol'nyi–Prasad–Sommerfield (BPS) states in a specific superstring compactification with $\mathcal{N} = 4$-extended supersymmetry in four dimensions, known as the $\mathbb{Z}_2$ Chaudhuri-Hockney-Lykken (CHL) model. Specifically, partition functions for quarter-BPS states in various charge sectors are derived and tested by combining worldsheet aspects, string dualities and modularity properties and the results are discussed in the light of black hole microstate counting.

After a brief and non-technical introduction to superstring theory and the problem of explaining the entropy of black holes, we introduce the specific string compactification with which we will work. Representations of the $\mathcal{N} = 4$ superalgebra are reviewed in view of BPS conditions before the central objects of our investigation, partition functions for quarter-BPS dyons in suitable electric-magnetic charge sectors of the theory, are introduced. These functions are identified in the chiral heterotic genus two orbifold partition function appropriate for the $\mathbb{Z}_2$ CHL model. We discuss parallels with electric half-BPS partition functions and test whether the derived quarter-BPS partition functions satisfy all physical requirements from charge quantization, wall-crossing and S-duality, which can be answered by investigating the transformation properties and pole structure of the corresponding Siegel modular forms. An alternative determination of the partition functions by reverse engineering the constraints is also briefly discussed, as well as the compatibility of our results with those derived by other means in the physics literature.

As the quarter-BPS states correspond to extremal dyonic black hole solutions in the low-energy effective supergravity theory, their microscopic degeneracy and hence statistical entropy can be compared to a macroscopic black hole entropy computed using the entropy function formalism in the two-derivative supergravity approximation plus model-specific higher-derivative corrections to the latter. This connection is explored in particular by comparing large-charge expansions of the entropy.

In the last part of the thesis we also compare our findings to closely related conjectures in enumerative geometry, in particular Donaldson-Thomas partition functions for CHL Calabi-Yau threefold geometries that correspond to the $\mathcal{N} = 4$ compactification space $(\mathrm{K3} \times T^2)/\mathbb{Z}_2$ of the dual type IIA string perspective.

List of Publications

This thesis is based on the following publication:

- F. Fischbach, A. Klemm and C. Nega,
  Lost chapters in CHL black holes: untwisted quarter-BPS dyons in the $\mathbb{Z}_2$ model,
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Especially, section 5.2 and chapters 4, 6, 7 and 9 as well as the appendix A of this thesis are, up to minor improvements or corrections, replications of sections 3, 4, 5 and 7 as well as the appendix A of the above journal publication. The mentioned portions of the journal publication are solely written by the author of the present thesis (with all sources used therein indicated).

Further publications and preprints co-authored by the author of the present thesis are:

- K. Bönisch, C. Duhr, F. Fischbach, A. Klemm and C. Nega,
  Feynman Integrals in Dimensional Regularization and Extensions of Calabi-Yau Motives,
  arXiv: `2108.05310 [hep-th]` (electronic preprint)

- K. Bönisch, F. Fischbach, A. Klemm, C. Nega and R. Safari,
  Analytic structure of all loop banana integrals,
  JHEP **05** (2021) 066
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- F. Fischbach, A. Klemm and C. Nega,
  WKB Method and Quantum Periods beyond Genus One,
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Contents

CHAPTER 1

Introduction

This doctoral thesis presents recent research in superstring theory, in particular, it studies the spectrum of supersymmetric states in a specific model that leads to $\mathcal{N} = 4$-extended supersymmetry in the effective four-dimensional spacetime description. One physical motivation to study these states is that they are supposed to have complementary descriptions, they can be regarded as microscopic configurations of spatially extended objects in a higher-dimensional spacetime with small extra-dimensions, or they can be regarded as macroscopic black hole configurations in a four-dimensional theory of (quantum) gravity. Understanding their spectrum in detail hence opens up a possibility to study the thermodynamics of black holes from a microscopic statistical point of view. Another physical motivation, indeed of a more theoretical nature, is to assess our understanding of string theory itself, more specifically the mathematics of compactifications and the dualities between different formulations of a conjecturally unified theory. In practice, a major part of this thesis deals with counting the relevant states, that is, finding the adequate partition functions in the microscopic description by making heavy use of the symmetries and dualities of the setup. For a certain part of the relevant spectrum the partition function was already known for some years [1], but only very recently conjectures were made about partition functions for the remaining parts of the spectrum [2, 3]. Our contribution to the counting problem at hand consists in providing another physical derivation of the desired partition functions, which follows ideas of [4–6] and is independent from the one recently given in the literature, and to make extensive consistency checks besides discussing the implications for the black hole entropy.

After this brief synopsis of the physical problem addressed in this thesis, we can now take a step back and provide some background information about string theory and black holes, and introduce more advanced concepts that will be needed in the later chapters. We do not intend to be comprehensive, nor historical, but try to stay brief and non-technical where possible. Unless stated otherwise, our exposition is based on [7–10] and [11–14], and some more specific entry points to the literature are given where adequate. Mild acquaintance with general relativity, quantum field theory and supersymmetry will be assumed. The ideal reader, a beginning researcher in string theory, should at the end of this introduction know which bits of string theory to study in order to be able to tackle the further chapters of this thesis.

Motivation

Theoretical physics is about mathematically describing and predicting processes in nature. Two major and very successful developments in theoretical physics in the twentieth century are the theory of general relativity (GR) and quantum field theory (QFT).

General relativity is considered to describe the physics at large length and time scales and as such can explain, for instance, the gravitational lensing caused by large masses like clusters of galaxies. It is moreover the mathematical basis for the current standard model of cosmology, the history of an expanding universe. It gives a differential geometrical description of a four-dimensional spacetime via a Lorentzian metric field, which in its interaction with matter and radiation becomes dynamical and generically non-flat (but curved). In the sense of a correspondence principle, GR reproduces Newtonian gravity and special relativity (flat Minkowski space) in suitable limiting cases. Like classical electromagnetism, it is formulated as a field theory whose equations of motions follow from a principle of least action, where the action is the Einstein-Hilbert action. The unification of GR with classical electromagnetism can be achieved by adding a local contribution from the electromagnetic field strength to the Lagrangian density entering the action.

Quantum field theory on the other hand goes beyond classical field theory and is used in describing physical phenomena at small length and time scales in a probabilistic fashion, and apart from its use in condensed matter physics (or for the description of quantum fluctuations in a very early, still tiny universe) it is *the* framework for describing matter and forces on subatomic scales. For the latter application, where spacetime is flat Minkowski space, it can loosely be seen as a unification of special relativity and quantum mechanics. The Standard Model of particle physics in particular is a quantum field theory. It contains quantum fields for matter particles, which are fermions (spin-$1/2$ fields), vector bosons (spin-1 fields) and the Higgs field (spin-0). Vector bosons can be thought of as the force carriers of the strong, weak and electromagnetic force, while the Higgs field gives via its non-zero vacuum expectation value mass to most of the matter fields (to the quarks and leptons, but not to the massless neutrinos) and to some of the vector bosons, especially leaving a single vector boson massless that describes the electromagnetic force on short distances. This model is experimentally well-tested, but leaves also some open ends. Apart from the non-vanishing of neutrino masses in the real world and the possible lack of viable candidate particles for the dark matter that the cosmological standard model postulates — to name at least two prominent possible shortcomings of the model — the Standard Model of particle physics does not describe a gravitational force, especially it does not contain a (hypothetical) graviton field, which is thought of as a spin-2 field. There are arguments that upon addition of such a quantum field the theory becomes perturbatively non-renormalizable and can at best be thought of as an effective description at sufficiently low energies, but not as a fundamental one. Independent of this issue, there are currently no experimental tests that could probe the quantum nature of gravity, as it is much weaker than the forces described by the Standard Model of particle physics. This dichotomy of quantum field theory and gravity is at least expected to eventually break down at the Planck scale $M_{\mathrm{Pl}} = \sqrt{\hbar c/G_N}$, where a more fundamental theory should come into play that reproduces the former two again by a correspondence principle.

One major problem in contemporary theoretical physics, whose solution is believed to necessitate a unification of quantum field theory and general relativity, is to describe the microstructure of black holes and to explain how their macroscopic properties, especially a non-zero entropy, can arise from these. We recall that in the theory of general relativity, (classical) black holes are basically non-trivial solutions to the Einstein field equations that exhibit an event horizon, a surface in spacetime characterized by the remarkable property that nothing, not even light, can escape from the interior due to the extreme gravitational pull. A black hole could for instance arise at the end of a stellar life cycle upon gravitational collapse and, astronomically, there is convincing observational evidence for the existence of black holes. One can formulate laws of black hole mechanics[1] analogous to the laws of thermodynamics, one of which asserts that the area of the horizon cannot decrease, not even when two black holes merge. In this respect the area behaves like the thermodynamic entropy, which by the second law of thermodynamics is non-decreasing in any closed system. By a postulate of Bekenstein and Hawking, the entropy of the black hole equals a quarter of the area of the horizon, $S_{\mathrm{BH}} = A \cdot k_{\mathrm{B}} c^3/(4G_N \hbar)$, involving only standard fundamental constants in the second factor. Note the occurrence of the reduced Planck constant in this formula, suggesting that quantum principles should play a role in the thermodynamics of black holes. That quantum effects play an important role in understanding the latter is also underpinned by ideas of Hawking, who studied quantum fields in a classical black hole background and predicted that black holes emit black-body radiation at a constant temperature. If the posed relation between the entropy and the area of the horizon is correct, and more than a formal analogy, the non-zero macroscopic entropy should arise from an underlying statistical, microscopical ensemble. This is where string theory comes into play.

Elements of String Theory

String theory is a candidate for a theory of quantum gravity. As the name suggests, the basic idea is that the fundamental constituents are not pointlike particles, but rather one-dimensional objects (strings), which can be either open or closed. Similar to a point particle tracing out a one-dimensional wordline in a d-dimensional (Minkowski) spacetime M_d, such strings trace out a two-dimensional surface Σ, commonly called the worldsheet. The perturbative starting point of string theory is to describe the string as a map $X : \Sigma \to M_d$ from the worldsheet to spacetime and to build a two-dimensional QFT of d (coordinate) scalar fields X^μ ($\mu = 1, \ldots, d$). As an action for this QFT one could take the most natural generalization of the action of a free relativistic particle moving in spacetime, which we recall is simply proportional to the proper length of the worldline of the particle. For the string this becomes the area of the embedded worldsheet, and the resulting action is known as the Nambu-Goto action, where the string tension $T = (2\pi\alpha')^{-1}$ (or mass per unit length, with α' being a dimensionful constant) takes over the role of the particle mass. There is an alternative action to start from, namely the Polyakov action, which after using the equations of motion reduces to the Nambu-Goto action (and hence they are said to be classically equivalent). The Polyakov action has the great advantage that it exhibits not only spacetime Poincaré invariance and general coordinate invariance with respect to the (auxiliary) worldsheet metric (world sheet diffeormorphism invariance), but also invariance under local rescalings of the worldsheet metric (Weyl invariance). Exploiting these invariances to gauge-fix the Polyakov action at the cost of introducing Faddeev-Popov ghost fields, one can arrive at a form that is invariant under two-dimensional conformal transformations. Hence one obtains a conformal field theory (CFT). For closed strings the X-fields can further be decomposed with respect to left- and right-moving modes, while for open strings this is not possible. It can be shown that in order to preserve the conformal symmetry at the quantum level, which is part of the gauge symmetry of the theory, the total central charge of the CFT must vanish, implying $d = 26$. This is the so-called critical dimension of the bosonic string theory and we comment on this high dimensionality shortly.

[1] See [15] for a review.

The quantized vibrational modes of a string behave at length scales much greater than the string length $\ell_s = \sqrt{\alpha'}$ like the elementary particles in QFT. This means at low energies the theory can be described by an effective quantum field theory, which amongst its massless fields especially contains a symmetric rank two tensor field, which is identified as a graviton field. While there is also an infinite tower of massive string states in the ultra-violet (with various other tensor structures), their mass is inversely proportional to the string scale ℓ_s and thus these states are too heavy to play a role in the low-energy effective theory.

Very roughly, for computing scattering cross-sections in string theory we have to specify the corresponding in- and outgouing asymptotic states, each containing string excitations and thus possibly representing different particles from the spacetime perspective. Moreover, similar to a sum over loop diagrams in QFT, we have to sum over all possible worldsheet topologies, which can also be interpreted as different possible splittings and joinings of strings during the scattering. The number of loops (or holes of the worldsheet) is formalized by the so-called genus g of the respective surface, and this perturbative loop expansion is controlled by the string coupling constant g_s. This constant, however, is not on the same footing as the string length ℓ_s, as the former is in fact related to the vacuum expectation value of the scalar dilaton field ϕ that is part of the massless string spectrum.

Two important shortcomings of the purely bosonic string theory are that the spectrum does not contain any spacetime fermions and that its ground state (no string excitations) has negative mass square. A possible cure for this is to supersymmetrize the worldsheet CFT by adding appropriate fermionic degrees of freedom. This idea leads to the five superstring theories, which are named the type IIA, type IIB, heterotic SO(32), heterotic $E_8 \times E_8$ and type I superstring theory, all possessing also excitations that behave like spacetime fermions. What they have in common is that they require the strings to propagate in a ten-dimensional spacetime, $d = 10$. Their construction, which will not be reviewed here, is discussed in depth in any string theory textbook, e. g. in [8]. They all possess some (non-zero) amount of supersymmetry on the worldsheet and in spacetime, especially their low-energy effective action is described by the respective eponymous ten-dimensional supergravity theory.

Compactification. Of course, the world we perceive so far — from the largest macroscopic scales down to the smallest scales currently probed by particle colliders — is four-dimensional, but the $d - 4 = 6$ extra-dimensions do not automatically render string theory useless. An idea commonly attributed to Kaluza and Klein and going under the name of compactification is to take the extra-dimensions to be compact, that is, of a finite and small size and to infer an effective four-dimensional description.

A simple example of a compactification is GR with a periodic fifth dimension such that the metric components are functions of the four-dimensional positions only. Here the radius of the internal spacetime circle associated with the fifth dimension gives rise to a scalar field in four dimensions and the mixed components of the metric field give rise to a vector field. The diffeomorphism invariance of the five-dimensional theory implies that the latter behaves like an Abelian gauge field from the four-dimensional point of view. Similarly, investigating the wave-equation of a free massless scalar field in 4+1 dimensions one finds that the quantized momentum along the circular direction effectively generates a mass term in the four-dimensional description. As this mass scales inversely with the squared radius of the circular direction, the massive (non-zero momentum) modes of the scalar field become heavy for a small radius. In other words, at length scales much larger than the internal radius the physics is well-described by a four-dimensional field theory involving the massless fields only.

These ideas carry over to superstring compactifications, although in a more elaborate manner. Spacetime is now described by a product manifold $M_{10} = M_4 \times M_6$, where M_4 is four-dimensional Minkowski space. Strings propagate in a potentially non-flat background metric, which we may think of as a coherent superposition of gravitons. The part of the background metric parametrizing the six compact dimensions of M_6 however is not arbitrary. In order to retain conformal invariance at the quantum level the beta function must vanish, and this in turn can be shown to imply Ricci-flatness. Note that this means that the metric of the compactification manifold thus solves the vacuum Einstein equations. The worldsheet CFT that describes the superstring propagation in ten-dimensional spacetime factors into a piece describing the superstring propagation along the four non-compact directions of M_4 and a piece describing the propagation along M_6. Properties of the latter CFT (factor) translate into properties of the compactification manifold.[2] Moreover, the massless field content of the four-dimensional low-energy effective theory on M_4 depends on the ten-dimensional string theory we started from (say type II or heterotic) and the geometry of M_6. A part of the massless scalar fields again parametrize the metric of the compact space. While massless scalars, so-called moduli, are not observed in nature and there are proposals for mechanisms in string theory to make these fields massive, we will not discuss this aspect in this thesis.

A very important aspect of string compactifications is the relationship between the choice of the six-dimensional compactification manifold and the resulting supersymmetry of the four-dimensional low-energy effective theory. While compactifying on a six-dimensional flat torus T^6 does not reduce the number of supercharges in comparison with the theory in ten-dimensional Minkowski space, a generic manifold M_6 will do so. The reason is that local supersymmetry requires the existence of a covariantly constant spinor. This in turn demands again that the Ricci tensor of the metric on M_6 vanishes and that the holonomy group of M_6 is contained in $\mathrm{SU}(3)$. One can then show that M_6 is automatically also a complex Kähler manifold. A compact Kähler manifold of complex dimension n (real dimension $2n$) and $\mathrm{SU}(n)$ holonomy is also called a Calabi-Yau n-fold. Specifically Calabi-Yau threefolds with holonomy group $\mathrm{SU}(3)$ (but not a proper subgroup thereof) break three-quarters of the supercharges when used as a compactification manifold. A heterotic string theory, which would have $\mathcal{N} = 1$ supersymmetry in ten-dimensional flat space (16 real supercharges), gives minimal $\mathcal{N} = 1$ supersymmetry in four dimensions when compactified on a Calabi-Yau threefold. For comparison, heterotic string theories compactified on a six-torus lead to $\mathcal{N} = 4$-extended supersymmetry in four dimensions (with all 16 supercharges preserved). On the other hand, type IIA and type IIB string compactifications on Calabi-Yau threefolds lead to $\mathcal{N} = 2$-extended supersymmetry in four dimensions (8 out of 32 supercharges preserved).

Such $\mathcal{N} = 1$- or $\mathcal{N} = 2$-theories will however not be considered in this thesis, rather we will only consider specific heterotic compactifications on a six-torus, type II compactifications on a Calabi-Yau two-fold of $\mathrm{SU}(2)$ holonomy (a K3 surface) times a Calabi-Yau one-fold (an elliptic curve, topologically a two-torus), i.e., on $\mathrm{K3} \times T^2$, and a specific orbifold of these. In all cases one obtains four-dimensional $\mathcal{N} = 4$ supersymmetry. For us an orbifold will simply mean the quotient space obtained by identifying the points of a manifold under a discrete (usually finite) group action. There is a closely related concept in CFT, where finite group actions can be used to construct new CFTs out of given ones, simply called orbifold theories.

[2] In fact, a more abstract CFT with no geometric interpretation would also be a possibility for the internal dynamics.

Extended objects and string dualities. In type II (and type I) superstring theories there are also open strings. At each end of an open string and for each spacetime direction one needs to impose a boundary condition, that is, either the endpoint is fixed (Dirichlet condition), or it is free to move but has vanishing momentum flow at its end (Neumann condition). An open string with $p + 1$ Neumann directions (inlcuding time) moves in a $(p + 1)$-dimensional subspace of spacetime, which is localized in the remaining $9 - p$ spatial dimensions. This is interpreted as a string ending on a so-called Dp-brane, which itself is a dynamical object extending in p spatial dimensions. The presence of D-branes in flat spacetime breaks the $\mathrm{SO}(1, 9)$ Lorentz invariance to a subgroup fixing the brane. For type IIA strings p must be even, for type IIB strings p must be odd. It is again possible to write down an action for these dynamical objects, which also takes into account the fact that D-branes have gauge fields living on their worldvolume, but this will not be needed in this thesis. We shall only mention that the brane tension scales with $1/g_s$.

While D-branes couple to the massless form fields in the R-R sector of type II strings, fundamental strings (which we can regard as 1-branes) couple electrically to the massless two-form $B_{\mu\nu}$ from the NS-NS sector.[3] There is also a (non-perturbative) five-dimensional object that couples magnetically to the B-field, the so-called NS5-brane. Its tension scales with $1/g_s^2$. The existence of the various branes is also supported by the existence of suitable brane-like solutions to the corresponding supergravity theories with electric and magnetic charges.[4] Charged branes are either of infinite extend in spacetime or wrap compact submanifolds of spacetime, or they end on another brane.

We mentioned above that there are in principle five superstring theories in ten dimensions. This is, however, not the whole story. There are several dualities between the superstring theories, which means that two seemingly different mathematical descriptions eventually may lead to the same physics. Let us mention some examples, especially those that also feature in the later chapters of this thesis.

A first simple example is T-duality. Given type IIA string theory compactified on a circle of radius R, this is T-dual to type IIB string theory on a circle of radius α'/R, with the roles of momentum and winding modes exchanged. Simultaneously, the boundary conditions for open strings interchange with respect to the circle along which T-duality is performed and Dp-branes transform into D$(p \pm 1)$-branes. In a similar fashion the two heterotic string theories become related when compactified on a circle, that is, the two theories actually possess a single moduli space, dissolving the strict distinction between the two. For compactifications on higher-dimensional tori T-duality generalizes to an infinite discrete T-duality group, establishing the physical equivalence of different classical backgrounds from the stringy point of view. In the heterotic case such backgrounds are specified by the metric and B-field on the torus as well as by Wilson lines. Closely related is the duality between type IIA an type IIB theories compactified on — in fact distinct — Calabi-Yau manifolds, called mirror symmetry.[5]

A second important example goes under the name of string-string duality, conjecturing that the heterotic string on a four-torus is dual to type IIA string theory on a K3 surface.[6] Both theories, when further compactified on another two-torus, lead to an $\mathcal{N} = 4$ supergravity theory as low-energy effective theory in four dimensions and their moduli spaces indeed coincide.

[3] As usual, R and NS abbreviate Ramond and Neveu-Schwarz (periodic and antiperiodic) boundary conditions of the worldsheet spinors, respectively. In type II theories a boundary condition needs to be chosen independently both for left- and right-movers $\psi^\mu_\pm$.

[4] See e.g. [8, section 18.3] for how (extended) objects couple electrically and magnetically to higher form gauge fields.

[5] See e.g. [16] for a thorough introduction.

[6] Strings on K3 surfaces are reviewed in [17].

The third important duality we will encounter is the so-called S-duality [18]. This duality relates a string theory with coupling constant g_s to another (or the same) string theory with coupling constant $1/g_s$. An immediate consequence of such a relation is that a strongly coupled theory with $g_s \gg 1$ has a dual description by a weakly coupled theory. For the models considered in this thesis S-duality is actually a self-duality and includes an exchange of electric and magnetic degrees of freedom.[7]

Conjecturally, all five superstring theories are limits of a unique eleven-dimensional theory, called M-theory. Its low-energy description is eleven-dimensional supergravity, whose dimensional reduction on a circle of radius R yields type IIA supergravity. Eleven-dimensional supergravity has a three-form gauge field, which can couple electrically to a two-dimensional object, the so-called M2-brane. The magnetic coupling in turn is possible for a five-dimensional object, the M5-brane. Similar to the D-branes in string theory, there are solutions to the supergravity equations of motion that support the existence of these objects.

From BPS states... It is especially hard to establish a duality when one of the two sides of the duality is in a non-perturbative (strongly coupled) regime, simply due to the loss of computational control. However, in theories with extended supersymmetry there are at least some states that are believed to provide reliable information in the strong-coupling regime although being constructed or studied initially in a weak-coupling regime. These are the supersymmetric so-called Bogomol'nyi–Prasad–Sommerfield (BPS) states, which are massive states transforming in smaller representations of the superalgebra than a generic massive state (that is, they preserve some of the supersymmetry). The reason for the latter is that their mass, given in terms of their charges and the moduli expectation values, is closely related to the central charge of the representation. Quantum corrections are not expected to spoil these relations. At generic points in moduli space these states are stable and at least for the cases we will be interested in the walls of marginal stability are sufficiently well understood to further constrain the BPS spectrum and to eventually allow for an extrapolation into a strongly coupled regime.

...to black holes. Especially, the application of BPS states we are interested in is where in the strong-coupling regime these states have the interpretation of macroscopic, charged black holes. Due to the BPS condition they are extremal, supersymmetric black holes. This means their mass is entirely fixed by their charges and they preserve some of the spacetime supersymmetry. Black holes bring us back to a central motivation for string theory, namely having a framework where quantum aspects of gravity can be studied. The quantum microstates of the black hole in particular depend on the string states, and likewise the states of the other extended objects, along the compact dimensions. Strominger and Vafa [20] considered a class of five-dimensional supersymmetric black holes in type IIB string theory compactified on $\mathrm{K3} \times S^1$ and successfully matched the entropy according to the area-law with that of the microscopic D-brane system along the compact dimensions. Apart from mentioning this historical milestone we will not review all the further developments that followed in the counting of BPS black holes in string theory, but rather point out that further compactification on an additional circle gives a four-dimensional $\mathcal{N} = 4$ theory, type IIB string theory on $\mathrm{K3} \times T^2$, dual to type IIA theory on $\mathrm{K3} \times T^2$, which in turn is also dual to heterotic string theory on a six-torus T^6. This setup is almost the one we will consider in this thesis and for this setup the relevant BPS spectrum is encoded in the famous Dijkgraaf-Verlinde-Verlinde (DVV) formula [21] that expresses the BPS

[7] For four-dimensional theories with sixteen supercharges, including the models considered in this thesis, this strong-weak coupling duality is also discussed in [19].

degeneracy of states carrying specified electric-magnetic charges in terms of the Fourier coefficients of the reciprocal Igusa cusp (Siegel modular) form.[8]

Moderately speaking, although the BPS setup might be a bit too simplistic to describe real astrophysical black holes, it is far from trivial to reproduce the Bekenstein-Hawking area law both from a macroscopic (low-energy) field theory point of view and from the microscopic statistical point of view (at least in a thermodynamic limit of large charges). Moreover, within string theory also corrections to the area law can be studied systematically by incorporating higher-derivative terms in the effective action, whereas the Einstein-Hilbert term alone, as occuring in GR, gives a two-derivative action. These corrections can in turn be compared to subleading terms for the microscopically determined entropy (say in a large charge expansion).

On the research presented in this thesis

For a class of $\mathbb{Z}_N$ orbifolds of heterotic strings on T^6 (or of type II strings on $\mathrm{K3} \times T^2$) preserving $\mathcal{N} = 4$ supersymmetry and being known as Chaudhuri-Hockney-Lykken (CHL) orbifolds, which includes the $\mathbb{Z}_2$ orbifold we are interested in, a natural generalization of the DVV formula [21] was proposed in [1], but it turns out that for the orbifold case the proposed formula only captures a specific sector of the BPS states in the theory. This sector is restricted by the orbit of the quantized charges carried by the BPS states under the T- and S-duality groups of the theory [11, 22].[9] A formula for the BPS index subject to a generic charge vector was recently proposed in [2], however, we will tackle the problem of finding (the partition functions for) BPS indices from a different point of view. That is, while the proposal of [2] relies on a conjectural six-derivative coupling in the effective action of a 3D CHL orbifold, expressed as a genus two modular integral that is asymptotically expanded in a decompactification limit to four dimensions, our ansatz is that of [4–6] and consists of mapping the BPS states of interest via a chain of dualities to the chiral states of a genus-two heterotic string. In particular, for the $\mathbb{Z}_2$ case the corresponding orbifold partition function should exhibit contributions that can be interpreted as representing the partition function of the dyonic quarter-BPS states in the different charge sectors (all satisfying the unit-torsion criterion). Apart from deriving these partition functions from a physically independent perspective, another main goal in this thesis is to provide extensive consistency checks for our results and to better understand their modular and polar structure, including the question in how far the latter structures already fix them by imposing the known physical constraints. We will compare our findings to that of [2, 22]. Moreover, similar to the analysis of [1] and along the lines of the previous paragraph we want to study their implication for the corresponding black hole entropy in the other charge sectors. Last but not least, the BPS indices studied in this thesis are believed to correspond to appropriate Donaldson-Thomas invariants of the type IIA geometry, for which partition functions have been conjectured recently as well [3]. This connection is explored at least on a simple and formal level, treating the algebro-geometric side necessarily as a black box.

The material presented in this thesis is organized as follows. In chapter 2 we first provide the required facts about two $\mathcal{N} = 4$ string compactifications, namely the well-known Narain compactification of heterotic string theory on a six-torus and its order two CHL orbifold. In chapter 3 we then address the

[8] Strictly speaking, it is a generating function for the sixth helicity supertraces (called quarter-BPS indices) that count quarter-BPS states of specified quadratic charge invariants, where the quantized charges are subject to an irreducibility criterion known as the unit-torsion condition [6].

[9] This holds even when considering only charges satisfying the unit-torsion condition that also the DVV formula of the parent theory underlies.

representation theory of the $\mathcal{N} = 4$ superalgebra, especially the half- and quarter-BPS representations, as well as the BPS indices (helicity supertraces) that count them. Based on this, we formally introduce partition functions for BPS indices that count quarter-BPS states in suitable charge sectors of the theory and further explain how charge quantization, S-duality and wall-crossing put strong constraints on these partition functions. Especially the wall-crossing constraints that describe the (dis-)appearance of bound states of two separately half-BPS components from the physical spectrum require a detailed understanding of the half-BPS spectrum, whose heterotic perturbative part is reviewed in chapter 5. In chapter 6, relying on a duality argument, we identify the desired quarter-BPS partition functions in a chiral heterotic orbifold partition function at genus two. Showing that the constraints set up earlier are indeed satisfied for the quarter-BPS partition functions thus obtained is the content of chapter 7. These microscopic partition functions then allows us to study the large charge behavior of the BPS index in the various charge sectors and we will compare this to the large charge expansion of the black hole entropy, as computed in the corresponding low-energy (higher-derivative) effective action, in chapter 8. We also compare our BPS partition functions to closely related partition functions of [3] for algebro-geometric Donaldson-Thomas invariants of the type IIA dual geometry (chapter 9). Our conclusions are presented in chapter 10. Throughout the thesis we rely on mathematical facts concerning Siegel modular forms collected in appendix A.

CHAPTER **2**

String compactifications with $\mathcal{N} = 4$ supersymmetry

In this chapter we discuss two specific string compactifications that lead to $\mathcal{N} = 4$ supersymmetry in four-dimensions. The first is the well-known Narain compactification of the heterotic string on a six-torus and the second is the $\mathbb{Z}_2$ CHL compactification to four dimensions. Discussing the Narain compactification first is worthwhile because it is the parent theory of the CHL orbifold and hence explains many features that are naturally inherited by the latter. Both theories possess the same amount of supersymmetry and enjoy an S- and T-duality symmetry group. The precise structure of these groups and the massless spectrum of course differ. Similarly, many technical and conceptual aspects in the counting of BPS states were first (or only) developed for the Narain compactification, for which it is a bit simpler. In other words, the orbifolding introduces a fair amount of additional complexity to the problem, especially regarding arithmetic aspects such as the properties of the charge lattices, the action of discrete duality groups on them and the consequences for the modular partition functions we will study in this work. Our review in this chapter will mostly cover those aspects that are relevant for the counting of BPS states.

2.1 The Narain compactification

We start with the ten-dimensional heterotic theories in Minkowski space and subsequently discuss the toroidal compactification to four dimensions. As briefly mentioned in the introduction, in ten non-compact dimensions there are two heterotic string theories, the $E_8 \times E_8$ and the $\mathrm{SO}(32)$ heterotic string. Both constructions, reviewed for instance in [8] (which we follow), base on treating the left- and right-moving sector of the string differently. The basic worldsheet degrees of freedom in either sector are as follows.

The left-moving sector consists of 26 bosonic fields $X_L^M(\tau + \sigma)$ (with τ and σ being standard worldsheet coordinates) and 16 of these bosons map to a sixteen-torus $T^{16} \cong \mathbb{R}^{16}/\Lambda_{16}$. The right-moving sector is that of 10 bosonic fields $X_R^M(\tau - \sigma)$ and their superpartners $\psi_R^M(\tau - \sigma)$. This sector can again be divided into the Neveu-Schwarz (NS) and the Ramond (R) sector. Modular invariance of the one-loop partition function requires in the left-moving sector that Λ_{16} must be an even and self-dual Euclidean lattice (with self-duality also being known as unimodularity), hence either two copies of the E_8 root lattice or a single copy of the weight lattice of $\mathrm{Spin}(32)/\mathbb{Z}_2$ (which contains the root lattice of SO(32)). In the right-moving sector modular invariance implies the GSO projection, removing the tachyon and leading to an $\mathcal{N} = 1$ spacetime supersymmetric spectrum. A bit more explicitly, the spectrum is obtained after tensoring the left- with the right-moving states and imposing the level-matching condition

$$N_L - 1 + \frac{\boldsymbol{p_L}^2}{2} = N_R - \delta_{NS}\frac{1}{2}\,, \tag{2.1}$$

where N_L is the left-moving level number getting contributions only from the bosonic oscillator modes $\bar{\alpha}^{\mu}_{-n}$ in the eight directions transverse to the light-cone, N_R its right-moving counterpart getting contributions from both bosonic (α^{μ}_{-n}) and fermionic modes (b^{μ}_{-r}) transverse to the light cone, and $\boldsymbol{p_L}$ is a lattice vector in Λ_{16}. The constant δ_{NS} gives unity for the NS sector, but vanishes for the R sector. Level-matching can also be understood as a matching of a left- and right-moving mass $m_L^2 = m_R^2$, that are defined by

$$\alpha' m_L^2 = \boldsymbol{p_L}^2 + 2(N_L - 1)\,, \qquad \alpha' m_R^2 = 2\left(N_R - \delta_{NS}\frac{1}{2}\right)\,. \tag{2.2}$$

Using these mass formulae together with $m^2 = m_L^2 + m_R^2$ it is easy to work out the massless spectrum in ten non-compact spacetime dimensions. States from the Ramond sector give rise to spacetime fermions since the R-ground states form a spacetime spinor $|S^{\alpha}\rangle_{\mathrm{R}}$, while states from the Neveu-Schwarz sector, whose ground state is a spacetime scalar $|0\rangle_{\mathrm{NS}}$, give rise to spacetime bosons. In this way one obtains a ten-dimensional graviton $G_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$, the scalar dilaton ϕ and their superpartners, namely the gravitino and the dilatino. Furthermore there are the gauge bosons of $G = E_8 \times E_8$ or $G = \mathrm{SO}(32)$, which are in the 496-dimensional adjoint representation of the gauge group G, plus their gaugini superpartners. The massless spectrum in ten dimensions arranges into an $\mathcal{N} = 1$ supergravity multiplet plus a G gauge multiplet, which are the fields of the low-energy effective theory.

However, we are not really interested in the ten-dimensional massless spectrum, but rather the four-dimensional one, so let us turn to the toroidal compactification of the heterotic string. Now for the compactified heterotic theory the fields $X^I_{R/L}$, $I = 5, \ldots 9$, map to a spacetime six-torus. Classical backgrounds for the worldsheet sigma-model action of the heterotic string are specified by the six-torus metric G_{IJ}, the antisymmetric two-form field B_{IJ} with legs along the torus directions and a gauge field background A^A_I (Wilson lines along the six-torus). The right- and left-moving momenta $\boldsymbol{p_{R/L}}$ along the compact directions will depend on discrete momentum and winding quantum numbers, but also on the just mentioned background fields (full expressions are not needed here, see for instance [8, 23]). They span the so-called Narain lattice [24, 25], in the following denoted as $\Lambda_{22,6}$, which is again even and self-dual but now of signature $(22, 6)$. Up to isomorphism there is a unique even unimodular lattice of signature (22,6), namely

$$E_8(1)^{\oplus 2} \oplus U^{\oplus 6} \cong \Lambda_{22,6}\,. \tag{2.3}$$

Here we denoted by U the hyperbolic lattice of signature $(1, 1)$ and $E_8(1)$ is the E_8 root lattice.[1]

The lattice U also arises in the simple example of a boson on a circle carrying integer momentum and winding quantum numbers (m, n). In that example, we can think of U as the lattice $\mathbb{Z}^2$ =

[1] Later we will encounter lattices with rescaled quadratic forms, which we will denote by displaying the rescaling factor in parentheses, e.g., we will write $E_8(2)$ if the quadratic form of the E_8 root lattice is rescaled by a factor of two.

$\{(m,n)|m,n \in \mathbb{Z}\}$ with bilinear form defined by $(m_1,n_1)\cdot(m_2,n_2) = m_1n_2 + m_2n_1$. More precisely, in the two-dimensional field theory one encounters this lattice (or better an isomorphic lattice) as the lattice of right- and left-momenta $p_{R/L}$, whose points take the form[2]

$$(p_R, p_L) = \frac{1}{\sqrt{2}}\left(m\frac{\sqrt{\alpha'}}{R} - n\frac{R}{\sqrt{\alpha'}}\ ,\ m\frac{\sqrt{\alpha'}}{R} + n\frac{R}{\sqrt{\alpha'}}\right) \in \Lambda_{1,1}\ , \tag{2.4}$$

where $m,n \in \mathbb{Z}$. A quadratic form on this lattice is given by

$$p_L^2 - p_R^2 = 2mn\ . \tag{2.5}$$

The radius R of the circle, which we may think of as parametrizing the metric along the circle, also parametrizes a one-parameter family of embeddings of $\mathbb{Z}^2$ into a two-dimensional Lorentzian space $\mathbb{R}^{1,1}$, such that the point $(m,n) \in U$ maps to $(p_L, p_R) \in \Lambda_{1,1}$ in (2.4). Due to (2.5) this is an isomorphism of lattices and we write $\Lambda_{1,1} \cong U$.

Leaving the example, we can think about (2.3) in a very similar way. At least locally the background fields parametrize the embedding of the abstract lattice on the left-hand side of eq. (2.3) into the pseudo-Riemannian space $\mathbb{R}^{22,6}$. That is, an abstract momentum-winding vector $\boldsymbol{p} \in E_8(1)^{\oplus 2} \oplus U^{\oplus 6}$ (carrying only the information about discrete momentum and winding quantum numbers) is decomposed into a left- and a right-moving part $p_{L,R}$ such that $\boldsymbol{p}^2 = \boldsymbol{p_L}^2 - \boldsymbol{p_R}^2$ holds. Formally it is the Grassmannian $\mathrm{Gr}_{r,s} := \mathrm{O}(r,s)/(\mathrm{O}(r)\times\mathrm{O}(s))$ that parametrizes splittings $\mathbb{R}^{r,s} \cong \mathbb{R}^{r,0} \oplus \mathbb{R}^{0,s}$, where in our notation $\mathrm{O}(r) = \mathrm{O}(r,\mathbb{R})$. Globally one also has to take into account an infinite discrete group of stringy symmetries, the T-duality group, that (amongst others) operates non-trivially on the background fields and the momentum-winding charges. It is given by the discrete automorphism group $\mathrm{O}(\Lambda_{22,6}) \cong \mathrm{O}(22,6;\mathbb{Z})$ of the Narain lattice $\Lambda_{22,6}$.

We note en passant that the mass formulae (2.2) get slightly modified when compactifying six dimensions on a torus. They now get contributions from the internal momentum-winding vectors $(\boldsymbol{p_L}, \boldsymbol{p_R}) \in \Lambda_{22,6}$:

$$\alpha' m_L^2 = \boldsymbol{p_L}^2 + 2(N_L - 1)\ , \qquad \alpha' m_R^2 = \boldsymbol{p_R}^2 + 2\left(N_R - \delta_{NS}\frac{1}{2}\right)\ . \tag{2.6}$$

Especially the total mass $m^2 = m_L^2 + m_R^2$ is sensitive to the background fields that enter $(\boldsymbol{p_L}, \boldsymbol{p_R})$. Also the level-matching is slightly modified and now reads

$$\frac{\boldsymbol{p_R}^2 - \boldsymbol{p_L}^2}{2} = N_L - 1 - \left(N_R - \delta_{NS}\frac{1}{2}\right)\ . \tag{2.7}$$

In contrast to the mass, the level-matching condition does not depend on the moduli (similar to (2.5), the left-hand side gives an integer). The mass formula for m_R^2 and the level-matching condition will become important for understanding the perturbative heterotic half-BPS spectrum (see chapter 5).

Let us turn to the low-energy effective theory. The (field theoretic) dimensional reduction of the massless fields on a six-torus, treating them as independent of the torus coordinates, is straightforward and is discussed in [23]. A strong hint that this will lead to an $\mathcal{N} = 4$ supergravity theory is already

[2] See for instance [23].

obtained from the decomposition of the supercharge in this construction [8]. Consider a supercharge Q in ten dimensions, transforming as a sixteen-component Majorana-Weyl spinor. We can decompose the Weyl representation of $\mathrm{SO}(1,9)$ under the $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$ subgroup, which yields

$$\mathbf{16} = (\mathbf{2}_L, \bar{\mathbf{4}}) + (\mathbf{2}_R, \mathbf{4}) \,, \tag{2.8}$$

with $\mathbf{2}_{L,R}$ denoting Weyl spinors of $\mathrm{SO}(1,3)$ and $\mathbf{4}, \bar{\mathbf{4}}$ denoting Weyl spinors of $\mathrm{SO}(6)$. All of the sixteen real supercharges are preserved and arrange into four Weyl spinors of either chirality.

Indeed [23], upon dimensional reduction of the massless fields one obtains the $\mathcal{N} = 4$ supergravity multiplet (a graviton, four Majorana gravitini, six graviphotons, four spin-$\frac{1}{2}$ Majorana fermions and the axio-dilaton complex scalar) and 22 vector multiplets (a vector, four spin-$\frac{1}{2}$ Majorana fermions and six scalars). The helicity content of these $\mathcal{N} = 4$ supermultiplets will be reviewed in chapter 3 when discussing BPS representations, see especially Tab. 3.1 there. For generic points in moduli space the gauge group is broken to $U(1)^{28}$, as the dimensional reduction gives rise to a Higgs-potential for the four-dimensional gauge fields, but there can be an enhancement to non-Abelian gauge groups at special points in moduli space.

Taking also into account the expectation value of the complex axio-dilaton scalar that takes values in the upper half-plane and transforms non-trivially under a non-perturbative S-duality, the global structure of the moduli space is

$$[\mathrm{O}(22,6;\mathbb{Z})\backslash \mathrm{O}(22,6)/(\mathrm{O}(22)\times \mathrm{O}(6))] \;\times\; [\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{U}(1)] \,. \tag{2.9}$$

The discrete groups acting from the left are the T- and S-duality group of the four-dimensional theory. Note that independent of whether we start with $E_8 \times E_8$ or $\mathrm{SO}(32)$ heterotic strings in ten dimensions, we get the same moduli space in four dimensions. The moduli space (2.9) is consistent with the moduli space and symmetries of $\mathcal{N} = 4$ supergravity, although the discrete nature of the T- and S-duality is of genuine stringy origin.

Although in this thesis we will mostly work with heterotic duality frames, it is occasionally useful to go to other duality frames. For a given factorization $T^6 = T^4 \times S^1 \times \hat{S}^1$, the dual type II description is IIA[$\mathrm{K3} \times S^1 \times \hat{S}^1$], or via T-duality on the last circle IIB[$\mathrm{K3} \times S^1 \times \tilde{S}^1$]. The complex structure modulus of $S^1 \times \tilde{S}^1$ in the type IIB theory, the complexified Kähler modulus of $S^1 \times \hat{S}^1$ in the type IIA theory and the heterotic axio-dilaton are dual to each other. Also the Narain lattice can be reinterpreted in the type IIA theory as

$$\Lambda_{22,6} \cong \Lambda_{20,4} \oplus \Lambda_{2,2} \,, \tag{2.10}$$

where $\Lambda_{20,4} \cong H^*(\mathrm{K3}, \mathbb{Z})$ is the integral cohomology lattice[3] of the K3 surface, while $\Lambda_{2,2}$ is the winding-momentum lattice for $S^1 \times \hat{S}^1$. As an abstract lattice, the latter is given by the direct sum of two hyperbolic lattices, i.e., $\Lambda_{2,2} \cong U^{\oplus 2}$.[4]

[3] In principle one should flip the sign of the quadratic form of the Narain lattice introduced earlier to make this statement formally correct. This is because the cohomology lattice of the K3 contains the piece $E_8(-1)^{\oplus 2}$ rather than $E_8(1)^{\oplus 2}$. However, we will henceforth be ignorant with respect to these sign issues, as they play no role in our analysis. Similarly, in some references, for instance in [19, 26], the quadratic form on the electric and magnetic lattice (which is essentially the momentum-winding lattice of the compactification and its dual) is rescaled by an overall factor of (-1), flipping the signature. We will follow the convention used in [2, 27].

[4] On subspaces of the Narain moduli space where the generic gauge group will be enhanced, with non-Abelian gauge bosons arising from additional root vectors in the Narain lattice, enhanced gauge symmetry occurs in the type IIA duality

2.2 The CHL orbifold of order two

In this section we briefly introduce the CHL compactification for which we want to investigate the BPS spectrum. This compactification is a $\mathbb{Z}_2$ orbifold of the heterotic toroidal compactification addressed in the previous section. A particular and important feature of this compactification is that it does not reduce the amount of supersymmetry, that is, it again leads to a four-dimensional theory with $\mathcal{N} = 4$-extended supersymmetry. However, the massless four-dimensional spectrum necessarily differs, as the rank of the gauge group will be reduced, and more differences will be pointed out in the following.

Although there are several $\mathbb{Z}_N$ orbifolds that preserve all sixteen supercharges, also known as CHL orbifolds [28–30], our focus lies on the $\mathbb{Z}_2$ case for simplicity.

Heterotic orbifold construction. The CHL compactification is most conveniently introduced as an asymmetric $\mathbb{Z}_2$ orbifold[5] of the $E_8 \times E_8$ heterotic string on a torus T^d [29, 32, 33]. The $\mathbb{Z}_2$ generator $g = g_R g_T$ acts freely by exchanging the two copies of the E_8-tori on which the 16 internal left-moving bosons live (g_R), and by simultaneously translating by half a period along a circle of the compactification torus T^d (g_T). This construction exists already in nine non-compact spacetime dimensions, i.e., for $T^d = S^1$.

We start with the nine-dimensional construction, closely following [29, 32, 33].[6] The circle shall have a radius of R, such that $x_9 \sim x_9 + 2\pi R$. Translating by half a period along the circle direction then means

$$g_T \ : \ X^9(\tau,\sigma) \mapsto X^9(\tau,\sigma) + \pi R \tag{2.11}$$

and $g_T^2 = 1$. The internal left-moving bosons on T^{16}, henceforth denoted as Y^I and $Y'^I = Y^{I+8}$ (where $I = 1, \dots, 8$) , experience a swap $g_R : (Y^I, Y'^I) \mapsto (Y'^I, Y^I)$, which can be diagonalized by introducing the (anti)symmetric combinations

$$Y^I_\pm = \frac{1}{\sqrt{2}}(Y^I \pm Y'^I) \ , \qquad g_R \ : \ Y^I_\pm \mapsto \pm Y^I_\pm \ . \tag{2.12}$$

Oscillator expansions of $Y^I_\pm(\tau+\sigma)$ are obtained from the standard oscillator expansion of the compact bosons $Y^I(\tau+\sigma), Y'^I(\tau+\sigma)$, where we similarly find $g_R : \alpha^I_\pm \mapsto \pm\alpha^I_\pm$ on oscillators. Note that the shift g_T does not affect the oscillators of $X^9(\tau,\sigma)$. In order to find the perturbative states invariant under g that descend to the orbifold theory we need to know, besides the action on the oscillators, the action on the momentum eigenstates (which are Fock vacua for the oscillators).

To describe the $\mathbb{Z}_2$ action on the momentum eigenstates we first recall the form of Narain momentum-winding vectors $(p_R; p_L, p^I) \in \Lambda_{17,1} \cong E_8(1)^{\oplus 2} \oplus U$ in nine dimensions, where $\Lambda_{17,1}$ is an even self-dual Lorentzian lattice of signature $(17,1)$, and in fact unique up to isomorphism. The lattice

frame for degenerations of the K3 surface (see, for instance, [17]).

[5] Asymmetric orbifolds are introduced in [31].

[6] In this section we set $\alpha' = 1$.

vectors are parametrized by the moduli and the discrete momentum-winding numbers as follows:

$$p_R = \frac{1}{\sqrt{2}R}\left[m - \left(R^2 + \frac{1}{2}A^2\right)n - \Pi \cdot A\right], \tag{2.13a}$$

$$p_L = \frac{1}{\sqrt{2}R}\left[m + \left(R^2 - \frac{1}{2}A^2\right)n - \Pi \cdot A\right], \tag{2.13b}$$

$$p^I = \Pi^I + A^I n, \tag{2.13c}$$

where I runs over $I = 1, ..., 16$ and the symbols $m, n \in \mathbb{Z}$ respectively denote momentum and winding quantum numbers of X^9. The expression $\Pi = (\pi, \pi') \in E_8(1) \oplus E_8(1)$ is a lattice vector. Note that the Wilson line modulus $A = (a, a)$ must be the same for both E_8 copies in order to be compatible with the orbifolding. Further define symmetric and antisymmetric combinations

$$p^I_+ = \frac{1}{\sqrt{2}}\left(p^I + p^{I+8}\right), \qquad p^I_- = \frac{1}{\sqrt{2}}\left(p^I - p^{I+8}\right), \quad I = 1, ..., 8, \tag{2.14}$$

and introduce the shorthand notation

$$\rho = \pi + \pi' \in E_8(1), \tag{2.15}$$

then we may recast (2.13) into the form

$$p_R = \frac{1}{\sqrt{2}R}\left[m - R^2 n - a^2 n - \rho \cdot a\right], \tag{2.16a}$$

$$p_L = \frac{1}{\sqrt{2}R}\left[m + R^2 n - a^2 n - \rho \cdot a\right] = p_R + \sqrt{2}Rn, \tag{2.16b}$$

$$p_+ = \frac{1}{\sqrt{2}}(\rho + 2an), \tag{2.16c}$$

$$p_- = \frac{1}{\sqrt{2}}(\pi - \pi'). \tag{2.16d}$$

Accordingly, for fixed moduli an element of $\Lambda_{17,1}$ can equivalently be specified by the quadruple $p = (p_R; p_L, p_+, p_-) =: (\boldsymbol{p_R}; \boldsymbol{p_L})$ or by (m, n, π, π'), which also label the momentum eigenstates of the heterotic theory before orbifolding. The generator g acts as

$$g|p_R, ; p_L, p_+, p_-\rangle = e^{2\pi i v \cdot p}|p_R, ; p_L, p_+, -p_-\rangle = e^{i\pi n}|p_R, ; p_L, p_+, -p_-\rangle \tag{2.17}$$

where we used the shift vector

$$v = (v_R; v_L, v_+, v_-) = \frac{1}{2\sqrt{2}}\left(-R - \frac{a^2}{R}; R - \frac{a^2}{R}, 2a, 0\right) \tag{2.18}$$

to rewrite the phase factor. Indeed, v is half a lattice vector of the form (2.16) with $\pi = \pi' = 0$, $m = 0$ and $n = 1$ (one unit of winding along x_9).

We now have all the ingredients needed for finding those perturbative states of the heterotic string that survive the orbifold projection and give rise to the untwisted sector states in the nine-dimensional CHL compactification. The invariant sector is spanned by states of the form

$$\frac{1}{\sqrt{2}}\left(f_{\text{osc}}|p_R,;p_L,p_+,p_-\rangle + e^{2\pi i v\cdot p}g(f_{\text{osc}})|p_R,;p_L,p_+,-p_-\rangle\right) \otimes |\psi_0\rangle \,, \tag{2.19}$$

where f_{osc} is a monomial in oscillators, $g(f_{\text{osc}})$ its image under g and the phase factor (a sign factor) is chosen such that the net effect of g is just an exchange of the two summands, leaving the whole state indeed invariant. Also we have explicitly shown a right-moving ground state $|\psi_0\rangle$ (that is, a lowest admissible state compatible with the GSO projection), which can either be an NS- or R-ground state. The NS-sector will give rise to spacetime bosons in the spectrum, while the R-sector will give rise to spacetime fermions. In both cases the right-ground state is eightfold degenerate. For the heterotic perturbative half-BPS states that we will discuss in chapter 5 this will imply that theses states always come with a total degeneracy of sixteen and they belong to a supermultiplet with 8 bosonic and 8 fermionic degrees of freedom (we will say more about supermultiples in chapter 3).

The orbifold construction also requires the introduction of twisted sector states. Boundary conditions for the twisted chiral bosons in the sixteen internal directions are now

$$Y^I(\sigma+2\pi) = Y'^I(\sigma) + y^I, \quad Y'^I(\sigma+2\pi) = Y^I(\sigma) + y'^I \tag{2.20}$$

for some root vectors $y, y' \in E_8$. We can again form the linear combinations $Y^I_\pm(\tau+\sigma)$ satisfying

$$Y^I_\pm(\sigma+2\pi) = \pm Y^I_\pm(\sigma) + \frac{1}{\sqrt{2}}\left(y^I \pm y'^I\right) \,, \tag{2.21}$$

i.e., they are up to lattice translations periodic or antiperiodic in σ, requiring the modes in the oscillator expansion to be integral or half-integral. The boundary condition for the boson associated with the spacetime circle reads in the twisted sector

$$X^9(\sigma+2\pi) = X^9(\sigma) + \pi R + 2\pi R\tilde{n} \tag{2.22}$$

for some integer $\tilde{n} \in \mathbb{Z}$ such that effectively the winding number $n = \tilde{n} + 1/2$ in the twisted sector is now half-integral. Twisted sector states do not have antisymmetric momentum p_- [33], but only p_+ and this takes the same form as in (2.16c) with $\rho = y + y' \in E_8$. For given values of the moduli, the twisted sector ground state is characterized by m, n and ρ only. All twisted sector states satisyfing level-matching survive the orbifold projection and further details about the twisted sector can again be found in the references [29, 32, 33]. For later use we would also like to mention the level-matching condition[7]

$$\frac{\boldsymbol{p_R}^2 - \boldsymbol{p_L}^2}{2} = \left(N_L - a_{\text{un/tw}}\right) - \left(N_R - \delta_{\text{NS}}\frac{1}{2}\right) \,, \tag{2.23}$$

where $a_{\text{un}} = 1$ for the untwisted orbifold sector (as in the parent theory) and $a_{\text{un}} = 1/2$ for the twisted orbifold sector (recall that in the twisted orbifold sector 8 chiral bosons satisfy antiperiodic boundary conditions), while δ_{NS} is again unity for the NS-sector states and vanishes for the R-sector.

The extension of the above discussion to CHL strings in four non-compact dimensions is straightforward, as the further compactification of the 9D CHL string on an additional T^5 only involves spectator

[7] The expressions $\boldsymbol{p_R}^2$ and $\boldsymbol{p_L}^2$ are separately computed with standard Euclidean inner product.

fields with regard to the orbifolding procedure. The background fields are almost the same as in the Narain compactification. We have the metric of the six-torus G_{IJ}, the Kalb-Ramond two-form B_{IJ} and Wilson lines A_I for each $S^1 \subset T^6$ $(I, J = 5, \ldots, 9)$. The only difference with respect to the parent theory is that the Wilson lines must be symmetric with respect to the two E_8 factors, $A_I = (a_I, a_I)$. As in [33] we define

$$E_{IJ} = G_{IJ} + B_{IJ} + a_I \cdot a_J \,, \qquad (I, J = 5, \ldots, 9) \,. \tag{2.24}$$

If n^I and m^I are, respectively,the winding and momentum quantum numbers along $I = 5, \ldots, 9$ then n^9 is the only one which may take half-integer values (being non-integer precisely for twisted sector states), the remaining ones are all integer. Introduce a vielbein e^I for the torus metric such that $e_I \cdot e_J = G_{IJ}$ and let $\hat{e}^I$ denote its inverse. The momentum-winding vectors $p = (p_R; p_L, p_+, p_-)$ generalize in the four-dimensional theory to

$$p_R = \frac{1}{\sqrt{2}} \left[m_I - E_{IJ} n^I \rho \cdot a_I \right] \,, \tag{2.25a}$$

$$p_L = \frac{1}{\sqrt{2}} \left[m_I + (2G_{IJ} - E_{IJ}) n^I \, \rho \cdot a_I \right] = p_R + \sqrt{2} n^I e_I \,, \tag{2.25b}$$

$$p_+ = \frac{1}{\sqrt{2}} \left(\rho + 2n^I a_I \right) \,, \tag{2.25c}$$

$$p_- = \frac{1}{\sqrt{2}} (\pi - \pi') \,. \tag{2.25d}$$

The lattice generated by the vectors $(p_R; p_L, p_+)$ (leaving out the antisymmetric part p_-) has rank 20 and signature $(14, 6)$. It is isomorphic to the lattice

$$E_8\left(\tfrac{1}{2}\right) \oplus U^{\oplus 5} \oplus U\left(\tfrac{1}{2}\right) \,, \tag{2.26}$$

where the summand $U^{\oplus 5}$ simply arises from the momentum-winding charges on the spectator T^5, while $U(\frac{1}{2})$ (respectively $E_8(\frac{1}{2})$) is the hyperbolic lattice U (the $E_8(1)$ root lattice) with quadratic form rescaled by a factor of $1/2$. The rescaling for U is equivalent to allowing the winding numbers n^9 to be half-integral, as it is the case for twisted sector states.

The shift vector v for orbifolding is again chosen such that $2v$ is formally a vector in the Narain lattice for a single unit of winding along the CHL circle and otherwise vanishing quantum numbers. Especially, $e^{2\pi i v \cdot p} = (-1)^{n^9}$ still holds. Up to re-interpreting $\boldsymbol{p_{R/L}}^2$ as products of higher-dimensional vectors, the level-matching condition (2.23) stays valid in the four-dimensional theory.

The one-loop partition function of the heterotic CHL orbifold (or rather a slight refinement thereof) will be presented in chapter 5 as it will be needed for the determination of the spectrum of Dabholkar-Harvey half-BPS states.

So far the discussion of the CHL compactification was from the heterotic (worldsheet) point of view. In the remainder of this section we broaden our perspective, aiming to collect results that are especially relevant for the counting of BPS states in the model. Our presentation for this part follows [2, 3, 11, 13, 22, 27, 34] (the corresponding discussion was presented also in section 2.1 of [35]).

Type IIA perspective. By virtue of $\mathcal{N} = 4$ string-string duality the CHL model has dual descriptions as freely acting $\mathbb{Z}_2$ orbifolds of heterotic string theory on T^6 or type IIA string theory on $\mathrm{K3} \times T^2$.

In the type IIA theory the orbifold group is generated by a pair (g, δ), consisting of an order two action g on the $\mathcal{N} = (4, 4)$ K3 non-linear sigma model (NLSM) and a simultaneous order-two shift in the direction $\delta/2$ on S^1, where $\delta \in \Lambda_{2,2}$ has square zero in order to satisfy level matching. The condition on g is to fix the superconformal algebra on the worldsheet and the spectral flow generators, see [36] for a precise characterization. The symmetry g is geometric in the sense that it describes an automorphism of the K3 surface that fixes the holomorphic-symplectic (2,0)-form (and thus preserves the $\mathrm{SU}(2)$ holonomy). Such symmetries are uniquely determined by their induced action on the lattice $H^2(\mathrm{K3}, \mathbb{Z})$. They are in fact, up to lattice automorphisms, already determined by the order $1 \le N \le 8$ of g and symplectic automorphisms of any order in that range do actually exist.[8]

As an aside, the former gives a possibility to construct further CHL models associated to a symplectic automorphism of a K3 surface, but we will not pursue this here. The closest generalization is obtained by considering a $\mathbb{Z}_N$ orbifold action. On the heterotic side, the $\mathbb{Z}_N$ orbifold action is again asymmetric, i.e., acts by a $\mathbb{Z}_N$ cyclic permutation and a shift on the left-moving coordinates while the right-moving coordinates are invariant up to shifts [30]. It is also possible to consider non-geometric symmetries of a K3 NLSM in the construction [19]. CHL models have been further investigated in [2, 3, 11, 19, 27, 30, 32–34, 40–53].

Coming back to the $\mathbb{Z}_2$ case, the middle cohomology lattice of the K3 surface

$$\Lambda \coloneqq H^2(\mathrm{K3}, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \tag{2.27}$$

contains an invariant $\Lambda^g = \{v \in \Lambda \,|\, gv = v\}$ and a coinvariant $\Lambda_g = \left(\Lambda^g\right)^\perp$ lattice with respect to g, i.e.,

$$\Lambda \supseteq \Lambda^g \ , \ \Lambda_g \ . \tag{2.28}$$

The geometric $\mathbb{Z}_2$ operation g on the K3 surface is also called a Nikulin involution. The induced action on Λ exchanges the $E_8(-1)$ sublattices and fixes $U^{\oplus 3}$ pointwise. The two sublattices in (2.28) become

$$\Lambda^g = U^{\oplus 3} \oplus E_8(-2) \ , \quad \Lambda_g = E_8(-2) \ , \tag{2.29}$$

with $E_8(-2) \subset E_8(-1)^{\oplus 2}$ denoting the diagonal or the antidiagonal, respectively.

Moduli space. The moduli G_{IJ}, B_{IJ} and a_I above may be taken as local coordinates for (a factor of) the moduli space of the four-dimensional $\mathbb{Z}_2$ CHL model. Globally the moduli space, inlcuding now also the heterotic axio-dilation modulus in the second factor, is conjectured to take the form

$$G_4(\mathbb{Z}) \backslash \ \big([\mathrm{O}(14, 6)/(\mathrm{O}(14) \times \mathrm{O}(6))] \ \times \ [\mathrm{SL}_2(\mathbb{R})/\mathrm{U}(1)] \big) \tag{2.30}$$

for some discrete group $G_4(\mathbb{Z})$ (the subscript indicates the compactification to four dimensions). This is again consistent with the local structure of the moduli space as demanded by the low-energy $\mathcal{N} = 4$ supergravity theory, which in addition to the supergravity multiplet now consists of 14 vector multiplets. Note the close analogy with the moduli space (2.9) of the Narain compactification. Orbifolding in this case has projected out the massless states that would correspond to 8 vector multiplets in the Narain

[8] See [37, 38] or [39, ch. 15].

compactification. The duality group group $G_4(\mathbb{Z})$ of the CHL compactification includes a T-duality group $\mathcal{T}$ acting (only) on the first factor of (2.30) and an S-duality group $\mathcal{S}$ acting on the second factor (via Möbius transformations on the heterotic axio-dilaton [18, 54]),

$$G_4(\mathbb{Z}) \supset \mathcal{T} \times \mathcal{S} \; . \tag{2.31}$$

Explicit formulae will be given shortly (eqs. (2.46) and (2.48)). The S-duality group for the $\mathbb{Z}_2$ orbifold turns out to be [41] (see also [19])

$$\mathcal{S} = \Gamma_1(2) \subset \mathrm{SL}_2(\mathbb{Z}) \; , \tag{2.32}$$

where we recall the definition

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| c \equiv 0 \, , a \equiv 1 \, , d \equiv 1 \mod N \right\} \; . \tag{2.33}$$

The T-duality group $\mathcal{T}$ should at least contain[9] the centralizer $C_{(g,\delta)}$ of the orbifold generator (g, δ) in $\mathrm{O}(\Lambda_{22,6})$,

$$\mathcal{T} \supset C_{(g,\delta)} \coloneqq \{ h \in \mathrm{O}(\Lambda_{22,6}) \mid h(\delta) = \delta, \; hg = gh \} \; . \tag{2.34}$$

A common way to parametrize the moduli associated with the Grassmanian (at least locally) in the low-energy effective action (see also (8.17)) is by means of a real 20×20 matrix M subject to

$$MLM^{\intercal} = L \, , \quad M = M^{\intercal} \tag{2.35}$$

where $L = L^{-1}$ is an $\mathrm{O}(14, 6)$-invariant matrix representing the non-degenerate bilinear form on $\mathbb{R}^{14,6}$. This means L has 14 eigenvalues -1 and 6 eigenvalues $+1$ counted with multiplicity and satisfies[10]

$$OLO^{\intercal} = L \, , \quad \text{for all } \; O \in \mathrm{O}(14, 6) \; . \tag{2.36}$$

As an aside, we mention that according to [19] there is evidence for a Fricke involution acting as $S_{\mathrm{het}} \mapsto -1/(NS_{\mathrm{het}})$ on the heterotic axio-dilaton and by an orthogonal, not necessarily integral, transformation on the other moduli (see for instance [2, 27] for further discussion of this additional duality). However, for simplicity we will mostly neglect this possible extra duality.

It will be useful for us in the following to think of elements in the T-duality group $\mathcal{T}$ as automorphisms of the electric charge lattice Λ_e defined next,

$$\mathcal{T} \subset \mathrm{O}(\Lambda_e) \; . \tag{2.37}$$

Electric-magnetic charges. Our main interest in the CHL model lies in the counting of (quarter-) BPS states. For the moment it is just important to know that these generically carry non-zero electric or magnetic charges. Especially, the quarter-BPS states we are mainly interested in (and which will be defined in later chapters) carry both types of charges and hence are called dyonic. An efficient way to organize the spectrum of BPS states is via their quantized charges taking values in an electric-magnetic lattice. Thus, we shall now collect facts about this lattice.

[9] In practice, we will take this to be an equality and do not rigorously draw distinctions.

[10] Equivalently we can write $O^{\intercal} LO = L$ for all $O \in \mathrm{O}(14, 6)$.

At generic points in moduli space the gauge group of the $\mathbb{Z}_2$ CHL model is completely broken to $U(1)^{20}$. That is, there are 20 Abelian gauge fields surviving the orbifold projection and accordingly the electric charges of the model are quantized in a lattice of rank 20. In the heterotic duality frame perturbative states (in the twisted and the untwisted sector) are purely electric and their electric charge is given by the vector $(p_R; p_L, p_+)$. Indeed, the electric charge lattice is[11]

$$\Lambda_e = E_8\left(\tfrac{1}{2}\right) \oplus U^{\oplus 5} \oplus U\left(\tfrac{1}{2}\right) . \tag{2.38}$$

There are also magnetically charged objects in the $\mathbb{Z}_2$ orbifold theory descending from the parent theory (we refer to [19] for details). In the heterotic frame these are NS5-branes along $T^6/\mathbb{Z}_2$, Kaluza-Klein monopoles with asymptotic circle along $T^6/\mathbb{Z}_2$ and magnetic monopoles for the surviving rank-8 part of the ten-dimensional heterotic gauge group (which was $E_8 \times E_8$ or SO(32) in ten dimensions before orbifolding). Magnetic charges are quantized in a lattice Λ_m of the same rank and signature as the electric lattice. Indeed, Λ_m is given by the dual of the electric charge lattice $\Lambda_m = \Lambda_e^*$. The inclusion $\Lambda_m \subset \Lambda_e^*$ is already required by the compatibility with the Dirac quantization condition. Take two tuples $(Q^{(1,2)}, P^{(1,2)})$ of electric (Q) and magnetic charges (P), then necessarily

$$P^{(1)} L Q^{(2)} - P^{(2)} L Q^{(1)} \in \mathbb{Z} , \tag{2.39}$$

where L is the metric in field space for the kinetic and θ-angle terms of the Abelian gauge fields in the effective action (see (8.17)). For the $\mathbb{Z}_2$ orbifold we explicitly have

$$\Lambda_m = E_8(2) \oplus U^{\oplus 5} \oplus U(2) = \Lambda_e^* . \tag{2.40}$$

The direct sum of the electric and the magnetic charge lattices gives the electric-magnetic lattice

$$\Lambda_{em} = \Lambda_e \oplus \Lambda_m . \tag{2.41}$$

Note that for the $\mathbb{Z}_2$ orbifold, different than in the Narain case, the lattices Λ_e and Λ_m are no longer self-dual (unimodular). Rather, they are 2-modular, meaning that $\Lambda_m^* \cong \Lambda_m(\frac{1}{2})$ or $\Lambda_m^*(2) \cong \Lambda_m$, i.e., they agree with their dual upon rotation and rescaling (see [27, eq. (2.10)] for a concrete example):

$$\exists\, \sigma_N \in \mathrm{O}(14, 6; \mathbb{R}) \ : \ \Lambda_m^* = \frac{\sigma_N}{\sqrt{N}} \Lambda_m . \tag{2.42}$$

Multiplying (2.42) by 2 from the left and using the natural inclusion $\Lambda_m \subset \Lambda_m^*$ it follows that[12]

$$2\Lambda_m \ \subset \ 2\Lambda_m^* \ = \ \sqrt{2}\sigma_2\, \Lambda_m \ \subset \Lambda_m \ \subset \ \Lambda_m^* . \tag{2.43}$$

The electrically and magnetically charged objects of the heterotic frame can be mapped to other duality frames and we say a little bit more about this in section 9.2, see especially Tab. 9.1 there. In

[11] In correlation with footnote 3 on page 14, there are different conventions for the overall sign of the quadratic form on the electric and magnetic charge lattice.

[12] The inclusion $2\Lambda_m^* \subset \Lambda_m$ is claimed in [2, 27], equivalent to $\langle 2v, w\rangle \in \mathbb{Z}$ for all $v, w \in \Lambda_e$.

the type IIA perspective the lattices are understood as

$$\Lambda_e = \left(H^*(\mathrm{K3}, \mathbb{Z})^g\right)^* \oplus U \oplus U\left(\tfrac{1}{N}\right) \tag{2.44}$$

and

$$\Lambda_m = H^*(\mathrm{K3}, \mathbb{Z})^g \oplus U \oplus U(N) = \Lambda_e^* \,. \tag{2.45}$$

Duality actions and charge invariants. The charge lattice Λ_{em} alone is not yet optimal as an organizing principle for the BPS-dyon spectrum. The reason is that the discrete duality groups $\mathcal{T}$ and $\mathcal{S}$ have a non-trivial action on them. As these dualities should be thought of physical equivalences of the theory (identifying points in the physical points in the moduli space), it is a natural question to ask how the BPS spectrum, and the charges the BPS states carry, transform under the duality groups. Indeed, our point of view will be to partition the electric-magnetic lattice into orbits of the duality groups and to find the BPS spectrum, practically the according partition function, for each such orbit. For technical reasons we will not determine the quarter-BPS spectrum in arbitrary charge orbits in Λ_{em}, but only for those which satisfy the so-called unit-torsion condition (introduced shortly). One reason is that this restriction will allow us to determine the (quarter-)BPS partition functions from a heterotic genus-two partition function in chapter 6. Even with this restriction, there will however be several disjoint charge orbits left and we want to characterize them using suitable duality invariants built using the charge vectors. Thus, we will now explain the action of the duality groups on the charges and introdcue a set of suitable invariants.

As mentioned, the heterotic string compactified on a torus and likewise its CHL orbifold enjoy the action of an infinite discrete T-duality group $\mathcal{T}$, which establishes the stringy equivalence between different classical backgrounds of the compactification and especially acts non-trivially on the massless fields entering the low-energy effective action. Besides a rotation of the field 20 strengths $F^{(i)}_{\mu\nu}$ (see (8.18)), which implies a rotation of the electric charges Q_i and the magnetic charges P_i, we have an action on the scalar moduli M (in (2.35)) associated with the Grassmanian. A T-transformation $\mathcal{T} \ni O$ fixes the heterotic axio-dilaton modulus S_{het} but acts on M and the charges as[13]

$$\begin{pmatrix} Q \\ P \end{pmatrix} \mapsto \begin{pmatrix} O^{-\intercal} Q \\ O^{-\intercal} P \end{pmatrix} , \quad M \mapsto OMO^{\intercal} \,. \tag{2.46}$$

Using the inner product on the electric-magnetic charge lattice (metric L) one can define quadratic expressions

$$Q^2 = Q^{\intercal} L Q \,, \qquad P^2 = P^{\intercal} L P \qquad \text{and} \qquad Q \cdot P = Q^{\intercal} L P \,, \tag{2.47}$$

which do not change under T-transformations and will thus be called the quadratic T-invariants.

An element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$ of the S-duality group acts on dyonic states with charge $(Q, P)^{\intercal} \in \Lambda_{em}$ in the standard way [19, eq. (2.8)]:

$$\begin{pmatrix} Q \\ P \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} Q \\ P \end{pmatrix} , \quad S_{\text{het}} \mapsto \frac{aS_{\text{het}} + b}{cS_{\text{het}} + d} \,. \tag{2.48}$$

[13] We use the notation $O^{-\intercal} = (O^{\intercal})^{-1}$.

The S-action of $\Gamma_1(N)$ on the quadratic T-invariants Q^2, P^2 and $Q \cdot P$ follows from (2.48). For later convenience let us also introduce the map[14]

$$t : \Lambda_{em} \to \mathbb{Q}^3 \ , \ (Q, P) \mapsto \left(\frac{P^2}{2}, Q \cdot P, \frac{Q^2}{2} \right) . \tag{2.49}$$

There are further, discrete T-duality invariants characterizing the duality orbit of a charge (Q, P). Following [6], take some basis of the lattice Λ_{em} and denote the integer coordinates of a charge (Q, P) with respect to this basis by Q_i and P_i, the greatest common divisor of the integers $(Q_i P_j - Q_j P_i)$, denoted as

$$I = \gcd(Q \wedge P) \ , \tag{2.50}$$

will then be a T-duality[15] and S-duality invariant, sometimes called torsion.[16] It has been shown that for heterotic strings on T^6 the quantity I and the above quadratic T-invariants are sufficient to uniquely determine a duality orbit under S- and T-transformations. If S-transformations are left out, apart from I and the quadratic T-invariants three *further* discrete T-invariants (on which the S-duality group acts non-trivially) are needed to characterize a T-orbit unambigously, see [55, 56] and [11, section 5.3] for details. Just in the special case $I = 1$, which fixes the remaining three discrete T-invariants to unity, there is a single T-orbit.

As was also pointed out in [3, app. B], the precise duality group $G_4(\mathbb{Z})$ of a four-dimensional $\mathbb{Z}_N$ CHL model with $N > 1$ is not yet (conclusively) determined, nor is a complete set of duality invariants that uniquely specifies the distinct charge orbits in Λ_{em} with respect to $G_4(\mathbb{Z})$. In any case, we expect that again finitely many duality invariants suffice to uniquely determine a duality orbit. Having several distinct duality orbits of charges means we should also expect several a priori distinct degeneracies associated to states with charge in the respective orbits. In this work we elaborate on this idea in the case of counting dyonic quarter-BPS states in the $\mathbb{Z}_2$ CHL model. For simplicity we will, as mentioned, focus on charges satisfying $I = 1$. However, in contrast to the unorbifolded theory (heterotic strings on T^6), this alone is not expected to uniquely specify a duality orbit, as there is at least one more discrete (candidate) charge invariant.

Following [3, app. B] the "residue" of a charge $(Q, P) \in \Lambda_{em}$ is defined as the class in the discriminant group[17]

$$r(Q, P) = [Q] \ \in \Lambda_e / \Lambda_e^* \ . \tag{2.51}$$

This quantity was shown to be invariant under $\Gamma_1(N) \times C_{(g,\delta)}$. For the $\mathbb{Z}_2$ model the discriminant group explicitly reads

$$\Lambda_e / \Lambda_e^* = \mathbb{Z}_2^2 \times \mathbb{Z}_2^8 \ , \tag{2.52}$$

where the first factor comes from $U(\frac{1}{2})/U(2)$ and the second factor from $E_8(\frac{1}{2})/E_8(2)$. In the

[14] Because of (2.44), (2.45) and (2.41) $P^2/2$ and $Q \cdot P$ are actually integral.

[15] As shown in [55, section 2] a change of basis given by an $\mathrm{SL}_r(\mathbb{Z})$ matrix leaves the gcd invariant (there rank $r = 22 + 6$ was considered). If $\mathcal{T} \subset \mathrm{O}(\Lambda_e) \subset \mathrm{SL}_r(\mathbb{Z})$ this argument also holds for the $\mathbb{Z}_2$ CHL orbifolds.

[16] We give some remarks. (1.) First note that (Q, P) being primitive in Λ_{em} does not imply that $Q \in \Lambda_e$ or $P \in \Lambda_m$ is primitive. In turn, if Q or P is primitive, then (Q, P) is primitive as well. (2.) If Q or P is non-primitive then $I > 1$. On the other hand, $I > 1$ does not imply that Q or P are non-primitive, as the example in [22, subsection 6.3] with $I = 2$ shows: there both Q and P are primitive (and $Q \pm P$ are both twice a primitive vector). So $I = 1$ is a sufficient, but not necessary condition for having both Q and P primitive.

[17] Recall $\Lambda_e / \Lambda_e^* \cong \Lambda_m^* / \Lambda_m$ so definition (2.51) is equivalent to the one given in [3, app. B].

perturbative heterotic description (purely electric states) the respective components of $[Q]$ can be interpreted in terms of the momentum-winding numbers along the CHL circle and the internal E_8 momentum. Especially, one $\mathbb{Z}_2$ component of $\mathbb{Z}_2^2$ distinguishes whether the state lies in the untwisted (i.e., integral CHL winding number) or twisted (strictly half-integral CHL winding number) orbifold sector.[18] Correspondingly, we will simply call electric charges untwisted sector charges or twisted sector charges. We will come back to the other components of $[Q]$ in section 5.2.

As already mentioned in the introduction, for a part of the spectrum of supersymmetric (quarter-BPS) states in the $\mathbb{Z}_2$ orbifold the partition was already known for some years [1]. The dyon partition function introduced in [1] counts unit-torsion quarter-BPS dyons whose electric charge belongs to the twisted sector in the above nomenclature (see, for instance, the discussion in [11, section 5.3]). Our goal is to propose partition functions belonging to other (unit-torsion) charge sectors (chapter 6), perform consistency checks (chapter 7) and study the entropy of the associated black hole configurations (chapter 8). First steps in this direction were undertaken in [22, section 6.5] for the $\mathbb{Z}_2$ model by analyzing a closed subsector of the untwisted sector of unit-torsion dyons. Although no closed formula for the respective partition function was given, strong constraints on the latter coming from charge quantization, wall-crossing and S-duality invariance were given. We will later verify this subsector result in section 7.

But before we can tackle these goals we shall give a proper introduction to BPS representations, introduce the BPS indices that count them (chapter 3) and discuss which properties we expect from the partition functions for BPS indices that specifically count quarter-BPS dyons (chapter 4).

[18] Occasionally, as for instance in [11, 22] one encounters somewhat different conventions, where twisted sector states have an odd winding number along the CHL circle, which has half the radius of parent theory. The untwisted sector states then have even winding number along the CHL circle.

CHAPTER 3

Supermultiplets and supertraces

This chapter collects necessary background material on representations of the $\mathcal{N} = 4$ supersymmetry algebra in four dimensions and defines suitable indices that characterize and count these (abstract) representations. In four-dimensional string theories with 16 real supercharges, such as the Narain compactification to four dimensions or its $\mathbb{Z}_2$ CHL orbifold, the spectrum of the effective four-dimensional theory is organized in terms of such representations (supermultiplets). But before diving into a technical description, we shall recall that our main interest lies in the so-called BPS representations, for which we first give a brief qualitative discussion.

Due to the extended supersymmetry a distinction between massive BPS and massive non-BPS representations can be made. The states in BPS representations enjoy some special properties that make them useful for establishing dualities or studying strongly coupled phases of string theory.[1] For BPS states the mass equals the (largest in absolute values) central charge of the supersymmetry algebra. The latter depends only on the quantized electric and magnetic charges as well as the non-renormalized coupling constants, the non-renormalization being due to the high amount of supersymmetry. So neither the central charge nor the mass gets renormalized in $\mathcal{N} = 4$ theories.

Furthermore, the relation between the mass and the central charge necessarily implies that BPS representations are shorter than generic massive representations. If now the mass increased relative to the central charge when varying the coupling constants, the length of the corresponding BPS multiplet would need to jump discontinuously. This can only happen when shorter multiplets combine into longer ones, subject to charge and energy conservation. So at generic points in moduli space a BPS state will be stable. If one is able to count only the shorter ones modulo the longer ones, which is what the various BPS indices (helicity supertraces) introduced in this chapter do, one expects this information to be extrapolable to say strong-coupling regimes. This, for instance, is especially useful if in the strong-coupling regime the BPS states describe black holes. Indeed, this approach is a common theme in string theory: much information about the non-perturbative structure of the theory, which usually is hard to extract, is obtained by studying this robust BPS part of the spectrum using duality arguments and perturbative computations.

For $\mathcal{N} = 4$ supersymmetry the BPS representations are divided into (short) half- and (intermediate) quarter-BPS representations. However, the quarter-BPS indices that count the quarter-BPS multiplets (modulo generic massive multiplets) we are eventually interested in are nevertheless not constant, but

[1] The reader will find a general discussion of the significance of BPS states in string compactifications for instance in [10, section 14.2, app. E, G], [8, section 18.4], [13, section 3.5] and [9, ch. 8].

only locally constant in moduli space. Discontinuities are known to occur in codimension one in moduli space, i.e., at the walls of marginal stability. In the black hole (supergravity) perspective this can be explained by the (dis-)appearance of two-centered bound states of two half-BPS black holes. At the walls of marginal stability their spatial separation becomes infinite and on the other side of the wall negative such that the corresponding states disappear from the physical BPS spectrum when crossing the wall [57]. Consequences of this wall-crossing for the partition functions of quarter-BPS indices will be discussed later in section 4.4. The upshot is that if these discontinuities are absent or sufficiently well understood, one may still extrapolate spectral BPS information for instance from weak to strong coupling, or from any corner of the moduli space where the computation is most easily done.

Let us now describe the BPS representation and BPS indices on a more technical level.

3.1 BPS representations of the $\mathcal{N} = 4$ superalgebra

In brief, BPS states are states that transform in an appropriate representation of the $\mathcal{N}$-extended supersymmetry algebra, characterized by certain relations between the mass of the state (or rather the mass of the representation) and the central charges of the algebra.

Recall that the supersymmetry algebra in four spacetime dimensions is an extension of the Poincaré algebra, obtained by introducing fermionic generators Q^I_α called supercharges. For each $I = 1, \ldots, \mathcal{N}$ the subscript $\alpha = 1, 2$ denotes the two components of a Weyl spinor (in the $(\frac{1}{2}, 0)$ spinor representation). It will be sufficient in the following to focus on a part of the algebra specified by anticommutators of the supercharges Q^I_α and their hermitian conjugates $\bar{Q}^I_{\dot\alpha} = (Q^I_\alpha)^\dagger$ (which transform in the $(0, \frac{1}{2})$ representation). For a more in-depth discussion we refer the reader to the textbooks [10, 58], which we follow, or the original paper [59]. The relevant relations for us are:

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ} \tag{3.1}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \tag{3.2}$$

$$\{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} \bar{Z}^{IJ} \; . \tag{3.3}$$

Here $\sigma^\mu = (-1_2, \sigma^i)$ are the Pauli matrices, P_μ denotes the momentum operator (the generator of translations), while $\epsilon_{\alpha\beta}$ (with $\epsilon_{12} = -1$) is the standard antisymmetric tensor. The central charges $Z^{IJ} = -Z^{JI}$ are complex and commute with all generators of the superalgebra. For $\mathcal{N} = 4$, which we assume in the following, we have $2\mathcal{N} = 8$ complex supercharges, or in other words sixteen real supercharges.

Let us consider a state transforming in a massive representation of the algebra, i.e., we may go to the rest frame with $P_\mu = (-M, 0, 0, 0)$ and $M > 0$. The mass square $M^2 = -P^2$ is the eigenvalue of a quadratic Casimir operator and constant along the representation. Upon a suitable unitary transformation of the central charge matrix it can be brought to block-diagonal form

$$\tilde{Z} = \begin{pmatrix} 0 & Z_1 & 0 & 0 \\ -Z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z_2 \\ 0 & 0 & -Z_2 & 0 \end{pmatrix} \tag{3.4}$$

with Z_1 and Z_2 being two positive numbers. Without loss of generality $Z_1 \geq Z_2$. The indices I, J

may be decomposed into a pair (a, m) with $a = 1, 2$ and $m = 1, \ldots, \mathcal{N}/2$, i.e., m labels the blocks. Transforming also the supercharges with a suitable unitary transformation the above anticommutation relations may be recast into the form

$$\{\tilde{Q}^{am}_{\alpha}, (\tilde{Q}^{bn}_{\beta})^{\dagger}\} = 2M\delta^{\beta}_{\alpha}\delta^{ab}\delta^{mn} \tag{3.5}$$

$$\{\tilde{Q}^{am}_{\alpha}, \tilde{Q}^{bn}_{\beta}\} = \epsilon_{\alpha\beta}\epsilon^{ab}\delta^{mn}Z_n \tag{3.6}$$

$$\{(\tilde{Q}^{am}_{\alpha})^{\dagger}, (\tilde{Q}^{bn}_{\beta})^{\dagger}\} = \epsilon_{\alpha\beta}\epsilon^{ab}\delta^{mn}Z_n \; . \tag{3.7}$$

Using the supercharges and their conjugates we can now define two sets of annihilation operators

$$a^{m}_{\alpha} = \frac{1}{\sqrt{2}}\left(\tilde{Q}^{1m}_{\alpha} + \epsilon_{\alpha\rho}(\tilde{Q}^{2m}_{\rho})^{\dagger}\right) \tag{3.8}$$

$$b^{m}_{\alpha} = \frac{1}{\sqrt{2}}\left(\tilde{Q}^{1m}_{\alpha} - \epsilon_{\alpha\rho}(\tilde{Q}^{2m}_{\rho})^{\dagger}\right) \; , \tag{3.9}$$

and their conjugates will consquently be regarded as creation operators. This interpreation as fermionic creation and annihilation operators is motivated by the algebra they satify:

$$\{a^{m}_{\alpha}, a^{n}_{\beta}\} = \{b^{m}_{\alpha}, b^{n}_{\beta}\} = \{a^{m}_{\alpha}, b^{n}_{\beta}\} = 0 \tag{3.10}$$

$$\{a^{m}_{\alpha}, (a^{n}_{\beta})^{\dagger}\} = \delta_{\alpha\beta}\delta^{mn}(2M + Z_n) \tag{3.11}$$

$$\{b^{m}_{\alpha}, (b^{n}_{\beta})^{\dagger}\} = \delta_{\alpha\beta}\delta^{mn}(2M - Z_n) \tag{3.12}$$

It can be shown that unitarity implies the so-called Bogomol'nyi-Prasad-Sommerfield (BPS) bound

$$M \geq \frac{Z_1}{2} \geq \frac{Z_2}{2} \; . \tag{3.13}$$

In the generic case (no equality in (3.13)) we have $2 \cdot 2 \cdot 2 = 8$ fermionic creation operators, which can be used to construct a 2^8 dimensional representation by acting on a Clifford vacuum, defined as being annihilated by all annihilation operators. On the other hand, if $M = \frac{Z_1}{2} > \frac{Z_2}{2}$, the b^{1}_{β} operators vanish identically and the representation is constructed from 6 remaining creation operators, hence is 2^6 dimensional. Lastly, if $M = \frac{Z_1}{2} = \frac{Z_2}{2}$, all b-operators vanish and we are left with the a-operators only. This representation is 2^4 dimensional. The representations of the three cases are respectively known as non-BPS, quarter-BPS and half-BPS supermultiplets, and sometimes also as long, intermediate and short multiplets.

We can relabel the non-trivial a- and b-operators as c-operators and extend the range for n appropriately, i.e., $n = 1, \ldots, N$ with $2N$ being 8, 6 or 4 for the cases of interest. As the distinct creation operators anticommute, a state constructed by acting with ν creation operators $(c^{n_i}_{\alpha_i})^{\dagger}$ is antisymmetric under exchange of any pair of indices (α_i, n_i). There are $\binom{2N}{\nu}$ such states, accordingly we have

$$\sum_{\nu=0}^{2N} \binom{2N}{\nu} = 2^{2N} \tag{3.14}$$

states in total, as was stated before. States with ν even are bosons (integer spin), while states with ν odd are fermions (strictly half-integer spin). There are as many bosonic as fermionic states in the supermultiplet, namely 2^{2N-1} each. The highest spin state is obtained by symmetrizing the maximal number of spinor indices α_i. Since we simultaneously need to antisymmetrize in the n_i indices to get a non-zero state, the maximal spin state of the supermultiplet is obtained for $\nu = N$, yielding a state of spin $N/2$. For the three types of multiplets discussed, this is a spin-2, a spin-3/2 and a spin-1 state, respectively.

In the above we implicitly assumed that the Clifford vacuum to start with is a Lorentz scalar (spin 0). In case it transforms in a spin $[j]$ representation, which has a degeneracy of $D_j = 2j + 1$, the resulting supermultiplet is a tensor product whose dimension is multiplied by same factor the D_j. For instance, for a half-BPS (short) massive multiplet the $2^4 D_j$ degrees of freedom are obtained from

$$S_j : \quad [j] \otimes (5[0] + 4[1/2] + [1]) \ , \tag{3.15}$$

while those for a quarter-BPS (intermediate) massive multiplet follow from

$$I_j : \quad [j] \otimes (14[0] + 14[1/2] + 6[1] + [3/2]) \ . \tag{3.16}$$

A generic non-BPS (long) massive multiplet has $2^8 D_j$ degrees of freedom:

$$L_j : \quad [j] \otimes (42[0] + 48[1/2] + 27[1] + 8[3/2] + [2]) \ . \tag{3.17}$$

Massless multiplets. Only in case of vanishing central charges we can have massless multiplets. An argument similar to the one above shows that multiplets can be constructed from 4 fermionic creation operators acting on a Clifford vacuum, which is the state of lowest helicity $\underline{\lambda}$ within the multiplet. The highest helicity within the multiplet will then be $\bar{\lambda} = \underline{\lambda} + \mathcal{N}/2$. To obtain a multiplet that closes under CPT, it might be necessary to double the degrees of freedom by adding the multiplet corresponding to the conjugate particles, which have the sign of the helicities reversed. For example, if the massless Clifford vacuum has helicity zero, the highest helicity obtained will be +2. The conjugate states then are constructed by starting from a second Clifford vacuum with helicity −2 such that the highest helicity obtained by acting with the creation operators will be 0. As a consequence, the completed multiplet, which is called supergravity multiplet since it contains the graviton states with $\lambda = \pm 2$, now has 32 degrees of freedom, 16 bosonic ones and 16 fermionic ones. Table 3.1 enumerates the contributing degrees of freedom sorted by their helicity λ. Also shown are the (CPT completed) gravitino multiplet and the vector multiplet. Note that the latter is already CPT complete with 16 states in total.

Let us also remark that massless particles with helicities greater than 2 are believed to be impossible to couple to gravity and do not occur in string theory [60, app. B]. Independent of that statement, such fields will not be needed in this thesis anyway. In fact, the formally constructed gravitino multiplet in Tab. 3.1 will not be needed either, as it does not arise in the four-dimensional supergravity theories obtained from the Narain or CHL compactification, but just serves as an example of the above construction.

Central charges. In string compactifications the central charges of a BPS representation depend on the electric and magnetic charges (Q, P) of the corresponding states and the moduli of the theory (i.e., the expectation values of the scalar fields). An explicit description can be found in [61], but it will not be needed here. What we only need is the fact that for quarter-BPS states with electric-magnetic

$\lambda\downarrow$ $\underline{\lambda}\rightarrow$	supergrav. m. 0	-2	gravitino m. $-\frac{3}{2}$	$-\frac{1}{2}$	vector m. -1
2	1				
$\frac{3}{2}$	4			1	
1	6			4	1
$\frac{1}{2}$	4		1	6	4
0	1	1	4	4	6
$-\frac{1}{2}$		4	6	1	4
-1		6	4		1
$-\frac{3}{2}$		4	1		
-2		1			

Table 3.1: The degrees of freedom of massless multiplets in $\mathcal{N} = 4$ supersymmetry.

charge $(Q, P) \in \Lambda_{em}$ the two components must satisfy $Q \nparallel P$, i.e., the electric and magnetic charges are not collinear as vectors in $\mathbb{R}^{20}$. For half-BPS states the charges obey the opposite charge condition, $Q \| P$ [61].

We also remark that in a given charge sector and for given moduli the BPS states are the states of the lowest mass. Charge and energy conservation prohibit their decay at generic points in the moduli space.

3.2 Helicity supertraces as BPS indices

Let us now define and discuss the indices that count the BPS states of interest, following [10, 62] (see also [63–65]).

Helicity supertraces are defined for a given representation R of the supersymmetry algebra. In four dimensions they involve Casimir operators of the little group of the Lorentz group. For massless representations this is the "helicity" taken to an even power. For massive representations this is the third component of the angular momentum in the rest frame. Both shall be denoted by λ. The helicity supertrace is then defined as

$$B_{2n}(R) := \mathrm{Tr}_R \left[(-1)^{2\lambda} \lambda^{2n} \right] . \tag{3.18}$$

Here $(-1)^{2\lambda} = (-1)^F$ is also called fermion number operator. Odd helicity supertraces vanish by CPT-invariance. These quantities can also be obtained from the generating function

$$Z(R; y) := \mathrm{Tr}_R \left[(-1)^{2\lambda} y^{2\lambda} \right] \tag{3.19}$$

via suitable derivatives:

$$B_{2n}(R) = \left(y^2 \frac{\partial}{\partial(y^2)} \right)^{2n} Z(R; y) \bigg|_{y=1} . \tag{3.20}$$

For a spin-$[j]$ Clifford vacuum (or equivalently, representing all supercharges by zero) the generating function is the Laurent polynomial of

$$Z_{[j]}(y) = \begin{cases} (-1)^{2j} \left(\frac{y^{2j+1}-y^{-2j-1}}{y-y^{-1}}\right) & : \text{massless rep.} \\ (-1)^{2j} \left(y^{2j} + y^{-2}\right) & : \text{massive rep.} \end{cases} . \tag{3.21}$$

If a supermultiplet is built from a spin-$[j]$ Clifford vacuum using precisely $2N$ non-trivial creation operators the generating function becomes

$$Z_{[j],N}(y) = Z_{[j]}(y)(1-y)^N(1-y^{-1})^N . \tag{3.22}$$

In general the generating function for a tensor product of two representations is simply the product of the two individual generating functions,

$$Z_{R\otimes R'}(y) = Z_R(y)Z_{R'}(y) . \tag{3.23}$$

Let us now specialize to the $\mathcal{N}=4$ case, where we are mainly interested in (counting) massive BPS states. Applying (3.20) to (3.22) we find that $B_0(R)$ and $B_2(R)$ vanish for any representation $R \in \{S_j, I_j, L_j\}$ (i.e., for any $2N = 4, 6, 8$ and any $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$). The fourth helicity supertrace in turn is non-trivial, but only sensitive to the short half-BPS representations:

$$B_4(L_j) = B_4(I_j) = 0, \qquad B_4(S_j) = (-1)^{2j}\frac{3}{2}D_j . \tag{3.24}$$

Going one step further, the sixth helicity supertrace is then non-vanishing for both half- and quarter-BPS multiplets while vanishing for non-BPS representations:

$$B_6(L_j) = 0, \quad B_6(I_j) = (-1)^{2j+1}\frac{45}{4}D_j, \qquad B_6(S_j) = (-1)^{2j}\frac{15}{8}D_j^3 . \tag{3.25}$$

Starting from the eight helicity supertrace, both BPS and non-BPS multiplets yield a non-trivial contribution. These results can in principle also be obtained from the definition (3.18) using the helicity content in eqs. (3.15)-(3.17). The fourth and sixth helicity supertrace are often simply called half- and quarter-BPS index, respectively.

Helicity supertraces in string compactifications. We have just learned that the fourth and sixth helicity supertrace are the best suited cases to study the BPS spectrum of a four-dimensional $\mathcal{N}=4$ supersymmetric theory.

More specifically, since a quarter-BPS dyon breaks 12 out of 16 supercharges a non-trivial index to "count" such states in a four-dimensional $\mathcal{N}=4$ string compactification is the sixth helicity supertrace. In analogy with the just defined trace on an (abstract) supersymmetry representation R, we would now like to consider traces in the Hilbert space of the 4D theory, more precisely in the subspace of the full Hilbert space that belongs to states of a fixed electric-magnetic charge $(Q,P) \in \Lambda_{em}$. This charge lattice gives a grading to the full Hilbert space of the theory. This sixth helicity supertrace is usually denoted by $\Omega_6(Q,P;\cdot)$ (e.g. in [2, 26]) and similar for the fourth helicity supertrace. The dot in $\Omega_6(Q,P;\cdot)$ represents the moduli of the theory. Recall that locally this index is constant, but it changes discontinously once the asymptotic moduli of the theory are varied across certain real codimension one subspaces, called walls of marginal stability. We will discuss wall-crossing in more detail in subsection 4.4.

Of course, computing $\Omega_6(Q, P; \cdot)$ is a somewhat daunting task, as it is not even clear how to describe the full Hilbert space of the 4D string theory, not least because we lack a fully non-perturbative description to work with. This is in principle why we have to resort to a combination perturbative worldsheet computations, duality arguments and insights from the supergravity approximation.

In order to nevertheless make the transition from abstract representations and supertraces evaluated on them to state counting in string compactifications a bit more plastic, the reader may wish to jump to section 5.2, where fourth helicity supertraces $\Omega_4(Q, 0)$ are computed in the perturbative heterotic Hilbert space of the $\mathbb{Z}_2$ CHL string. These count half-BPS states, which moreover are purely electric of charge Q. The procedure there is in close analogy with the one presented above, namely we first give a generating function Z and then take suitable derivatives with respect to the auxiliary fugacities (or chemical potentials) to obtain the supertraces. An additional step (taking Fourier coefficients) is required in this case, as we then still have to specify the charge Q (or actually Q^2).

In the following chapter we introduce partition functions for quarter-BPS indices.

CHAPTER 4

The structure of quarter-BPS partition functions

In this chapter[1] we turn to a discussion of quarter-BPS partition functions in $\mathcal{N} = 4$ string compactifications, following [22]. Many details will be omitted and can be found in the reference. The highlighted properties are mostly generalizations of observations made for specific instances of such partition functions, notably the (unit-torsion) dyon partition function of [21] (and charge subsector truncations) and the (unit-torsion) twisted sector partition functions introduced in [1] for the CHL orbifolds. Collecting these properties serves a dual purpose. First, they put strong consistency checks on a any quarter-BPS partition function to be derived in chapter 6 using the genus two heterotic computation. Second, as we will discuss in parallel when going through these checks in chapter 7, they are (almost) sufficient to "bootstrap" the desired partition functions in closed form.

4.1 Charge sectors for quarter-BPS dyon counting

For the purpose of analyzing or constraining a (quarter-BPS) dyon partition function it may be convenient to reduce the problem to analyzing charge subsectors, for which the counting problem simplifies. Let us introduce some notation. For a set of electric-magnetic charges $\mathcal{Q} \subset \Lambda_{em}$ we define the following conditions:

(Q1) *Quarter-BPS condition*:
For all $(Q, P) \in \mathcal{Q}$ we have $Q \nparallel P$.

(Q2) *Unit-torsion condition*:
For all $(Q, P) \in \mathcal{Q}$ we have $I = \gcd(Q \wedge P) = 1$.

(Q3) *T-closure condition*:
For any given triplet (q_1, q_2, q_3) of the quadratic T-invariants the set

$$\left\{ (Q, P) \in \mathcal{Q} \,\middle|\, \left(\frac{P^2}{2}, Q \cdot P, \frac{Q^2}{2} \right) = (q_1, q_2, q_3) \right\}, \tag{4.1}$$

if not empty, maps to itself under the action of the T-duality group $\mathcal{T}$.

[1] This chapter appeared as section 2.2 in the publication [35].

(Q4) *T-transitivity condition*:
Any two elements of subsets of the form (4.1) are related via $\mathcal{T}$.

(Q5) *Unboundedness condition*:
Any of the quadratic T-invariants takes arbitrarily large absolute values on $\mathcal{Q}$.

(Q6) *Quantization condition*:
There are rational numbers $q_i \in \mathbb{Q}^+$ such that for any $(Q, P) \in \mathcal{Q}$ we can find integers $\nu_i \in \mathbb{Z}$ satisfying

$$\frac{P^2}{2} = \nu_1 q_1 \ , \ \ Q \cdot P = \nu_2 q_2 \ , \ \ \frac{Q^2}{2} = \nu_3 q_3 \ . \tag{4.2}$$

Some remarks are in order. If $Q \parallel P$, then $Q \wedge P = 0$, so (Q2) implies (Q1). The T-closure condition (Q3) obviously transfers to the whole set $\mathcal{Q} = \mathcal{T}\mathcal{Q}$. Condition (Q4) especially implies that any (further) T-invariants become constant functions on sets of the form (4.1). Under both assumptions (Q3) and (Q4) a unique representative can be chosen for any non-empty set of the form (4.1) and the remaining elements of that set are precisely all $\mathcal{T}$-images of it. Furthermore, condition (Q6) is always satisfied for *some* rational numbers $q_i \in \mathbb{Q}^+$ (c.f. eqs. (2.44) and (2.45)) and from now on we consider the *maximal* numbers $\mathsf{q}_i \in \mathbb{Q}^+$ for which (4.2) is satisied.[2] If (Q3) to (Q6) are satisfied the T-orbits (4.1) are in one-to-one correspondence with points in $t(\mathcal{Q})$, which form a subset of some rank-three lattice shifted by a non-zero vector, $\mathsf{L} \subset \mathbb{Q}^3$. The charge examples in [22] are constructed such that already the T-representatives form a shifted rank-three lattice $\mathsf{L}_{\mathcal{Q}} \subset \Lambda_{em}$ which then bijects to its T-invariants $t(\mathcal{Q}) = \mathsf{L}$ and $\mathcal{Q}$ is obtained by simply taking all T-images, $\mathcal{Q} = \mathcal{T}\mathsf{L}_{\mathcal{Q}}$. In this way (Q1)-(Q6) are satisfied simultaneously.

We make the standard assumption that the sixth helicity supertrace $\Omega_6(Q, P; \cdot)$ (or simply BPS index in the following) is invariant under T-transformations, i.e., at a given generic point in the moduli space it only depends on the duality orbit of $(Q, P) \in \Lambda_{em}$. It is also S-invariant if charges and moduli are transformed simultaneously. Given $\mathcal{Q}$ satisfying (Q1), (Q3) and (Q4), because of the T-invariance the BPS index of dyons with charge $(Q, P) \in \mathcal{Q}$ will already be uniquely determined by specifying the quadratic T-invariants of the charge and for some appropriate $f_{\mathcal{Q}}$ we have

$$\Omega_6(Q, P; \cdot) = f_{\mathcal{Q}}(P^2, Q \cdot P, Q^2 \, ; \cdot) \ . \tag{4.3}$$

One can also introduce a partition function for these numbers via[3]

$$\mathsf{Z}_{\mathcal{Q}}(\tau, z, \sigma) = \frac{1}{\Phi_{\mathcal{Q}}(\tau, z, \sigma)} := \sum_{P^2, Q\cdot P, Q^2} (-1)^{Q\cdot P+1} f_{\mathcal{Q}}(P^2, Q\cdot P, Q^2\, ; \cdot)\, e^{2\pi i\left(\tau \frac{P^2}{2} + z\, Q\cdot P + \sigma \frac{Q^2}{2}\right)} , \tag{4.4}$$

where a sign factor has been introduced to follow conventions in [22] and the sum runs over all quadratic values belonging to charge vectors $(Q, P) \in \mathcal{Q}$.

[2] This becomes relevant when the charges in $\mathcal{Q}$ satisfy coarser quantization conditions than Λ_{em}, as applying to the charge sets considered in [22, section 6]. In their simplest example one has a charge set $\mathcal{Q} \subset \Lambda_{em}$ for which $Q^2/2$ only takes even values, leading to $\mathsf{q}_3 = 2$ in that case, while $\Lambda_e = U^{\oplus 6} \oplus E_8(1)^{\oplus 2}$ (considering charges of Het[T^6]) also allows for odd values of $Q^2/2$ (corresponding to $q_3 = 1$).

[3] Following [22], we also introduced $\Phi_{\mathcal{Q}} := (\mathsf{Z}_{\mathcal{Q}}(\tau, z, \sigma))^{-1}$. Writing the partition function in the form $\mathsf{Z}_{\mathcal{Q}}(\tau, z, \sigma) = \frac{1}{\Phi_{\mathcal{Q}}(\tau, z, \sigma)}$ is alluding to the original DVV result $1/\chi_{10}$ and the CHL orbifold analogs considered by Sen et al.

Under the condition (Q5) the partition function is expected to have infinitely many non-zero terms.[4] Typically the generalized chemical potentials τ, z, σ conjugate to $P^2/2$, $Q \cdot P$ and $Q^2/2$, must lie in a suitable domain of the Siegel upper half plane $\mathbb{H}_2$ for this series to converge (see appendix A for a definition) and we will assume that this is the case. Different domains of convergence admit different Fourier expansions, which in turn give BPS indices valid for different regions of the moduli space. As $\mathcal{Q}$ satisfies (Q6), the partition function will be periodic:

$$\forall n_1, n_2, n_3 \in \mathbb{Z}: \quad \mathsf{Z}_{\mathcal{Q}}\left(\tau + \frac{n_1}{\mathsf{q}_1}, z + \frac{n_2}{\mathsf{q}_2}, \sigma + \frac{n_3}{\mathsf{q}_3}\right) = \mathsf{Z}_{\mathcal{Q}}(\tau, z, \sigma)\ . \tag{4.5}$$

BPS indices can be extracted from $\mathsf{Z}_{\mathcal{Q}}$ by taking an appropriate contour integral

$$f_{\mathcal{Q}}(P^2, Q \cdot P, Q^2; \cdot) = \frac{(-1)^{Q\cdot P+1}}{(\mathsf{q}_1\mathsf{q}_2\mathsf{q}_3)^{-1}} \oint_C \frac{e^{-2\pi i\left(\tau\frac{P^2}{2} + z\, Q\cdot P + \sigma\frac{Q^2}{2}\right)}}{\Phi_{\mathcal{Q}}(\tau, \sigma, z)}\, \mathrm{d}\tau \wedge \mathrm{d}\sigma \wedge \mathrm{d}z \tag{4.6}$$

over a (minimal) period in each direction at some fixed, large imaginary part. In this work we will stay schematic with regard to the choice of integration contour, which could in principle be analyzed more carefully as in [61], see also [22, 66]. As mentioned before, we are mainly concerned with quarter-BPS dyons of unit-torsion, and for these dyons we assume the validity of the moduli-dependent contour proposed in [61].

For quarter-BPS dyons of unit-torsion we expect that a *finite* number of discrete T-invariants provides a partition of the set

$$\left\{ (Q, P) \in \Lambda_{em} \,\middle|\, \gcd(Q \wedge P) = 1 \right\} \tag{4.7}$$

into a *finite* number of pairwise disjoint subsets $\mathcal{Q}$, each obeying (Q1) to (Q6). The important point is that this yields a finite set of (a priori different) quarter-BPS partition functions $\mathsf{Z}_{\mathcal{Q}}$.

We remark that for any two of such disjoint charge sets $\mathcal{Q}, \mathcal{Q}'$ with quarter-BPS partition functions $\mathsf{Z}_{\mathcal{Q}}, \mathsf{Z}_{\mathcal{Q}'}$, respectively, one can formally define the sum $\mathsf{Z}_{\mathcal{Q}} + \mathsf{Z}_{\mathcal{Q}'}$. If there are no common triplets of quadratic T-invariants, $t(\mathcal{Q}) \cap t(\mathcal{Q}') = \emptyset$, hence no common triple exponents in the respective expansion of the type (4.4), $\mathcal{Q} \cup \mathcal{Q}'$ again satisfies (Q1) to (Q6) and $\mathsf{Z}_{\mathcal{Q}} + \mathsf{Z}_{\mathcal{Q}'}$ can be interpreted as $\mathsf{Z}_{\mathcal{Q}\cup\mathcal{Q}'}$. No information is lost upon addition. On the other hand, if $t(\mathcal{Q}) \cap t(\mathcal{Q}') \neq \emptyset$, condition (Q4) is no longer satisfied. Extracting from $\mathsf{Z}_{\mathcal{Q}} + \mathsf{Z}_{\mathcal{Q}'}$ Fourier coefficients analogously to (4.6) in this case yields numbers for which the interpretation (4.3) does not hold, as there is no unique charge orbit (or orbit representative) given the quadratic invariants. Rather it is a sum of two BPS indices. However, such a "compound" BPS index can still be a well-behaved object, inheriting for instance the wall-crossing properties of its components that we discuss below (mostly due to linearity), and $\mathsf{Z}_{\mathcal{Q}} + \mathsf{Z}_{\mathcal{Q}'}$ exhibits modular transformation properties consistent with that. Similar remarks can be made for the half-BPS partition functions in section 5.

[4] Eventually we want $\mathsf{Z}_{\mathcal{Q}}$ to be a Siegel modular form (for some congruence subgroup) and we expect that this requires infinitely many non-zero "Fourier modes" $\exp(2\pi i k x)$, for each $x \in \{\tau, \sigma, z\}$.

4.2 Constraints from S-duality symmetry

Generically a subset $\mathcal{Q}$ will not be preserved (setwise) under the full S-duality group $\mathcal{S}$ but only under a subgroup $\mathcal{S}_{\mathcal{Q}} \subset \mathcal{S}$ and transformations in $\mathcal{S} \backslash \mathcal{S}_{\mathcal{Q}}$ map to other subsets $\mathcal{Q}'$. This is in line with the discussion after (2.50) and further examples can be found in [22]. In any case, the invariance under $\mathcal{S}_{\mathcal{Q}} \subset \mathcal{S}$ has important consequences for $\mathsf{Z}_{\mathcal{Q}}$, as we will now discuss.

Recall that the S-duality group acts on the charges via (2.48). Those transformations which map $\mathcal{Q}$ to itself form a subgroup $\mathcal{S}_{\mathcal{Q}}$ and for such transformations $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ S-duality invariance of the BPS indices can be recast into the (suggestive) form (see [22] for a derivation)

$$\Phi_{\mathcal{Q}}((AZ+B)(CZ+D)^{-1}) = \det(CZ+D)^k\, \Phi_{\mathcal{Q}}(Z) \tag{4.8}$$

for some k, where

$$Z := \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}, \qquad \text{and} \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix}. \tag{4.9}$$

At this point k is undetermined, since the determinant is unity. However, k is determined by wall-crossing and modular invariance (more on this later). The 4×4 matrix in (4.9) is symplectic and takes the form given in (A.5) for $U = \left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$.

4.3 Constraints from charge quantization

We can also rewrite the periodicity property of $\Phi_{\mathcal{Q}}$ (stemming from (4.5)) in the form (4.8), but now with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & r_1 & r_2 \\ 0 & 1 & r_2 & r_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.10}$$

and suitable periods r_1, r_2, r_3 subject to the choice of $\mathcal{Q}$. This is also a special case of a symplectic matrix, see eq. (A.4) with $S = \left(\begin{smallmatrix} r_1 & r_2 \\ r_2 & r_3 \end{smallmatrix}\right)$.

4.4 Constraints from wall-crossing

Let us now explain how wall-crossing puts additional modular constraints on $\Phi_{\mathcal{Q}}$.

Each wall is associated to a specific decay of the quarter-BPS dyon into a pair of half-BPS dyons that only exists on one side of the wall. This wall-crossing phenomenon [67–72] is best understood in the case where the decay products carry primitive charges and for simplicity we restrict us to this case. Considering a quarter-BPS dyon with charge $(Q, P) \in \Lambda_{em}$ that decays at a certain (generically present) wall into two half-BPS states via

$$(Q, P) \longrightarrow (Q, 0)\ +\ (0, P)\,, \tag{4.11}$$

it is clear that we should hence restrict us to dyons where both $Q \in \Lambda_e$ and $P \in \Lambda_m$ are primitive lattice vectors. Furthermore we restrict to the case $I = 1$. According to [6] this is also a necessary condition for the dyon partition function to be related to a chiral genus two partition function of the heterotic string, as we will discuss later.

In principle there can also be decays where at least one decay product is quarter-BPS, however [73], if Q and P are both primitive charges these occur in the moduli space at codimension two or higher. Thus generic points in this space can be connected by paths that do not cross these loci and the BPS index is not affected by such decay channels.

A general parametrization for the decay of a quarter-BPS dyon into a pair of half-BPS dyons is given by

$$(Q,P) \rightarrow (a_0d_0Q - a_0b_0P, c_0d_0Q - c_0b_0P) + (-b_0c_0Q + a_0b_0P, -c_0d_0Q + a_0d_0P) \tag{4.12}$$

with $a_0d_0 - b_0c_0 = 1$. The decay products on the right hand side of (4.12),

$$(Q_1, P_1) \coloneqq (a_0Q', c_0Q') = \left(\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} Q' \\ 0 \end{pmatrix}\right)^{\intercal} \tag{4.13}$$

$$(Q_2, P_2) \coloneqq (b_0P', d_0P') = \left(\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} 0 \\ P' \end{pmatrix}\right)^{\intercal} , \tag{4.14}$$

where we have set

$$Q' \coloneqq d_0Q - b_0P \quad \text{and} \quad P' \coloneqq -c_0Q + a_0P , \tag{4.15}$$

again have to belong to the charge lattice Λ_{em}. Note that a charge set $\mathbf{Q} \subset \Lambda_{em}$ always comes along with its allowed decays (4.12) and thus determines charges (Q', P') and (Q_i, P_i).

Following the ansatz that the jump in the BPS index due the decay (4.12) is determined by a second order pole of $\Phi_{\mathbf{Q}}^{-1}$ at

$$z' \coloneqq c_0d_0\,\tau + a_0b_0\,\sigma + (a_0d_0 + b_0c_0)\,z = 0 , \tag{4.16}$$

the contour integral (4.6) for the Fourier coefficient of (4.4) needs to pick up a residue[5]

$$(-1)^{Q'\cdot P'+1}\;\; Q'\cdot P'\;\; d_h(a_0Q', c_0Q')\; d_h(b_0P', d_0P') \tag{4.17}$$

up to a sign. In this expression $d_h(\tilde{Q}, \tilde{P}) = \Omega_4(\tilde{Q}, \tilde{P})$ denotes the fourth helicity supertrace, i.e., the index only sensitive to half-BPS multiplets of dyonic charge $(\tilde{Q}, \tilde{P})$. As in [22] we want to restrict to those cases where the half-BPS indices again can be written as Fourier coefficients of a suitable partition function,

$$d_h(a_0Q', c_0Q') = \frac{1}{T}\int_{iM-T/2}^{iM+T/2} \frac{e^{-i\pi Q'^2\sigma'}}{\phi_e(\sigma'; a_0, c_0)}\,\mathrm{d}\sigma' \tag{4.18}$$

$$d_h(b_0P', d_0P') = \frac{1}{T'}\int_{iM-T'/2}^{iM+T'/2} \frac{e^{-i\pi P'^2\tau'}}{\phi_m(\tau'; b_0, d_0)}\,\mathrm{d}\tau' . \tag{4.19}$$

Here the integration contour lies parallel to the real axis and extends over a unit period $T(T')$ of

[5] This wall-crossing formula is only valid for *primitive* charges in the decay products, see [22, p. 7] and [72].

$\phi_e(\phi_m)$ and $M \gg 0$ is large enough to ensure convergence. Half-BPS partition functions for purely electrically charged states (in the heterotic frame) can, for instance, be found by counting perturbative, heterotic Dabholkar-Harvey (DH) states of the corresponding charge (as reviewed in chapter 5 for the $\mathbb{Z}_2$ model). Requiring the existence of functions $\phi_e(\phi_m)$ as stated imposes constraints[6] on Q:

(Q7) For any (Q', P') appearing as above, the values $(Q')^2$ takes for fixed $(P')^2$ are independent of the latter. The same holds for their roles reversed.[7]

(Q8) For fixed "decay code" $\left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right)$, all the decay products (Q_1, P_1) obtained from letting (Q, P) run over Q need to fall into a *single* T-orbit for each value of Q'^2 . The same holds for (Q_2, P_2) and P'^2.

Without (Q8), i.e., if there were several orbits, the half-BPS indices would not be functions of the mere quadratic T-invariants.

The property (Q8) is similar to (Q4) above. In accordance with the remarks on page 35 for compound quarter-BPS indices obtained from unions of charge orbits the half-BPS indices (or partition functions) occuring in the wall-crossing formula are again sums, coming from the decay products of the component orbits.

A sufficient condition for the jump is that near $z' = 0$ the function Φ_Q behaves as

$$\Phi_Q^{-1}(\tau, \sigma, z) \propto \Big(\phi_e(\sigma'; a_0, c_0)^{-1} \ \phi_m(\tau'; b_0, d_0)^{-1} \ z'^{-2} + O(z'^0) \Big) \tag{4.20}$$

in the transformed variables

$$Z' := \begin{pmatrix} \tau' & z' \\ z' & \sigma' \end{pmatrix} = (AZ + B)(CZ + D)^{-1}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}. \tag{4.21}$$

More explicitly, z' is as defined in (4.16) while

$$\tau' = d_0^2\,\tau + b_0^2\,\sigma + 2b_0 d_0\, z \quad \text{and} \quad \sigma' = c_0^2\,\tau + a_0^2\,\sigma + 2a_0 c_0\, z\,. \tag{4.22}$$

Note that (Q7) is generically required for the factorization in (4.20).

Given that the functions $\phi_m(\tau; b_0, d_0)$ and $\phi_e(\tau; a_0, c_0)$ transform as weight $k + 2$ modular forms under fractional linear transformations (a.k.a. Möbius transformations) of τ encoded by $\mathrm{SL}_2(\mathbb{Z})$-matrices $\left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} p_1 & q_1 \\ r_1 & s_1 \end{smallmatrix}\right)$, respectively, we can map these to symplectic transformations of the form

$$\begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1 & 0 & \beta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_1 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix} \tag{4.23}$$

[6] These are the subtleties mentioned in [22, pp. 19 f. and p. 21 f. n. 8].

[7] An example of an excluded case: $(Q')^2/2$ is odd iff $(P')^2/2$ is even and vice versa.

and

$$\begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p_1 & 0 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & r_1 & 0 & s_1 \end{pmatrix} \begin{pmatrix} d_0 & b_0 & 0 & 0 \\ c_0 & a_0 & 0 & 0 \\ 0 & 0 & a_0 & -c_0 \\ 0 & 0 & -b_0 & d_0 \end{pmatrix}, \tag{4.24}$$

respectively. These in turn act as $Z \mapsto (AZ + B)(CZ + D)^{-1}$ when written in the usual block form. Typically such a half-BPS partition function is a modular form for some congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. In some cases (4.23) and (4.24) lift to modular symmetries of Φ_Q in the sense of (4.8). Also notice the simple relation between the modular weights $k + 2$ of the functions $\phi_{e,m}$ and the weight k of the function Φ_Q. Hence, wall-crossing determines the *location* and *coefficients* of quadratic poles in our quarter-BPS partition function together with candidate Siegel *modular symmetries* and the *modular weight*.[8]

We remark that the middle matrix in each (4.23), (4.24) preserves the locus $z = 0$, while the conjugated matrix preserves the locus $z' = 0$.

Formally, (4.21) resembles an embedded S-duality transformation (c. f. (4.9)), but the matrix $\left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right)$ does not need to lie in $\mathcal{S}_Q \subset \mathrm{SL}_2(\mathbb{Z})$. Indeed, S-duality can be shown to act on a decay code $\left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right)$ from the left. In this way S-duality symmetry of the theory and the behavior at $z = 0$, which is related to the decay $(Q, P) \to (Q, 0) + (0, P)$ with the identity matrix as decay code, already imply the location and coefficients of an infinite set of quadratic poles. Furthermore, multiplying a decay code by $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ from the right for any real $\lambda \neq 0$ leads to an equivalent decay. The same holds for $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This makes it clear that for heterotic strings on T^6 with the weight 10 Igusa cusp form taking the role of Φ_Q all decays are related to the one at $z = 0$ by an $\mathrm{SL}_2(\mathbb{Z})$ transformation, which is known to be the S-duality group of that theory. However, in CHL orbifolds we may find inequivalent walls after modding out the mentioned redundancies.

As was multiply exemplified in [22], the expected properties of Φ_Q just described lead to a heuristics for finding quarter-BPS counting functions subject to a charge set $\mathcal{Q}$. By the same token, they provide a set of highly non-trivial tests for any given candidate counting function. Since the half-BPS partition functions form a key ingredient of this approach, we will now recall some facts about the latter in case of the heterotic $\mathbb{Z}_2$ CHL model.

[8] There might be additional ("accidental") modular symmetries as in [22, subsection 6.4] or some of the (genus one) modular symmetries do not lift to the full quarter-BPS partition function, see, for instance, the example in [22, subsection 6.2].

CHAPTER 5

Half-BPS spectra from Dabholkar-Harvey states in the $\mathbb{Z}_2$ model

In this chapter[1] we review the computation of electric half-BPS partition functions in the heterotic $\mathbb{Z}_2$ CHL orbifold that appear in wall-crossing relations for quarter-BPS partition functions. Our main reference is [26].[2] Doing so we set the notation and collect relevant wall-crossing data for section 7. The genus two analysis of section 6 will eventually go along similar lines, so this review section also serves as a warm-up exercise.

5.1 Heterotic Dabholkar-Harvey states and their half-BPS property

Electric half-BPS partition functions are generating functions for fourth helicity supertraces that count perturbative heterotic Dabholkar-Harvey (DH) states [75, 76] of a given purely electric charge. These are half-BPS states.[3] They are constructed by restricting the superconformal side of the heterotic string, which in our convention is the right-moving one, to the oscillator ground state. One possibility to see why this gives a half-BPS state is as follows [8] (see also [60]). In ten spacetime dimensions the heterotic string or its CHL orbifold exhibit $\mathcal{N} = (0,1)$ supersymmetry and there is a single supercharge Q. This charge transforms as Majorana-Weyl spinor of $\mathrm{SO}(1,9)$ (note that neither toroidal compactification nor the $\mathbb{Z}_2$ shift along a circle of the six-torus affect the holonomy). The sixteen real components Q_α satisfy

$$\{Q_\alpha, Q_\beta\} = -p_M (\Gamma^M C^{-1})_{\alpha\beta} \,, \tag{5.1}$$

where in the chosen Majorana representation $C = \Gamma^0$ is the charge conjugation matrix and Majorana spinors are real. It is further useful to split the momentum with index $M = 0, ..., 9$ into a four-dimensional part p_μ, $\mu = 0, \ldots, 3$ and a six-dimensional internal part p_I, $I = 4, \ldots, 9$. For massive representations we have $p_\mu = (m, 0, 0, 0)$ upon Lorentz rotation to the rest frame. The

[1] Section 5.2 in this chapter appeared as section 3 in the publication [35].

[2] See also [32] and [27, app. A.1] for closely related results. For the prime order CHL models these half-BPS partition functions, or rather those of the singly twisted sector, have also recently been revisited in [74] from a macroscopic point of view.

[3] When we speak of DH states in the following, we will always mean the perturbative heterotic half-BPS states. Otherwise, DH states are not always half-BPS [77, f.n. 6].

anticommutator (5.1) then may be recast into the form

$$\{Q_\alpha, Q_\beta\} = m\delta_{\alpha\beta} + Z_{\alpha\beta}\,, \qquad Z = p_I \Gamma^I \Gamma^0\,. \tag{5.2}$$

Since $\mathrm{tr}\, Z = 0$ and $Z^2 = (p_I)^2$, the eigenvalues of Z are $\pm\sqrt{(p_I)^2}$, each of which has a multiplicity of eight. With a little bit of algebra the anticommutator relations for the sixteen supercharges Q_α in (5.2) can be recast into the algebra of eight complex fermionic creation and annilation operators built from linear combinations of the charges, $\{q_i, q_j^\dagger\} = (1 \pm z)\delta_{ij}$, where $mz = \sqrt{(p_I)^2}$ defines z and the upper sign is for $i \leq 4$, while the lower sign for $4 < i \leq 8$. As $q_i q_i^\dagger$ should be positive semi-definite for any i, the BPS bound $z \leq 1$ follows. What we learn from this is that for massive half-BPS representations, which saturate the bound $z = 1$, the mass must be equal to the length of the internal momentum vector. Let us now see what constraint this puts in the worldsheet description of half-BPS states. Spacetime supersymmetry is due to the right-moving side of the heterotic string only. The supercharges of the heterotic string are obtained from the gravitino vertex operator and the internal momentum appearing in (5.1) is the holomorphic momentum operator $p_R^M = \frac{2}{\alpha'}\oint \partial X^M$, whose eigenvalues are p^μ and p_R^I. The four-momentum in Lorentzian signature gives $p^2 = -m^2$. In the BPS condition we may hence replace the internal momentum by the right-moving internal momentum and squaring the relation we find $m^2 = (p_R^I)^2$. Recall from chapter 2 that the total mass m^2 on the other hand consists of contributions $m_{L/R}^2$ from the right- and left-moving sides of the heterotic string , i.e., $m^2 = m_R^2 + m_L^2$, where level-matching requires $m_L^2 = m_R^2$. It follows $m_R^2 = (p_R^I)^2/2$. This relation can be unwrapped further by considering the formula for the right-moving mass:[4]

$$\alpha' m_R^2 = \frac{\alpha'}{2}(p_R^I)^2 + 2N_R - \delta_{\mathrm{NS}}\,. \tag{5.3}$$

As usual, the right-moving number operator N_R counts both bosonic and fermionic oscillator excitations transverse to the light-cone. The constant δ_{NS} is unity for the NS-sector of the fermionic string, but vanishes for the Ramond sector. A half-BPS state thus has to satisfy the simple constraint $N_R = 0$ and $N_R = 1/2$ in the R- and NS-sector, respectively. In fact, these are the lowest values compatible with the GSO-projection in the right-moving side of the heterotic string. As announced before, the right-moving degrees of freedom of the heterotic half-BPS state are thus restricted to the oscillator ground state.

On the left-moving side the BPS condition does not pose any constraint and any left-moving momenta and oscillator excitations compatible with the usual mass-shell condition are allowed. What the latter precisely looks like is subject to the specific string compactification, i.e., may slightly differ between the Narain compactification and its CHL orbifold. However, the above arguments only concern the right-moving side of the heterotic string and are valid in either of the two compactifications.

We can now proceed to the enumeration of heterotic DH half-BPS states in the $\mathbb{Z}_2$ CHL model.

[4] The conventions here are slightly different than in (2.2), where p_R was dimensionless. This expains the extra factor of $\frac{\alpha'}{2}$ in front of $(p_R^I)^2$ here.

5.2 Computation of half-BPS partition functions in the CHL model

Degeneracies for heterotic DH states can be computed both by direct enumeration of the relevant orbifold-invariant bosonic oscillator configurations or by making use of the helicity supertrace method. We make use of the latter.

Consider the generating function [10]

$$\mathsf{Z}(q,\bar{q};v,\bar{v}) = \mathrm{Tr}_{\mathcal{H}}\left[(-1)^F e^{2\pi i v J_3^L} e^{2\pi i \bar{v} J_3^R} q^{L_0}\bar{q}^{\bar{L}_0}\right] , \tag{5.4}$$

where the trace is taken over the Hilbert space $\mathcal{H}$ of the perturbative heterotic $E_8 \times E_8$ string compactified on T^6 or its CHL orbifold. The spacetime fermion number is denoted by F and the physical helicity in the four non-compact spacetime dimensions $J_3 = J_3^R + J_3^L$ is a sum of the left-helicity J_3^L coming from left-movers and the right-helicity J_3^R coming from right-movers. More precisely, the oscillators that contribute to the right-helicity J_3^R come from the right-moving light-cone bosons $\partial\bar{X}^{\pm} = \partial\bar{X}^3 \pm i\partial\bar{X}^4$, contributing helicity ± 1, respectively, and the light-cone fermions $\psi^{\pm}$, again contributing ± 1 to the right helicity. On the other hand, only $\partial X^{\pm}$ contribute to the left-helicity J_3^L. For instance, the 2+2 chiral light-cone bosons contribute a factor of

$$\frac{\xi(v)}{\eta^2}\frac{\bar{\xi}(\bar{v})}{\bar{\eta}^2} = q^{-2/24}\bar{q}^{-2/24}\prod_{n=1}^{\infty}\frac{1}{(1-q^n e^{2\pi i v})(1-q^n e^{-2\pi i v})}\frac{1}{(1-\bar{q}^n e^{2\pi i \bar{v}})(1-\bar{q}^n e^{-2\pi i \bar{v}})} . \tag{5.5}$$

Note that by putting $v = \bar{v} = 0$ the generating function $\mathsf{Z}(q,\bar{q};0,0) = \mathrm{Tr}_{\mathcal{H}}\left[(-1)^F q^{L_0}\bar{q}^{\bar{L}_0}\right]$ reduces to the ordinary one-loop partition function of the heterotic string (or its orbifold) including the GSO projection. Apart from the above mentioned light-cone fields, the extra fugacities in (5.4) have no effect on the computation of the partition function.

Taking all together the generating function of helicity supertraces for the $\mathbb{Z}_2$ CHL orbifold is[5]

$$\begin{aligned}\mathsf{Z}(q,\bar{q};v,\bar{v}) = \frac{1}{\tau_2}\frac{\xi(v)\bar{\xi}(\bar{v})}{\eta^2\bar{\eta}^2}&\left(\frac{1}{2}\sum_{\alpha,\beta=0}^{1}(-1)^{\alpha+\beta+\alpha\beta}\frac{\bar{\theta}\left[{\alpha/2 \atop \beta/2}\right](\bar{v})}{\bar{\eta}}\frac{\bar{\theta}^3\left[{\alpha/2 \atop \beta/2}\right](0)}{\bar{\eta}^3}\right)\\ &\times\left(\frac{1}{2}\sum_{g,h=0}^{1}\frac{\mathcal{Z}_{6,6}\left[{h \atop g}\right]}{\eta^6\bar{\eta}^6}\mathcal{Z}_8\left[{h \atop g}\right]\right) .\end{aligned} \tag{5.6}$$

Here $\alpha,\beta = 0,1$ run over the four spin structures, $h = 0,1$ indicates the untwisted or twisted sector and $g = 0,1$ indicates an insertion of the orbifold involution into the trace. In the above expression we have the partition function of the (shifted) Narain lattice

$$\mathcal{Z}_{6,6}\left[{h \atop g}\right] = \sum_{Q\in\Lambda_{6,6}^{[h]}}(-1)^{g\,\delta\cdot Q}\, e^{i\pi Q_L\tau Q_L - i\pi Q_R\bar{\tau}Q_R} , \tag{5.7}$$

[5] We keep the $q,\bar{q}$ dependence implicit, where $q = \exp(2\pi i\tau)$.

where the subscript L/R denotes the left- and right-part of the lattice vectors and

$$\Lambda_{6,6}^{[h]} = \left(\Lambda_{6,6} + \frac{h}{2}\delta\right) \tag{5.8}$$

is the Narain lattice associated with T^6, depending on the value of h shifted by half of the null vector $\delta = (0^6\,;\,0^{6-1}, 1)$ (i.e., vanishing momentum quantum numbers but a single unit of winding charge along the CHL circle). This in accordance with the CHL action on T^6, which is just a translation along the last circle in T^6 (the CHL circle) by half a period. We already use the symbol Q in the lattice summation since the momentum-winding vectors along the compact dimensions eventually give rise to electric charges of the four-dimensional BPS state (the remaining components of the electric charge being sourced by momentum in the E_8-directions). Thus $h = 0$ means summation over untwisted sector charges, $Q \in \Lambda_{6,6} \cong U^{\oplus 6}$, the winding number along the CHL circle taking integral values. On the other hand, $h = 1$ gives twisted sector charges $Q \in \Lambda_{6,6} + \frac{\delta}{2}$ with the winding number along the CHL circle taking values in $\mathbb{Z} + \frac{1}{2}$. The for $g = 1$ inserted phase factor $(-1)^{\delta\cdot Q}$ then becomes (-1) for an odd number of momentum quanta along the CHL circle and $(+1)$ for an even number of such quanta. Furthermore, in (5.6) we introduced the orbifold blocks $\mathcal{Z}_8\left[\begin{smallmatrix} h \\ g \end{smallmatrix}\right]$ for the 16 chiral bosons compactified on the $E_8 \times E_8$ root lattice, where the orbifold involution exchanges the two E_8 factors[6] and one finds

$$\mathcal{Z}_8\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = \frac{\left[\theta_{E_8(1)}(\tau)\right]^2}{\eta^{16}(\tau)}\,, \qquad \mathcal{Z}_8\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \frac{\theta_{E_8(1)}(2\tau)}{\eta^{8}(2\tau)}\,, \tag{5.9}$$

$$\mathcal{Z}_8\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \frac{\theta_{E_8(1)}(\frac{\tau}{2})}{\eta^{8}(\frac{\tau}{2})} \quad \text{and} \quad \mathcal{Z}_8\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = e^{-2\pi i/3}\frac{\theta_{E_8(1)}(\frac{\tau+1}{2})}{\eta^{8}(\frac{\tau+1}{2})}\,. \tag{5.10}$$

Especially, the E_8 theta series $\theta_{E_8(1)}(\tau) = \sum_{v\in E_8(1)} e^{i\pi\tau v^2} = E_4(\tau)$ is the weight four Eisenstein series.

As a remark, the terms in the first line of (5.6) should arise for all heterotic $\mathbb{Z}_N$ CHL orbifolds (including the trivial one), as the superconformal sector of the heterotic string is unaffected by the orbifold action. On the other hand, the terms in the second line of (5.6) are the orbifold blocks specific to the order $N = 2$ shift along one of the circles of T^6 and the order N permutation on the left-moving chiral bosons.

Helicity supertraces can be obtained from the generating function (5.4) by taking appropriate derivatives with respect to the generalized chemical potentials v and $\bar{v}$ coupling to the left and right helicity, respectively:

$$B_n(q,\bar{q}) = \left(\frac{1}{2\pi i}\frac{\partial}{\partial v} + \frac{1}{2\pi i}\frac{\partial}{\partial \bar{v}}\right)^n \mathsf{Z}(q,\bar{q};v,\bar{v})\Big|_{v=0=\bar{v}}\,. \tag{5.11}$$

We now want to obtain the fourth helicity supertrace B_4.[7] The fermion terms in the first line of (5.6)

[6] Recall from chapter 2 that upon diagonalization this gives eight invariant chiral bosons and eight chiral bosons that pick up a minus sign under the $\mathbb{Z}_2$ action.

[7] Strictly speaking, this is rather another generating function, not yet a helicity supertrace $\Omega_4(Q, 0)$ for fixed charge (orbit).

can be rewritten using the Riemann identity to give $\bar{\theta}_1^4(\bar{v}/2)$. This implies that the only combination of v- and $\bar{v}$-derivatives that does not vanish when evaluated at $v = \bar{v} = 0$ is taking four $\bar{v}$-derivatives, since $\theta_1(0|\tau) = 0$. Using further

$$\partial_{\tilde{v}}\bar{\theta}\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](\tilde{v}|\tau)\Big|_{\tilde{v}=0} = 2\pi\bar{\eta}(\tau)^3 \tag{5.12}$$

and $\xi(0) = \bar{\xi}(0) = 1$ we obtain[8]

$$B_4(q,\bar{q}) = \frac{3}{2}\frac{1}{\tau_2}\frac{1}{\eta^{2+6}} \times \left(\frac{1}{2}\sum_{g,h=0}^{1} \mathcal{Z}_{6,6}\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]\mathcal{Z}_8\left[\begin{smallmatrix}h\\g\end{smallmatrix}\right]\right) . \tag{5.13}$$

Inserting the identities (5.9) and (5.10) we can also write

$$\begin{aligned} B_4(q,\bar{q}) = \frac{3}{2\tau_2}\,\frac{1}{2}\Bigg[&\frac{\theta^2_{E_8(1)}(\tau)}{\eta^{24}(\tau)}\mathcal{Z}_{6,6}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] + \frac{\theta_{E_8(1)}(2\tau)}{\eta^8(\tau)\eta^8(2\tau)}\mathcal{Z}_{6,6}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] + \frac{\theta_{E_8(1)}(\frac{\tau}{2})}{\eta^8(\tau)\eta^8(\frac{\tau}{2})}\mathcal{Z}_{6,6}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \\ &+ e^{-2\pi i/3}\frac{\theta_{E_8(1)}(\frac{\tau+1}{2})}{\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\mathcal{Z}_{6,6}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\Bigg] . \end{aligned} \tag{5.14}$$

Before interpreting the result (5.14), we interlude with a reminder of the unorbifolded case. The contribution with $\mathcal{Z}_{6,6}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ corresponds (up to the factor $1/2$) to helicity supertraces of perturbative states in the unorbifolded theory Het$[T^6]$:

$$B_4^{\text{unorb}}(q,\bar{q}) = \frac{1}{\tau_2} \times \mathcal{Z}_{6,6}\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right](q,\bar{q})\,\theta^2_{E_8(1)}(\tau) \times \frac{3}{2}\frac{1}{\eta^{24}(\tau)} . \tag{5.15}$$

Let us pause to recall the semantics of this factorization. We have a continuous degeneracy due to the momenta p_3, p_4 in the non-compact directions transverse to the light-cone, leading to a factor of $1/\tau_2$. Also we have the Narain lattice sum of vectors $Q \in \Lambda_{22,6} \cong E_8(1)^{\oplus 2} \oplus U^{\oplus 6}$ and a factor $\eta^{-24}(\tau)$ corresponding to oscillator modes of 24 chiral bosons (transverse to the light-cone). As seen from the four-dimensional spacetime perspective for each momentum (p_3, p_4) and electric charge vector $Q \in \Lambda_{22,6}$ (momentum and winding) we have the full tower of DH states generated by allowing arbitrary left-moving oscillators while keeping the superconformal sector in the ground state. The latter is, due to the GSO projection, a Weyl spinor with $2^{8/2} = 16$ components. Hence for fixed (p_3, p_4) we can relate the fourth helicity supertrace of states with charge Q to the absolute degeneracy of states with charge Q as[9]

$$d_h(Q,0) = \Omega_4(Q,0) = \frac{3}{32}\Omega_{\text{abs}}(Q,0) = \frac{3}{2}p_{24}(N) . \tag{5.16}$$

[8] The factor $3/2$ arises as $24 \times (1/2)^4$ coming from the $4! = 24$ permutations of $\bar{v}$-derivatives and the inner derivative, c.f. the argument $\tilde{v} = \bar{v}/2$.

[9] Recall that $p_{24}(N)$ is the number of ways of writing the non-negative integer N as a sum of 24 non-negative integers. This is also the Fourier coefficient of q^{N-1} in $\eta^{-24}(\tau)$. For any $\tau \in \mathbb{H}$ the Fourier series of the latter converges, so there is no ambiguity, i.e., no wall-crossing for these half-BPS states and no moduli dependence in Ω_4.

The (left-moving) level number N (not to be confused with the order of the CHL orbifold group) is related to the charge $Q \in \Lambda_{22,6}$ via the level matching condition

$$N - 1 = \frac{1}{2}\left(Q_R^2 - Q_L^2\right) = \frac{1}{2}Q^2 . \tag{5.17}$$

Note that the result (5.16) has the structure demanded by (3.24).

As another remark, we recall that in the unorbifolded case the discriminant function $\Delta(\sigma) = \eta^{24}(\sigma)$ appears in the diagonal divisor limit $z \to 0$ of $\chi_{10}^{-1}(Z)$, which is the (complete $I = 1$) quarter-BPS partition function of heterotic strings on T^6 (c. f. the discussion of poles in 4.4):

$$\chi_{10}^{-1}(\tau, \sigma, z) \propto \left(\Delta(\sigma)^{-1} \, \Delta(\tau)^{-1} \; z^{-2} + O\big(z^0\big) \right) . \tag{5.18}$$

Historically the appearance of this perturbative half-BPS partition function and its magnetic counterpart, together with manifest electric-magnetic (S-)duality between them, was a crucial point in the proposal of [21].

We return to the CHL orbifold and apply a similar logic to $B_4(q, \bar{q})$ in eq. (5.14), which we split into the untwisted and twisted sector contribution,

$$B_4 = B_4^{\text{untw}} + B_4^{\text{tw}} . \tag{5.19}$$

Untwisted sector. To read off the degeneracies of DH states with fixed electric charge, the Narain lattice vectors $(P_1, P_2) \in E_8(1)^{\oplus 2}$ are decomposed[10] with respect to their sum — which is invariant under $\mathbb{Z}_2$ and hence a physical charge — and their difference. That is,

$$P_1 \pm P_2 = 2P_\pm \pm \mathcal{P} \tag{5.20}$$

for some root lattice vectors $P_+, P_- \in E_8(1)$ and a shift vector $\mathcal{P} \in E_8(1)/(2E_8(1))$. The latter represents an element of a finite group of rank 2^8, which is by a simple rescaling by $1/\sqrt{2}$ isomorphic to the residue component from $E_8(1/2)/E_8(2)$ in eq. (2.52). In terms of $E_8(2)$ theta functions with characteristics $\mathcal{P}$, defined as

$$\theta_{E_8(2),\mathcal{P}}(\tau) := \sum_{\Delta \in E_8(1)} \exp\left[\pi i \tau \left(\sqrt{2}\Delta - \frac{\mathcal{P}}{\sqrt{2}}\right)^2\right] , \tag{5.21}$$

the theta function for $E_8(1)^{\oplus 2}$ may be expressed as

$$\theta_{E_8(1)}^2 = \theta_{E_8(2),1}^2 + 120\, \theta_{E_8(2),248}^2 + 135\, \theta_{E_8(2),3875}^2 . \tag{5.22}$$

Here it has been used that $\theta_{E_8(2),\mathcal{P}}$ only depends on the orbit O_* of $\mathcal{P}$ under the Weyl group of E_8. There are three such orbits, namely the orbit of the fundamental weight of the trivial, of the adjoint and of the 3875 representation of respective lengths $1 + 120 + 135 = 2^8$, i.e.,

$$\frac{E_8(1)}{2E_8(1)} = O_1 \cup O_{248} \cup O_{3875} , \tag{5.23}$$

[10] Also see [78] for a relation to numerators of affine characters of $\hat{E}_8$ at level two.

where the subscript labels the dimension of the respective representation. In general, any vector Q' in $E_8(\frac{1}{2}) = \frac{1}{\sqrt{2}}E_8(1)$ decomposes as

$$Q' = \frac{1}{\sqrt{2}}(2Q'' + \mathcal{P}) \tag{5.24}$$

for appropriate elements $Q'' \in E_8(1)$ and $\mathcal{P} \in E_8(1)/(2E_8(1))$, and therefore one also has

$$E_8(\tfrac{1}{2}) = E_8(2) \cup \big(E_8(2) + O_{248}\big) \cup \big(E_8(2) + O_{3875}\big) \tag{5.25}$$

$$\theta_{E_8(\frac{1}{2})}(\tau) = \theta_{E_8(2),1} + 120\,\theta_{E_8(2),248} + 135\,\theta_{E_8(2),3875}\,. \tag{5.26}$$

Both (5.22) and (5.26) are easily checked by writing $\theta_{E_8(2),\mathcal{P}}(\tau)$ in terms of theta constants (see appendix A). Note that under $\tau \mapsto \tau + 1$ only the sign of the term corresponding to the 248-orbit in (5.26) flips, since $\mathcal{P}^2 \equiv 2 \pmod 4$ for this orbit, while $\mathcal{P}^2 \equiv 0 \pmod 4$ for the other two orbits.

The untwisted sector contribution reads in terms of the $\theta_{E_8(2),\mathcal{P}}(\tau)$ functions[11]

$$\begin{aligned} B_4^{\text{untw}}(q,\bar{q}) = \frac{3}{2\tau_2} \times \sum_{\epsilon\in\{+1,-1\}} \frac{\mathcal{Z}_{6,6}\big[\begin{smallmatrix}0\\0\end{smallmatrix}\big] + \epsilon\mathcal{Z}_{6,6}\big[\begin{smallmatrix}0\\1\end{smallmatrix}\big]}{2} \Bigg[\theta_{E_8(2),1} \times \frac{1}{2}\left(\frac{\theta_{E_8(2),1}}{\eta^{24}} + \epsilon\frac{1}{\eta^8(\tau)\eta^8(2\tau)}\right) \\ + 120\,\theta_{E_8(2),248} \times \left(\frac{\theta_{E_8(2),248}}{2\eta^{24}}\right) + 135\,\theta_{E_8(2),3875} \times \left(\frac{\theta_{E_8(2),3875}}{2\eta^{24}}\right)\Bigg]\,. \end{aligned} \tag{5.27}$$

In this form B_4^{untw} corresponds to the non-orbifold counterpart (5.15), with the modular form on the right-hand side of each "×"-sign playing the role of η^{-24}. The E_8 theta series inside the parentheses sums only over the unphysical charge $(\frac{P_1-P_2}{2})^2$. The sign ϵ corresponds to two kinds of DH states in the untwisted sector. It specifies the sign picked up by the oscillator monomial under $\mathbb{Z}_2$ (c.f. section 2.2). This goes along with an even (+1) or odd (−1) number of momentum quanta along the CHL circle, such that the two phases coming from the (left-moving) oscillators and the (left-moving) zero-mode cancel out to give an invariant state. As can be seen, e.g., from the explicit form of the T-transformations on charges in [11], this "momentum parity" along the CHL circle is also invariant under T-transformations, so we have a splitting into two disjoint T-orbits. Correspondingly, we find that untwisted sector $\mathcal{P} = 0$ DH states with odd (even) momentum parity possess a separate half-BPS partition function

$$\frac{1}{2}\left(\frac{\theta_{E_8(2)}}{\eta^{24}} + \epsilon\frac{1}{\eta^8(\tau)\eta^8(2\tau)}\right) \tag{5.28}$$

with $\epsilon = -1$ ($\epsilon = +1$), as was implicitly used in writing down [22, eq. (6.5.12)]. For untwisted sector states with $\mathcal{P} \neq 0$ the parity of the CHL momentum does not play a role in the counting, as seen from eq. (5.27).

To get the half-BPS index for states with fixed electric charge from these partition functions we reformulate the level matching condition (5.17), as the quantity $Q \in \Lambda_{22,6}$ in (5.17) is no longer the physical electric charge in the orbifold theory. In the untwisted sector we introduce the modified level

[11] This corrects a typo in [26, eq. (3.42)], where $\frac{2^4}{\theta_2^4\eta^{12}} = \frac{\theta_3^4\theta_4^4}{\eta^{24}} = \frac{1}{\eta^8(\tau)\eta^8(2\tau)}$.

number

$$N' := N - \frac{(P_1 - P_2)^2}{4} = N - \left(P_- - \frac{\mathcal{P}}{2}\right)^2 \tag{5.29}$$

and with a physical electric charge $Q \in E_8(\frac{1}{2}) \oplus U \oplus U^{\oplus 5}$ we find again $N' - 1 = \frac{1}{2}Q^2$. Thus, when expanding eq. (5.28) in terms of $q = e^{2\pi i \tau}$, the exponent of $q^{N'-1}$ in each term gives $\frac{Q^2}{2}$, while the coefficient gives the desired index $\Omega_4(Q, 0)$ for Q in the respective charge sector (re-installing the universal factor 3/2). In the example of (5.28) the charge sector is $Q \in E_8(2) \oplus U(2) \oplus U^{\oplus 5}$ for $\epsilon = +1$ and $Q \in E_8(2) \oplus (U \backslash U(2)) \oplus U^{\oplus 5}$ for $\epsilon = -1$. Here we have identified $U(2) \subset U$ as the (non-shifted) momentum-winding vectors with an even number of momentum quanta along the CHL circle.

Twisted sector. The twisted sector part of B_4 is

$$\begin{aligned} B_4^{\mathrm{tw}}(q, \bar{q}) = \frac{3}{2\tau_2} \times \sum_{\epsilon \in \{+1,-1\}} \frac{\mathcal{Z}_{6,6}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \epsilon \mathcal{Z}_{6,6}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]}{2} \Bigg[& \theta_{E_8(2),1} \times \frac{1}{2}\left(\frac{1}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} + \epsilon \frac{e^{-2\pi i/3}}{\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\right) \\ & + 120\, \theta_{E_8(2),248} \times \left(\frac{1}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} - \epsilon \frac{e^{-2\pi i/3}}{\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\right) \\ & + 135\, \theta_{E_8(2),3875} \times \left(\frac{1}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} + \epsilon \frac{e^{-2\pi i/3}}{\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\right) \Bigg] . \end{aligned} \tag{5.30}$$

Note that the relative sign between the two terms in each pair of parentheses is that of $(-1)^{Q^2} = (-1)^{\mathcal{P}^2/2}(-1)^{Q\cdot\delta}$ with $(-1)^{Q\cdot\delta} = \epsilon$. The twisted sector level-matching equates the exponents in the q-expansion of the functions in parentheses in (5.30) to the value of $\frac{1}{2}(Q_8^2 + Q_1^2 + Q_5^2) \in \frac{1}{2}\mathbb{Z}$, where $(Q_8, Q_1, Q_5) \in E_8(\frac{1}{2}) \oplus (U + \frac{\delta}{2}) \oplus U^{\oplus 5}$ is a physical electric charge vector in the twisted sector.

Comparison of the sectors. In accordance with the analysis of the perturbative spectrum in [32] (see also [33]), the degeneracies for certain subsectors of the untwisted sector agree with twisted sector degeneracies. This is due to the modular identities

$$\frac{1}{2}\left(\frac{\theta_{E_8(2),1}}{\eta^{24}} + \frac{1}{\eta^8(\tau)\eta^8(2\tau)}\right) = \frac{1}{2}\left(\frac{1}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} + \frac{e^{-2\pi i/3}}{\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\right) + \frac{1}{\eta^8(\tau)\eta^8(2\tau)} \tag{5.31}$$

$$\frac{1}{2}\left(\frac{\theta_{E_8(2),1}}{\eta^{24}} - \frac{1}{\eta^8(\tau)\eta^8(2\tau)}\right) = \frac{1}{2}\left(\frac{1}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} + \frac{e^{-2\pi i/3}}{\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\right) \tag{5.32}$$

$$\frac{\theta_{E_8(2),248}}{2\,\eta^{24}} = \frac{1}{2}\left(\frac{1}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} - \frac{e^{-2\pi i/3}}{\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\right) \tag{5.33}$$

$$\frac{\theta_{E_8(2),3875}}{2\,\eta^{24}} = \frac{1}{2}\left(\frac{1}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} + \frac{e^{-2\pi i/3}}{\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\right) . \tag{5.34}$$

Note that on the right-hand-side of eqs. (5.32) to (5.34) the second term is, up to sign, the first term shifted by $\tau \mapsto \tau + 1$, which is

$$\frac{1}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} = \frac{1}{\sqrt{q}} + 8 + 52\sqrt{q} + 256q + 1122q^{3/2} + 4352q^2 + 15640q^{5/2} + O\left(q^3\right) . \tag{5.35}$$

Hence, adding the second term to the first projects to terms with even (eqs. (5.32) and (5.34)) or odd (eq. (5.33)) exponents of $\sqrt{q} = e^{2\pi i \frac{\tau}{2}}$. The parity of this exponent modulo two matches the parity of $Q^2/2$ and due to this one might simply regard $\eta^{-8}(\tau)\eta^{-8}(\frac{\tau}{2})$ as the half-BPS partition function for twisted sector DH states — and in fact as the half-BPS partition function for DH states with charge in any of the sectors listed in eqs. (5.32) to (5.34). The only charge sector that is not covered by this is that of even momentum $\mathcal{P} = 0$ untwisted states with $\epsilon = +1$ in (5.28), i.e., electric charges $Q \in E_8(2) \oplus U(2) \oplus U^{\oplus 5} = \Lambda_m \subset \Lambda_e$. Their degeneracy is not just given by the coefficient of $q^{Q^2/2}$ in $\eta^{-8}(\tau)\eta^{-8}(\frac{\tau}{2})$ but gets an extra contribution from the coefficient of $q^{Q^2/2}$ in $\eta^{-8}(\tau)\eta^{-8}(2\tau)$, as also observed in [27]. Another way to arrive at the same conclusion is via the following identity. Since the exchange of the two E_8 factors alone without the shift along a circle of the torus gives back an equivalent theory, there is an equality between the partition functions of the two theories [5, app. B]:

$$\frac{E_4(\tau)^2}{\eta^{16}(\tau)} = \frac{1}{2}\frac{E_4(\tau)^2}{\eta^{16}(\tau)} + \frac{1}{2}\frac{E_4(2\tau)}{\eta^8(2\tau)} + \frac{1}{2}\frac{E_4(\frac{\tau}{2})}{\eta^8(\frac{\tau}{2})} + \frac{e^{-2\pi i/3}}{2}\frac{E_4(\frac{\tau+1}{2})}{\eta^8(\frac{\tau+1}{2})} . \tag{5.36}$$

Using this observation eq. (5.14) can be re-expressed as

$$B_4(q,\bar{q}) = \frac{3}{2\tau_2}\left[\frac{\Gamma_{\Lambda_e^*}}{\eta^8(\tau)\eta^8(2\tau)} + \frac{1}{2}\frac{\Gamma_{\Lambda_e}}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} + \frac{1}{2}\frac{\Gamma_{\Lambda_e}[(-1)^{Q^2}]}{e^{2\pi i/3}\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\right] , \tag{5.37}$$

where the notation [27]

$$\Gamma_{\Lambda_0}[\mathcal{X}] = \sum_{Q\in\Lambda_0} \mathcal{X}\, q^{\frac{1}{2}Q_L^2}\, \bar{q}^{\frac{1}{2}Q_R^2} \tag{5.38}$$

was introduced. Pairs of (Narain) theta functions multiplying the same eta-quotient have been recast into a single lattice sum for the electric lattice Λ_e or the magnetic lattice $\Lambda_e^* \subset \Lambda_e$, as defined in (2.40). An equivalent representation is

$$B_4(q,\bar{q}) = \frac{3}{2\tau_2}\sum_{Q\in\Lambda_e} q^{\frac{Q^2}{2}}\left[\frac{\delta_{Q\in\Lambda_e^*}}{\eta^8(\tau)\eta^8(2\tau)} + \frac{1}{2}\frac{1}{\eta^8(\tau)\eta^8(\frac{\tau}{2})} + \frac{1}{2}\frac{(-1)^{Q^2}}{e^{2\pi i/3}\eta^8(\tau)\eta^8(\frac{\tau+1}{2})}\right] , \tag{5.39}$$

where $(-1)^{Q^2} = (-1)^{\frac{\mathcal{P}^2}{2}}(-1)^{h\,Q\cdot\delta}$ with h as in (5.8). This also nicely fits the assertion that the DH states are electrically charged with respect to Λ_e as given in (2.40). In chapter 6 a genus two analog of the identity (5.36) will become important.

CHAPTER 6

Quarter-BPS spectra from genus two partition function in the $\mathbb{Z}_2$ model

Our analysis in chapter 4 mostly concerned generic quarter-BPS partition functions. We now turn specifically to unit-torsion quarter-BPS dyons in the $\mathbb{Z}_2$ CHL model, the prime interest being dyons whose electric charge in the heterotic frame belongs to the untwisted sector. The goal of this chapter[1] is to obtain closed expressions for the relevant partition functions by relating them to a genus two chiral partition function for the four-dimensional heterotic $\mathbb{Z}_2$ CHL model.[2] Properties of the candidate dyon partition functions thus obtained will be addressed in chapter 7.

6.1 From string webs to heterotic strings

According to [4, 5, 79, 80] quarter-BPS dyons can be represented as string webs [81, 82], which via an M-theory lift are related to a chiral genus two partition function of the heterotic string. As was argued in [6], the genus g of the M-theory lift of the string web is actually given by $g = I + 1$, so the genus two partition function is expected to only capture unit-torsion dyons ($I = 1$). Indeed, in [5] the *twisted* sector dyon partition function of [1, 83] was re-derived by identifying appropriate contributions to the genus two orbifold partition function that can be interpreted as arising from states of the relevant charge type.[3] Our *untwisted* sector quarter-BPS partition functions should in a similar fashion be found in this heterotic genus two partition function. The latter was recently revisited in [2, section B.2], expanding the results of [5] by, for instance, also writing down the remaining orbifold blocks. For the sake of a clear and coherent presentation, we will first reproduce the orbifold partition function of [2] and collect the relevant formulae that are needed in our subsequent analysis.

[1] This chapter appeared as section 4 in the publication [35].

[2] Left- and right-moving partition function should be understood as in [5, f. n. 2].

[3] The contour prescription and wall-crossing phenomenon can also be studied in the genus two picture [80, 84], though the analysis was mostly spelled out for the maximal rank theory.

6.2 Derivation of the quarter-BPS partition functions

6.2.1 Genus two orbifold blocks for the $\mathbb{Z}_2$ CHL orbifold

As in the one-loop case, the chiral partition function is given by a sum of orbifold blocks, each associated to a choice of periodicity conditions $[h_1, h_2]$ and $[g_1, g_2]$ along the A- and B-cycles of a genus two surface with period matrix $\Omega = \left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} = \Omega_1 + i\Omega_2 \in \mathbb{H}_2$, i.e.,

$$\mathcal{Z}(\Omega) = \frac{1}{2^2} \sum_{\substack{h_1,h_2 \in \{0,1\} \\ g_1,g_2 \in \{0,1\}}} \mathcal{Z}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] . \tag{6.1}$$

At least on the locus of the moduli space where the Narain lattice splits as $E_8 \oplus E_8 \oplus \Lambda_{6,6}$ we may factorize the orbifold blocks into a contribution of the ten-dimensional $E_8 \times E_8$ string and the contribution of the bosonic zero-modes of the chiral bosons on T^6,

$$\mathcal{Z}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] = \mathcal{Z}_8\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] \mathcal{Z}_{6,6}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] . \tag{6.2}$$

Here we have

$$\mathcal{Z}_{6,6}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] = \sum_{(Q_1,Q_2)\in\Lambda_{6,6}^{[h_1,h_2]}} (-1)^{\delta\cdot(g_1 Q_1 + g_2 Q_2)}\, e^{i\pi Q_L^r \Omega_{rs} Q_L^s - i\pi Q_R^r \bar{\Omega}_{rs} Q_R^s} \tag{6.3}$$

with summation over $r, s \in \{0, 1\}$ here and in the following (no distinction between upper and lower indices made). Let us abbreviate the exponential by $e_{Q_1,Q_2}(\Omega)$. Also we have

$$\Lambda_{6,6}^{[h_1,h_2]} = \left(\Lambda_{6,6} + \frac{h_1}{2}\delta\right) \oplus \left(\Lambda_{6,6} + \frac{h_2}{2}\delta\right) , \tag{6.4}$$

the genus two analog of the Narain lattice associated with T^6, shifted by half of the null vector $\delta = (0^6\, ; 0^{6-1}, 1)$.[4]

In view of the twisted sector dyon states, the authors of [5] computed the orbifold block

$$\mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \left(\left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)\right) = \frac{\Theta_{E_8}^{(2)}(2\tau, 2z, 2\sigma)}{\Phi_{6,0}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2}, z, 2\sigma)}{16\,\Phi_{6,1}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2}, z, 2\sigma)}{16\,\Phi_{6,2}} \tag{6.5}$$

building on the results of [85]. Here we have the genus two theta series for the E_8 root lattice,

$$\Theta_{E_8}^{(2)}(\Omega) = \sum_{(Q_1,Q_2)\in E_8\oplus E_8} e^{i\pi\, Q^r \Omega_{rs} Q^s} = E_4^{(2)}(\Omega) , \tag{6.6}$$

[4] We interpret $\delta \cdot Q_i$ as the momentum of the "CHL circle boson" flowing along the i-th B-cycle of the genus two worldsheet. This should correct a typo below [2, eq. (B.52)] (there: "winding" instead of "momentum") and restore consistency with [27, section A.1]. Also note that we have dropped a factor of $(\det \Omega_2)^{6/2}$ in $\mathcal{Z}_{6,6}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right]$, which will not be relevant in our discussion.

agreeing with the Siegel-Eisenstein series $E_4^{(2)}(\Omega)$, as well as the weight six Siegel modular forms $\Phi_{6,k}$ defined in appendix A (one of which is given by a multiplicative lift of the K3 twining genera of class $2A$). Rescalings and shifts in the arguments of the E_8 theta series can be rewritten in terms of the theta series for 2-modular lattices and insertions of sign factors, for instance:[5]

$$\Theta^{(2)}_{E_8}(2\tau, 2z, 2\sigma) = \sum_{\substack{(Q_1,Q_2)\in \\ E_8(2)\oplus E_8(2)}} e^{i\pi Q^r \Omega_{rs} Q^s} = E_4^{(2)}(2\Omega) \tag{6.7}$$

$$\Theta^{(2)}_{E_8}(\tfrac{\tau}{2}, z, 2\sigma) = 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8(2)^*\oplus E_8(2)}} e^{i\pi Q^r \Omega_{rs} Q^s} \tag{6.8}$$

$$\Theta^{(2)}_{E_8}(\tfrac{\tau+1}{2}, z, 2\sigma) = 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8(2)^*\oplus E_8(2)}} (-1)^{Q_1^2}\, e^{i\pi Q^r \Omega_{rs} Q^s} . \tag{6.9}$$

Further expressions of this kind we will encounter below are moved to appendix A. The third and second term in (6.5) turn out to be modular images of the first under the Petersson slash operator,

$$\mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \sum_{\gamma\in\Gamma^{(2)}_{e_1}(2)/\Gamma^{(2)}_{0,e_1}(2)} \left(\frac{\Theta^{(2)}_{E_8}(2\tau, 2z, 2\sigma)}{\Phi_{6,0}}\right)\Bigg|_\gamma , \tag{6.10}$$

see (A.65) and (A.66) for explicit γ. Per definition $\Gamma^{(2)}_{e_1}(2) \subset \mathrm{Sp}_4(\mathbb{Z})$ is the index 15 subgroup that preserves the periodicity conditions (characteristics) $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ modulo 2, while the group $\Gamma^{(2)}_{0,e_1}(2)$ is its intersection with the level two congruence subgroup $\Gamma^{(2)}_0(2) \subset \mathrm{Sp}_4(\mathbb{Z})$. This intersection has index 3 in $\Gamma^{(2)}_{e_1}(2)$, the three cosets correspond to three terms in eq. (6.10) or eq. (6.5). Since $E_4^{(2)}(2\Omega)$ and $\Phi_{6,0}(\Omega)$ are Siegel modular forms for $\Gamma^{(2)}_0(2)$ (they are invariant under $(\cdot)|_\gamma$ for $\gamma \in \Gamma^{(2)}_0(2)$), the other summands in (6.10) are Siegel modular forms with respect to subgroups conjugate to $\Gamma^{(2)}_0(2)$. From (6.10) it is clear that $\mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is indeed invariant under the group $\Gamma^{(2)}_{e_1}(2) \subset \mathrm{Sp}_4(\mathbb{Z})$. An analogous formula also holds when the torus contribution is taken into account,

$$\mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\mathcal{Z}_{6,6}\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] = \sum_{\gamma\in\Gamma^{(2)}_{e_1}(2)/\Gamma^{(2)}_{0,e_1}(2)} \left(\frac{\Gamma^{(2)}_{U^{\oplus 6}\oplus E_8(2)}[(-1)^{\delta\cdot Q_2}]}{\Phi_{6,0}}\right)\Bigg|_\gamma , \tag{6.11}$$

where we adopted the notation

$$\Gamma^{(2)}_{\Lambda_0}[\mathcal{X}] = \sum_{(Q_1,Q_2)\in(\Lambda_0)^{\oplus 2}} \mathcal{X}\, e^{i\pi Q_{r,L}\Omega_{rs}Q_{s,L} - i\pi Q_{r,R}\bar{\Omega}_{rs}Q_{s,R}} \tag{6.12}$$

for $\Lambda_0 = U^{\oplus 6} \oplus E_8(2)$, $\mathcal{X} = (-1)^{\delta\cdot Q_2}$ (the E_8 charges being "left-moving", consistent with (6.6)).

[5] Later we need theta series related to the ones in (6.5) by an exchange in the roles of (τ, Q_1), (σ, Q_2), see eqs. (A.44) and (A.45).

Further modular transformations on the above block (6.11) with $\tilde{\gamma} \in \mathrm{Sp}_4(\mathbb{Z})/\Gamma^{(2)}_{e_1}(2)$ generate the remaining 14 of the $2^4 - 1 = 15$ orbifold blocks with non-trivial boundary conditions. The respective part from $\mathcal{Z}_8\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right]$ is displayed in Table 6.1 for convenience. The orbifold block $\mathcal{Z}\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ forms a separate orbit, which is the genus two chiral partition function of the parent model, the (left-moving) heterotic string on T^6 and the same holds for $\mathcal{Z}\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ in eq. (6.13) below.

As in the one-loop partition function there is a modular identity arising from the equivalence of the $E_8 \times E_8$ theory with its orbifold obtained by exchange of the E_8 factors (without any shift along T^6):

$$\mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \frac{\left[\Theta^{(2)}_{E_8}(\Omega)\right]^2}{\chi_{10}} = \sideset{}{'}\sum_{\substack{h_1,h_2\in\{0,1\} \\ g_1,g_2\in\{0,1\}}} \mathcal{Z}_8\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] . \tag{6.13}$$

This is the genus two analog of (5.36).

Using the behavior of $E^{(2)}_4(\Omega)$ and the genus two Thetanullwerte $\theta_{a_1a_2b_1b_2}(\Omega)$ (which appear in $\Phi_{6,k}$) in the diagonal limit $z \longrightarrow 0$, together with some simple theta identities (see appendix A), it is straightforward to verify that each orbifold block factorizes into two genus one orbifold blocks:

$$\mathcal{Z}_8\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] \xrightarrow{z\to 0} -\frac{1}{4\pi z^2}\,\mathcal{Z}_8\left[\begin{smallmatrix} h_1 \\ g_1 \end{smallmatrix}\right](\tau)\,\mathcal{Z}_8\left[\begin{smallmatrix} h_2 \\ g_2 \end{smallmatrix}\right](\sigma) + O(z^0) . \tag{6.14}$$

This limiting behavior mirrors the wall-crossing constraints of quarter-BPS partition functions.

6.2.2 Identification of quarter-BPS partition functions

In the following we will identify the genus two period matrix $\Omega \in \mathbb{H}_2$ with the chemical potentials conjugate to the quadratic T-duality invariants obtained from the electric and magnetic components of a dyonic charge,

$$\Omega \stackrel{!}{=} Z \quad \left(= \left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)\right) . \tag{6.15}$$

This means τ is conjugate to the magnetic charge $\frac{1}{2}P^2$, whereas σ is conjugate to the electric charge $\frac{1}{2}Q^2$.[6] It has important consequences for finding the contributions in $\mathcal{Z}$ that can be interpreted as arising from appropriate dyonic charges $(Q, P) = (Q_2, Q_1)$ in the lattice sums. The most convenient way to write $\mathcal{Z}$ for the following discussion is

$$\mathcal{Z} = \frac{1}{2^2} \sideset{}{'}\sum_{\substack{h_1,h_2\in\{0,1\} \\ g_1,g_2\in\{0,1\}}} \mathcal{Z}_8\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] \left(\mathcal{Z}_{6,6}\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] + \mathcal{Z}_{6,6}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right]\right) . \tag{6.16}$$

The prime denotes omission of the trivial characteristics. We first address the toroidal part $\mathcal{Z}_{6,6}\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right]$ and recall (6.4) and (2.40), finding that the summation over $Q_1 = P$ in the lattice sums must go

[6] The roles of the chemical potentials τ, σ on the diagonal of the 2×2 period matrix are switched with respect to [5, p. 8]. As a remark, switching the diagonal entries of a period matrix corresponds to the action of the symplectic matrix (A.5), $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, switching the periodicity conditions along the pairs of cycles (A_1, B_1), (A_2, B_2), i.e., $\left(\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} h_2 & h_1 \\ g_2 & g_1 \end{smallmatrix}\right)$. However, in the sequel paper [6] the authors also use the convention employed here.

$\left[\begin{smallmatrix} h_1 h_2 \\ g_1 g_2 \end{smallmatrix}\right]$	$\mathcal{Z}_8\left[\begin{smallmatrix} h_1 h_2 \\ g_1 g_2 \end{smallmatrix}\right]$	$\tilde{\gamma} \in \mathrm{Sp}_4(\mathbb{Z})/\Gamma_{e_1}^{(2)}(2)$
$\left[\begin{smallmatrix} 00 \\ 01 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(2\tau,2\sigma,2z)}{\Phi_{6,0}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\sigma,z)}{2^4\Phi_{6,1}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\sigma,z)}{2^4\Phi_{6,2}}$	$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$
$\left[\begin{smallmatrix} 00 \\ 10 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(2\tau,2\sigma,2z)}{\Phi_{6,0}} + \frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\sigma}{2},z)}{2^4\Phi_{6,3}} + \frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\sigma+1}{2},z)}{2^4\Phi_{6,4}}$	$\begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 01 \\ 00 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\sigma}{2},z)}{2^4\Phi_{6,3}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma}{2},\frac{z}{2})}{2^8\Phi_{6,5}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma}{2},\frac{z}{2})}{2^8\Phi_{6,6}}$	$\begin{pmatrix} 1&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 10 \\ 00 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},2\sigma,z)}{2^4\Phi_{6,1}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma}{2},\frac{z}{2})}{2^8\Phi_{6,5}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma+1}{2},\frac{z}{2})}{2^8\Phi_{6,7}}$	$\begin{pmatrix} 0&0&0&-1 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 11 \\ 00 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma}{2},\frac{z}{2})}{2^8\Phi_{6,5}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma+1}{2},\frac{z+1}{2})}{2^8\Phi_{6,9}} + \frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\sigma-2z+\tau}{2},z-\tau)}{2^4\Phi_{6,13}}$	$\begin{pmatrix} 1&0&0&0 \\ -1&0&0&-1 \\ 0&1&1&0 \\ 0&1&0&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 01 \\ 01 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\sigma+1}{2},z)}{2^4\Phi_{6,4}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma+1}{2},\frac{z}{2})}{2^8\Phi_{6,7}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma+1}{2},\frac{z}{2})}{2^8\Phi_{6,8}}$	$\begin{pmatrix} 1&0&0&0 \\ 0&1&0&-1 \\ 0&0&1&0 \\ 0&1&0&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 10 \\ 10 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},2\sigma,z)}{2^4\Phi_{6,2}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma}{2},\frac{z}{2})}{2^8\Phi_{6,6}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma+1}{2},\frac{z}{2})}{2^8\Phi_{6,8}}$	$\begin{pmatrix} 0&1&0&-1 \\ 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 01 \\ 10 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\sigma}{2},z)}{2^4\Phi_{6,3}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma}{2},\frac{z+1}{2})}{2^8\Phi_{6,10}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma}{2},\frac{z+1}{2})}{2^8\Phi_{6,11}}$	$\begin{pmatrix} 0&1&0&0 \\ 0&0&-1&0 \\ 0&0&1&1 \\ 1&-1&0&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 10 \\ 11 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},2\sigma,z)}{2^4\Phi_{6,2}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma+1}{2},\frac{z+1}{2})}{2^8\Phi_{6,9}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma}{2},\frac{z+1}{2})}{2^8\Phi_{6,11}}$	$\begin{pmatrix} 0&1&-1&-1 \\ 1&-1&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 10 \\ 01 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},2\sigma,z)}{2^4\Phi_{6,1}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma}{2},\frac{z+1}{2})}{2^8\Phi_{6,10}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma+1}{2},\frac{z+1}{2})}{2^8\Phi_{6,12}}$	$\begin{pmatrix} 0&0&-1&-1 \\ 1&-1&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 01 \\ 11 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\sigma+1}{2},z)}{2^4\Phi_{6,4}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma+1}{2},\frac{z+1}{2})}{2^8\Phi_{6,9}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma+1}{2},\frac{z+1}{2})}{2^8\Phi_{6,12}}$	$\begin{pmatrix} 0&1&0&0 \\ 1&-1&-1&0 \\ 0&0&1&1 \\ 1&-1&0&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 00 \\ 11 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(2\tau,2\sigma,2z)}{\Phi_{6,0}} + \frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\tau-2z+\sigma}{2},z-\tau)}{2^4\Phi_{6,13}} + \frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\tau-2z+\sigma+1}{2},z-\tau)}{2^4\Phi_{6,14}}$	$\begin{pmatrix} 0&1&0&0 \\ 1&-1&0&0 \\ 0&0&1&1 \\ 0&0&1&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 11 \\ 01 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma+1}{2},\frac{z}{2})}{2^8\Phi_{6,7}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma}{2},\frac{z+1}{2})}{2^8\Phi_{6,11}} + \frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\tau-2z+\sigma+1}{2},z-\tau)}{2^4\Phi_{6,14}}$	$\begin{pmatrix} 1&0&0&0 \\ -1&1&0&-1 \\ 0&1&1&0 \\ 0&1&0&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 11 \\ 10 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma}{2},\frac{z}{2})}{2^8\Phi_{6,6}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma+1}{2},\frac{z+1}{2})}{2^8\Phi_{6,12}} + \frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\tau-2z+\sigma+1}{2},z-\tau)}{2^4\Phi_{6,14}}$	$\begin{pmatrix} 1&1&1&0 \\ -1&0&0&-1 \\ 0&1&1&0 \\ 0&1&0&0 \end{pmatrix}$
$\left[\begin{smallmatrix} 11 \\ 11 \end{smallmatrix}\right]$	$\frac{\Theta_{E_8}^{(2)}(\frac{\tau+1}{2},\frac{\sigma+1}{2},\frac{z}{2})}{2^8\Phi_{6,8}} + \frac{\Theta_{E_8}^{(2)}(\frac{\tau}{2},\frac{\sigma}{2},\frac{z+1}{2})}{2^8\Phi_{6,10}} + \frac{\Theta_{E_8}^{(2)}(2\tau,\frac{\sigma-2z+\tau}{2},z-\tau)}{2^4\Phi_{6,13}}$	$\begin{pmatrix} 1&1&1&0 \\ -1&1&0&-1 \\ 0&1&1&0 \\ 0&1&0&0 \end{pmatrix}$

Table 6.1: Chiral genus two orbifold blocks for the heterotic $\mathbb{Z}_2$ CHL model (taken from [2]).

over the non-shifted lattice for the interpretation as a magnetic charge being possible, i.e., we must consider terms with $h_1 = 0$. Also a magnetic charge $P \in \Lambda_m$ has components only along the subsets $E_8(\frac{1}{2}) \subset E_8(1/2)$ and $U(2) \subset U$. The latter restriction will naturally be satisfied for terms in the $\mathcal{Z}_8\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right]$ blocks that appear both for $g_1 = 0$ and $g_1 = 1$. The reason is that this effectively means the presence of the desired projector $\frac{1}{2}(1 + (-1)^{Q_1 \cdot \delta})$ to $U(2) \subset U$ in the toroidal lattice sum. For untwisted sector charges $Q \in E_8(\frac{1}{2}) \oplus U \oplus U^{\oplus 5}$, i.e. $h_2 = 0$, such terms can only arise for

$$\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] \in \left\{ \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right\}, \tag{6.17}$$

while for twisted sector charges $Q \in E_8(\frac{1}{2}) \oplus (U + \frac{\delta}{2}) \oplus U^{\oplus 5}$, i.e., $h_2 = 1$, the analogous statement is

$$\left[\begin{smallmatrix} h_1 & h_2 \\ g_1 & g_2 \end{smallmatrix}\right] \in \left\{ \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \right\}. \tag{6.18}$$

Note that due to the replacement (6.13) the characteristic $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is not listed in (6.17). Inspecting Tab. 6.1, suitable terms in the above blocks that appear for both g_1 cases are the ones with denominators

$$\Phi_{6,0},\ \Phi_{6,3},\ \Phi_{6,4} \quad \text{and} \quad \Phi_{6,3},\ \Phi_{6,4}\ , \tag{6.19}$$

respectively. Collecting these and writing out the sum over the torus lattice gives for the untwisted case ($h_2 = 0$)

$$\begin{aligned} \frac{1}{2^2} \sum_{\substack{Q_1 \in U^{\oplus 6} \\ Q_2 \in U^{\oplus 6}}} e_{Q_1,Q_2}(\Omega) \Bigg[& \frac{\Theta^{(2)}_{E_8}(2\tau, 2z, 2\sigma)}{\Phi_{6,0}} (1 + (-1)^{\delta \cdot Q_1} + (-1)^{\delta \cdot Q_2} + (-1)^{\delta \cdot Q_1 + \delta \cdot Q_2}) \\ & + \frac{\Theta^{(2)}_{E_8}(2\tau, 2z, \frac{\sigma}{2})}{16\, \Phi_{6,3}} (1 + (-1)^{\delta \cdot Q_1}) + \frac{\Theta^{(2)}_{E_8}(2\tau, 2z, \frac{\sigma+1}{2})}{16\, \Phi_{6,4}} (1 + (-1)^{\delta \cdot Q_1}) \Bigg] \end{aligned} \tag{6.20}$$

and

$$\frac{1}{2^2} \sum_{\substack{Q_1 \in U^{\oplus 6} \\ Q_2 \in (U + \frac{\delta}{2}) \oplus U^{\oplus 5}}} e_{Q_1,Q_2}(\Omega) \Bigg[\frac{\Theta^{(2)}_{E_8}(2\tau, 2z, \frac{\sigma}{2})}{16\, \Phi_{6,3}} (1 + (-1)^{\delta \cdot Q_1}) + \frac{\Theta^{(2)}_{E_8}(2\tau, 2z, \frac{\sigma+1}{2})}{16\, \Phi_{6,4}} \tag{6.21}$$

$$((-1)^{\delta \cdot Q_2} + (-1)^{\delta \cdot Q_1 + \delta \cdot Q_2}) \Bigg] \tag{6.22}$$

for the twisted case ($h_2 = 1$). As announced, we may factor a projector $\frac{1}{2}(1 + (-1)^{Q_1 \cdot \delta})$ and henceforth restrict to summation over $U(2) \subset U$.

Let us look at the E_8 part more closely. Recall that the charge components $Q' = \sqrt{2}Q'' + \frac{\mathcal{P}}{\sqrt{2}}$ along $E_8(\frac{1}{2}) \subset \Lambda_e$ come in three classes , where $Q'' \in E_8(1)$ and $\mathcal{P} \in E_8(1)/(2E_8(1))$, labelled by the orbit O_1, O_{248}, O_{3875} under the Weyl group of E_8 that $\mathcal{P}$ belongs to. For these orbits O_x define

$$\Theta_x := \sum_{\mathcal{P}\in O_x} \sum_{\substack{(Q_1,Q_2)\in \\ E_8(2)\oplus\left(E_8(2)+\frac{\mathcal{P}}{\sqrt{2}}\right)}} e^{i\pi Q^r\Omega_{rs}Q^s} . \tag{6.23}$$

The Siegel theta functions in the numerators of (6.20) may be re-expressed as

$$\Theta^{(2)}_{E_8}(2\tau, 2z, 2) = \sum_{\substack{(Q_1,Q_2)\in \\ E_8(2)\oplus E_8(2)}} e^{i\pi Q^r\Omega_{rs}Q^s} = \Theta_1 \tag{6.24}$$

$$\Theta^{(2)}_{E_8}(2\tau, z, \tfrac{\sigma}{2}) = 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8(2)\oplus E_8(2)^*}} e^{i\pi Q^r\Omega_{rs}Q^s} = \Theta_1 + \Theta_{248} + \Theta_{3875} \tag{6.25}$$

$$\Theta^{(2)}_{E_8}(2\tau, z, \tfrac{\sigma+1}{2}) = 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8(2)\oplus E_8(2)^*}} e^{i\pi Q^r\Omega_{rs}Q^s} (-1)^{Q_2^2} = \Theta_1 - \Theta_{248} + \Theta_{3875} . \tag{6.26}$$

The second of these relations is a genus two analog of (5.26). Collecting Θ_x gives

$$\begin{aligned}\sum_{\substack{Q_1\in U(2)\oplus U^{\oplus 5}\\ Q_2\in U^{\oplus 6}}} e_{Q_1,Q_2}(\Omega)\Bigg[\Theta_1 \times \left(\frac{\frac{1}{2}(1+(-1)^{\delta\cdot Q_2})}{\Phi_{6,0}} + \frac{1}{2}\left(\frac{1}{16\,\Phi_{6,3}} + \frac{1}{16\,\Phi_{6,4}}\right)\right) \\ +\Theta_{248}\times\frac{1}{2}\left(\frac{1}{16\,\Phi_{6,3}} - \frac{1}{16\,\Phi_{6,4}}\right) + \Theta_{3875}\times\frac{1}{2}\left(\frac{1}{16\,\Phi_{6,3}} + \frac{1}{16\,\Phi_{6,4}}\right)\Bigg].\end{aligned} \tag{6.27}$$

This in turn is the genus two analog of (5.27), the terms on the right-hand-side of each "×"-symbol give the quarter-BPS partition function in the respective subsector of the untwisted charge sector, depending on $\mathcal{P}$ and $(-1)^{\delta\cdot Q_2}$ of the electric charge of the dyon. The sign between $\Phi^{-1}_{6,3}$ and $\Phi^{-1}_{6,4}$ matches $(-1)^{\frac{\mathcal{P}^2}{2}}$. Note also the presence of a projector in the term with $\Phi_{6,0}$. It is zero unless the winding along the CHL circle is even, and since this term only occurs for $\mathcal{P} = 0 = h_2$, we can equivalently say that it only arises for $Q \in \Lambda_m \subset \Lambda_e$ (or $r(Q,P) = [Q] = [0]$). With the identities (5.31) to (5.34) we recognize pairs of corresponding modular forms

$$\left(\Phi_{6,0}\,,\,\eta^8(\sigma)\eta^8(2\sigma)\right)\,, \qquad \left(\Phi_{6,3}\,,\,\eta^8(\sigma)\eta^8(\tfrac{\sigma}{2})\right)\,, \qquad \left(\Phi_{6,4}\,,\,\eta^8(\sigma)\eta^8(\tfrac{\sigma+1}{2})\right)\,. \tag{6.28}$$

The first pair contains the cusp form for the level two congruence subgroup $\Gamma_0(2)$ of the (Siegel) modular group, the second is obtained from it via an (embedded) S-duality transformation $\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ on the variable σ and the third pair is the $\sigma \mapsto \sigma + 1$ translate of the latter. Besides this we have

$$\frac{1}{\Phi_{6,0}} = \frac{1}{(2\pi i z)^2} \frac{1}{\eta^8(\tau)\eta^8(2\tau)} \frac{1}{\eta^8(\sigma)\eta^8(2\sigma)} \left(1 + O(z^0)\right) \tag{6.29}$$

$$\frac{1}{16\,\Phi_{6,3}} = \frac{1}{(2\pi i z)^2} \frac{1}{\eta^8(\tau)\eta^8(2\tau)} \frac{1}{\eta^8(\sigma)\eta^8(\frac{\sigma}{2})} \left(1 + O(z^0)\right) \tag{6.30}$$

$$\frac{1}{16\,\Phi_{6,4}} = \frac{1}{(2\pi i z)^2} \frac{1}{\eta^8(\tau)\eta^8(2\tau)} \frac{e^{-2\pi i/3}}{\eta^8(\sigma)\eta^8(\frac{\sigma+1}{2})} \left(1 + O(z^0)\right) , \tag{6.31}$$

by the help of which we immediately see that (5.31) to (5.34) re-appear in the linear combinations of (6.27) near the diagonal locus $z = 0$. The same holds for twisted sector electric charges and

$$\sum_{\substack{Q_1 \in U(2) \oplus U^{\oplus 5} \\ Q_2 \in (U + \frac{\delta}{2}) \oplus U^{\oplus 5}}} e_{Q_1,Q_2}(\Omega) \Bigg[\Theta_1 \times \frac{1}{2}\left(\frac{1}{16\,\Phi_{6,3}} + \frac{(-1)^{\delta\cdot Q_2}}{16\,\Phi_{6,4}}\right) + \Theta_{248} \times \frac{1}{2}\left(\frac{1}{16\,\Phi_{6,3}} - \frac{(-1)^{\delta\cdot Q_2}}{16\,\Phi_{6,4}}\right) + \Theta_{3875} \times \frac{1}{2}\left(\frac{1}{16\,\Phi_{6,3}} + \frac{(-1)^{\delta\cdot Q_2}}{16\,\Phi_{6,4}}\right) \Bigg], \tag{6.32}$$

corresponding in turn to (5.30). As in the genus one case, these linear combinations of $\Phi_{6,3}^{-1}$ and $\Phi_{6,4}^{-1}$ basically imply the projection to Fourier modes with even or odd exponents of $e^{2\pi i \sigma}$, depending on the parity of the momentum along the CHL circle encoded in $(-1)^{\delta\cdot Q_2}$. One may thus argue that the quarter-BPS partition function of unit-torsion dyons with twisted sector electric charge $Q \in E_8(\frac{1}{2}) \oplus (U + \frac{\delta}{2}) \oplus U^{\oplus 5}$ is simply $2^{-4}\Phi_{6,3}^{-1}$, in agreement with the result of [1, 5, 86].[7]

Comparing the untwisted and twisted sector results (eqs. (6.27) and (6.32)) and following the logic of chapter 5, we similarly find that the quarter-BPS index of unit-torsion dyons is given by the Fourier coefficient of $\Phi_{6,3}^{-1}$ (understood with the moduli-dependent contour prescription in (4.6)) plus an extra contribution in case that $Q \in \Lambda_m \subset \Lambda_e$, coming from the Fourier coefficient of $\Phi_{6,0}^{-1}$. By analogy with eq. (5.39) we can write

$$\sum_{\substack{Q_1 \in \Lambda_m \\ Q_2 \in \Lambda_e}} e_{Q_1,Q_2}(\Omega) \left[\frac{\delta_{Q_2 \in \Lambda_e^*}}{\Phi_{6,0}} + \frac{1}{2}\left(\frac{1}{16\,\Phi_{6,3}} + \frac{(-1)^{Q_2^2}}{16\,\Phi_{6,4}}\right) \right] . \tag{6.33}$$

For later convenience let us introduce some notation for the basic partition functions that are encountered here

$$\mathsf{Z}^{(0)} := \frac{1}{2}\left(\frac{1}{16\,\Phi_{6,3}} + \frac{1}{16\,\Phi_{6,4}}\right) + \frac{1}{\Phi_{6,0}} \tag{6.34}$$

$$\mathsf{Z}^{(\pm)} := \frac{1}{2}\left(\frac{1}{16\,\Phi_{6,3}} \pm \frac{1}{16\,\Phi_{6,4}}\right) . \tag{6.35}$$

[7] Note that $\Phi_{6,3}(\tau, \sigma, z) = \Phi_{6,1}(\sigma, \tau, z)$ upon swapping the diagonal elements, so this is the same Siegel modular form as in [5] once the meaning of the chemical potentials is properly matched.

The forms $\mathsf{Z}^{(0)}$ and $\mathsf{Z}^{(+)}$ may also be rewritten in terms of modular forms W, Y, T for the Iwahori subgroup $B(2) \subset \mathrm{Sp}_4(\mathbb{Z})$ (see appendix A) via

$$\frac{1}{W} = \frac{1}{\Phi_{6,0}}\,, \qquad \frac{1}{16\,\Phi_{6,3}} + \frac{1}{16\,\Phi_{6,4}} = \frac{16T}{YW}\,, \tag{6.36}$$

where $YW = \chi_{10}$ is the Igusa cusp form.

Our findings are compatible with the findings of [2, eq. (2.14)], if our partition functions are subject to the condition $P \in \Lambda_m \backslash 2\Lambda_e$. In section 7 we will support this statement by considering wall-crossing. To conclude, the M-theory lift of string webs argument of [4–6], which lead us to analyzing the chiral fluctuations of the genus two heterotic CHL string, provides quarter-BPS indices for a large class of unit-torsion dyons that are compatible with indices obtained from suitable six-derivative couplings in the 3D CHL vacuum in the circle decompactification limit considered in [2]. As we have just shown, by analyzing the genus two orbifold partition function in greater detail, one can make the previous point not just for states with twisted sector electric charge (to which [5] was limited), but also for untwisted electric charge sectors.

CHAPTER 7

Modular and polar constraints in the $\mathbb{Z}_2$ model

In the previous chapter we have proposed quarter-BPS partition functions for unit-torsion dyons in various subsectors of the untwisted and twisted charge sector of the $\mathbb{Z}_2$ orbifold. In light of chapter 4 there are non-trivial constraints, especially from S-duality symmetry and wall-crossing, such a partition function must satisfy. These constraints will be addressed in this chapter.[1] In fact, this analysis already highly constrains these partition functions. With only few assumptions one might already "guess" the form of the latter.

7.1 Quantization of the charge invariants

First recall that for $Q \in \Lambda_e$ the parity of

$$Q^2 \equiv \frac{\mathcal{P}^2}{2} + h\, Q \cdot \delta \mod 2 \tag{7.1}$$

depends on the Weyl orbit of the shift vector $\frac{\mathcal{P}}{\sqrt{2}} \in E_8(\frac{1}{2})/E_8(2)$, on the twistedness $h \in \{0, 1\}$ and on the CHL circle momentum $Q \cdot \delta \in \mathbb{Z}$. In fact $Q^2 \equiv [Q]^2$ modulo two. This parity is fixed within each of the charge subsectors considered in chapter 6 and determines the periodicity the respective partition function must obey in the variable σ. For even Q^2 the period is 1, for odd Q^2 the period is 2. For each charge sector of eqs. (6.27) and (6.32) (also see (6.33)) this expected periodity is indeed satisfied by the respective partition function, as $\mathsf{Z}^{(0)}$ and $\mathsf{Z}^{(+)}$ have period 1, while $\mathsf{Z}^{(-)}$ is only periodic under $\sigma \mapsto \sigma + 2$. Thus, according to (4.10), all symplectic matrices of the form

$$\begin{pmatrix} 1 & 0 & r_1 & r_2 \\ 0 & 1 & r_2 & r_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \begin{pmatrix} 1 & 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.2}$$

acting on $Z \in \mathbb{H}_2$ in the usual way leave the former two partition functions invariant, while for $\mathsf{Z}^{(-)}$ this is only the case if also r_3 is even. For $r_3 = 1$ the form $\mathsf{Z}^{(-)}$ picks up a minus sign. In all cases $\frac{P^2}{2}, Q \cdot P \in \mathbb{Z}$, so the period in both the τ and z direction is unity.

[1] This chapter appeared as section 5 in the publication [35].

7.2 S-duality symmetry

As a second constraint, the S-duality group for the $\mathbb{Z}_2$ CHL model is $\Gamma_1(2) = \Gamma_0(2)$ and leaves unchanged the residue $r(Q,P) = [Q]$ of a dyon charge, so for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(2)$ the embedded S-transformation (4.9), i.e., $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in the form

$$\begin{pmatrix} d & b & 0 & 0 \\ c & a & 0 & 0 \\ 0 & 0 & a & -c \\ 0 & 0 & -b & d \end{pmatrix} \in \begin{pmatrix} 2\mathbb{Z}+1 & \mathbb{Z} & 0 & 0 \\ 2\mathbb{Z} & 2\mathbb{Z}+1 & 0 & 0 \\ 0 & 0 & 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 0 & 0 & \mathbb{Z} & 2\mathbb{Z}+1 \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) , \tag{7.3}$$

should describe the symmetry (4.8) of each quarter-BPS partition function, here simply

$$\mathsf{Z}(Z) = \mathsf{Z}(Z'') , \quad Z'' = (AZ+B)(CZ+D)^{-1} . \tag{7.4}$$

For $\mathsf{Z}^{(0)}$ and $\mathsf{Z}^{(+)}$ eq. (6.36) shows that (7.3) indeed is a valid modular symmetry as the matrix lies in $B(2)$, and for Sen's partition function $2^{-4}\Phi_{6,3}^{-1}$ this symmetry is also known. The combination of these facts then demonstrates that $\mathsf{Z}^{(-)}$ is also $\Gamma_1(2)$ S-duality invariant.

7.3 Wall-crossing relations

We now apply the general lessons from section 4.4 and study the implications of wall-crossing.

First wall. Regarding unit-torsion dyon charges $(Q,P) \in \Lambda_e \oplus \Lambda_m$ for any of the subsectors of chapter 6 we first consider the decay into half-BPS states

$$(Q,P) \to (Q,0) + (0,P) . \tag{7.5}$$

This decay is encoded by the matrix $\left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and demands that the respective partition function Z exhibits a quadratic pole at $z=0$, with coefficient given by

$$\mathsf{Z}\left(\left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)\right) \propto \frac{1}{z^2}\, \phi_e^{-1}(\sigma)\, \phi_m^{-1}(\tau) + O(z^0) . \tag{7.6}$$

The functions $\phi_e^{-1}(\sigma)$ and $\phi_m^{-1}(\tau)$ are the half-BPS counting functions of the decay products $(Q,0)$ and $(0,P)$, respectively.

We start with the magnetic part. On page 59 we have already made the assertion that our dyon partition functions are subject to the restriction $P \in \Lambda_m \backslash 2\Lambda_e$ on the magnetic charges. This was required by matching the results of [2]. It is also consistent with wall-crossing. To give some background, we first remark that in [27], in accordance with $\Gamma_1(2)$ S-duality and Fricke symmetry, it was shown that the half-BPS index (fourth helicity supertrace) for primitive charges $(Q,P) \in (\Lambda_e \oplus \Lambda_m) \backslash (\Lambda_m \oplus 2\Lambda_e)$ is given by

$$\Omega_4(Q,P) = c_8\left(-\frac{\gcd(2Q^2, P^2, Q\cdot P)}{2}\right) , \tag{7.7}$$

while for charges in the complement $(Q, P) \in (\Lambda_m \oplus 2\Lambda_e)$ it is given by

$$\Omega_4(Q, P) = c_8\left(-\frac{\gcd(2Q^2, P^2, Q \cdot P)}{2}\right) + c_8\left(-\frac{\gcd(2Q^2, P^2, Q \cdot P)}{2 \cdot 2}\right) . \tag{7.8}$$

The numbers $c_8(\ldots)$ are the (always positive) Fourier coefficients of the $\Gamma_0(2)$ modular form

$$\frac{1}{\eta^8(\tau)\eta^8(2\tau)} = \sum_{m=-1}^{\infty} c_8(m)\, q^m = \frac{1}{q} + 8 + 52q + 256q^2 + O(q^3) . \tag{7.9}$$

Since $\frac{P^2}{2} \in \mathbb{Z}$ in general, for purely magnetic charges $(0, P)$ the first term in (7.8) always contributes $c_8(\frac{P^2}{2})$ while the second term $c_8(\frac{P^2}{4})$ in (7.8) vanishes unless $P^2 \in 4\mathbb{Z}$. Furthermore, considering $Q = 0$ states where trivially $Q \in \Lambda_m$, the condition $P \in \Lambda_m \backslash 2\Lambda_e$ holds if and only if $(Q, P) \in (\Lambda_e \oplus \Lambda_m)\backslash(\Lambda_m \oplus 2\Lambda_e)$, which in turn is equivalent to (7.7). Otherwise (7.7) holds. Hence $P \in \Lambda_m \backslash 2\Lambda_e$ is a sufficient condition for (7.7), and half-BPS states of charge $(0, P)$ being counted by

$$\phi_m^{-1}(\tau) = \frac{1}{\eta^8(\tau)\eta^8(2\tau)} . \tag{7.10}$$

Indeed, the latter occurs on the diagonal divisor of all our quarter-BPS partition functions by eqs. (6.29) to (6.31), suggesting that our counting formula should be understood as holding for states with $P \in \Lambda_m \backslash 2\Lambda_e$.

The magnetic charge assumption for our unit-torsion quarter-BPS partition functions in chapter 6 is also consistent with results in the literature that rely on charge configurations for which this magnetic condition is explicitly satisfied. Regarding twisted sector unit-torsion dyons, the derivation in [11, 86], which is independent from the ansatz pursued here and in [5], starts indeed from charge representatives (Q, P) satisfying $P \in \Lambda_m \backslash 2\Lambda_e$ and arrives at the quarter-BPS counting function $2^{-4}\Phi_{6,3}^{-1}$. This clearly exhibits (7.10) at $z = 0$, counting half-BPS states with charge $(0, P)$. Regarding untwisted sector unit-torsion dyons in a certain (sub-)subsector, an analysis starting from explicit charge representatives (Q, P) satisfying $P \in \Lambda_m \backslash 2\Lambda_e$ was presented in [22, section 6.5] and again leads to constraints consistent with our untwisted sector quarter-BPS partition functions (discussed further below).

Now we turn to the electric part. Here we can refer to eq. (6.33). Via the identities (6.29) to (6.31), the half-BPS partition function of states with charge $(Q, 0)$ always reduces to the respective one in (5.39). This consistently works out for all types $[Q]$ of electric charge.

Since $\eta^{-8}(\tau)\eta^{-8}(2\tau)$ and $\theta_{E_8(2),1}(\sigma)/\eta^{24}(\sigma)$ transform as weight -8 modular forms for $\Gamma_0(2) = \Gamma_1(2)$ (recall (5.31), (5.32) and (5.34)), the weight of $\mathsf{Z}^{(0)}$ and $\mathsf{Z}^{(+)}$ must be -6, which is indeed the case. They should also transform as Siegel modular forms under modular transformations given by

$$\begin{pmatrix} 2\mathbb{Z}+1 & 0 & \mathbb{Z} & 0 \\ 0 & 1 & 0 & 0 \\ 2\mathbb{Z} & 0 & 2\mathbb{Z}+1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) , \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\mathbb{Z}+1 & 0 & \mathbb{Z} \\ 0 & 0 & 1 & 0 \\ 0 & 2\mathbb{Z} & 0 & 2\mathbb{Z}+1 \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) , \tag{7.11}$$

the first coming from the magnetic and the second coming from the electric part. Indeed, as these matrices belong to $B(2)$, the correct transformation of $\mathsf{Z}^{(+)}$ and $\mathsf{Z}^{(0)}$ is guaranteed.

The form $\mathsf{Z}^{(-)}$, where $\phi_e^{-1}(\sigma) = \frac{1}{2}(\eta^8(\sigma)\eta^8(\frac{\sigma}{2}) - \eta^8(\sigma)\eta^8(\frac{\sigma+1}{2}))$ is a modular form for $\Gamma(2)$ (or for $\Gamma_0(2)$ with multiplier $(-1)^{q_1}$), transforms correctly under

$$\begin{pmatrix} 2\mathbb{Z}+1 & 0 & \mathbb{Z} & 0 \\ 0 & 1 & 0 & 0 \\ 2\mathbb{Z} & 0 & 2\mathbb{Z}+1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) \, , \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\mathbb{Z}+1 & 0 & 2\mathbb{Z} \\ 0 & 0 & 1 & 0 \\ 0 & 2\mathbb{Z} & 0 & 2\mathbb{Z}+1 \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) \, , \tag{7.12}$$

which of course must be the case as this is formally just a projection of the partition function $2^4\Phi_{6,3}^{-1}$ to odd $Q^2/2$. The latter is known to satisfy the modular and polar constraints mentioned in chapter 4 (see [22, section 6.4] or [11]), and $\mathsf{Z}^{(-)}$ will inherit this property.

Second wall. Next we want to investigate the decay into half-BPS states

$$(Q, P) \to (Q - P, 0) + (P, P) \, . \tag{7.13}$$

This decay is now encoded by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and demands that Z exhibits a quadratic pole at $z' = 0$ (recall eq. (4.16)), with coefficient given by

$$\mathsf{Z}\left(\left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)\right) \;\propto\; \frac{1}{z'^2}\, \phi_e^{-1}(\sigma'; 1, 0)\, \phi_m^{-1}(\tau'; 1, 1) \;+\; O(z'^0) \, . \tag{7.14}$$

The variables for this decay are related via (4.21), explicitly

$$Z' = \begin{pmatrix} \tau' & z' \\ z' & \sigma' \end{pmatrix} = \left(\begin{smallmatrix} \tau+\sigma+2z & z+\sigma \\ z+\sigma & \sigma \end{smallmatrix}\right) . \tag{7.15}$$

Even though this decay is related to the previous one by an S-duality transformation in $\Gamma_1(2)$, we shall briefly analyze it to further illustrate the appearance of the Iwahori subgroup $B(2)$ for $\mathsf{Z}^{(0)}$ and $\mathsf{Z}^{(+)}$ on physical grounds. Furthermore, it allows to better test the untwisted sector partition functions against the analysis presented in [22, section 6.5].

Now note that adding any vector from Λ_m to $Q \in \Lambda_e$ cannot change the residue $[Q] \in \Lambda_e \backslash \Lambda_e^*$. As we have seen in chapter 5, the residue selects the half-BPS partition function of purely electric states, so the partition function for decay products $(Q - P, 0) \in \Lambda_e$ will be the same as the one for decay products $(Q, 0)$, i.e.,

$$\phi_e^{-1}(\sigma'; 1, 0) = \phi_e^{-1}(\sigma') \, . \tag{7.16}$$

For unit-torsion dyons the electric component $(Q, 0)$ must be primitive and consistency requires $(Q - P, 0)$ to be primitive as well. Namely, if $Q - P = nQ'$ for some integer n and primitive $Q' \in \Lambda_e$, then $Q \wedge P = n(Q' \wedge P)$, but we know that $I = \gcd(Q \wedge P) = 1$ for unit-torsion dyons, so $n = 1$. Similarly, in the new duality frame obtained by the S-duality transformation $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ we still count dyons of unit-torsion, so again $Q - P$ must be primitive in Λ_e.

Since S-duality also relates $(P, P)^{\intercal} = (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)(0, P))^{\intercal}$ to $(0, P)$, we know that

$$\phi_m^{-1}(\tau'; 1, 1) = \frac{1}{\eta^8(\tau')\eta^8(2\tau')} \, . \tag{7.17}$$

Both (7.16) and (7.17) also follow from S-duality invariance for elements (7.3) by combining (7.4), (7.6) and (7.15).

We have already mentioned that the functions appearing here are, in the case of $\mathsf{Z}^{(0)}$ and $\mathsf{Z}^{(+)}$, modular forms for the congruence subgroup $\Gamma_0(2) = \Gamma_1(2) \ni \left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} p_1 & q_1 \\ r_1 & s_1 \end{smallmatrix}\right)$. Employing (4.23) and (4.24), each Z is required to transform as a Siegel modular form with respect to

$$\begin{pmatrix} \alpha_1 & \alpha_1 - 1 & \beta_1 & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_1 & \gamma_1 & \delta_1 & 0 \\ \gamma_1 & \gamma_1 & \delta_1 - 1 & 1 \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) \,, \qquad \begin{pmatrix} 1 & 1 - p_1 & q_1 & -q_1 \\ 0 & p_1 & -q_1 & q_1 \\ 0 & 0 & 1 & 0 \\ 0 & r_1 & 1 - s_1 & s_1 \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) \,, \tag{7.18}$$

where γ_1 and r_1 are even, while α_1, δ_1, p_1 and s_1 are odd. For $\mathsf{Z}^{(-)}$ the integer q_1 must be even.

Again we compare these constraints to the explicit form of $\mathsf{Z}^{(0)}$, $\mathsf{Z}^{(+)}$ and $\mathsf{Z}^{(-)}$ proposed before. Since (4.21) describes an S-duality transformation for this decay code, $\mathsf{Z}(Z) = \mathsf{Z}(Z')$ holds via (7.4). This immediately translates (7.6) into

$$\mathsf{Z}(Z) \propto \frac{1}{z'^2}\, \phi_e^{-1}(\sigma')\, \frac{1}{\eta^8(\tau')\eta^8(2\tau')} + O(z'^0) \,, \tag{7.19}$$

and therefore matches (7.14) with (7.16) and (7.17).

Third wall. There is one decay channel only possible for dyons with untwisted sector charge (Q, P) subject to the extra condition $Q \in \Lambda_m \subset \Lambda_e$, namely

$$(Q, P) \longrightarrow (Q, Q) + (0, -Q + P) \,. \tag{7.20}$$

The decay code $\left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ does not lie in $\Gamma_0(2)$ and coordinates appropriate for this pole are now by (4.21)

$$Z' = \begin{pmatrix} \tau' & z' \\ z' & \sigma' \end{pmatrix} = \begin{pmatrix} \tau & z + \tau \\ z + \tau & \tau + \sigma + 2z \end{pmatrix} \quad = M_{3w} Z \tag{7.21}$$

with

$$M_{3w} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma_0^{(2)}(2) \setminus B(2) \,. \tag{7.22}$$

Recall that Q is primitive in Λ_e since we consider unit-torsion, so the first decay product $(Q, Q) \in (\Lambda_m \backslash 2\Lambda_e)^{\oplus 2}$ is again counted by

$$\phi_e^{-1}(\sigma'; 1, 1) = \frac{1}{\eta^8(\sigma')\eta^8(2\sigma')} \,, \tag{7.23}$$

in accordance with (7.7). Note that $(Q, Q)^\intercal = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} 0 \\ Q \end{pmatrix}$ is related via an S-transformation in $\Gamma_0(2)$ to a purely magnetic charge of the form $(0, \tilde{P}) \in \Lambda_m \backslash 2\Lambda_e$.[2]

[2] So the subscript "e" in $\phi_e^{-1}(\sigma'; 1, 1)$ is a notational artifact inherited from [22].

It was mentioned before that $Q - P$ is primitive for unit-torsion dyons, so the second decay product, a purely magnetic half-BPS state of charge $(0, -Q + P) \in \Lambda_m \backslash 2\Lambda_e$, also corresponds to the eta-quotient

$$\phi_m^{-1}(\tau'; 0, 1) = \frac{1}{\eta^8(\tau')\eta^8(2\tau')} \,. \tag{7.24}$$

Combining the ingredients we infer that wall-crossing demands that for $z' \to 0$ the quadratic pole in $\mathsf{Z}^{(0)}$ becomes

$$\mathsf{Z}^{(0)}\left(\left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)\right) \propto \frac{1}{z'^2}\, \phi_e^{-1}(\sigma'; 1, 1)\, \phi_m^{-1}(\tau'; 0, 1) + O(z'^0) \tag{7.25}$$

with the given eta-quotients. As a consequence, by (4.23) and (4.24) the partition function $\mathsf{Z}^{(0)}$ should also transform as a Siegel modular form with respect to the embedded transformations

$$\begin{pmatrix} \alpha_1 & 0 & \beta_1 & \beta_1 \\ -\alpha_1 - 1 & 1 & -\beta_1 & -\beta_1 \\ \gamma_1 & 0 & \delta_1 & \delta_1 + 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ p_1 - 1 & p_1 & 0 & q_1 \\ r_1 & r_1 & 1 & s_1 - 1 \\ r_1 & r_1 & 0 & s_1 \end{pmatrix} \tag{7.26}$$

where $\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \left(\begin{smallmatrix} p_1 & q_1 \\ r_1 & s_1 \end{smallmatrix}\right) \in \Gamma_1(2)$.

Let us see whether (7.25) is also satisfied for the concrete $\mathsf{Z}^{(0)}$ proposed before. Starting from (6.36), we consider the tautology $\mathsf{Z}^{(0)}(Z) = \mathsf{Z}^{(0)}(M_{3w}^{-1} Z')$. Since $M_{3w}^{-1} \in \Gamma_0^{(2)}(2) \setminus B(2)$, only the $B(2)$ modular form T transforms non-trivially, so for $T(M_{3w}^{-1} Z')$ we may use the transformation formula for theta characteristics (see appendix A) to find a new characteristic

$$M_{3w}^{-1} \left\{ \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} \right\} = \begin{pmatrix} a_1 + a_2 \\ a_2 \\ b_1 \\ b_1 + b_2 \end{pmatrix} . \tag{7.27}$$

This means that

$$16^2\, T(M_3^{-1} Z') = \theta_{1100}^4(Z')\, \theta_{1111}^4(Z') \tag{7.28}$$

and thus

$$\mathsf{Z}^{(0)}(Z) = -\frac{1}{2}\frac{1}{W(Z')} - \frac{1}{32}\frac{\theta_{1100}^4(Z')\, \theta_{1111}^4(Z')}{Y W(Z')} \,. \tag{7.29}$$

Now use the behavior of the theta constants $\theta_{a_1 a_2 b_1 b_2}(Z')$ under $z' \to 0$ (again see appendix A) to find that the second term in (7.29), being proportional to $\theta_{1111}^2(Z')$, vanishes quadratically in z' for $z' \to 0$. Only the first term contributes to the quadratic pole in z' which is relevant for the BPS indices. More precisely, for $z' \to 0$ we have

$$\mathsf{Z}^{(0)}(Z) = -\frac{1}{2}\frac{1}{(2\pi i)^2}\frac{1}{z'^2}\frac{1}{\eta^8(\tau')\eta^8(2\tau')}\frac{1}{\eta^8(\sigma')\eta^8(2\sigma')} + O(z'^0) \,, \tag{7.30}$$

nicely matching our wall-crossing expectations. The calculation also shows that only $\Phi_{6,0}^{-1}$ contributes to the pole, while $\mathsf{Z}^{(+)}$ does not.

Modular symmetry group. In close parallel to [22, section 6.5] the question emerges, whether the symplectic matrices (7.2), (7.3), (7.11), (7.18) and (7.26) fit into a subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ defined by some congruence relation. Affirmative answer can be given for the group

$$\begin{pmatrix} 2\mathbb{Z}+1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z}+1 & \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z}+1 & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & \mathbb{Z} & 2\mathbb{Z}+1 \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & \mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{Sp}_4(\mathbb{Z}) \,, \tag{7.31}$$

where the group on the right hand side is in fact the Iwahori subgroup $B(2)$. To see why (7.31) is an equality, we use eq. (A.3). Then for $M, M^{-1} \in B(2) \subset \mathrm{Sp}_4(\mathbb{Z})$ we inspect entries in $MM^{-1} - 1_4 \equiv 0$ mod 2 to find what is claimed.

Thus, by analyzing the polar and modular constraints one is led naturally to $B(2)$ as the symmetry group for the charge sectors with even $Q^2 = Q_2^2$. As we know by chapter 6 (recall eqs. (6.27) and (6.32) or (6.33)), for these sectors the partition function is either given by $\mathsf{Z}^{(0)}$ (if $[Q] = 0$) or $\mathsf{Z}^{(+)}$ (if $[Q] \neq 0$) and both of them indeed are modular forms for $B(2)$.

Remark on a subsector. Let us comment on the relation between our findings and that of [22, section 6.5].

The unit-torsion quarter-BPS partition function considered there concerns dyons with untwisted sector electric charge subject to the constraints

$$\begin{aligned} &\frac{1}{2}Q^2 \in 2\mathbb{Z}+1, && \frac{1}{2}P^2 \in 2\mathbb{Z}+1, && Q \cdot P \in 2\mathbb{Z}, \\ &h = 0, && \mathcal{P} = 0, && Q \cdot \delta \in 2\mathbb{Z}+1 \,. \end{aligned} \tag{7.32}$$

As these restrictions are only preserved for S-transformations in $\Gamma(2) \subset \Gamma_1(2)$, the partition function for this subsector does not need to be invariant under *all* elements in (7.3), but only under those where b is even.

The partition function for unit-torsion dyons that have odd $Q^2/2$ and satisfy all constraints in the second line of (7.32), but have generic values of $\frac{1}{2}P^2 \in \mathbb{Z}$ and $Q \cdot P \in \mathbb{Z}$, are counted by $\mathsf{Z}^{(-)}$. The additional parity restrictions on the latter quadratic T-invariants can be implemented in $\mathsf{Z}^{(-)}$ by applying suitable projections. For odd $P^2/2$, for instance, one has the lower sign in

$$\mathsf{Z}^{(-)}(\tau, \sigma, z) \to \frac{1}{2}\left(\mathsf{Z}^{(-)}(\tau, \sigma, z) \pm \mathsf{Z}^{(-)}(\tau + \tfrac{1}{2}, \sigma, z)\right) \,. \tag{7.33}$$

With this and the properties of $\mathsf{Z}^{(-)}$ it is straightforward to check that also on the subset (7.32) the modular and polar constraints discussed in [22, section 6.5] are met.

As mentioned already, for magnetic half-BPS states $(0, P)$ counted at $z = 0$, our assumption (7.10) is compatible with the explicit representatives P chosen in [22, section 6.5]. These are primitive vectors $P \in \Lambda_m \backslash 2\Lambda_e$. Indeed, these are also the same magnetic charges as occuring in the twisted sector quarter-BPS states [22, section 6.4] (up to restriction to odd "K" quantum number there, causing $P^2/2$ to be odd for the untwisted case).

Befor proceeding, we remark that affirmative consistency checks starting from charge representatives in other subsectors can be performed in complete analogy to [22, section 6.5], however, these are mostly straightforward and in light of our preceeding analysis rather redundant so we will not display them here.

7.4 Modular reverse engineering

In summary, the constraints from quantization laws, S-duality and wall-crossing suggest that $\mathsf{Z}^{(+)}$ and $\mathsf{Z}^{(0)}$ transform as Siegel modular forms for the Iwahori subgroup $B(2)$ with weight -6. As announced earlier, we may now conclude that the modular and polar constraints alone are (almost) restrictive enough to guess the respective Z in closed form. Of course, the analysis of chapter 6 already provides explicit expressions, which we have shown to satisfy all constraints, nevertheless, it is instructive to have an alternative approach that gives consistent results.

Since explicit generators for the ring of even (positive) weight Siegel modular forms for $\Gamma_0^{(2)}(2)$, $K(2)$ and $B(2) = \Gamma_0^{(2)}(2) \cap K(2)$ are known in the mathematics literature (references are given in appendix A), a suitable ansatz might reduce the problem of fixing Z to a determination of a finite number of coefficients. This Siegel modular form must exhibit the quadratic poles in (7.6), (7.14) and (7.25). Indeed, any decay code $\left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ is related to either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (first wall) or $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ (third wall) by an S-duality transformation in $\Gamma_1(2)$, which has index two in $\mathrm{SL}_2(\mathbb{Z})$. We can therefore demand that $\mathsf{Z}(Z)$ must exhibit a quadratic pole at all images of the diagonal locus $\left(\begin{smallmatrix} \tau & 0 \\ 0 & \sigma \end{smallmatrix}\right)$ under the group generated by $\mathrm{SL}_2(\mathbb{Z})$-transformations (4.9) and integer translations (7.2).[3] The arguably simplest compatible ansatz one might choose for $\mathsf{Z}(Z)$ is $F(Z)/\chi_{10}(Z)$, where the Igusa cusp form χ_{10}, i.e., the product of the square of the ten even genus two Thetanullwerte, vanishes quadratically at all $\mathrm{Sp}_4(\mathbb{Z})$-images of the diagonal. The latter is also the partition function for unit-torsion quarter-BPS dyons in the parent theory and at least the untwisted sector dyons of interest might be regarded as an invariant subset thereof. In this ansatz $F(Z)$ is a weight four Siegel modular form for $B(2)$, which is expected to be holomorpic in $\mathbb{H}_2$ such that there are no additional, spurious poles. Zeroes in $F(Z)$ however might cancel any additional, spurious poles in χ_{10}^{-1} (if there are such). Working, for instance, with the ring generators given in (A.38) and the properties of the theta constants, the behavior of Z at the wall-crossing divisors fixes $F(Z)$ eventually to $F^{(+)} = 8T$ or $F^{(0)} = Y + 8T$. This gives precisely back $\mathsf{Z}^{(+)}$ and $\mathsf{Z}^{(0)}$ found via the chiral genus two partition function in chapter 6.[4]

[3] As an aside, motivated by CHL dyon counting functions Cléry and Gritsenko [87] classified and constructed all so-called *dd-modular forms*, i.e., Siegel modular forms for the Hecke congruence subgroups $\Gamma_0^{(2)}(N)$ which vanish precisely along the $\Gamma_0^{(2)}(N)$-translates of the diagonal divisor $z = 0$ (with vanishing order one; possibly with a multiplier system). Especially, this includes the square roots of the Igusa cusp form and the Siegel modular form $\Phi_{6,0}$ appearing in the $N = 1, 2$ CHL models. However, this does not characterize the partition functions $\mathsf{Z}^{(0)}$, $\mathsf{Z}^{(+)}$ or $\mathsf{Z}^{(-)}$.

[4] A similar analysis could be done for $\mathsf{Z}^{(-)}$, but we are not aware of similar results for the ring of Siegel modular forms for the corresponding congruence subgroup in this case. Of course, this can be worked out, but for convenience we restricted to the two cases leading to modular forms for $B(2)$ for which results are readily available. As we are more interested in giving a proof of principle in this section, and as we already know the partition functions from chapter 6, the argument for $\mathsf{Z}^{(-)}$ can safely be skipped.

CHAPTER 8

BPS state counting and black hole entropy

In this chapter we interpret the dyon partition functions obtained in chapter 6 as microscopic partition functions for extremal dyonic black holes in four dimensions. Especially, we want to compute the microscopic degeneracy for a fixed charge configuration and compare this with a macroscopic computation of the black hole entropy in the four-dimensional $\mathcal{N} = 4$ theory. The matching of the two can, in principle, also be regarded as a (further) constraint on the dyon partition functions.

Our analysis in this chapter will focus on the least intricate features, namely the Bekenstein-Hawking term and the first correction in inverse powers of the charges (considering a large charge expansion). Similar discussions for the twisted sector alone were given [1, 11, 21, 22, 86, 88].

8.1 Macroscopic determination of the black hole entropy

For an observer in the four non-compact spacetime dimensions, the microscopic quarter-BPS configurations of the $\mathcal{N} = 4$ compactification appear as black hole configurations, which carry electric and magnetic charges [89, 90]. They can be regarded as generalizations of the Reissner-Nordström solutions of Einstein-Maxwell gravity describing (non-rotating) electrically charged black holes, that is, of their extremal limit $M^2 = Q^2$. This generalization allows for both non-zero electric and magnetic charges, i.e., these black holes are dyonic. Moreover, the effective action, whose details depend on the string compactification, generically gets higher-derivative corrections. A formula for the entropy of a black hole in (field) theories with higher-derivative terms is the so-called Wald entropy [91–94]. For the spherically symmetric extremal black holes we will discuss, the Wald entropy can also be obtained by Sen's entropy function formalism [95, 96]. As the latter is easier to deal with, it will be considered here exclusively.

8.1.1 Entropy function for extremal black holes

For spherically symmetric extremal black holes, which have an $AdS_2 \times S^2$ near-horizon geometry, the Wald entropy can as mentioned also be computed using the entropy function formalism [95, 96] that we now briefly explain.

Assume we are given a Lagrangian density $\sqrt{-\det g}\mathcal{L}$ for a four-dimensional theory of gravity, expressed in terms of the metric $g_{\mu\nu}$, the field strengths $F^{(i)}_{\mu\nu}$ for r Abelian gauge fields $A^{(i)}_\mu$, some neutral scalar fields ϕ_s $(s = 1, \ldots, n_s)$ and the covariant derivatives of these fields. Consistency with the $SO(2,1) \times SO(3)$ symmetry of $AdS_2 \times S^2$ requires the field configuration at the horizon to be of the following form:

$$\mathrm{d}s^2 = v_1 \left(-r^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} \right) + v_2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \right) \tag{8.1}$$

$$\phi_s = u_s \tag{8.2}$$

$$F^{(i)}_{rt} = e_i \,, \qquad F^{(i)}_{\theta\phi} = \frac{p_i}{4\pi} \sin\theta \,, \tag{8.3}$$

where v_1, v_2, u_s and e_i, p_i are constants. It can be shown that for this background the covariant derivatives of the scalars, the field strengths and the Riemann tensor vanish and hence any terms in the Langrangian involving these objects do not contribute to the equations of motion.

Up to an overall constant, the entropy function is now defined by integrating the Lagrangian density over the S^2 factor in the near horizon geometry (8.1) and subsequently taking a Legendre transform of the integral with respect to the electric field parameters e_i. To this end, define

$$f(u, v, e, p) := \int \mathrm{d}\theta \mathrm{d}\phi \sqrt{-\det g}\, \mathcal{L} \,. \tag{8.4}$$

Then the entropy function is given by the expression

$$\mathcal{E}(u, v, e; q, p) := 2\pi \left(e_i q_i - f \right) \,, \tag{8.5}$$

where for $i = 1, \ldots, r$ the quantity

$$q_i := \frac{\partial f}{\partial e_i} \tag{8.6}$$

is interpreted as the electric charge vector of the black hole solution (and the index i is summed over in (8.5)). The last identity is equivalent to

$$\frac{\partial \mathcal{E}}{\partial e_i} = 0 \,, \qquad i = 1, \ldots, r \,, \tag{8.7}$$

and the analogously formed equations

$$\frac{\partial \mathcal{E}}{\partial v_a} = 0 \,, \qquad a = 1, 2 \,, \tag{8.8}$$

$$\frac{\partial \mathcal{E}}{\partial u_s} = 0 \,, \qquad s = 1, \ldots, n_s \,, \tag{8.9}$$

can be shown to be equivalent to the equations of motions for the near horizon fields.

Taking the electric and magnetic charges (q, p) as independent parameters of a near-horizon solution, the field parameters u_s, v_a and e_i are implicitly determined by the eqs. (8.7) to (8.9). At this stationary point, defined by the vanishing derivatives above, the value of the entropy function $\mathcal{E}$ generically only depends on the charges (q, p) and we shall denote this by $\mathcal{E}^*(q, p)$. After some algebra, which the reader can find in the reference [11], one finds that the Wald entropy of the extremal black hole configuration (8.1) indeed agrees with the stationary value of the just defined entropy function, that is,

$$S_{\text{Wald}} = \mathcal{E}^*(q, p) \ . \tag{8.10}$$

We stress that the Lagrangian-density is allowed to contain higher derivative terms and thus possibly going beyond the Einstein-Maxwell theory of gravity and electromagnetism.

Example: Entropy of an extremal Reissner-Nordström black hole. Let us apply the entropy function formalism to pure Einstein-Maxwell theory and the field configuration of an extremal Reissner-Nordström black hole. That is, we have a single Abelian gauge field A_μ ($i = 1$) and no scalar fields, otherwise we can take the field configuration in eqs. (8.1) and (8.3).[1] The action is given by

$$\mathcal{S} = \int \mathrm{d}^4x \sqrt{-\det g}\,\mathcal{L} \ , \qquad \mathcal{L} = \frac{1}{16\pi G_N} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \ . \tag{8.11}$$

The integral over the angular coordinates then yields

$$f(v_1, v_2, e, p) = 4\pi v_1 v_2 \left[\frac{1}{16\pi G_N} \left(-\frac{2}{v_1} + \frac{2}{v_2} \right) + \frac{1}{2} \left(\frac{e^2}{v_1^2} - \frac{p^2}{(4\pi v_2)^2} \right) \right] \ , \tag{8.12}$$

whose Legendre transform becomes

$$\mathcal{E}(v_1, v_2, e; q, p) = 2\pi \left[qe - \frac{v_1 - v_2}{2G_N} - 2\pi \left(\frac{v_2}{v_1} e^2 - \frac{v_1}{v_2} \left(\frac{p}{4\pi} \right)^2 \right) \right] \ . \tag{8.13}$$

Solving the equations (8.7) and (8.8) in the present case gives

$$v_1 = v_2 = G_N \frac{q^2 + p^2}{4\pi} \ , \qquad e = \frac{q}{4\pi} \ . \tag{8.14}$$

Plugging the values for v_1, v_2 and e back into the entropy function (8.13) finally yields

$$\mathcal{E}^*(q, p) = \frac{q^2 + p^2}{4} \ . \tag{8.15}$$

Indeed, this agrees with the Bekenstein-Hawking entropy

$$S_{\text{BH}} = \frac{\text{Area of the horizon}}{4G_N} \ , \tag{8.16}$$

as the horizon sphere S^2 has radius $\sqrt{v_1} = \sqrt{v_2}$, and consequently an area of $4\pi v_1$.

In the rest of this section we will turn to the four-dimensional effective theory relevant for $\mathcal{N} = 4$ string compactifions and repeat the above computation of the extremal black hole entropy, treating the leading supergravity approximation as well as a specific correction term to the Lagrangian.

[1] This (ansatz) metric is slightly more general than the original extremal Reissner-Nordström black hole metric (see e.g. [11, section 2.1]), as v_1 and v_2 are a priori independent. However, the equations of motion, or equivalently the extremization conditions of the entropy function, eventually relate the two.

8.1.2 The supergravity approximation at two-derivative level

In our discussion of the $\mathcal{N} = 4$ supergravity theory and the derivation of the extremal black hole entropy within this theory we now follow [11], see also [96, 97].

Up to and including two derivatives the action for $\mathcal{N} = 4$ supergravity is completely fixed by supersymmetry and the action for the massless bosonic fields reads

$$\begin{aligned}\mathcal{S} = \frac{1}{2\pi\alpha'} \int \mathrm{d}^4x \,\sqrt{-\det G}\, S \big[R_G + \frac{1}{S^2} G^{\mu\nu} \left(\partial_\mu S\, \partial_\nu S - \frac{1}{2}\partial_\mu a\, \partial_\nu a \right) + \frac{1}{8} G^{\mu\nu}\, \mathrm{Tr}\left(\partial_\mu M L \partial_\nu M L \right) \\ - G^{\mu\mu'} G^{\nu\nu'} F^{(i)}_{\mu\nu} (LML)_{ij} F^{(j)}_{\mu'\nu'} - \frac{a}{S} G^{\mu\mu'} G^{\nu\nu'} F^{(i)}_{\mu\nu} L_{ij} \tilde{F}^{(j)}_{\mu'\nu'} \big] ,\end{aligned} \tag{8.17}$$

where L is the $\mathrm{O}(14, 6)$-invariant bilinear form introduced in section 2.2 (similarly $\mathrm{O}(22, 6)$ for the Narain case), S is the imaginary part of the complex scalar $a + iS$ that takes values in the upper half-plane and $G_{\mu\nu} = S\, g_{\mu\nu}$ is the string metric, conformally related to the Einstein frame metric $g_{\mu\nu}$. Recall also that the 20×20 scalar moduli matrix M (or 28×28 for the Narain case) enjoys the T-transformation law (2.46), which is accompanied by an analogous rotation of the vector of two-form field strength tensors of the generic $U(1)^{14+6}$ Abelian gauge group ($i = 1, \ldots, 20$),

$$F^{(i)}_{\mu\nu} \to O_{ij} F^{(j)}_{\mu\nu} \, . \tag{8.18}$$

The action stays invariant under this simultaneous transformation of the moduli matrix M and the field strengths $F^{(i)}_{\mu\nu}$.

For a dyonic quarter-BPS state in this theory the near-horizon region is described by a spherically symmetric extremal black hole solution, labelled by electric and magnetic charges (q_i, p_i). Near the horizon the metric, the scalar fields and the field strengths become

$$\mathrm{d}s^2 = \frac{\alpha'}{16} v_1 \left(-r^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} \right) + \frac{\alpha'}{16} v_2 \left(\mathrm{d}\theta^2 + \sin^2\,\theta \mathrm{d}\phi^2 \right) \tag{8.19}$$

$$S = u_S \, , \qquad a = u_a \, , \qquad M_{ij} = u_{M\, ij} \tag{8.20}$$

$$F^{(i)}_{rt} = \frac{\sqrt{\alpha'}}{4} e_i \, , \qquad F^{(i)}_{\theta\phi} = \frac{p_i \sqrt{\alpha'}}{16\pi} \sin\theta \, , \tag{8.21}$$

where u_S, u_a, u_M as well as v_1 and v_2 are constants to be determined from the stationarity of the entropy function $\mathcal{E}$. A straightforward computation (see [11]) yields, after elimination of the electric field parameters (eq. (8.6)), the entropy function

$$\mathcal{E}(u_S, u_a, u_M, \vec{v}; \vec{q}, \vec{p}) = \frac{\pi}{2} \left[u_S (v_2 - v_1) + \frac{v_1}{v_2 u_S} \left(Q^T u_M Q + (u_S^2 + u_a^2) P^T u_M P - 2 u_a Q^T u_M P \right) \right] , \tag{8.22}$$

where the shorthands

$$Q_i \coloneqq 2 q_i \, , \qquad P_i \coloneqq \frac{1}{4\pi} L_{ij} p_j \tag{8.23}$$

have been introduced for the electric and magnetic charges. Transforming the scalar moduli matrix M and the field strengths $F^{(i)}_{\mu\nu}$ as in eqs. (2.46) and (8.18), respectively, the just defined electric and magnetic charges (Q, P) in (8.23) indeed transform as stated in eq. (2.46). The entropy function $\mathcal{E}$

is invariant under this transformation, which can be used to simplify the further computation of $\mathcal{E}^*$. Solving the extremization conditions on $\mathcal{E}$ with respect to u_a, u_S, u_M, v_1, v_2 will lead to

$$S_{\mathrm{BH}} = \pi\sqrt{\left|Q^2P^2 - (Q\cdot P)^2\right|}\,. \tag{8.24}$$

This result gives us the entropy of a spherically symmetric extremal black hole in the two-derivative approximation of the effective $\mathcal{N} = 4$ supergravity theory that describes the physics in the four non-compact spacetime dimensions. It is T- and S-duality invariant and has the same functional dependence on the charge bilinears $Q^2, P^2, Q\cdot P$ for both the original heterotic string theory on T^6 and the CHL orbifold. However, the last point will change once corrections induced by higher-derivative terms in the Lagrangian are taken into account. Elaborating on this will be the subject of the next subsection.

8.1.3 Entropy corrections due to a Gauss-Bonnet term

The one-loop effective action for the type IIB string on $\mathrm{K3}\times T^2$ or its orbifold, where $a + iS =: u$ now corresponds to the complex structure modulus of the 2-torus of the compactification manifold, contains a Gauss-Bonnet term [65, 98, 99]

$$\int \mathrm{d}^4x\,\sqrt{-\det g}\;\phi(a,S)\,\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right)\,, \tag{8.25}$$

where the Riemann tensor $R_{\mu\nu\rho\sigma}$ is the one obtained from the Einstein frame metric $g_{\mu\nu}$, which is related to the string frame metric $G_{\mu\nu}$ via a rescaling $g_{\mu\nu} = S^{-1}G_{\mu\nu}$. The gravitational coupling ϕ satisfies the differential identity [65]

$$\partial_u\phi = \int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\tau_2}\partial_u B_4^{(\mathrm{II})}\,. \tag{8.26}$$

On the right hand side of this equation we have an integral over the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$, the integrand is the derivative of the fourth helicity supertrace $B_4^{(\mathrm{II})}$ of the type II superstring on $\mathrm{K3}\times T^2$ (or its CHL orbifold). The latter is the type II analog of the supertrace computed in chapter 5, which concerned the perturbative heterotic string. We will not review its computation here, and the reader is referred to [65, 97]. However, it is worth noting that this index is invariant under deformations of the moduli, and the computation can be done e.g. at points where the K3 is realized as an orbifold of a suitable four-torus T^4, making the computation tractable. Also we note that because of (8.26) and (3.24) only half-BPS states (in the perturbative type II string spectrum) contribute to the effective coupling ϕ. Depending on the $\mathcal{N} = 4$ theory we are considering, we will obtain a result of the form

$$\phi(a,S) = -\frac{1}{64\pi^2}\left[(k+2)\log S + \log g(a+iS) + \log g(-a+iS)\right] + \text{const.} \tag{8.27}$$

More precisely, we have

$$g(\tau) = \begin{cases}\eta^{24}(\tau) & : k = 10\\ \eta^8(\tau)\eta^8(2\tau) & : k = 6\end{cases} \tag{8.28}$$

with $k = 10$ corresponding to the unorbifolded theory and $k = 6$ corresponding to the $\mathbb{Z}_2$ CHL orbifold, respectively. The function g makes the Gauss-Bonnet term manifestly invariant with respect to the appropriate S-duality group, which for instance is the $\Gamma_1(2)$ congruence subroup for the $\mathbb{Z}_2$ CHL model.

From eqs. (8.4) and (8.5) it is clear that the inclusion of the Gauss-Bonnet term enhances the entropy function by a term $\Delta\mathcal{E} = 64\pi^2\phi$ to the new expression

$$\mathcal{E} = \frac{\pi}{2}\left[u_S(v_2 - v_1) + \frac{v_1}{v_2 u_S}\left(Q^T u_M Q + (u_S^2 + u_a^2)P^T u_M P - 2u_a Q^T u_M P\right) + 128\pi\,\phi(u_a, u_S)\right] . \tag{8.29}$$

The extremization conditions[2] for the complex scalar modulus now become

$$P^2 u_a - Q \cdot P + 64\pi u_S \frac{\partial\phi}{\partial u_a} = 0 \tag{8.30}$$

$$-\frac{1}{u_S^2}\left(Q^2 - 2u_a Q \cdot P + P^2 u_a^2\right) + P^2 + 128\pi \frac{\partial\phi}{\partial u_S} = 0\,. \tag{8.31}$$

No exact analytic solution is known for these equations. However, what we can nevertheless compare to the statistical entropy obtained from the microscopic quarter-BPS spectrum is the large charge regime of the black hole entropy. To this end, note that under a simultaneous rescaling of all the charges the terms in the extremization conditions and in $\mathcal{E}$ which come from a non-trivial ϕ are scaling invariant. At the same time these terms are suppressed by two powers of the charges with respect to terms coming from the two-derivative supergravity approximation alone. The upshot[3] is that in order to get the leading entropy correction in an expansion in terms of inverse powers of the charges, one can simply add $\Delta\mathcal{E}$, evaluated at the previous stationary values for a and S (corresponding absent ϕ), to the result from the two-derivative supergravity approximation (8.24), that is,

$$S_{\mathrm{BH}} = \pi\sqrt{Q^2P^2 - (Q \cdot P)^2} + 64\pi^2\,\phi\left(\frac{Q \cdot P}{P^2}, \frac{\sqrt{Q^2P^2 - (Q \cdot P)^2}}{P^2}\right) + \cdots\,. \tag{8.32}$$

It will be our task in the following to compare this macroscopically determined black hole entropy to the corresponding statistical entropy based on the large charge expansion of the BPS index (4.6).

8.2 Matching with the microscopic statistical entropy

From the macroscopic analysis in the previous section we get a quadratically growing black hole entropy in the limit of large charges. The statistical Boltzmann entropy in turn is proportional to the logarithm of the (micro)state degeneracy, so identifying the two requires for consistency that the degeneracy scales like the exponential of the quadratic charges. This is what one roughly gets from the exponential in the Fourier integral (4.6), as the chemical potentials τ, z, σ have positive imaginary part. However, for large charges also the phase of the integrand is rapidly oscillating along

[2] These conditions are at least valid for the case, $P^2 > 0,\ Q^2 > 0,\ P^2Q^2 > (Q \cdot P)^2$, which is discussed in detail in section 3 of [11].

[3] Besides [11], the reader will also find a discussion of this point in [13].

the contour such that the value of the integral itself cannot be estimated based on the absolute value of the integrand. The common strategy[4] to circumvent this difficulty is to deform the original integration contour to a new contour for which the Fourier integral will be exponentially suppressed compared with the original Fourier integral. Hence the dominant contribution (which could account for the quadratic growth of the entropy) must come from the residues of the integrand that are picked up when deforming the contour. Taking the residue of the integrand at such a pole eliminates one out of the three integration variables (say z) and the integral over the remaining two variables (τ, σ) is treated in a saddle-point approximation. As we will argue below, it will in fact be sufficient for us to identify the dominant contribution amongst the contributing residues.

The dyon partition functions considered in this thesis have infinitely many poles, described by certain quadratic divisors in the Siegel upper half space. This follows from the fact that using eqs. (6.34) and (A.58)-(A.62) they can be written as F/χ_{10} with F being a holomorphic Siegel modular form of the appropriate congruence subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ and the appropriate weight (weight four for the $\mathbb{Z}_2$ CHL orbifold). Now the zeroes of χ_{10} occur for

$$\begin{aligned} n_2(\sigma\tau - z^2) + jz + n_1\sigma - m_1\tau + m_2 &= 0 \,, \\ m_1 n_1 + m_2 n_2 + \frac{j^2}{4} &= \frac{1}{4} \,, \\ \text{where} \quad m_1 \in \mathbb{Z} \,, \quad n_1 \in \mathbb{Z} \,, \quad j \in 2\mathbb{Z}+1 \,, \quad m_2 \in \mathbb{Z} \,, \quad n_2 \in \mathbb{Z} \,. \end{aligned} \tag{8.33}$$

Indeed, these divisors are simply the $\mathrm{Sp}_2(\mathbb{Z})$ images of the diagonal divisor $z = 0$. In other words, upon a suitable $\mathrm{Sp}_2(\mathbb{Z})$ (coordinate) transformation of the Siegel matrix $\left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)$ the divisor (8.33) maps to the standard diagonal divisor $z' = 0$ in the transformed coordinates. Except for the divisors where the numerator F vanishes as well, a (double) zero of the Igusa cusp form χ_{10} leads to a (double) pole of the dyon partition function in the integral. For $\Phi_{6,3}^{-1} = Y'/\chi_{10}$ for instance (the twisted sector partition of [1], which we recall does not resolve the fine dependence on the charge residue $[Q]$) the presence of the Siegel modular form Y' in the numerator has the effect that some of the divisors of (8.33) do not lead to a double pole of $\Phi_{6,3}^{-1}$. This happens when any of the four distinct theta functions appearing quadratically in Y' (recall the identity (A.63)) maps under the just mentioned coordinate transformation via eq. (A.23) to the theta function θ_{1111}. By virtue of (A.27) the latter vanishes on the new diagonal $z' = 0$. Thus, the quadratic zeroes of numerator and denominator cancel at such a divisor and $\Phi_{6,3}^{-1} = Y'/\chi_{10}$ will not have a pole there. In this simple case, and similarly for the other prime order CHL orbifolds considered by [1, 86], it is known that the subset of (8.33) that descends to true poles of $\Phi_{6,3}^{-1} = Y'/\chi_{10}$ is obtained by simply restricting to $m_1 \in N\mathbb{Z}$ with $N = 2$ being the order of the $\mathbb{Z}_2$ orbifolding CHL group in consideration. As all the Siegel modular forms $\Phi_{6,i}$ introduced in eq. (A.57) are non-trivial modular images of $\Phi_{6,3}$ under $\mathrm{Sp}_4(\mathbb{Z})$ (or equivalently of any fixed $\Phi_{6,j}$), an analogous discussion can be applied to them. For the next paragraph we can just pretend that the generic dyon partition function Z has poles simply given by (8.33) and explain afterwards why, for the purpose of finding the leading large charge behavior of the BPS index, this does not introduce a significant error.

Following [86] we introduce

$$A = n_2 \,, \quad \vec{B} = (n_1, m_1, \frac{j}{2}) \,, \quad \vec{y} = (\tau, \sigma, -z) \,, \quad C = m_2 \,, \quad \vec{q} = (Q^2, P^2, Q\cdot P) \,, \tag{8.34}$$

[4] See [1, 21, 86, 88, 100] or the review [11] for extensive discussions.

where the three-component vectors are considered as elements of a vector space with the SO(2, 1) invariant bilinear form

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_2 + a_2 b_1 - 2a_3 b_3 \ . \tag{8.35}$$

Noting that

$$y^2 = 2(\tau\sigma - z^2) \ , \qquad \vec{b} \cdot \vec{y} = jz + n_1\sigma - m_1\tau \ , \tag{8.36}$$

the pole condition (8.33) turns into

$$\frac{1}{2} A y^2 + \vec{B} \cdot \vec{y} + C = 0 \ . \tag{8.37}$$

On the other hand, the exponent in (4.6) now reads

$$-2\pi i \left(\tau \frac{P^2}{2} + zQ \cdot P + \sigma \frac{Q^2}{2} \right) = -i\pi \vec{q} \cdot \vec{y} \ . \tag{8.38}$$

According to the large charge evaluation strategy outlined above, the saddle-point approximation requires us to extremize (8.38) under the condition (8.37). This simple optimization problem can be solve by the method of Lagrange multipliers. Skipping directly to the result we have

$$\exp(-i\pi\vec{q} \cdot \vec{y}) = \exp\left(\frac{\pi}{A} \sqrt{\frac{q^2}{2}} + \frac{i\pi}{A} \vec{q} \cdot \vec{B} \right) \ . \tag{8.39}$$

The second term just leads to a phase factor, while the first can be written as

$$\frac{1}{n_2} \pi \sqrt{Q^2 p^2 - (Q \cdot P)^2} \ , \tag{8.40}$$

resembling the leading term in (8.32). For $n_2 = 1$ this gives the domimant contribution to the integral and using the shift symmetries in τ, σ, z one can bring the divisor to the form

$$\mathcal{D} := \tau\sigma - z^2 + z = 0 \ . \tag{8.41}$$

Now coming back to the issue of having a non-trivial numerator in our dyon partion function F/χ_{10}, the preceeding analysis only required us to know that the poles are a subset of (8.33). Amongst all the (candidate) poles (8.33), the pole described by (8.41) will give the dominant contribution. What we have to check is that this really is a pole of our dyon partition function, which is equivalent to F being non-vanishing there.

Before proceeding, we interlude with the remark that in the above argument we have implicitly used that neglecting the dependence of the original integration contour on the asymptotic scalar moduli only amounts to introducing exponentially suppressed ambiguities in the BPS degeneracy, which hence are not relevant for obtaining the leading large charge behavior. Such contributions come from divisors with $n_2 = 0$ and only grow as exponentials of linear powers of the charges [66, 100].

We shall now study the behavior of $\mathsf{Z} \in \{\mathsf{Z}^{(0)}, \mathsf{Z}^{(+)}, \mathsf{Z}^{(-)}\}$ near the divisor $\mathcal{D} = 0$. In all three cases the numerator F in $\mathsf{Z} = F/\chi_{10}$ is a linear combination of the modular forms Y, Y' and Y'' introduced in appendix A. In order to find out how Z behaves near $\mathcal{D} = 0$, we hence have to find out how they behave there. First, an $\mathrm{Sp}_4(\mathbb{Z})$ transformation on (τ, σ, z) with

$$M_{\mathcal{D}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{8.42}$$

defines new coordinates

$$\tau' = \frac{\tau\sigma - z^2}{\sigma} \,, \quad \sigma' = \frac{\tau\sigma - (z-1)^2}{\sigma} \quad \text{and} \quad z' = \frac{\tau\sigma - z^2 + z}{\sigma} \,, \tag{8.43}$$

such that the condition $\mathcal{D} = 0$ becomes equivalent to $z' = 0$. But the matrix $M_{\mathcal{D}}$ is not an element of $\Gamma_0^{(2)}(2)$ or $B(2)$, for which we know at least how $\mathsf{Z}^{(+)}$ and $Z^{(0)}$ transform, so we better express Z in terms of genus two theta functions using (A.32)-(A.36) and study explicitly how they transform. Using eq. (A.22) theta characteristics transform as

$$M_{\mathcal{D}}^{-1}\{(a_1, a_2, b_1, b_2)^\intercal\} = (a_1 + a_2, b_1 + b_2, b_1, a_2)^\intercal \,. \tag{8.44}$$

As an example we find

$$\theta_{0010}(Z) = \theta_{0010}(M_{\mathcal{D}}^{-1} Z') \;\propto\; (2z' - \tau' - \sigma')^{1/2} \theta_{1111}(Z') \,. \tag{8.45}$$

Amongst the occuring Siegel modular forms Y, Y' and Y'' only Y' is non-vanishing in the limit $z' \to 0$, as is easily found using (A.26) and (A.27).

Whatever $\mathsf{Z} \in \{\mathsf{Z}^{(0)}, \mathsf{Z}^{(+)}, \mathsf{Z}^{(-)}\}$ we consider, only the term with Y' in the numerator, which is formally the same as the twisted sector partition function $2^{-4}\Phi_{6,3}$ of [1], contributes to the double pole. The other two Siegel modular forms that may contribute to the chosen Z stay finite for $z' \to 0$. Explicit calculation furthermore gives

$$\mathsf{Z} \propto \frac{1}{z'^2} \frac{1}{(2z' - \tau' - \sigma')^6} \frac{1}{\eta^{12}(\rho')\theta_2^4(\tau')} \frac{1}{\eta^{12}(\sigma')\theta_2^4(\sigma')} + O(z'^4) \,, \tag{8.46}$$

which indeed reproduces

$$\mathsf{Z} \propto \frac{1}{(2z' - \tau' - \sigma')^6} \left(\frac{1}{z'^2} \frac{1}{g(\tau')} \frac{1}{g(\sigma')} + O(z'^4) \right) . \tag{8.47}$$

Here we have used the first eta-product identity in (A.29). In other words, we find exactly the same behavior (8.47) that was found earlier in the literature [1] when studying the twisted sector partition function $2^{-4}\Phi_{6,3}^{-1}$. The consequence of this is that in the chosen saddle-point approximation our generic unit-torsion quarter-BPS dyon partition function $\mathsf{Z} \in \{\mathsf{Z}^{(0)}, \mathsf{Z}^{(+)}, \mathsf{Z}^{(-)}\}$ will be consistent with the large charge behavior of the black hole entropy (8.32), if this is also true for the saddle-point approximation based on the Siegel modular form $2^{-4}\Phi_{6,3}^{-1}$ alone. Indeed, this is now formally the same problem as considered already in [1] and subsequent works. This brings us into the comfortable situation that in order to compute the large charge statistical entropy for our new (or charge refined) dyon partition functions Z, we can simply jump to the result [100]:

$$S_{\text{stat.}} = \log |f_Q(P^2, Q \cdot P, Q^2; \cdot)| \simeq S^{(0)} + S^{(1)} + O(q^{-2}) , \tag{8.48}$$

where

$$S^{(0)} = \pi \sqrt{Q^2 p^2 - (Q \cdot P)^2} \tag{8.49}$$

$$S^{(1)} = -\log g(\alpha_{(0)}) - \log g(-\bar{\alpha}_{(0)}) - 8 \log(\alpha_{(0)}) + \text{const.} \tag{8.50}$$

and

$$\alpha_{(0)} \coloneqq \alpha_{(0)1} + i\alpha_{(0)2} , \qquad \alpha_{(0)1} = \frac{Q \cdot P}{P^2} , \qquad \alpha_{(0)2} = \frac{\sqrt{Q^2 P^2 - (Q \cdot P)^2}}{P^2} . \tag{8.51}$$

Equation (8.48) gives the statistical entropy, approximated by the dominant saddle-point contribution from $\mathcal{D} = 0$, up to terms that are suppressed by square power of the charges. Indeed, up to unidentified constant terms (that are likely to not matter in the large charge regime), this is the same large charge behavior as in (8.32).

In summary, any untwisted (or twisted) sector partition function Z gives rise not just to the leading Bekenstein-Hawking term in (8.32), but also to the correct subleading correction in inverses powers of the charges. This clearly aligns with physical intuition, as in the limit of large charges the fine details of the microscopic charge sector encapsulated by $[Q] \in \Lambda_e/\Lambda_e^*$ (which can already change by adding single charge quanta), should not affect the macroscopic entropy.

We leave it as an open problem to perform more careful and extensive analyses as, e.g., in [100, 101] and to check whether a difference in the entropy of twisted sector and untwisted sector (quarter-BPS unit-torsion) dyons can be found in further subleading terms (say at exponentially suppressed orders). If so, one might ask for a macroscopic explanation in the quantum entropy function [102, 103] (say as certain subleading saddles to the supergravity path integral), see [100–110] for research in this line of thought.

Let us close this chapter with some remarks. The approach we have chosen here is to consider the dyon partition functions obtained earlier from a heterotic genus two partition function and to use their saddle-point approximation to determine the large charge behavior up to exponentially and power suppressed corrections. This was then compared to the macroscopic computation of the black hole entropy. One can, extending the approach of section 7.4, in principle also turn around the argument and obtain a constraint on the microscopic dyon partition function, which could be helpful in bootstrapping the latter. In this direction one can demand that, similar to what is observed for the known examples of dyon partition functions in the $\mathbb{Z}_N$ CHL models, in the saddle-point approximation the dominant contribution comes from evaluating the residue at the divisor (8.41) and that there we have coefficients as in (8.47). Because then the correct macroscopic black hole entropy will be recovered, including the model dependent subleading term originating from the Gauss-Bonnet term (where ϕ is now model-specific, but known from other calculations). Such an argument was indeed also put forward in [22, section 6.5], yielding a predicition about the behavior of a specific untwisted sector (unit-torsion) dyon partition function, without knowing the relevant Siegel modular form explicitly.

An alert reader might also have noticed that in the microscopic prescription we compute an index, the sixth helicity supertrace, which treats the fermionic and bosonic contributions with different signs. The main advantage of considering such an index, sensititive to BPS states only, is that it is largely protected by supersymmetry and allows for an interpolation between weak- and strong-coupling regimes. On the other hand, a statistical entropy is given by the logarithm of an absolute degeneracy of states. However, it can be argued [111] that for the extremal BPS black holes considered here this distinction dissolves and the logarithm of the microscopic index nevertheless computes the correct entropy.

Having successfully passed the test of black hole entropy, we finally compare the dyon partition functions to the Donaldson-Thomas partition functions.

CHAPTER 9

Comparison to results from Donaldson-Thomas theory

The spectrum of quarter-BPS states in four-dimensional $\mathcal{N} = 4$ string theories has been linked to the enumerative geometry of algebraic curves in Calabi-Yau threefolds. Predictions from string duality have thus led to precise mathematical conjectures [3, 112], some of which have been proven in recent years [113, 114]. Here[1] we explore the connection between quarter-BPS indices and reduced Donaldson-Thomas (DT) invariants by comparing the BPS partition functions in the $\mathbb{Z}_2$ CHL model with recently conjectured formulas for (tentative) DT counterparts [3].

9.1 A brief summary of the DT result

For the comparison let us briefly collect some definitions and (conjectural) formulas for DT invariants of the $\mathbb{Z}_2$ CHL model from [3]. The geometric $N = 2$ CHL model is given by the Calabi-Yau threefold $X = (S \times E)/\mathbb{Z}_N$, where S is a non-singular projective K3 surface and E is a non-singular elliptic curve. In accordance with section 2.2 the orbifold group $\mathbb{Z}_2$ acts by a symplectic involution $g : S \to S$ on S and a translation in E by some two-torsion point e_0. Correspondingly, there is a projection operator

$$\Pi = \frac{1}{2}(1 + g_*) : H^*(S, \mathbb{Q}) \to H^*(S, \mathbb{Q}) \tag{9.1}$$

and an isomorphism [3, app. B]

$$\Pi(H^*(S, \mathbb{Z})) \;\cong\; \left(H^*(S, \mathbb{Z})^g\right)^* \;\cong\; E_8(\tfrac{1}{2}) \oplus U^{\oplus 4}. \tag{9.2}$$

By the *divisibility* of a curve class $\gamma \in \mathrm{Image}(\Pi|_{N_1(S)})$ one means the biggest integer $m \in \mathbb{N}_{>0}$ for which

$$\frac{\gamma}{m} \in \mathrm{Image}(\Pi|_{N_1(S)}) \subset \frac{1}{2} H_2(S, \mathbb{Z}) \tag{9.3}$$

is satisfied, where $N_1(\cdot)$ denotes the group of algebraic one-cycles. If its divisibility is 1, γ is called a *primitive* class, which is further called *untwisted* if $\gamma \in H_2(S, \mathbb{Z})$, or *twisted* if $\gamma \in \frac{1}{2} H_2(S, \mathbb{Z}) \backslash H_2(S, \mathbb{Z})$.

[1] This chapter appeared as section 7 in the publication [35].

We consider the curve class[2]

$$\beta = (\gamma, d) \in N_1(X) \subset H_2(X, \mathbb{Z}) \tag{9.4}$$

for some primitive, non-zero γ with self-intersection

$$\langle \gamma, \gamma \rangle = 2s, \quad s \in \begin{cases} \mathbb{Z} & \text{if } \gamma \text{ untwisted} \\ \frac{1}{2}\mathbb{Z} & \text{if } \gamma \text{ twisted} \end{cases} . \tag{9.5}$$

The reduced Donaldson–Thomas invariant $\mathsf{DT}^X_{n,(\gamma,d)}$ only depends on n, s, d and whether γ is untwisted or twisted, so one may also write $\mathsf{DT}^{\text{untw}}_{n,s,d}$ and $\mathsf{DT}^{\text{tw}}_{n,s,d}$ for the two cases. Introducing respective partition functions

$$\mathsf{Z}^{\text{untw}}(q,t,p) := \sum_{\substack{s\in\mathbb{Z} \\ s\geq -1}} \sum_{d\geq 0} \sum_{n\in\mathbb{Z}} \mathsf{DT}^{\text{untw}}_{n,s,d}\, q^{d-1}\, t^s\, (-p)^n \tag{9.6}$$

$$\mathsf{Z}^{\text{tw}}(q,t,p) := \sum_{\substack{s\in\frac{1}{2}\mathbb{Z} \\ s\geq -1/2}} \sum_{d\geq 0} \sum_{n\in\mathbb{Z}} \mathsf{DT}^{\text{tw}}_{n,s,d}\, q^{d-1}\, t^s\, (-p)^n \tag{9.7}$$

and writing

$$q = e^{2\pi i \tau}, \quad t = e^{2\pi i \sigma}, \quad p = e^{2\pi i z}, \quad \text{and } Z = \left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right) \in \mathbb{H}_2 \tag{9.8}$$

one obtains tentative Siegel modular forms.

The partition function for the twisted primitive DT invariants on X is conjecturally given by the negative reciprocal of the Borcherds lift of the corresponding twisted-twined elliptic genera,

$$\mathsf{Z}^{\text{tw}}(q,t,p) = -\frac{1}{\tilde{\Phi}_2(Z)} , \tag{9.9}$$

and thus agrees with the quarter-BPS counting function obtained in [1, 83], which is (possibly up to a multiplicative constant) the function $2^{-4}\Phi^{-1}_{6,3}$.

On the other hand, the untwisted primitive DT invariants are determined by

$$\mathsf{Z}^{\text{untw}}(q,t,p) = \frac{-8F_4(Z) + 8G_4(Z) - \frac{7}{30}E^{(2)}_4(2Z)}{\chi_{10}(Z)} , \tag{9.10}$$

where χ_{10} is the weight ten Igusa cusp form appearing already in the partition function of the unorbifolded model, namely DT theory on $S \times E$, physically IIA[$S \times E$] or Het[T^6]. In the numerator we have two Siegel modular forms $G_4(Z)$ and $E^{(2)}_4(2Z)$, both of weight four for the level two congruence subgroup $\Gamma^{(2)}_0(2) \subset \mathrm{Sp}_4(\mathbb{Z})$. The function $F_4(Z)$ is a weight four Siegel paramodular form of degree two for the paramodular group $K(2)$. All of them can be expressed within the ring of even genus two theta constants, see appendix A. Thus, Z^{untw} is a weight -6 Siegel modular form for the level two Iwahori subgroup $B(2) = K(2) \cap \Gamma^{(2)}_0(2)$. We remark that with the help of (A.33),

[2] By [3, eq. (9), Lemma 1.4] we have $H_2(X,\mathbb{Z}) = \mathrm{Im}(\Pi) \oplus \mathbb{Z}[E/\mathbb{Z}_2]$ and $N_1(X) = \Pi(N_1(S)) \oplus \mathbb{Z}[E/\mathbb{Z}_2]$, both modulo torsion.

(A.36), (A.63) and (A.68), Z^{untw} might be recast into the form

$$\begin{aligned} \mathsf{Z}^{\text{untw}} &= -\frac{1}{2}\left(\frac{1}{W} + \frac{16T}{YW}\right) \\ &= -\frac{1}{2}\frac{Y + \frac{1}{16}Y' + \frac{1}{16}Y''}{YW} \\ &= -\frac{1}{2}\left(\frac{1}{\Phi_{6,0}} + \frac{1}{2^4\,\Phi_{6,3}} + \frac{1}{2^4\,\Phi_{6,4}}\right) \ . \end{aligned} \tag{9.11}$$

9.2 DT invariants as BPS indices

A connection to physics was already outlined in the appendix of [3], which we shall reproduce and build on.[3]

DT invariants on Calabi-Yau threefolds are believed to give virtual counts of D6-D2-D0 bound states in type IIA theory, which in turn engineer dyonic BPS states. Recall that a BPS D$(2n)$-brane wraps an algebraic n-cycle in X and especially has support in $H_{2n}(X,\mathbb{Z})$. These D-branes source various components of the dyon charge (Q,P). The translation to the heterotic duality frame and others is given in Table 9.1, which we have adapted from the K3$\times T^2$ case described in [115].[4] The magnetic charges are sourced by the non-perturbative objects of the parent theory surviving the orbifolding procedure (see [19, section 4], for instance). Those D4-branes supported on the elliptic curve times a curve in the K3 which survive the orbifold projection are charged in the invariant lattice $H^2(S,\mathbb{Z})^g = E_8(-2) \oplus U^{\oplus 3}$. Since the sympletic involution on the K3 leaves invariant the H^0 and H^4 components of the cohomology spanning a U summand, we have simply kept the notation of [115] for the D0- and D4(K3)-charges. The fundamental (heterotic) string winding number F1(3) along the CHL circle $S^1_{(3)}$ is quantized in units of $\frac{1}{2}$ and the momentum p(3) along the CHL circle in integer units, giving rise to $U(\frac{1}{2}) \subset \Lambda_e$. Moreover, a configuration of two NS5-branes localized in $S^1_{(3)}$, denoted by NS5($\hat{3}$), with a separation of $\delta/2$ (half the circumference) survives the orbifolding, so this charge will be quantized in units of 2 and gives rise to the $U(2) \subset \Lambda_m$ summand. An integer unit of KKM($\hat{3}$) charge belongs to a Kaluza-Klein monopole with the CHL circle $S^1_{(3)}/\mathbb{Z}_2$ as asymptotic circle.

Now in the case of primitive DT invariants on $S \times E$ and unit-torsion dyons of IIA$[S \times E]$ (or of Het$[T^6]$) an explicit charge assignment (Q,P) subject to the requirement[5]

$$\mathsf{DT}^{S\times E}_{n,(\gamma,d)} = f(P^2, Q\cdot P, Q^2) \tag{9.12}$$

for $(\gamma,d) \in H_2(S\times E,\mathbb{Z})$ is given by

$$Q = (ne_1, 0, 0, \gamma) \quad \text{and} \quad P = ((d-1)e_1 + e_2, 0, 0, 0) \ . \tag{9.13}$$

[3] For better comparison with the geometric aspects of the type IIA compactification, we have flipped the signature of the electric and magnetic charge lattice in this chapter. As also stated in footnote 3 on page 14, this is unproblematic and mostly due to notational conventions. Here we now use the conventions of [19, 26].

[4] See also section 2.1 of [26] for a map between the heterotic and type IIA charges.

[5] We suppress the dependence on the moduli domain. Also note the relative overall minus sign between eqs. (9.6)-(9.7) and (4.4).

Electric and magnetic charges $(Q,P) \in \Lambda_e \oplus \Lambda_m$				
	Het	IIA	M	IIB
$\mathbb{Z}_2\backslash$	$S^1_{(2)}{\times}S^1_{(3)}{\times}S^1_{(4)}{\times}T^3$	$S^1_{(2)}{\times}S^1_{(3)}{\times}\mathrm{K3}$	$S^1_{(1)}{\times}S^1_{(2)}{\times}S^1_{(3)}{\times}\mathrm{K3}$	$S^1_{(1)}{\times}S^1_{(3)}{\times}\mathrm{K3}$
U	p(4) F1(4)	D0 D4(K3)	p(1) M5(1,K3)	F1(1) NS5(1,K3)
U	p(2) F1(2)	p(2) NS5(2,K3)	p(2) M5(2,K3)	D1(1) D5(1,K3)
$U(\frac{1}{2})$	p(3) F1(3)	p(3) NS5(3,K3)	p(3) M5(3,K3)	p(3) KKM($\hat{1}$)
$E_8(-\frac{1}{2}){\oplus}U^{\oplus 3}$	q_A	D2(α^A)	M2(α^A)	D3($1,\alpha^A$)
U	NS5($\hat{4}$) KKM($\hat{4}$)	D2(2,3) D6(2,3,K3)	M2(2,3) TN(2,3,K3)	F1(3) NS5(3,K3)
U	NS5($\hat{2}$) KKM($\hat{2}$)	F1(3) KKM($\hat{2}$)	M2(1,3) KKM($\hat{2}$)	D1(3) D5(3,K3)
$U(2)$	NS5($\hat{3}$) KKM($\hat{3}$)	F1(2) KKM($\hat{3}$)	M2(1,2) KKM($\hat{3}$)	p(1) KKM($\hat{3}$)
$E_8(-2){\oplus}U^{\oplus 3}$	p^A	D4$(2,3,C_{AB}\alpha^B)$	M5$(1,2,3,C_{AB}\alpha^B)$	D3$(3,C_{AB}\alpha^B)$

Table 9.1: Sources of the dyon charge (Q,P) in different duality frames of the four-dimensional $\mathcal{N}=4$ $\mathbb{Z}_2$ CHL model. The α_A's are a basis of the 14-dimensional lattice $E_8(-2)\oplus U^{\oplus 3} \cong H^2(S,\mathbb{Z})^g$ with bilinear form denoted by C_{AB}. (Table adapted from [115, Table 3.1].)

Here e_1 and e_2 denote the generators of the hyperbolic lattice U, n is the D0-charge, γ the D2-charge. We have a single unit of D6-charge. These charges have been highlighted in Table 9.1, where $E_8(-\frac{1}{2})\oplus U^{\oplus 3}$ should be understood as $E_8(-1)^{\oplus 2}\oplus U^{\oplus 3}$ before orbifolding and similar for other sublattices. Again f expresses the sixth helicity supertrace (the quarter-BPS index) of unit-torsion states in terms of the quadratic T-invariants

$$Q^2 = \gamma^2 = 2s\,, \quad P^2 = 2(d-1)\,, \quad Q\cdot P = n\,. \tag{9.14}$$

Matching notations, we are lead to identify the Siegel coordinate Z in (9.8) with the chemical potentials Z in (4.9) conjugate to the quadratic T-invariants. In the non-orbifold theory on $S\times E$ the quarter-BPS index of the D6-D2-D0 configuration and the DT invariant are both obtained from $1/\chi_{10}$.

Now returning to the CHL model X, note that if $\gamma\in\Pi(H_2(S,\mathbb{Z}))$ then already $\gamma\in\Lambda_e$ since $\Pi(H_2(S,\mathbb{Z}))\subset\left(H^*(S,\mathbb{Z})^g\right)^*\subset\Lambda_e$ (c. f. eq. (9.2) and eqs. (2.44), (2.40)). Thus, the charges assigned in (9.13) indeed belong to the CHL electric lattice (2.44) and CHL magnetic lattice (2.45), respectively. In other words, the assignment is still meaningful.

Moreover, for primitive untwisted $\gamma\in H_2(S,\mathbb{Z})^g = E_8(-2)\oplus U^{\oplus 3}$, the charge assignment (9.13) gives electric charge with $\mathcal{P}=0$. So regarding DT invariants $\mathsf{DT}^X_{n,(\gamma,d)}$, we may expect that the charge formulas (9.13) are still valid for the orbifold case $X = (S \times E)/\mathbb{Z}_2$ if $\gamma \in H_2(S, \mathbb{Z})^g$. However, the function (9.11) is not found amongst the untwisted sector partition functions in (6.27) (nor amongst those of the twisted sectors in (6.32)). Formally, the function (9.11) is the *average* of the modular forms $\mathsf{Z}^{(0)}$ and $\mathsf{Z}^{(+)}$. In (6.27) these two functions belong to orbits $(-1)^{Q\cdot\delta} = +1$ and -1, respectively (but both with $\mathcal{P} = 0$ and $h = 0$). Alternatively, for fixed value $(-1)^{Q\cdot\delta} = +1$ the functions $\mathsf{Z}^{(0)}$ and $\mathsf{Z}^{(+)}$ distinguish between the $h = 0$ and $h = 1$ case, respectively (i.e., the $\mathcal{P} = 0$ terms of (6.27) and (6.32), respectively). Note also that the charge residue component $((-1)^h, (-1)^{Q\cdot\delta}) \in U(\frac{1}{2})/U(2)$ is apparently independent of any D-brane charges in the type IIA theory (c.f. Table 9.1) and especially the heterotic CHL winding number is not seen by the type II D-branes (nor in the data specifying the DT invariant). In any case, there does not seem to be a unique charge (orbit) whose partition function reduces to $\mathsf{Z}^{\mathrm{untw}}$, but rather a pair (union) thereof.

For a primitive twisted class $\gamma \in E_8(-\frac{1}{2}) \oplus U^{\oplus 3}$ (with $\mathcal{P} \neq 0$) the DT formula for Z^{tw} is not in tension with the results of (6.27) for the respective quarter-BPS generating functions $\mathsf{Z}^{\mp}$, since the two possible cases for $\mathcal{P} \in O_{248} \cup O_{3875}$ via (7.1) belong to different modes in the Fourier expansion of Z^{tw}, collected in $\mathsf{Z}^{\mp}$. Formally, this again agrees with the (in this case trivial) average over $(-1)^{Q\cdot\delta}$ ($h = 0$ fixed) for each Weyl orbit of $\mathcal{P}$ or, alternatively, the average over $h = 0, 1$ ($(-1)^{Q\cdot\delta} = +1$ fixed).

Whether the DT invariants computed in [3] really should be interpreted as averages of suitable quarter-BPS indices or whether the relation is more subtle than that remains an interesting open question to be clarified by future research.

CHAPTER 10

Conclusion and outlook

In this thesis we have investigated the spectrum of supersymmetric BPS states in the four-dimensional $\mathbb{Z}_2$ CHL compactification exhibiting $\mathcal{N} = 4$ supersymmetry. In particular, our first main goal was to find the partition functions for the BPS indices (sixth helicity supertraces) of dyonic quarter-BPS states with generic unit-torsion charge from the perspective of the genus two heterotic string. We have provided — physically independently from previous approaches in [2, 3] — solutions to the dyon counting problem and the results have been compared to that of [1–3, 22]. Specifically, relying on the M-theory lift of string webs proposed in [4–6] and refining the computation of [5], explicit expressions for partition functions for unit-torsion dyons in the remaining charge sectors have been obtained from a chiral genus two orbifold partition function of the heterotic string. The expressions found are Siegel modular forms for congruence subgroups of the Siegel modular group. Via the contour prescription of [61] our results for the partition functions are compatible with the BPS index formula of [2]. Comparing with the older results in the literature for the twisted sector, the dyon partition functions derived here exhibit additional dependence on the discrete charge residue and may therefore be considered as a refinement of the expression proportional to $\Phi_{6,3}^{-1}$ that was introduced in [1] (although the contour-based extraction of BPS indices yields equivalent results). In addition to matching [2] and [1], we have performed extensive physical consistency checks of our results by verifying the modular and polar constraints coming from charge quantization, S-duality and wall-crossing appropriate for the respective charge sector. This includes a confirmation of the expected properties of the partition function specific to a small charge subsector discussed in [22]. Moreover, improving the analysis of [22] and extending it to other charge sectors, in this thesis we have argued that these constraints naturally explain the role played by (Iwahori) congruence subgroups of the Siegel modular group which govern the transformation behavior of the dyon partition functions and, in fact, we have briefly explained how this (almost) fixes them in terms of the elements of the respective ring of Siegel modular forms. We also found a remarkably simple correspondence between the half- (eq. (5.39)) and quarter-BPS partition functions (eq. (6.33)).

Furthermore, in this work we have elaborated on the black hole interpretation of the found dyon partition functions. The microscopic quarter-BPS states are expected to give rise to extremal dyonic black hole solutions in the $\mathcal{N} = 4$ low-energy effective theory. We have briefly reviewed the macroscopic computation of the black hole entropy for such a configuration using Sen's entropy function formalism, taking also into account the Gauss-Bonnet term in the one-loop effective action. On the microscopic side we have argued that within the standard saddle-point approximation to the contour integral that extracts the BPS index from the dyon partition function the dominant contribution always comes from the same universal divisor in the Siegel upper half-plane (eq. (8.41)), independent of the specific charge sector (charge residue) and thus specific dyon partition function under consideration. This sector universality relies on the precise modular transformation behavior and pole locations of the dyon partition functions and effectively reduces the approximation scheme to that for the partition function of [1]. As a consequence of the latter fact, the microscopic statistical entropy starting from any sector is immediately consistent with the macroscopic black hole entropy obtained in the entropy function formalism, at least in the large charge limit considered here and to the given precision. The second immediate consequence is that in any sector one also recovers the leading, semi-classical Bekenstein-Hawking area term and a power-suppressed correction (suppressed in terms of the charges) due to the model-specific Gauss-Bonnet term. It is an interesting open problem to perform a more careful analysis of the large-charge behavior of the BPS index, identifying further (e.g. exponentially suppressed) corrections to the statistical entropy. Such corrections could likewise be studied from the macroscopic perspective. Apart from higher-derivative corrections in the effective action there are also *quantum* corrections to the dyonic extremal black hole entropy. Based on the AdS/CFT correspondence, the proposed *quantum* entropy function of [102, 103] (which goes beyond the entropy function formalism of section 8.1) can capture both kinds of corrections to the entropy and especially accounts for exponentially suppressed contributions as demanded by the microscopic index formula [100, 101, 104]. See [12, 111, 116] for reviews and [105–110] for more recent studies of the (quarter-BPS) quantum entropy that rely on localization of the supergravity path integral.

Last but not least, Donaldson-Thomas invariants are supposed to count D6-D2-D0 bound states on the type IIA geometry. Although the agreement between the non-orbifold counting theories (quarter-BPS indices for unit-torsion dyons on $\mathrm{K3}\times T^2$ and reduced primitve DT invariants on $\mathrm{K3}\times T^2$) that we briefly reviewed in chapter 9 supports this supposition, the relation between the BPS indices and the DT invariants is less clear for the $\mathbb{Z}_2$ orbifold. What has been called untwisted sector DT partition function in [3] is not literally found amongst the quarter-BPS partition functions presented here. Rather, it is formally a sum or overage of two such BPS partition functions. If the translation of the dyon charges between the various string duality frames in Tab. 9.1 is correct and if the lattice data specifying the DT invariant is properly represented by the highlighted D6-D2-D0 charges, then these charges do not uniquely specify the discrete dyon charge residue in the discriminant group of the electric lattice. In particular, averaging over thus unspecified components, which can be done in two ways, gives the untwisted sector DT result, and likewise the twisted sector DT result for appropriate curve class. Clearly, it would be desirable to understand the relation between the DT invariants and the BPS indices for the $\mathbb{Z}_2$ CHL model better on a conceptual level, potentially resolving the slight mismatch between the two. This, however, we leave as an open problem for future investigation.

APPENDIX A

Siegel modular forms

In this appendix[1] we collect basic definitions and useful formulae for the Siegel modular forms appearing in the main text. Our main references are [117, chapter VII], [3, section 2] and [2, appendix A], also see [77] for a review that emphasizes the relation between the theory of Siegel modular forms, mock modular forms and quantum black holes.

Preliminaries. By $\mathrm{Sp}_4(\mathbb{Z})$ we denote the symplectic group of integer 4×4 matrices $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ that satisfy

$$M^{\mathsf{T}} J M = J \qquad \text{and} \qquad J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} , \tag{A.1}$$

which is equivalent to

$$A^{\mathsf{T}} C = C^{\mathsf{T}} A \,, \quad B^{\mathsf{T}} D = D^{\mathsf{T}} B \quad \text{and} \quad A^{\mathsf{T}} D - C^{\mathsf{T}} B = 1_2 \tag{A.2}$$

for the 2×2 block matrices in M. The groups $\mathrm{Sp}_4(\mathbb{Q})$ and $\mathrm{Sp}_4(\mathbb{R})$ are defined analogously. If $M \in \mathrm{Sp}_4(\mathbb{Z})$ as above then the inverse of M is given by

$$M^{-1} = \begin{pmatrix} D^{\mathsf{T}} & -B^{\mathsf{T}} \\ -C^{\mathsf{T}} & A^{\mathsf{T}} \end{pmatrix} \tag{A.3}$$

and by using this in (A.1) we see that also $M^{\mathsf{T}} \in \mathrm{Sp}_4(\mathbb{Z})$. Taking the Pfaffian and using $\mathrm{Pf}(M^{\mathsf{T}} J M) = \det(M)\mathrm{Pf}(J)$ one concludes that $\det(M) = 1$, which more conceptually is equivalent to the fact that symplectic transformations are orientation preserving.

Special examples of symplectic matrices that also play a role for the quarter-BPS partition functions are (for $\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, respectively)

$$\begin{pmatrix} 1_2 & S \\ 0 & 1_2 \end{pmatrix} \qquad \text{with} \quad S^{\mathsf{T}} = S \tag{A.4}$$

$$\text{and} \quad \begin{pmatrix} U^{\mathsf{T}} & 0 \\ 0 & U^{-1} \end{pmatrix} \qquad \text{with} \quad U \in \mathrm{GL}_2(\mathbb{K}) \,. \tag{A.5}$$

[1] This appendix appeared as appendix A in the publication [35].

Any symplectic matrix with $C = 0$ can be written as a product of the form "(A.5) times (A.4)". The prinicipal congruence subgroup $\Gamma^{(2)}(N)$ (with $N \geq 1$) is defined by

$$\Gamma^{(2)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) \middle| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \mod N \right\} . \tag{A.6}$$

A congruence subgroup $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ is a subgroup that contains a principal congruence subgroup, for instance,

$$\Gamma_0^{(2)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) \middle| C \equiv 0 \mod N \right\} \supset \Gamma^{(2)}(N) . \tag{A.7}$$

For a prime number $p \geq 1$ the group $K(p)$ is defined by [118, 119]

$$K(p) = \mathrm{Sp}_4(\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}, \tag{A.8}$$

while the Iwahori subgroup is defined by the intersection

$$B(p) = K(p) \cap \Gamma_0^{(2)}(p) = \mathrm{Sp}_4(\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} . \tag{A.9}$$

By conjugation in $\mathrm{GL}_4(\mathbb{Q})$ (see [118] for references) the group $K(p)$ is related to the Siegel paramodular group $\Gamma^{\mathrm{para}}(p)$, formed by integer 4×4 matrices that obey (A.1) with J replaced by $J_2(p) = \left(\begin{smallmatrix} 0 & P \\ -P & 0 \end{smallmatrix}\right)$ with $P = \mathrm{diag}(1, p)$.

Let $\mathbb{H}_2$ be the (genus two) Siegel upper half space, i.e., the set of 2×2 symmetric complex matrices

$$Z = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \tag{A.10}$$

with positive definite imaginary part, explicitly

$$\Im(\tau) > 0, \quad \Im(\sigma) > 0, \qquad \text{and} \quad \Im(\tau)\Im(\sigma) - \Im(z)^2 > 0 . \tag{A.11}$$

A group action of $\mathrm{Sp}_4(\mathbb{R}) \ni M, M'$ on $\mathbb{H}_2 \ni Z$ is defined by

$$MZ := (AZ + B)(CZ + D)^{-1} , \tag{A.12}$$

where M and M' define the same action if and only if they differ by their sign. The special examples (A.4) and (A.5) above act via

$$Z \mapsto Z + S \qquad \text{and} \qquad Z \mapsto U^{\intercal} Z U , \tag{A.13}$$

respectively. Important for wall-crossing relations are the following embedded, commuting $\mathrm{SL}_2(\mathbb{R})$

subgroups of $\mathrm{Sp}_4(\mathbb{R})$:

$$\mathrm{SL}_2(\mathbb{R})_\tau \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix}_\tau = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.14}$$

$$\mathrm{SL}_2(\mathbb{R})_\sigma \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix}_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}. \tag{A.15}$$

Their action on the Siegel coordinate Z is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\tau Z = \begin{pmatrix} \frac{a\tau+b}{c\tau+d} & \frac{z}{c\tau+d} \\ \frac{z}{c\tau+d} & \sigma - \frac{cz^2}{c\tau+d} \end{pmatrix} \tag{A.16}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\sigma Z = \begin{pmatrix} \tau - \frac{cz^2}{c\sigma+d} & \frac{z}{c\sigma+d} \\ \frac{z}{c\sigma+d} & \frac{a\sigma+b}{c\sigma+d} \end{pmatrix} , \tag{A.17}$$

respectively. From these expressions it follows that the diagonal locus $z = 0$ is preserved under the two embedded subgroups, where they operate componentwise on $\tau \in \mathbb{H}_1$ and $\sigma \in \mathbb{H}_1$, respectively. Another symplectic transformation preserving the diagonal locus is given by (A.5) with $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which exchanges the diagonal entries of Z.

Now let $f : \mathbb{H}_2 \longrightarrow \mathbb{C}$ be a holomorphic function, k be an integer and $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ be a congruence subgroup (or a discrete subgroup $\Gamma \subset \mathrm{Sp}_4(\mathbb{R})$ with finite covolume [118, 120]). If

$$f(MZ) = \det(CZ + D)^k f(Z) \tag{A.18}$$

for all $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, then f is called a Siegel modular form of weight k for Γ. As in [3] denote by $\mathrm{Mod}_k^{(2)}(\Gamma)$ the space of Siegel modular forms of weight k for Γ and by

$$\mathrm{Mod}^{(2)}(\Gamma) = \bigoplus_k \mathrm{Mod}_k^{(2)}(\Gamma) \tag{A.19}$$

the $\mathbb{C}$-algebra of Siegel modular forms for Γ. Also introduce the Petersson slash operator for a function $f : \mathbb{H}_2 \to \mathbb{C}$, an element $M \in \mathrm{Sp}_4(\mathbb{R})$ and an integer k via

$$(f\big|_k M)(Z) = \det(CZ + D)^{-k} f((AZ + B)(CZ + D)^{-1}) \, . \tag{A.20}$$

Then $f \in \mathrm{Mod}_k^{(2)}(\Gamma)$ is equivalent to $f\big|_k M = f$ for all $M \in \Gamma$.[2] One often simply writes $f\big|M$. If (A.5) lies in Γ for $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, such $f(Z)$ is invariant under exchange of the diagonal entries of Z (possibly up to a root of unity).

[2] Here we only deal with the case of a trivial multiplier system.

Modular forms for level two subgroups. Generators for rings of modular forms can often be expressed in terms of genus two theta constants (german Thetanullwerte), which we introduce now. For column vectors $m' = a = \left(\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}\right)$, $m'' = b = \left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right) \in \mathbb{Z}^2$ and $m = \left(\begin{smallmatrix} m' \\ m'' \end{smallmatrix}\right)$ consider the genus two theta constant of characteristic m

$$\theta_m(Z) = \sum_{x\in\mathbb{Z}^2} e\left(\frac{1}{2}\left(x + \frac{1}{2}m'\right)^{\intercal} Z \left(x + \frac{1}{2}m'\right) + \left(x + \frac{1}{2}m'\right)^{\intercal} \frac{m''}{2}\right) \tag{A.21}$$

with shorthand $e(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}$. This is also written as $\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \theta_{a_1 a_2 b_1 b_2}$. The theta constants vanish identically iff $a^{\intercal} b \mod 2$ is odd. For genus two there are precisely ten "even" non-trivial theta constants. There is a useful transformation formula under $M \in \mathrm{Sp}_4(\mathbb{Z})$,

$$\theta_{M\{m\}}(MZ) = v(M,m)\det(CZ+D)^{1/2}\theta_m(Z)\,, \tag{A.22}$$

where $v(M,m)$ is an eigth root of unity and, denoting by $(\ldots)_0$ the diagonal as a column vector,

$$M\{m\} = M\{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)\} = M^{-\intercal}\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) + \begin{pmatrix} (CD^{\intercal})_0 \\ (AB^{\intercal})_0 \end{pmatrix} \mod 2\,. \tag{A.23}$$

As special cases we have for the elements in (A.4) and (A.5) simplified formulas

$$\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](Z+S) = \theta\left[\begin{smallmatrix} a \\ b+Sa+S_0 \end{smallmatrix}\right](Z)\cdot e^{\frac{i\pi}{4}a^{\intercal}Sa} \tag{A.24}$$

$$\text{and} \qquad \theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](U^{\intercal}ZU) = \theta\left[\begin{smallmatrix} Ua \\ U^{-\intercal}b \end{smallmatrix}\right](Z)\,. \tag{A.25}$$

On the diagonal $z = 0$ the theta constants factorize as

$$\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\left(\left(\begin{smallmatrix} \tau & 0 \\ 0 & \sigma \end{smallmatrix}\right)\right) = \theta\left[\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right](\tau)\,\theta\left[\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix}\right](\sigma)\,. \tag{A.26}$$

For $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ this vanishes linearly in $z \to 0$, more precisely

$$\theta\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]\left(\left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)\right) \to \frac{z}{2\pi i}\,\theta'\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](\tau)\,\theta'\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](\sigma)\,, \qquad \text{with} \quad \theta'\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = 2\pi\eta^3\,. \tag{A.27}$$

In these expressions we used standard genus one theta constants defined in complete analogy to (A.21) (read: sum over $x \in \mathbb{Z}$, $Z \in \mathbb{H}_1$, $m', m'' \in \mathbb{Z}$). Special instances, labelled by

$$\theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \theta_2, \qquad \theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = \theta_3, \qquad \text{and} \quad \theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \theta_4, \tag{A.28}$$

relate to the eta-products of the $\mathbb{Z}_2$ CHL orbifold partition function via

$$\theta_2^4(\tau)\eta^4(\tau) = 2^4\eta^8(2\tau)\,,\quad \theta_3^4(\tau)\eta^4(\tau) = -e^{2\pi i/3}\eta^8\left(\tfrac{\tau+1}{2}\right) \text{ and } \theta_4^4(\tau)\eta^4(\tau) = \eta^8\left(\tfrac{\tau}{2}\right)\,. \tag{A.29}$$

and satisfy

$$\theta_2^4(\tau) - \theta_3^4(\tau) + \theta_4^4(\tau) = 0 \qquad \text{(Riemann identity)}\,, \tag{A.30}$$

$$\theta_2(\tau)\theta_3(\tau)\theta_4(\tau) - 2\eta^3(\tau) = 0 \qquad \text{(Jacobi triple product identity)}\,. \tag{A.31}$$

Now set[3]

$$X = 2^{-2}\left(\theta_{0000}^4 + \theta_{0001}^4 + \theta_{0010}^4 + \theta_{0011}^4\right) \tag{A.32}$$

$$Y = \left(\theta_{0000}\theta_{0001}\theta_{0010}\theta_{0011}\right)^2 \tag{A.33}$$

$$Z = 2^{-14}(\theta_{0100}^4 - \theta_{0110}^4)^2 \tag{A.34}$$

$$W = 2^{-12}(\theta_{0100}\theta_{0110}\theta_{1000}\theta_{1001}\theta_{1100}\theta_{1111})^2 \tag{A.35}$$

$$T = 2^{-8}\,(\theta_{0100}\theta_{0110})^4\,. \tag{A.36}$$

As was proven in [120] (see also [121]), the functions X, Y, Z, W are Siegel modular forms for $\Gamma_0^{(2)}(2)$ of respective weight $2, 4, 4$ and 6 and they generate the ring of even weight modular forms for $\Gamma_0^{(2)}(2)$, i.e.,

$$\mathrm{Mod}^{(2)}_{\mathrm{even}}(\Gamma_0^{(2)}(2)) = \mathbb{C}[X, Y, Z, W]\,. \tag{A.37}$$

The function W agrees with the function "K" defined in [87]. On the other hand, the function T is a weight four modular form for the Iwahori subgroup $B(2)$ and by [120] the structure of the ring of even weight modular forms for $B(2)$ is known to be

$$\mathrm{Mod}^{(2)}_{\mathrm{even}}(B(2)) = \mathbb{C}[X, Y, Z, W, T] \cong \mathbb{C}[x, y, z, w, t]/j\,, \tag{A.38}$$

where x, y, z, w, t are five algebraically independent variables and j is the ideal of $\mathbb{C}[x, y, z, w, t]$ spanned by

$$64w^2 + 16xtw + t(-16yz + t[x^2 - y - 1024z - 64t])\,. \tag{A.39}$$

For the structure of the ring of modular forms for $K(2) \supset B(2)$ we refer to the results given in [118] and just mention that the function $F_4(Z)$ appearing in the untwisted sector quarter-BPS partition function is the unique weight four Siegel modular form for $K(2)$, which may be defined as

$$F_4(Z) = \frac{1}{960}(X^2 + 3Y + 3072Z + 960T)\,. \tag{A.40}$$

Also the Siegel modular form $G_4(Z)$ appears in the untwisted sector partition function, which satisfies

$$G_4(Z) = \frac{1}{120}X^2 - \frac{3}{80}Y - \frac{12}{5}Z \quad \in \mathrm{Mod}_4^{(2)}(\Gamma_0^{(2)}(2))\,. \tag{A.41}$$

As in the genus one case, the theta function $\Theta^{(2)}_{E_8}$ for the E_8 root lattice yields a (Siegel) Eisenstein

[3] It should be clear from the context whether the symbol $Z = \left(\begin{smallmatrix}\tau & z\\ z & \sigma\end{smallmatrix}\right)$ is referring to a coordinate for $\mathbb{H}_2$ or the Siegel modular form Z defined in (A.34).

series and we have the following expressions in terms of theta constants:

$$E_4^{(2)}(Z) = 4X^2 - 3Y + 12288Z \quad \in \mathrm{Mod}_4^{(2)}(\mathrm{Sp}_4(\mathbb{Z})) \tag{A.42}$$

$$E_4^{(2)}(2Z) = \frac{1}{4}X^2 + \frac{3}{4}Y - 192Z \quad \in \mathrm{Mod}_4^{(2)}(\Gamma_0^{(2)}(2))\ . \tag{A.43}$$

These both appear in section 6, along with closely related functions $(E_4^{(2)}(2Z))|_M$ for appropriate $M \in \mathrm{Sp}_4(\mathbb{Z})\backslash\Gamma_0^{(2)}(2)$, which we give in the form

$$\Theta_{E_8}^{(2)}(2\tau, z, \tfrac{\sigma}{2}) = 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8(2)\oplus E_8(2)^*}} e^{i\pi Q^r \Omega_{rs} Q^s} \tag{A.44}$$

$$\Theta_{E_8}^{(2)}(2\tau, z, \tfrac{\sigma+1}{2}) = 2^{-4} \sum_{\substack{(Q_1,Q_2)\in \\ E_8(2)\oplus E_8(2)^*}} (-1)^{Q_2^2}\, e^{i\pi Q^r \Omega_{rs} Q^s}\ . \tag{A.45}$$

All of these may again be expressed in terms of theta constants. We note that in the limit $z = 0$ these reduce to products of the genus one theta series for the E_8 root lattice or related functions, which we list here for convenience:

$$\theta_{E_8(1)}(\tau) = \frac{1}{2}\left(\theta_2^8 + \theta_3^8 + \theta_4^8\right) \tag{A.46}$$

$$\theta_{E_8(1)}(2\tau) = \frac{1}{2^4}\left(\theta_3^8 + \theta_4^8 + 14\theta_3^4\theta_4^4\right) \tag{A.47}$$

$$\theta_{E_8(1)}(\tfrac{\tau}{2}) = \theta_2^8 + \theta_3^8 + 14\theta_2^4\theta_3^4 \tag{A.48}$$

$$\theta_{E_8(1)}(\tfrac{\tau+1}{2}) = \theta_2^8 + \theta_4^8 - 14\theta_2^4\theta_3^4\ . \tag{A.49}$$

Besides those, of interest are also

$$\theta_{E_8(2),1}(\tau) = \frac{1}{2^4}\left(\theta_3^8 + \theta_4^8 + 14\theta_3^4\theta_4^4\right) \quad = \frac{1}{2^4}\left(\theta_2^4\theta_3^4 + 16\,\theta_3^4\theta_4^4 - \theta_2^4\theta_4^4\right) \tag{A.50}$$

$$\theta_{E_8(2),248}(\tau) = \frac{1}{2^4}\left(\theta_3^8 - \theta_4^8\right) \quad = \frac{1}{2^4}\left(\theta_2^4\theta_3^4 + \theta_2^4\theta_4^4\right) \tag{A.51}$$

$$\theta_{E_8(2),3875}(\tau) = \frac{1}{2^4}\ \theta_2^8 \quad = \frac{1}{2^4}\left(\theta_2^4\theta_3^4 - \theta_2^4\theta_4^4\right)\ , \tag{A.52}$$

with the notation of eq. (5.21), and the two sets are related via

$$\theta_{E_8(1)}(2\tau) = \theta_{E_8(2),1}(\tau) \tag{A.53}$$

$$\theta_{E_8(1)}(\tfrac{\tau}{2}) = \theta_{E_8(2),1}(\tau) + 120\,\theta_{E_8(2),248}(\tau) + 135\,\theta_{E_8(2),3875}(\tau) \tag{A.54}$$

$$\theta_{E_8(1)}(\tfrac{\tau+1}{2}) = \theta_{E_8(2),1}(\tau) - 120\,\theta_{E_8(2),248}(\tau) + 135\,\theta_{E_8(2),3875}(\tau)\ . \tag{A.55}$$

Coming back to Siegel modular forms, the Igusa cusp form $\chi_{10} \in \mathrm{Mod}_{10}^{(2)}(\mathrm{Sp}_4(\mathbb{Z}))$, whose reciprocal counts unit-torsion dyons in Het[T^6], is given by the well-known product of the squares of all even genus two theta constants

$$\chi_{10}(Z) = YW\ . \tag{A.56}$$

In the $\mathbb{Z}_2$ orbifold we also encounter the $\Gamma_0^{(2)}(2)$ cusp form $\Phi_{6,0} = W$ and its modular images

$$\Phi_{6,i} \coloneqq \Phi_{6,0}|_{M_i} \tag{A.57}$$

under $M_i \in \mathrm{Sp}_4(\mathbb{Z})\backslash\Gamma_0^{(2)}(2)$ (the M_i being specified in (A.65) and (A.66)):

$$\frac{1}{\Phi_{6,0}} = \frac{\theta_{0000}^2\theta_{0001}^2\theta_{0010}^2\theta_{0011}^2}{\chi_{10}} = \frac{1}{W} \tag{A.58}$$

$$\frac{1}{\Phi_{6,1}} = \frac{\theta_{0000}^2\theta_{0001}^2\theta_{1000}^2\theta_{1001}^2}{\chi_{10}} \tag{A.59}$$

$$\frac{1}{\Phi_{6,2}} = -\frac{\theta_{1000}^2\theta_{1001}^2\theta_{0010}^2\theta_{0011}^2}{\chi_{10}} \tag{A.60}$$

$$\frac{1}{\Phi_{6,3}} = \frac{\theta_{0000}^2\theta_{0010}^2\theta_{0100}^2\theta_{0110}^2}{\chi_{10}} = \frac{Y'}{YW} \tag{A.61}$$

$$\frac{1}{\Phi_{6,4}} = -\frac{\theta_{0001}^2\theta_{0011}^2\theta_{0100}^2\theta_{0110}^2}{\chi_{10}} = \frac{Y''}{YW}\ . \tag{A.62}$$

Here we have kept the notation of [2] and introduced

$$Y' = (\theta_{0000}\theta_{0010}\theta_{0100}\theta_{0110})^2 \quad \text{and} \quad Y'' = -(\theta_{0001}\theta_{0011}\theta_{0100}\theta_{0110})^2\ . \tag{A.63}$$

With the help of (A.24) one easily checks that[4]

$$Y'(Z + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)) = Y''(Z) \quad \Rightarrow \quad \Phi_{6,3}(Z + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)) = \Phi_{6,4}(Z)\ . \tag{A.64}$$

The corresponding elements $M_i \in \mathrm{Sp}_4(\mathbb{Z})\backslash\Gamma_0^{(2)}(2)$ are in the notation of (A.14) and (A.15)

$$M_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)_\tau\ ,\ \ M_2 = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)_\tau\ ,\ \ M_3 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)_\sigma\ ,\ \text{and}\ \ M_4 = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)_\sigma\ . \tag{A.65}$$

Indeed $\Phi_{6,1/2}$ and $\Phi_{6,3/4}$ map to each other under exchange of the diagonal elements of $Z \in \mathbb{H}_2$, for instance, M_1 is conjugate to M_3 by the element (A.5) with $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. For the other Siegel modular forms $\Phi_{6,k}$, with $k \in \{5, 6, 10, 11\}$, that appear in section 6 we do not need explicit expressions and just give

$$M_5 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)_\tau\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)_\sigma\ ,\ \ M_6 = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)_\tau\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)_\sigma\ ,\ \ M_{10} = \left(\begin{smallmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)M_5\ ,\ \ M_{11} = \left(\begin{smallmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)M_5\ . \tag{A.66}$$

There are many quadratic relations that the squares of the theta constants satisfy and which have, for instance, been reviewed in [122]. One particular identity important for our untwisted partition

[4] The minus sign in (A.60) and (A.62) is imporant for reproducing the result for the orbifold block $\mathcal{Z}_8\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ obtained in [5, eq. (4.38)], c. f. the relative signs between the terms in (6.5).

functions is the relation [122, eq. (5.1)]

$$\theta_{0100}^2\theta_{0110}^2 = \theta_{0000}^2\theta_{0010}^2 - \theta_{0001}^2\theta_{0011}^2\,, \tag{A.67}$$

which implies for the above Siegel modular forms

$$16^2\, T = Y' + Y''\,. \tag{A.68}$$

Finally, we remark that the quadratic divisors, on which χ_{10} and its orbifold analog $\Phi_{6,1}$ (or $\Phi_{6,3}$ with the roles of the diagonal entries swapped) vanish quadratically, can, for instance, be found in [86, section 4] or [11, appendix D]. By an appropriate $\mathrm{Sp}_4(\mathbb{Z})$-transformation they can be mapped to the standard diagonal divisor, as was used in [101].

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