Heat flow aspects of synthetic Ricci bounds in the extended Kato class

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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Bonn

Bonn, September 2021

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen
Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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2. Gutachter: Prof. Dr. Matthias Erbar

Tag der Promotion: 14.01.2022
Erscheinungsjahr: 2022

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Acknowledgments

Er ragt aus fahlem Grau hervor,
Der Mast in seinem tristen Schein.
Gar viele stehen dort empor,
Ein Mastbetrieb muss das wohl sein.
The author.

Getting lost in the fog and drawing nonsense conclusions are probably the two most dangerous compagnons in pursuing mathematical research, especially at the early stage of being a Ph.D. student, not to mention in times of the great pandemic. During my work on this thesis, I have been most fortunate to be supported by so many people who prevented me from these risks. Be it their help in escaping the grey times when I was stuck with thematic or private issues, or be it their continuous support I had the pleasure to enjoy at uncountably many occasions.

First and foremost, I am grateful to my supervisor Karl-Theodor Sturm for having trusted in me, for his advice, guidance, help, patience, and numerous remarks, and for sharing his immense experience with me at so many occasions.

To Matthias Erbar for his kind agreement to be the second supervisor of my thesis and his valuable help during his time in Bonn.

To Thomas Litt and Martin Rumpf for their kind consent to take part in my Ph.D. committee.

To Sergio Conti for having been my BIGS mentor.

To all my inspiring teachers during my studies in Bonn for academically having made me who I am.

To Batu Güneysu, Karen Habermann, Chiara Rigoni and Karl-Theodor Sturm for our cooperative work on [BG20, BHS21, BR21] and their most valuable input that constantly made me learn new mathematical and stylistic details. To Lorenzo Dello Schiavo, Matthias Erbar, Batu Güneysu, Chiara Rigoni, Christian Rose, Karl-Theodor Sturm and Luca Tamanini for their helpful comments on any of the works, especially those I authored on my own, contained in this thesis.

To Matthias Erbar, Eva Kopfer, André Schlichting, Karl-Theodor Sturm and Luca Tamanini for their great affability in organizing teaching together.

To Lorenzo Dello Schiavo for patiently answering my mathematical questions and the (shared successful organization of our) nice and impressive trips to Kyōto and Fukuoka, and to Kohei Suzuki for his kind guidance through the best places in Fukuoka.

To all other members of the research group during my Ph.D. studies I did not mention so far, for their help at various opportunities and our nice beer garden visits.

To Thilo Baumann and Friedrich Klaus for our common time in Bonn, inside or outside Mathematics, and in our personal or Zoom meetings with numerous units of deep conversations, hard-fought games and high quality music.

To Marianne and Wolfgang Steinig, for their everyday happiness, for always making me feel home in the most beautiful part of Bonn already since the end of my Bachelor studies, and of course for freely allowing me to use their pool.

To my family, especially my parents Andreas and Tatjana Braun and my brothers Alexander and Johannes Braun, for their most valuable constant support, help and care since the very beginning of my life.

Funding by the European Research Council, through the ERC AdG "RicciBounds", Grant Agreement 694405, and the Bonn International Graduate School of Mathematics (BIGS) is gratefully acknowledged.

Summary

This thesis studies heat flows acting on different objects on possibly singular spaces that admit synthetic lower Ricci curvature bounds by constants, functions, or signed measures. Geometric properties of such spaces and probabilistic features of diffusion processes on these are related to functional inequalities for the involved semigroups. Moreover, heat flow methods are used to set up a second order calculus in the general presence of such measure-valued lower Ricci bounds.

The introductory Chapter 0 describes the subject of this thesis in more detail. It reviews the relevant literature, results and historical background.

The results from Chapter 1, obtained in collaboration with Karen Habermann and Karl-Theodor Sturm, have been published in [BHS21]. For an RCD space and a lower semicontinuous, lower bounded function k on it, we prove the equivalence of the following synthetic characterizations (w.r.t. k) of the "Ricci curvature at every $x \in M$ being bounded from below by $k(x)$": geodesic semiconvexity of the relative entropy, the evolution variational inequality, Bochner's inequality, gradient bounds for the functional heat flow $(\mathsf{P}_t)_{t\geq 0}$, transport estimates, and the pathwise coupling property. These formulations are partly novel and have partly been initiated in [Stu15]. A key ingredient is our variable version of Kuwada's duality [Kuw10].

The results from Chapter 2, obtained in collaboration with Batu Güneysu, have appeared in the preprint [BG20]. On arbitrary weighted Riemannian manifolds, we prove the equivalence of the pathwise coupling property w.r.t. k from Chapter 1 and pointwise lower boundedness of the Bakry–Émery Ricci tensor by k, only assuming continuity of k. Under an additional exponential integrability condition on k, which holds if k is in the functional Kato class of the weighted manifold, we prove conservativeness and Bismut–Elworthy–Li's derivative formula for $(\mathsf{P}_t)_{t\geq 0}$.

The results from Chapter 3 have appeared in the preprint [Bra21]. We extend the second order calculus for RCD spaces from [Gig18] to Dirichlet spaces which are tamed by a signed extended Kato class measure in the sense of [ER$^+$20]. Inter alia, based on the analysis of $(\mathsf{P}_t)_{t\geq 0}$ in [ER$^+$20], nonsmooth analogues of Hessians, covariant and exterior derivatives, and the Ricci curvature are defined. Employing these objects, in turn, we define heat flows on 1-forms and vector fields and, along with their basic properties, prove domination of the latter by certain semigroups acting on functions.

The results from Chapter 4 have appeared in the preprint [Bra20]. In the setting of RCD spaces, we obtain functional inequalities and regularization properties of the heat flow $(\mathsf{H}_t)_{t\geq 0}$ on 1-forms. The spectrum of its generator, the Hodge Laplacian, is studied as well. Finally, we construct a heat kernel for $(\mathsf{H}_t)_{t\geq 0}$ and prove Gaussian upper bounds on its pointwise operator norm.

All these four publications or preprints identify the author of this thesis by his nickname *Mathias Braun* instead of his full name.

Chapter Zero

Introduction

Since the second half of the twentieth century, by the respective concepts of heat flow, Brownian motion, and curvature, three important areas of mathematics have revealed numerous deep connections with one another: *analysis*, *probability*, and *geometry*. A prominent role is played by uniform lower bounds on the *Ricci* curvature. Their classical importance stems from their close link to estimates on heat kernels, Green's functions, and eigenvalues as well as inequalities à la Harnack, Sobolev, or functional or isoperimetric ones, cf. [BGL14, Stu06a] and the references therein: the latter properties are by now well-known to hold in — or even to be equivalent to — the presence of such constant bounds. More recently, for Riemannian manifolds one has developed a quite complete picture of how analytic concepts to describe uniform Ricci bounds are related to properties of underlying diffusion processes, cf. [Wan14] and the references therein.

The goal of this thesis is to study these cross connections on possibly singular spaces which admit generalized notions of *nonconstant* Ricci curvature bounds, namely, by a function or a signed measure. We focus both on *consequences* as well as equivalent *characterizations* of such bounds in terms of heat flows (at different levels, namely functions, probability measures, and 1-forms) and Brownian motion.

Section 0.3 outlines the results of our thesis in more detail, followed by a short survey in Section 0.4 over basic notations we constantly use in this thesis. Before, in Section 0.1 and Section 0.2 we give a historical account on relevant well-known relations between heat flow, Brownian motion, and Ricci curvature. As a side effect, we softly push the reader towards which kinds of precise cross connections we eventually aim to study. In Section 0.1, we first present three scenarios in which these links to (notions of) constant lower Ricci bounds show up from different directions, and then summarize in Subsection 0.1.4 how these are related. Section 0.2 contains a glimpse on related results beyond the constant situation and initiates the evidence of studying the nonconstant and nonuniform framework in larger generality.

In the course of this introduction, for illustrative reasons we generously neglect technical details, identifications, etc. All these are treated in more detail or referred to relevant literature from Chapter 1 on.

0.1 Uniform lower Ricci bounds

0.1.1 Scenario I. Heat flow on 1-forms, smooth manifolds

Setting Let M be a compact, connected Riemannian manifold without boundary, endowed with the metric tensor $\langle\cdot,\cdot\rangle$. Let $\vec{\Delta} := \mathrm{d}\,\delta + \delta\,\mathrm{d}$ be the corresponding *Hodge*

Laplacian on 1-forms, where δ is the formal L^2-adjoint of the exterior differential d w.r.t. the volume measure $\mathfrak{m} := \mathfrak{v}$.

Hess–Schrader–Uhlenbrock inequality The Hodge Laplacian can be expressed by the Bochner Laplacian $\square$ and the Ricci tensor Ric of M by *Weitzenböck's identity* $-\vec{\Delta} = \square - \mathrm{Ric}$. This fact relates the heat flows $(\mathsf{H}_t)_{t\geq 0}$ on 1-forms and $(\mathsf{P}_t)_{t\geq 0}$ on functions, respectively defined by spectral calculus through $\mathsf{H}_t := \mathrm{e}^{-t\vec{\Delta}}$ and $\mathsf{P}_t := \mathrm{e}^{t\Delta}$, where Δ is the negative Laplace–Beltrami operator. That is, $(\mathsf{H}_t)_{t\geq 0}$ has a natural interpretation as *Feynman–Kac*-type semigroup "$\mathsf{H}_t = \mathrm{e}^{-t\mathrm{Ric}+t\square}$" with potential Ric, and if Ric is uniformly bounded from below by $K \in \mathbf{R}$, the pointwise inequality

$$|\mathsf{H}_t\omega| \leq \mathrm{e}^{-Kt}\,\mathsf{P}_t|\omega| \tag{0.1.1}$$

should thus hold for every appropriate 1-forms ω over M.

Using stochastic calculus and based on Weitzenböck's identity, (0.1.1) was proven by [Air75, Mal74], cf. Theorem 2.2.1 below. An analytic access to the implication from lower boundedness of Ric by K to (0.1.1) has been due to [HSU77, HSU80], which is why (0.1.1) is often termed *Hess–Schrader–Uhlenbrock inequality*.

Form domination vs. semigroup domination Heuristically, the argument from [HSU77, HSU80] is based on Bochner's technique [Pet06, Ch. 7] as follows. By local computations, Weitzenböck's identity implies the vector Bochner formula

$$\Delta\frac{|X|^2}{2} + \big\langle X, \vec{\Delta}X\big\rangle = \mathrm{Ric}(X,X) + \big|\nabla X\big|_{\mathrm{HS}}^2 \tag{0.1.2}$$

with pointwise Hilbert–Schmidt norm $|\cdot|_{\mathrm{HS}}$ for appropriate vector fields X over M. Merging, at every point where $|X| \neq 0$, the chain rule $\Delta|X|^2 = 2\,|X|\,\Delta|X| + 2\,\big|\nabla|X|\big|^2$ for Δ together with the Kato inequality [Kat72]

$$\big|\nabla|X|\big| \leq |\nabla X|_{\mathrm{HS}} \tag{0.1.3}$$

then reduces (0.1.2) to

$$\Delta|X| - K\,|X| \geq |X|^{-1}\,\big\langle X, \vec{\Delta}X\big\rangle, \tag{0.1.4}$$

a strong form of which we later call *vector* 1-*Bochner inequality* (cf. Theorem 3.6.21 and also Definition 1.1.5 and Theorem 1.3.6). It can be phrased as a certain inequality between the quadratic forms corresponding to the Schrödinger operator $\Delta - K$ and $\vec{\Delta}$, a principle which is known in more general situations as *form domination*. (Hence, pedantically speaking, the heat flow $(\mathsf{P}_t)_{t\geq 0}$ only shows up in (0.1.1) since constant lower Ricci bounds are considered, and one should generally rather think of (0.1.1) as an inequality between *Schrödinger* semigroups.) This inequality is in fact equivalent to the domination — *semigroup domination* — of the associated semigroups just as in (0.1.1) [HSU77, Sim77]. This insight generalizes the Beurling–Deny criteria for positivity preservingness of a single scalar semigroup [BD58], has been studied later in large generalities [Ouh99, Shi97, Shi00], and has many applications in mathematical physics, e.g. spectral questions about magnetic Schrödinger operators and Yang–Mills theory [AHS78, CSS78, Sch78, Sim77]. The most recent article about this equivalence we are aware of is [LSW21], see also the references therein. Works generalizing the concrete estimate (0.1.1) are given in Section 0.2 below.

We emphasize that (0.1.3) essentially follows from metric compatibility of the Levi-Civita connection ∇ and Cauchy–Schwarz's inequality. The term "Kato's inequality" is henceforth used for it, albeit it should be mentioned for completeness that only a consequence of it — namely an induced inequality between the quadratic forms of Δ and $\Box$ — has been named that way in [HSU80, Kat72].

0.1.2 Scenario II. Heat flow on functions, Bakry–Émery theory

Setting Let $(M, \mathscr{E}, \mathfrak{m})$ be a Dirichlet space, i.e. a Lusin measure space $(M, \mathfrak{m})$ endowed with a symmetric, quasi-regular and strongly local Dirichlet form $\mathscr{E}$ with domain $\mathscr{F} \subset L^2(M)$. (All Dirichlet spaces here are understood symmetric, but it will be useful later to stress this aspect.) Inter alia, influential works on these have been [AMR93, BD58, Che92, Fuk71, Fuk79, LJ76, LJ77, Sil74]. Many books have been devoted to their theory [BH91, CF12, FOT11, MR92, Sil76] that largely generalizes various notions from Subsection 0.1.1, cf. Section 2.2 and Subsection 3.2.2 for details. For instance, they canonically come with a Laplacian Δ, whose domain $\mathscr{D}(\Delta) \subset \mathscr{F}$ is characterized by the integration by parts formula

$$-\int_M g\,\Delta f\,\mathrm{d}\mathfrak{m} = \mathscr{E}(g, f) = \int_M \Gamma(g, f)\,\mathrm{d}\mathfrak{m}, \tag{0.1.5}$$

hence with an $\mathfrak{m}$-symmetric heat flow $(\mathsf{P}_t)_{t\geq 0}$ on $L^2(M)$. The latter object extends to a sub-Markovian semigroup on $L^p(M)$ for every $p \in [1, \infty]$, strongly continuous if $p < \infty$ and weakly* continuous if $p = \infty$. It is usually either part of our assumptions or provided by default in the setting we work with that a symmetric, bilinear, continuous carré du champ operator $\Gamma\colon \mathscr{F}^2 \to L^1(M)$ satisfying the second identity of (0.1.5) exists, cf. Remark 3.2.36. Moreover, there exists an $\mathfrak{m}$-reversible, continuous process $\mathsf{b}^{\cdot}$ on M with the strong Markov property and explosion time $\zeta^{\cdot}$ such that

$$\mathsf{P}_t f = \mathbf{E}\big[f(\mathsf{b}^{\cdot}_{2t})\,1_{\{t<\zeta^{\cdot}/2\}}\big] \tag{0.1.6}$$

for every appropriate $f\colon M \to \mathbf{R}$. By some abuse of terminology (cf. Chapter 2), the process b^x is called Brownian motion starting in $x \in M$.

Bakry–Émery curvature-dimension condition Trying to define a Ricci tensor on general Dirichlet spaces "as usual" is hopeless, since this would require M to have a smooth structure (or at least C^2). However, the present setting provides a convenient framework to set up a condition under which its — fictive, non-existent — Ricci curvature is *bounded from below* by $K \in \mathbf{R}$ (and its dimension is bounded from above by $N \in [1, \infty]$ which, to streamline the presentation, is only discussed for the case $N = \infty$ in more detail, also in Subsection 0.1.3). This curvature-dimension condition was initiated in [Bak85, BE85], still, however, under some technical smoothness assumptions removed later [Sav14]. It relies on the observation that, in Subsection 0.1.1, inserting $X := \nabla f$ for an appropriate function $f\colon M \to \mathbf{R}$ into (0.1.2) yields

$$\Delta\frac{|\nabla f|^2}{2} - \big\langle \nabla f, \nabla\Delta f\big\rangle \geq K\,|\nabla f|^2 \tag{0.1.7}$$

which can be weakly made sense of on every Dirichlet space, cf. Definition 1.1.4 and Definition 3.2.63, and carries the name *Bakry–Émery condition* $\mathrm{BE}_2(K, \infty)$. (The subscript 2 becomes apparent in Chapter 1.)

As indicated, for smooth Riemannian M with $\mathfrak{m} := \mathrm{e}^{-2\varphi}\,\mathfrak{v}$, $\varphi \in \mathrm{C}^2(M)$, one can always construct a canonical Dirichlet space. It satisfies $\mathrm{BE}_2(K, \infty)$ if and only if the *Bakry–Émery Ricci tensor* $\mathrm{Ric} + 2\,\mathrm{Hess}\,\varphi$ is bounded from below by K.

Self-improvement Back again in all generality, $\mathrm{BE}_2(K,\infty)$, $K \in \mathbf{R}$, can be characterized by functional inequalities for $(\mathsf{P}_t)_{t\geq 0}$ [Bak85, BE85, BL06]: it is equivalent to the validity of the gradient estimate

$$|\mathrm{d}\mathsf{P}_t f|^2 \leq \mathrm{e}^{-2Kt}\, \mathsf{P}_t\big(|\mathrm{d}f|^2\big) \tag{0.1.8}$$

for sufficiently many $f\colon M \to \mathbf{R}$. This is surprising at least in view of Subsection 0.1.1 and what we just have mentioned. Indeed, since $\mathsf{H}_t \mathrm{d}f = \mathrm{d}\mathsf{P}_t f$ for every appropriate $f\colon M \to \mathbf{R}$, see Section 2.2 for technical subtleties, this equivalence must mean that (0.1.8) implies the a priori *stronger* inequality (0.1.1) for gradients along $(\mathsf{P}_t)_{t\geq 0}$. This remains true in all generality which, at the (equivalent, cf. e.g. Theorem 1.3.4 below) level of the Bochner inequalities (0.1.7) and (0.1.4), is known as self-improvement property [Bak85]. Roughly speaking, the reason is the great diversity of calculus rules for Γ on Dirichlet spaces which allow one to "regain" the Hessian term — or at least the term $\big|\nabla|\nabla f|\big|^2$ in light of (0.1.3) — that has been thrown away in the passage from (0.1.2) to (0.1.7) by some manipulations of (0.1.7), thus obtaining (a variant of) form domination for "exact 1-forms". This powerful self-improvement strategy will be revisited in Chapter 1 and Chapter 3 in more detail. The proofs of "improved Bochner inequalities" implying "improved functional inequalities" we sometimes use (see e.g. Theorem 1.3.4, Theorem 3.4.26 and Theorem 3.6.33 below) are variants of the arguments for the above mentioned implication from form to semigroup domination.

The Bakry–Émery condition has been extensively studied over the last years. We refer to [BL06, BL06, Stu18a], the book [BGL14] and the references therein.

0.1.3 Scenario III. Heat flow on measures, Otto calculus, Lott–Sturm–Villani theory

Setting Let $(M,\mathsf{d},\mathfrak{m})$ be a metric measure space, always understood to be a triple consisting of a complete and separable metric space (M,d) endowed with a fully supported, locally finite Borel measure $\mathfrak{m}$.

Why another curvature concept? In high generality — but not always — metric measure spaces give rise to a canonical symmetric Dirichlet space in terms of the Cheeger energy defined in (0.1.12) below. For those which do, the Bakry–Émery theory from Subsection 0.1.2 can be used to define when the Ricci curvature of such $(M,\mathsf{d},\mathfrak{m})$ is bounded from below by $K \in \mathbf{R}$.

Besides the uncertainty of whether $(M,\mathsf{d},\mathfrak{m})$ induces a Dirichlet structure, another serious issue arises when treating stability questions. Although some works deal with convergence of Dirichlet spaces [CMT21, KS03], for metric measure spaces the latter depend in a highly nontrivial way on the basic data $(M,\mathsf{d},\mathfrak{m})$ [Stu06a], which makes the Bakry–Émery theory impractical for e.g. measured Gromov–Hausdorff (mGH) convergence [Fuk87, GMS15]. The desire of a well-behaved notion of Ricci lower bounds has been enforced by the influential works of Cheeger and Colding [CC97, CC00a, CC00b] on Ricci limit spaces, i.e. mGH-limits of sequences of Riemannian manifolds along which the respective Ricci curvatures are uniformly bounded from below. Since the involved objects depend more directly on the quantities involved in the definition of this convergence, stability questions for mGH-limits of Dirichlet metric measure structures should be studied at the Bochner level (0.1.7). However, (0.1.7) contains higher order objects whose behavior is unclear under a zeroth order topology such as the mGH one.

Optimal transport These issues have been addressed satisfactorily using optimal transport theory, basic notions of which are reviewed first. See [Vil09] for details.

Let $\mathscr{P}(M)$ be the space of Borel probability measures on M. For $p \in [1,\infty)$, define the *p-Kantorovich–Wasserstein distance* between $\mu, \nu \in \mathscr{P}(M)$ by

$$W_p(\mu,\nu) := \inf \Big[\int_{M^2} \mathsf{d}^p(x,y)\,\mathrm{d}\pi(x,y)\Big]^{1/p}, \tag{0.1.9}$$

and we set $W_\infty(\mu,\nu) := \lim_{p\to\infty} W_p(\mu,\nu)$, which will be well-defined by Hölder's inequality. The infimum is taken over all *couplings* π of μ and ν, i.e. all $\pi \in \mathscr{P}(M^2)$ such that $\pi[A\times M] = \mu[A]$ and $\pi[M\times B] = \nu[B]$ for every Borelian $A, B \subset M$. At least one admissible such π does always exist, namely the product measure $\mu\otimes\nu$. This definition of W_p appeared in [Vas69]. $W_p^p(\mu,\nu)$ is nothing but a special case — with cost function d^p — of the famous *Monge–Kantorovich problem* [Kan42], leaned on a more restrictive and demanding version of (0.1.9) [Mon81]. Pictorially said, $W_p^p(\mu,\nu)$ describes the minimal total cost for transporting a mass distribution μ (e.g. snow) to a target configuration ν (e.g. a snowman) subject to the cost $\mathsf{d}^p(x,y)$ of transporting the respective infinitesimal mass portions $\mathrm{d}x$ to $\mathrm{d}y$.

Since $\mathsf{d}^p : M^2 \to [0,\infty)$ is nonnegative and continuous, a minimizer of (0.1.9) for $p\in[1,\infty)$ always exists. $W_p(\mu,\nu)$ may be infinite, but is finite — in fact, giving rise to a metric W_p on the space to follow — if μ and ν belong to the *p-Wasserstein space*

$$\mathscr{P}_p(M) := \Big\{\iota \in \mathscr{P}(M) : \int_M \mathsf{d}^p(\cdot,o)\,\mathrm{d}\iota < \infty \text{ for some } o\in M\Big\}.$$

Since (M,d) is Polish, so is $(\mathscr{P}_p(M), W_p)$ for every $p\in[1,\infty)$. Moreover, if (M,d) is *geodesic* (i.e. every two points in M can be joined by a geodesic, a continuous curve $\gamma\colon [0,1]\to M$ with $\mathsf{d}(\gamma_s,\gamma_t) = |t-s|\,\mathsf{d}(\gamma_0,\gamma_1)$ for every $s,t\in[0,1]$), the same property is inherited by $(\mathscr{P}_p(M), W_p)$. An important result [Lis07] which we often use in Chapter 1 connects geodesics in both spaces as soon as $p>1$: for every geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_p(M)$, there exists a probability measure $\boldsymbol{\pi}$ on the space of geodesics in M whose law at time t precisely coincides with μ_t for every $t\in[0,1]$ and whose joint law of the endpoints is a W_p-optimal coupling of μ_0 and μ_1.

Lott–Sturm–Villani curvature-dimension condition In [Bre91] for Euclidean and [McC97] for Riemannian M, see also [Gig11, Gig12], geodesics between measures in $\mathscr{P}_2(M)$ absolutely continuous w.r.t. the respective reference "Lebesgue" measure were shown to have a particularly nice form, thereby indicating the importance of the 2-Wasserstein space. This was underlined by [JKO98, Ott01] where solutions ρ to the Fokker–Planck and the porous medium equation were interpreted as the gradient flow "$\dot{\boldsymbol{\rho}} = -\nabla\,\mathrm{Ent}(\boldsymbol{\rho})$" of entropy-type functionals in $\mathscr{P}_2(\mathbf{R}^d)$, $d\in\mathbf{N}$. (We refer to the introduction of [AGS08] for good heuristics of how gradient flows are set up in general metric spaces, and to [AGS14b, Def. 2.14] for the precise definition which we omit here.) In [Ott01], the 2-Wasserstein space was moreover endowed with the structure (*Otto calculus*) of a formal, infinite-dimensional Riemannian manifold.

One of the many powerful applications of this calculus was a heuristic hint [OV00] to geodesic K-convexity of the Boltzmann (or relative) entropy

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \displaystyle\int_M \rho\log\rho\,\mathrm{d}\mathfrak{m} & \text{if } \mu \ll \mathfrak{m},\ \mu = \rho\,\mathfrak{m},\\ \infty & \text{otherwise} \end{cases}$$

w.r.t. $\mathfrak{m} := \mathfrak{v}$ on $\mathscr{P}_2(M)$ over a Riemannian manifold M with Ricci curvature bounded from below by $K \in \mathbf{R}$. That is, for every $\mu_0, \mu_1 \in \mathscr{P}_2(M)$ with finite entropy w.r.t. $\mathfrak{m}$ there exists a geodesic $(\mu_t)_{t\in[0,1]}$ in $\mathscr{P}_2(M)$ such that for every $t \in [0,1]$,

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \leq (1-t)\,\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{Kt(1-t)}{2}\,W_2^2(\mu_0,\mu_1). \tag{0.1.10}$$

This kind of convexity is also called *displacement convexity* [McC97]. It was proven in [CE⁺01] ("if" part for $K = 0$) and [vRS05] (in the general case) that $\mathrm{Ent}_{\mathfrak{v}}$ on $\mathscr{P}_2(M)$ is *strongly* K-convex if and only if Ric is no smaller than K. More generally, the entropy $\mathrm{Ent}_{\mathfrak{m}}$ w.r.t. the weighted measure $\mathfrak{m} := \mathrm{e}^{-2\varphi}\,\mathfrak{v}$, $\varphi \in \mathrm{C}^2(M)$, is strongly K-convex if and only if $\mathrm{Ric} + 2\,\mathrm{Hess}\,\varphi$ is bounded from below by $K \in \mathbf{R}$ [Stu06a], which suggests a link to Bakry–Émery's theory in Subsection 0.1.2. (Strong K-convexity asks (0.1.10) to hold for *every* geodesic $(\mu_t)_{t\in[0,1]}$ as above. However, on Riemannian manifolds, geodesics between $\mathfrak{m}$-absolutely continuous measures in $\mathscr{P}_2(M)$ are unique [Bre91, McC97], and this will in fact always hold in the relevant spaces in Chapter 1 [RS14].)

Remarkably, the involved objects W_2 and $\mathrm{Ent}_{\mathfrak{m}}$ only depend on d and $\mathfrak{m}$, but not on the smooth structure of M. This motivated Sturm [Stu06a], and independently Lott and Villani [LV09], to introduce the curvature-dimension condition $\mathrm{CD}(K,\infty)$ for general metric measure spaces $(M,\mathsf{d},\mathfrak{m})$ in terms of geodesic K-convexity of $\mathrm{Ent}_{\mathfrak{m}}$ in $\mathscr{P}_2(M)$ as a synthetic replacement of Ricci lower boundedness by K.

Heat flow on probability measures Inspired by Otto's calculus, the *heat flow* on probability measures defined as the gradient flow [AGS08] of $\mathrm{Ent}_{\mathfrak{m}}$ over $(\mathscr{P}_2(M), W_2)$ attracted high interest soon. Existence and uniqueness were known in various special cases [Erb10, JKO98, Oht09, OS09, Sav07, Vil09] for large sets of initial conditions before this being accomplished for $\mathrm{CD}(K,\infty)$ spaces [AGS14a]. In these works, the close link of $(\mathscr{H}_t)_{t\geq 0}$ to the functional heat flow $(\mathsf{P}_t)_{t\geq 0}$ — consistently defined as the gradient flow of the (halved) Cheeger energy from (0.1.12) below if the latter does not satisfy the parallelogram identity — was highlighted: $(\mathsf{P}_t)_{t\geq 0}$ is mass-preserving, and for every probability density $f \in L^2(M)$ such that $\mu := f\,\mathfrak{m}$ has finite entropy, the heat flow $(\mathscr{H}_t\mu)_{t\geq 0}$ starting in μ exists uniquely with $\mathscr{H}_t\mu = \mathsf{P}_t f\,\mathfrak{m}$.

However, the naive guess perhaps motivated from the gradient flow theory of semiconvex functionals that in $\mathrm{CD}(K,\infty)$ spaces, $K \in \mathbf{R}$, for every $\mu, \nu \in \mathscr{P}_2(M)$ from which a $\mathscr{P}_2$-heat flow can be started, one has the W_2-contraction estimate

$$W_2(\mathscr{H}_t\mu, \mathscr{H}_t\nu) \leq \mathrm{e}^{-K(t-s)}\,W_2(\mathscr{H}_s\mu, \mathscr{H}_s\nu), \tag{0.1.11}$$

for every $s, t \geq 0$, $s \leq t$, is *false*. Indeed, Euclidean Minkowski spaces are $\mathrm{CD}(0,\infty)$ spaces where (0.1.11) does not hold for *any* $K \in \mathbf{R}$ [OS11]. Thus, though compact Finsler manifolds obey $\mathrm{CD}(K,\infty)$ for some $K \in \mathbf{R}$ [Oht09], in general (0.1.11) is not implied by — and in particular does not characterize — lower Ricci bounds on spaces where even (generalized) notions of Ricci curvature exist.

0.1.4 Relation between these scenarios

Now we finally clarify how the above Eulerian (i.e. Bakry–Émery) and Lagrangian (i.e. Lott–Sturm–Villani) viewpoints on Ricci curvature lower bounds, in particular those of Subsection 0.1.2 and Subsection 0.1.3, are related for a given metric measure space $(M,\mathsf{d},\mathfrak{m})$, keeping in mind the mentioned negative results.

Cheeger energy We first outline the definition of an "energy" on $(M, \mathsf{d}, \mathfrak{m})$. This theory was initiated in [Che99] and developed further in [AGS13, AGS14a, AGS14b, Sha00]. We follow [AGS14a], which unified the approaches [Che99, Sha00].

Define the *local Lipschitz slope* of a Lipschitz function $f \in \mathrm{Lip}(M)$ by

$$\mathrm{lip}(f)(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)}$$

and the *Cheeger energy* $\mathscr{E}(f) \colon L^2(M) \to [0, \infty]$ by

$$\mathscr{E}(f) := \inf \liminf_{n \to \infty} \int_M \mathrm{lip}^2(f_n)\, \mathrm{d}\mathfrak{m}, \tag{0.1.12}$$

where the infimum is taken over all sequences $(f_n)_{n \in \mathbf{N}}$ of bounded Lipschitz functions converging to f in $L^2(M)$. $\mathscr{E}$ is strictly convex and L^2-lower semicontinuous. For every $f \in W^{1,2}(M)$, where $W^{1,2}(M)$ is the finiteness domain of $\mathscr{E}$, there exists a unique element $|\mathrm{d}f| \in L^2(M)$ whose L^2-norm w.r.t. $\mathfrak{m}$ coincides with $\mathscr{E}(f)$. The function $|\mathrm{d}f|$ is called *minimal weak upper gradient*. For every such f, there exists a sequence $(f_n)_{n \in \mathbf{N}}$ of bounded Lipschitz functions in $W^{1,2}(M)$ such that $f_n \to f$ and $\mathrm{lip}(f_n) \to |\mathrm{d}f|$ in $L^2(M)$ as $n \to \infty$. In particular, the set of bounded Lipschitz functions in $W^{1,2}(M)$ is dense both in $W^{1,2}(M)$ and $L^2(M)$.

Riemannian spaces The breakthrough in connecting $\mathrm{BE}_2(K, \infty)$ and $\mathrm{CD}(K, \infty)$, $K \in \mathbf{R}$, was achieved in [AGS14b, AGS15] by adding the condition of *infinitesimal Hilbertianity* [Gig15]. A metric measure space is termed that way if its Cheeger energy satisfies the parallelogram identity (or equivalently, it admits a linear heat flow or Laplacian, both a priori defined by basic gradient flow theory). This rules out "badly behaved" Finsler spaces: indeed, a Finsler manifold is infinitesimally Hilbertian if and only if it is Riemannian. An infinitesimally Hilbertian $\mathrm{CD}(K, \infty)$ space is thus usually called $\mathrm{RCD}(K, \infty)$ space, the letter R standing for "Riemannian". Modulo technicalities, $\mathrm{BE}_2(K, \infty)$ and $\mathrm{RCD}(K, \infty)$ were then shown to be equivalent as follows. A $\mathrm{BE}_2(K, \infty)$ Dirichlet space $(M, \mathscr{E}, \mathfrak{m})$ endowed with the *intrinsic metric*

$$\mathsf{d}_{\mathscr{E}}(x, y) := \sup\bigl\{\psi(x) - \psi(y) : \psi \in \mathscr{F} \cap \mathrm{C_b}(M),\ \Gamma(\psi) \leq 1\ \mathfrak{m}\text{-a.e.}\bigr\}$$

is an $\mathrm{RCD}(K, \infty)$ space (and $\mathscr{E}$ equals the Cheeger energy induced by $\mathsf{d}_{\mathscr{E}}$) [AGS15]. Conversely, the Cheeger energy $\mathscr{E}$ of a given $\mathrm{RCD}(K, \infty)$ space $(M, \mathsf{d}, \mathfrak{m})$ is a quasi-regular, strongly local Dirichlet form obeying $\mathrm{BE}_2(K, \infty)$ (and $\mathsf{d} = \mathsf{d}_{\mathscr{E}}$) [AGS14b].

The tails in our above outlines were soon closed by providing other equivalent characterizations of the $\mathrm{RCD}(K, \infty)$ condition, stated again modulo technicalities: existence of $\mathrm{EVI}(K)$ gradient flows [AGS14b], q-gradient estimates for $(\mathsf{P}_t)_{t \geq 0}$, $q \in [1, \infty]$, based on an adaptation of Bakry–Émery's self-improvement property [Sav14], W_p-contraction estimates for $(\mathsf{H}_t)_{t \geq 0}$, $p \in [1, \infty]$, such as (0.1.9) [Sav14] by *Kuwada's duality* [Kuw10], and pathwise coupling properties of Brownian motions [Stu15]. (In Chapter 1, we treat these equivalences in a more complicated setting which, though, can be simplified in the constant case and thus may be consulted for details on the previously mentioned facts.) A byproduct of these works is that $(\mathscr{H}_t)_{t \geq 0}$ extends continuously (w.r.t. the weak topology) to $\mathscr{P}(M)$ still satisfying the indicated W_p-contraction estimates for every $p \in [1, \infty]$ on RCD spaces [Sav14].

Based on this large set of equivalent viewpoints on the RCD condition, as well as similar characterizations of soon developed finite-dimensional analogues [EKS15, Gig15] based on the seminal works [BS10, Stu06b, LV09], many beautiful and powerful results could be achieved. The literature is too large to be cited exhaustively, we only mention [BS20, CM17, ES21, Gig13, Ket15b, Ket15c, KS18, MN19, Stu18c] here as key cornerstones, some of which will be put into a context later.

0.2 Nonuniform or irregular lower Ricci bounds

We have outlined a quite complete picture of how functional inequalities for $(\mathsf{P}_t)_{t\geq 0}$ and contraction estimates for $(\mathscr{H}_t)_{t\geq 0}$ are related to synthetic notions of constant lower Ricci bounds. However, until recently few of these were studied for nonconstant lower Ricci bounds, and if so, mostly in the smooth case. The various technical issues that appear in this business as well as the evidence of studying synthetic lower Ricci bounds beyond constant ones is explained in this section, before then moving to the contributions of our thesis towards these questions in Section 0.3.

Interior Ricci curvature The first, seemingly elementary, motivation arises from the case of a possibly noncompact Riemannian manifold without boundary. The best possible lower bound on Ric is, of course, its pointwise lowest eigenvalue, given as a continuous function $\mathscr{k}\colon M \to \mathbf{R}$. For instance, in the setting of Subsection 0.1.1, the results from [Air75, Mal74] yield the improved Hess–Schrader–Uhlenbrock inequality

$$|\mathsf{H}_t\omega| \leq \mathbf{E}\big[\mathrm{e}^{-\int_0^{2t} \mathscr{k}(\mathsf{b}_r^{\cdot})/2\,\mathrm{d}r}\,|\omega|(\mathsf{b}_{2t}^{\cdot})\big] \tag{0.2.1}$$

with Brownian motion b^x starting in $x \in M$ for the 1-form heat flow $(\mathsf{H}_t)_{t\geq 0}$.

Even for bounded $\mathscr{k}$, (0.2.1) is usually better than (0.1.1) [Stu20, p. 1650]. Though, unlike the compact case, $\mathscr{k}$ may generally be unbounded (see Chapter 2 and Chapter 3 for examples). It is thus a priori not clear whether (and how) the r.h.s. of (0.2.1) makes sense. This problem, along with finding versions of (0.2.1) for unbounded $\mathscr{k}$, has been addressed in the noncompact case from two different directions.

In probabilistic approaches, possible irregularities are usually catched up simply by weak exponential integrability assumptions on $\mathscr{k}$ such as (0.3.1) below. If Ric is bounded from below by such $\mathscr{k}$, then (0.2.1) makes sense and holds e.g. for bounded ω [DT01, EL94a, EL94b, Li92, Li94, Ros88, Tha97] (these also contain the case of weighted Riemannian manifolds). Among others, the impressive set of functional inequalities for general scalar diffusion semigroups [Wan14] yields the equivalence of (0.2.1) — for exact 1-forms — with Ric being no smaller than $\mathscr{k}$ under suitable, more geometric conditions. Such estimates can also be obtained by coupling methods [Cra91, Ken86, Qia97]. A delicate issue in all these cases is the possible explosion of Brownian motion in finite time. The nature of the mentioned exponential integrability assumptions is quite general, in the sense that formal Schrödinger operators associated to such $\mathscr{k}$ (see below) do not necessarily have the desired good analytic properties. The price one pays is that the approach by stochastic differential geometry is exclusively provided by the *smooth* framework until today.

Analytically, (0.2.1) has recently been derived in [Gün12, Gün17a, MO20]. There, as indicated in Subsection 0.1.1 and the previous lines, the r.h.s. of (0.2.1) is interpreted as Feynman–Kac semigroup with potential $\mathscr{k}$. [Gün12, Gün17a, MO20] studied $\mathscr{k}$ belonging to the (*extended*) *Kato class*, which has recently attracted some research attention and also constitutes *the* relevant class of lower Ricci bounds in our work, see (the references in) Chapter 2 and Chapter 3. It was introduced in [AS82] following [Kat72] and turned out to be just the right class w.r.t. whose induced Feynman–Kac semigroups can be defined and have good L^p-properties [SV96, Stu94]. (In particular, this typically allows one to enlarge the class of admissible ω in (0.2.1).) Bounded — in particular constant — functions are elementary examples of Kato class elements, which thus cover the frameworks from Section 0.1. We refer to Section 2.2 and Section 3.1 for an overview over the large literature on the Kato class.

Rough conformal changes Furthermore, somewhat interpolating between the previous and the next paragraph, one generally expects irregular behavior of the Ricci curvature Ric_ϕ induced by a conformal change of $\langle\cdot,\cdot\rangle$ — i.e. Ric_ϕ being induced by the metric $\langle\cdot,\cdot\rangle_\phi := \mathrm{e}^{2\phi}\,\langle\cdot,\cdot\rangle$ — through a not necessarily smooth function $\phi\colon M \to (0,\infty)$. Indeed, if $\phi \in \mathrm{C}^\infty(M)$ then

$$\mathrm{Ric}_\phi = \mathrm{Ric} - (\dim M - 2)\,\big[\nabla \mathrm{d}\phi - \mathrm{d}\phi \otimes \mathrm{d}\phi\big] - \big[\Delta\phi + (\dim M - 2)\,|\mathrm{d}\phi|^2\big]\,\langle\cdot,\cdot\rangle,$$

see e.g. [Bes87, p. 59], while the r.h.s. amounts to make sense only in a measure- or distribution-valued sense if ϕ has (interior) singularities.

The affection of such irregular conformal changes have recently been studied for RCD spaces in [Han19, HS21, Stu20].

Boundaries Another situation where measure-valued Ricci bounds come into play is the presence of boundaries or boundary singularities [ER⁺20, Wan14]. (Here, $(\mathsf{P}_t)_{t\geq 0}$ and $(\mathsf{H}_t)_{t\geq 0}$ have to be endowed with Neumann and absolute boundary conditions, respectively, cf. Chapter 3.) For instance, on a compact Riemannian manifold M with not necessarily convex boundary ∂M, the *signed Borel measure*

$$\kappa := \Bbbk\,\mathfrak{v} + \ell\,\mathfrak{s} \tag{0.2.2}$$

plays the natural role of a lower "Ricci" bound [Hsu02a, Stu20, Wan14]. Here $\mathfrak{s}$ is the surface measure of ∂M, and $\Bbbk$ and ℓ are the pointwise lowest eigenvalues of Ric and the second fundamental form $\mathbb{I}$. The corresponding version of (0.2.1) then reads

$$|\mathsf{H}_t\omega| \leq \mathbf{E}\big[\mathrm{e}^{-\int_0^{2t} \Bbbk(\mathsf{b}_r^\cdot)/2\,\mathrm{d}r - \int_0^{2t}\ell(\mathsf{b}_r^\cdot)/2\,\mathrm{d}L_r^\cdot}\,|\omega|(\mathsf{b}_{2t}^\cdot)\big], \tag{0.2.3}$$

where L^x is the *local time* of (reflecting) Brownian motion b^x starting in $x \in M$ at ∂M.

By probabilistic means, for compact Riemannian M, (0.2.3) was proven in [Hsu02b] building upon [IW81] for $\Bbbk$ and ℓ being any functions no larger than the pointwise lowest eigenvalues of Ric and $\mathbb{I}$, respectively. If M is noncompact, (0.2.3) still holds for exact 1-forms if ∂M is convex [Qia97, Wan14] under an exponential integrability assumption on $\Bbbk$. The nonconvex case is much more subtle and not yet completely understood. The problem here is that the local time is generally unclear to be exponentially integrable. At least, this is ensured under certain geometric conditions on the tubular neighborhood of the boundary [Wan05b, Wan09, Wan14], in which case (0.2.3) holds for exact 1-forms [Wan14] or more general ω [AL17]. (In fact, under these conditions one can *convexify* ∂M by a conformal change, a concept that, as mentioned above, has recently been revisited for subsets of RCD spaces by [Stu20].)

Approaches to boundary theory for Dirichlet forms are e.g. due to [Che92, Ebe99, Sil74, Sil76] and the book [CF12] which can be consulted for more relevant works. However, from curvature aspects, analytic approaches were poorly spread until recently, since it is generally unclear how to treat the local time appropriately. The only older work we are aware of is [Shi00], where (0.2.3) has been proven on compact M with convex boundary, but the local time itself in (0.2.3) does not show up in [Shi00].

Tamed spaces Spaces with boundary have recently become intersting in the context of the curvature-dimension conditions from Section 0.1 [BNS20, Han20, HS21, Stu20]. By [Han20], it is known that Riemannian manifolds with uniformly lower bounded Ricci curvature and *convex* boundary are still RCD spaces. (Convexity ensures that sets of geodesics stay in M and do not constrict at the boundary.) However, already the appearance of a small boundary concavity makes it generally impossible for the relative entropy to be K-convex [Stu20] or for (0.1.7) to hold [Wan14] for any $K \in \mathbf{R}$. The rich set of equivalent characterizations of lower Ricci bounds as in Section 0.1 for spaces with boundary is thus unlikely to exist.

Still, (0.1.7) is still quite flexible, for it can be formally rewritten as

$$\Delta^{2K}\frac{|\nabla f|^2}{2} - \big\langle \nabla f, \nabla\Delta f\big\rangle \geq 0 \tag{0.2.4}$$

with Schrödinger operator $\Delta^{2K} := \Delta - 2K$. This was first observed in [Stu15], continued in [BHS21, Stu20] (see Chapter 1), and finally lead [ER+20] to introduce the notion of *tamed spaces*. These are Dirichlet spaces satisfying (0.2.4) in a weak sense, in which K can even be a function, a measure, or a distribution κ. [ER+20] proved the equivalence of a weak version of (0.2.4) to gradient estimates for $(\mathsf{P}_t)_{t\geq 0}$ in terms of the Schrödinger semigroup associated with $\Delta^{2\kappa}$ (which had to be made sense of as well). Among the many examples — with possibly singular behavior — given in [ER+20], we stress two particular ones: *any* compact Riemannian manifold with boundary, with κ as in (0.2.2), and, in all generality, spaces satisfying (0.2.4) in a weak sense for *any* signed measure κ in the extended Kato class of M, see Chapter 3.

0.3 Main contributions

Now we briefly summarize the main results of our thesis. Related literature, further details, and possible extensions, though, are often outsourced into the introductory sections of the respective chapters.

Roughly speaking, we address the following two goals that have been delineated in our previous discussion. First, we study the equivalences from Section 0.1 on possibly singular spaces for a class of lower Ricci bounds in a generality as large as possible (Chapter 1 and Chapter 2). Second, we close the nonsmooth circle to the framework of 1-forms which was mostly left after Subsection 0.1.1, and derive the indicated functional inequalities in large generality for an appropriate notion of heat flow on 1-forms (Chapter 3 and Chapter 4).

0.3.1 Main results of Chapter 1

The results of Chapter 1 are obtained in the author's joint work [BHS21] with Karen Habermann and Karl-Theodor Sturm.

On an RCD metric measure space $(M, \mathsf{d}, \mathfrak{m})$, we prove the equivalence of synthetic approaches to *variable* lower Ricci bounds — i.e. of "the Ricci curvature of M at every $x \in M$ being bounded from below by $k(x)$" — *all* of which are analogues of the conditions outlined in Section 0.1, introduced in Section 1.1. (We point out that in Chapter 1, the a priori RCD assumption will provide all necessary existence and regularity results about heat flows — especially at the level of probability measures — and further identifications obtained by highly nontrivial means in [AGS14a, AGS14b, AGS15, Gig15, Sav14].) Here $k\colon M \to \mathbf{R}$ is a lower semicontinuous, lower bounded function. The respective variable counterparts of the Lott–Sturm–Villani curvature-dimension condition $\mathrm{CD}(k, \infty)$, the evolution variational inequality $\mathrm{EVI}(k)$, the 2-Bakry–Émery condition $\mathrm{BE}_2(k, \infty)$, and the 2-gradient estimate $\mathrm{GE}_2(k)$ have been introduced in [Stu15]. Moreover, the first and the last two properties have been shown to be equivalent therein, respectively. On the other hand, our variable notions of *Wasserstein contractivity* and *pathwise coupling estimate* — i.e. the existence of coupled Brownian motions satisfying certain pathwise estimates — are entirely new even in the smooth setting, cf. Chapter 2. Both involve a novel "geodesic average function" $\underline{k}$, defined in (1.1.2), whose form is probably best motivated by its natural appearance from the Cranston–Kendall coupling construction on Riemannian manifolds in Subsection 2.4.2. This metric measure framework of variable curvature bounds seems to be the only extension of the RCD theory in which all characterizations outlined in Section 0.1 have analogues so far.

The large web of implications to prove the main result in Chapter 1, i.e. Theorem 1.1.1, is unbundled in Section 1.1 and will not be detailed here. We only outline the following two guiding principles that appear at various places. Unlike the constant case, the quantities in our respective conditions in Section 1.1 involving k are usually no multiples of heat flows, Wasserstein distances, etc. Hence, the constant arguments from [AGS14a, AGS14b, AGS15, Kuw10] mostly do not carry over. However, first, under certain mild restrictions, *localization arguments* often allow us to regard k as "approximately constant", so that suitable arguments — partly inspired by [AGS14b, AGS15] — entail connections of two respectively considered conditions "locally with constant Ricci bounds" (e.g. towards our variable Kuwada duality). Then, second, *local-to-global properties* allow us to extend these local conditions to their global counterparts.

0.3.2 Main results of Chapter 2

The results of Chapter 2 are obtained in the author's joint work [BG20] with Batu Güneysu.

We will study both consequences, see Theorem 2.1.1, and characterizations, see Theorem 2.1.6, of a continuous function $k\colon M \to \mathbf{R}$ on a complete, connected, noncompact Riemannian manifold M without boundary — obeying the condition

$$\sup_{x\in M} \mathbf{E}\big[\mathrm{e}^{-\int_0^{2t} k(\mathsf{b}_r^x)/2\,\mathrm{d}r}\,1_{\{t<\zeta^x/2\}}\big] < \infty \tag{0.3.1}$$

as indicated in Section 0.2 to bound the Ricci tensor of M from below. Here b^x is a Brownian motion on M starting in $x \in M$ generated by the (halved) Laplace–Beltrami operator $\Delta/2$. But, as clarified at the end of Section 2.1, the results from Chapter 2 hold for general *gradient* drift diffusions. We further highlight the (functional) *Kato class* as a special class of possibly unbounded functions for which (0.3.1) holds, and discuss examples of manifolds with Kato lower Ricci bounds in Subsection 2.5.1.

Consequences, to be proven in Section 2.3, concern the stochastic completeness of M, i.e. the $\mathbf{P}$-a.s. nonexplosion of b^x for every $x \in M$. Furthermore, for appropriate functions $f\colon M \to \mathbf{R}$ we prove the Bismut–Elworthy–Li derivative formula

$$\big\langle \nabla \mathsf{P}_t f(x), \xi\big\rangle = \frac{1}{\sqrt{2}\,t}\,\mathbf{E}\Big[f(\mathsf{b}_{2t}^x)\int_0^t \big\langle \mathsf{Q}_s^x\xi, \mathrm{d}W_s^x\big\rangle\Big].$$

Here, ξ is an arbitrary tangent vector at $x \in M$, Q^x can be formally interpreted as a stochastic $\mathrm{Ent}(T_xM)$-valued version of the matrix exponential with potential Ric, see

(2.1.2), and W^x — the Brownian *anti-development* — is a canonically given Brownian motion in T_xM. The particular strength of the above formula is that at the r.h.s., no derivative of f shows up. In particular, under (0.3.1) we are able to directly deduce the L^∞-Lip-regularization of $(\mathsf{P}_t)_{t\geq 0}$ by duality. An important role in our arguments is played by the heat flow $(\mathsf{H}_t)_{t\geq 0}$ on 1-forms and its probabilistic features.

The characterization part, cf. Section 2.4, partly extends the results from Chapter 1 and proves the equivalence of lower boundedness of Ric and the pathwise coupling estimate from Chapter 1. Modulo the small additional issue of stochastic completeness, this equivalence even holds without (0.3.1). Unlike Chapter 1, however, $\mathscr{k}$ is not necessarily bounded from below any more. Hence, the backward implication above requires a delicate short-time analysis for Brownian motion, somewhat similar to the localization arguments from Chapter 1. The forward implication, on the other hand, is provided by a standard technique, namely the well-known *Cranston–Kendall coupling*, cf. Subsection 2.4.2.

0.3.3 Main results of Chapter 4

The results of Chapter 4 are obtained in the author's work [Bra20]. Chronologically, they were obtained before those presented in Chapter 3. However, the latter extends some of the results from [Bra20]. Moreover, since [Bra20] itself heavily relies on the tensor calculus developed in [Gig18], which in turn we generalize in Chapter 3, we decided to order the corresponding chapters in the mentioned way to give a widely self-contained introduction into the machinery needed for our results in Chapter 4. In particular, Chapter 3 and Chapter 4 close the circle to the heat flow on 1-forms initially considered smoothly in Subsection 0.1.1.

Let $(M, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, \infty)$ space, $K \in \mathbf{R}$. The above indicated powerful (first and) second order calculus from [Gig18] for RCD spaces, along with natural nonsmooth analogues to the notions of e.g. Hessian, covariant and exterior derivative, and a measure-valued Ricci curvature, introduced a nonsmooth *Hodge Laplacian* $\vec{\Delta}$ and an associated heat flow $(\mathsf{H}_t)_{t\geq 0}$ on 1-forms. By an interpolation argument following [Bak85, BE85, Sav14], it was shown in [Gig18] that

$$|\mathsf{H}_t\omega|^2 \leq \mathrm{e}^{-2Kt}\,\mathsf{P}_t\big(|\omega|^2\big) \tag{0.3.2}$$

for appropriate 1-forms ω on M. Further properties of $(\mathsf{H}_t)_{t\geq 0}$, beyond those coming from the definition of it, were not discussed in [Gig18]. However, among others — see Section 4.1 for more motivations — the importance of the heat flow on 1-forms in Chapter 2 suggests a detailed study of its nonsmooth counterpart from [Gig18].

In [Bra20], restated in Theorem 4.1.1, we improved (0.3.2) to a nonsmooth *Hess–Schrader–Uhlenbrock inequality* (0.1.1) in the setting of [Gig18], i.e.

$$|\mathsf{H}_t\omega| \leq \mathrm{e}^{-Kt}\,\mathsf{P}_t|\omega| \tag{0.3.3}$$

Our strategy was inspired from the smooth argument in Subsection 0.1.1. The required Bochner identity has been established — in a measure-valued way — in [Gig18], while Kato's inequality (0.1.3) was proven in [DGP21] for RCD spaces. Carefully adapting the arguments from Subsection 0.1.1 in a weak way, we obtained the desired inequality (0.3.3). As said, though, for convenience we oursourced this part into Subsection 3.6.4, where (0.3.3) is stated and proven in slightly higher generality, rather leading to (0.2.1).

The rest of Chapter 4 consists of applications of (0.3.3). We study L^p- as well as hyper- and ultracontractivity properties of $(\mathsf{H}_t)_{t\geq 0}$, and the relation of the latter to logarithmic Sobolev inequalities for 1-forms, in Section 4.3. We give explicit examples of these by "lifting" known logarithmic Sobolev inequalities for functions to those for forms, making use again of Kato's inequality. Then, in Section 4.4, we study spectral properties of $\vec{\Delta}$, i.e. spectral inclusions, spectral bottom estimates, and — under additional assumptions — the independence of the L^p-spectrum of $\vec{\Delta}$ on p. Both chapters widely follow the respective smooth treatises [Cha05, Cha07], cf. Section 4.1 for further references especially for the functional case.

The main consequence of (0.3.3) treated in Section 4.5 is an appropriate axiomatization and existence proof of a *heat kernel* h for $(\mathsf{H}_t)_{t\geq 0}$, Theorem 4.5.5. It requires the existence of a functional heat kernel p [AGS14b] satisfying Gaussian bounds [Tam19], which is the main reason for restricting ourselves to RCD spaces in Chapter 4. The construction relies on a perturbation argument which has been partly inspired by [Cha07, Thm. 4.3], and an "L^∞-module version" (in the language of [Gig18]) of Dunford–Pettis' theorem. Finally, we discuss its basic properties, e.g. the pointwise version $|\mathsf{h}_t|_\otimes \leq \mathrm{e}^{-Kt}\,\mathsf{p}_t$ of (0.3.3) on M^2 for the "pointwise operator norm" $|\mathsf{h}_t|_\otimes$ of h_t.

0.3.4 Main results of Chapter 3

The results of Chapter 3 are obtained in the author's work [Bra21].

Our goal of this chapter is to introduce a (first and a) second order calculus on Dirichlet spaces $(M, \mathscr{E}, \mathfrak{m})$ which are tamed by a signed measure κ in the extended Kato class $\mathbf{K}_{1-}(M)$ [ER+20], cf. Section 0.2 above. This extends the RCD treatise [Gig18] whose exposition we closely follow. A crucial ingredient is the fine functional heat flow analysis from [ER+20], which — such as the one from [Sav14] in [Gig18] — is employed to provide the desired rich second order calculus. Indeed, it yields the existence of a large class of "test functions" which — by an integrated Bochner inequality and a self-improvement variant of [BE85, Sav14] in the spirit of [Gig18] — will all be proven to have a *Hessian* in Theorem 3.3.11. This gets a first order calculus on vector fields and differential forms going, as thoroughly discussed from Section 3.4 on by introducing covariant and exterior derivatives. In particular, we define the heat flow $(\mathsf{H}_t)_{t\geq 0}$ on 1-forms and derive a Hess–Schrader–Uhlenbrock inequality under an additional "convexity" assumption which still includes the variable case of Chapter 1. In Section 3.6, the main result of which is Theorem 3.6.9, we finally introduce (drifted and non-drifted) *Ricci measures* $\mathbf{Ric}^\kappa$ and $\mathbf{Ric}$. As for Chapter 1 above, we refer the reader to Section 3.1 for a detailed outline of our arguments of how we intend to make sense of all mentioned calculus objects. In particular, the various technical challenges compared to the seminal work [Gig18] caused by the possible singularity of extended Kato lower Ricci bounds is explained there. Let us briefly outline for now the main differences of our treatise compared to the work [Gig18].

The first, evident, difference is the larger setting. Dirichlet spaces are more general than metric measure spaces: they cover e.g. certain noncomplete spaces, extended metric measure spaces such as configuration spaces [AKR98, EH15], etc. (In fact, the latter are one of the main examples of tamed spaces which are technically not covered by [Gig18].) From many perspectives, they seem to be the correct framework in which elements of a vector calculus should be studied [BK19, HRT13]. This is why we believe that it might be useful to translate the RCD treatise from [Gig18] to the Dirichlet setting. In particular, this setting could be the correct one to develop first nonsmooth notions of stochastic differential geometry. For instance, proving a Feynman–Kac-type representation for $(\mathsf{H}_t)_{t\geq 0}$ as in Theorem 2.2.1 should require a vector-valued extension of the connection between measure-valued perturbations of Laplace-type operators

— such as "$-\vec{\Delta} = \square - \mathbf{Ric}$", cf. Lemma 3.6.16 and compare with Subsection 0.1.1 — on functions provided by *Revuz correspondence* [CF12], keeping in mind that **Ric** is a signed *smooth* measure. Indications about some connections between tensor and stochastic calculus have been made in [HRT13, Ch. 9], and it is subject to our near future plans to explore these connections in our second order framework that has not been available to [HRT13] yet.

Second, the considered lower Ricci bounds, examples of which are due to [BR21, ER+20, GvR20], may be highly irregular. In fact [EKS15, Hon18b], already for uniform lower bounds, the Bakry–Émery setting is strictly larger than the RCD one if the Sobolev-to-Lipschitz property (which is currently unknown for configuration spaces [DSS20]) is dropped. See Subsection 3.2.6.

Third, we want to pursue a thorough discussion of how the appearance of a "boundary" — or more precisely, an $\mathfrak{m}$-negligible, non-$\mathscr{E}$-polar set — in M affects the calculus objects that are introduced similarly as in [Gig18]. Besides the need of measure-valued Ricci bounds to describe curvature of $\mathfrak{m}$-singular sets as by (0.2.2), boundaries play an increasing role in recent research, as outlined in Section 0.2. This motivated us to make sense, in all generality and apart from extrinsic structures, of measure-valued boundary objects such as *normal components* of vector fields (inspired by a similar approach by [BCM19]). In fact, our guiding example is the case of compact Riemannian manifolds with boundary which, unlike only partly in the RCD setting, is fully covered by tamed spaces. Returning to this setting from time to time also provides us with a negative insight on an open question in [Gig18], namely whether "$H = W$", see e.g. Subsection 3.3.4. (This does not conflict with the smooth "$H = W$" results [Sch95] as our "H-spaces" are different from the smooth ones.)

0.4 Notations

In this section, we briefly list the main notations used all over this thesis for a given topological space (M, τ). In every chapter to follow, we consider a fixed reference measure $\mathfrak{m}$, w.r.t. which we downsize corresponding notations as described below.

Measures All very elementary measure-theoretic terminologies are agreed upon [Bog07a, Bog07b, Hal50]. More specific points are shortly addressed now.

The Borel σ-algebra induced by τ — chosen to be the topology induced by the metric in Chapter 1, Chapter 2 and Chapter 4, and the Lusin topology in Chapter 3, and hence not further reflected in our notation — is denoted by $\mathscr{B}(M)$, while its Carathéodory completion w.r.t. a Borel measure μ on M is denoted by $\mathscr{B}^\mu(M)$. (If not explicitly stated otherwise, we identify certain subsets of M with their equivalence classes in $\mathscr{B}^\mu(M)$.) The *support* of every Borel measure μ on M is defined [MR92, Sec. V.1] and denoted by $\operatorname{spt}\mu$. By $\mathfrak{M}_{\mathrm{f}}^+(M)$, $\mathfrak{M}_\sigma^+(M)$, $\mathfrak{M}_{\mathrm{f}}^\pm(M)$ and $\mathfrak{M}_\sigma^\pm(M)$, we intend the spaces of Borel measures on M which are finite, σ-finite, signed and finite, as well as signed and σ-finite, respectively. Here, σ-finiteness of $\mu \in \mathfrak{M}_\sigma^\pm(M)$ refers to the existence of an increasing sequence of *open* subsets of M on whose elements μ is finite. The subscripts R, such as in $\mathfrak{M}_{\mathrm{fR}}^+(M)$, or $\mathscr{E}$, such as in $\mathfrak{M}_{\mathrm{f}}^+(M)_{\mathscr{E}}$, indicate the respective subclass of (signed) measures which are Radon or do not charge $\mathscr{E}$-polar sets, see below.

Given any $\mu \in \mathfrak{M}_\sigma^\pm(M)$, denote by $\mu^+, \mu^- \in \mathfrak{M}_\sigma^+(M)$ the positive and the negative parts of μ in its Jordan decomposition. Note that μ^+ or μ^- is finite — hence $\mu^+ - \mu^-$ is well-defined — while they are both finite if $\mu \in \mathfrak{M}_{\mathrm{f}}^\pm(M)$ [Hal50, Thm. 29.B].

The *total variation* $|\mu| \in \mathfrak{M}_\sigma^+(M)$ of $\mu \in \mathfrak{M}_\sigma^\pm(M)$ is defined by

$$|\mu| := \mu^+ + \mu^-.$$

This gives rise to the *total variation norm*

$$\|\mu\|_{\mathrm{TV}} := |\mu|[M]$$

of μ. Note that $(\mathfrak{M}_\mathrm{f}^\pm(M), \|\cdot\|_{\mathrm{TV}})$ is indeed a normed vector space.

In this notation, given a not necessarily nonnegative $\nu \in \mathfrak{M}_\sigma^\pm(M)$, we write $\nu \ll \mu$ if $\nu \ll |\mu|$, or equivalently $|\nu| \ll |\mu|$ [Hal50, Thm. 30.A], for *absolute continuity*. Note that if $\mu, \nu \in \mathfrak{M}_\mathrm{f}^+(M)$ are *singular* to each other, written $\mu \perp \nu$, then

$$|\mu + \nu| = |\mu| + |\nu|.$$

Lastly, recall that given any $\mu, \nu \in \mathfrak{M}_\sigma^\pm(M)$, μ admits a *Lebesgue decomposition* w.r.t. ν [Hal50, Thm. 32.C] — that is, there exist unique $\mu_\ll, \mu_\perp \in \mathfrak{M}_\sigma^\pm(M)$ whose sum is well-defined and with the property that

$$\begin{aligned} \mu_\ll &\ll \nu, \\ \mu_\perp &\perp \nu, \\ \mu &= \mu_\ll + \mu_\perp. \end{aligned}$$

Functions Let $L_0(M)$ and $L_\infty(M)$ be the spaces of real-valued and bounded real-valued $\mathscr{B}(M)$-measurable functions defined everywhere on M. Write $\mathrm{SF}(M)$ for the space of *simple functions*, i.e. of those $f \in L_\infty(M)$ with finite range. $\mathrm{C}(M)$ and $\mathrm{C_b}(M)$ designate the spaces of (bounded) continuous functions $f\colon M \to \mathbf{R}$.

Let μ be a Borel measure on M. Let $L^0(M,\mu)$ be the real vector space of equivalence classes of elements in $L_0(M)$ w.r.t. μ-a.e. equality. We mostly make neither notational nor descriptional distinction between (μ-a.e. properties of) functions $f \in L^0(M,\mu)$ and (properties of) its equivalence class $[f]_\mu \in L^0(M,\mu)$. In the only case where this difference matters, i.e. when speaking about quasi-notions in Chapter 1 and Chapter 3, we use the distinguished notations $\widetilde{f}$ or $f_\sim$.

The *support* of $f \in L^0(M,\mu)$ is defined as

$$\mathrm{spt}_\mu f := \mathrm{spt}(|f|\,\mu)$$

and we briefly write spt for $\mathrm{spt}_\mathfrak{m}$. Let $L^0_\mathrm{c}(M,\mu)$ be the class of $f \in L^0(M,\mu)$ such that $\mathrm{spt}_\mu f$ is compact. We write $L^0(M)$ and $L^0_\mathrm{c}(M)$ for $L^0(M,\mathfrak{m})$ and $L^0_\mathrm{c}(M,\mu)$. Given any partition $(E_j)_{j\in\mathbf{N}}$ of M into Borel sets of finite and positive $\mathfrak{m}$-measure, it is a complete and separable metric space w.r.t. the metric $\mathsf{d}_{L^0(M)}$ defined through

$$\mathsf{d}_{L^0(M)}(f,g) := \sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]} \int_{E_j} \min\big\{|f-g|, 1\big\}\,\mathrm{d}\mathfrak{m}.$$

The induced topology on $L^0(M)$ does not depend on the choice of $(E_j)_{j\in\mathbf{N}}$ — indeed, $(f_n)_{n\in\mathbf{N}}$ is a Cauchy sequence w.r.t. $\mathsf{d}_{L^0(M)}$ if and only if it is Cauchy w.r.t. convergence in $\mathfrak{m}$-measure on any Borel set $B \subset M$ with $\mathfrak{m}[B] < \infty$. Similar facts hold — and definitions are used — for the space $L^0(M^2)$ of $\mathfrak{m}^{\otimes 2}$-measurable $f\colon M^2 \to \mathbf{R}$.

Given $p \in [1,\infty]$ we denote the (local) p-th order Lebesgue spaces w.r.t. μ by $L^p(M,\mu)$, with the usual norm $\|\cdot\|_{L^p(M,\mu)}$, and $L^p_\mathrm{loc}(M,\mu)$. We always abbreviate $L^p(M,\mathfrak{m})$ and $L^p_\mathrm{loc}(M,\mathfrak{m})$ by $L^p(M)$ and $L^p_\mathrm{loc}(M)$, respectively.

If (M, d) is a metric space, we write $\mathrm{Lip}(M)$ for the space of real-valued Lipschitz functions w.r.t. d. $\mathrm{Lip}_{\mathrm{b}}(M)$ and $\mathrm{Lip}_{\mathrm{bs}}(M)$ are the classes of bounded and boundedly supported elements in $\mathrm{Lip}(M)$. The Lipschitz constant of an $f \in \mathrm{Lip}(M)$ is

$$\mathrm{Lip}(f) := \sup_{\substack{x,y \in M,\\ x \neq y}} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)}.$$

For $i \in \{1, 2\}$, the projection maps $\mathrm{pr}_i \colon M^2 \to M$ are given by

$$\mathrm{pr}_i(x_1, x_2) := x_i.$$

Similarly, we define $\mathrm{pr}_i \colon M^3 \to M$, $i \in \{1, 2, 3\}$.

Chapter One

Optimal transport, gradient estimates, and pathwise Brownian coupling on spaces with variable Ricci bounds

This chapter is based on the author's joint work [BHS21] with Karen Habermann and Karl-Theodor Sturm, from which large parts are taken over *verbatim*.

Throughout this chapter, the triple $(M, \mathsf{d}, \mathfrak{m})$ is a metric measure space according to Section 0.1. Let $\kappa\colon M \to \mathbf{R}$ be a lower semicontinuous function which is bounded from below. We say that $(M, \mathsf{d}, \mathfrak{m})$ is an RCD space if it satisfies the $\mathrm{RCD}(K, \infty)$ condition for some $K \in \mathbf{R}$. This will be our standing assumption throughout. We refer to Section 1.2 below for technical details about RCD spaces which are needed in this chapter and have not yet been mentioned in Section 0.1.

If it exists, the limit $|\dot{\gamma}_t| := \lim_{h\to 0} \mathsf{d}(\gamma_{t+h}, \gamma_t)/|h|$ is called *metric speed* of the curve $\gamma \in \mathrm{C}([0,1]; M)$ at $t \in [0,1]$, and we write $|\dot{\gamma}|$ if $|\dot{\gamma}_t| = |\dot{\gamma}_s|$ for every $s, t \in [0,1]$. Moreover, $\mathrm{Geo}(M)$ denotes the space of *geodesics* on M, i.e. the set of all $\gamma \in \mathrm{C}([0,1]; M)$ with $\mathsf{d}(\gamma_t, \gamma_s) = |t-s|\, \mathsf{d}(\gamma_0, \gamma_1)$ for every $s, t \in [0,1]$. Similarly, we define $\mathrm{Geo}(\mathscr{P}_p(M))$ as the space of W_p-geodesics in the space of probability measures, where $\mathscr{P}_p(M)$ and W_p are defined in Section 0.1. We say that $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(M))$ *represents* the W_p-geodesic $(\mu_t)_{t\in[0,1]}$ if $\mu_t = (\mathsf{e}_t)_\sharp \boldsymbol{\pi}$ for all $t \in [0,1]$, where $\mathsf{e}_t\colon \mathrm{C}([0,1]; M) \to M$ is the evaluation map defined by $\mathsf{e}_t(\gamma) := \gamma_t$. By [Lis07, Thm. 6], every W_p-geodesic can be represented by some $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(M))$, $p \in (1, \infty)$.

1.1 Main result and definitions

This chapter presents various synthetic approaches to the notion of "Ricci curvature at every $x \in M$ bounded from below by $\kappa(x)$" and proves their equivalence. These characterizations are suitable extensions, concretized after the main Theorem 1.1.1, of the curvature-dimension condition, the evolution variational inequality, Bochner's inequality, gradient estimates and transport estimates to nonconstant curvature bounds. To this list, we add a description in terms of pathwise coupling of Brownian motions. In total, our main result is the following.

Theorem 1.1.1. *Let* $(M, \mathsf{d}, \mathfrak{m})$ *be an* RCD *space, and let* $\kappa\colon M \to \mathbf{R}$ *be a lower semicontinuous, lower bounded function. For all exponents* $p \in (1, \infty)$ *and* $q \in [1, \infty)$*, the following properties are equivalent:*

(i) *the curvature-dimension condition* $\mathrm{CD}(\kappa, \infty)$*,*

(ii) *the evolution variational inequality* $\mathrm{EVI}(k)$,
(iii) *the q-Bochner inequality* $\mathrm{BE}_q(k,\infty)$,
(iv) *the q-gradient estimate* $\mathrm{GE}_q(k)$,
(v) *the p-transport estimate* $\mathrm{PTE}_p(k)$, *and*
(vi) *the pathwise coupling property* $\mathrm{PCP}(k)$.

Moreover, any of these properties yields (iii), (iv) *and* (v) *for every* $p, q \in [1,\infty)$.

Lagrangian formulation Here and in the sequel, $\mathrm{g}\colon [0,1]^2 \to [0,1]$ denotes the Green's function $\mathrm{g}(s,t) := \min\{s(1-t), t(1-s)\}$ of the unit interval $[0,1]$. Moreover, define the *Boltzmann entropy* $\mathrm{Ent}_{\mathfrak{m}}\colon \mathscr{P}(M) \to [-\infty,\infty]$ as

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \displaystyle\int_M \rho\log\rho\,\mathrm{d}\mathfrak{m} & \text{if } \mu \ll \mathfrak{m},\ \mu = \rho\,\mathfrak{m},\\ \infty & \text{otherwise.}\end{cases}$$

We set $\mathscr{D}(\mathrm{Ent}_{\mathfrak{m}}) := \{\mu \in \mathscr{P}(M) : \mathrm{Ent}_{\mathfrak{m}}(\mu) \in \mathbf{R}\}$. For technically very precise definitions of $\mathrm{Ent}_{\mathfrak{m}}$, see [Stu06a, AG+15].

The next two definitions are given in [Stu15, Def. 3.2, Def. 3.3].

Definition 1.1.2. *An* RCD *space* $(M,\mathsf{d},\mathfrak{m})$ *is said to satisfy the* curvature-dimension condition *with variable curvature bound k, briefly* $\mathrm{CD}(k,\infty)$, *if for every $\mu_0,\mu_1 \in \mathscr{P}_2(M)\cap\mathscr{D}(\mathrm{Ent}_{\mathfrak{m}})$ there exists a measure $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(M))$ representing some W_2-geodesic $(\mu_t)_{t\in[0,1]}$ connecting μ_0 and μ_1 such that, for every $t \in [0,1]$,*

$$\begin{aligned}\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \leq (1-t)\,\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1)\\ - \int_0^1\int_{\mathrm{Geo}(M)} \mathrm{g}(s,t)\,k(\gamma_s)\,|\dot\gamma|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}s.\end{aligned}$$

Definition 1.1.3. *An* RCD *space* $(M,\mathsf{d},\mathfrak{m})$ *is said to satisfy the* evolution variational inequality *with variable curvature bound k, briefly* $\mathrm{EVI}(k)$, *if for every $\mu_0 \in \mathscr{P}_2(M)$ there exists a locally absolutely continuous curve $(\mu_t)_{t>0}$ in $\mathscr{P}_2(M)\cap\mathscr{D}(\mathrm{Ent}_{\mathfrak{m}})$ with $W_2(\mu_t,\mu_0) \to 0$ as $t \to 0$, and for every $t > 0$ and every $\nu \in \mathscr{P}_2(M)$ there exists $\boldsymbol{\pi}_t \in \mathscr{P}(\mathrm{Geo}(M))$ representing some W_2-geodesic connecting μ_t and ν such that*

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu) + \int_0^1\int_{\mathrm{Geo}(M)} (1-s)\,k(\gamma_s)\,|\dot\gamma|^2\,\mathrm{d}\boldsymbol{\pi}_t(\gamma)\,\mathrm{d}s \leq \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu_t).$$

From [Stu15, Thm. 3.4], it is already known that $\mathrm{CD}(k,\infty)$ is equivalent to $\mathrm{EVI}(k)$ on RCD spaces, which establishes the equivalence of (i) and (ii) in Theorem 1.1.1.

Eulerian formulation Let us now switch to the Eulerian picture which, to shorten the presentation, is directly presented for arbitrary exponents. Recall the definition of the Cheeger energy $\mathscr{E}$ with domain $\mathscr{D}(\mathscr{E}) := W^{1,2}(M)$, carré du champ a.k.a. squared minimal weak upper gradient $\Gamma := |\mathrm{d}\cdot|^2$ and generator Δ from Subsection 0.1.3.

Definition 1.1.4. *Given $q \in [1,\infty)$, we say that an* RCD *space* $(M,\mathsf{d},\mathfrak{m})$ *satisfies the* q-Bochner inequality *or* q-Bakry–Émery estimate *with variable curvature bound k, briefly* $\mathrm{BE}_q(k,\infty)$, *if*

$$\int_M \Big[\frac{1}{q}\Gamma(f)^{q/2}\,\Delta\phi - \Gamma(f)^{q/2-1}\,\Gamma(f,\Delta f)\,\phi\Big]\,\mathrm{d}\mathfrak{m} \geq \int_M k\,\Gamma(f)^{q/2}\,\phi\,\mathrm{d}\mathfrak{m}$$

holds for every $f \in \mathscr{D}(\Delta)$ with $\Delta f \in W^{1,2}(M)$ as well as $\Gamma(f) \in L^\infty(M)$ and for every nonnegative $\phi \in \mathscr{D}(\Delta) \cap L^\infty(M)$ with $\Delta\phi \in L^\infty(M)$.

The equivalence of (i) and (iii) for $q = 2$ in Theorem 1.1.1 states that the variable Eulerian and Lagrangian approaches to synthetic lower Ricci bounds coincide, i.e. $\mathrm{CD}(k, \infty)$ is equivalent to $\mathrm{BE}_2(k, \infty)$. For constant k, this is due to the groundbreaking works of Ambrosio, Gigli and Savaré, see [AGS14b, Thm. 6.2], which follows [GKO13], for (i) implying (iii), and [AGS15, Thm. 1.1] for (iii) implying (i). In the nonconstant case, this remained open in previous works [Ket15a, Ket17, Stu15].

The implication from $\mathrm{BE}_2(k, \infty)$ to $\mathrm{CD}(k, \infty)$ follows from Theorem 1.3.4 and Theorem 1.4.5. The proof of the converse is a consequence of Proposition 1.4.6, Proposition 1.5.6, Theorem 1.5.19 and eventually Theorem 1.3.4. This requires a detailed heat flow analysis, both at the level of functions and measures (cf. Section 1.2), and in particular an extension of Kuwada's duality [Kuw10, Thm. 2.2] between q-gradient estimates and p-transport estimates for dual $p, q \in (1, \infty)$. This is quite demanding — indeed, until now not even a formulation of an appropriate p-transport estimate with nonconstant curvature bound existed.

The "self-improvement property" of the q-Bochner inequality will be another key result. Indeed, the $\mathrm{BE}_q(k, \infty)$ condition is *independent* of q, see Theorem 1.3.6, which provides the equivalence of (i) and (iii) in Theorem 1.1.1 for general $q \in [1, \infty)$.

Improved gradient estimates Following [Stu15], see Subsection 3.2.6 and [CF12] for more details, let $(\mathsf{P}_t^{qk})_{t \geq 0}$ be the Schrödinger semigroup on $L^2(M)$ associated to the generator $\Delta - qk$ for $q \in [1, \infty)$. It extends to a semigroup on $L^r(M)$ for every $r \in [1, \infty]$, strongly continuous if $r < \infty$ and weakly* continuous if $r = \infty$. In terms of Brownian motion b^x on M starting in $x \in M$, for every $f \in L^r(M)$, $r \in [1, \infty]$, it can be expressed through the Feynman–Kac formula

$$\mathsf{P}_t^{qk} f(x) = \mathbf{E}\big[\mathrm{e}^{-\int_0^{2t} qk(\mathsf{b}_r^x)/2\,\mathrm{d}r}\, f(\mathsf{b}_{2t}^x)\big]. \tag{1.1.1}$$

Definition 1.1.5. *We say that a* q-gradient estimate *with variable curvature bound k, briefly $\mathrm{GE}_q(k)$, holds if for every $f \in W^{1,2}(M)$ and every $t \geq 0$,*

$$\Gamma(\mathsf{P}_t f)^{q/2} \leq \mathsf{P}_t^{qk}\big(\Gamma(f)^{q/2}\big) \quad \mathfrak{m}\text{-a.e.}$$

Adapting the arguments for constant Ricci bounds from [BE85, Sav14], one can prove, cf. Theorem 1.3.4, that $\mathrm{BE}_q(k, \infty)$ holds if and only if $\mathrm{GE}_q(k)$ is satisfied. This yields the equivalence of (iii) and (iv) in Theorem 1.1.1 for general $q \in [1, \infty)$.

Variable transport estimates In order to formulate a dual p-transport estimate for $p \in [1, \infty)$, we consider evolutions on the product space M^2. Denoting by $\mathrm{G}_\varepsilon(x, y)$, $\varepsilon \geq 0$, the set of geodesics $\gamma \in \mathrm{Geo}(M)$ with $\gamma_0 \in \overline{B}_\varepsilon(x)$ and $\gamma_1 \in \overline{B}_\varepsilon(y)$, we introduce the function $\underline{k}\colon M^2 \to \mathbf{R}$ defined by

$$\underline{k}(x, y) := \lim_{\varepsilon \to 0} \inf_{\gamma \in \mathrm{G}_\varepsilon(x,y)} \int_0^1 k(\gamma_s)\,\mathrm{d}s. \tag{1.1.2}$$

Its properties are summarized in Section 1.2. As we will see in Remark 1.5.12, Theorem 1.6.1 and Theorem 1.5.17, it turns out that $\underline{k}$ can equivalently be replaced in all relevant quantities in Theorem 1.1.1 by the larger function $\overline{k}\colon M^2 \to \mathbf{R}$, where

$$\overline{k}(x, y) := \liminf_{\substack{x' \to x,\\ y' \to y}} \sup_{\gamma \in \mathrm{G}_0(x', y')} \int_0^1 k(\gamma_s)\,\mathrm{d}s. \tag{1.1.3}$$

Definition 1.1.6. *A pair* $(\mathsf{b}^1, \mathsf{b}^2)$ *of stochastic processes* $\mathsf{b}^i := (\mathsf{b}^i_t)_{t\geq 0}$*,* $i \in \{1,2\}$*, on* M *is called* coupling of Brownian motions *if it is defined on a common probability space* $(\Omega, \mathscr{A}, \mathbf{P})$ *and each of the processes* b^1 *and* b^2 *is a Brownian motion, in the sense of Section 1.2 below, on* M*.*

The common probability space $(\Omega, \mathscr{A}, \mathbf{P})$ — in particular the measure $\mathbf{P}$ — will always be implicitly fixed when we speak about pairs of coupled Brownian motions.

Given $\mu_1, \mu_2 \in \mathscr{P}_p(M)$, we define the *perturbed p-transport cost* at time $t \geq 0$ by

$$W_p^{\underline{k}}(\mu_1, \mu_2, t) := \inf \mathbf{E}\big[\mathrm{e}^{\int_0^{2t} p\underline{k}(\mathsf{b}^1_r, \mathsf{b}^2_r)/2\,\mathrm{d}r}\,\mathsf{d}^p\big(\mathsf{b}^1_{2t}, \mathsf{b}^2_{2t}\big)\big]^{1/p},$$

where the infimum is taken over all pairs $(\mathsf{b}^1, \mathsf{b}^2)$ of coupled Brownian motions on M, restricted to $[0, 2t]$ and modeled on some common probability space, with initial distributions μ_1 and μ_2, respectively. Note that $W_p^{\underline{k}}(\mu_1, \mu_2, 0) = W_p(\mu_1, \mu_2)$ and that for general $t \geq 0$, if k is constant, say $k = K$, the perturbed p-transport cost can be expressed in terms of the usual p-transport cost, cf. Lemma 1.5.1, through

$$W_p^{\underline{k}}(\mu_1, \mu_2, t) = \mathrm{e}^{Kt}\, W_p(\mathscr{H}_t \mu_1, \mathscr{H}_t \mu_2).$$

Definition 1.1.7. *Given any* $p \in [1, \infty)$*, we say that a* p-transport estimate *with variable curvature bound* k*, briefly* $\mathrm{PTE}_p(k)$*, holds if the map* $t \mapsto W_p^{\underline{k}}(\mu_1, \mu_2, t)$ *is nonincreasing on* $[0, \infty)$ *for every pair* $\mu_1, \mu_2 \in \mathscr{P}_p(M)$*.*

Having at our disposal appropriate replacements for the terms $\mathrm{e}^{-qKt}\,\mathsf{P}_t\big(\Gamma(f)^{q/2}\big)$ as well as $\mathrm{e}^{Kt}\, W_p(\mathscr{H}_t \mu_1, \mathscr{H}_t \mu_2)$ in terms of Feynman–Kac formulas with potentials qk for Brownian motion on M and $-p\underline{k}$ for pairs of coupled Brownian motions on M^2, respectively (compare with Section 0.1), we are in a position to formulate and prove a generalization of the above indicated fundamental Kuwada duality in the case of nonconstant k. This addresses the equivalence of (iv) and (v) in Theorem 1.1.1 (for any dual exponents).

Theorem 1.1.8. *For every* $p, q \in (1, \infty)$, $1/p + 1/q = 1$*, the following are equivalent:*

- (iv) *the q-gradient estimate* $\mathrm{GE}_q(k)$*, and*
- (v) *the p-transport estimate* $\mathrm{PTE}_p(k)$*.*

This result is a consequence of Theorem 1.5.16 and Theorem 1.5.19. For both of these, we use a localization argument in regions where k or $\underline{k}$ are "approximately constant" and then use tail estimates for Brownian paths to control the remainder terms.

Suitable extensions to the case $q = 1$ and $p = \infty$ will be discussed, and eventually shown to be equivalent, in Theorem 1.5.10, Theorem 1.5.17 and Theorem 1.6.1. Therefore, making sense of an appropriate "$\mathrm{PTE}_p(k)$ condition for $p = \infty$" is the content of the paragraph after the next Remark 1.1.9.

Remark 1.1.9. We often use the characterization of $\mathrm{PTE}_p(k)$, which is zeroth-order in nature, by a first-order condition via the *differential p-transport inequality* in the spirit of the connection between $\mathrm{BE}_q(k, \infty)$ and $\mathrm{GE}_q(k)$: for every $x, y \in M$,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_0 W_p^p(\mathscr{H}_t \delta_x, \mathscr{H}_t \delta_y) \leq -p\,\underline{k}(x, y)\,\mathsf{d}^p(x, y).$$

The equivalence of $\mathrm{PTE}_p(k)$ and the previous estimate, which for constant k is essentially Gronwall's lemma and a standard coupling technique, is treated in Theorem 1.5.7.

A posteriori, for every $p \in (1,\infty)$, any of the conditions (i) to (vi) in Theorem 1.1.1 will indeed give the stronger estimate

$$\frac{\mathrm{d}^+}{\mathrm{d}t} W_p^p(\mathscr{H}_t\mu_1, \mathscr{H}_t\mu_2) \leq -p \int_0^1 \int_{\mathrm{Geo}(M)} k(\gamma_s)\,|\dot\gamma|^p \,\mathrm{d}\boldsymbol{\pi}_t(\gamma)\,\mathrm{d}s$$

for every $t \geq 0$, where $\mu_1, \mu_2 \in \mathscr{P}(M)$ have finite W_p-distance to each other, and $\boldsymbol{\pi}_t \in \mathscr{P}(\mathrm{Geo}(M))$ is an *arbitrary* measure representing a W_p-geodesic from $\mathscr{H}_t\mu_1$ to $\mathscr{H}_t\mu_2$, see Corollary 1.5.11. ■

Pathwise coupling of Brownian motions Finally, we reinforce the p-transport estimate from Definition 1.1.7 by passing to the limit $p \to \infty$ and by replacing the mean value estimates by a pathwise one.

Definition 1.1.10. *We say that the* pathwise coupling property *with variable curvature bound k, briefly* PCP(k)*, holds if for every $\mu_1, \mu_2 \in \mathscr{P}(M)$ there exists a pair* $(\mathsf{b}^1, \mathsf{b}^2)$ *of coupled Brownian motions on M with initial distributions μ_1 and μ_2, respectively, such that* **P***-a.s., we have*

$$\mathsf{d}(\mathsf{b}_t^1, \mathsf{b}_t^2) \leq \mathrm{e}^{-\int_s^t \underline{k}(\mathsf{b}_r^1, \mathsf{b}_r^2)/2\,\mathrm{d}r}\, \mathsf{d}(\mathsf{b}_s^1, \mathsf{b}_s^2)$$

for every $s, t \in [0,\infty)$ with $s \leq t$.

By [ACT11, Thm. 4.1], complete Riemannian manifolds with Ricci curvature bounded from below by $K \in \mathbf{R}$ satisfy PCP(k) with constant $k = K$. [Stu15, Thm. 2.9] extended this to general $\mathrm{RCD}(K,\infty)$ spaces. A first result into the nonconstant direction is due to [Vey11, Thm. 6]. Again on Riemannian manifolds with a uniform lower bound on the Ricci curvature, it claims the existence of a pair $(\mathsf{b}^1, \mathsf{b}^2)$ of coupled Brownian motions starting in (x, y) obeying for every $t \geq 0$, on the event that $(\mathsf{b}_r^1, \mathsf{b}_r^2)$ does not belong to the cut-locus of M for every $r \in [0,t]$ (see Subsection 2.4.2 for detailed definitions about cut-loci), the pathwise estimate from Definition 1.1.10 holds *with equality* and $\underline{k}$ replaced by Ollivier's *coarse curvature* κ, where $\kappa(x,y) := -\mathrm{d}^+ \log W_1(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y)/\mathrm{d}t\big|_0$ for $x, y \in M$, $x \neq y$ [Vey11, Def. 10]. For x, y close to each other, say $y = \exp_x(\varepsilon\xi)$, where $\varepsilon > 0$ and $\xi \in T_xM$,

$$\kappa(x,y) = \mathrm{Ric}(x)(\xi,\xi) + \mathrm{o}(1) \quad \text{as } \varepsilon \downarrow 0,$$

see [Vey11, Thm. 19, Rem. 20]. The construction of this process deeply relies on smooth calculus tools, which are unavailable in our setting and thus cannot be adopted.

Theorem 1.1.1 extends these results in terms of $\underline{k}$ and circumvents regularity issues involving the variable curvature bound. Indeed, already in the setting of (weighted) Riemannian manifolds M, the definition (1.1.2) of $\underline{k}$ provides a convenient way to avoid cut-loci. See Chapter 2 where, in Theorem 2.1.6, we extend this equivalence of PCP(k) to lower boundedness of the Bakry–Émery Ricci tensor by k for continuous $k\colon M \to \mathbf{R}$ apart from any lower boundedness assumption on k.

In the setting of this chapter, the existence of a process satisfying the PCP(k) condition is equivalent to $\mathrm{CD}(k,\infty)$. Indeed, given $\mathrm{PTE}_p(k)$ for every large enough $p \in (1,\infty)$, we deduce PCP(k) by means of Theorem 1.6.1, the content of which is the implication from (v) to (vi) in Theorem 1.1.1. According to the previous Theorem 1.1.8 and nestedness of q-gradient estimates, see Lemma 1.3.3, the 1-gradient estimate $\mathrm{GE}_1(k)$ implies $\mathrm{PTE}_p(k)$ for every $p \in (1,\infty)$ and thus PCP(k). The converse of this, i.e. the implication from PCP(k) to $\mathrm{GE}_1(k)$, is addressed in Theorem 1.5.17.

1.2 Preliminaries

The Riemannian curvature-dimension condition We say that $(M, \mathsf{d}, \mathfrak{m})$ satisfies the *Riemannian curvature-dimension condition* $\mathrm{RCD}(k, \infty)$ if it is infinitesimally Hilbertian (recall Section 0.1) and obeys $\mathrm{CD}(k, \infty)$ according to Definition 1.1.2. As said, we always assume that $(M, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ space for some $K \in \mathbf{R}$. The value of K does not enter any of our results. Without restriction $k \geq K$ on M. Indeed, one should think of k as being much larger than K everywhere on M.

The $\mathrm{RCD}(K, \infty)$ assumption carries numerous important consequences for the metric measure space $(M, \mathsf{d}, \mathfrak{m})$. Further details on the subsequent results can be found in the cited references and [AG+15, AGS14a]. For more basic notions about the Dirichlet space part, we refer to [Sav14] or Subsection 3.2.2 below.

a. *Volume growth* [Stu06a, Thm. 4.24]. For every $z \in M$ there exists a constant $C > 0$ such that $\mathfrak{m}[B_r(z)] \leq \mathrm{e}^{Cr^2}$ for every $r > 0$.
b. *Nondegeneracy of entropy* [AG+15, p. 4665]. $\mathrm{Ent}_{\mathfrak{m}}$ is well-defined and does not attain the value $-\infty$ on $\mathscr{P}_2(M)$.
c. *Uniqueness of W_2-geodesics* [RS14, Cor. 1.4]. For every pair of $\mathfrak{m}$-absolutely continuous measures $\mu_0, \mu_1 \in \mathscr{P}_2(M)$, there exists a unique W_2-geodesic connecting them.
d. *Heat flow regularity* [AGS14b, Thm. 6.1, Thm. 6.8]. The heat flow $(\mathsf{P}_t)_{t\geq 0}$ from Section 0.1 can be chosen to be strong Feller, i.e. P_t maps $L^\infty(M)$ to $\mathrm{Lip}(M)$, $t > 0$, with $\mathrm{Lip}(\mathsf{P}_t f) \leq \|f\|_{L^\infty(M)}/\sqrt{t}$ if $K = 0$, while if $K \neq 0$, for every $f \in L^\infty(M)$, we have

$$\mathrm{Lip}(\mathsf{P}_t f)^2 \leq \frac{K}{\mathrm{e}^{2Kt} - 1} \big\|f\big\|^2_{L^\infty(M)}. \tag{1.2.1}$$

The semigroup $(\mathsf{P}_t)_{t\geq 0}$ is in duality with the semigroup $(\mathscr{H}_t)_{t\geq 0}$ defined as the gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ in $\mathscr{P}_2(M)$ and extended to $\mathscr{P}(M)$ by continuity, i.e. for every $f \in \mathrm{C_b}(M)$ and every $\mu \in \mathscr{P}(M)$,

$$\int_M f \,\mathrm{d}\mathscr{H}_t\mu = \int_M \mathsf{P}_t f \,\mathrm{d}\mu.$$

Hence, $\mathscr{H}_t(g\,\mathfrak{m}) = (\mathsf{P}_t g)\,\mathfrak{m}$ for every nonnegative $g \in L^1(M)$, $\|g\|_{L^1(M)} = 1$.
e. *Uniqueness of* EVI *curves* [AGS14b, Thm. 5.1], [Stu15, Thm. 3.4]. Every curve $(\mu_t)_{t\geq 0}$ in $\mathscr{P}_2(M)$ satisfying the obstructions from Definition 1.1.3 with arbitrary choice of $k \geq K$ necessarily coincides with the heat flow $(\mathscr{H}_t\mu_0)_{t\geq 0}$ starting at μ_0.
f. *Brownian motion* [AGS14b, Thm. 6.5]. For every $\mu \in \mathscr{P}(M)$, there exists a conservative Markov process $(\mathbf{P}, \mathsf{b})$, or briefly b, unique in law, with continuous sample paths and transition semigroup given by

$$\mathbf{E}\big[f(\mathsf{b}_{t+s})\,\big|\,\mathsf{b}_s\big] = \mathsf{P}_{t/2} f(\mathsf{b}_s)$$

for every $s, t \in [0, \infty)$ and every $f \in \mathrm{C_b}(M)$, and with $(\mathsf{b}_0)_\sharp\mathbf{P} = \mu$. This process is called *the* Brownian motion on M with initial distribution μ.
g. *Test functions* [Sav14, Sec. 3.2]. The set

$$\mathrm{Test}(M) := \big\{f \in \mathscr{D}(\Delta) \cap L^\infty(M) : \Gamma(f) \in L^\infty(M),\ \Delta f \in W^{1,2}(M)\big\}$$

is a core for $\mathscr{E}$ and an algebra w.r.t. pointwise multiplication.

h. *Twice differentiability* [DGP21, Lem. 3.5], [Gig18, Cor. 3.3.9]. We have $\Gamma(f)^{1/2} \in W^{1,2}(M)$ for every $f \in \mathscr{D}(\Delta)$ and

$$\mathscr{E}\big(\Gamma(f)^{1/2}\big) \le \big\|\Delta f\big\|^2_{L^2(M,\mathfrak{m})} - K\,\mathscr{E}(f).$$

i. *Sobolev-to-Lipschitz property* [AGS14b, Thm. 6.2]. Every $f \in W^{1,2}(M)$ with $|\mathrm{d}f| \in L^\infty(M)$ has a Lipschitz representative $\overline{f}$ with $\mathrm{Lip}(\overline{f}) \le \||\mathrm{D}f|\|_{L^\infty(M)}$.

Hopf–Lax semigroup For later use, we summarize the main properties of the p-Hopf–Lax (or Hamilton–Jacobi) semigroup $(Q_s)_{s\ge 0}$, $p \in (1,\infty)$. A detailed account on this topic in general metric spaces can be found in [AGS13, AGS14a, GRS15].

Given any $f \in \mathrm{Lip}(M)$, its p-Hopf–Lax evolution $(Q_s f)_{s\ge 0}$ is defined by

$$Q_s f(x) := \begin{cases} \inf\limits_{y\in M} \Big\{ f(y) + \dfrac{\mathsf{d}^p(x,y)}{p s^{p-1}} \Big\} & \text{if } s > 0,\\ f(x) & \text{otherwise.}\end{cases}$$

The map $s \mapsto Q_s f$ belongs to $\mathrm{Lip}([0,\infty);\mathrm{C}(M))$ [AGS13, Prop. 3.3]. We also have $Q_s f \in \mathrm{Lip}(M)$ with $\mathrm{Lip}(Q_s f) \le p\,\mathrm{Lip}(f)$ for every $s > 0$ [AGS13, Prop. 3.4]. Denoting by $q \in (1,\infty)$ the dual exponent to p and by lip the local Lipschitz slope acting on $\mathrm{Lip}(M)$, for every $x \in M$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} Q_s f(x) + \frac{1}{q}\mathrm{lip}(Q_s f)^q(x) \le 0$$

for all but at most countably many $s > 0$ [AGS13, Thm. 3.5, Thm. 3.6]. Equality holds if (M,d) is geodesic, which here follows from the RCD condition [AG13, Rem. 2.14].

In terms of the p-Hopf–Lax semigroup, we have the following form of the duality for the p-Kantorovich–Wasserstein distance [Kuw10, Vil09]: for every $\mu,\nu \in \mathscr{P}(M)$,

$$\frac{1}{p} W_p^p(\mu,\nu) = \sup\Big\{ \int_M Q_1 f \,\mathrm{d}\mu - \int_M f\,\mathrm{d}\nu : f \in \mathrm{Lip_b}(M)\Big\}. \tag{1.2.2}$$

The function $\underline{k}$ and Lipschitz approximation Recall that k is lower semicontinuous and bounded from below by K, and so is $\underline{k}$ by construction. If k is also bounded from above, say by $C \in \mathbf{R}$, then so is $\underline{k}$. By reparameterization of geodesics, we get $\underline{k}(x,y) = \underline{k}(y,x)$ for every $x,y \in M$. Note that k can be reconstructed from $\underline{k}$, since $k(x) = \underline{k}(x,x)$. Lastly, the function $\underline{k}$ defined in (1.1.2) is the pointwise monotone limit from below of a sequence $(\underline{k}_n)_{n\in\mathbf{N}}$ of bounded Lipschitz functions on M^2, and so is the function k by considering $\underline{k}_n$, $n \in \mathbf{N}$, on the diagonal. Here, we intend Lipschitz continuity on M^2 w.r.t. the product metric d_{M^2} defined by $\mathsf{d}_{M^2}\big((x,y),(x',y')\big) := \big[\mathsf{d}^2(x,x') + \mathsf{d}^2(y,y')\big]^{1/2}$. This fact will be used frequently. Following [AGS08], we can, for instance, define $\underline{k}_n : M^2 \to \mathbf{R}$, $n \in \mathbf{N}$, by

$$\underline{k}_n(x,y) := \inf\big\{ \min\{\underline{k}(x',y'),n\} + n\,\mathsf{d}_{M^2}\big((x,y),(x',y')\big) : x',y' \in M\big\}.$$

Lemma 1.2.1. *The above functions $\underline{k}_n$, $n \in \mathbf{N}$, have the following properties.*

(i) *For every $n \in \mathbf{N}$, the function $\underline{k}_n$ is Lipschitz on M^2 with $\mathrm{Lip}(\underline{k}_n) \le n$.*
(ii) *For every $x \in M$ and every $n \in \mathbf{N}$, we have $K \le \underline{k}_n(x) \le \underline{k}_{n+1}(x) \le n+1$.*
(iii) *The sequence $(\underline{k}_n)_{n\in\mathbf{N}}$ converges pointwise from below to $\underline{k}$.*

1.3 Gradient estimates, Bochner's inequality, and their self-improvements

In this section, we adapt the well-known arguments from [Bak85, BE85, Sav14] for constant curvature lower bounds to derive the equivalence of the q-Bochner inequality with the q-gradient estimate with exponent $q \in [1, \infty)$. Moreover, we prove that these properties are independent of q.

Up to replacing $\mathscr{k}$ by $\mathscr{k}_n := \min\{\mathscr{k}, n\}$, $n \in \mathbf{N}$, we may and will assume throughout this section that $\mathscr{k}$ is bounded. In the general case, each of the subsequent results still holds for $\mathscr{k}$ since $\mathrm{BE}_q(\mathscr{k}, \infty)$ and $\mathrm{GE}_q(\mathscr{k})$ trivially imply $\mathrm{BE}_q(\mathscr{k}_n, \infty)$ and $\mathrm{GE}_q(\mathscr{k}_n)$ for every $n \in \mathbf{N}$, respectively, and conversely, if $\mathrm{BE}_q(\mathscr{k}_n, \infty)$ and $\mathrm{GE}_q(\mathscr{k}_n)$ hold for every $n \in \mathbf{N}$, Levi's theorem implies $\mathrm{BE}_q(\mathscr{k}, \infty)$ and $\mathrm{GE}_q(\mathscr{k})$, respectively.

1.3.1 Equivalence of Bochner and gradient estimate

First, we review the measure-valued Laplacian $\mathbf{\Delta}$ and the measure-valued Γ_2-operator $\mathbf{\Gamma}_2$ as introduced and analyzed in [Gig15, Sav14], defined by means of

$$\begin{aligned} \int_M g \, \mathrm{d}\mathbf{\Delta} u &:= -\int_M \Gamma(g, u) \, \mathrm{d}\mathfrak{m}, \qquad (1.3.1) \\ \mathbf{\Gamma}_2(f) &:= \mathbf{\Delta}\frac{\Gamma(f)}{2} - \Gamma(f, \Delta f)\, \mathfrak{m} \end{aligned}$$

for every $g \in \mathrm{Lip}_{\mathrm{bs}}(M)$ and suitable $u, f \in W^{1,2}(M)$. We write $u \in \mathscr{D}(\mathbf{\Delta})$ if the signed measure $\mathbf{\Delta} u$ exists, which is then uniquely determined by (1.3.1) and does not charge sets of zero $\mathscr{E}$-capacity. We denote the density of the $\mathfrak{m}$-absolutely continuous part of $\mathbf{\Gamma}_2(f)$ by $\gamma_2(f)$. The singular part of $\mathbf{\Gamma}_2(f)$ w.r.t. $\mathfrak{m}$ is nonnegative. If $f \in \mathrm{Test}(M)$, we have $\Gamma(f) \in W^{1,2}(M)$, and $\mathbf{\Gamma}_2(f)$ is well-defined and has finite total variation [Sav14, Lem. 3.2]. In this case, (1.3.1) holds for every $g \in W^{1,2}(M)$. Lastly, a consequence of generic calculus rules of Γ, cf. Proposition 3.2.9 below, is the following chain rule for $\mathbf{\Delta}$ proven in [BHS21, Lem. 3.1].

Lemma 1.3.1. *Fix $u \in \mathscr{D}(\mathbf{\Delta}) \cap L^\infty(M)$, an interval $I \subset \mathbf{R}$ with $0 \in I$ containing the image of u, and a function $\varphi \in \mathrm{C}^2(I)$ such that $\varphi(0) = 0$. Then $\varphi \circ u \in \mathscr{D}(\mathbf{\Delta})$ and*

$$\mathbf{\Delta}\big[\varphi \circ u\big] = \big[\varphi' \circ \widetilde{u}\big]\, \mathbf{\Delta} u + \big[\varphi'' \circ u\big]\, \Gamma(u)\, \mathfrak{m}.$$

Once $\mathrm{BE}_2(\mathscr{k}, \infty)$ holds, as in the proof of [Sav14, Lem. 3.2] we get

$$\begin{aligned} \mathscr{E}\big(\Gamma(f)\big) &\leq -\int_M \big[2\mathscr{k}\, \Gamma(f)^2 + \Gamma(f)\, \Gamma(f, \Delta f)\big] \, \mathrm{d}\mathfrak{m}, \\ \mathscr{k}\, \Gamma(f)\, \mathfrak{m} &\leq \mathbf{\Delta}\frac{\Gamma(f)}{2} - \Gamma(f, \Delta f)\, \mathfrak{m} \end{aligned}$$

for every $f \in \mathrm{Test}(M)$. Taking these estimates into account, one can argue exactly as in [Bak85, Prop. 1] and [Sav14, Thm. 3.4] to obtain that, for every $f \in \mathrm{Test}(M)$,

$$\Gamma(\Gamma(f)) \leq 4\big[\gamma_2(f) - \mathscr{k}\, \Gamma(f)\big]\, \Gamma(f) \quad \mathfrak{m}\text{-a.e.} \qquad (1.3.2)$$

Using this, we deduce the whole range of q-Bochner inequalities from $\mathrm{BE}_2(\mathscr{k}, \infty)$.

Proposition 1.3.2. *The condition $\mathrm{BE}_2(\mathscr{k}, \infty)$ implies $\mathrm{BE}_q(\mathscr{k}, \infty)$ for every $q \in [1, \infty)$.*

Proof. Fix $f \in \mathrm{Test}(M)$ and a nonnegative $\phi \in \mathscr{D}(\Delta) \cap L^\infty(M)$ with $\Delta\phi \in L^\infty(M)$. Given $\varepsilon > 0$, consider the function $\varphi_\varepsilon \in \mathrm{C}^\infty([0,\infty))$ with $\varphi_\varepsilon(r) := (r+\varepsilon)^{q/2} - \varepsilon^{q/2}$. Since $2 - q \leq 1$, we obtain

$$\begin{aligned}-\Gamma(\Gamma(f))\,\big[\varphi_\varepsilon'' \circ \Gamma(f)\big] &\leq \frac{q}{4}\Gamma(\Gamma(f))\,\big[\Gamma(f)+\varepsilon\big]^{q/2-2}\\ &\leq 2\big[\gamma_2(f) - \Bbbk\,\Gamma(f)\big]\,\big[\varphi_\varepsilon' \circ \Gamma(f)\big] \quad \mathfrak{m}\text{-a.e.}\end{aligned}$$

by means of (1.3.2). Multiplying this by ϕ and integrating, one gets

$$\begin{aligned}&-\int_M \Gamma(\Gamma(f))\,\big[\varphi_\varepsilon''\circ\Gamma(f)\big]\,\phi\,\mathrm{d}\mathfrak{m}\\ &\qquad\leq 2\int_M \big[\varphi_\varepsilon'\circ\Gamma(f)_\sim\big]\,\widetilde{\phi}\,\mathrm{d}\mathbf{\Gamma}_2(f) - 2\int_M \Bbbk\,\Gamma(f)\,\big[\varphi_\varepsilon'\circ\Gamma(f)\big]\,\phi\,\mathrm{d}\mathfrak{m}\\ &\qquad= \int_M \big[\varphi_\varepsilon'\circ\Gamma(f)_\sim\big]\,\widetilde{\phi}\,\mathrm{d}\mathbf{\Delta}\Gamma(f)\\ &\qquad\qquad - 2\int_M \big[\varphi_\varepsilon'\circ\Gamma(f)\big]\,\big[\Gamma(f,\Delta f) + \Bbbk\,\Gamma(f)\big]\,\phi\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Invoking Lemma 1.3.1 with $u := \Gamma(f)$, this amounts to

$$\begin{aligned}2\int_M \big[\varphi_\varepsilon'\circ\Gamma(f)\big]\,\big[\Gamma(f,\Delta f)+\Bbbk\,\Gamma(f)\big]\,\phi\,\mathrm{d}\mathfrak{m} &\leq \int_M \widetilde{\phi}\,\mathrm{d}\mathbf{\Delta}\big[\varphi_\varepsilon\circ\Gamma(f)\big]\\ &= \int_M \big[\varphi_\varepsilon\circ\Gamma(f)\big]\,\Delta\phi\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

By Lebesgue's theorem, letting $\varepsilon \downarrow 0$ in the preceding inequality gives the $\mathrm{BE}_q(\Bbbk,\infty)$ inequality for $f \in \mathrm{Test}(M)$.

To extend this to general $f \in \mathscr{D}(\Delta)$ with $\Delta f \in W^{1,2}(M)$ and $\Gamma(f) \in L^\infty(M)$, we approximate it in $W^{1,2}(M)$ by means of its heat flow regularizations $\mathsf{P}_t f_n \in \mathrm{Test}(M)$ as $n \to \infty$ and $t \downarrow 0$, where we set $f_n := \min\{n, \max\{-n, f\}\} \in W^{1,2}(M) \cap L^\infty(M)$, $n \in \mathbf{N}$. Since $\mathsf{P}_t f_n \to \mathsf{P}_t f$ in $W^{1,2}(M)$ as $n \to \infty$ for every $t > 0$, we have $\Gamma(\mathsf{P}_t f_n)^{q/2}\phi \to \Gamma(\mathsf{P}_t f)^{q/2}\phi$ in $L^1(M)$. Using (standard a priori estimates for) the heat flow on 1-forms, see [Gig18, Ch. 6] and also Lemma 3.5.15 and Theorem 3.6.23, we furthermore have, for every $t > 0$,

$$\lim_{n\to\infty}\Gamma(\mathsf{P}_t f_n)^{q/2-1}\,\Gamma(\mathsf{P}_t f_n, \Delta\mathsf{P}_t f_n) = \Gamma(\mathsf{P}_t f)^{q/2-1}\,\Gamma(\mathsf{P}_t f, \Delta\mathsf{P}_t f)$$

in $L^1(M)$. Now, since $\Gamma(\mathsf{P}_t f) \to \Gamma(f)$ and $\Gamma(\mathsf{P}_t f, \Delta\mathsf{P}_t f) \to \Gamma(f,\Delta f)$ in $L^1(M)$ as $t \downarrow 0$, $\Gamma(\mathsf{P}_t f)$ is uniformly bounded in $L^\infty(M)$ for small enough t, and $\Gamma(\Delta\mathsf{P}_t f)^{1/2}$ is uniformly bounded in $L^2(M)$ for small enough t, we easily get

$$\begin{aligned}\lim_{t\downarrow 0}\Gamma(\mathsf{P}_t f)^{q/2} &= \Gamma(f)^{q/2},\\ \lim_{t\downarrow 0}\Gamma(\mathsf{P}_t f)^{q/2-1}\,\Gamma(\mathsf{P}_t f, \Delta\mathsf{P}_t f) &= \Gamma(f)^{q/2-1}\,\Gamma(f,\Delta f)\end{aligned}$$

in $L^2(M)$ respectively $L^1(M)$. This yields the claim. □

By the Feynman–Kac representation (1.1.1) of $\mathsf{P}_t^{q\Bbbk}$, $t \geq 0$, and Jensen's inequality, the hierarchy from Lemma 1.3.3 is immediate. This and the above self-improvement property of $\mathrm{BE}_2(\Bbbk,\infty)$ are then used in the proof of Theorem 1.3.4. The forward implication is a standard interpolation argument due to Bakry and Émery, the backward one follows by differentiating the q-gradient estimate at zero [BHS21, Thm. 3.4]. Compare e.g. with Theorem 3.4.26 and Theorem 3.6.33 as well as Theorem 1.5.19 and Theorem 2.5.1 below.

Lemma 1.3.3. *If* $\mathrm{GE}_q(\Bbbk)$ *holds for some* $q \in [1,\infty)$*, then* $\mathrm{GE}_{q'}(\Bbbk)$ *is satisfied for every* $q' \in [q,\infty)$*.*

Theorem 1.3.4. *For every* $q \in [1,\infty)$*, the properties* $\mathrm{BE}_q(\Bbbk,\infty)$ *and* $\mathrm{GE}_q(\Bbbk)$ *are equivalent to each other.*

1.3.2 Independence of the q-Bochner inequality of q

In this section, we prove the independence of the q-Bochner inequality of $q \in [1,\infty)$. See Theorem 1.3.6 below for the precise statement.

We start with the following result. For its proof, we adapt the arguments of [Han18b]. A crucial point in this argument is that our a priori RCD assumption guarantees that $\Gamma(f)^{q/2} \in W^{1,2}(M)$ for every $f \in \mathrm{Test}(M)$ and *every* $q \in [1,\infty)$, and that $\mathrm{Test}(M)$ is dense in $W^{1,2}(M)$, cf. [Sav14, Lem. 2.6, Cor. 4.3] and Proposition 3.2.79 below.

Proposition 1.3.5. *The condition* $\mathrm{BE}_q(\Bbbk,\infty)$ *implies* $\mathrm{BE}_2(\Bbbk,\infty)$ *for every* $q \in (2,\infty)$*.*

Proof. Thanks to Theorem 1.3.4, it suffices to show the claimed implication starting from $\mathrm{GE}_q(\Bbbk)$ with $q \in (2,\infty)$.

Arguing exactly as in the constant situation in [Han18b, Lem. 3.2] (see also [Sav14, Thm. 3.4]), one can show that for every $r \in (2,\infty)$, $\mathrm{BE}_r(\Bbbk,\infty)$ holds if and only if

$$\begin{aligned} &\Gamma(f)\,\delta\frac{\Gamma(f)}{2} + \frac{r-2}{4}\,\Gamma(\Gamma(f)) \geq \Gamma(f)\,\Gamma(f,\Delta f) + \Bbbk\,\Gamma(f)^2 \quad \mathfrak{m}\text{-a.e.},\\ &\Gamma(f)_\sim\,\mathbf{\Delta}_\perp\Gamma(f) \geq 0 \end{aligned} \tag{1.3.3}$$

are valid for every $f \in \mathrm{Test}(M)$. Here, $\delta\Gamma(f)/2$ is the density of the $\mathfrak{m}$-absolutely continuous part of $\mathbf{\Delta}\Gamma(f)$ w.r.t. $\mathfrak{m}$, and $\mathbf{\Delta}_\perp\Gamma(f)/2$ stands for the corresponding $\mathfrak{m}$-singular part. In particular, note that $\mathrm{GE}_q(\Bbbk)$ already yields $\Gamma(f)_\sim\,\mathbf{\Delta}_\perp\Gamma(f) \geq 0$ by (1.3.3) which is independent of q.

The crucial point is now to show that

$$\Gamma(f)\,\delta\frac{\Gamma(f)}{2} + \varepsilon\,\Gamma(\Gamma(f)) \geq \Gamma(f)\,\Gamma(f,\Delta f) + \Bbbk\,\Gamma(f)^2 \quad \mathfrak{m}\text{-a.e.} \tag{1.3.4}$$

for every $\varepsilon > 0$. Given the observation (1.3.3), this will imply $\mathrm{BE}_{2+4\varepsilon}(\Bbbk,\infty)$ for every $\varepsilon > 0$, and eventually letting $\varepsilon \downarrow 0$ and applying Levi's theorem, we get the claimed $\mathrm{BE}_2(\Bbbk,\infty)$ condition.

Given $\mathrm{BE}_{q'}(\Bbbk,\infty)$ for arbitrary $q' \geq q$, it is straightforward to follow the proof of [Han18b, Thm. 3.6], which relies on generic calculus rules for $\mathbf{\Gamma}_2$, cf. Lemma 3.2.76 below, and closely follows the strategy presented in [Sav14], to prove (1.3.4) with ε replaced by $q' - (q'+1)^{-1}/4$. Now, according to [Han18b, Lem. 3.3], given any $\varepsilon > 0$ there exist $n \in \mathbf{N}$ and $q' \geq q$ so that $P^n(q') = \varepsilon$, where $P(r) := r - (r+1)^{-1}/4$ and P^n is the n-fold composition of P. Since $\mathrm{BE}_q(\Bbbk,\infty)$ yields $\mathrm{BE}_{q'}(\Bbbk,\infty)$, iterating the foregoing reasoning allows us to finally reach the inequality (1.3.4). □

Proposition 1.3.2, Lemma 1.3.3 and Proposition 1.3.5 thus yield the following.

Theorem 1.3.6. *If the q-Bakry–Émery estimate* $\mathrm{BE}_q(\Bbbk,\infty)$ *holds for some* $q \in [1,\infty)$*, then it holds for every* $q \in [1,\infty)$*.*

1.3.3 Localization of Bochner's inequality

To study a suitable local-to-global behavior of the q-Bochner inequality, we present a reformulation of it where we enlarge the class of admissible functions ϕ, compare with Definition 1.1.4 above. Recall that our standing assumption $\mathrm{RCD}(K,\infty)$ implies $\Gamma(f)^{q/2} \in W^{1,2}(M)$ for every $f \in \mathrm{Test}(M)$ and every $q \in [1,\infty)$.

Lemma 1.3.7. *Given $q \in [1,\infty)$, the $\mathrm{BE}_q(\Bbbk,\infty)$ property holds if and only if for every $f \in \mathrm{Test}(M)$ and every nonnegative $\phi \in W^{1,2}(M) \cap L^\infty(M)$,*

$$\begin{aligned} -\int_M \Big[\frac{1}{q}\Gamma\big(\Gamma(f)^{q/2},\phi\big) + \Gamma(f)^{q/2-1}\,\Gamma(f,\Delta f)\,\phi\Big]\,\mathrm{d}\mathfrak{m} \\ \geq \int_M \Bbbk\,\Gamma(f)^{q/2}\,\phi\,\mathrm{d}\mathfrak{m}. \end{aligned} \tag{1.3.5}$$

Proof. Obtaining $\mathrm{BE}_q(\Bbbk,\infty)$ from (1.3.5) through integration by parts and the density of $\mathrm{Test}(M)$ in $W^{1,2}(M)$ is easy, thus we focus on the converse. Trivially, the inequality (1.3.5) holds for every $\phi \in \mathscr{D}(\Delta)\cap L^\infty(M)$ with $\Delta\phi \in L^\infty(M)$. Approximating any function $\phi \in W^{1,2}(M)\cap L^\infty(M)$ in $W^{1,2}(M)$ by a mollified heat flow, cf. e.g. Lemma 3.2.73 below or [Sav14, Sec. 2.3] allows us to extend the class of admissible ϕ. ☐

Definition 1.3.8. *We say that the* local q-Bakry–Émery condition *with variable curvature bound $\Bbbk$, briefly* $\mathrm{BE}_{q,\mathrm{loc}}(\Bbbk,\infty)$, *with $q \in [1,\infty)$ holds if for every $z \in M$ there exists $\delta > 0$ such that*

$$-\int_M \Big[\frac{1}{q}\Gamma\big(\Gamma(f)^{q/2},\phi\big) + \Gamma(f)^{q/2-1}\,\Gamma(f,\Delta f)\,\phi\Big]\,\mathrm{d}\mathfrak{m} \geq \int_M \Bbbk\,\Gamma(f)^{q/2}\,\phi\,\mathrm{d}\mathfrak{m}$$

for every $f \in \mathrm{Test}(M)$ and every nonnegative $\phi \in W^{1,2}(M)\cap L^\infty(M)$ with the property that $\operatorname{spt}\phi \subset B_\delta(z)$.

It is elementary to pass from the global $\mathrm{BE}_q(\Bbbk,\infty)$ condition to $\mathrm{BE}_{q,\mathrm{loc}}(\Bbbk,\infty)$. The converse is more involved. The proof of the following result is similar to the one of [AMS16, Thm. 6.12], but uses a more elementary partition of unity and does not require local compactness or upper dimension bounds of the base space.

Theorem 1.3.9. *For every $q \in [1,\infty)$, the property $\mathrm{BE}_{q,\mathrm{loc}}(\Bbbk,\infty)$ implies the $\mathrm{BE}_q(\Bbbk,\infty)$ condition.*

Proof. Let $\{z_i : i \in \mathbf{N}\}$ be a countable dense subset of M and consider the collection of metric balls $B_{\delta_i}(z_i)$ with $\delta_i > 0$ chosen in such a way that the local q-Bakry–Émery inequality is satisfied around z_i. For $i \in \mathbf{N}$, define functions on M by

$$\begin{aligned} \eta_i^0 &:= \frac{2}{\delta_i}\mathsf{d}\big(\cdot, B_{\delta_i}(z_i)^{\mathrm{c}}\big),\\ \eta_i^* &:= \min\Big\{\sum_{j=1}^{i}\eta_j^0, 1\Big\},\\ \eta_i &:= \eta_i^* - \eta_{i-1}^*. \end{aligned}$$

Then $\eta_i \in \mathrm{Lip}_{\mathrm{b}}(M)$ with support in $B_{\delta_i}(z_i)$, $i \in \mathbf{N}$, and $\sum_{i\in\mathbf{N}}\eta_i = 1_M$ on M. Thus, for arbitrary nonnegative $\phi \in W^{1,2}(M)\cap L^\infty(M)$, the assumption $\mathrm{BE}_{q,\mathrm{loc}}(\Bbbk,\infty)$ yields

$$-\int_M \Big[\frac{1}{q}\Gamma\big(\Gamma(f)^{q/2},\phi\big) + \Gamma(f)^{q/2-1}\,\Gamma(f,\Delta f)\,\phi\Big]\,\mathrm{d}\mathfrak{m}$$

$$
\begin{aligned}
&= -\sum_{i\in\mathbf{N}} \int_M \Big[\frac{1}{q}\Gamma\big(\Gamma(f)^{q/2}, \phi\,\eta_i\big) + \Gamma(f)^{q/2-1}\,\Gamma(f,\Delta f)\,\phi\,\eta_i\Big]\,\mathrm{d}\mathfrak{m}\\
&\geq \sum_{i\in\mathbf{N}} \int_M \Bbbk\,\Gamma(f)^{q/2}\,\phi\,\eta_i\,\mathrm{d}\mathfrak{m}\\
&= \int_M \Bbbk\,\Gamma(f)^{q/2}\,\phi\,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

We conclude the assertion using Lemma 1.3.7 above. □

1.4 From 2-gradient estimates to CD and differential 2-transport estimates

Our goal now is to derive the evolution variational inequality $\mathrm{EVI}(\Bbbk)$ with variable curvature bound $\Bbbk$ from the 2-gradient estimate $\mathrm{GE}_2(\Bbbk)$. In [Stu15] there is a first part of the proof for this implication. With some extra arguments, we complete it.

The key point is a localization argument. Indeed, it suffices to prove the $\mathrm{EVI}(\Bbbk)$ “locally”, that is, for measures supported in a given small neighborhood. The heat flow of these will neither stay within this neighborhood nor in any other bounded region. We thus modify it by truncating its tails. Due to the Gaussian behavior of the heat flow, the difference is of arbitrary polynomial order for small times. This will imply the $\mathrm{CD}(\Bbbk,\infty)$ inequality locally. However, the latter is already known to give the $\mathrm{CD}(\Bbbk,\infty)$ inequality globally, and this in turn yields the global version of the $\mathrm{EVI}(\Bbbk)$.

1.4.1 Tail estimates for the heat flow

Given any ball $B_\delta(z) \subset M$, $\delta > 0$ and $z \in M$, and any $\rho \in \mathscr{P}(M)$, we set

$$\mathscr{H}_t^*\rho := 1_{B_{2\delta}(z)}\,\mathscr{H}_t\rho + \mathscr{H}_t\rho\big[B_{2\delta}(z)^{\mathrm{c}}\big]\,\delta_z.$$

Lemma 1.4.1. *Assume that $\rho \in \mathscr{P}(M)$ is $\mathfrak{m}$-absolutely continuous with density $f \in L^2(M)$ and $\operatorname{spt}\rho \subset B_\delta(z)$. Then for every $a > 0$ there exists $t_* > 0$ such that for every $t \in [0,t_*]$ and every $\phi \in L_\infty(M)$,*

$$
\begin{aligned}
W_2^2(\mathscr{H}_t^*\rho, \mathscr{H}_t\rho) &\leq t^a,\\
\Big|\int_M \phi\,\mathrm{d}\mathscr{H}_t^*\rho - \int_M \phi\,\mathrm{d}\mathscr{H}_t\rho\Big| &\leq t^a\,\sup|\phi|(M).
\end{aligned}
$$

Proof. To see the first assertion for $t > 0$, the case $t = 0$ being trivial, observe that

$$
\begin{aligned}
W_2^2(\mathscr{H}_t^*\rho, \mathscr{H}_t\rho) &\leq \int_{B_{2\delta}(z)^{\mathrm{c}}} \mathsf{d}^2(z,x)\,\mathrm{d}\mathscr{H}_t\rho(x)\\
&\leq \sum_{n=3}^{\infty} (n\delta)^2 \int_{B_{n\delta}(z)\setminus B_{(n-1)\delta}(z)} \mathsf{P}_t f\,\mathrm{d}\mathfrak{m}\\
&\leq \|f\|_{L^2(M)} \sum_{n=3}^{\infty} (n\delta)^2\,\mathfrak{m}\big[B_{n\delta}(z)\setminus B_{(n-1)\delta}(z)\big]^{1/2}\,\mathrm{e}^{-(n-2)^2\delta^2/4t}
\end{aligned}
$$

where the last inequality comes from the integrated Gaussian heat kernel estimate of [Stu95, Thm. 1.8] applied with $K = k = 1$, $\gamma = \lambda = 0$, $A = B_{n\delta}(z)\setminus B_{(n-1)\delta}(z)$ and $B = B_\delta(z)$ and replacing 1_B and $\mathfrak{m}[B]^{1/2}$ by f and $\|f\|_{L^2(M)}$ therein, respectively.

Therefore, by the volume growth property in $\mathrm{RCD}(K,\infty)$ spaces and finally assuming that t is small enough, we obtain

$$\begin{aligned} W_2^2(\mathscr{H}_t^*\rho,\mathscr{H}_t\rho) &\leq \|f\|_{L^2(M)}\,\Big[\sum_{n=3}^\infty \mathfrak{m}\big[B_{n\delta}(z)\setminus B_{(n-1)\delta}(z)\big]\,\mathrm{e}^{-n^2\delta^2/72t}\Big]^{1/2}\,\mathrm{e}^{-\delta^2/8t}\\ &\leq \|f\|_{L^2(M)}\,\Big[\int_M \mathrm{e}^{-\mathsf{d}^2(z,\cdot)/72t}\,\mathrm{d}\mathfrak{m}\Big]^{1/2}\,\mathrm{e}^{-\delta^2/8t} \leq t^a. \end{aligned}$$

The second assertion follows from the first one, since

$$\begin{aligned} \Big|\int_M \phi\,\mathrm{d}\mathscr{H}_t^*\rho - \int_M \phi\,\mathrm{d}\mathscr{H}_t\rho\Big| &\leq \sup|\phi|(M)\,\mathscr{H}_t\rho\big[B_{2\delta}(z)^{\mathrm{c}}\big]\\ &\leq \frac{\sup|\phi|(M)}{\delta^2}\,W_2^2(\mathscr{H}_t^*\rho,\mathscr{H}_t\rho). \end{aligned}$$ □

In Section 1.5, we need the following result, which is a consequence of Lemma 1.4.1.

Lemma 1.4.2. *For every $z\in X$, every $\delta>0$ and every $a>0$ there exists $t_*>0$ such that for every $x\in B_\delta(z)$ and every $t\in[0,t_*]$,*

$$\mathbf{P}\big[\mathsf{b}_t^x\notin B_{3\delta}(z)\big]\leq t^a,$$

where b^x denotes Brownian motion on M starting in x.

Proof. Let $\rho\in\mathscr{P}(M)$ be the uniform distribution of $B_{\delta/2}(z)$. Choose a pair $(\mathsf{b}^x,\mathsf{b})$ of coupled Brownian motions with initial distributions δ_x and ρ, respectively, such that $\mathsf{d}(\mathsf{b}_t^x,\mathsf{b}_t)\leq \mathrm{e}^{-Kt}\,\mathsf{d}(x,\mathsf{b}_0)$ $\mathbf{P}$-a.s. for every $t\geq 0$, see [Stu15, Thm. 2.9] for the construction. Thus in particular, $\mathbf{P}$-a.s. we have

$$\mathsf{d}(\mathsf{b}_t^x,\mathsf{b}_t)\leq\delta$$

for every $t\in[0,t_*']$ and a suitable $t_*'>0$. According to the previous Lemma 1.4.1,

$$\mathbf{P}\big[\mathsf{b}_t\notin B_{2\delta}(z)\big]\leq t^a$$

for every $t\in[0,t_*]$ and some $t_*>0$ depending only on $\mathfrak{m}[B_{\delta/2}(z)]$ and a. Combining both estimates yields that

$$\mathbf{P}\big[\mathsf{b}_t^x\notin B_{3\delta}(z)\big]\leq\mathbf{P}\big[\mathsf{b}_t\notin B_{2\delta}(z)\big]\leq t^a.$$

uniformly in $x\in B_\delta(z)$ for small enough times. □

1.4.2 From 2-gradient estimates to CD

In this subsection, we assume that κ is Lipschitz and bounded. The general case follows using the approximation scheme via the sequence $(\kappa_n)_{n\in\mathbf{N}}$ with $\kappa_n(x):=\underline{\kappa}_n(x,x)$ for $x\in M$ derived from Lemma 1.2.1. Indeed, $\mathrm{GE}_2(\kappa)$ trivially implies $\mathrm{GE}_2(\kappa_n)$ for every $n\in\mathbf{N}$, which will imply both $\mathrm{CD}(\kappa_n,\infty)$ and $\mathrm{EVI}(\kappa_n)$. Since W_2-geodesics between $\mathfrak{m}$-absolutely continuous measures and $\mathrm{EVI}(\kappa)$-curves are unique, we may then pass to the limit $n\to\infty$ by Levi's theorem.

We present a modification of [Stu15, Lem. 3.5] which is proven in exactly the same way as the previous version subject to the choice of parameterization from [AGS15, Thm. 4.16] involving the additional parameter κ. Throughout this subsection, denote by $(Q_s)_{s\geq 0}$ the 2-Hopf–Lax semigroup.

Lemma 1.4.3. *Assume the* 2*-gradient estimate* $\mathrm{GE}_2(k)$ *with variable curvature bound* k*, and let* $\kappa \in \mathbf{R}$ *be a constant. Let* $(\rho_s)_{s\in[0,1]}$*,* $\rho_s = f_s\,\mathfrak{m}$*, be a regular curve in the sense of* [AGS15, Def. 4.10]*, and set* $\vartheta_{\kappa,t}(s) := (\mathrm{e}^{\kappa st}-1)\,(\mathrm{e}^{\kappa t}-1)^{-1}$ *if* $\kappa \neq 0$ *and* $\vartheta_{0,t}(s) := s$ *as well as* $\mathrm{R}_\kappa(t) := \kappa t\,(\mathrm{e}^{\kappa t}-1)^{-1}$ *if* $\kappa\neq 0$ *and* $\mathrm{R}_0(t) := 1$*,* $t>0$*. Then*

$$\begin{aligned}\int_M Q_1\phi\,\mathrm{d}\mathscr{H}_t\rho_1 - \int_M \phi\,\mathrm{d}\rho_0 - \frac{1}{2}\mathrm{R}_\kappa^2(t)\int_0^1 \left|\dot\rho_{\vartheta_{\kappa,t}(s)}\right|^2\mathrm{d}s&\\ + t\left[\mathrm{Ent}_{\mathfrak{m}}(\mathscr{H}_t\rho_1) - \mathrm{Ent}_{\mathfrak{m}}(\rho_0)\right]&\\ \leq -\int_0^1\int_0^{st}\int_X \mathsf{P}_r\big((k-\kappa)\,\mathsf{P}_{st-r}^{2(k-\kappa)}\Gamma(Q_s\phi)\big)\,\mathrm{d}\rho_{\vartheta_{\kappa,t}(s)}\,\mathrm{d}r\,\mathrm{d}s&\end{aligned}$$

for every $\phi\in\mathrm{Lip}_{\mathrm{bs}}(M)$ *and every* $t>0$*. Here, the term* $\left|\dot\rho_{\vartheta_{\kappa,t}(s)}\right|$ *has to be understood as the metric speed of the original curve* $(\rho_s)_{s\in[0,1]}$ *evaluated at* $\vartheta_{\kappa,t}(s)$*.*

The same holds for every W_2*-geodesic* $(\rho_s)_{s\in[0,1]}$ *with* $\mathfrak{m}$*-absolutely continuous measures, in which case, independently of* κ *and* t*,*

$$\int_0^1 \left|\dot\rho_{\vartheta_{\kappa,t}(s)}\right|^2\mathrm{d}s = W_2^2(\rho_0,\rho_1).$$

Lemma 1.4.4. *Assume the* 2*-gradient estimate* $\mathrm{GE}_2(k)$ *with variable curvature bound* k*. Suppose that* $k \geq K_z$ *in* $B_{2\delta}(z)$ *for some* $z\in M$*,* $K_z\in\mathbf{R}$ *and* $\delta>0$*. Then for every* $\rho_0,\rho_1\in\mathscr{P}_2(X)\cap\mathscr{D}(\mathrm{Ent}_{\mathfrak{m}})$ *with support in* $B_\delta(z)$ *and bounded densities w.r.t.* $\mathfrak{m}$*,*

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 \frac{1}{2}W_2^2(\mathscr{H}_t\rho_1,\rho_0) + \frac{K_z}{2}W_2^2(\rho_0,\rho_1) \leq \mathrm{Ent}_{\mathfrak{m}}(\rho_0) - \mathrm{Ent}_{\mathfrak{m}}(\rho_1).$$

Proof. We follow the reasoning for [Stu15, Lem. 3.6] and [AGS15, Thm. 4.16] with a subtle modification. While the curve $(\mathscr{H}_{ts}\rho_{\vartheta_{K_z,t}(s)})_{s\in[0,1]}$ connects ρ_0 and $\mathscr{H}_t\rho_1$, the potentials $Q_s\phi_t$, $s\in[0,1]$, will be Hopf–Lax interpolations of optimal Kantorovich potentials for the transport from ρ_0 to $\mathscr{H}_t^*\rho_1$, $t>0$. Thus, we have to match these two different situations and then use the nice behavior of the remainder terms.

We know by [AGS14a, Prop. 3.9] that for any W_2-optimal coupling $\pi_t\in\mathscr{P}(M^2)$ of ρ_0 and $\mathscr{H}_t^*\rho_1$, and any Kantorovich potential $\phi_t\colon M\to\mathbf{R}$ relative to π_t, we have $|\mathrm{d}\phi_t| \leq \mathsf{d}(x,y)\leq 4\delta$ for π_t-a.e. $(x,y)\in M^2$. Hence, by (1.2.2) and boundedness of $\operatorname{spt}\rho_0$, by [AGS14a, Prop. 2.12] there exists $\phi_t\in\mathrm{Lip}_{\mathrm{bs}}(M)$ with $\mathrm{Lip}(\phi_t)\leq 4\delta$ and

$$\frac{1}{2}W_2^2(\mathscr{H}_t^*\rho_1,\rho_0) = \int_M Q_1\phi_t\,\mathrm{d}\mathscr{H}_t^*\rho_1 - \int_M \phi_t\,\mathrm{d}\rho_0.$$

Possibly adding constants and invoking a cutoff argument, we may and will assume that $|\phi_t|\leq C$ everywhere on M for some $C>0$ independent of t. Thus, $|Q_s\phi_t|$ is bounded on M and $\mathrm{Lip}(Q_s\phi_t)\leq 8\delta$, uniformly in $s\in[0,1]$ and small $t>0$.

Let $(\rho_s)_{s\in[0,1]}$ be the W_2-geodesic joining ρ_0 and ρ_1. Note that the measures $\rho_s = f_s\,\mathfrak{m}$, $s\in[0,1]$, are supported in $B_{2\delta}(z)$. The background $\mathrm{CD}(K,\infty)$ condition furthermore ensures that the f_s are bounded uniformly in $s\in[0,1]$ [Raj12b, Thm. 1.3]. Applying Lemma 1.4.3 with $\kappa := K_z$ we get

$$\begin{aligned}&\frac{1}{2t}\big[W_2^2(\mathscr{H}_t\rho_1,\rho_0) - W_2^2(\rho_0,\rho_1)\big]\\ &\quad = \frac{1}{2t}\Big[W_2^2(\mathscr{H}_t\rho_1,\rho_0) - W_2^2(\mathscr{H}_t^*\rho_1,\rho_0) + 2\int_M Q_1\phi_t\,\mathrm{d}\mathscr{H}_t^*\rho_1\end{aligned}$$

$$
\begin{aligned}
&\qquad - 2\int_M \phi_t \,\mathrm{d}\rho_0 - W_2^2(\rho_0,\rho_1)\Big] \\
&\leq \frac{1}{2t}\Big[W_2^2(\mathscr{H}_t\rho_1,\rho_0) - W_2^2(\mathscr{H}_t^*\rho_1,\rho_0) + 2\int_M Q_1\phi_t \,\mathrm{d}\mathscr{H}_t^*\rho_1 \\
&\qquad - 2\int_M Q_1\phi_t\,\mathrm{d}\mathscr{H}_t\rho_1\Big] \\
&\qquad + \frac{1}{2t}\big[\mathrm{R}^2_{K_z}(t) - 1\big]\, W_2^2(\rho_0,\rho_1) + \mathrm{Ent}_{\mathfrak{m}}(\rho_0) - \mathrm{Ent}_{\mathfrak{m}}(\mathscr{H}_t\rho_1) \\
&\qquad - \frac{1}{t}\int_0^1 s\int_0^t\int_M \Gamma(Q_s\phi_t)\,\mathsf{P}^{2(k-K_z)}_{s(t-r)}\big((k-K_z)\,\mathsf{P}_{sr}f_{\vartheta_t(s)}\big)\,\mathrm{d}\mathfrak{m}\,\mathrm{d}r\,\mathrm{d}s,
\end{aligned}
$$

where we have set $\vartheta_t := \vartheta_{K_z,t}$. Observe that the limsup as $t\downarrow 0$ of the last term is nonnegative since $(k - K_z)\,f_s \geq 0$ $\mathfrak{m}$-a.e. on M for every $s\in[0,1]$ and

$$
\lim_{t\downarrow 0}\frac{1}{t}\int_0^t \mathsf{P}^{2(k-K_z)}_{s(t-r)}\big((k-K_z)\,\mathsf{P}_{sr}f_{\vartheta_t(s)}\big)\,\mathrm{d}r = (k-K_z)\,f_s
$$

w.r.t. convergence in $L^1(M)$. Indeed, $\vartheta_t(s)\to s$ as $t\downarrow 0$ for every $s\in[0,1]$ and therefore $f_{\vartheta_t(s)}\to f_s$ pointwise $\mathfrak{m}$-a.e. As all considered functions are nonnegative and $\int_M f_{\vartheta_t(s)}\,\mathrm{d}\mathfrak{m} = \int_M f_s\,\mathrm{d}\mathfrak{m}$ for every $t>0$, we have $f_{\vartheta_t(s)}\to f_s$ in $L^1(M)$ as $t\downarrow 0$. We conclude by strong continuity of the heat and the Schrödinger semigroup with potential $2(k-K_z)$ in $L^1(M)$.

Lower semicontinuity of $\mathrm{Ent}_{\mathfrak{m}}$ yields $-\liminf_{t\downarrow 0}\mathrm{Ent}_{\mathfrak{m}}(\mathscr{H}_t\rho_1)\leq -\mathrm{Ent}_{\mathfrak{m}}(\rho_1)$. Also, $\mathrm{R}^2_{K_z}(t) = 1 - K_z t + \mathrm{o}(t)$ as $t\downarrow 0$. Lastly, $\big[W_2^2(\mathscr{H}_t\rho_1,\rho_0) - W_2^2(\mathscr{H}_t^*\rho_1,\rho_0)\big]/2t\to 0$ as $t\downarrow 0$ according to Lemma 1.4.1 applied with $a := 2$. The claim follows. □

Theorem 1.4.5. *The* 2*-gradient estimate* $\mathrm{GE}_2(k)$ *implies* $\mathrm{CD}(k,\infty)$.

Proof. Given $\varepsilon>0$, Lemma 1.4.4 translates into a "local" $\mathrm{EVI}(k-\varepsilon)$ property at time 0: for every $z\in M$, choosing $\delta>0$ and $K_z\in\mathbf{R}$ such that $K_z\leq k\leq K_z+\varepsilon$ in $B_{2\delta}(z)$, for every $\mu,\nu\in\mathscr{P}_2(M)\cap\mathscr{D}(\mathrm{Ent}_{\mathfrak{m}})$ with support in $B_\delta(z)$ and bounded densities w.r.t. $\mathfrak{m}$, for $\boldsymbol{\pi}\in\mathscr{P}(\mathrm{Geo}(M))$ representing the W_2-geodesic from μ to ν,

$$
\begin{aligned}
&\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 \frac{1}{2}W_2^2(\mathscr{H}_t\mu,\nu) + \int_0^1\int_{\mathrm{Geo}(M)}(1-s)\,\big[k(\gamma_s)-\varepsilon\big]\,|\dot\gamma|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}s \\
&\qquad\leq \mathrm{Ent}_{\mathfrak{m}}(\nu) - \mathrm{Ent}_{\mathfrak{m}}(\mu).
\end{aligned}
$$

With the same argument used in the proof of [Stu15, Thm. 3.4] for the equivalence of $\mathrm{CD}(k,\infty)$ and $\mathrm{EVI}(k)$ (based on previous work [DS08] in the case of constant k), we conclude that this local $\mathrm{EVI}(k-\varepsilon)$ implies a "local" $\mathrm{CD}(k-\varepsilon,\infty)$ condition in the following sense: for every $z\in M$ there exists $\delta>0$ such that for every $\mu_0,\mu_1\in\mathscr{P}_2(M)\cap\mathscr{D}(\mathrm{Ent}_{\mathfrak{m}})$ with support in $B_\delta(z)$ and bounded densities w.r.t. $\mathfrak{m}$, if $\boldsymbol{\pi}\in\mathscr{P}(\mathrm{Geo}(M))$ represents the W_2-geodesic from μ_0 to μ_1, for every $t\in[0,1]$,

$$
\begin{aligned}
\mathrm{Ent}_{\mathfrak{m}}(\mu_t) &\leq (1-t)\,\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) \\
&\qquad - \int_0^1\int_{\mathrm{Geo}(M)} \mathrm{g}(s,t)\,\big[k(\gamma_s)-\varepsilon\big]\,|\dot\gamma|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}s.
\end{aligned}
$$

Using the local-to-global property from [Stu15, Thm. 3.7] and taking the limit $\varepsilon\downarrow 0$, noticing again that the choice of W_2-geodesics does not depend on ε, allows us to pass from this local $\mathrm{CD}(k-\varepsilon,\infty)$ property to $\mathrm{CD}(k-\varepsilon,\infty)$ and finally to $\mathrm{CD}(k,\infty)$. □

1.4.3 From EVI to a differential 2-transport estimate

It has already been observed in [Ket15a] that $\mathrm{EVI}(k)$ yields contraction estimates for the 2-Kantorovich–Wasserstein distance along two heat flows starting at regular measures. For irregular initial data, we now aim in deducing a weak version of it, compare with Remark 1.1.9 above.

Proposition 1.4.6. *The* $\mathrm{EVI}(k)$ *implies the following differential* 2*-transport estimates.*

(i) *For every* $\mu_1, \mu_2 \in \mathscr{P}_2(M) \cap \mathscr{D}(\mathrm{Ent}_{\mathrm{m}})$*, one has*

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_2^2(\mathscr{H}_t\mu_1, \mathscr{H}_t\mu_2) \leq -2\int_0^1\int_{\mathrm{Geo}(M)} k(\gamma_s)\,|\dot\gamma|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}s,$$

where $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(M))$ *represents the* W_2*-geodesic from* μ_1 *to* μ_2*.*

(ii) *For every* $x, y \in M$*,*

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_2^2(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) \leq -2\underline{k}(x,y)\,\mathsf{d}^2(x,y).$$

Proof. Concerning (i), up to truncating k and using Levi's theorem afterwards, we may and will assume that k is bounded. Naively, the claim follows by applying the $\mathrm{EVI}(k)$ to $(\mathscr{H}_t\mu_1)_{t\geq 0}$ and $(\mathscr{H}_t\mu_2)_{t\geq 0}$, respectively. Some care, however, is needed to deal with the double t-dependency of the nonsmooth function $t \mapsto W_2^2(\mathscr{H}_t\rho_0, \mathscr{H}_t\rho_1)$. To deal with this, one adds up the $\mathrm{EVI}(k)$, integrated from t to $t+h$, $h > 0$, for the flow $(\mathscr{H}_t\mu_1)_{t\geq 0}$ with observation point $\mathscr{H}_{t+h}\mu_2$ and for the flow $(\mathscr{H}_t\mu_2)_{t\geq 0}$ with observation point $\mathscr{H}_t\mu_1$. The entropy terms cancel out, and we obtain the desired estimate after dividing by h and letting $h \downarrow 0$. See [Ket15a, Thm. 6.1] for details.

Next, we show (ii). Denote by $(\underline{k}_n)_{n\in\mathbf{N}}$ a sequence in $\mathrm{Lip_b}(M^2)$ converging pointwise from below in a monotone way to $\underline{k}$, see Lemma 1.2.1, and set $k_n(x) := \underline{k}_n(x,x)$ for $x \in M$. Given any $x, y \in M$ and any $t > 0$, select $\tau_* > 0$ sufficiently small to ensure that, for every $\tau \in (0, \tau_*)$,

$$W_2^2(\mathscr{H}_\tau\delta_x, \mathscr{H}_\tau\delta_y) \leq \mathsf{d}^2(x,y) + 2t^2.$$

The local absolute continuity of the curves $(\mathscr{H}_t\delta_x)_{t\geq 0}$ and $(\mathscr{H}_t\delta_y)_{t\geq 0}$ on $(0,\infty)$ w.r.t. W_2 and property (i) with k_n in place of k, since $k_n \leq k$ on M, yield

$$\begin{aligned}
&\frac{1}{2t}\big[W_2^2(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) - \mathsf{d}^2(x,y)\big]\\
&\qquad\leq t + \frac{1}{2t}\big[W_2^2(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) - W_2^2(\mathscr{H}_\tau\delta_x, \mathscr{H}_\tau\delta_y)\big]\\
&\qquad\leq t - \frac{1}{t}\int_\tau^t\int_0^1\int_{\mathrm{Geo}(M)} k_n(\gamma_s)\,|\dot\gamma|^2\,\mathrm{d}\boldsymbol{\pi}_r(\gamma)\,\mathrm{d}s\,\mathrm{d}r,
\end{aligned}$$

where $\boldsymbol{\pi}_r \in \mathscr{P}(\mathrm{Geo}(M))$ represents the W_2-geodesic from $\mathscr{H}_r\delta_x$ to $\mathscr{H}_r\delta_y$. As $n \to \infty$, by Levi's theorem, the above inequality still holds with k in place of k_n. The definition of $\underline{k}$ and the inequality $\underline{k}_n \leq \underline{k}$ on M for every $n \in \mathbf{N}$ give, setting $\pi_r := (\mathsf{e}_0, \mathsf{e}_1)_\sharp\boldsymbol{\pi}_r$,

$$\begin{aligned}
&\frac{1}{2t}\big[W_2^2(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) - \mathsf{d}^2(x,y)\big]\\
&\qquad\leq t - \frac{1}{t}\int_\tau^t\int_{M^2} \underline{k}_n(x',y')\,\mathsf{d}^2(x',y')\,\mathrm{d}\pi_r(x',y')\,\mathrm{d}r.
\end{aligned}$$

Since $\mathscr{H}_r\delta_x \to \delta_x$ and $\mathscr{H}_r\delta_y \to \delta_y$ w.r.t. W_2 as $r \to 0$ and since $W_2(\mathscr{H}_r\delta_x, \mathscr{H}_r\delta_y)$ is uniformly bounded for small r, stability of optimal couplings, see e.g. [AGS08, Prop. 7.1.3], and uniqueness of the W_2-optimal coupling $\pi_0 := \delta_x \otimes \delta_y$ of δ_x and δ_y imply that $\pi_r \to \pi_0$ weakly as $r \to 0$. Thus, the map $r \mapsto \int_{M^2} \underline{k}_n \mathsf{d}^2 \,\mathrm{d}\pi_r$ is continuous at 0 by [Vil09, Lem. 4.3]. The claim follows by taking successively $\tau \downarrow 0$, $t \downarrow 0$ and $n \to \infty$ in the above inequality. □

A posteriori, knowing from Theorem 1.1.1 that $\mathrm{EVI}(k)$ implies $\mathrm{GE}_1(k)$, we will be able to improve the bound (ii) from Proposition 1.4.6 even for exponents different from 2, see Remark 1.5.12 below.

1.5 Duality of p-transport estimates and q-gradient estimates

Throughout the rest of this chapter, given $t \geq 0$, we use the short-hand notation

$$\Pi_t := \mathrm{C}([0,t]; M^2).$$

Moreover, at several instances we consider a function $\underline{\ell} \colon M^2 \to \mathbf{R}$ which, unless stated otherwise, is assumed lower semicontinuous and lower bounded. However, it should practically rather be thought of as a bounded Lipschitz function "approximating" $\underline{k}$ from below without being of the particular form (1.1.2). This often allows us to assume that $\underline{\ell} \in \mathrm{Lip}_{\mathrm{b}}(M^2)$, while $\underline{k}$ is not continuous in general, even if k is Lipschitz.

1.5.1 Perturbed costs and coupled Brownian motions

Given any $p \in [1,\infty)$ and $\mu_1, \mu_2 \in \mathscr{P}_p(M)$, let us define the *perturbed p-transport cost with potential $-p\underline{\ell}$* at $t \geq 0$ by

$$W_p^{\underline{\ell}}(\mu_1,\mu_2,t) := \inf \mathbf{E}\big[\mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1,\mathsf{b}_r^2)/2\,\mathrm{d}r}\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1,\mathsf{b}_{2t}^2\big)\big]^{1/p}, \tag{1.5.1}$$

where the infimum is taken over all pairs $(\mathsf{b}^1,\mathsf{b}^2)$ of coupled Brownian motions on M, restricted to $[0,2t]$ and modeled on a common probability space, with initial distributions μ_1 and μ_2, respectively. In more analytic words,

$$W_p^{\underline{\ell}}(\mu_1,\mu_2,t) = \inf \Big[\int_{\Pi_{2t}} \mathrm{e}^{\int_0^{2t} p\underline{\ell}(\gamma_r^1,\gamma_r^2)/2\,\mathrm{d}r}\,\mathsf{d}^p\big(\gamma_{2t}^1,\gamma_{2t}^2\big)\,\mathrm{d}\boldsymbol{\nu}(\gamma)\Big]^{1/p}, \tag{1.5.2}$$

where the infimum now is taken over all measures $\boldsymbol{\nu} \in \mathscr{P}(\Pi_{2t})$ whose marginals $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2 \in \mathscr{P}(\mathrm{C}([0,2t];M))$ are the laws of Brownian motions on M, restricted to $[0,2t]$, with initial distribution μ_1 and μ_2, respectively. If $\underline{\ell} = \underline{k}$, this is the perturbed p-transport cost from Definition 1.1.

A natural, albeit nontrivial identity relates the perturbed p-transport cost in the case of constant k with the usual p-transport cost. Lemma 1.5.2, in turn, follows by a standard minimization argument [BHS21, Lem. 5.2].

Lemma 1.5.1. *If $\underline{\ell}$ is constantly equal to $L \in \mathbf{R}$, then for every $t \geq 0$,*

$$W_p^{\underline{\ell}}(\mu_1,\mu_2,t) = \mathrm{e}^{Lt}\, W_p(\mathscr{H}_t\mu_1, \mathscr{H}_t\mu_2).$$

Proof. Since $W_p(\mathscr{H}_t\mu_1,\mathscr{H}_t\mu_2)^{1/p} = \inf \mathbf{E}\big[\mathsf{d}^p(\mathsf{x},\mathsf{y})\big]^{1/p}$, the infimum ranging over all pairs of random variables $\mathsf{x} \sim \mathscr{H}_t\mu_1$ and $\mathsf{y} \sim \mathscr{H}_t\mu_2$ which are defined on a common probability space $(\Omega, \mathscr{A}, \mathbf{P})$, and since $\mathsf{b}_{2t} \sim \mathscr{H}_t\mu$ for every Brownian motion b with initial distribution $\mu \in \mathscr{P}(M)$, we get

$$W_p^{\underline{\ell}}(\mu_1, \mu_2, t) \geq \mathrm{e}^{Lt}\, W_p(\mathscr{H}_t\mu_1, \mathscr{H}_t\mu_2).$$

For the converse inequality, let $\pi_t \in \mathscr{P}(M^2)$ be a W_p-optimal coupling of $\mathscr{H}_t\mu_1$ and $\mathscr{H}_t\mu_2$. Consider Brownian motions b^1 and b^2, restricted to $[0, 2t]$, starting at μ_1 and μ_2, defined on probability spaces $(\Omega_1, \mathscr{A}_1, \mathbf{P}_1)$ and $(\Omega_2, \mathscr{A}_2, \mathbf{P}_2)$, respectively. We define the "bridge measures" $\mathbf{P}_1^x$ for $x \in M$ by disintegrating $\mathbf{P}_1$ w.r.t. $\mathscr{H}_t\mu_1(\mathrm{d}x)$ or, in other words, by conditioning b^1 on the event $\{\mathsf{b}^1_{2t} = x\}$. Similarly, let $\mathbf{P}_2^y$ for $y \in M$ be the disintegration of $\mathbf{P}_2$ w.r.t. $\mathscr{H}_t\mu_2(\mathrm{d}y)$. Consider the "glued measure" $\widetilde{\mathbf{P}}$ with

$$\widetilde{\mathbf{P}} := \int_{M^2} \mathbf{P}_1^x \otimes \mathbf{P}_2^y \,\mathrm{d}\pi_t(x, y)$$

on $\Omega := \Omega_1 \times \Omega_2$. Then $(\widetilde{\mathbf{P}}, \mathsf{b}^1)$ and $(\widetilde{\mathbf{P}}, \mathsf{b}^2)$ is a pair of coupled Brownian motions with joint distribution π_t at time $2t$. The desired inequality then follows directly, since

$$\widetilde{\mathbf{E}}\big[\mathsf{d}^p(\mathsf{b}^1_{2t}, \mathsf{b}^2_{2t})\big] = \int_{M^2} \mathsf{d}^p(x, y)\,\mathrm{d}\pi_t(x, y) = W_p^p(\mathscr{H}_t\mu_1, \mathscr{H}_t\mu_2). \qquad \square$$

Lemma 1.5.2. *For every $p \in [1, \infty)$, $t \geq 0$ and $\mu_1, \mu_2 \in \mathscr{P}_p(M)$ as above, the infima in* (1.5.1) *and in* (1.5.2) *are attained. Moreover, for every sequence $(\underline{\ell}_n)_{n\in\mathbf{N}}$ of lower semicontinuous functions $\underline{\ell}_n : M^2 \to \mathbf{R}$ converging pointwise to $\underline{\ell}$ from below in an increasing way, we have*

$$\lim_{n\to\infty} W_p^{\underline{\ell}_n}(\mu_1, \mu_2, t) = W_p^{\underline{\ell}}(\mu_1, \mu_2, t).$$

Let us denote by $\mathscr{B}^\nu(M^2)$ the completion of the Borel σ-field on M^2 w.r.t. a given $\nu \in \mathscr{P}(M^2)$. The σ-field of all *universally measurable* subsets of M^2 is

$$\mathscr{B}^{\mathrm{univ}}(M^2) := \bigcap_{\nu\in\mathscr{P}(M^2)} \mathscr{B}^\nu(M^2).$$

Lemma 1.5.3. *For every $t \geq 0$ and every $p \in [1, \infty)$, there exists a universally measurable map $\boldsymbol{\eta}^t : M^2 \to \mathscr{P}(\Pi_{2t})$ such that for every $x, y \in X$, the marginals of $\boldsymbol{\eta}^t_{x,y} := \boldsymbol{\eta}^t(x, y)$ are laws of Brownian motions, restricted to $[0, 2t]$, starting in x and y, respectively, and $\boldsymbol{\eta}^t_{x,y}$ is a minimizer in the definition* (1.5.2) *of $W_p^{\underline{\ell}}(\delta_x, \delta_y, t)$.*

Proof. According to Lemma 1.5.2, for every pair $(x, y) \in M^2$ there exists an admissible measure in $\mathscr{P}(\Pi_{2t})$ which attains the infimum in (1.5.2). The class of all probability measures with this property is closed. Then a measurable selection argument, see [Bog07b, Thm. 6.9.2] and [Stu15, Lem. 2.2], allows us to produce a family of measures $\boldsymbol{\eta}^t_{x,y}$ still satisfying the minimality property so that $(x, y) \mapsto \boldsymbol{\eta}^t_{x,y}$ is universally measurable in $(x, y) \in M^2$. $\square$

An important consequence of these observations is a type of Markov property which will be crucial in the proof of Proposition 1.5.6. For this and also for later use, fix $s, t \geq 0$, $\boldsymbol{\nu} \in \mathscr{P}(\Pi_s)$ and a universally measurable map $\boldsymbol{\mu} : M^2 \to \mathscr{P}(\Pi_t)$ such that $(\mathsf{e}_0)_\sharp\boldsymbol{\mu}_{x,y} = \delta_x \otimes \delta_y$ for every $x, y \in M$. Define their composition $\boldsymbol{\mu} \circ \boldsymbol{\nu} \in \mathscr{P}(\Pi_{s+t})$ by

$$\int_{\Pi_{s+t}} f\,\mathrm{d}(\boldsymbol{\mu} \circ \boldsymbol{\nu}) := \int_{\Pi_s}\int_{\Pi_t} f \circ \Phi_{s,t}(\alpha, \beta)\,\mathrm{d}\boldsymbol{\mu}_{\alpha_s^1, \alpha_s^2}(\beta)\,\mathrm{d}\boldsymbol{\nu}(\alpha)$$

for every $f \in \mathrm{C_b}(\Pi_{s+t})$, where

$$\Phi_{s,t}(\alpha,\beta)_r := \begin{cases} \alpha_r & \text{if } r \in [0,s], \\ \beta_{r-s} & \text{if } r \in (s, s+t] \end{cases}$$

is the *concatenation map* "gluing" together the curves $(\alpha_\sigma)_{\sigma\in[0,s]}$ and $(\beta_\tau)_{\tau\in[0,t]}$.

Proposition 1.5.4. *For every $p \in [1,\infty)$, every $s,t \geq 0$ and every $\mu_1, \mu_2 \in \mathscr{P}_p(M)$, there exists a pair $(\mathsf{b}^1, \mathsf{b}^2)$ of coupled Brownian motions on M with initial distributions μ_1 and μ_2, respectively, which minimizes* (1.5.1) *for the given time t and such that*

$$W_p^{\underline{\ell}}(\mu_1,\mu_2,t+s)^p \leq \mathbf{E}\big[\mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1,\mathsf{b}_r^2)/2\,\mathrm{d}r}\, W_p^{\underline{\ell}}\big(\delta_{\mathsf{b}_{2t}^1},\delta_{\mathsf{b}_{2t}^2},s\big)^p\big].$$

Proof. Denote the map from Lemma 1.5.3 with s in place of t by $\boldsymbol{\eta}^s$, denote a minimizer of (1.5.2) for time t by $\boldsymbol{\nu}_t$ (recall Lemma 1.5.2), and define $\boldsymbol{\eta}^{t+s} := \boldsymbol{\eta}^s \circ \boldsymbol{\nu}_t \in \mathscr{P}(\Pi_{2(s+t)})$. This defines a coupling of the laws of two Brownian motions with initial distributions μ_1 and μ_2, respectively, restricted to $[0, 2(t+s)]$ such that

$$\begin{aligned}&\int_{\Pi_{2(s+t)}} \mathrm{e}^{\int_0^{2(t+s)} p\underline{\ell}(\gamma_r^1,\gamma_r^2)/2\,\mathrm{d}r}\, \mathsf{d}^p\big(\gamma_{2(t+s)}^1,\gamma_{2(t+s)}^2\big)\,\mathrm{d}\boldsymbol{\nu}^{t+s}(\gamma)\\ &\qquad = \int_{\Pi_{2t}} \mathrm{e}^{\int_0^{2t} p\underline{\ell}(\alpha_r^1,\alpha_r^2)/2\,\mathrm{d}r}\, W_p^{\underline{\ell}}\big(\delta_{\alpha_{2t}^1},\delta_{\alpha_{2t}^2},s\big)^p\,\mathrm{d}\boldsymbol{\nu}_t(\alpha).\end{aligned}$$

This proves the claim. □

Less formally, the previous construction can be described as follows. To estimate the perturbed p-transport cost at time $t+s$, we construct the required process by first choosing a pair process $(\mathsf{b}^1,\mathsf{b}^2)$ of Brownian motions with given initial distributions μ_1 and μ_2 which realizes the minimum for $W_p^{\underline{\ell}}(\mu_1,\mu_2,t)$. Then we switch to a pair of Brownian motions starting in b_{2t}^1 and b_{2t}^2, respectively, which minimizes the perturbed p-transport cost at time s.

1.5.2 Differential p-transport inequalities and p-transport estimates

To deduce a p-transport estimate $\mathrm{PTE}_p(\underline{k})$, we have to control the upper derivatives of the function $t \mapsto W_p^{\underline{k}}(\delta_x,\delta_y,t)^p$ or, more generally, of $t \mapsto W_p^{\underline{\ell}}(\delta_x,\delta_y,t)^p$, $x,y \in M$.

Lemma 1.5.5. *If $\underline{\ell} \in \mathrm{C_b}(M^2)$, then for every $x,y \in M$ and every $p \in [1,\infty)$,*

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_p^{\underline{\ell}}(\delta_x,\delta_y,t)^p \leq p\,\underline{\ell}(x,y)\,\mathsf{d}^p(x,y) + \frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_p^p(\mathscr{H}_t\delta_x,\mathscr{H}_t\delta_y).$$

Proof. Choose any exponent $p' \in (p,\infty)$ with dual exponent $q' \in (1,\infty)$. For every $t > 0$, denote by $(\mathsf{b}^1,\mathsf{b}^2)$ a pair of coupled Brownian motions starting in (x,y) and such that the law of $(\mathsf{b}_{2t}^1,\mathsf{b}_{2t}^2)$ constitutes a $W_{p'}$-optimal coupling of $\mathscr{H}_t\delta_x$ and $\mathscr{H}_t\delta_y$. Albeit this process still depends on t, we suppress this dependency in the sequel to simplify the notation. For a precise construction of such a process, we refer to the proof of Lemma 1.5.1. Then

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_p^{\underline{\ell}}(\delta_x,\delta_y,t)^p \leq \limsup_{t\downarrow 0}\frac{1}{t}\mathbf{E}\big[\mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1,\mathsf{b}_r^2)/2\,\mathrm{d}r}\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1,\mathsf{b}_{2t}^2\big) - \mathsf{d}^p\big(\mathsf{b}_0^1,\mathsf{b}_0^2\big)\big]$$

$$\leq \limsup_{t\downarrow 0} \frac{1}{t}\mathbf{E}\Big[\big[\mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1,\mathsf{b}_r^2)/2\,\mathrm{d}r} - 1\big]\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1,\mathsf{b}_{2t}^2\big)\Big]$$
$$+ \frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_{p'}^p(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y).$$

Each of the last two limits will be estimated separately. The last term will converge to the upper derivative of $W_p^p(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y)$ at 0 as $p' \downarrow p$ by Levi's theorem. Moreover, since $\underline{\ell}$ is bounded, the former term can be estimated through

$$\limsup_{t\downarrow 0} \frac{1}{t}\mathbf{E}\Big[\big[\mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1,\mathsf{b}_r^2)/2\,\mathrm{d}r} - 1\big]\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1,\mathsf{b}_{2t}^2\big)\Big]$$
$$\leq \limsup_{t\downarrow 0} \frac{p}{2t}\mathbf{E}\Big[\int_0^{2t} \underline{\ell}\big(\mathsf{b}_r^1,\mathsf{b}_r^2\big)\,\mathrm{d}r\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1,\mathsf{b}_{2t}^2\big)\Big].$$

Now we split the expectation into a term where $(\mathsf{b}^1,\mathsf{b}^2)$ behaves well and a remainder term. Let $\varepsilon > 0$ and choose $\delta > 0$ such that for every $x' \in B_\delta(x)$ and every $y' \in B_\delta(y)$,

$$\max\big\{\big|\underline{\ell}(x',y') - \underline{\ell}(x,y)\big|, \big|\mathsf{d}^p(x',y') - \mathsf{d}^p(x,y)\big|\big\} \leq \varepsilon,$$

and define the exceptional set $E_{r,2t}$, $r \in (0,2t)$, by

$$E_{r,2t} := \big\{\mathsf{b}_r^1 \notin B_\delta(x)\big\} \cup \big\{\mathsf{b}_{2t}^1 \notin B_\delta(x)\big\} \cup \big\{\mathsf{b}_r^2 \notin B_\delta(y)\big\} \cup \big\{\mathsf{b}_{2t}^2 \notin B_\delta(y)\big\}.$$

By these definitions and Fubini's theorem, since $\underline{\ell}$ is bounded,

$$\limsup_{t\downarrow 0} \frac{p}{2t}\mathbf{E}\Big[\int_0^{2t} \underline{\ell}\big(\mathsf{b}_r^1,\mathsf{b}_r^2\big)\,1_{E_{r,2t}^{\mathrm{c}}}\,\mathrm{d}r\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1,\mathsf{b}_{2t}^2\big)\Big]$$
$$\leq p\,\big[\underline{\ell}(x,y) + \varepsilon\big]\,\big[\mathsf{d}^p(x,y) + \varepsilon\big]\,\limsup_{t\downarrow 0} \frac{1}{2t}\int_0^{2t} \mathbf{P}\big[E_{r,2t}^{\mathrm{c}}\big]\,\mathrm{d}r.$$

According to Lemma 1.4.2, we have $\mathbf{P}[E_{r,2t}] \to 0$ as $r \downarrow 0$ and $t \downarrow 0$, therefore the latter limsup is equal to 1. On the other hand, if $C > 0$ denotes an upper bound for $\underline{\ell}$, using Hölder's inequality the second term can be bounded through

$$\limsup_{t\downarrow 0} \Big|\frac{p}{2t}\mathbf{E}\Big[\int_0^{2t} \underline{\ell}\big(\mathsf{b}_r^1,\mathsf{b}_r^2\big)\,1_{E_{r,2t}}\,\mathrm{d}r\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1,\mathsf{b}_{2t}^2\big)\Big]\Big|$$
$$\leq p\,C\,\limsup_{t\downarrow 0} \mathbf{E}\big[\mathsf{d}^{p'}\big(\mathsf{b}_{2t}^1,\mathsf{b}_{2t}^2\big)\big]^{p/p'}\,\limsup_{t\downarrow 0}\Big[\frac{1}{2t}\int_0^{2t} \mathbf{P}\big[E_{r,2t}\big]\,\mathrm{d}r\Big]^{1-p/p'}.$$

By the choice of the pair process $(\mathsf{b}^1,\mathsf{b}^2)$, the first limsup equals $\mathsf{d}^p(x,y)$ while the second one is 0, as already observed above. Since ε was arbitrary, the claim follows. □

Proposition 1.5.6. *Fix $p \in [1,\infty)$ and assume the differential p-transport estimate*

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_p^p(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) \leq -p\,\underline{k}(x,y)\,\mathsf{d}^p(x,y) \tag{1.5.3}$$

for every $x, y \in M$. Then the p-transport estimate $\mathrm{PTE}_p(k)$ *is satisfied.*

Proof. We first show that for every $\mu_1, \mu_2 \in \mathscr{P}_p(M)$, the function $t \mapsto W_p^{\underline{\ell}}(\mu_1, \mu_2, t)$ is nonincreasing on $[0, \infty)$ whenever $\underline{\ell} \in \mathrm{C_b}(M^2)$ with $\underline{\ell} \leq \underline{k}$ on M^2.

To get started, we demonstrate that its p-th power $t \mapsto W_p^{\underline{\ell}}(\mu_1, \mu_2, t)^p$ is upper Lipschitz continuous on $[0, \infty)$. To see this, fix $h \in (0, 1]$ and $t > 0$, and consider the pair process $(\mathsf{b}^1, \mathsf{b}^2)$ as provided by Proposition 1.5.4. By the estimate from this proposition, Lemma 1.5.1 and contractivity of the Wasserstein heat flow, we have

$$\begin{aligned}
&\frac{1}{h}\big[W_p^{\underline{\ell}}(\mu_1, \mu_2, t+h)^p - W_p^{\underline{\ell}}(\mu_1, \mu_2, t)^p\big] \\
&\qquad \leq \frac{1}{h}\mathbf{E}\Big[\mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1, \mathsf{b}_r^2)/2\,\mathrm{d}r}\big[W_p^{\underline{\ell}}\big(\delta_{\mathsf{b}_{2t}^1}, \delta_{\mathsf{b}_{2t}^1}, h\big)^p - \mathsf{d}^p\big(\mathsf{b}_{2t}^1, \mathsf{b}_{2t}^2\big)\big]\Big] \\
&\qquad \leq \frac{1}{h}\mathbf{E}\Big[\mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1, \mathsf{b}_r^2)/2\,\mathrm{d}r}\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1, \mathsf{b}_{2t}^2\big)\,\big[\mathrm{e}^{pCh} - 1\big]\Big] \\
&\qquad \leq C'\, W_p^{\underline{\ell}}(\mu_1, \mu_2, t)^p
\end{aligned} \tag{1.5.4}$$

for suitable constants $C, C' > 0$. This proves upper Lipschitz continuity of the p-th power of the perturbed p-transport cost with potential $-p\underline{\ell}$, which in turn implies

$$W_p^{\underline{\ell}}(\mu_1, \mu_2, \tau)^p - W_p^{\underline{\ell}}(\mu_1, \mu_2, \sigma)^p \leq \int_\sigma^\tau \frac{\mathrm{d}^+}{\mathrm{d}t} W_p^{\underline{\ell}}\big(\mu_1, \mu_2, t\big)^p\,\mathrm{d}t \tag{1.5.5}$$

for every $\sigma, \tau \in [0, \infty)$ with $\sigma \leq \tau$. Letting $h \downarrow 0$ in (1.5.4) and observing that

$$\mathbf{E}\big[\mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1, \mathsf{b}_r^2)/2\,\mathrm{d}r}\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1, \mathsf{b}_{2t}^2\big)\big] < \infty,$$

which justifies to apply Fatou's lemma, gives

$$\frac{\mathrm{d}^+}{\mathrm{d}t} W_p^{\underline{\ell}}(\mu_1, \mu_2, t)^p \leq \mathbf{E}\Big[\mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1, \mathsf{b}_r^2)/2\,\mathrm{d}r}\,\frac{\mathrm{d}^+}{\mathrm{d}h}\Big|_0 W_p^{\underline{\ell}}\big(\delta_{\mathsf{b}_{2t}^1}, \delta_{\mathsf{b}_{2t}^2}, h\big)^p\Big].$$

Finally, the inequality (1.5.5) for the upper derivative inside the latter expectation, Lemma 1.5.5 and then the assumed estimate (1.5.3), noting that $-\underline{k} \leq -\underline{\ell}$ everywhere on M^2, yield the initial claim.

The nonincreasingness of $t \mapsto W_p^{\underline{k}}(\mu_1, \mu_2, t)$ on $[0, \infty)$ is then immediate due to an easy approximation argument using Lemma 1.2.1 and Lemma 1.5.2. □

Theorem 1.5.7. *For every $p \in [1, \infty)$,* $\mathrm{PTE}_p(\underline{k})$ *and the differential p-transport estimate* (1.5.3) *are equivalent.*

Proof. According to Proposition 1.5.6, it suffices to prove that $\mathrm{PTE}_p(\underline{k})$ implies

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_p^p(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) \leq -p\,\underline{k}(x, y)\,\mathsf{d}^p(x, y)$$

for every $x, y \in M$. For every $t > 0$ and every $p' \in (p, \infty)$, we denote by $(\mathsf{b}^1, \mathsf{b}^2)$ a pair of coupled Brownian motions which realizes the minimum in the definition of $W_{p'}^{\underline{\ell}}(\delta_x, \delta_y, t)$. This process does depend on t and p', but we leave out these dependencies from the notation. Arguing as in the proof of Lemma 1.5.5, we get

$$\begin{aligned}
&\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_p^p(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) \\
&\qquad \leq \limsup_{t\downarrow 0} \frac{1}{t}\mathbf{E}\Big[\big[1 - \mathrm{e}^{\int_0^{2t} p\underline{\ell}(\mathsf{b}_r^1, \mathsf{b}_r^2)/2\,\mathrm{d}r}\big]\,\mathsf{d}^p\big(\mathsf{b}_{2t}^1, \mathsf{b}_{2t}^2\big)\Big] + \frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_{p'}^{\underline{k}}(\delta_x, \delta_y, t)^p
\end{aligned}$$

$$\leq -p\,\underline{\ell}(x,y)\,\mathsf{d}^p(x,y) + \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_0 W_{p'}^{\underline{k}}(\delta_x,\delta_y,t)^p$$

for every $\underline{\ell} \in \mathrm{C_b}(M^2)$ with $\underline{\ell} \leq \underline{k}$ on M^2. Letting $p' \downarrow p$, the last upper derivative becomes nonpositive due to $\mathrm{PTE}_p(k)$, and approximating $\underline{k}$ from below using Lemma 1.2.1 gives the conclusion. □

Using this equivalence, Hölder's inequality and the chain rule, the subsequent nestedness of $\mathrm{PTE}_p(k)$, which is the Lagrangian analogue of Lemma 1.3.3, is shown.

Corollary 1.5.8. *If* $\mathrm{PTE}_p(k)$ *holds for some* $p \in [1,\infty)$, *then* $\mathrm{PTE}_{p'}(k)$ *is satisfied for every* $p' \in [1,p]$.

1.5.3 Transport estimates via vertical Brownian perturbations

We prove the variable Kuwada duality from Theorem 1.1.8. We start by first showing the implication from $\mathrm{GE}_q(k)$ to $\mathrm{PTE}_p(k)$, where $p,q \in (1,\infty)$ are dual to each other. Since the behavior of Brownian trajectories can only be controlled for small times, we show the equivalent infinitesimal first-order description of $\mathrm{PTE}_p(k)$ in terms of a differential p-transport estimate. This is done by a localization argument.

In addition, in the extremal case $q = 1$, the argument mentioned above can actually be circumvented and we are able to derive the contraction estimate

$$\frac{\mathrm{d}^+}{\mathrm{d}t} W_p^p(\mathscr{H}_t\mu, \mathscr{H}_t\nu) \leq -p\int_0^1\int_{\mathrm{Geo}(M)} k(\gamma_s)\,|\dot\gamma|^p\,\mathrm{d}\boldsymbol{\pi}_t(\gamma)\,\mathrm{d}s$$

for every $t \geq 0$ and every $\mu,\nu \in \mathscr{P}(M)$ of finite W_p-distance to each other, for *every* $p \in (1,\infty)$. The measure $\boldsymbol{\pi}_t \in \mathscr{P}(\mathrm{Geo}(M))$ induces an *arbitrary* W_p-optimal coupling of $\mathscr{H}_t\mu$ and $\mathscr{H}_t\nu$. This is discussed now, see Theorem 1.5.10 and Corollary 1.5.11, where, possibly replacing k by $\min\{k,n\}$ for $n \in \mathbf{N}$, we assume that k is bounded. This is not restrictive as, given these results for every $n \in \mathbf{N}$, they easily pass to the limit $n \to \infty$ by Levi's theorem.

Recall that $\mathrm{G}_0(x,y)$ denotes the set of geodesics from $x \in M$ to $y \in M$. Given $p \in (1,\infty)$ and $t \geq 0$, we define the function $\mathsf{d}^0_{p,k,t} : M^2 \to [0,\infty)$ by

$$\mathsf{d}^0_{p,k,t}(x,y) := \inf_{\gamma\in\mathrm{G}_0(x,y)} \Big[\int_0^1 \mathbf{E}\big[\mathrm{e}^{-\int_0^{2t} p k(\mathsf{b}_r)/2\,\mathrm{d}r}\big]\,|\dot\gamma|^p\,\mathrm{d}s\Big]^{1/p}.$$

Here b denotes Brownian motion starting in γ_s for every $s \in [0,1]$. We will not explicitly mention the dependency of the process b on s. The function $\mathsf{d}^0_{p,k,t}$ can be turned into a metric $\mathsf{d}_{p,k,t}$ on M by defining

$$\mathsf{d}_{p,k,t}(x,y) := \inf\Big\{\sum_{i=1}^n \mathsf{d}^0_{p,k,t}(x_{i-1},x_i) : n \in \mathbf{N},\ x_0,\dots,x_n \in M,\ x_0 = x,\ x_n = y\Big\}.$$

It is equivalent to d by boundedness of k since d is a length metric. Let us denote by $W^0_{p,k,t}$ and $W_{p,k,t}$ the transport "distances" w.r.t. $\mathsf{d}^0_{p,k,t}$ and $\mathsf{d}_{p,k,t}$, respectively. Then $W_{p,k,t}$ is a metric on $\mathscr{P}_p(M)$, which is equivalent to the usual p-Kantorovich–Wasserstein metric W_p. Compared to the perturbed p-transport cost W_p^k which measures Brownian evolutions "horizontally" by following their trajectories with *fixed* starting points, the distance $W_{p,k,t}$ varies the initial points along a geodesic and may thus be seen as a "vertical" counterpart of W_p^k.

Let $(Q_s)_{s\geq 0}$ be the p-Hopf–Lax semigroup, $p \in (1,\infty)$, and $q \in (1,\infty)$ such that $1/p + 1/q = 1$. Similarly to [Kuw10, Prop. 3.7], the key point will be the following Lipschitz regularity along geodesics.

Lemma 1.5.9. *Let $f \in \mathrm{Lip_b}(M)$. Then for every $x, y \in M$, every $t > 0$ and every $\gamma \in \mathrm{G}_0(y,x)$, the map $s \mapsto \mathsf{P}_t Q_s f(\gamma_s)$ belongs to $\mathrm{Lip}([0,1])$, and*

$$\mathsf{P}_t Q_1 f(x) - \mathsf{P}_t f(y) \leq \int_0^1 \Big[\limsup_{h\downarrow 0} \frac{1}{h}\big[\mathsf{P}_t Q_s f(\gamma_{s+h}) - \mathsf{P}_t Q_s f(\gamma_s)\big] - \frac{1}{q}\mathsf{P}_t\big(\mathrm{lip}(Q_s f)^q\big)(\gamma_s)\Big]\,\mathrm{d}s.$$

Proof. Let $h \in (0,1)$ and $s \in [0, 1-h]$. Notice that

$$\begin{aligned}
&\frac{1}{h}\big|\mathsf{P}_t Q_{s+h} f(\gamma_{s+h}) - \mathsf{P}_t Q_s f(\gamma_s)\big| \\
&\qquad \leq \frac{1}{h}\big|\mathsf{P}_t Q_{s+h} f(\gamma_{s+h}) - \mathsf{P}_t Q_{s+h} f(\gamma_s)\big| + \frac{1}{h}\Big|\int_M \big[Q_{s+h} f - Q_s f\big]\,\mathrm{d}\mathscr{H}_t\delta_{\gamma_s}\Big| \\
&\qquad \leq \frac{\mathsf{d}(x,y)}{h}\int_s^{s+h} |\mathrm{d}\mathsf{P}_t Q_{s+h} f|(\gamma_v)\,\mathrm{d}v + \int_M \frac{1}{h}|Q_{s+h} f - Q_s f|\,\mathrm{d}\mathscr{H}_t\delta_{\gamma_s}.
\end{aligned}$$

The latter is bounded uniformly in s and h since the first integral can be controlled using the Lipschitz regularization estimate (1.2.1) of the heat flow while the second one exploits the fact that the map $s \mapsto Q_s f$ is Lipschitz from $[0,\infty)$ to $\mathrm{C}(M)$.

It follows that $\mathsf{P}_t Q_1 f(x) - \mathsf{P}_t f(y)$ is bounded from above by

$$\int_0^1 \Big[\limsup_{h\downarrow 0} \frac{1}{h}\big[\mathsf{P}_t Q_s f(\gamma_{s+h}) - \mathsf{P}_t Q_s f(\gamma_s)\big] + \limsup_{h\downarrow 0} \frac{1}{h}\int_M \big[Q_{s+h} f - Q_s f\big]\,\mathrm{d}\mathscr{H}_t\delta_{\gamma_{s+h}}\Big]\,\mathrm{d}s.$$

The Kantorovich–Rubinstein formula (1.2.2) for W_1, the W_1-contractivity of the heat flow and the duality of P_t and $\mathscr{H}_t$ give us the following upper bound for the second limsup in the previous expression:

$$\begin{aligned}
&\limsup_{h\downarrow 0} \frac{1}{h}\int_M \big[Q_{s+h} f - Q_s f\big]\,\mathrm{d}\big[\mathscr{H}_t\delta_{\gamma_{s+h}} - \mathscr{H}_t\delta_{\gamma_s}\big] \\
&\qquad\qquad + \limsup_{h\to 0} \frac{1}{h}\int_M \big[Q_{s+h} f - Q_s f\big]\,\mathrm{d}\mathscr{H}_t\delta_{\gamma_s} \\
&\qquad \leq \mathrm{Lip}(Q_\cdot f)\,\limsup_{h\downarrow 0} W_1\big(\mathscr{H}_t\delta_{\gamma_{s+h}}, \mathscr{H}_t\delta_{\gamma_s}\big) + \int_M \frac{\mathrm{d}}{\mathrm{d}s} Q_s f\,\mathrm{d}\mathscr{H}_t\delta_{\gamma_s} \\
&\qquad = -\frac{1}{q}\int_M \mathrm{lip}(Q_s f)^q\,\mathrm{d}\mathscr{H}_t\delta_{\gamma_s} \\
&\qquad = -\frac{1}{q}\mathsf{P}_t\big(\mathrm{lip}(Q_s f)^q\big)(\gamma_s).
\end{aligned}$$

Here we used $\mathrm{Lip}(Q_\cdot f)$ as a shorthand for the Lipschitz constant of the map $s \mapsto Q_s f$ from $[0,\infty)$ to $\mathrm{C}(M)$. These estimates conclude the proof. □

Theorem 1.5.10. *Assume the* 1*-gradient estimate* $\mathrm{GE}_1(\Bbbk)$*. Then for every* $p \in (1,\infty)$*,* $t \geq 0$ *and* $\mu, \nu \in \mathscr{P}(M)$*,*

$$W_p(\mathscr{H}_t\mu, \mathscr{H}_t\nu) \leq W_{p,\Bbbk,t}(\mu,\nu) \leq W^0_{p,\Bbbk,t}(\mu,\nu).$$

Proof. The second inequality is trivial since by definition $\mathsf{d}_{p,\Bbbk,t} \leq \mathsf{d}^0_{p,\Bbbk,t}$, thus we concentrate on the first one.

Let us initially consider the case $\mu := \delta_x$ and $\nu := \delta_y$ for $x, y \in M$, and $t > 0$. By the duality (1.2.2), we have to estimate $\mathsf{P}_t Q_1 f(x) - \mathsf{P}_t f(y)$ from above for every $f \in \mathrm{Lip}_{\mathrm{b}}(M)$. Pick a geodesic $\gamma \in \mathrm{G}_0(y,x)$. By the upper gradient property of $|\mathrm{d}\mathsf{P}_t Q_s f|$ and the $\mathrm{GE}_1(\Bbbk)$ inequality, we deduce for $\mathscr{L}^1$-a.e. $s \in [0,1]$ that

$$\begin{aligned}
&\limsup_{h\downarrow 0} \frac{1}{h}\big[\mathsf{P}_t Q_s f(\gamma_{s+h}) - \mathsf{P}_t Q_s f(\gamma_s)\big]\\
&\qquad \leq \limsup_{h\downarrow 0} \frac{\mathsf{d}(x,y)}{h}\int_s^{s+h} \mathsf{P}_t^{\Bbbk}|\mathrm{d}Q_s f|(\gamma_v)\,\mathrm{d}v\\
&\qquad \leq \mathsf{d}(x,y)\,\mathbf{E}\big[\mathrm{e}^{-\int_0^{2t} p\Bbbk(\mathsf{b}_r)/2\,\mathrm{d}r}\big]^{1/p}\,\mathsf{P}_t\big(\mathrm{lip}(Q_s f)^q\big)^{1/q}(\gamma_s),
\end{aligned}$$

denoting by b Brownian motion on M starting in γ_s. Invoking Lemma 1.5.9 and Young's inequality, we infer that

$$\mathsf{P}_t Q_1 f(x) - \mathsf{P}_t f(y) \leq \frac{\mathsf{d}^p(x,y)}{p}\int_0^1 \mathbf{E}\big[\mathrm{e}^{-\int_0^{2t} p\Bbbk(\mathsf{b}_r)/2\,\mathrm{d}r}\big]\,\mathrm{d}s. \tag{1.5.6}$$

Taking the supremum over $f \in \mathrm{Lip}_{\mathrm{b}}(M)$ and then infimizing over all geodesics γ connecting y to x, we deduce the inequality $W_p(\mathscr{H}_t\mu, \mathscr{H}_t\nu) \leq W^0_{p,\Bbbk,t}(\mu,\nu) = \mathsf{d}^0_{p,\Bbbk,t,}(x,y)$. By the triangle inequality and the definition of $\mathsf{d}_{p,\Bbbk,t}$, this already implies that $W_p(\mathscr{H}_t\mu, \mathscr{H}_t\nu) \leq W_{p,\Bbbk,t}(\mu,\nu) = \mathsf{d}_{p,\Bbbk,t,}(x,y)$ under the above assumptions.

The case $t = 0$ follows by letting $t \downarrow 0$ in (1.5.6) and then concluding as above.

Lastly, the inequality $W_p(\mathscr{H}_t\mu, \mathscr{H}_t\nu) \leq W_{p,\Bbbk,t}(\mu,\nu)$ for general $\mu, \nu \in \mathscr{P}(M)$ follows by a standard coupling argument as in the proof of [BHS21, Thm. 5.10]. □

With this in hand, we can proceed to what we have indicated in Remark 1.1.9.

Corollary 1.5.11. *Assume* $\mathrm{GE}_1(\Bbbk)$*. Let* $\mu, \nu \in \mathscr{P}(M)$ *such that* $W_p(\mu,\nu) < \infty$*, let* $t \geq 0$*, and let* $\boldsymbol{\pi}_t \in \mathscr{P}(\mathrm{Geo}(M))$ *represent an arbitrary* W_p*-optimal coupling between* $\mathscr{H}_t\mu$ *and* $\mathscr{H}_t\nu$*, i.e.* $(\mathsf{e}_0, \mathsf{e}_1)_\sharp\boldsymbol{\pi}_t$ *is a* W_p*-optimal coupling of* $\mathscr{H}_t\mu$ *and* $\mathscr{H}_t\nu$*. Then*

$$\frac{\mathrm{d}^+}{\mathrm{d}t} W_p^p(\mathscr{H}_t\mu, \mathscr{H}_t\nu) \leq -\int_0^1\int_{\mathrm{Geo}(M)} \Bbbk(\gamma_s)\,|\dot\gamma|^p\,\mathrm{d}\boldsymbol{\pi}_t(\gamma)\,\mathrm{d}s.$$

Proof. Given any optimal geodesic plan $\boldsymbol{\pi}_t$ as above, using Theorem 1.5.10 gives

$$\begin{aligned}
&\limsup_{h\to 0}\frac{1}{ph}\big[W_p^p(\mathscr{H}_{t+h}\mu, \mathscr{H}_{t+h}\nu) - W_p^p(\mathscr{H}_t\mu, \mathscr{H}_t\nu)\big]\\
&\qquad \leq \limsup_{h\downarrow 0}\frac{1}{ph}\big[W^0_{p,\Bbbk,h}(\mathscr{H}_t\mu, \mathscr{H}_t\nu)^p - W_p^p(\mathscr{H}_t\mu, \mathscr{H}_t\nu)\big]\\
&\qquad \leq \limsup_{h\downarrow 0}\frac{1}{ph}\int_{\mathrm{Geo}(M)}\Big[\int_0^1 \mathbf{E}\big[\mathrm{e}^{-\int_0^{2h} p\Bbbk(\mathsf{b}_r)/2\,\mathrm{d}r}\big]\,\mathrm{d}s - 1\Big]\,\mathsf{d}^p(\gamma_0,\gamma_1)\,\mathrm{d}\boldsymbol{\pi}_t(\gamma)
\end{aligned}$$

$$= -\int_0^1 \int_{\mathrm{Geo}(M)} \mathscr{k}(\gamma_s)\,|\dot\gamma|^p\,\mathrm{d}\boldsymbol{\pi}_t(\gamma)\,\mathrm{d}s,$$

where b denotes Brownian motion on M starting in γ_s. In the very last step, we used the assumed boundedness of $\mathscr{k}$ together with Lebesgue's theorem. □

Remark 1.5.12. Corollary 1.5.11 applied to $\mu := \delta_x$ and $\nu := \delta_y$ for $x, y \in M$ at $t = 0$, choosing $\boldsymbol{\pi}_0$ as the Dirac mass on an arbitrary $\gamma \in \mathrm{G}_0(x, y)$, yields the estimate

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_p^p(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) \leq -p \sup_{\gamma\in\mathrm{G}_0(x,y)} \int_0^1 \mathscr{k}(\gamma_s)\,\mathrm{d}s\,\mathsf{d}^p(x,y) \leq -p\,\overline{\mathscr{k}}(x,y)\,\mathsf{d}^p(x,y),$$

where, as in (1.1.3), the function $\overline{\mathscr{k}}\colon M^2 \to \mathbf{R}$ is defined by

$$\overline{\mathscr{k}}(x,y) := \liminf_{\substack{x'\to x,\\ y'\to y}} \sup_{\gamma\in\mathrm{G}_0(x,y)} \int_0^1 \mathscr{k}(\gamma_s)\,\mathrm{d}s.$$

Note that $\overline{\mathscr{k}}$ is lower semicontinuous and bounded from below.

This improves the differential p-transport estimate (1.5.3), since $\underline{\mathscr{k}} \leq \overline{\mathscr{k}}$ on M^2, see also Proposition 1.4.6. In Section 1.6, we shall construct a coupling of Brownian motions obeying pathwise bounds involving the larger function $\overline{\mathscr{k}}$ in place of $\underline{\mathscr{k}}$. In particular, using Theorem 1.5.17, all equivalences from Theorem 1.1.1 and Theorem 1.1.8 are still valid when replacing the function $\underline{\mathscr{k}}$ by $\overline{\mathscr{k}}$ in all relevant quantities. ■

The proof of the $\mathrm{PTE}_p(\mathscr{k})$ property starting from $\mathrm{GE}_q(\mathscr{k})$ with dual $p, q \in (1,\infty)$ is slightly more involved as a control of the error terms is only possible "locally" for small times. A crucial ingredient is the subsequent result.

Lemma 1.5.13. *Let $u, v \in L_\infty(M)$ such that $u \leq v$ on a ball $B_\delta(z)$, $z \in M$ and $\delta > 0$. Then for every $p \in (1,\infty)$ and every $\varepsilon > 0$, there exists $t_* > 0$ such that for every $t \in [0, t_*]$, every nonnegative Borel function $g\colon M \to \mathbf{R}$, and every Brownian motion b^x on M starting in $x \in B_{\delta/2}(z)$, we have*

$$\mathbf{E}\big[\mathrm{e}^{\int_0^t u(\mathsf{b}_r^x)\,\mathrm{d}r} g(\mathsf{b}_t^x)\big] \leq \mathbf{E}\big[\mathrm{e}^{p\int_0^t (v(\mathsf{b}_r^x)+\varepsilon)\,\mathrm{d}r} g^p(\mathsf{b}_t^x)\big]^{1/p}.$$

Proof. The condition on u and v guarantees that for fixed $T > 0$ and every $t \in [0,T]$,

$$\begin{aligned}\mathrm{e}^{\int_0^t u(\mathsf{b}_r^x)\,\mathrm{d}r} - \mathrm{e}^{\int_0^t v(\mathsf{b}_r^x)\,\mathrm{d}r} &= \int_0^t \mathrm{e}^{\int_0^s u(\mathsf{b}_r^x)\,\mathrm{d}r + \int_s^t v(\mathsf{b}_r^x)\,\mathrm{d}r} (u-v)(\mathsf{b}_s^x)\,\mathrm{d}s\\ &\leq C\int_0^t 1_{\{\mathsf{b}_s^x\notin B_\delta(z)\}}\,\mathrm{d}s.\end{aligned}$$

Here, $C > 0$ is a constant depending only on u, v and T. Therefore,

$$\begin{aligned}\mathbf{E}\big[\mathrm{e}^{\int_0^t u(\mathsf{b}_r^x)\,\mathrm{d}r} g(\mathsf{b}_t^x)\big] &\leq \mathbf{E}\big[\mathrm{e}^{\int_0^t v(\mathsf{b}_r^x)\,\mathrm{d}r} g(\mathsf{b}_t^x)\big]\\ &\qquad + C\int_0^t \mathbf{E}\big[\mathrm{e}^{\int_0^t v(\mathsf{b}_r^x)\,\mathrm{d}r} g(\mathsf{b}_t^x)\,1_{\{\mathsf{b}_s^x\notin B_\delta(z)\}}\big]\,\mathrm{d}s\\ &\leq \mathbf{E}_x\big[\mathrm{e}^{\int_0^t pv(\mathsf{b}_r^x)\,\mathrm{d}r} g^p(\mathsf{b}_t^x)\big]^{1/p}\Big[1 + C\int_0^t \mathbf{P}\big[\mathsf{b}_s^x \notin B_\delta(z)\big]^{1/q}\,\mathrm{d}s\Big],\end{aligned}$$

where $q \in (1,\infty)$ denotes the dual exponent to p. Thanks to Lemma 1.4.2, we know that $\mathbf{P}[\mathsf{b}_s^x \notin B_\delta(z)] \leq s^q$ for every $s \in [0,t]$ and small enough t. Thus, $1 + C\int_0^t \mathbf{P}[\mathsf{b}_s^x \notin B_\delta(z)]^{1/q}\,\mathrm{d}s \leq \mathrm{e}^{\varepsilon t}$, which directly proves the claim. □

Remark 1.5.14. With the very same strategy, also estimates for Feynman–Kac-type expressions in terms of pairs of Brownian motions can be derived, every component being required to start within $B_{\delta/2}(z)$. Moreover, the integrands u and v are then supposed to be functions on M^2 with $u \leq v$ on $B_\delta(z)^2$. ■

Proposition 1.5.15. *Let $p, q \in (1,\infty)$ such that $1/p+1/q = 1$ and assume the q-gradient estimate $\mathrm{GE}_q(\underline{k})$. Assume that $\underline{\ell} \in \mathrm{C_b}(M^2)$ with $\underline{\ell} \leq \underline{k}$ on M^2, and set $\ell(x) := \underline{\ell}(x,x)$ for $x \in M$. Then for every $\varepsilon > 0$, every $p' \in (1, p)$ and every $z \in M$, there exist $\delta > 0$ and $t_* > 0$ such that for every $x, y \in B_\delta(z)$, every $\gamma \in \mathrm{G}_0(y,x)$ and every $t \in [0, t^*]$,*

$$W_{p'}^{p'}(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) \leq \mathsf{d}(x,y)\,\mathrm{e}^{-[\int_0^1 \ell(\gamma_r)\,\mathrm{d}r - \varepsilon]t},$$

and thus in particular,

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_{p'}(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) \leq -\mathsf{d}(x,y)\,\Big[\int_0^1 \ell(\gamma_r)\,\mathrm{d}r - \varepsilon\Big].$$

Proof. We adapt the proof of Theorem 1.5.10 by adding a localization argument. Given $z \in M$ and $\varepsilon > 0$, choose $\delta > 0$ and $L_z \in \mathbf{R}$ such that $L_z \leq \ell \leq L_z + \varepsilon/2$ on $B_{3\delta}(z)$. Let $x, y \in B_\delta(z)$ and $\gamma \in \mathrm{G}_0(y,x)$, and note that

$$L_z \leq \int_0^1 \ell(\gamma_r)\,\mathrm{d}r \leq L_z + \frac{\varepsilon}{2}.$$

Denote by Q_s the p'-Hopf–Lax semigroup with dual exponent $q' \in (q,\infty)$. Since $|\mathrm{d}\mathsf{P}_t Q_s f|$ is a weak upper gradient and using $\mathrm{GE}_q(\underline{k})$, which clearly implies $\mathrm{GE}_q(\underline{\ell})$, we directly obtain, for $\mathscr{L}^1$-a.e. $s \in [0,1]$,

$$\limsup_{h\downarrow 0} \frac{1}{h}\big[\mathsf{P}_t Q_s f(\gamma_{s+h}) - \mathsf{P}_t Q_s f(\gamma_s)\big] \leq \mathsf{d}(x,y)\,\big[\mathsf{P}_t^{q\ell}|\mathrm{d}Q_s f|^q\big]^{1/q}(\gamma_s).$$

Applying Lemma 1.5.13 with $\varepsilon/2$ and $t/2$ in place of ε and t, respectively, we get, for small enough t,

$$\big[\mathsf{P}_t^{q\ell}|\mathrm{d}Q_s f|^q\big]^{1/q}(\gamma_s) \leq \mathrm{e}^{-(L_z-\varepsilon/2)t}\,\mathsf{P}_t\big(\mathrm{lip}(Q_s f)^{q'}\big)^{1/q'}(\gamma_s),$$

and thus

$$\mathsf{d}(x,y)\,\big[\mathsf{P}_t^{q\ell}|\mathrm{d}Q_s f|\big]^{1/q}(\gamma_s) \leq \frac{\mathsf{d}^{p'}(x,y)}{p'}\,\mathrm{e}^{-p'(L_z-\varepsilon/2)t} + \frac{1}{q'}\mathsf{P}_t\big(\mathrm{lip}(Q_s f)^{q'}\big)(\gamma_s)$$

for $\mathscr{L}^1$-a.e. $s \in [0,1]$ by Young's inequality. Lemma 1.5.9 with q' in place of q yields

$$\mathsf{P}_t Q_1 f(x) - \mathsf{P}_t f(y) \leq \frac{\mathsf{d}^{p'}(x,y)}{p'}\,\mathrm{e}^{-p'(L_z-\varepsilon/2)t} \leq \frac{\mathsf{d}^{p'}(x,y)}{p'}\,\mathrm{e}^{-p'[\int_0^1 \ell(\gamma_r)\,\mathrm{d}r-\varepsilon]t}.$$

Taking the supremum over $f \in \mathrm{Lip_b}(M)$, we conclude by (1.2.2). □

Theorem 1.5.16. *Given $p, q \in (1,\infty)$ with $1/p + 1/q = 1$, the q-gradient estimate $\mathrm{GE}_q(\underline{k})$ implies the p-transport estimate $\mathrm{PTE}_p(\underline{k})$.*

Proof. Fix $x, y \in M$, an arbitrary geodesic $\gamma \in \mathrm{G}_0(y,x)$ and ℓ as in Proposition 1.5.15. Given $\varepsilon > 0$, choose a finite covering of $\gamma([0,1])$ by metric balls $B_{\delta_i/2}(\gamma_{s_i})$, $i \in \{1,\dots,n\}$ and $n \in \mathbf{N}$, such that each of the enlarged balls $B_{\delta_i}(\gamma_{s_i})$ satisfies the assumption of the previous Proposition 1.5.15. Without restriction, we may and will assume $s_1 = 0$ and $s_n = 1$. Applying this proposition to pairs of intermediate points $\gamma_{s_{i-1}}$ and γ_{s_i} and the reparameterized geodesics $\gamma^i \in \mathrm{G}_0(\gamma_{s_{i-1}}, \gamma_{s_i})$ defined by $\gamma^i_r := \gamma_{s_{i-1}+r(s_i-s_{i-1})}$, $r \in [0,1]$, yields

$$\begin{aligned}
\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_{p'}(\mathscr{H}_t\delta_x, \mathscr{H}_t\delta_y) &\le \sum_{i=1}^n \frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_{p'}\big(\mathscr{H}_t\delta_{\gamma_{s_{i-1}}}, \mathscr{H}_t\delta_{\gamma_{s_i}}\big)\\
&\le -\sum_{i=1}^n \mathsf{d}(\gamma_{s_{i-1}}, \gamma_{s_i})\Big[\int_0^1 \ell\big(\gamma^i_r\big)\,\mathrm{d}r - \varepsilon\Big]\\
&= -\mathsf{d}(x,y)\Big[\int_0^1 \ell(\gamma_r)\,\mathrm{d}r - \varepsilon\Big].
\end{aligned}$$

Since ℓ is arbitrary, this bound holds with $\mathscr{k}$ in place of ℓ by Lemma 1.2.1. Moreover, by definition of $\underline{\mathscr{k}}$ and the arbitrariness of $\varepsilon > 0$, we deduce the differential transport estimate (1.5.3) with p replaced by p'. Since this true for every $p' \in (1,p)$, this finally yields $\mathrm{PTE}_p(\mathscr{k})$ by Proposition 1.5.6 and Levi's theorem. $\square$

1.5.4 Gradient estimates out of pathwise and transport estimates

A modification of the arguments given in [Kuw10, Prop. 3.1] allows us to prove the converse direction of Theorem 1.1.8, i.e. that the p-transport estimate $\mathrm{PTE}_p(\mathscr{k})$ implies the q-gradient estimate $\mathrm{GE}_q(\mathscr{k})$, where $1/p + 1/q = 1$. As in the previous subsection, a control of the error terms can only be achieved for small times. Therefore, instead of deriving $\mathrm{GE}_q(\mathscr{k})$ directly, it is more convenient to establish a local version of the q-Bochner inequality $\mathrm{BE}_q(\mathscr{k}, \infty)$.

As in the preceding Subsection 1.5.3, the extremal version $q = 1$ is much easier to treat: in this case, the condition "$\mathrm{PTE}_\infty(\mathscr{k})$" is to be interpreted as "$\mathrm{PTE}_p(\mathscr{k})$ holds for every $p \in [1,\infty)$", which translates into $\mathrm{PCP}(\mathscr{k})$ as discussed in Section 1.6. In the proof of the similar manifold statement from Theorem 2.1.6 below, see Subsection 2.4.3, when no lower bound on $\underline{\mathscr{k}}$ is available we will again have to use a short-time argument, using the good behavior of Brownian paths within small time regions.

Theorem 1.5.17. *The property* $\mathrm{PCP}(\mathscr{k})$ *implies the 1-gradient estimate* $\mathrm{GE}_1(\mathscr{k})$*, that is, for every* $f \in W^{1,2}(M)$ *and every* $t \ge 0$*, we have*

$$\Gamma(\mathsf{P}_t f)^{1/2} \le \mathsf{P}^{\mathscr{k}}_t\big(\Gamma(f)^{1/2}\big) \quad \mathfrak{m}\text{-a.e.}$$

Proof. Fix $f \in \mathrm{Lip}_{\mathrm{bs}}(M)$ and $x \in M$. Pick a function $\underline{\ell} \in \mathrm{Lip}_{\mathrm{b}}(M^2)$ with $\underline{\ell} \le \underline{\mathscr{k}}$ on M^2, and set $\ell(x) := \underline{\ell}(x,x)$ for $x \in M$. By $\mathrm{PCP}(\mathscr{k})$, given any $\varrho > 0$ and $y \in B_\varrho(x)$, we may and will choose a pair $(\mathsf{b}^1, \mathsf{b}^2)$ of coupled Brownian motions starting in x and y, respectively, in such a way that $\mathbf{P}$-a.s., we have

$$\mathsf{d}\big(\mathsf{b}^1_t, \mathsf{b}^2_t\big) \le \mathrm{e}^{-\int_0^t \underline{\mathscr{k}}(\mathsf{b}^1_r, \mathsf{b}^2_r)/2\,\mathrm{d}r}\,\mathsf{d}(x,y) \le \mathrm{e}^{-\int_0^t \underline{\ell}(\mathsf{b}^1_r, \mathsf{b}^2_r)/2\,\mathrm{d}r}\,\mathsf{d}(x,y) \tag{1.5.7}$$

for every $t \ge 0$. With this in hand, we can estimate

$$|\mathrm{d}\mathsf{P}_{t/2} f|(x) \le \lim_{\varrho\downarrow 0}\,\sup_{y\in B_\varrho(x)} \frac{|\mathsf{P}_{t/2} f(x) - \mathsf{P}_{t/2} f(y)|}{\mathsf{d}(x,y)}$$

$$\leq \lim_{\varrho\downarrow 0} \sup_{y\in B_\varrho(x)} \mathbf{E}\Big[\frac{|f(\mathsf{b}_t^1) - f(\mathsf{b}_t^2)|}{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)} \frac{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}{\mathsf{d}(x,y)} \big[1_{U_{\varrho,t}} + 1_{V_{\varrho,t}} + 1_{W_{\varrho,t}}\big]\Big],$$

where $V_{\varrho,t} := \big\{\mathsf{d}\big(\mathsf{b}_t^1,\mathsf{b}_t^2\big) \geq \varrho^{1/2}\big\}$, $W_{\varrho,t} := \big\{\int_0^t \mathsf{d}\big(\mathsf{b}_r^1,\mathsf{b}_r^2\big)\,\mathrm{d}r/t \geq \varrho^{1/2}\big\}$ as well as $U_{\varrho,t} := V_{\varrho,t}^{\mathrm{c}} \cap W_{\varrho,t}^{\mathrm{c}}$.

Let us consider this upper bound for the weak upper gradient $|\mathrm{d}\mathsf{P}_{t/2}f|(x)$ term by term, starting with the contribution coming from $U_{\varrho,t}$. We have the inequality

$$\int_0^t \underline{\ell}(\mathsf{b}_r^1,\mathsf{b}_r^2)\,\mathrm{d}r \geq \int_0^t \ell(\mathsf{b}_r^1)\,\mathrm{d}r - \mathrm{Lip}(\underline{\ell})t\varrho^{1/2} \quad \text{on } W_{\varrho,t}^{\mathrm{c}},$$

which gives

$$\begin{aligned}
&\lim_{\varrho\downarrow 0} \sup_{y\in B_\varrho(x)} \mathbf{E}\Big[\frac{|f(\mathsf{b}_t^1) - f(\mathsf{b}_t^2)|}{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)} \frac{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}{\mathsf{d}(x,y)} 1_{U_{\varrho,t}}\Big]\\
&\quad\leq \lim_{\varrho\downarrow 0} \sup_{y\in B_\varrho(x)} \mathbf{E}\Big[\mathrm{e}^{-\int_0^t \ell(\mathsf{b}_r^1)/2\,\mathrm{d}r + \mathrm{Lip}(\underline{\ell})t\varrho^{1/2}/2} \sup_{z\in B_{\varrho^{1/2}}(\mathsf{b}_t^1)} \Big|\frac{f(\mathsf{b}_t^1) - f(z)}{\mathsf{d}(\mathsf{b}_t^1, z)}\Big|\Big]\\
&\quad= \lim_{\varrho\downarrow 0} \widetilde{\mathbf{E}}\Big[\mathrm{e}^{-\int_0^t \ell(\mathsf{x}_r)/2\,\mathrm{d}r + \mathrm{Lip}(\underline{\ell})t\varrho^{1/2}/2} \sup_{z\in B_{\varrho^{1/2}}(\mathsf{x}_t)} \Big|\frac{f(\mathsf{x}_t) - f(z)}{\mathsf{d}(\mathsf{x}_t, z)}\Big|\Big]\\
&\quad= \widetilde{\mathbf{E}}\big[\mathrm{e}^{-\int_0^t \ell(\mathsf{x}_r)/2\,\mathrm{d}r}\, |\mathrm{d}f|(\mathsf{x}_t)\big] = \mathsf{P}_{t/2}^{\ell}\big(\Gamma(f)^{1/2}\big)(x).
\end{aligned}$$

We point out the intermediate change from the process b^1, which in general also depends on y, to a Brownian motion x on M starting in x under an appropriate probability $\widetilde{\mathbf{P}}$, chosen independently of y.

Next we consider the term involving $1_{V_{\varrho,t}}$. Denoting by $C > 0$ a suitable upper bound on $\underline{\ell}$, we obtain by (1.5.7) that

$$\begin{aligned}
&\lim_{\varrho\downarrow 0} \sup_{y\in B_\varrho(x)} \mathbf{E}\Big[\frac{|f(\mathsf{b}_t^1) - f(\mathsf{b}_t^2)|}{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)} \frac{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}{\mathsf{d}(x,y)} 1_{V_{\varrho,t}}\Big]\\
&\quad\leq \mathrm{Lip}(f) \lim_{\varrho\downarrow 0} \frac{1}{\varrho^{1/2}} \sup_{y\in B_\varrho(x)} \mathbf{E}\Big[\frac{\mathsf{d}^2(\mathsf{b}_t^1,\mathsf{b}_t^2)}{\mathsf{d}(x,y)}\Big]\\
&\quad\leq \mathrm{Lip}(f)\,\mathrm{e}^{Ct} \lim_{\varrho\downarrow 0} \frac{1}{\varrho^{1/2}} \sup_{y\in B_\varrho(x)} \mathsf{d}(x,y) = 0.
\end{aligned}$$

Similarly, the last expression which involves $W_{\varrho,t}$ can be bounded through

$$\begin{aligned}
&\lim_{\varrho\downarrow 0} \sup_{y\in B_\varrho(x)} \mathbf{E}\Big[\frac{|f(\mathsf{b}_t^1) - f(\mathsf{b}_t^2)|}{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)} \frac{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}{\mathsf{d}(x,y)} 1_{W_{\varrho,t}}\Big]\\
&\quad\leq \mathrm{Lip}(f) \lim_{\varrho\downarrow 0} \frac{1}{t\varrho^{1/2}} \sup_{y\in B_\varrho(x)} \int_0^t \mathbf{E}\Big[\frac{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)\mathsf{d}(\mathsf{b}_r^1,\mathsf{b}_r^2)}{\mathsf{d}(x,y)}\Big]\,\mathrm{d}r\\
&\quad\leq \mathrm{Lip}(f)\,\mathrm{e}^{Ct} \lim_{\varrho\downarrow 0} \frac{1}{\varrho^{1/2}} \sup_{y\in B_\varrho(x)} \mathsf{d}(x,y) = 0.
\end{aligned}$$

Finally, we have to extend the class of admissible functions f and pass to $\mathrm{GE}_1(\mathscr{k})$. By uniform convexity of $\mathscr{E}$, every $f \in W^{1,2}(M)$ can be approximated strongly in

$W^{1,2}(M)$ by a sequence of Lipschitz functions f_n with bounded support [AGS14a]. Thus, possibly passing to a subsequence, we get, for some suitable $c \in \mathbf{R}$, that

$$\lim_{n\to\infty} \mathsf{P}_t^{\ell}\big(\Gamma(f-f_n)^{1/2}\big) \le \mathrm{e}^{ct} \lim_{n\to\infty} \mathsf{P}_t\big(\Gamma(f-f_n)^{1/2}\big) = 0 \quad \mathfrak{m}\text{-a.e.}$$

Moreover, $\Gamma(\mathsf{P}_t f_n) \to \Gamma(\mathsf{P}_t f)$ in $L^1(M)$ as $n \to \infty$ and thus, up to a subsequence, this convergence holds $\mathfrak{m}$-a.e., which then proves $\mathrm{GE}_1(\ell)$ for arbitrary $f \in W^{1,2}(M)$. By the arbitrariness of $\underline{\ell}$, Lemma 1.2.1 and the identity $\Bbbk(x) = \underline{\Bbbk}(x,x)$ for every $x \in M$, we deduce $\mathrm{GE}_1(\Bbbk)$ by Levi's theorem. □

Proposition 1.5.18. *Let $\varepsilon > 0$, $z \in M$ and $q \in (1,\infty)$. Assume the transport estimate $\mathrm{PTE}_p(\Bbbk)$, where $1/p + 1/q = 1$. Suppose that $\underline{\ell} \in \mathrm{C_b}(M^2)$ with $\underline{\ell} \le \underline{\Bbbk}$ on M^2. Then for every $q' \in (q,\infty)$, there exist $t_* > 0$ and $\delta > 0$ such that*

$$\Gamma(\mathsf{P}_t f)^{q'/2} \le \mathsf{P}_t^{q'(\ell-\varepsilon)}\big(\Gamma(f)^{q'/2}\big) \quad \mathfrak{m}\text{-a.e.} \quad \text{on } B_\delta(z)$$

for every $t \in [0,t_*]$ and every $f \in \mathrm{Lip_b}(M)$.

Proof. Fix $T > 0$. Given $\varepsilon > 0$, choose $\delta > 0$ and $L_z \in \mathbf{R}$ such that $L_z \le \underline{\ell}(x,y) \le L_z + \varepsilon/3$ for every $x, y \in B_{3\delta}(z)$. Given $t \in [0,T]$, $x \in B_\delta(z)$ and $y \in B_\varrho(z)$ with $\varrho \in (0,\delta]$, select a pair $(\mathsf{b}^1, \mathsf{b}^2)$ of coupled Brownian motions starting in (x,y) which attains the minimum in the definition of $W_p^{\Bbbk}(\delta_x, \delta_y, t/2) \le \mathsf{d}(x,y)$. The choice of this pair does depend on x, y and t, but these dependencies are suppressed in the notation. Similarly to the proof of Theorem 1.5.17, for every $f \in \mathrm{Lip_b}(M)$, we have

$$|\mathsf{dP}_{t/2} f|(x) \le \lim_{\varrho\downarrow 0} \sup_{y \in B_\varrho(x)} \mathbf{E}\Big[\frac{|f(\mathsf{b}_t^1) - f(\mathsf{b}_t^2)|}{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)} \frac{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}{\mathsf{d}(x,y)} \big[1_{V_{\varrho,t}} + 1_{V_{\varrho,t}^{\mathrm{c}}}\big]\Big]$$

where $V_{\varrho,t} := \big\{\mathsf{d}\big(\mathsf{b}_t^1,\mathsf{b}_t^2\big) \ge \varrho^{1/2q}\big\}$. The contribution of $V_{\varrho,t}$ vanishes as $\varrho \downarrow 0$ due to

$$\begin{aligned}
&\lim_{\varrho\downarrow 0} \sup_{y \in B_\varrho(x)} \mathbf{E}\Big[\frac{|f(\mathsf{b}_t^1) - f(\mathsf{b}_t^2)|}{\mathsf{d}(x,y)} 1_{V_{\varrho,t}}\Big] \\
&\qquad \le \mathrm{Lip}(f)\, \mathrm{e}^{Ct} \lim_{\varrho\downarrow 0} \Big[\varrho^{(1-p)/2q} \\
&\qquad\qquad \times \sup_{y \in B_\varrho(x)} \frac{1}{\mathsf{d}(x,y)} \mathbf{E}\Big[\mathrm{e}^{\int_0^t p\underline{\Bbbk}(\mathsf{b}_r^1,\mathsf{b}_r^2)/2\,\mathrm{d}r} \mathsf{d}^p\big(\mathsf{b}_t^1,\mathsf{b}_t^2\big)\Big]\Big] \\
&\qquad \le \mathrm{Lip}(f)\, \mathrm{e}^{Ct} \lim_{\varrho\downarrow 0} \varrho^{(1-p)/2q} \sup_{y \in B_\varrho(x)} \mathsf{d}^{p-1}(x,y) = 0
\end{aligned}$$

for a suitable $C > 0$, where we used the assumption that $\underline{\ell} \le \underline{\Bbbk}$ in the first inequality and the $\mathrm{PTE}_p(\Bbbk)$ condition in the last inequality.

Next we study the influence coming from $V_{\varrho,t}^{\mathrm{c}}$. Choosing some exponents $q'' \in (q,q')$ and $p'' \in (1,p')$ dual to each other, using Hölder's inequality, Lemma 1.5.13 with $\varepsilon/3$ and $t/2$ in place of ε and t, respectively, and eventually assumption $\mathrm{PTE}_p(\Bbbk)$, we obtain for sufficiently small t that

$$\begin{aligned}
&\lim_{\varrho\downarrow 0} \sup_{y \in B_\varrho(x)} \mathbf{E}\Big[\frac{|f(\mathsf{b}_t^1) - f(\mathsf{b}_t^2)|}{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)} \frac{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}{\mathsf{d}(x,y)} 1_{V_{\varrho,t}^{\mathrm{c}}}\Big] \\
&\qquad \le \mathrm{e}^{-(L_z-\varepsilon/3)t/2} \lim_{\varrho\downarrow 0} \sup_{y \in B_\varrho(x)} \mathbf{E}\Big[\Big|\frac{f(\mathsf{b}_t^1) - f(\mathsf{b}_t^2)}{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}\Big|^{q''} 1_{V_{\varrho,t}^{\mathrm{c}}}\Big]^{1/q''}
\end{aligned}$$

$$
\begin{aligned}
&\times \lim_{\varrho\downarrow 0}\sup_{y\in B_\varrho(x)} \mathbf{E}\Big[\mathrm{e}^{p''(L_z-\varepsilon/3)t/2}\,\Big|\frac{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}{\mathsf{d}(x,y)}\Big|^{p''}\Big]^{1/p''}\\
&\leq \mathrm{e}^{-(L_z-\varepsilon/3)t/2}\,\lim_{\varrho\downarrow 0}\widetilde{\mathbf{E}}\Big[\sup_{z\in B_{\varrho^{1/2q}}(\mathsf{x}_t)}\Big|\frac{f(\mathsf{x}_t)-f(z)}{\mathsf{d}(\mathsf{x}_t,z)}\Big|^{q''}\Big]^{1/q''}\\
&\times \frac{1}{\mathsf{d}(x,y)}\,W_p^{\underline{k}}(\delta_x,\delta_y,t)\\
&\leq \mathrm{e}^{-(L_z-\varepsilon/3)t/2}\,\widetilde{\mathbf{E}}\big[|\mathrm{d}f|^{q''}(\mathsf{x}_t)\big]^{1/q''}.
\end{aligned}
$$

Here x is a Brownian motion (under $\widetilde{\mathbf{P}}$) on M starting in x, chosen independently of y. Once again using Lemma 1.5.13 as above to estimate the last expression, we obtain

$$
\lim_{\varrho\downarrow 0}\sup_{y\in B_\varrho(x)}\mathbf{E}\Big[\frac{|f(\mathsf{b}_t^1)-f(\mathsf{b}_t^2)|}{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}\,\frac{\mathsf{d}(\mathsf{b}_t^1,\mathsf{b}_t^2)}{\mathsf{d}(x,y)}\,1_{V_{\varrho,t}^{\mathrm{c}}}\Big] \leq \mathsf{P}_t^{q'(\ell-\varepsilon)}\big(|\mathrm{d}f|^{q'}\big)^{1/q'}(x). \qquad \square
$$

Theorem 1.5.19. *Given $p,q\in(1,\infty)$ with $1/p+1/q=1$, the p-transport estimate* $\mathrm{PTE}_p(k)$ *implies the q-gradient estimate* $\mathrm{GE}_q(k)$.

Proof. Let $\underline{\ell}$ be as in Proposition 1.5.18 and set $\ell(x):=\underline{\ell}(x,x)$ for $x\in M$. First, we assume that $q\in[2,\infty)$. Given $\varepsilon>0$, $z\in M$, $t_*>0$, $q'\in(q,\infty)$ and the associated time $t_*>0$ from in Proposition 1.5.18, straightforwardly arguing as in the proof of [BHS21, Thm. 3.4], the function $F\colon [0,t_*]\to\mathbf{R}$ defined by

$$
F(t) := \int_M \Big[\mathsf{P}_t^{q'(\ell-\varepsilon)}\big(\Gamma(f)^{q'/2}\big) - \Gamma(\mathsf{P}_t f)^{q'/2}\Big]\,\phi\,\mathrm{d}\mathfrak{m}
$$

belongs to $\mathrm{C}^1([0,t_*])$ for every $f\in\mathrm{Test}(M)$ and every nonnegative function $\phi\in W^{1,2}(M)\cap L^\infty(M)$ supported in $B_\delta(z)$. The function F itself and its derivative at 0 are nonnegative by Proposition 1.5.18. The latter translates into

$$
-\int_M\Big[\frac{1}{q'}\Gamma\big(\Gamma(f)^{q'/2},\phi\big)+\Gamma(f)^{q'/2}\,\Gamma(f,\Delta f)\,\phi\Big]\,\mathrm{d}\mathfrak{m} \geq \int_M(\ell-\varepsilon)\,\Gamma(f)^{q'/2}\,\phi\,\mathrm{d}\mathfrak{m}.
$$

Approximating k from below by the sequence $k_n\in\mathrm{Lip}_{\mathrm{b}}(M)$ of functions $k_n(x):=\underline{k}_n(x,x)$ for $x\in M$, or in other words, replacing $\underline{\ell}$ by $\underline{k}_n$ for every $n\in\mathbf{N}$, where $\underline{k}_n$ tends to $\underline{k}$ from below as provided by Lemma 1.2.1, and letting $q'\downarrow q$ and $\varepsilon\downarrow 0$, we obtain precisely the local q-Bakry–Émery inequality $\mathrm{BE}_{q,\mathrm{loc}}(k,\infty)$ according to Definition 1.3.8. Since the latter implies $\mathrm{BE}_q(k,\infty)$ by Theorem 1.3.9, the equivalence with $\mathrm{GE}_q(k)$ finishes the proof in the case $q\in[2,\infty)$.

If $q\in[1,2)$, choosing $q':=2$ in Proposition 1.5.18 and arguing as above, we obtain $\mathrm{BE}_2(k,\infty)$, which in turn implies $\mathrm{BE}_q(k,\infty)$. $\square$

1.6 A pathwise coupling estimate

It remains to treat the pathwise coupling property w.r.t. k to finish the proof of Theorem 1.1.1. By Theorem 1.5.17, we know that $\mathrm{PCP}(k)$ implies $\mathrm{GE}_1(k)$. Conversely, letting $\overline{k}$ be the function from Remark 1.5.12 which is even larger than $\underline{k}$, $\mathrm{GE}_1(k)$ implies that, for every $p\in(1,\infty)$ and every $x,y\in M$,

$$
\frac{\mathrm{d}^+}{\mathrm{d}t}\Big|_0 W_p^p(\mathscr{H}_t\delta_x,\mathscr{H}_t\delta_y) \leq -p\,\overline{k}(x,y)\,\mathsf{d}^p(x,y).
$$

The same argument as for Proposition 1.5.6 then shows that $t \mapsto W_p^{\overline{\mathscr{k}}}(\delta_x, \delta_y, t)$ is nonincreasing for every $p \in (1, \infty)$ and every $x, y \in M$. Therefore, $\mathrm{PCP}(\mathscr{k})$ follows once having proven the subsequent stronger statement which has a weaker assumption.

Theorem 1.6.1. *Suppose that, for every large enough $p \in (1, \infty)$, the map $t \mapsto W_p^{\overline{\mathscr{k}}}(\delta_x, \delta_y, t)$ is nonincreasing on $[0, \infty)$ for every $x, y \in M$. Then for every $\mu_1, \mu_2 \in \mathscr{P}(M)$ there exists a pair $(\mathsf{b}^1, \mathsf{b}^2)$ of coupled Brownian motions on M with initial distributions μ_1 and μ_2, respectively, such that $\mathbf{P}$-a.s., we have*

$$\mathsf{d}(\mathsf{b}_t^1, \mathsf{b}_t^2) \leq \mathrm{e}^{-\int_s^t \overline{\mathscr{k}}(\mathsf{b}_r^1, \mathsf{b}_r^2)/2\,\mathrm{d}r}\, \mathsf{d}(\mathsf{b}_s^1, \mathsf{b}_s^2)$$

for every $s, t \in [0, \infty)$ with $s \leq t$. In particular, the pathwise coupling property $\mathrm{PCP}(\mathscr{k})$ is satisfied.

For the proof of Theorem 1.6.1, it is necessary to adapt the arguments from [Stu15, Sec. 2] in a nontrivial way, since our pathwise estimate requires control of the entire path of $(\mathsf{b}^1, \mathsf{b}^2)$ on the interval $[s, t]$ and not just at the endpoints.

The proof of Theorem 1.6.1 will be subdivided into multiple steps. Firstly, we construct a coupled process starting in $\delta_x \otimes \delta_y$, $x, y \in M$, satisfying the desired pathwise contraction estimate on the interval $[0, 1]$. Secondly, a gluing procedure will let us extend the process to $[0, \infty)$. Finally, we use a coupling technique to allow for arbitrary initial distributions.

Proposition 1.6.2. *Under the same assumptions as in Theorem 1.6.1, for every $t \geq 0$, there exists a universally measurable map $\boldsymbol{\mu}^t : M^2 \to \mathscr{P}(\Pi_t)$ such that for every $x, y \in X$, the marginals of $\boldsymbol{\mu}_{x,y}^t := \boldsymbol{\mu}^t(x, y)$ are laws of Brownian motions, restricted to $[0, t]$, starting in x and y, respectively, and for $\boldsymbol{\mu}_{x,y}^t$-a.e. $\gamma \in \Pi_t$,*

$$\mathsf{d}(\gamma_t^1, \gamma_t^2) \leq \mathrm{e}^{-\int_0^t \overline{\mathscr{k}}(\gamma_r^1, \gamma_r^2)/2\,\mathrm{d}r}\, \mathsf{d}(x, y).$$

Proof. Given $x, y \in M$ and an increasing sequence $(p_n)_{n\in\mathbf{N}}$ tending to ∞, denote by $\underline{\boldsymbol{\eta}}_{x,y}^{t,n} \in \mathscr{P}(\Pi_t)$ the measure obtained by Lemma 1.5.3 for the exponent p_n, $\underline{\ell}$ replaced by $\overline{\mathscr{k}}$, and time $t/2$ in place of t. As for Lemma 1.5.2, we see that the sequence $(\boldsymbol{\eta}_{x,y}^{t,n})_{n\in\mathbf{N}}$ is tight. Hence it converges weakly to some $\boldsymbol{\eta}_{x,y}^t \in \mathscr{P}(\Pi_t)$ along a subsequence which we do not relabel.

Let $p \in (1, \infty)$ arbitrary, and fix $\overline{\ell} \in \mathrm{C_b}(M^2)$ with $\overline{\ell} \leq \overline{\mathscr{k}}$ on M^2. Then by Hölder's inequality and the nonincreasingness of $t \mapsto W_{p_n}^{\overline{\mathscr{k}}}(\delta_x, \delta_y, t)$ for large enough n,

$$\begin{aligned}
&\Big[\int_{\Pi_t} \mathrm{e}^{\int_0^t p\overline{\ell}(\gamma_r^1, \gamma_r^2)/2\,\mathrm{d}r}\, \mathsf{d}^p(\gamma_t^1, \gamma_t^2)\, \mathrm{d}\boldsymbol{\eta}_{x,y}^t(\gamma)\Big]^{1/p} \\
&\qquad \leq \liminf_{n\to\infty} \Big[\int_{\Pi_t} \mathrm{e}^{\int_0^t p\overline{\ell}(\gamma_r^1, \gamma_r^2)/2\,\mathrm{d}r}\, \mathsf{d}^p(\gamma_t^1, \gamma_t^2)\, \mathrm{d}\boldsymbol{\eta}_{x,y}^{t,n}(\gamma)\Big]^{1/p} \\
&\qquad \leq \limsup_{n\to\infty} \Big[\int_{\Pi_t} \mathrm{e}^{\int_0^t p_n\overline{\mathscr{k}}(\gamma_r^1, \gamma_r^2)/2\,\mathrm{d}r}\, \mathsf{d}^{p_n}(\gamma_t^1, \gamma_t^2)\, \mathrm{d}\boldsymbol{\eta}_{x,y}^{t,n}(\gamma)\Big]^{1/p_n} \leq \mathsf{d}(x, y).
\end{aligned}$$

Sending $p \to \infty$ and then approximating $\overline{\mathscr{k}}$ from below by Lemma 1.2.1 gives

$$\mathsf{d}(\gamma_t^1, \gamma_t^2) \leq \mathrm{e}^{-\int_0^t \overline{\mathscr{k}}(\gamma_r^1 \gamma_r^2)/2\,\mathrm{d}r}\, \mathsf{d}(x, y)$$

for $\boldsymbol{\eta}_{x,y}^t$-a.e. $\gamma \in \Pi_t$. A measurable selection argument as in the proof of Lemma 1.5.3 establishes the claim. □

The next goal is to obtain a measure which obeys such pathwise bound at *every* initial and terminal time instance in, say, $[0,1]$. Indeed, this is the point where the main work has to be done.

Theorem 1.6.3. *Under the same assumptions as in Theorem 1.6.1, there exists a universally measurable map $\boldsymbol{\mu}\colon M^2 \to \mathscr{P}(\Pi_1)$ such that for every $x, y \in M$, we have that the marginals of $\boldsymbol{\mu}_{x,y} := \boldsymbol{\mu}(x,y)$ are laws of Brownian motions, restricted to $[0,1]$, starting in x and y, respectively, and that there exists a $\boldsymbol{\mu}_{x,y}$-negligible Borel set $E \subset \Pi_1$ with the property that*

$$\mathsf{d}\big(\gamma_t^1, \gamma_t^2\big) \le \mathrm{e}^{-\int_s^t \overline{k}(\gamma_r^1,\gamma_r^2)/2\,\mathrm{d}r}\,\mathsf{d}\big(\gamma_s^1,\gamma_s^2\big)$$

for every $s, t \in [0,1]$ with $s \le t$, for every $\gamma \in \Pi_1 \setminus E$.

Proof. The strategy relies on patching the laws obtained in the previous proposition together on small dyadic partitions of $[0,1]$. Denote by $\boldsymbol{\mu}^{2^{-n}}$ the map from Proposition 1.6.2 and define $\boldsymbol{\mu}_{n,x,y} \in \mathscr{P}(\Pi_1)$ by

$$\boldsymbol{\mu}_{n,x,y} := \underbrace{\boldsymbol{\mu}^{2^{-n}} \circ \cdots \circ \boldsymbol{\mu}^{2^{-n}}}_{2^{n-1}\text{ kernels}} \circ\, \boldsymbol{\mu}_{x,y}^{2^{-n}},$$

that is, at every dyadic partition point of $[0,1]$ at scale 2^{-n}, we attach a new random curve evolving according to the law obtained in Proposition 1.6.2 to the random endpoint of the previous curve. The marginals of $\boldsymbol{\mu}_{n,x,y}$ are the laws of Brownian motions on M, restricted to $[0,1]$, starting in x and y, respectively. As in the proof of Lemma 1.5.2, we may exhibit a (non-relabeled) subsequence weakly converging to some $\boldsymbol{\mu}_{x,y} \in \mathscr{P}(\Pi_1)$.

The key point lies in proving that for every $s, t \in \mathbf{Q} \cap [0,1]$ with $s \le t$, there exists a $\boldsymbol{\mu}_{x,y}$-negligible Borel set $E_{s,t} \subset \Pi_1$ such that, for every $\gamma \in \Pi_1 \setminus E_{s,t}$,

$$\mathsf{d}\big(\gamma_t^1, \gamma_t^2\big) \le \mathrm{e}^{-\int_s^t \overline{k}(\gamma_r^1,\gamma_r^2)/2\,\mathrm{d}r}\,\mathsf{d}\big(\gamma_s^1,\gamma_s^2\big). \tag{1.6.1}$$

By continuity, the desired requirements are then satisfied by the $\boldsymbol{\mu}_{x,y}$-null set

$$E := \bigcup_{\substack{s,t\in\mathbf{Q}\cap[0,1],\\ s\le t}} E_{s,t}.$$

Let $\overline{\ell} \in \mathrm{C_b}(M^2)$ as above, i.e. $\overline{\ell} \le \overline{k}$ on M^2. Pick s and t as above and notice that the sequences $(s_m)_{m\in\mathbf{N}}$ and $(t_m)_{m\in\mathbf{N}}$, $s_m := 2^{-m}\lfloor 2^m s\rfloor$ and $t_m := 2^{-m}\lfloor 2^m t\rfloor$, tend to s and t, respectively. Fix $m \in \mathbf{N}$ and an arbitrary $n \ge m$. Given any $i \in \{1, \dots, 2^n - 1\}$, for every path $\widetilde{\gamma} \in \Pi_{2^{-n}}$ one gets, for $\boldsymbol{\mu}^{2^{-n}}_{\widetilde{\gamma}^1_{2^{-n}},\widetilde{\gamma}^2_{2^{-n}}}$-a.e. $\gamma \in \Pi_{2^{-n}}$,

$$\mathsf{d}\big(\gamma^1_{2^{-n}}, \gamma^2_{2^{-n}}\big) \le \mathrm{e}^{-\int_0^{2^{-n}} \overline{\ell}(\gamma_r^1,\gamma_r^2)/2\,\mathrm{d}r}\,\mathsf{d}\big(\widetilde{\gamma}^1_{2^{-n}}, \widetilde{\gamma}^2_{2^{-n}}\big).$$

Observing that the dyadic partition of $[0,1]$ of step size 2^{-n} contains the one at scale 2^{-m} and then integrating the resulting $\boldsymbol{\mu}_{n,x,y}$-a.e. valid estimate, truncated at large enough $C > 0$, against an arbitrary nonnegative function $\phi \in \mathrm{C_b}(\Pi_1)$, we obtain

$$\begin{aligned}&\int_{\Pi_1} \phi(\gamma)\,\mathsf{d}_C\big(\gamma^1_{t_m}, \gamma^2_{t_m}\big)\,\mathrm{d}\boldsymbol{\mu}_{n,x,y}(\gamma)\\ &\qquad\le \int_{\Pi_1} \phi(\gamma)\,\mathrm{e}^{-\int_{2^{-n}\lfloor 2^n s_m\rfloor}^{2^{-n}\lfloor 2^n t_m\rfloor} \overline{\ell}(\gamma_r^1,\gamma_r^2)/2\,\mathrm{d}r}\,\mathsf{d}_C\big(\gamma^1_{s_m}, \gamma^2_{s_m}\big)\,\mathrm{d}\boldsymbol{\mu}_{n,x,y}(\gamma),\end{aligned}$$

where $\mathsf{d}_C := \min\{\mathsf{d}, C\}$. Since $\overline{\ell}$ is bounded, for every $m \in \mathbf{N}$, every $\varepsilon > 0$ and every large enough n this yields

$$\begin{aligned}\int_{\Pi_1} \phi(\gamma)\, \mathsf{d}_C\big(\gamma^1_{t_m}, \gamma^2_{t_m}\big)\, \mathrm{d}\boldsymbol{\mu}_{n,x,y}(\gamma) &\\ \leq \int_{\Pi_1} \phi(\gamma)\, \mathrm{e}^{-\int_{s_m}^{t_m} \overline{\ell}(\gamma^1_r,\gamma^2_r)/2\,\mathrm{d}r}\, \mathsf{d}_C\big(\gamma^1_{s_m}, \gamma^2_{s_m}\big)\, \mathrm{d}\boldsymbol{\mu}_{n,x,y}(\gamma) &\\ + \varepsilon \int_{\Pi_1} \phi(\gamma)\, \mathsf{d}_C\big(\gamma^1_{s_m}, \gamma^2_{s_m}\big)\, \mathrm{d}\boldsymbol{\mu}_{n,x,y}(\gamma).&\end{aligned}$$

Letting $n \to \infty$, $\varepsilon \downarrow 0$ and then $C \to \infty$ in the previous estimate as well as extending the class of ϕ to nonnegative, bounded Borel functions by a routine approximation argument, we get, for $\boldsymbol{\mu}_{x,y}$-a.e. $\gamma \in \Pi_1$,

$$\mathsf{d}\big(\gamma^1_{t_m}, \gamma^2_{t_m}\big) \leq \mathrm{e}^{-\int_{s_m}^{t_m} \overline{\ell}(\gamma^1_r,\gamma^2_r)/2\,\mathrm{d}r}\, \mathsf{d}\big(\gamma^1_{s_m}, \gamma^2_{s_m}\big). \tag{1.6.2}$$

Let us now set

$$\widetilde{E}_{s,t} := \bigcup_{m\in\mathbf{N}} \big\{\gamma \in \Pi_1 : \gamma \text{ does not satisfy (1.6.2)}\big\},$$

which clearly satisfies $\boldsymbol{\mu}_{x,y}\big[\widetilde{E}_{s,t}\big] = 0$, and (1.6.1) holds on $\Pi_1 \setminus \widetilde{E}_{s,t}$ with $\overline{\ell}$ in place of $\overline{k}$ by the convergences $s_m \to s$ and $t_m \to t$ as $m \to \infty$. Finally, denoting by $(\overline{k}_n)_{n\in\mathbf{N}}$ a sequence in $\mathrm{Lip_b}(M)$ approximating $\overline{k}$ from below as provided by Lemma 1.2.1, the above reasoning gives Borel subsets $\widetilde{E}^n_{s,t}$ of Π_1 such that $\boldsymbol{\mu}_{x,y}\big[\widetilde{E}^n_{s,t}\big] = 0$ and

$$\mathsf{d}\big(\gamma^1_t, \gamma^2_t\big) \leq \mathrm{e}^{-\int_s^t \overline{k}_n(\gamma^1_r,\gamma^2_r)/2\,\mathrm{d}r}\, \mathsf{d}\big(\gamma^1_s, \gamma^2_s\big)$$

for every $\gamma \in \Pi_1 \setminus \widetilde{E}^n_{s,t}$. Then $\boldsymbol{\mu}_{x,y}\big[E_{s,t}\big] = 0$ for

$$E_{s,t} := \bigcup_{n\in\mathbf{N}} \widetilde{E}^n_{s,t},$$

and that (1.6.1) holds for every $\gamma \in \Pi_1 \setminus E_{s,t}$ by Levi's theorem.

A similar argument and arguing as for Lemma 1.5.3 shows that we can then select the obtained measures in a universally measurable way. □

The cases of arbitrary initial distributions $\mu \in \mathscr{P}(M^2)$ and an infinite time horizon are immediate given the construction in the proof of Theorem 1.6.3.

Proof of Theorem 1.6.1. By iteratively composing copies of $\boldsymbol{\mu}$ with $\boldsymbol{\mu} \circ \mu$, we obtain a measure $\boldsymbol{\rho}_\mu \in \mathscr{P}(\mathrm{C}([0,\infty); M^2))$ such that $(\mathsf{e}_0)_\sharp \boldsymbol{\rho}_\mu = \mu$. The pathwise coupling properties on each interval $[n-1, n]$, $n \in \mathbf{N}$, which are inherited by $\boldsymbol{\mu}$ carry over to the entire space.

By considering the canonical process $(\mathsf{b}^1, \mathsf{b}^2)$ defined by $\mathsf{b}^1_t(\gamma) := \gamma^1_t$ and $\mathsf{b}^2_t(\gamma) := \gamma^2_t$ under the measure $\boldsymbol{\rho}_\mu$, we immediately obtain the assertion of Theorem 1.6.1, which is just a stochastic rephrasing of the previous considerations. □

Chapter Two

Heat flow regularity, Bismut–Elworthy–Li's derivative formula, and pathwise couplings on Riemannian manifolds with Kato bounded Ricci curvature

This chapter is based on the author's joint work [BG20] with Batu Güneysu, from which large parts are taken over *verbatim*.

In this chapter, let M be a smooth, geodesically complete, noncompact, connected Riemannian manifold without boundary according to Subsection 3.2.1 below. Let d and $\mathfrak{m} := \mathfrak{v}$ denote the Riemannian distance and the Riemannian volume induced by the metric tensor $\langle\cdot,\cdot\rangle$, respectively. With the usual abuse of notation, the fiberwise norm both on TM and T^*M is $|\cdot| := \langle\cdot,\cdot\rangle^{1/2}$. Let ∇ be the Levi-Civita connection on M and Ric be the induced Ricci curvature. Unlike the other chapters and following a common abuse of terminology, here by *Brownian motion* starting in $x \in M$ we mean the M-valued diffusion process b^x, defined on a suitable probability space $(\Omega, \mathscr{A}, \mathbf{P})$ with explosion time ζ^x, generated by the (halved, drift-free, essentially self-adjoint when initially considered on $\mathrm{C}_\mathrm{c}^\infty(M)$ [Str83]) *Laplace–Beltrami* operator $\Delta/2$ [Elw82, Hsu02a, IW81, Wan14]. $(\mathsf{P}_t)_{t\geq 0}$ is the heat flow generated by Δ. However, our results will hold in larger generality, see the end of the next Section 2.1.

Throughout, we fix a continuous function $k\colon M \to \mathbf{R}$. We write "Ric $\geq k$ on M" if, for every $x \in M$ and every $\xi \in T_xM$,

$$\mathrm{Ric}(x)(\xi,\xi) \geq k(x)\,|\xi|^2.$$

2.1 Main results

The goal of this chapter is to study the previous condition, where the negative part k^- of k obeys the following integrability assumption for every $t > 0$:

$$\mathsf{C}_t := \sup_{x\in M} \mathbf{E}\big[\mathrm{e}^{\int_0^{2t} k^-(\mathsf{b}_r^x)/2\,\mathrm{d}r}\, 1_{\{t<\zeta^x/2\}}\big] < \infty. \tag{2.1.1}$$

Our main results come in two groups. First, we study analytic and probabilistic *consequences* of the assumption Ric $\geq k$ on M if k satisfies (2.1.1), as described below and stated in Theorem 2.1.1 and Theorem 2.1.5. Along with this, we treat an explicit class of k for which (2.1.1) holds, the so-called *Kato decomposable* ones, and highlight a general condition for k to obey the latter property, Theorem 2.1.3. Second, as outlined in Chapter 1 we give equivalent *characterizations* of the condition Ric $\geq k$

on M, which are summarized below as well, see Theorem 2.1.6 therein, and mostly do not even require (2.1.1).

Besides [ER+20, GvR20], this work is among the first to systematically study analytic and probabilistic consequences of variable lower Ricci bounds — and equivalent characterizations of these — which are not uniformly bounded from below and do not underlie geometric growth conditions. We also stress our novel general sufficient condition from Theorem 2.1.3 to determine whether a given variable Ricci curvature lower bound is Kato decomposable, while the — albeit more general — condition (2.1.1) is in general hard to verify directly. Lastly, our equivalence result improves upon previously known ones especially because it involves a pathwise coupling estimate which has been introduced in Chapter 1, i.e. [BHS21].

Consequences of variable lower Ricci bounds To formulate our first result, given an initial point $x \in M$, let $/\!\!/^x$ denote the stochastic parallel transport w.r.t. ∇ along the sample paths of b^x, i.e. $/\!\!/^x_r \in \mathrm{Hom}(T_xM; T_{\mathsf{b}^x_r}M)$ for every $r \in [0, \zeta^x)$, let the $\mathrm{End}(T_xM)$-valued process Q^x be defined as the unique solution, a priori up to ζ^x, to the pathwise ordinary differential equation

$$\begin{aligned} \mathrm{d}\mathsf{Q}^x_s &= -\mathsf{Q}^x_s\,(/\!\!/^x_{2s})^{-1}\,\mathrm{Ric}(\mathsf{b}^x_{2s})\,/\!\!/^x_{2s}\,\mathrm{d}s,\\ \mathsf{Q}^x_0 &= \mathrm{Id}_{T_xM}, \end{aligned} \tag{2.1.2}$$

where $\mathrm{Ric}(\mathsf{b}^x_{2s})$ is canonically regarded as an element of $\mathrm{End}(T_{\mathsf{b}^x_{2s}}M)$. Let W^x denote the anti-development of b^x, a canonically given Euclidean Brownian motion on T_xM. See [Elw82, Hsu02a, IW81, Wan14] for details and precise definitions.

Theorem 2.1.1. *Let* $k\colon M \to \mathbf{R}$ *be a continuous function satisfying* (2.1.1) *and assume that* $\mathrm{Ric} \geq k$ *on* M*. Then the following properties hold.*

(i) *M is* stochastically complete*, i.e. for every $x \in M$,*

$$\mathbf{P}\big[\zeta^x = \infty\big] = 1.$$

(ii) *For every $f \in L^\infty(M)$ and every $t > 0$, the heat operator P_t obeys* Bismut–Elworthy–Li's derivative formula

$$\big\langle \nabla \mathsf{P}_t f(x), \xi\big\rangle = \frac{1}{\sqrt{2}\,t}\,\mathbf{E}\Big[f(\mathsf{b}^x_{2t})\int_0^t \big\langle \mathsf{Q}^x_s\xi, \mathrm{d}W^x_s\big\rangle\Big]$$

for every $x \in M$ and every $\xi \in T_xM$, where the stochastic integral inside the expectation is understood in Itô's sense.

(iii) *For every $t > 0$, one has the L^∞-*Lip-regularization property $\mathsf{P}_t(L^\infty(M)) \subset \mathrm{Lip}(M)$ *and, for every $f \in L^\infty(M)$,*

$$\mathrm{Lip}(\mathsf{P}_t f) \leq \frac{2}{\sqrt{t}}\,\sup_{x\in M}\mathbf{E}\big[\mathrm{e}^{\int_0^t k^-(\mathsf{b}^x_r)/2\,\mathrm{d}r}\big]\,\|f\|_{L^\infty(M)}.$$

Before further commenting on Theorem 2.1.1 and its proof, in order to make more refined statements, we introduce the following definition.

Definition 2.1.2. *The* (functional) Kato class $\mathrm{K}(M)$ *of M is the linear space of all Borel functions* $\mathsf{v}\colon M \to \mathbf{R}$ *such that*

$$\lim_{t\downarrow 0}\,\sup_{x\in M}\int_0^t \mathbf{E}\big[|\mathsf{v}|(\mathsf{b}^x_r)\,1_{\{r<\zeta^x\}}\big]\,\mathrm{d}r = 0.$$

A Borel function $\mathsf{v}\colon M \to \mathbf{R}$ *is called* Kato decomposable *if it belongs to* $L^1_{\mathrm{loc}}(M)$ *and* v^- *belongs to* $\mathrm{K}(M)$.

Kato (decomposable) functions have been studied in great detail in the literature in the context of (scalar) Schrödinger operators, see [AS82, BG78, CZ95, Gün17a, SV96, Stu94] and the references therein. We also refer to Subsection 3.2.6 below for a more general account on Kato classes of signed measures. The survey [RS20] provides a concise overview over the use of Kato decomposability in the context of Riemannian manifolds and its connections to semigroup domination. A detailed study of the Kato class and the induced Schrödinger semigroups corresponding to a large class of Hunt processes can be found in [DvC00]. In connection with lower Ricci bounds, Kato decomposable functions have been introduced in [GP15] in the context of BV functions. They have been considered further recently in [Car19, Ros19] in the context of heat kernel, Betti number and eigenvalue estimates, and in [MO20] within the study on L^p-properties of heat semigroups on forms. See also [GvR20], which treats some probabilistic and geometric aspects of molecular Schrödinger operators under Kato assumptions.

In particular, note that in view of

$$\mathbf{E}\big[|\mathsf{v}|(\mathsf{b}_r^x)\,1_{\{r<\zeta^x\}}\big] \leq \|\mathsf{v}\|_{L^\infty(M)}$$

for every $x \in M$ and every $r \geq 0$, it follows that $L^\infty(M) \subset \mathrm{K}(M)$. More generally, in view of an explicit Example 2.5.5, we provide the following criterion in Subsection 2.5.2, for which we denote by $\Xi\colon M \to \mathbf{R}$ the function $\Xi(x) := \mathfrak{v}[B_1(x)]^{-1}$.

Theorem 2.1.3. *Assume that* $\dim M \geq 2$*, that* M *is quasi-isometric to a complete Riemannian manifold whose Ricci curvature is bounded from below by a constant, and that* $k^- \in L^p(M, \Xi\,\mathfrak{v}) + L^\infty(M)$ *for some* $p \in (\dim M/2, \infty)$*. Then* k *is Kato decomposable.*

One key feature for us about functions $\mathsf{v} \in \mathrm{K}(M)$ is that they always satisfy

$$\sup_{x\in M} \mathbf{E}\big[\mathrm{e}^{\int_0^{2t} \mathsf{v}(\mathsf{b}_r^x)/2\,\mathrm{d}r}\,1_{\{t<\zeta^x/2\}}\big] < \infty$$

locally uniformly in $t \in [0,\infty)$. This is known as *Khasminskii's lemma*, see [Gün17a, Lem. VI.8] for a comprehensive proof. In particular, since $\mathrm{K}(M)$ is a linear space, we have the following link of Kato decomposability to (2.1.1).

Lemma 2.1.4. *Assume that* k *is a Kato decomposable function. Then for every* $q \in [1,\infty)$*, the exponential integrability* (2.1.1) *holds with* k *replaced by* qk.

This is ultimately the key behind the following result which states that in this case, Bismut–Elworthy–Li's derivative formula holds on an L^p-scale.

Theorem 2.1.5. *Assume* $k\colon M \to \mathbf{R}$ *is a continuous Kato decomposable function satisfying* $\mathrm{Ric} \geq k$ *on* M*. Then* (2.1.1) *is satisfied for every* $t \geq 0$*, and moreover, Bismut–Elworthy–Li's derivative formula from Theorem 2.1.1 holds for every* $p \in (1,\infty]$*, every* $f \in L^p(M)$ *and every* $t > 0$.

The proof of (i) in Theorem 2.1.1 can be found in Subsection 2.3.1, while (ii) and (iii) as well as Theorem 2.1.5 are studied in Subsection 2.3.2.

Let us collect some bibliographical comments on Theorem 2.1.1 and Theorem 2.1.5.

In the framework of uniform bounds from below on the Ricci curvature, (i) in Theorem 2.1.1 is due to [Yau78]. On weighted Riemannian manifolds — on which the Ricci tensor is always replaced by the corresponding Bakry–Émery Ricci tensor, see the last paragraph below — the non-explosion for the induced diffusion processes under uniform lower Ricci bounds has been obtained by [Bak86]. In connection with (2.1.1), also for weighted Riemannian manifolds, the latter result has been extended by [Li94] using an approach via stochastic and Hessian flows. In fact, the corresponding condition at [Li94, p. 423] is implied by our condition (2.1.1). Once we have established all necessary intermediate results, our proof then closely follows the lines in [Bak86] (which is also worked towards in [Li94]). For different, more geometric non-explosion criteria in terms of distance functions, see [Wan14] and the references therein. A nonsmooth result similar to (i) — assuming a Kato- or rather a Dynkin-type [SV96, Stu94] lower bound instead of only (2.1.1) — has recently been shown in [ER+20]. This includes the corresponding result for RCD spaces [AGS14a, Thm. 4.20].

Formula (ii) in Theorem 2.1.1 has first appeared in [Bis84a] in the compact case. In the noncompact case, this result, as well as Theorem 2.1.5, have been proven in [EL94a, EL94b] under more general assumptions than (2.1.1) using the slightly different technique of stochastic derivative flows. We also refer to [DT01] for similar treatises for heat semigroups over vector bundles, and also [Hsu02a, Wan14] for similar results under more geometric conditions on the lower bound of Ric. Remarkably, localized versions of the Bismut–Elworthy–Li derivative formula hold without any assumptions on the geometry of the manifold, see e.g. [Tha97, TW98, TW11].

The L^∞-Lip-regularization (iii) from Theorem 2.1.1 is a corollary of (ii), thus indicating the importance of the latter in studying further regularity properties of $(\mathsf{P}_t)_{t\geq 0}$. In fact, *local* versions of (iii) are already known even without the assumption (2.1.1) on k [TW98, Wan14], with slightly different estimates on $\operatorname{Lip}(\mathsf{P}_t f)$ involving locally uniform lower bounds on Ric. (The proof uses the above mentioned local derivative formula.) Outside the smooth scope, a similar property as (iii) is known on $\mathrm{RCD}(K,\infty)$ spaces [AGS14b] (recall Section 1.2). Of course, this setting allows for more flexibility in the variety of spaces (metric measure spaces), but is still restricted to (synthetic) uniform lower Ricci bounds.

Characterizations of variable lower Ricci bounds We now come to our second main result, i.e. several equivalent characterizations of lower Ricci bounds, which we shortly introduce in the next lines.

The closest characterization of $\mathrm{Ric} \geq k$ on M is the (smooth pointwise version, recall Definition 1.1.4, of the) L^1-*Bochner inequality* which is related to the Ricci curvature of M by the following well-known Bochner formula: given any open $U \subset M$ and denoting by $|\cdot|_{\mathrm{HS}}$ the usual fiberwise Hilbert–Schmidt norm, for every $f \in \mathrm{C}^\infty(U)$,

$$\Delta\frac{|\nabla f|^2}{2} = \langle \nabla\Delta f, \nabla f\rangle + \big|\mathrm{Hess}\, f\big|_{\mathrm{HS}}^2 + \mathrm{Ric}(\nabla f, \nabla f) \quad \text{on } U. \tag{2.1.3}$$

We also derive a one-to-one connection between lower boundedness of Ric by k and the existence of certain couplings of Brownian motions on M similar to (vi) in Theorem 1.1.1. Recall the definition (1.1.2) of the average $\underline{k}\colon M^2 \to \mathbf{R}$ of k which, by local compactness of M, simplifies to

$$\underline{k}(x,y) = \inf_{\gamma\in \mathrm{G}_0(x,y)} \int_0^1 k(\gamma_r)\,\mathrm{d}r. \tag{2.1.4}$$

Theorem 2.1.6. *Let $k\colon M \to \mathbf{R}$ be a continuous function satisfying* (2.1.1). *Then the following conditions are equivalent.*

(i) *We have* Ric $\geq k$ *on M.*
(ii) *The L^1*-Bochner inequality w.r.t. *k is satisfied, i.e. for every $f \in \mathrm{C}_\mathrm{c}^\infty(M)$,*

$$\Delta|\nabla f| - |\nabla f|^{-1}\,\langle\nabla\Delta f, \nabla f\rangle \geq k\,|\nabla f| \quad \text{on } \{|\nabla f| \neq 0\}. \tag{2.1.5}$$

(iii) *We have the* pathwise coupling property w.r.t. *k, i.e. M is stochastically complete and for every $x, y \in M$, there exists a coupling $(\mathsf{b}^x, \mathsf{b}^y)$ of Brownian motions on M (recall Definition 1.1.6) starting in x and y, respectively, such that $\mathbf{P}$-a.s.,*

$$\mathsf{d}\big(\mathsf{b}_t^x, \mathsf{b}_t^y\big) \leq \mathrm{e}^{-\int_s^t \underline{k}(\mathsf{b}_r^x, \mathsf{b}_r^y)/2\,\mathrm{d}r}\,\mathsf{d}\big(\mathsf{b}_s^x, \mathsf{b}_s^y\big)$$

for every $s, t \geq 0$ with $s \leq t$.

Remark 2.1.7. Thanks to the local, respectively pathwise, nature of the statements, the implications from (iii) to (ii), from (ii) to (i) and from (i) to (ii) are even true without (2.1.1). Moreover, under the *a priori* assumption of stochastic completeness, (iii) follows from (i) without (2.1.1). For a slightly more general version of (iii) implying (ii), see Remark 2.4.7 below. ■

We prove (ii) implying (i) in Subsection 2.4.1, (i) implying (iii) in Subsection 2.4.2 and (iii) implying (ii) in Subsection 2.4.3. For Kato decomposable functions k, another equivalent characterization of Ric $\geq k$ on M in terms of the *L^1-gradient estimate* similar to Definition 1.1.5 above is discussed in Subsection 2.5.1.

Again, some bibliographical comments are in order.

In the abstract framework of [ER+20], the equivalence between (i) and (ii) — with (ii) in a weak formulation — together with their equivalence to (a nonsmooth version of) the L^1-gradient estimate from Theorem 2.5.1 has been shown independently. More details about this are to be found in Subsection 3.2.7 below.

The pathwise estimate appearing in (iii), as well as the equivalence of (iii) to lower Ricci bounds, extends the results from Chapter 1. Even for the smooth case, the stated pathwise inequality involving the function $\underline{k}$ has been firstly introduced in the corresponding work [BHS21]. (Although it is quite straightforward to detect the place where $\underline{k}$ enters from the construction of the coupling, see Subsection 2.4.2, the function $\underline{k}$ was seemingly never mentioned explicitly in the literature before [BHS21].) In the Riemannian case, Theorem 2.1.6 establishes a similar result in full generality without any lower boundedness assumption on k. We point out that, in contrast to Chapter 1, the coupling technique on manifolds does not require any notion of "Wasserstein contractivity" for the dual heat flow to $(\mathsf{P}_t)_{t\geq 0}$ on the space of Borel probability measures on M such as in Definition 1.1.7. It is rather provided in a direct way by the method of *coupling by parallel displacement* [Cra91, Ken86].

Extensions to possible other settings Apart from well-known geometric and topological applications [Bis86, Bue99, Li94], recent results [ER+20, GvR20] suggest a detailed study of *weighted* Riemannian manifolds having Kato-type lower bounds on their Bakry–Émery Ricci tensor. In this context, Theorem 2.1.1, Theorem 2.1.5 and Theorem 2.1.6 remain valid if for $\varphi \in \mathrm{C}^2(M)$, we replace

- $\mathfrak{v}$ by the weighted measure $\mathfrak{m} := \mathrm{e}^{-2\varphi}\,\mathfrak{v}$,
- Δ by the drift Laplacian $\Delta - 2\,\langle\nabla\varphi, \nabla\cdot\rangle$,

- Ric by the Bakry–Émery Ricci tensor $\mathrm{Ric} + 2\,\mathrm{Hess}\,\varphi$,
- $(\mathsf{P}_t)_{t\geq 0}$ by the semigroup generated by $\Delta - 2\,\langle\nabla\varphi, \nabla\cdot\rangle$, noting that the latter is again essentially self-adjoint [Li92, Sec. 2.1] on the L^2-space w.r.t. $\mathrm{e}^{-2\varphi}\,\mathfrak{v}$,
- b^x by the diffusion generated by the operator $\Delta/2 - \langle\nabla\varphi, \nabla\cdot\rangle$, see e.g. [Wan14, Ch. 3] for the particular form of the corresponding stochastic differential equation and the construction of its solution, and
- I by the weighted index form stated in Remark 2.4.4.

Other appropriate changes compared to the non-weighted setting, if needed, will always be indicated in the sequel.

It would be interesting to study our main results, Theorem 2.1.1, Theorem 2.1.5 and Theorem 2.1.6, in the context of lower bounds on the Bakry–Émery Ricci curvature $\mathrm{Ric}_Z := \mathrm{Ric} + 2\,\nabla Z$ which is associated to a C^1-vector field Z on M not necessarily of gradient-type. See [Wan05a, Wan14] and the references therein for a summary of similar statements under different, more geometric conditions. Given appropriate interpretations of the involved analytic objects, see [Wan05a, Wan14] for details, some of the results immediately carry over with trivial modifications (for instance, (iii) implying (ii), or the equivalence between (i) and (ii) in Theorem 2.1.6). On the other hand, many of our arguments, e.g. Theorem 2.2.1 and thus (i) in Theorem 2.1.1, or Theorem 2.5.1, are implicitly based on self-adjointness of the semigroup $(\mathsf{P}_t)_{t\geq 0}$ and the heat flow on 1-forms. The latter properties lack in this generality, which is why we restricted ourselves to gradient vector fields.

Finally, a further possible (but highly nontrivial) direction of investigation is the case of manifolds with boundary, taking the heat flow with Neumann boundary conditions. See [CF12, Wan14] and the references therein for an account on diffusion processes on these. The key difficulty in this context will be to take into account the local time of the boundary appropriately. Results that are relevant in this context have been obtained in [Air75, AL17, DZ05, Hsu02b, IW81, Mér79, Wan14].

2.2 Preliminaries

For more details on the following standard facts about the heat flows on functions and on 1-forms collected in this section, we refer the reader to [Dav89, Gri09, Gün17a, Hsu02a, Ros97, Str83] and the references therein. For details on their connection with the underlying stochastic processes, see [IW81, Mal97, Wan14]. All objects and results presented here have counterparts in the weighted case outlined above: the heat flow on functions [Gri09], Brownian motion (or rather the corresponding Ornstein–Uhlenbeck process) [IW81, Li92, Wan14], and the heat flow on 1-forms [Li92].

Heat flow on functions The operator Δ is the generator of the standard strongly local, regular Dirichlet form $\mathscr{E}\colon L^2(M) \to [0,\infty]$ induced by exterior differentiation d with domain $W^{1,2}(M)$, cf. Example 3.2.13. Note that under our standing assumption on M, $\mathrm{C}_\mathrm{c}^\infty(M)$ is dense in the Sobolev space $W^{1,2}(M)$ w.r.t. its natural norm [Aub76] — in other words, $W^{1,2}_0(M) = W^{1,2}(M)$. For what concerns the associated heat flow $(\mathsf{P}_t)_{t\geq 0}$, powerful L^2-L^∞-regularization properties of it on relatively compact subsets of M, an exhaustion procedure and bootstrapping of regularity imply the existence of the so-called *minimal heat kernel* $\mathsf{p} \in \mathrm{C}^\infty((0,\infty)\times M^2;(0,\infty))$ on M, the smallest positive fundamental solution to the heat operator $\partial/\partial t - \Delta$. It has the property that for every $f \in L^2(M)$ and $t > 0$, (a version of) $\mathsf{P}_t f$ can be represented by

$$\mathsf{P}_t f := \int_M \mathsf{p}_t(\cdot, y)\, f(y)\, \mathrm{d}\mathfrak{m}(y).$$

The previous representation formula is still valid for every $p \in [1,\infty]$ and every $f \in L^p(M)$. For such f, the above properties of the heat kernel show that $\mathsf{P}_\cdot f \in \mathrm{C}^\infty((0,\infty)\times M)$ solves the heat equation on M in the classical sense. In addition, we have $\mathsf{P}_\cdot f \in \mathrm{C}^\infty([0,\infty)\times M)$ if f is also smooth.

Heat flow on 1-forms In the sequel, Borel equivalence classes of 1-forms on M with L^p-regularity w.r.t. $\mathfrak{m}$, $p \in [1,\infty]$, are denoted by $L^p(T^*M)$. Similarly, $L^p(TM)$ stands for Borel equivalence classes of p-integrable vector fields. Let $\Gamma(T^*M)$ and $\Gamma(TM)$ denote the spaces of smooth (co-)vector fields; we use the subscript c to designate the respective subclasses of compactly supported elements. Let $\vec{\Delta} := \mathrm{d}\,\delta + \delta\,\mathrm{d}$ denote the Hodge Laplacian. When defined initially on $\Gamma_{\mathrm{c}}(T^*M)$, by geodesic completeness this operator has a unique self-adjoint extension in the Hilbert space $L^2(T^*M)$, which will be denoted with the same symbol again. Note our sign convention: $\vec{\Delta}$ is nonnegative, while Δ is nonpositive. The heat semigroup $(\mathsf{H}_t)_{t\geq 0}$ on 1-forms given by $\mathsf{H}_t := \mathrm{e}^{-t\vec{\Delta}/2}$ in $L^2(T^*M)$ is smooth, in the sense for every $\omega \in L^2(T^*M)$ one has a jointly smooth representative $\mathsf{H}_\cdot\omega$ which solves the heat equation

$$\frac{\partial}{\partial t}\mathsf{H}_t\omega = -\frac{1}{2}\vec{\Delta}\mathsf{H}_t\omega \quad \text{in } (0,\infty)\times M$$

on 1-forms with initial condition ω (and in $[0,\infty)\times M$ if ω is also smooth).

On exact forms, H_t can be represented by the heat operator P_t, $t \geq 0$; more precisely, for every $f \in W^{1,2}(M)$ one has [DT01, Li92]

$$\mathsf{H}_t\mathrm{d}f = \mathrm{d}\mathsf{P}_t f. \tag{2.2.1}$$

If one drops geodesic completeness of M, such a commutation relation becomes subtle (cf. [Tha98] for a negative and [Gün17a] for a positive result in this direction).

Lastly, a key result is the Feynman–Kac formula taken from [DT01, Thm. B.4], for which we recall the process Q^x from (2.1.2). Compare with Section 2.5.1. Note that the first asserted inequality in the theorem follows from Gronwall's inequality, cf. e.g. (2.3.1) below. See also [EL94b, Mal74] for the compact case.

Theorem 2.2.1. *Suppose that* Ric $\geq k$ *on M for some continuous $k\colon M \to \mathbf{R}$ satisfying* (2.1.1). *Then for every $t > 0$ and every $\omega \in L^\infty(T^*M)$ with compact support, the* Feynman–Kac formula

$$\mathsf{H}_t\omega(x) = \mathbf{E}\big[\mathsf{Q}_t^x\,(/\!\!/_{2t}^x)^{-1}\,\omega^\sharp(\mathsf{b}_{2t}^x)\,1_{\{t<\zeta^x/2\}}\big]^\flat$$

holds for every $x \in M$, and in particular

$$|\mathsf{H}_t\omega|(x) \leq \mathbf{E}\big[\mathrm{e}^{-\int_0^{2t} k(\mathsf{b}_r^x)/2\,\mathrm{d}r}\,|\omega|(\mathsf{b}_{2t}^x)\,1_{\{t<\zeta^x/2\}}\big] \leq \mathsf{C}_t\,\|\omega\|_{L^\infty(T^*M)}$$

for every $x \in M$, where C_t is defined in (2.1.1).

Remark 2.2.2. On weighted Riemannian manifolds, in the notation at the end of Section 2.1 one has to replace $(\mathsf{H}_t)_{t\geq 0}$ by the semigroup — defined on the Hilbert space of 1-forms that are L^2 w.r.t. $\mathrm{e}^{-2\varphi}\,\mathfrak{v}$ — which is generated by the essentially self-adjoint operator $-\vec{\Delta} - 2\,\mathrm{d}\,i_{\nabla\varphi} - 2\,i_{\nabla\varphi}\,\mathrm{d}$. Here $i_{\nabla\varphi}$ denotes interior multiplication of differential forms with the vector field $\nabla\varphi$ [Li92, Sec. 1.5]. ■

2.3 Proof of Theorem 2.1.1 and Theorem 2.1.5

This section treats the stochastic completeness of M, Bismut–Elworthy–Li's derivative formula, and the L^∞-Lip-regularization of the heat semigroup $(\mathsf{P}_t)_{t\geq 0}$ if we have $\mathrm{Ric} \geq k$ on M for some continuous function $k\colon M \to \mathbf{R}$ satisfying (2.1.1).

2.3.1 Stochastic completeness

A key tool for proving stochastic completeness under geodesic completeness, already used in [Bak86], are sequences of first-order cutoff-functions [Str83, Ch. 2]. Their existence is equivalent to the geodesic completeness of M [Gün16, PS14]. The proof of [ER$^+$20, Thm. 3.11] follows similar lines.

Lemma 2.3.1. *There exists a sequence* $(\psi_n)_{n\in\mathbf{N}}$ *in* $\mathrm{C}_\mathrm{c}^\infty(M)$ *satisfying*

(i) $\psi_n(M) \subset [0,1]$ *for every* $n \in \mathbf{N}$,
(ii) *for every compact set* $C \subset M$, *there exists* $N \in \mathbf{N}$ *such that* $\psi_n\big|_C = 1_C$ *for every* $n \geq N$, *and*
(iii) $\|\mathrm{d}\psi_n\|_{L^\infty(T^*M)} \to 0$ *as* $n \to \infty$.

Proof of (i) *in Theorem 2.1.1.* By (0.1.6), it clearly suffices show that $\mathsf{P}_t 1_M = 1_M$ for every $t > 0$. Let $\phi \in \mathrm{C}_\mathrm{c}^\infty(M)$, and let $(\psi_n)_{n\in\mathbf{N}}$ be a sequence of first-order cutoff functions provided by Lemma 2.3.1 above. Then Theorem 2.2.1 applied to the 1-form $\omega := \mathrm{d}\psi_n$ for every $n \in \mathbf{N}$ gives

$$\big\|\mathsf{H}_s \mathrm{d}\psi_n\big\|_{L^\infty(T^*M)} \leq \mathsf{C}_s\, \|\mathrm{d}\psi_n\|_{L^\infty(M)} \leq \mathsf{C}_t\, \|\mathrm{d}\psi_n\|_{L^\infty(T^*M)},$$

uniformly in $s \in [0,t]$. Since $\mathsf{P}_\cdot \psi_n$ solves the heat equation on M, also using Fubini's theorem, integration by parts as well as the commutation rule (2.2.1) we arrive at

$$\begin{aligned}\int_M \big[\mathsf{P}_t\psi_n - \psi_n\big]\,\phi\,\mathrm{d}\mathfrak{m} &= \int_M \int_0^t \phi\,\Delta\mathsf{P}_s\psi_n\,\mathrm{d}s\,\mathrm{d}\mathfrak{m}\\ &= -\int_0^t\int_M \big\langle \mathrm{d}\phi, \mathrm{d}\mathsf{P}_s\psi_n\big\rangle\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s\\ &= -\int_0^t\int_M \big\langle \mathrm{d}\phi, \mathsf{H}_s\mathrm{d}\psi_n\big\rangle\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\Big|\int_M \big[\mathsf{P}_t 1_M - 1_M\big]\,\phi\,\mathrm{d}\mathfrak{m}\Big| &= \lim_{n\to\infty}\Big|\int_M \big[\mathsf{P}_t\psi_n - \psi_n\big]\,\phi\,\mathrm{d}\mathfrak{m}\Big|\\ &\leq \limsup_{n\to\infty}\int_0^t\int_M |\mathrm{d}\phi|\,\big|\mathsf{H}_s\mathrm{d}\psi_n\big|\,\mathrm{d}\mathfrak{m}\,\mathrm{d}s\\ &\leq \mathsf{C}_t\, t\, \|\mathrm{d}\phi\|_{L^1(T^*M)}\, \limsup_{n\to\infty}\|\mathrm{d}\psi_n\|_{L^\infty(T^*M)} = 0.\end{aligned}$$

Since ϕ was arbitrary, this proves the claim. □

2.3.2 Bismut–Elworthy–Li's derivative formula and the Lipschitz smoothing property

In view of proving Bismut–Elworthy–Li's derivative formula and the L^∞-Lip-regularization property of $(\mathsf{P}_t)_{t\geq 0}$, for convenience we state the following version of the Burkholder–Davis–Gundy inequality for $q \in [1,\infty)$ proven in [Ren08, Thm. 2] (although we only need the upper bounds, respectively), which improves the classically known constants to better ones.

Lemma 2.3.2. *Let* M *be a real-valued continuous local martingale with* $\mathsf{M}_0 = 0$*, and let* $q \in [1,\infty)$*. Then for every stopping time* τ*,*

$$(8q)^{-q/2}\,\mathbf{E}\big[[\mathsf{M}]_\tau^{q/2}\big] \leq \mathbf{E}\Big[\sup_{r\in[0,\tau]} |\mathsf{M}_r|^q\Big] \leq (8q)^{q/2}\,\mathbf{E}\big[[\mathsf{M}]_\tau^{q/2}\big],$$

where $([\mathsf{M}]_r)_{r\geq 0}$ *denotes the quadratic variation process of* $(\mathsf{M}_r)_{r\geq 0}$.

Recall the process Q^x defined by (2.1.2) and taking values in T_xM, $x\in M$.

Proof of (ii) *in Theorem 2.1.1.* Fix $x\in M$, $t>0$ and $\xi \in T_xM$. It suffices to treat the case that $|\xi|\leq 1$. We first assume that $f\in \mathrm{C}_{\mathrm{c}}^\infty(M)$. By [DT01, Prop. 3.2] and keeping in mind that $\zeta^x = \infty$ $\mathbf{P}$-a.s., the process N^x given by

$$\mathsf{N}_r^x := \Big\langle \mathsf{Q}_r^x\,(/\!\!/_{2r}^x)^{-1}\,\nabla \mathsf{P}_{t-r}f(\mathsf{b}_{2r}^x), \frac{t-r}{t}\,\xi\Big\rangle + \frac{1}{\sqrt{2}\,t}\,\mathsf{P}_{t-r}f(\mathsf{b}_{2r}^x)\int_0^r \big\langle \mathsf{Q}_s^x\xi, \mathrm{d}W_s^x\big\rangle,$$

$r\in[0,t]$, is a local martingale. We show that under the given assumption (2.1.1) on k, this process is even a true martingale.

As already indicated before Theorem 2.2.1, given any $s\geq 0$, it follows from Gronwall's inequality and $\mathrm{Ric}\geq k$ on M that

$$\big|\mathsf{Q}_s^x\big| \leq \mathrm{e}^{-\int_0^s k(\mathsf{b}_{2r}^x)\,\mathrm{d}r} \leq \mathrm{e}^{\int_0^{2s} k^-(\mathsf{b}_r^x)/2\,\mathrm{d}r} \quad \mathbf{P}\text{-a.s.} \tag{2.3.1}$$

Hence, for every $q\in[1,\infty)$, by Lemma 2.3.2 we obtain

$$\begin{aligned}\mathbf{E}\Big[\sup_{r\in[0,t]}\Big|\int_0^r \big\langle \mathsf{Q}_s^x\xi, \mathrm{d}W_s^x\big\rangle\Big|^q\Big] &\leq (8q)^{q/2}\,\mathbf{E}\Big[\Big[\int_0^t |\mathsf{Q}_s^x|^2\,\mathrm{d}s\Big]^{q/2}\Big]\\ &\leq (8q)^{q/2}\,t^{q/2}\sup_{y\in M}\mathbf{E}\big[\mathrm{e}^{\int_0^{2t} qk^-(\mathsf{b}_r^y)/2\,\mathrm{d}r}\big].\end{aligned} \tag{2.3.2}$$

(This estimate will only be needed for $q=1$ in this proof, but is derived for arbitrary q as above for later convenience.) Now, estimating $|\mathsf{Q}_r^x|$ as in (2.3.1) above and using the commutation relation (2.2.1) as well as Theorem 2.2.1, for every $r\in[0,t]$,

$$\begin{aligned}\big|\mathsf{N}_r^x\big| &\leq \mathrm{e}^{\int_0^{2r} k^-(\mathsf{b}_s^x)/2\,\mathrm{d}s}\,|\mathsf{H}_{t-r}\,\mathrm{d}f|(\mathsf{b}_{2r}^x) + \frac{1}{\sqrt{2}\,t}\,\|f\|_{L^\infty(M)}\Big|\int_0^r \big\langle \mathsf{Q}_s^x\xi,\,\mathrm{d}W_s^x\big\rangle\Big|\\ &\leq \mathrm{e}^{\int_0^{2t} k^-(\mathsf{b}_s^x)/2\,\mathrm{d}s}\,\mathsf{C}_{t-r}\,\|\mathrm{d}f\|_{L^\infty(T^*M)} + \frac{1}{\sqrt{2}\,t}\,\|f\|_{L^\infty(M)}\Big|\int_0^r \big\langle \mathsf{Q}_s^x\xi, \mathrm{d}W_s^x\big\rangle\Big|\\ &\leq \mathrm{e}^{\int_0^{2t} k^-(\mathsf{b}_s^x)/2\,\mathrm{d}s}\,\mathsf{C}_t\,\|\mathrm{d}f\|_{L^\infty(T^*M)} + \frac{1}{\sqrt{2}\,t}\,\|f\|_{L^\infty(M)}\Big|\int_0^r \big\langle \mathsf{Q}_s^x\xi, \mathrm{d}W_s^x\big\rangle\Big| \quad \mathbf{P}\text{-a.s.}\end{aligned}$$

It follows that

$$\mathbf{E}\Big[\sup_{r\in[0,t]}\big|\mathsf{N}_r^x\big|\Big] \leq \mathsf{C}_t^2\,\|\mathrm{d}f\|_{L^\infty(T^*M)} + \frac{1}{\sqrt{2}\,t}\,\|f\|_{L^\infty(M)}\,\mathbf{E}\Big[\sup_{r\in[0,t]}\Big|\int_0^r \big\langle \mathsf{Q}_s^x\xi, \mathrm{d}W_s^x\big\rangle\Big|\Big].$$

The first summand on the right-hand side is finite thanks to (2.1.1). Estimating the second summand by (2.3.2) above for $q = 1$, also the second summand is finite again by (2.1.1). It follows that N^x is a true martingale, and thus

$$\big\langle \nabla \mathsf{P}_t f(x), \xi \big\rangle = \mathbf{E}\big[\mathsf{N}_0^x\big] = \mathbf{E}\big[\mathsf{N}_t^x\big] = \frac{1}{\sqrt{2}\,t}\,\mathbf{E}\Big[f(\mathsf{b}_{2t}^x) \int_0^t \big\langle \mathsf{Q}_s^x \xi, \mathrm{d}W_s^x \big\rangle\Big]. \tag{2.3.3}$$

The claimed equality for bounded $f \in \mathrm{C}^\infty(M)$ follows by replacing f by $\psi_n f$ in (2.3.3), $n \in \mathbf{N}$, where $(\psi_n)_{n\in\mathbf{N}}$ is as in Lemma 2.3.1, and letting $n \to \infty$ (together with Lebesgue's theorem on the right-hand side). In turn, if only $f \in L^\infty(M)$, a similar procedure works by replacing f by $\mathsf{P}_\varepsilon f$ in (2.3.3), where $\varepsilon > 0$, and letting $\varepsilon \downarrow 0$. □

Proof of (iii) *in Theorem 2.1.1.* Using the previous Bismut–Elworthy–Li formula and (2.3.2) above for $q = 1$, for every $x \in M$, every $t > 0$ and arbitrary $\xi \in T_xM$ with $|\xi| \leq 1$, we obtain

$$\big\langle \nabla \mathsf{P}_t f(x), \xi \big\rangle \leq \frac{1}{\sqrt{2}\,t}\,\mathbf{E}\Big[\Big|\int_0^t \big\langle \mathsf{Q}_s^x \xi,\, \mathrm{d}W_s^x \big\rangle\Big|\Big]\, \|f\|_{L^\infty(M)} \leq \frac{2}{\sqrt{t}}\,\mathsf{C}_t\, \|f\|_{L^\infty(M)},$$

and duality gives the claim. □

Now we assume Kato decomposability of $\mathscr{k}$ in the rest of this subsection, devoting ourselves to the proof of Theorem 2.1.5. In this situation, one has to guarantee that the right-hand side of Bismut–Elworthy–Li's formula is well-defined for $f \in L^p(M)$, where $p \in (1, \infty)$, which is essentially the content of the following lemma.

Lemma 2.3.3. *Let $t \geq 0$ and $V \in L^\infty(TM)$. Then for every $f \in L^\infty(M)$ and every $x \in M$, the random variable $f(\mathsf{b}_{2t}^x) \int_0^t \big\langle \mathsf{Q}_s^x V(x), \mathrm{d}W_s^x \big\rangle$ is integrable. Moreover, for every $p \in (1, \infty]$, the operator E_t^V given on functions $f \in L^\infty(M) \cap L^p(M)$ by*

$$\mathsf{E}_t^V f(x) := \mathbf{E}\Big[f(\mathsf{b}_{2t}^x) \int_0^t \big\langle \mathsf{Q}_s^x V(x), \mathrm{d}W_s^x \big\rangle\Big]$$

extends to a bounded linear operator from $L^p(M)$ into $L^p(M)$, and the previous representation is valid and well-defined for every $f \in L^p(M)$.

Proof. Let $V \in L^\infty(TM)$ and $f \in L^\infty(M)$, for which we assume without loss of generality that $\|V\|_{L^\infty(TM)} \leq 1$ and $\|f\|_{L^\infty(M)} \leq 1$. The inequality (2.3.2) for $q = 1$ and Lemma 2.1.4 directly show the claimed integrability of $f(\mathsf{b}_{2t}^x) \int_0^t \big\langle \mathsf{Q}_s^x V(x),\, \mathrm{d}W_s^x \big\rangle$, and they also show that E_t^V is a bounded linear operator from $L^\infty(M)$ into $L^\infty(M)$.

If $p \in (1, \infty)$, successively using Hölder's inequality, (2.3.2) for $q = p/(p-1)$, Lemma 2.1.4 again and mass preservation of $(\mathsf{P}_t)_{t\geq 0}$, we infer the existence of a constant $\mathsf{C} > 0$ depending only on $\mathscr{k}^-$, t and p such that for every $f \in L^p(M) \cap L^\infty(M)$,

$$\begin{aligned}
\big\|\mathsf{E}_t^V f\big\|_{L^p(M)}^p &= \int_M \Big|\mathbf{E}\Big[f(\mathsf{b}_{2t}^x) \int_0^t \big\langle \mathsf{Q}_s^x V(x), \mathrm{d}W_s^x \big\rangle\Big]\Big|^p \,\mathrm{d}\mathfrak{m}(x) \\
&\leq \int_M \mathbf{E}\big[|f|^p(\mathsf{b}_{2t}^x)\big]\, \mathbf{E}\Big[\Big|\int_0^t \big\langle \mathsf{Q}_s^x V(x), \mathrm{d}W_s^x \big\rangle\Big|^q\Big]^{p/q} \,\mathrm{d}\mathfrak{m}(x) \\
&\leq (8q)^{p/2}\, t^{p/2} \sup_{y\in M} \mathbf{E}\big[\mathrm{e}^{\int_0^{2t} q\mathscr{k}^-(\mathsf{b}_r^x)/2\,\mathrm{d}r}\big]^{p-1} \int_M \mathsf{P}_t\big(|f|^p\big)(x)\,\mathrm{d}\mathfrak{m}(x) \\
&\leq \mathsf{C}\,(8q)^{p/2}\, t^{p/2}\, \big\|f\big\|_{L^p(M)}^p.
\end{aligned}$$

We conclude the statement by a standard approximation argument. □

Proof of Theorem 2.1.5. Trivially, $L^\infty(M) \cap L^p(M)$ is dense in $L^p(M)$. Note that, given $p \in (1,\infty)$, and $f \in L^p(M)$, it follows from the divergence theorem as well as Lemma 2.3.3 — replacing ξ by an appropriate smooth and bounded vector field $V \in \Gamma(TM)$ such that $V(x) = \xi$ — that both sides of (2.3.3) are continuous in f w.r.t. convergence in $L^p(M)$. In particular, the desired pointwise identity follows. □

2.4 Proof of Theorem 2.1.6

We turn to characterizations of continuous lower Ricci curvature bounds in terms of functional inequalities and existence of couplings. Throughout this section, we assume that $k\colon M \to \mathbf{R}$ is continuous, and only state explicitly if we need (2.1.1).

2.4.1 From the L^1-Bochner inequality to lower Ricci bounds

As already hinted, the point in showing the implication from (ii) to (i) in Theorem 2.1.6 is Bochner's formula (2.1.3), subject to a clever choice of f as granted by the subsequent lemma [vRS05, Lem. 3.2], together with the chain rule to deduce $\mathrm{Ric} \geq k$ on M.

It is well-known in Riemannian geometry that, given any $x \in M$, there exists an open subset $O_x \subset T_xM$ such that the restriction of the exponential map to O_x provides a diffeomorphism $\exp_x\colon O_x \to \exp_x(O_x)$. We denote its inverse by $\exp_x^{-1}$.

Lemma 2.4.1. *Let $x \in M$ and $\xi \in T_xM$ with unit norm. Let $H := \{\exp_x \eta : \eta \in O_x,\ \langle \eta, \xi\rangle = 0\}$ be the* $(\dim M - 1)$*-dimensional hypersurface in M orthogonal to ξ at x. Then there exists an open neighborhood $U \subset \exp_x(O_x)$ of x such that the* signed distance function $\rho_H : U \to \mathbf{R}$ *given by*

$$\rho_H(y) := \mathsf{d}(y, H)\ \mathrm{sgn}\big\langle \xi, \exp_x^{-1} y\big\rangle,$$

where $\mathsf{d}(\cdot, H)$ is the distance function from H, obeys $\rho_H \in \mathrm{C}^\infty(U)$, $\nabla\rho_H(x) = \xi$, $|\nabla\rho_H|(U) = \{1\}$ and $\mathrm{Hess}\,\rho_H(x) = 0$.

Proof of (ii) *implying* (i) *in Theorem 2.1.6.* Let $x \in M$ and $\xi \in T_xM$ in Lemma 2.4.1, in whose notation we consider the function $f := \rho_H$. By Lemma 2.4.1, Bochner's formula (2.1.3) and the chain rule for Δ, at the given point x we have

$$\begin{aligned}\mathrm{Ric}(\xi,\xi) &= \Delta\frac{|\nabla f|^2}{2} - \langle \nabla\Delta f, \nabla f\rangle\\ &= |\nabla f|\,\Delta|\nabla f| + \big|\nabla|\nabla f|\big|^2 - \langle \nabla\Delta f, \nabla f\rangle\\ &\geq k\,|\nabla f|^2 = k.\end{aligned}$$

The arbitrariness of ξ concludes the proof. □

Remark 2.4.2. In the weighted setting outlined at the end of Section 2.1 — retaining the notation therein — the only essential change needed to modify the previous proof is to replace the unweighted Bochner identity (2.1.3) by its weighted counterpart

$$\Delta^\varphi \frac{|\nabla f|^2}{2} = \langle \nabla\Delta^\varphi f, \nabla f\rangle + \big|\mathrm{Hess}\, f\big|_{\mathrm{HS}}^2 + \mathrm{Ric}^\varphi(\nabla f, \nabla f),$$

where $\Delta^\varphi := \Delta - 2\,\langle\nabla\varphi, \nabla\cdot\rangle$ and $\mathrm{Ric}^\varphi := \mathrm{Ric} + 2\,\mathrm{Hess}\,\varphi$. The latter follows from (2.1.3), the definition of $\mathrm{Hess}\,\varphi$ and metric compatibility of ∇, see e.g. [Pet06, p. 28]:

$$2\,\mathrm{Hess}\,\varphi(\nabla f, \nabla f) = 2\,\big\langle \nabla_{\nabla f}\nabla\varphi, \nabla f\big\rangle$$

$$= 2\,\big\langle \nabla\langle\nabla\varphi, \nabla f\rangle, \nabla f\big\rangle - \big\langle \nabla\varphi, \nabla|\nabla f|^2\big\rangle.$$

The chain rule for Δ^φ is analogous to the one for Δ, see Lemma 3.2.11 below. ■

2.4.2 From lower Ricci bounds to pathwise couplings

We start with the existence of a suitable coupling of Brownian motions under the inequality $\mathrm{Ric} \geq k$ on M, also assuming (2.1.1) in this subsection. (Note that the stochastic completeness of M is already known by Theorem 2.1.1.) The coupling technique is well-known and called *coupling by parallel displacement*, see [Cra91, Ken86, Wan05a, Wan14] and the references therein. See also [Wan94] for a "local" treatise on regular Riemannian subdomains.

We first collect some notation. Denote by cut_v the cut-locus of $v \in M$, by diag the diagonal of M^2, and by R the Riemann curvature tensor of M. Abbreviate $d := \dim M$ and define $\mathrm{cut} := \{(u,v) \in M^2 : u \in \mathrm{cut}_v\}$. Given any $(u,v) \in M^2 \setminus (\mathrm{diag} \cup \mathrm{cut})$, let $J_1, \dots, J_{d-1}$ be Jacobi fields along the unique minimal geodesic $\gamma\colon [0, \mathsf{d}(u,v)] \to M$ from u to v such that $\{J_1(s), \dots, J_{d-1}(s), \dot\gamma_s\}$ is an orthonormal basis of $T_{\gamma_s}M$ both for $s = 0$ as well as $s = \mathsf{d}(u,v)$. Define the *index form* by

$$I(u,v) := \sum_{i=1}^{d-1} \int_0^{\mathsf{d}(u,v)} \big[\big|\nabla_{\dot\gamma_s} J_i(s)\big|^2 - \big\langle \mathrm{R}(\dot\gamma_s, J_i(s))\dot\gamma_s, J_i(s)\big\rangle\big]\,\mathrm{d}s.$$

Theorem 2.4.3. *For every $x, y \in M$ with $x \neq y$, there exists a coupling $(\mathsf{b}^x, \mathsf{b}^y)$ of Brownian motions on M starting in (x,y) which coincide past their* coupling time

$$T(\mathsf{b}^x, \mathsf{b}^y) := \inf\big\{t \geq 0 : \mathsf{b}^x_t = \mathsf{b}^y_t\big\}$$

such that for every $I' \in \mathrm{C}(M^2)$ for which $I' \geq I$ holds outside diag ∪ cut, *before $T(\mathsf{b}^x, \mathsf{b}^y)$ we have the differential inequality*

$$\mathrm{d}\mathsf{d}\big(\mathsf{b}^x_t, \mathsf{b}^y_t\big) \leq \frac{1}{2} I'\big(\mathsf{b}^x_t, \mathsf{b}^y_t\big)\,\mathrm{d}t.$$

The construction of this coupling is thoroughly carried out in [Wan05a, Thm. 2.1.1, Prop. 2.5.1], see also [Wan14, Thm. 2.3.2]. The key to deduce (iii) from (i) in Theorem 2.1.6 using Theorem 2.4.3 now is to construct an appropriate function $I' \in \mathrm{C}(M^2)$ with $I' \geq I$ outside diag ∪ cut, hence circumventing cut-locus issues. This is the place where the definition of $\underline{k}$ enters.

Proof of (i) *implying* (iii) *in Theorem 2.1.6.* Let $u, v \in M^2 \setminus (\mathrm{diag} \cup \mathrm{cut})$. As in the proof of [Wan05a, Thm. 2.1.4], let $U_1, \dots, U_{d-1}$ be parallel vector fields along γ such that $\{U_1(s), \dots, U_{d-1}(s), \dot\gamma_s\}$ is an orthonormal basis of $T_{\gamma_s}M$ for every $s \in [0, \mathsf{d}(u,v)]$. By the index lemma [CE75, Lem. 1.21], we have

$$\begin{aligned} I(u,v) &\leq -\int_0^{\mathsf{d}(u,v)} \Big[\sum_{i=1}^{d-1} \big\langle \mathrm{R}(\dot\gamma_s, U_i(s))\dot\gamma_s, U_i(s)\big\rangle\Big]\,\mathrm{d}s \\ &= -\int_0^{\mathsf{d}(u,v)} \mathrm{Ric}(\gamma_s)(\dot\gamma_s, \dot\gamma_s)\,\mathrm{d}s \\ &\leq -\int_0^{\mathsf{d}(u,v)} k(\gamma_s)\,\mathrm{d}s \leq -\mathsf{d}(u,v)\,\underline{k}(u,v). \end{aligned} \tag{2.4.1}$$

As $\underline{k}$ is lower-semicontinuous, a well-known consequence of Baire's theorem yields the existence of a pointwise increasing sequence $(\underline{k}_n)_{n\in\mathbf{N}}$ in $\mathrm{C}(M^2)$ converging pointwise to $\underline{k}$. Applying Theorem 2.4.3 with I' replaced by $I'_n \in \mathrm{C}(M^2)$ given by $I'_n(u,v) := -\mathsf{d}(u,v)\,\underline{k}_n(u,v)$ and integrating the resulting differential inequality, $\mathbf{P}$-a.s. we have the pathwise estimate

$$\mathsf{d}\big(\mathsf{b}_t^x, \mathsf{b}_t^y\big) \leq \mathrm{e}^{-\int_s^t \underline{k}_n(\mathsf{b}_r^x,\mathsf{b}_r^y)/2\,\mathrm{d}r}\,\mathsf{d}\big(\mathsf{b}_s^x, \mathsf{b}_s^y\big)$$

for every $s,t \geq 0$ with $s \leq t$ and for every $n \in \mathbf{N}$. (Recall that b^x and b^y coincide past their coupling time.) Letting $n \to \infty$ with the aid of Levi's theorem, we obtain the desired pathwise estimate. □

Remark 2.4.4. In the weighted case from the end of Section 2.1, the quantity I has to be replaced by its weighted counterpart

$$I_\varphi(u,v) := I(u,v) - (\nabla\varphi)\mathsf{d}(\cdot,v)(u) - (\nabla\varphi)\mathsf{d}(\cdot,u)(v).$$

Theorem 2.4.3 remains true for the corresponding diffusion process, and the weighted adaptation of the estimates (2.4.1) follows the proof of [Wan05a, Thm. 2.1.4]. ■

2.4.3 From pathwise couplings to the L^1-Bochner inequality

Similarly as for the proof of the corresponding implication in Theorem 1.1.1, Theorem 1.5.17, we need to approximate $\underline{k}$ by Lipschitz functions on M^2. To this aim, we present the following local version of Lemma 1.2.1 whose elementary proof is omitted.

Lemma 2.4.5. *Let $D \subset M$ be a compact subset. Then, in D^2, $\underline{k}$ is the pointwise limit of a pointwise increasing sequence of functions in* $\mathrm{Lip_b}(M^2)$ *which are everywhere no smaller than* $\inf \underline{k}(D^2)$.

In particular, note that since $k(x) = \underline{k}(x,x)$ for every $x \in M$, in D, k is the pointwise limit of a pointwise increasing sequence of functions in $\mathrm{Lip_b}(M)$ which are everywhere no smaller than $\inf k(D)$.

The step from the pathwise coupling property w.r.t. k towards (2.1.5) requires a nontrivial extension of the arguments for Theorem 1.5.17 (which makes crucial use of uniform lower boundedness of the Ricci curvature) for short times instead of fixed ones. This kind of localization argument was indeed used Chapter 1 in different variants at different instances, and as in the previous chapter (recall Lemma 1.4.2), a certain short-time behavior of Brownian motion as subsequently recorded plays a crucial role.

Given any $x \in M$ and $\varepsilon > 0$, let τ_ε^x be the first exit time of Brownian motion starting in a fixed $x \in M$ from $B_\varepsilon(x)$. The following estimate for τ_ε^x is a variant of [Wan14, Lem. 2.1.4], noting that by Laplacian comparison, compare with [Hsu02a, Cor. 3.4.4, Cor. 3.4.5, Thm. 3.6.1], the constant c_1 therein can be chosen uniformly in x off its respective cut-locus as long as long as x belongs to a compact subset of M. (An analogous version of Lemma 2.4.6 holds for general gradient diffusions, taking — in the notation from Section 2.1 — into account the continuity of $\nabla\varphi$.)

Lemma 2.4.6. *For every compact $D \subset M$ and every $\varepsilon > 0$, there exists a constant $c > 0$ such that for every $x \in D$ and every $t \in (0,1]$,*

$$\mathbf{P}\big[\tau_\varepsilon^x \leq t\big] \leq \mathrm{e}^{-c/t}.$$

Proof of (iii) *implying* (ii) *in Theorem 2.1.6.* **Step 1.** *Initial preparations.* Let $f \in \mathrm{C}_\mathrm{c}^\infty(M)$ and $x \in M$ with $|\nabla f|(x) \neq 0$. Let $\varepsilon \in (0, 1/4]$, which is kept fixed throughout this proof, be such that $|\nabla f|$ is bounded away from zero — in particular smooth — on $\overline{B}_{4\varepsilon}(x)$. Moreover, let γ be the unique geodesic starting in x with initial velocity $\dot\gamma_0 = \nabla f(x)/|\nabla f|(x)$. The continuity of $\Bbbk$ yields $\Bbbk \geq K$ on $\overline{B}_6(x)$ for some negative real number K. Define the set of points in M with distance at most 1 to γ by

$$D := \bigcup_{s\in[0,1]} \overline{B}_1(\gamma_s).$$

By the definition (2.1.4) of $\underline{\Bbbk}$ and since $\mathsf{d}(x, \gamma_s) \leq 1$ for every $s \in [0, 1]$, we have

$$\underline{\Bbbk} \geq K \quad \text{on } D^2. \tag{2.4.2}$$

Finally, let $\underline{\ell} \in \mathrm{Lip}_\mathrm{b}(M^2)$ be any function with $K \leq \underline{\ell} \leq \underline{\Bbbk}$ on D^2, cf. Lemma 2.4.5.

Step 2. *Rewriting the quantities to consider.* The key idea to derive the L^1-Bochner inequality (2.1.5) for f from the given pathwise coupling estimates is to consider certain difference quotients of the map $(t, s) \mapsto \mathsf{P}_t f(\gamma_s)$ near $(0, 0)$, and to express the involved heat semigroups in terms of coupled Brownian motions. To address the first point, we note that, given $t > 0$, by the smoothness of $s \mapsto \mathsf{P}_t f(\gamma_s)$ on $[0, \infty)$, Taylor's theorem in its mean value remainder form and the geodesic equation for γ, given any $s > 0$ there exists $\nu \in [0, s]$ such that

$$\mathsf{P}_t f(\gamma_s) - \mathsf{P}_t f(x) = s \left\langle \nabla \mathsf{P}_t f(x), \dot\gamma_0 \right\rangle + \frac{s^2}{2} \operatorname{Hess} \mathsf{P}_t f(\gamma_\nu)\big(\dot\gamma_\nu, \dot\gamma_\nu\big).$$

Dividing by s, subtracting $\langle \nabla f(x), \dot\gamma_0\rangle$ and dividing by t, respectively, yields

$$\begin{aligned} &\frac{1}{t}\Big[\frac{1}{s}\big[\mathsf{P}_t f(\gamma_s) - \mathsf{P}_t f(x)\big] - \big\langle \nabla f(x), \dot\gamma_0\big\rangle\Big] \\ &\qquad = \frac{1}{t}\left\langle \nabla \mathsf{P}_t f(x) - \nabla f(x), \dot\gamma_0\right\rangle + \frac{s}{2t} \operatorname{Hess} \mathsf{P}_t f(\gamma_\nu)\big(\dot\gamma_\nu, \dot\gamma_\nu\big). \end{aligned} \tag{2.4.3}$$

To now invoke the coupled Brownian motions, given $s \in (0, 1]$ let us denote by $(\mathsf{b}^x, \mathsf{b}^{\gamma_s})$ a process starting in (x, γ_s) given by the pathwise coupling property w.r.t. $\Bbbk$. Let τ_ε^x and $\tau_\varepsilon^{\gamma_s}$ denote the first exit times of the marginal Brownian motions b^x and b^{γ_s} from $B_\varepsilon(x)$ and $B_\varepsilon(\gamma_s)$, respectively. Since $\overline{B}_\varepsilon(x), \overline{B}_\varepsilon(\gamma_s) \subset D$, for every $s \in [0, 1]$ $\mathbf{P}$-a.s. on the event $\big\{\tau_\varepsilon^x > t,\ \tau_\varepsilon^{\gamma_s} > t\big\}$, for arbitrary $t \in (0, 1]$, we have

$$\mathsf{d}\big(\mathsf{b}_t^x, \mathsf{b}_t^{\gamma_s}\big) \leq \mathrm{e}^{-\int_0^t \underline{\Bbbk}(\mathsf{b}_r^x, \mathsf{b}_r^{\gamma_s})/2\,\mathrm{d}r}\, s \leq \mathrm{e}^{-\int_0^t \underline{\ell}(\mathsf{b}_r^x, \mathsf{b}_r^{\gamma_s})/2\,\mathrm{d}r}\, s. \tag{2.4.4}$$

Step 3. *Estimating* (2.4.3) *via coupled Brownian motions.* Given $s \in (0, \mathrm{e}^{-1/2}]$, define $t_s := -c/\log s^2 \in (0, c]$, where $c > 0$ is the constant from Lemma 2.4.6 associated to D and ε. Observe that $t_s \to 0$ and $s^\alpha/t_s \to 0$ as $s \to 0$ for $\alpha \in \{1/2, 1\}$. We shall consider the events $A_s := \big\{\tau_\varepsilon^x > t_s,\ \tau_\varepsilon^{\gamma_s} > t_s\big\}$, $V_s := A_s \cap \big\{\mathsf{d}\big(\mathsf{b}_{t_s}^x, \mathsf{b}_{t_s}^{\gamma_s}\big) \geq s^{1/2}\big\}$, $W_s := A_s \cap \big\{\int_0^{t_s} \mathsf{d}\big(\mathsf{b}_r^x, \mathsf{b}_r^{\gamma_s}\big)\,\mathrm{d}r/t_s \geq s^{1/2}\big\}$ and $U_s := A_s \cap V_s^\mathrm{c} \cap W_s^\mathrm{c}$. Since the function $(t, s) \mapsto \operatorname{Hess} \mathsf{P}_t f(\gamma_s)(\dot\gamma_s, \dot\gamma_s)$ is locally bounded on $[0, \infty)^2$ by joint smoothness of the heat semigroup, by (2.4.3) with $t_s/2$ in place of t we have

$$\begin{aligned} &\frac{1}{2}\,|\nabla f|(x)^{-1}\left\langle \nabla\Delta f(x), \nabla f(x)\right\rangle \\ &\qquad = \lim_{s\downarrow 0} \frac{1}{t_s}\left\langle \nabla \mathsf{P}_{t_s/2} f(x) - \nabla f(x), \dot\gamma_0\right\rangle \end{aligned}$$

$$\begin{aligned}
&\leq \limsup_{s\to 0} \frac{1}{t_s}\Big[\frac{1}{s}\big[\mathsf{P}_{t_s/2}f(\gamma_s) - \mathsf{P}_{t_s/2}f(x)\big] - \big\langle \nabla f(x), \dot\gamma_0\big\rangle\Big]\\
&\leq \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\frac{1}{s}\mathbf{E}\big[\big|f(\mathsf{b}^x_{t_s}) - f(\mathsf{b}^{\gamma_s}_{t_s})\big|\big] - |\nabla f|(x)\Big]\\
&= \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\frac{1}{s}\mathbf{E}\big[\big|f(\mathsf{b}^x_{t_s}) - f(\mathsf{b}^{\gamma_s}_{t_s})\big|\,\big[1_{V_s} + 1_{W_s} + 1_{U_s} + 1_{A_s^{\mathrm{c}}}\big]\big] - |\nabla f|(x)\Big].
\end{aligned}$$

Now we estimate the contributions of the events defined above separately.

Step 3.1. The contribution of A_s^{c} becomes negligible thanks to

$$\begin{aligned}
&\limsup_{s\downarrow 0} \frac{1}{t_s\, s}\,\mathbf{E}\big[\big|f(\mathsf{b}^x_{t_s}) - f(\mathsf{b}^{\gamma_s}_{t_s})\big|\,1_{A_s^{\mathrm{c}}}\big]\\
&\qquad\leq \|f\|_{L^\infty(M)} \limsup_{s\downarrow 0} \frac{2}{t_s\, s}\,\mathbf{P}\big[A_s^{\mathrm{c}}\big]\\
&\qquad\leq \|f\|_{L^\infty(M)} \limsup_{s\downarrow 0} \frac{2}{t_s\, s}\,\big[\mathbf{P}\big[\tau^x_\varepsilon \leq t_s\big] + \mathbf{P}\big[\tau^{\gamma_s}_\varepsilon \leq t_s\big]\big]\\
&\qquad\leq \|f\|_{L^\infty(M)} \limsup_{s\downarrow 0} \frac{4}{t_s\, s}\,\mathrm{e}^{-c/t_s} = 0,
\end{aligned}$$

where the last inequality is granted by Lemma 2.4.6.

Step 3.2. By (2.4.4) and (2.4.2), the contribution of V_s is controlled by

$$\begin{aligned}
&\limsup_{s\downarrow 0} \frac{1}{t_s\, s}\,\mathbf{E}\big[\big|f(\mathsf{b}^x_{t_s}) - f(\mathsf{b}^{\gamma_s}_{t_s})\big|\,1_{V_s}\big]\\
&\qquad= \limsup_{s\downarrow 0} \frac{1}{t_s\, s}\,\mathbf{E}\Big[\frac{|f(\mathsf{b}^x_{t_s}) - f(\mathsf{b}^{\gamma_s}_{t_s})|}{\mathsf{d}(\mathsf{b}^x_{t_s}, \mathsf{b}^{\gamma_s}_{t_s})}\,\mathsf{d}(\mathsf{b}^x_{t_s}, \mathsf{b}^{\gamma_s}_{t_s})\,1_{V_s}\Big]\\
&\qquad\leq \mathrm{Lip}(f)\,\limsup_{s\downarrow 0} \frac{1}{t_s\, s^{3/2}}\,\mathbf{E}\big[\mathsf{d}(\mathsf{b}^x_{t_s}, \mathsf{b}^{\gamma_s}_{t_s})^2\,1_{A_s}\big]\\
&\qquad\leq \mathrm{Lip}(f)\,\limsup_{s\downarrow 0} \frac{s^2}{t_s\, s^{3/2}}\,\mathbf{E}\big[\mathrm{e}^{-\int_0^{t_s} \underline{\ell}(\mathsf{b}^x_r, \mathsf{b}^{\gamma_s}_r)\,\mathrm{d}r}\,1_{A_s}\big]\\
&\qquad\leq \mathrm{Lip}(f)\,\limsup_{s\downarrow 0} \frac{s^{1/2}}{t_s}\,\mathrm{e}^{-K t_s} = 0.
\end{aligned}$$

Step 3.3. In a similar way, we can ignore the influence of W_s by

$$\begin{aligned}
&\limsup_{s\downarrow 0} \frac{1}{t_s\, s}\,\mathbf{E}\big[\big|f(\mathsf{b}^x_{t_s}) - f(\mathsf{b}^{\gamma_s}_{t_s})\big|\,1_{W_s}\big]\\
&\quad= \limsup_{s\downarrow 0} \frac{1}{t_s\, s}\,\mathbf{E}\Big[\frac{|f(\mathsf{b}^x_{t_s}) - f(\mathsf{b}^{\gamma_s}_{t_s})|}{\mathsf{d}(\mathsf{b}^x_{t_s}, \mathsf{b}^{\gamma_s}_{t_s})}\,\mathsf{d}(\mathsf{b}^x_{t_s}, \mathsf{b}^{\gamma_s}_{t_s})\,1_{W_s \cap \{\mathsf{b}^x_{t_s} \neq \mathsf{b}^{\gamma_s}_{t_s}\}}\Big]\\
&\quad\leq \mathrm{Lip}(f)\,\limsup_{s\downarrow 0} \frac{1}{t_s^2\, s^{3/2}}\,\mathbf{E}\Big[\int_0^{t_s} \mathsf{d}\big(\mathsf{b}^x_{t_s}, \mathsf{b}^{\gamma_s}_{t_s}\big)\,\mathsf{d}(\mathsf{b}^x_r, \mathsf{b}^{\gamma_s}_r)\,1_{A_s}\,\mathrm{d}r\Big]\\
&\quad\leq \mathrm{Lip}(f)\,\limsup_{s\downarrow 0} \frac{s^2}{t_s^2\, s^{3/2}}\,\mathbf{E}\Big[\int_0^{t_s} \mathrm{e}^{-\int_0^{t_s} \underline{\ell}(\mathsf{b}^x_a, \mathsf{b}^{\gamma_s}_a)/2\,\mathrm{d}a - \int_0^r \underline{\ell}(\mathsf{b}^x_a, \mathsf{b}^{\gamma_s}_a)/2\,\mathrm{d}a}\,1_{A_s}\,\mathrm{d}r\Big]\\
&\quad\leq \mathrm{Lip}(f)\,\limsup_{s\downarrow 0} \frac{s^{1/2}}{t_s}\,\mathrm{e}^{-K t_s} = 0.
\end{aligned}$$

Step 3.4. Finally we turn to the most delicate part, namely the study of the effect of U_s. To this aim, we first note that, defining the function $\ell \in \mathrm{Lip_b}(M)$ by $\ell(x) := \underline{\ell}(x,x)$, as in the proof of Theorem 1.5.17, we have

$$\int_0^{t_s} \underline{\ell}(\mathsf{b}_r^x, \mathsf{b}_r^{\gamma_s})\,\mathrm{d}r - \int_0^{t_s} \ell(\mathsf{b}_r^x)\,\mathrm{d}r \geq -\mathrm{Lip}(\underline{\ell})\, t_s\, s^{1/2} \quad \text{on } A_s \cap W_s^{\mathrm{c}}.$$

Together with (2.4.4) and since $\mathsf{d}(\mathsf{b}_{t_s}^x, \mathsf{b}_{t_s}^{\gamma_s}) < s^{1/2}$ on $A_s \cap V_s^{\mathrm{c}}$, we thus obtain

$$\begin{aligned}
&\limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\frac{1}{s}\mathbf{E}\big[\big|f(\mathsf{b}_{t_s}^x) - f(\mathsf{b}_{t_s}^{\gamma_s})\big|\,1_{U_s}\big] - |\nabla f|(x)\Big]\\
&\quad\leq \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\frac{1}{s}\mathbf{E}\Big[\mathsf{d}(\mathsf{b}_{t_s}^x, \mathsf{b}_{t_s}^{\gamma_s})\,\frac{|f(\mathsf{b}_{t_s}^x) - f(\mathsf{b}_{t_s}^{\gamma_s})|}{\mathsf{d}(\mathsf{b}_{t_s}^x, \mathsf{b}_{t_s}^{\gamma_s})}\,1_{U_s\cap\{\mathsf{b}_{t_s}^x \neq \mathsf{b}_{t_s}^{\gamma_s}\}}\Big] - |\nabla f|(x)\Big]\\
&\quad\leq \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\mathbf{E}\Big[\mathrm{e}^{-\int_0^{t_s}\ell(\mathsf{b}_r^x)/2\,\mathrm{d}r}\,\mathrm{e}^{\mathrm{Lip}(\underline{\ell})\,t_s\,s^{1/2}/2}\\
&\qquad\qquad \times \sup_{z\in B_{s^{1/2}}(\mathsf{b}_{t_s}^x)\setminus\{\mathsf{b}_{t_s}^x\}} \frac{|f(\mathsf{b}_{t_s}^x) - f(z)|}{\mathsf{d}(\mathsf{b}_{t_s}^x, z)}\,1_{A_s}\Big] - |\nabla f|(x)\Big].
\end{aligned}$$

For small enough $s > 0$, on the event A_s we have $B_{2s^{1/2}}(\mathsf{b}_{t_s}^x) \subset B_{2\varepsilon}(x)$. In this case, by applying the mean value theorem twice,

$$\begin{aligned}
&\sup_{z\in B_{s^{1/2}}(\mathsf{b}_{t_s}^x)\setminus\{\mathsf{b}_{t_s}^x\}} \frac{|f(\mathsf{b}_{t_s}^x) - f(z)|}{\mathsf{d}(\mathsf{b}_{t_s}^x, z)}\\
&\qquad\leq \sup_{y\in B_{2s^{1/2}}(\mathsf{b}_{t_s}^x)} |\nabla f|(y)\\
&\qquad\leq |\nabla f|(\mathsf{b}_{t_s}^x) + \sup_{y\in B_{2s^{1/2}}(\mathsf{b}_{t_s}^x)} \big||\nabla f|(y) - |\nabla f|(\mathsf{b}_{t_s}^x)\big|\\
&\qquad\leq |\nabla f|(\mathsf{b}_{t_s}^x) + 2\,s^{1/2} \sup_{v\in \overline{B}_{4\varepsilon}(x)} \big|\nabla|\nabla f|\big|(v).
\end{aligned}$$

Let $\psi \in \mathrm{C}_\mathrm{c}^\infty(M)$ be nonnegative with $\psi = |\nabla f|$ on $B_{2\varepsilon}(x)$. Invoking Lebesgue's theorem, (0.1.6) and the smoothness of the heat semigroup up to zero, the terms containing $s^{1/2}$ above become negligible as $s \downarrow 0$, and we are left with

$$\begin{aligned}
&\limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\frac{1}{s}\mathbf{E}\big[\big|f(\mathsf{b}_{t_s}^x) - f(\mathsf{b}_{t_s}^{\gamma_s})\big|\,1_{U_s}\big] - |\nabla f|(x)\Big]\\
&\quad\leq \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\mathbf{E}\big[\mathrm{e}^{-\int_0^{t_s}\ell(\mathsf{b}_r^x)/2\,\mathrm{d}r}\,|\nabla f|(\mathsf{b}_{t_s}^x)\,1_{A_s}\big] - |\nabla f|(x)\Big]\\
&\quad= \limsup_{s\downarrow 0} \frac{1}{t_s}\Big[\mathbf{E}\big[\mathrm{e}^{-\int_0^{t_s}\ell(\mathsf{b}_r^x)/2\,\mathrm{d}r}\,\psi(\mathsf{b}_{t_s}^x)\,1_{A_s}\big] - \psi(x)\Big]\\
&\quad\leq \limsup_{s\downarrow 0} \frac{1}{t_s}\big[\mathbf{E}\big[\psi(\mathsf{b}_{t_s}^x)\big] - \psi(x)\big]\\
&\qquad\qquad + \limsup_{s\downarrow 0} \frac{1}{t_s}\mathbf{E}\Big[\big[\mathrm{e}^{-\int_0^{t_s}\ell(\mathsf{b}_r^x)/2\,\mathrm{d}r} - 1\big]\,\psi(\mathsf{b}_{t_s}^x)\Big]\\
&\quad= \frac{1}{2}\Delta\psi(x) - \frac{1}{2}\,\ell(x)\,\psi(x) = \frac{1}{2}\Delta|\nabla f|(x) - \frac{1}{2}\,\ell(x)\,|\nabla f|(x),
\end{aligned}$$

where in second last identity, we used that the marginal law of b^x is independent of s. Since $\underline{\ell}$ was arbitrary, we conclude (2.1.5) by Lemma 2.4.5. □

Remark 2.4.7. Without (2.1.1), a careful inspection of the previous proof shows that if (iii) in Theorem 2.1.6 holds for any symmetric lower semicontinuous function $\underline{k}\colon M^2 \to \mathbf{R}$ which is not necessarily the average of some function as in (2.1.4), then the L^1-Bochner inequality holds for $k\colon M \to \mathbf{R}$ defined by $k(x) := \underline{k}(x,x)$. ■

2.5 Kato decomposable lower Ricci bounds

2.5.1 The L^1-gradient estimate

In this subsection, we present a last equivalent characterization of the condition $\mathrm{Ric} \geq k$ on M for the class of Kato decomposable k in terms of gradient estimates for $(\mathsf{P}_t)_{t\geq 0}$. A similar result can be found in [Wu20, Cor. 2.2], compare also with Theorem 1.1.1 again. See [Wan14, Thm. 2.3.1] for more geometric growth conditions on k^-.

Theorem 2.5.1. *Assume that $k\colon M \to \mathbf{R}$ is a continuous and Kato decomposable function. Then any of the equivalent conditions in Theorem 2.1.6 is equivalent to the* L^1-gradient estimate w.r.t. k, *i.e. for every $f \in \mathrm{C}_\mathrm{c}^\infty(M)$, every $x \in M$ and every $t > 0$,*

$$|\nabla \mathsf{P}_t f|(x) \leq \mathbf{E}\big[\mathrm{e}^{-\int_0^{2t} k(\mathsf{b}_r^x)/2\,\mathrm{d}r}\,|\nabla f|(\mathsf{b}_{2t}^x)\,1_{\{t<\zeta^x/2\}}\big].$$

Proof. If k obeys $\mathrm{Ric} \geq k$ on M, then the claimed L^1-gradient estimate is just a restatement of Theorem 2.2.1 for exact 1-forms together with (2.2.1).

Conversely, assume the L^1-gradient estimate. A similar argument as in the proof of (i) in Theorem 2.1.1 in Subsection 2.3.1 — directly employing the given gradient estimate instead of Theorem 2.2.1 — shows that M is stochastically complete. Let $f \in \mathrm{C}_\mathrm{c}^\infty(M)$ and $x \in M$ with $|\nabla f|(x) \neq 0$. Let $\varepsilon > 0$ such that $|\nabla f|$ is bounded away from zero — in particular smooth — on $\overline{B}_\varepsilon(x)$. By Kato's inequality for the Bochner Laplacian [HSU80, Prop. 2.2], we have $|\nabla f| \in W^{1,2}(M)$. Thus, given a nonnegative $\phi \in \mathrm{C}_\mathrm{c}^\infty(M)$ with support in $B_\varepsilon(x)$, by the chain rule, (0.1.6) and a standard representation of the quadratic form $\mathcal{E}$ in terms of $(\mathsf{P}_t)_{t\geq 0}$, see e.g. [Dav89, p. 15],

$$\begin{aligned}
&\frac{1}{2}\int_M |\nabla f|(y)^{-1}\,\big\langle \nabla f(y), \nabla\Delta f(y)\big\rangle\,\phi(y)\,\mathrm{d}\mathfrak{m}(y)\\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_0 \int_M |\nabla \mathsf{P}_{t/2} f|^2(y)^{1/2}\,\phi(y)\,\mathrm{d}\mathfrak{m}(y)\\
&\quad = \lim_{t\downarrow 0}\frac{1}{t}\int_M \big[|\nabla \mathsf{P}_{t/2} f|(y) - |\nabla f|(y)\big]\,\phi(y)\,\mathrm{d}\mathfrak{m}(y)\\
&\quad \leq \limsup_{t\downarrow 0}\frac{1}{t}\int_M \Big[\mathbf{E}\big[\mathrm{e}^{-\int_0^t k(\mathsf{b}_r^y)/2\,\mathrm{d}r}\,|\nabla f|(\mathsf{b}_t^y)\big] - |\nabla f|(y)\Big]\,\phi(y)\,\mathrm{d}\mathfrak{m}(y)\\
&\quad \leq \limsup_{t\downarrow 0}\frac{1}{t}\int_M \big[\mathbf{E}\big[|\nabla f|(\mathsf{b}_t^y)\big] - |\nabla f|(y)\big]\,\phi(y)\,\mathrm{d}\mathfrak{m}(y)\\
&\qquad\qquad + \limsup_{t\downarrow 0}\frac{1}{t}\int_M \mathbf{E}\Big[\big[\mathrm{e}^{-\int_0^t k(\mathsf{b}_r^y)/2\,\mathrm{d}r} - 1\big]\,|\nabla f|(\mathsf{b}_t^y)\Big]\,\phi(y)\,\mathrm{d}\mathfrak{m}(y)\\
&\quad = -\frac{1}{2}\int_M \big\langle \nabla|\nabla f|(y), \nabla\phi(y)\big\rangle\,\mathrm{d}\mathfrak{m}(y) \qquad\qquad (2.5.1)\\
&\qquad\qquad + \limsup_{t\downarrow 0}\frac{1}{t}\int_M \mathbf{E}\Big[\big[\mathrm{e}^{-\int_0^t k(\mathsf{b}_r^y)/2\,\mathrm{d}r} - 1\big]\,|\nabla f|(\mathsf{b}_t^y)\Big]\,\phi(y)\,\mathrm{d}\mathfrak{m}(y).
\end{aligned}$$

It remains to estimate the latter limit. Let τ_ε^y be the first exit time of b^y from $B_\varepsilon(y)$. Since $\mathscr{k}$ is bounded on the bounded set $\bigcup_{y\in\overline{B}_\varepsilon(x)} \overline{B}_\varepsilon(y)$, and by continuity of Brownian sample paths, Lebesgue's theorem gives

$$\begin{aligned}
&\limsup_{t\downarrow 0}\frac{1}{t}\int_M \mathbf{E}\Big[\big[\mathrm{e}^{-\int_0^t \mathscr{k}(\mathsf{b}_r^y)/2\,\mathrm{d}r}-1\big]\,|\nabla f|(\mathsf{b}_t^y)\,1_{\{t<\tau_\varepsilon^y\}}\Big]\,\phi(y)\,\mathrm{d}\mathfrak{m}(y)\\
&\qquad = -\frac{1}{2}\int_M \mathscr{k}(y)\,|\nabla f|(y)\,\phi(y)\,\mathrm{d}\mathfrak{m}(y).
\end{aligned}$$

Using the Cauchy-Schwarz inequality, Lemma 2.1.4 and Lemma 2.4.6, for arbitrary $T>0$ we obtain

$$\begin{aligned}
&\limsup_{t\downarrow 0}\frac{1}{t}\int_M \mathbf{E}\Big[\big[\mathrm{e}^{-\int_0^t \mathscr{k}(\mathsf{b}_r^y)/2\,\mathrm{d}r}-1\big]\,|\nabla f|(\mathsf{b}_t^y)\,1_{\{t\geq\tau_\varepsilon^y\}}\Big]\,\phi(y)\,\mathrm{d}\mathfrak{m}(y)\\
&\qquad\leq \|\nabla f\|_{L^\infty(TM)}\limsup_{t\downarrow 0}\int_M \Big[\mathbf{E}\Big[\big|\mathrm{e}^{\int_0^T \mathscr{k}^-(\mathsf{b}_r^y)/2\,\mathrm{d}r}-1\big|^2\Big]^{1/2}\\
&\qquad\qquad\times \mathbf{P}\big[\tau_\varepsilon^y\leq t\big]^{1/2}\,\phi(y)\Big]\,\mathrm{d}\mathfrak{m}(y)\\
&\qquad\leq \sqrt{2}\,\|\nabla f\|_{L^\infty(TM)}\Big[\sup_{y\in M}\mathbf{E}\big[\mathrm{e}^{\int_0^T \mathscr{k}^-(\mathsf{b}_r^y)\,\mathrm{d}r}\big]^{1/2}+1\Big]\\
&\qquad\qquad\times\limsup_{t\downarrow 0}\frac{1}{t}\int_M \mathbf{P}\big[\tau_\varepsilon^y\leq t\big]^{1/2}\,\phi(y)\,\mathrm{d}\mathfrak{m}(y)=0,
\end{aligned}$$

and the L^1-Bochner inequality (2.1.5) follows after integrating (2.5.1) by parts and using the arbitrariness of ϕ. □

Remark 2.5.2. One can replace $\mathrm{C}_\mathrm{c}^\infty(M)$ by $W^{1,2}(M)$ in Theorem 2.5.1. This follows from (2.2.1) and the fact that under Kato decomposability, the Feynman–Kac formula for the heat semigroup on 1-forms, Theorem 2.2.1, holds for all square integrable 1-forms. (This formula has been shown in [Gün12] in a more general context, and of course it also follows from Theorem 2.2.1 by approximating forms in $L^2(T^*M)$ by elements of $\Gamma_\mathrm{c}(T^*M)$ using Khasminskii's lemma.) In view of the Cauchy–Schwarz inequality it seems unlikely that the Feynman–Kac formula on 1-forms holds for all square integrable 1-forms under the weaker assumption (2.1.1), although we are not aware of a counterexample (which would be interesting to have). We refer the reader also to the recent [BG21], where Feynman–Kac formulas for general perturbations of order no larger than 1 — rather than just zeroth order perturbations — of Bochner Laplacians on vector bundles have been treated. ■

Remark 2.5.3. Somewhat in line with the previous remark, assume that $\mathscr{k}$ satisfies (2.1.1) instead of Kato decomposability. Of course, if $\mathrm{Ric}\geq\mathscr{k}$ on M, the L^1-gradient estimate from Theorem 2.5.1 then still holds by virtue of Theorem 2.2.1. However, as it becomes apparent from the above proof, the converse implication seems to be more involved and to require at least some higher order exponential integrability of $\mathscr{k}^-$. ■

2.5.2 Proof of Theorem 2.1.3

Now, we present one possible step-by-step analysis in order to check the existence of (continuous) Kato decomposable lower Ricci bounds for M, along with proving Theorem 2.1.3. Let us abbreviate $d:=\dim M$.

Proof of Theorem 2.1.3. Let $\Xi : M \to (0, \infty)$ be a Borel function such that, up to a certain uniform constant $C > 0$,

$$\sup_{y \in M} \mathsf{p}_t(x, y) \leq C\, \Xi(x) \left[t^{-d/2} + 1\right] \tag{2.5.2}$$

for every $x \in M$, and every $t \in (0,1]$. (Using a parabolic L^1-mean value inequality, it has been shown in [Gün17b, Thm. 2.9], see also [Gün17a, Rem. IV.17], that every Riemannian manifold admits a canonical choice of a function Ξ as above. So does any *Lipschitz* Riemannian manifold, as will be discussed in the forthcoming work [BR21].) [Gün17a, Prop. VI.10] states that for every $p \in [1, \infty)$, if $d = 1$, and every $p \in (d/2, \infty)$, if $d \geq 2$, we have $L^p(M, \Xi\, \mathfrak{v}) + L^\infty(M) \subset \mathrm{K}(M)$. Thus, any locally $\mathfrak{v}$-integrable function $k \colon M \to \mathbf{R}$ such that $k^- \in L^p(M, \Xi\, \mathfrak{v}) + L^\infty(M)$ for some Ξ and p as above is Kato decomposable.

Now let $\langle\cdot,\cdot\rangle$ be quasi-isometric to a complete metric on M whose Ricci curvature is bounded from below by constant. Then, as the Li–Yau heat kernel estimate, the Cheeger–Gromov volume estimate and the local volume doubling property are qualitatively stable under quasi-isometry, it follows from the considerations in [Gün17a, Ex. IV.18] that there exists a constant $C > 0$ such that, for every $x \in M$ and every $t \in (0, 1]$,

$$\sup_{y \in M} \mathsf{p}_t(x, y) \leq C\, \mathfrak{v}[B_1(x)]^{-1} \left[t^{-d/2} + 1\right].$$

Thus every $k \colon M \to \mathbf{R}$ such that, choosing $\Xi := \mathfrak{v}[B_1(\cdot)]^{-1}$, one has $k^- \in L^p(M, \Xi\, \mathfrak{v}) + L^\infty(M)$ for some p as in the previous step is Kato decomposable. □

Remark 2.5.4. The previous proof shows that the assertion of Theorem 2.1.3 remains valid if the inverse volume function is replaced by any function obeying (2.5.2). ■

Example 2.5.5. Assume that M is a model manifold in the sense of [Gri09], meaning that $M = \mathbf{R}^d$ as a manifold with $d \geq 2$, and that the Riemannian metric $\langle\cdot,\cdot\rangle$ is given in polar coordinates as $\mathrm{d}r^2 + \psi(r)\, \mathrm{d}\theta^2$, where $\psi \in \mathrm{C}^\infty((0, \infty))$ is a positive function. The volume of balls on such manifolds does not depend on the center, and the Ricci curvature behaves in the radial direction like $\psi''/\psi - (d-1)(\psi')^2/\psi^2$, see e.g. [Bes87, p. 266]. Assume now that, for some $p \in (d/2, \infty)$,

$$\left(\psi''/\psi - (d-1)(\psi')^2/\psi^2\right)^- \in L^p((0,\infty), \psi^{d-1}\, \mathscr{L}^1\big|_{(0,\infty)}) + L^\infty((0,\infty)).$$

Since the volume measure behaves in the radial direction as $\psi^{d-1}(r)\, \mathrm{d}r$, the Ricci curvature is lower bounded by a function with negative part in $L^p(M) + L^\infty(M)$.

To ensure that the latter function space is included in $\mathrm{K}(M)$ it suffices from the above considerations to assume that there exists a smooth positive function ψ_0 defined on $(0, \infty)$ such that

a. $\psi_0(0) = 0$, $\psi_0'(0) = 1$ and $\psi_0''(0) = 0$,
b. $\psi_0''/\psi_0 - (d-1)(\psi_0')^2/\psi_0^2$ is uniformly bounded from below by a constant, and
c. $\psi_0/C \leq \psi \leq C\psi_0$ for some constant $C > 1$.

Indeed, a. guarantees that there exists a complete metric g_0 on M which — in polar coordinates — is written as $g_0 = \mathrm{d}r^2 + \psi_0(r)\, \mathrm{d}\theta^2$. Assumption b. guarantees that the Ricci curvature associated to g_0 is bounded from below by a constant, and c. implies that $\langle\cdot,\cdot\rangle$ is quasi-isometric to g_0. For instance, one can take the Euclidean metric corresponding to $\psi_0(r) := r$ or the hyperbolic metric corresponding to $\psi_0(r) = \sinh(r)$ as reference metrics. ■

Chapter Three

Second order calculus for tamed Dirichlet spaces

This chapter is based on the author's work [Bra21], from which large parts are taken over *verbatim*.

In this chapter, let M be a topological Lusin space (i.e. a continuous injective image of a Polish space) with a σ-finite Borel measure $\mathfrak{m}$ on M. Let $\mathscr{E}$ be a quasi-regular, strongly local Dirichlet form on $L^2(M)$ with domain $\mathscr{F}$ and *extended* domain $\mathscr{F}_{\mathrm{e}}$. The triple $(M, \mathscr{E}, \mathfrak{m})$ is called *Dirichlet space*. In this framework, the previous topological assumption on M is not restrictive [MR92, Rem. IV.3.2]; further details on Dirichlet forms are given in Section 3.2 below. We assume that $\mathscr{E}$ admits a carré du champ Γ, although this is not always required, see Remark 3.2.36. Denote by Δ the generator of $\mathscr{E}$, the *Laplacian*, with L^2-dense domain $\mathscr{D}(\Delta) \subset \mathscr{F}$.

3.1 Main results

Objective Inspired by and following [Gig18], our goal in this chapter is to construct a functional first and second order calculus if the Dirichlet space $(M, \mathscr{E}, \mathfrak{m})$ is *tamed* by a signed measure κ in the extended Kato class $\mathbf{K}_{1-}(M)$. (There are various reasons for working with $\kappa \in \mathbf{K}_{1-}(M)$ rather than with general $\mathscr{E}$*-quasi-local distributions* $\kappa \in \mathscr{F}_{\mathrm{qloc}}^{-1}$ [ER+20], which are summarized in an own paragraph below. Still, already in the former case, many arguments become technically more challenging compared to [Gig18].) These types of spaces have been introduced in [ER+20], relevant definitions will be surveyed in Subsection 3.2.6. Already the "function" part in $\mathbf{K}_{1-}(M)$, cf. Definition 2.1.2, is of particular interest already for Riemannian manifolds without boundary [Car19, GP15, Gün17a, GvR20, MO20, Ros19, RS20] or their Ricci limits [CMT21], see also the previous Chapter 2. This is just the right class of measure-valued potentials for which the associated Feynman–Kac semigroup has good properties [SV96, Stu94].

In turn, such a second order calculus will induce a first order calculus on vector-valued objects. A functional first order structure for Dirichlet spaces is, of course, well-known to exist [BK19, CS03, Ebe99, HRT13, HT15, IRT12]. In [Bra21], we have put it into the picture of the approach through L^∞-modules [Gig18] and have shown its compatibility with the previous works. To streamline the presentation and to lay the focus on the really relevant second order calculus, we only briefly recapitulate those results from [Bra21] which concern first order objects, see Subsection 3.2.4, without proofs. On the other hand, besides [Gig18] higher order objects are only studied in one-dimensional cases [BK19, HT15] or under restrictive structural assumptions [LLW02]. In our general approach, the two most important quantities will be

- the *Hessian* operator on appropriate functions, along with proving that sufficiently many of these do exist, and
- a measure-valued *Ricci curvature*.

In addition, we concisely incorporate the tamed analogue of the finite-dimensional $\mathrm{BE}_2(K,N)$ condition [BGL14, EKS15, ER+20], $K \in \mathbf{R}$ and $N \in [1,\infty)$, following the $\mathrm{RCD}^*(K,N)$-treatise [Han18a] which is not essentially different from [Gig18].

Possible extensions Besides the tamed space versions of possible extensions mentioned in Section 4.1, we moreover hope that the toolbox provided by Chapter 3 becomes helpful in further investigations of tamed spaces. Possible directions could include

- the study of covariant Schrödinger operators [Gün17a], see also Chapter 4,
- rigidity results for and properties of finite-dimensional tamed spaces [BNS20, BS20],
- the study of bounded variation functions under Kato conditions [BPS19, BCM19, GP15],
- super-Ricci flows [KS18, Stu18b], noting that the Kato condition, in contrast to L^p-conditions, on the Ricci curvature along Kähler–Ricci flows is stable [TZ16],
- a structure theory for Kato Ricci limit or tamed spaces [CMT21, MN19].

First order calculus To speak about vector-valued objects, we employ the theory of *L^p-normed L^∞-modules*, $p \in [1,\infty]$, w.r.t. a given measure — here $\mathfrak{m}$ — introduced in [Gig18], see Subsection 3.2.3. This is a Banach space $\mathscr{M}$ endowed with a group action by $L^\infty(M)$ and a map $|\cdot|\colon \mathscr{M} \to L^p(M)$, the *pointwise norm*, such that

$$\|\cdot\|_{\mathscr{M}} = \big\||\cdot|\big\|_{L^p(M)}.$$

In terms of $|\cdot|$, all relevant $\mathfrak{m}$-a.e. properties of elements of $\mathscr{M}$, e.g. their $\mathfrak{m}$-a.e. vanishing outside some given Borel set $A \subset M$, can be rigorously made sense of. $L^\infty(M)$ is chosen as acting group given that multiplying vector-valued objects by functions should preserve the initial object's $\mathfrak{m}$-integrability. Thus, to some extent L^∞-modules allow us to speak of generalized *sections* without any vector bundle (which we will also not define). We believe that this interpretation is more straightforward and better suited for analytic purposes than the *fiber* one by measurable Hilbert fields from [BK19, CS03, Ebe99, HRT13, HT15, IRT12] — albeit the approaches are equivalent, see Remark 3.2.25 — where such a bundle is actually constructed.

The space $L^2(T^*M)$ of L^2-1-forms w.r.t. $\mathfrak{m}$, termed *cotangent module* [Gig18], is explicitly constructed in [Bra21] following [Gig18]. By duality, the *tangent module* $L^2(TM)$ of L^2-vector fields w.r.t. $\mathfrak{m}$ is then defined in Definition 3.2.43. In Subsection 3.2.4, we will outline the main result of this treatise, namely that $L^2(T^*M)$ and $L^2(TM)$ are both L^2-normed L^∞-modules with pointwise norms both denoted by $|\cdot|$. They come with a linear *differential* $\mathrm{d}\colon \mathscr{F}_\mathrm{e} \to L^2(T^*M)$ and a linear *gradient* $\nabla\colon \mathscr{F}_\mathrm{e} \to L^2(TM)$ such that for every $f \in \mathscr{F}_\mathrm{e}$,

$$|\mathrm{d}f| = |\nabla f| = \Gamma(f)^{1/2} \quad \mathfrak{m}\text{-a.e.}$$

Both d and ∇ obey all expected locality and calculus rules, cf. Proposition 3.2.37. Moreover, polarization of $|\cdot|$ induces a *pointwise scalar product* $\langle\cdot,\cdot\rangle$ on $L^2(T^*M)^2$ and $L^2(TM)^2$ which, by integration w.r.t. $\mathfrak{m}$, turns the latter into Hilbert spaces, respectively.

Measure-valued divergence Recall the *Gauß–Green formula*

$$-\int_M \mathrm{d}h(X)\,\mathrm{d}\mathfrak{v} = \int_M h\,\mathrm{div}_{\mathfrak{v}} X\,\mathrm{d}\mathfrak{v} - \int_{\partial M} h\,\langle X, \mathsf{n}\rangle\,\mathrm{d}\mathfrak{s}, \tag{3.1.1}$$

valid for every compact Riemannian manifold M with boundary ∂M, every $X \in \Gamma_c(TM)$ and every $h \in \mathrm{C}_c^\infty(M)$. Here, n is the outward-pointing unit normal vector field at ∂M, and $\mathfrak{v}$ and $\mathfrak{s}$ are the usual volume and surface measure on M and ∂M, respectively. This motivates our first key differential object, the *measure-valued divergence* of appropriate vector fields, which in turn is suitable to define the *normal component* of the latter.

Leaned on [BCM19], we thus propose the following in Definition 3.2.46.

Definition 3.1.1. *We say that $X \in L^2(TM)$ has a* measure-valued divergence*, briefly $X \in \mathscr{D}(\mathbf{div})$, if there exists a σ-finite signed Borel measure $\mathbf{div}\,X$ charging no $\mathscr{E}$-polar sets such that for sufficiently many $h \in \mathscr{F}$,*

$$-\int_M \mathrm{d}h(X)\,\mathrm{d}\mathfrak{m} = \int_M \widetilde{h}\,\mathrm{d}\mathbf{div}\,X.$$

In turn, keeping in mind (3.1.1) and using Lebesgue's decomposition

$$\mathbf{div}\,X = \mathbf{div}_{\ll} X + \mathbf{div}_{\perp} X$$

of $\mathbf{div}\,X$ w.r.t. $\mathfrak{m}$, we define the normal component of $X \in \mathscr{D}(\mathbf{div})$ by

$$\mathbf{n}\,X := -\,\mathbf{div}_{\perp} X,$$

see Definition 3.2.47. Calculus rules for $\mathbf{div}\,X$ and $\mathbf{n}\,X$, $X \in \mathscr{D}(\mathbf{div})$, are listed in Subsection 3.2.5. In our generality, we do not know more about the support of $\mathbf{n}\,X$ than its $\mathfrak{m}$-singularity. Nevertheless, these notions are satisfactorily compatible with other recent *extrinsic* approaches to Gauß–Green's formula and boundary components on (subsets of) RCD spaces [BPS19, BCM19, Stu20] as outlined in Section 3.7.

The advantage of this measure point of view compared to the L^2-one from [Gig18], see Definition 3.2.45, is its ability to "see" the normal component of $X \in \mathscr{D}(\mathbf{div})$ rather than the latter being left out in the relevant integration by parts formulas and interpreted as zero. This distinction does mostly not matter: matching with the interpretation of the generator Δ of $\mathscr{E}$ as *Neumann Laplacian*, on tamed spaces, for many $g \in \mathscr{F} \cap L^\infty(M)$ and $f \in \mathscr{D}(\Delta)$ — e.g. for $g, f \in \mathrm{Test}(M)$, cf. Lemma 3.2.54 and (3.1.4) below — the vector field $X := g\,\nabla f \in L^2(TM)$ belongs to $\mathscr{D}(\mathbf{div})$ with

$$\begin{aligned} \mathbf{div}_{\ll} X &= \big[\mathrm{d}g(\nabla f) + g\,\Delta f\big]\,\mathfrak{m},\\ \mathbf{n}\,X &= 0. \end{aligned} \tag{3.1.2}$$

(In fact, many relevant spaces will be defined in terms of such vector fields, hence all Laplace-type operators considered in this chapter, see Definition 3.4.20 and Definition 3.5.21, implicitly obey Neumann boundary conditions in certain senses.) By now, it is however not even clear if there exist (m)any $f \in \mathscr{F}$ with

a. $|\nabla f|^2 \in \mathscr{F}$, not to say with
b. $\nabla|\nabla f|^2 \in \mathscr{D}(\mathbf{div})$.

These issues appear similarly when initially trying to define higher order differential operators, as briefly illustrated now along with addressing a. and b.

Second order calculus The subsequent pointwise formulas hold on the interior M° of any Riemannian manifold M with boundary, for every $f, g_1, g_2 \in \mathrm{C}^\infty(M)$, every $X, X_1, X_2 \in \Gamma(TM)$ and every $\omega \in \Gamma(T^*M)$ [Lee18, Pet06]:

$$\begin{aligned}2\,\mathrm{Hess}\, f(\nabla g_1, \nabla g_2) &= \big\langle \nabla\langle\nabla f, \nabla g_1\rangle, \nabla g_2\big\rangle + \big\langle \nabla\langle\nabla f, \nabla g_2\rangle, \nabla g_1\big\rangle\\ &\quad - \big\langle \nabla\langle\nabla g_1, \nabla g_2\rangle, \nabla f\big\rangle,\\ \big\langle \nabla_{\nabla g_1} X, \nabla g_2\big\rangle &= \big\langle \nabla\langle X, \nabla g_1\rangle, \nabla g_2\big\rangle - \mathrm{Hess}\, g_2(X, \nabla g_1),\\ \mathrm{d}\omega(X_1, X_2) &= \mathrm{d}\big[\omega(X_2)\big](X_1) - \mathrm{d}\big[\omega(X_1)\big](X_2)\\ &\quad - \omega(\nabla_{X_1} X_2 - \nabla_{X_2} X_1).\end{aligned} \tag{3.1.3}$$

The first identity characterizes the Hessian $\mathrm{Hess}\, f$ of f, the second is a definition of the covariant derivative ∇X of X in terms of that Hessian, and in turn, the exterior derivative $\mathrm{d}\omega$ of ω can be defined with the help of ∇. (A similar formula is true for the exterior differential acting on forms of any degree, see Example 3.5.1.) Hence, we may and will axiomatize these three differential operators in the previous order. In the sequel, we only outline how we paraphrase the first identity in (3.1.3) nonsmoothly. The operators ∇ and d can then be defined by similar (integration by parts) procedures and, as for the Hessian, satisfy a great diversity of expected calculus rules, see Subsection 3.3.4, Section 3.4 and Section 3.5 for details.

Up to the small point of defining the two-fold tensor product $L^2((T^*)^{\otimes 2}M)$ of $L^2(T^*M)$, see Subsection 3.2.3, and keeping in mind (3.1.2), the following, stated in Definition 3.3.2, is naturally motivated by (3.1.3).

Definition 3.1.2. *The space* $\mathscr{D}(\mathrm{Hess})$ *consists of all* $f \in \mathscr{F}$ *such that there exists some* $\mathrm{Hess}\, f \in L^2((T^*)^{\otimes 2}M)$ *such that for every* $g_1, g_2 \in \mathrm{Test}(M)$*,*

$$\begin{aligned}&2\int_M h\,\mathrm{Hess}\, f(\nabla g_1, \nabla g_2)\,\mathrm{d}\mathfrak{m}\\ &\quad = -\int_M \langle\nabla f, \nabla g_1\rangle\,\mathrm{d}\mathbf{div}_{\ll}(h\,\nabla g_2) - \int_M \langle\nabla f, \nabla g_2\rangle\,\mathrm{d}\mathbf{div}_{\ll}(h\,\nabla g_1)\\ &\qquad - \int_M h\,\big\langle\nabla f, \nabla\langle\nabla g_1, \nabla g_2\rangle\big\rangle\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

The advantage of this definition is that the r.h.s. of the defining property only contains one derivative of f. All terms make sense if, as stated, g_1 and g_2 are in

$$\mathrm{Test}(M) := \big\{f \in \mathscr{D}(\Delta) \cap L^\infty(M) : |\nabla f| \in L^\infty(M),\ \Delta f \in \mathscr{F}\big\}, \tag{3.1.4}$$

cf. the part about test functions in Subsection 3.2.7 and Subsection 3.3.1. $\mathrm{Test}(M)$ is dense in $\mathscr{F}$, and is a cornerstone of our discussion, playing the role of *smooth* functions (recall also a similar definition in Section 1.2). For instance, $|\nabla f|^2 \in \mathscr{F}$ for $f \in \mathrm{Test}(M)$ by Proposition 3.2.75, which also addresses a. above. (In fact, $|\nabla f|^2$ is in the domain of the *measure-valued Schrödinger operator* $\mathbf{\Delta}^{2\kappa}$, Definition 3.2.74. For possible later extensions, κ will mostly not be separated from the considered operators. Hence, b. will be answered quite late, but positively, in Lemma 3.6.12.) The latter technical grounds have been laid in [ER+20] following [Sav14], are summarized in Subsection 3.2.7, and are one key place where taming by $\kappa \in \mathbf{K}_{1-}(M)$ is needed.

In Theorem 3.3.11, we show that $\mathscr{D}(\mathrm{Hess})$ is nonempty, in fact, dense in $L^2(M)$.

Theorem 3.1.3. *Every* $f \in \mathrm{Test}(M)$ *belongs to* $\mathscr{D}(\mathrm{Hess})$ *with*

$$\int_M \big|\mathrm{Hess}\, f\big|_{\mathrm{HS}}^2\,\mathrm{d}\mathfrak{m} \leq \int_M (\Delta f)^2\,\mathrm{d}\mathfrak{m} - \big\langle\kappa\,\big|\,|\nabla f|^2\big\rangle.$$

Here $|\cdot|_{\mathrm{HS}}$ is the pointwise Hilbert–Schmidt-type norm on $L^2((T^*)^{\otimes 2}M)$ — as well as the two-fold tensor product $L^2(T^{\otimes 2}M)$ of $L^2(TM)$ — see the tensor product paragraph in Subsection 3.2.3.

The key ingredient for the proof of the previous result is Lemma 3.3.9. It results from a variant of the famous *self-improvement* technique [Bak85], which has already played a role in Chapter 1 above and will be outlined in more detail in this chapter. Here we follow [Gig18], see also [ER+20, Sav14, Stu18a]. The idea is to replace $f \in \mathrm{Test}(M)$ in the taming condition

$$\mathbf{\Delta}^{2\kappa} \frac{|\nabla f|^2}{2} - \big\langle \nabla f, \nabla \Delta f \big\rangle\, \mathfrak{m} \geq 0$$

from Proposition 3.2.75 by a polynomial in appropriate test functions. By optimizing over the coefficients, Theorem 3.3.11 follows by integrating the resulting inequality

$$\mathbf{\Delta}^{2\kappa} \frac{|\nabla f|^2}{2} - \big\langle \nabla f, \nabla \Delta f \big\rangle\, \mathfrak{m} \geq \big|\mathrm{Hess}\, f\big|_{\mathrm{HS}}^2\, \mathfrak{m}.$$

Ricci curvature The second main result of this chapter is the existence of the named measure-valued *curvature tensors*. Both are defined by *Bochner's identity*. The latter requires some work to be made sense of at least for the large class $\mathrm{Reg}(TM)$ of *regular vector fields*, i.e. all linear combinations of elements of the form $X := g\,\nabla f \in L^2(TM)$, $g \in \mathrm{Test}(M) \cup \mathbf{R}\,1_M$ and $f \in \mathrm{Test}(M)$, see Subsection 3.2.8. (It is generally larger than the one of *test vector fields* $\mathrm{Test}(TM)$ considered in [Gig18].) Such X, first, obey $|X|^2 \in \mathscr{D}(\mathbf{\Delta}^{2\kappa})$ by Lemma 3.6.2, second, have a covariant derivative $\nabla X \in L^2(T^{\otimes 2}M)$ by Theorem 3.4.3, and third, have a 1-form counterpart $X^\flat \in L^2(T^*M)$ in the domain of the Hodge Laplacian $\vec{\Delta}$ by Lemma 3.6.1. Therefore, for $X \in \mathrm{Reg}(TM)$ the definition

$$\mathbf{Ric}^\kappa(X,X) := \mathbf{\Delta}^{2\kappa} \frac{|X|^2}{2} + \vec{\Delta} X^\flat(X)\, \mathfrak{m} - \big|\nabla X\big|_{\mathrm{HS}}^2\, \mathfrak{m}$$

makes sense. In fact, a variant of which is Theorem 3.6.9, we have the following.

Theorem 3.1.4. *The previous map $\mathbf{Ric}^\kappa$ extends continuously to the closure $H^{1,2}_\sharp(TM)$ of $\mathrm{Reg}(TM)$ w.r.t. an appropriate $H^{1,2}$-norm, see Definition 3.6.7, with values in the space of Borel measures on M with finite total variation charging no $\mathscr{E}$-polar sets.*

The nonnegativity implicitly asserted therein comes precisely from the taming condition. Abusing terminology, the map $\mathbf{Ric}^\kappa$ will be called *κ-Ricci measure*.

Finally, in Subsection 3.6.2 we separate the measure κ from $\mathbf{Ric}^\kappa$. To this aim, in Lemma 3.6.2 and Lemma 3.6.12 we discover that $\nabla|X|^2 \in \mathscr{D}(\mathbf{div})$ together with the relation $\mathbf{div}\,\nabla|X|^2 = \mathbf{\Delta}^{2\kappa}|X| + 2\,|X|_\sim^2\,\kappa$ — for an $\mathscr{E}$-quasi-continuous $\mathfrak{m}$-version $|X|_\sim^2$ of $|X|^2$ — for every $X \in \mathrm{Reg}(TM)$, linking the operator $\mathbf{div}$ to the κ-Ricci measure $\mathbf{Ric}^\kappa$ (recall b. above). Based on this observation we then set, for $X \in \mathrm{Reg}(TM)$,

$$\begin{aligned} \mathbf{Ric}(X,X) &:= \mathbf{Ric}^\kappa(X,X) + |X|_\sim^2\,\kappa \\ &= \mathbf{div}\,\nabla\frac{|X|^2}{2} + \vec{\Delta}X^\flat(X)\,\mathfrak{m} - \big|\nabla X\big|_{\mathrm{HS}}^2\,\mathfrak{m}. \end{aligned} \tag{3.1.5}$$

In fact, the first identity makes sense for general $X \in H^{1,2}_\sharp(TM)$.

Other interesting results Our treatise comes with further beautiful results that are worth mentioning here and hold in great generality. Examples are

- metric compatibility of the covariant derivative ∇ w.r.t. the "Riemannian metric" $\langle\cdot,\cdot\rangle$, see Proposition 3.4.11, and
- a nonsmooth analogue of the Hodge theorem, see Theorem 3.5.23.

Moreover, we address various points that have not been treated in [Gig18], but rather initiated in [Bra20, Han18a] (and part of which are studied in more detail in Chapter 4 below), among others

- semigroup domination of the heat flow on vector fields w.r.t. the functional one, see Theorem 3.4.26, as well as of the heat flow on 1-forms w.r.t. the Schrödinger semigroup with potential κ, see Theorem 3.6.33,
- spectral bottom estimates for the Bochner Laplacian, see Corollary 3.4.21, and the Hodge Laplacian, see Corollary 3.6.28,
- a vector version of the measure-valued q-Bochner inequality, $q \in [1,2]$, see Theorem 3.6.21 and compare with Chapter 1, and
- the boundedness of the "local dimension" of $L^2(TM)$ by $\lfloor N \rfloor$, see Proposition 3.3.14.

Comments on the extended Kato condition Finally, we comment on the assumption $\kappa \in \mathbf{K}_{1-}(M)$ and technical issues, compared to [Gig18], which arise later.

In [Gig18, Cor. 3.3.9, Cor. 3.6.4], the following "integrated Bochner inequality" for $\mathrm{RCD}(K,\infty)$ spaces, $K \in \mathbf{R}$, is derived for suitable $X \in L^2(TM)$:

$$\int_M \left|\nabla X\right|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} \leq \int_M |\mathrm{d}X^\flat|^2 \,\mathrm{d}\mathfrak{m} + \int_M |\delta X^\flat|^2 \,\mathrm{d}\mathfrak{m} - K\int_M |X|^2 \,\mathrm{d}\mathfrak{m}. \tag{3.1.6}$$

Here δ is the codifferential operator. The interpretation of (3.1.6) is that *an appropriate first order norm on* 1*-forms controls the first order topology on vector fields qualitatively and quantitatively*. Indeed, first, for gradient vector fields, by heat flow regularization (3.1.6) implies that $\mathscr{D}(\Delta) \subset \mathscr{D}(\mathrm{Hess})$, and (3.1.6) is stable under this procedure. Second, (3.1.6) is crucial in the RCD version of our second main Theorem 3.6.9 [Gig18, Thm. 3.6.7], for extending (3.1.5) beyond $\mathrm{Test}(TM)$ requires continuous dependency of the covariant term w.r.t. a contravariant norm. In both cases, the curvature term is clearly continuous, even in $\mathscr{F}$ or $L^2(TM)$, respectively.

The latter is wrong in our situation: already on a compact Riemannian manifold M with boundary and $\ell \neq 0$, the pairing $\left\langle \kappa \,\middle|\, |X|^2 \right\rangle$ according to (0.2.2) does not even make sense for general $X \in L^2(TM)$. Hence, we will have to deal with two correlated problems: controlling our calculus by stronger continuity properties of $X \mapsto \left\langle \kappa \,\middle|\, |X|^2 \right\rangle$, but also vice versa. (The fact that certain first order norms on 1-forms bound covariant ones on compact Riemannian manifolds with boundary, a classical result by Gaffney [Sch95], see Remark 3.5.20, is already nontrivial.)

The key property of $\kappa \in \mathbf{K}_{1-}(M)$ in this direction is that $f \mapsto \left\langle \kappa \,\middle|\, f^2 \right\rangle$ is (well-defined and) $\mathscr{E}$-form bounded on $\mathscr{F}$ with form bound smaller than 1, see Lemma 3.2.60. That is, there exist $\rho' \in [0,1)$ and $\alpha' \in \mathbf{R}$ such that for every $f \in \mathscr{F}$,

$$\left|\left\langle \kappa \,\middle|\, f^2 \right\rangle\right| \leq \rho' \int_M |\nabla f|^2 \,\mathrm{d}\mathfrak{m} + \alpha' \int_M f^2 \,\mathrm{d}\mathfrak{m}. \tag{3.1.7}$$

Now, from (3.1.7), we first note that the pairing $\langle \kappa \,|\, |X|^2 \rangle$ is well-defined for all X in a covariant first order space termed $H^{1,2}(TM)$, see Definition 3.4.5, since for every $X \in H^{1,2}(TM)$ we have $|X| \in \mathcal{F}$ by *Kato's inequality*

$$\big|\nabla|X|\big| \le \big|\nabla X\big|_{\mathrm{HS}} \quad \mathfrak{m}\text{-a.e.}, \tag{3.1.8}$$

as proven in Lemma 3.4.13. The latter is essentially a consequence of metric compatibility of ∇, cf. Proposition 3.4.11, and Cauchy–Schwarz's inequality. In particular, combining (3.1.7) with (3.1.8) will imply that $X \mapsto \langle \kappa \,|\, |X|^2 \rangle$ is even continuous in $H^{1,2}(TM)$, see Corollary 3.4.14. For completeness, we also mention here that Kato's inequality is useful at other places as well, e.g. in proving the above mentioned semigroup domination results. On $\mathrm{RCD}(K,\infty)$ spaces, $K \in \mathbf{R}$ — on which (3.1.8) has been proven in [DGP21] in order to find "quasi-continuous representatives" of vector fields — this has been observed in [Bra20] (and will also be used in the corresponding Chapter 4 below).

However, extending the inequality

$$\int_M \big|\nabla X\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} \le \int_M |\mathrm{d}X^\flat|^2 \,\mathrm{d}\mathfrak{m} + \int_M |\delta X^\flat|^2 \,\mathrm{d}\mathfrak{m} - \langle \kappa \,|\, |X|^2 \rangle \tag{3.1.9}$$

similar to (3.1.6), see Lemma 3.6.8, from $X \in \mathrm{Reg}(TM)$ — for which it is valid by many careful computations, see Lemma 3.3.9 and Lemma 3.6.2, and the $\mathrm{BE}_1(\kappa,\infty)$ condition, see Proposition 3.2.79 and Corollary 3.6.6 — continuously to more general $X \in H^{1,2}_\sharp(TM)$ requires better control on the curvature term. Here is where the form bound $\rho' \in [0,1)$ comes into play. Indeed, using (3.1.7) and (3.1.8),

$$-\langle \kappa \,|\, |X|^2 \rangle \le \rho' \int_M \big|\nabla X\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} + \alpha' \int_M |X|^2 \,\mathrm{d}\mathfrak{m},$$

and this can be merged with (3.1.9) to obtain the desired continuous control of the covariant by a contravariant first order norm. In fact, this kind of argumentation, without already having Kato's inequality at our disposal, will also be pursued in our proof that $\mathscr{D}(\Delta) \subset \mathscr{D}(\mathrm{Hess})$, see Corollary 3.3.12.

In view of this key argument, we believe that the extended Kato framework is somewhat maximal possible for which a second order calculus, at least with the presented diversity of higher order differential operators, as below can be developed.

Lastly, it is worth to spend few words on a different technical issue. Namely, to continuously extend $\mathbf{Ric}^\kappa$ in Theorem 3.6.9 w.r.t. a meaningful target topology, we need to know in advance that $\mathbf{\Delta}^{2\kappa}|X|^2$ has finite total variation for $X \in \mathrm{Reg}(TM)$. Even for gradient vector fields, this is not discussed in [ER+20]. On the other hand, the corresponding RCD space result [Sav14, Lem. 2.6] uses their stochastic completeness [AGS14a]. In this chapter, the latter is neither assumed nor generally known to be a consequence of the condition $\kappa \in \mathbf{K}_{1-}(M)$. Compare with the intrinsic completeness paragraph in Subsection 3.2.6. In Proposition 3.2.79, we give an alternative, seemingly new proof of the above finiteness which relies instead on the $\mathrm{BE}_1(\kappa,\infty)$ condition.

3.2 Preliminaries

Before coming to the second order calculus from Section 3.3 on, we review basic notions of manifolds with boundary in Subsection 3.2.1, Dirichlet spaces in Subsection 3.2.2, and Gigli's L^∞-modules in Subsection 3.2.3. Moreover, we briefly outline the first order differential structure from [Bra21, Ch. 2, Ch. 3]. From Subsection 3.2.6 on, the content from [ER+20] is summarized, and some technical results are proven for later use.

3.2.1 Riemannian manifolds with boundary

One family of guiding examples for our constructions pursued from Section 3.3 on are Riemannian manifolds, possibly noncompact and possibly with boundary. Here we collect basic terminologies on these. See [Lee97, Lee18, Pet06, Sch95] for details.

Setting Unless explicitly stated otherwise, any Riemannian manifold M is understood to have topological dimension $d \geq 2$ and smooth, i.e. to be locally homeomorphic to $\mathbf{R}^d$ or $\mathbf{R}^{d-1} \times [0, \infty)$ depending on whether $\partial M = \emptyset$ or $\partial M \neq \emptyset$ and with smooth transition functions. Recall that a function $f : M \to \mathbf{R}$ is smooth [Lee18, Ch. 1] if $f \circ \mathsf{x}^{-1} : \mathsf{x}(U) \to \mathbf{R}$ is smooth in the ordinary Euclidean sense for every chart (U, x) on M — if (U, x) is a boundary chart, i.e. $U \cap \partial M \neq \emptyset$, this means that $f \circ \mathsf{x}^{-1}$ has a smooth extension to an open subset of $\mathbf{R}^d$. For simplicity, any Riemannian manifold is assumed to be connected and (metrically) complete. Set $M^\circ := M \setminus \partial M$.

If $\partial M \neq \emptyset$, then ∂M is a smooth codimension 1 submanifold of M. It naturally becomes Riemannian when endowed with the pullback metric

$$\langle \cdot, \cdot \rangle_\jmath := \jmath^* \langle \cdot, \cdot \rangle$$

under the natural inclusion $\jmath : \partial M \to M$. The map $\jmath$ induces a natural inclusion $\mathrm{d}\jmath : T\partial M \to TM\big|_{\partial M}$ which is *not* surjective. In particular, the vector bundles $T\partial M$ and $TM\big|_{\partial M}$ do *not* coincide. Rather, $T\partial M$ is identifiable with the codimension 1 subbundle $\mathrm{d}\jmath(T\partial M)$ of $TM\big|_{\partial M}$.

Sobolev spaces on vector bundles Denote the space of smooth sections of a real vector bundle $\mathbf{F}$ over M (or ∂M) by $\Gamma(\mathbf{F})$. (This is a slight abuse of notation since we will also denote carré du champs associated to Dirichlet energies by Γ. However, it will always be clear from the context which meaning is intended.) With a connection ∇ on $\mathbf{F}$ — always chosen to be the Levi-Civita one if $\mathbf{F} := TM$ — one can define Sobolev spaces $W^{k,p}(\mathbf{F})$, $k \in \mathbf{N}_0$ and $p \in [1, \infty)$, in various ways, e.g. by completing $\Gamma_c(\mathbf{F})$ w.r.t. an appropriate norm w.r.t. $\mathfrak{v}$ (or $\mathfrak{s}$), or in a weak sense [Sch95, Sec. 1.3]. The "natural" approaches all coincide if M is compact [Sch95, Thm. 1.3.6], but for noncompact M, without further geometrical restrictions ambiguities may occur [Eic88]. Here, it is always either clear from the context which definition is intended, or precise meanings are simply irrelevant when only the compact setting is considered.

Throughout, it is useful to keep in mind the following *trace theorem* for compact M [Sch95, Thm. 1.3.7]. If M is noncompact, it only holds true locally [Sch95, p. 39].

Proposition 3.2.1. *For every $k \in \mathbf{N}_0$ and every $p \in [1, \infty)$, the natural restriction map $\cdot\big|_{\partial M} : \Gamma(\mathbf{F}) \to \Gamma(\mathbf{F}\big|_{\partial M})$ in the spirit of the next paragraph extends to a continuous — in fact, compact — map from $W^{k+1,p}(\mathbf{F})$ to $W^{k,p}(\mathbf{F}\big|_{\partial M})$.*

Normal and tangential components Denote by $\mathsf{n} \in \Gamma(TM\big|_{\partial M})$ the outward pointing unit normal vector field at ∂M. It can be smoothly extended to a (non-relabeled) vector field n on an open neighborhood of ∂M [Sch95, Thm. 1.1.7].

The restriction $X\big|_{\partial M}$ of a given $X \in \Gamma(TM)$ to ∂M decomposes into a *normal part* $X^\perp \in \Gamma(TM\big|_{\partial M})$ and a *tangential part* $X^\parallel \in \Gamma(TM\big|_{\partial M})$ defined by

$$\begin{aligned} X^\perp &:= \langle X, \mathsf{n} \rangle\, \mathsf{n}, \\ X^\parallel &:= X\big|_{\partial M} - X^\perp. \end{aligned} \tag{3.2.1}$$

Under a slight abuse of notation, in a unique way every $X^\parallel \in \Gamma(T\partial M)$ can be identified with a tangential element $X \in \Gamma(TM\big|_{\partial M})$, i.e. $X^\perp = 0$ [Sch95, p. 16]. In turn, such an X can be smoothly extended to an open neighborhood of ∂M [Lee18, Lem. 8.6]. That is, knowing the restrictions to ∂M of all $X \in \Gamma(TM)$ with purely tangential boundary components suffices to recover the entire intrinsic covariant structure of ∂M.

Similarly, the tangential part $\mathrm{t}\,\omega \in \Gamma(\Lambda^k T^*M\big|_{\partial M})$ and the normal part $\mathrm{n}\,\omega \in \Gamma(\Lambda^k T^*M\big|_{\partial M})$, $k \in \mathbf{N}$, of a k-form $\omega \in \Gamma(\Lambda^k T^*M)$ at ∂M are defined by

$$\begin{aligned} \mathrm{t}\,\omega(X_1, \dots, X_k) &:= \omega(X_1^\parallel, \dots, X_k^\parallel), \\ \mathrm{n}\,\omega &:= \omega\big|_{\partial M} - \mathrm{t}\,\omega \end{aligned} \tag{3.2.2}$$

for every $X_1, \dots, X_k \in \Gamma(TM\big|_{\partial M})$. Taking tangential and normal parts of differential forms is dual to each other through the Hodge $\star$-operator [Sch95, Prop. 1.2.6].

3.2.2 Dirichlet forms

In this subsection, we summarize various important notions of Dirichlet spaces. This survey is enclosed by two guiding examples that we frequently use for illustrative reasons in the sequel.

For more detailed accounts, we refer to the books [BH91, CF12, FOT11, MR92].

Basic definitions We always fix a symmetric, quasi-regular [MR92, Def. III.3.1] and strongly local Dirichlet form $(\mathscr{E}, \mathscr{F})$ with linear domain $\mathscr{F} := \mathscr{D}(\mathscr{E})$ which is dense in $L^2(M)$. Recall that strong locality [CF12, Def. 1.3.17] means that for every $f, g \in \mathscr{F} \cap L^0_{\mathrm{c}}(M)$ such that f is constant on a neighborhood of $\operatorname{spt} g$,

$$\mathscr{E}(f, g) = 0.$$

Remark 3.2.2. Strong locality is not strictly necessary to run the construction of the cotangent module in [Bra21]. One could more generally assume $\mathscr{E}$ to have trivial killing part in its Beurling–Deny decomposition, see [CF12, Thm. 4.3.4], [FOT11, Thm. 3.2.1] and [Kuw98, Thm. 5.1]. However, if $\mathscr{E}$ has nontrivial jump part — which precisely distinguishes it from being strongly local [CF12, Prop. 4.3.1] — the important calculus rules from Proposition 3.2.9 below typically fail. ■

If we say that a property holds for $\mathscr{E}$, we usually mean that it is satisfied by the pair $(\mathscr{E}, \mathscr{F})$. We use the abbreviations

$$\begin{aligned} \mathscr{F}_{\mathrm{b}} &:= \mathscr{F} \cap L^\infty(M), \\ \mathscr{F}_{\mathrm{c}} &:= \mathscr{F} \cap L^0_{\mathrm{c}}(M), \\ \mathscr{F}_{\mathrm{bc}} &:= \mathscr{F}_{\mathrm{b}} \cap \mathscr{F}_{\mathrm{c}}. \end{aligned}$$

By definition, $\mathscr{E}$ is *closed* [MR92, Def. I.2.3], i.e. $(\mathscr{F}, \|\cdot\|_{\mathscr{F}})$ is complete, where

$$\big\|f\big\|_{\mathscr{F}}^2 := \big\|f\big\|_{L^2(M)}^2 + \mathscr{E}(f),$$

and *Markovian* [MR92, Def. I.4.5], i.e. for every $f \in \mathscr{F}$, $\min\{f^+, 1\} \in \mathscr{F}$ and

$$\mathscr{E}\big(\min\{f^+, 1\}\big) \leq \mathscr{E}(f).$$

A densely defined, quadratic form on $L^2(M)$ — for notational convenience, we concentrate on $\mathscr{E}$ — is called *closable* if for every $\mathscr{E}$-Cauchy sequence $(f_n)_{n\in\mathbf{N}}$ in

$\mathscr{D}(\mathscr{E})$ with $\|f_n\|_{L^2(M)} \to 0$ as $n \to \infty$, we have $\mathscr{E}(f_n) \to 0$ as $n \to \infty$. $\mathscr{E}$ is closable if and only if it has a closed extension [FOT11, p. 4]. Here, we term a sequence $(f_n)_{n\in\mathbf{N}}$ in $\mathscr{F}$ *$\mathscr{E}$-Cauchy* if $\mathscr{E}(f_n - f_m) \to 0$ as $n, m \to \infty$, and *$\mathscr{E}$-bounded* if

$$\sup_{n\in\mathbf{N}} \mathscr{E}(f_n) < \infty.$$

Basic properties For all relevant quasi-notions evolving around the definition of quasi-regularity, we refer to [CF12, Ch. 1] or [MR92, Ch. III]. Here, we solely state the following useful properties [MR92, Prop. IV.3.3] frequently used in our work. (Here and in the sequel, "$\mathscr{E}$-q.e." and "$\mathscr{E}$-q.c." abbreviate *$\mathscr{E}$-quasi-everywhere* [MR92, Def. III.2.1] and *$\mathscr{E}$-quasi-continuous* [MR92, Def. III.3.2], respectively.)

For a quasi-regular Dirichlet form $\mathscr{E}$, $\mathscr{F}$ is a separable Hilbert space w.r.t. $\|\cdot\|_{\mathscr{F}}$. Every $f \in \mathscr{F}$ has an $\mathscr{E}$-q.c. $\mathfrak{m}$-version denoted $\widetilde{f}$ or $f_\sim$. Moreover, if a function f is $\mathscr{E}$-q.c. and is nonnegative $\mathfrak{m}$-a.e. on an open set $U \subset M$, then f is nonnegative $\mathscr{E}$-q.e. on U. In particular, an $\mathscr{E}$-q.c. $\mathfrak{m}$-respresentative $\widetilde{f}$ of any $f \in \mathscr{F}$ is $\mathscr{E}$-q.e. unique. Lastly, if $f \in \mathscr{F} \cap L^\infty(M)$, then any $\mathscr{E}$-q.c. $\mathfrak{m}$-version $\widetilde{f}$ of f obeys

$$|\widetilde{f}| \leq \|f\|_{L^\infty(M)} \quad \mathscr{E}\text{-q.e.}$$

Extended domain We now define the main space around which Theorem 3.2.32 is built. In view of that result, the point is our goal to speak about L^2-differentials $\mathrm{d}f$ of appropriate $f \in L^0(M)$ without imposing any integrability assumption on f.

The following definition can be found in [Kuw98, p. 690].

Definition 3.2.3. *The* extended domain $\mathscr{F}_{\mathrm{e}}$ *of $\mathscr{E}$ is defined to consist of all $f \in L^0(M)$ for which there exists an $\mathscr{E}$-Cauchy sequence $(f_n)_{n\in\mathbf{N}}$ of elements of $\mathscr{F}$ such that $f_n \to f$ pointwise $\mathfrak{m}$-a.e. as $n \to \infty$.*

Remark 3.2.4. In the metric measure terminology of [AGS14a, Gig18], up to a possible "non-Riemannian structure" of M (recall Section 0.1) $\mathscr{F}_{\mathrm{e}}$ is contained in the *Sobolev class* $\mathrm{S}^2(M)$ [Gig18, Def. 2.1.4], see [AGS14a, Thm. 6.2], with no equality in general. Hence, the cotangent module constructed in [Bra21, Ch. 2], see Subsection 3.2.4 below, could a priori be smaller than its counterpart from [Gig18, Ch. 2] on infinitesimally Hilbertian metric measure spaces. However, no ambiguity occurs, see Remark 3.2.35 and Remark 3.2.41. In our treatise, the extended domain will suffice as reference domain for our entire first and second order calculus to be developed, for all key results will only need certain elements to belong to $\mathscr{F}$ or $\mathscr{F}_{\mathrm{e}}$. (And some finer results, e.g. Lemma 3.2.39, do not extend beyond $\mathscr{F}_{\mathrm{e}}$ in general.) ■

We say that a sequence $(f_n)_{n\in\mathbf{N}}$ converges to f in $\mathscr{F}_{\mathrm{e}}$ if $f_n \in \mathscr{F}$ for every $n \in \mathbf{N}$, $(f_n)_{n\in\mathbf{N}}$ is $\mathscr{E}$-Cauchy, and $f_n \to f$ $\mathfrak{m}$-a.e. as $n \to \infty$. The following Proposition 3.2.5 is provided thanks to [Kuw98, Prop. 3.1, Prop. 3.2]. Let us also set

$$\begin{aligned}
\mathscr{F}_{\mathrm{eb}} &:= \mathscr{F}_{\mathrm{e}} \cap L^\infty(M),\\
\mathscr{F}_{\mathrm{ec}} &:= \mathscr{F} \cap L^0_{\mathrm{c}}(M),\\
\mathscr{F}_{\mathrm{ebc}} &:= \mathscr{F}_{\mathrm{eb}} \cap \mathscr{F}_{\mathrm{ec}}.
\end{aligned}$$

Proposition 3.2.5. *The extended domain $\mathscr{F}_{\mathrm{e}}$ has the following properties.*

(i) $\mathcal{E}$ *uniquely extends to a (non-relabeled) real-valued bilinear form on* $\mathcal{F}_{\mathrm{e}}^2$ *in such a way that for every* $f \in \mathcal{F}_{\mathrm{e}}$,

$$\mathcal{E}(f) = \lim_{n\to\infty} \mathcal{E}(f_n)$$

for every sequence $(f_n)_{n\in\mathbf{N}}$ *that converges to* f *in* $\mathcal{F}_{\mathrm{e}}$.

(ii) $\mathcal{E}^{1/2}$ *is a seminorm on* $\mathcal{F}_{\mathrm{e}}$.

(iii) *A function* $f \in L^0(M)$ *belongs to* $\mathcal{F}_{\mathrm{e}}$ *if and only if there exists an* $\mathcal{E}$*-bounded sequence* $(f_n)_{n\in\mathbf{N}}$ *in* $\mathcal{F}_{\mathrm{e}}$ *such that* $f_n \to f$ $\mathfrak{m}$*-a.e. as* $n \to \infty$*. In other words, the functional* $\mathcal{E}_1 \colon L^0(M) \to [0,\infty]$ *defined by*

$$\mathcal{E}_1(f) := \begin{cases} \mathcal{E}(f) & \text{if } f \in \mathcal{F}_{\mathrm{e}}, \\ \infty & \text{otherwise} \end{cases}$$

is lower semicontinuous w.r.t. pointwise $\mathfrak{m}$*-a.e. convergence. In particular,* $\mathcal{E}_1$ *viewed as a functional from* $L^2(M)$ *with values in* $[0,\infty]$ *is convex and* L^2*-lower semicontinuous.*

(iv) *If* $f, g \in \mathcal{F}_{\mathrm{e}}$*, then* $\min\{f,g\}, \min\{f^+,1\} \in \mathcal{F}_{\mathrm{e}}$ *with*

$$\begin{aligned} \mathcal{E}\big(\min\{f,g\}\big) &\leq \mathcal{E}(f) + \mathcal{E}(g), \\ \mathcal{E}\big(\min\{f^+,1\}\big) &\leq \mathcal{E}\big(f, \min\{f^+,1\}\big). \end{aligned}$$

(v) *We have*

$$\mathcal{F} = \mathcal{F}_{\mathrm{e}} \cap L^2(M).$$

(vi) *Every* $f \in \mathcal{F}_{\mathrm{e}}$ *has an* $\mathcal{E}$*-q.c.* $\mathfrak{m}$*-version* $\widetilde{f}$ *which is* $\mathcal{E}$*-q.e. unique.*

In general, unfortunately, the constant function 1_M does not belong to $\mathcal{F}_{\mathrm{e}}$. Since we nevertheless need approximations to 1_M in $\mathcal{F}$ at various instances, we record the following result due to [Kuw98, Thm. 4.1].

Lemma 3.2.6. *There exists a sequence* $(G_n)_{n\in\mathbf{N}}$ *of* $\mathcal{E}$*-quasi-open Borel subsets of* M *such that* $G_n \subset G_{n+1}$ $\mathcal{E}$*-q.e. for every* $n \in \mathbf{N}$*,* $\bigcup_{n\in\mathbf{N}} G_n$ *covers* M *up to an* $\mathcal{E}$*-polar set, and for every* $n \in \mathbf{N}$ *there exists* $g_n \in \mathcal{F}_{\mathrm{b}}$ *such that*

$$g_n = 1 \quad \mathfrak{m}\text{-a.e.} \quad \text{on } G_n.$$

Remark 3.2.7. In the terminology of [Kuw98], Lemma 3.2.6 asserts that 1_M belongs to the *local space* $\dot{\mathcal{F}}_{\mathrm{loc}}$. It contains $\mathcal{F}_{\mathrm{e}}$ [Kuw98, Thm. 4.1], but this inclusion may be strict [DSS20, Rem. 2.13]. ■

Remark 3.2.8. Unlike the setting of Example 3.2.14 which is mostly worked upon in [Gig18], Lemma 3.2.6 does *not* provide a global control on $(\mathcal{E}(g_n))_{n\in\mathbf{N}}$. In particular, we do not know in general whether $\mathcal{E}(g_n) \to 0$ as $n \to \infty$, a property which is closely related to recurrence of $(\mathsf{P}_t)_{t\geq 0}$ [FOT11, Thm. 1.6.3]. ■

Carré du champ and calculus rules The class $\mathcal{F}_{\mathrm{b}}$ is an algebra w.r.t. pointwise multiplication [BH91, Prop. I.2.3.2] which is dense in $\mathcal{F}$ [Kuw98, Cor. 2.1]. Moreover, as mentioned in the beginning of this chapter, we assume that $\mathcal{E}$ admits a carré du champ. That is, there exists bilinear *carré du champ* $\Gamma \colon \mathcal{F}_{\mathrm{e}}^2 \to L^1(M)$ with nonnegative values on the diagonal of $\mathscr{F}_{\mathrm{e}}^2$, which is continuous in both arguments w.r.t. convergence in $\mathscr{F}_{\mathrm{e}}$ such that, for every $f \in \mathscr{F}_{\mathrm{e}}$,

$$\mathscr{E}(f) = \big\|\Gamma(f)\big\|_{L^1(M)}.$$

The following facts about Γ and Δ are well-known, see e.g. [BH91, Sec. I.5, Sec. I.6, Sec. I.7], [Kuw98, Lem. 5.2, Thm. 6.1] or [DSS20, Thm. 2.8].

Proposition 3.2.9. *The map Γ satisfies the following obstructions.*

(i) Cauchy–Schwarz inequality. *For every $f, g \in \mathscr{F}_{\mathrm{e}}$,*

$$\Gamma(f,g)^2 \leq \Gamma(f)\,\Gamma(g) \quad \mathfrak{m}\text{-a.e.}$$

(ii) Truncation. *For every $f, g, h \in \mathscr{F}_{\mathrm{e}}$, $\min\{f,g\} \in \mathscr{F}_{\mathrm{e}}$ and*

$$\Gamma(\min\{f,g\}, h) = 1_{\{f \leq g\}}\,\Gamma(f,h) + 1_{\{f>g\}}\,\Gamma(g,h) \quad \mathfrak{m}\text{-a.e.}$$

(iii) Locality. *For every $f \in \mathscr{F}_{\mathrm{e}}$,*

$$1_{\{f=0\}}\,\Gamma(f) = 0 \quad \mathfrak{m}\text{-a.e.}$$

(iv) Chain rule. *For every $k, l \in \mathbf{N}$, every $f \in \mathscr{F}_{\mathrm{eb}}^k$ and $g \in \mathscr{F}_{\mathrm{eb}}^l$ as well as every $\varphi \in \mathrm{C}^1(\mathbf{R}^k)$ and $\psi \in \mathrm{C}^1(\mathbf{R}^l)$ — with $\varphi(0) = \psi(0) = 0$ if $\mathfrak{m}[M] = \infty$ — we have $\varphi \circ f, \psi \circ g \in \mathscr{F}_{\mathrm{e}}$ with*

$$\Gamma(\varphi \circ f, \psi \circ g) = \sum_{i=1}^{k}\sum_{j=1}^{l} \big[\partial_i\varphi \circ f\big]\,\big[\partial_j\psi \circ g\big]\,\Gamma(f_i, g_j) \quad \mathfrak{m}\text{-a.e.}$$

(v) Leibniz rule. *For every $f, g, h \in \mathscr{F}_{\mathrm{eb}}$, we have $f\,g \in \mathscr{F}_{\mathrm{eb}}$ and*

$$\Gamma(f\,g, h) = f\,\Gamma(g,h) + g\,\Gamma(f,h) \quad \mathfrak{m}\text{-a.e.}$$

Remark 3.2.10. In particular, by approximation of Lipschitz by C^1-functions, see e.g. the proof of [Gig18, Thm. 2.2.6], the same conclusion as in (iv) in Proposition 3.2.9 holds for $f \in \mathscr{F}_{\mathrm{e}}^k$ and $g \in \mathscr{F}_{\mathrm{e}}^l$ under the hypotheses that $\varphi \in \mathrm{Lip}(\mathbf{R}^k)$ and $\psi \in \mathrm{Lip}(\mathbf{R}^l)$, where the partial derivatives $\partial_i\varphi$ and $\partial_j\psi$ are defined arbitrarily on their respective sets of non-differentiability points. ■

Lemma 3.2.11. *If $\mathscr{E}$ admits a carré du champ, the following hold.*

(i) Leibniz rule. *For every $f, g \in \mathscr{D}(\Delta) \cap L^\infty(M)$ with $\Gamma(f), \Gamma(g) \in L^\infty(M)$,*

$$\Delta(f\,g) = f\,\Delta g + 2\,\Gamma(f,g) + g\,\Delta f \quad \mathfrak{m}\text{-a.e.}$$

(ii) Chain rule. *For every $f \in \mathscr{D}(\Delta) \cap L^\infty(M)$ with $\Gamma(f) \in L^\infty(M)$ and every $\varphi \in \mathrm{C}^\infty(I)$ for some interval $I \subset \mathbf{R}$ which contains 0 and the image of f — with $\varphi(0) = 0$ if $\mathfrak{m}[M] = \infty$ — we have*

$$\Delta(\varphi \circ f) = \big[\varphi' \circ f\big]\,\Delta f + \big[\varphi'' \circ f\big]\,\Gamma(f) \quad \mathfrak{m}\text{-a.e.}$$

Finally, at various occasions we will need the subsequent standard a priori estimates for the heat flow, see e.g. [Bre73] or the arguments in [Gig18, Subsec. 3.4.4].

Lemma 3.2.12. *For every $f \in L^2(M)$ and every $t > 0$,*

$$\mathscr{E}(\mathsf{P}_t f) \leq \frac{1}{2t}\,\|f\|_{L^2(M)}^2,$$
$$\big\|\Delta\mathsf{P}_t f\big\|_{L^2(M)}^2 \leq \frac{1}{2t^2}\|f\|_{L^2(M)}^2.$$

Two guiding examples The following frameworks frequently serve as guiding examples throughout our treatise and are mainly listed to fix notation.

Example 3.2.13 (Riemannian manifolds with boundary)**.** Let M be a Riemannian manifold with boundary as in Subsection 3.2.1, and let $\mathfrak{m}$ be a Borel measure on M which is locally equivalent to $\mathfrak{v}$. Let $W^{1,2}(M^\circ)$ be the Sobolev space w.r.t. $\mathfrak{m}$ defined in the usual sense on M°. Define $\mathscr{E}\colon W^{1,2}(M^\circ) \to [0,\infty)$ through

$$\mathscr{E}(f) := \int_{M^\circ} |\nabla f|^2 \,\mathrm{d}\mathfrak{m}$$

and the quantity $\mathscr{E}(f,g)$, $f,g \in W^{1,2}(M)$, by polarization. Then $(\mathscr{E}, W^{1,2}(M))$ is a Dirichlet form which is strongly local and regular, since $\mathrm{C}_\mathrm{c}^\infty(M)$ is a dense set of $W^{1,2}(M)$ which is also uniformly dense in $\mathrm{C}_0(M)$. (The latter is the space of continuous functions on M vanishing at ∞.) $\mathscr{E}$ admits a carré du champ which is precisely given by $|\nabla\cdot|^2$. See [CF12, Dav89, FOT11, Stu21] for details.

Furthermore, suppose that $\mathfrak{m} := \mathrm{e}^{-2w}\,\mathfrak{v}$ for some $w \in \mathrm{C}^2(M)$. By Green's formula, see e.g. [Lee97, p. 44], Δ is the self-adjoint realization of the drift Laplacian $\Delta_0 - 2\,\langle\nabla w, \nabla\cdot\rangle$ w.r.t. *Neumann boundary conditions*, where Δ_0 is the Laplace–Beltrami operator on M, initially defined on functions $f \in \mathrm{C}_\mathrm{c}^\infty(M)$ with

$$\mathrm{d}f(\mathsf{n}) = 0 \quad \text{on } \partial M.$$

This equation makes sense $\mathfrak{s}$-a.e. for every $f \in \mathscr{D}(\Delta)$ by the local trace theorem. ■

Example 3.2.14 (Infinitesimally Hilbertian metric measure spaces)**.** Let $(M,\mathsf{d},\mathfrak{m})$ be an infinitesimally Hilbertian metric measure space, according to Section 0.1, for which $\mathfrak{m}$ satisfies the growth condition (4.2) in [AGS14a], compare with Section 1.2. In this case $\mathscr{E}$ is the *Cheeger energy* introduced in [AGS14a, Thm. 4.5], with domain denoted by $W^{1,2}(M)$. By [Sav14, Thm. 4.1], $\mathscr{E}$ is quasi-regular and strongly local, and it admits a carré du champ which $\mathfrak{m}$-a.e. coincides with the *minimal weak upper gradient* from [AGS14a, Def. 4.2], see also Section 1.2. ■

Remark 3.2.15 (Subsets)**.** Let $(M,\mathsf{d},\mathfrak{m})$ be as in Example 3.2.14 and $E \subset M$ be a closed subset. Assume that $E = \overline{E^\circ}$, $\mathfrak{m}[E] > 0$, $\mathfrak{m}[\partial E] = 0$, and that the length distance d_E on E^2 induced by d is nondegenerate. Then $(E, \mathsf{d}_E, \mathfrak{m}_E)$, where $\mathfrak{m}_E := \mathfrak{m}[\,\cdot \cap E]$, induces a quasi-regular, strongly local Dirichlet space $(E, \mathscr{E}_E, \mathfrak{m}_E)$ [Stu20, Stu21] whose carré du champs coincide $\mathfrak{m}_E$-a.e. on E°. Moreover,

$$W^{1,2}(M)\big|_E \subset W^{1,2}(E) \tag{3.2.3}$$

in the sense of restrictions of functions, see e.g. (47) in [Stu20]. ■

Whenever M fits into Example 3.2.13 or Example 3.2.14, we intend the canonically induced Dirichlet space $(M, \mathscr{E}, \mathfrak{m})$ without further notice.

3.2.3 L^∞-modules

Now we recall the theory of L^∞*-modules* introduced in [Gig18]. We emphasize for completeness that the reference measure to run the machinery from [Gig18] will always chosen to be $\mathfrak{m}$. Other choices are possible [Bra21, Gig18], but are irrelevant for this chapter. In particular, we neither express M nor $\mathfrak{m}$ in the terminology of an L^p-normed L^∞-module according to the next Definition 3.2.16.

Definition and basic properties

Definition 3.2.16. *Given $p \in [1,\infty]$, a real Banach space $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$ or simply $\mathscr{M}$ is termed an L^p-*normed L^∞-module (over M) *if it comes with*

a. *a bilinear map $\cdot\colon L^\infty(M)\times\mathscr{M}\to\mathscr{M}$ satisfying*

$$\begin{aligned}(f\,g)\cdot v &= f\cdot(g\cdot v),\\ 1_M\cdot v &= v,\end{aligned}$$

b. *a nonnegatively valued map $|\cdot|\colon\mathscr{M}\to L^p(M)$ such that*

$$\begin{aligned}|f\cdot v| &= |f|\,|v| \quad \mathfrak{m}\text{-a.e.},\\ \|v\|_{\mathscr{M}} &= \big\||v|\big\|_{L^p(M)},\end{aligned}$$

for every $f,g\in L^\infty(M)$ and every $v\in\mathscr{M}$. If only item a. *is satisfied, we call $(\mathscr{M},\|\cdot\|_{\mathscr{M}})$ or simply $\mathscr{M}$ an L^∞-*premodule.

Remark 3.2.17. In [Gig18, Def. 1.2.10], spaces obeying Definition 3.2.16 are called *L^p-normed premodules* and are as such a priori more general than L^p-normed L^∞-modules. However, these notions coincide for $p\in[1,\infty)$ [Gig18, Prop. 1.2.12]. What only might be missing in the case $p=\infty$ is the gluing property [Gig18, Def. 1.2.1, Ex. 1.2.5], whose lack will never occur in this chapter, hence the minor change of terminology. ■

Example 3.2.18. $L^p(M)$ is an L^p-normed L^∞-module, $p\in[1,\infty]$. ■

We call $\mathscr{M}$ an *L^∞-module* if it is L^p-normed for some $p\in[1,\infty]$ — which is assumed throughout the rest of this subsection — and *separable* if it is a separable Banach space. We term $v\in\mathscr{M}$ ($\mathfrak{m}$-*essentially*) *bounded* if $|v|\in L^\infty(M)$. If $\mathscr{M}$ is separable, it admits a countable dense subset of bounded elements. We drop the $\cdot$ sign if the multiplication on $\mathscr{M}$ is understood. By [Gig18, Prop. 1.2.12], $|\cdot|$ is local in the sense that $|v|=0$ $\mathfrak{m}$-a.e. on E if and only if $1_E\,v=0$ for every $v\in\mathscr{M}$ and every $E\in\mathscr{B}^{\mathfrak{m}}(M)$. We shall then write

$$\begin{aligned}\{v=0\} &:= \{|v|=0\},\\ \{v\neq 0\} &:= \{v=0\}^{\mathrm{c}}\end{aligned}$$

If M is a metric space, we write $\mathscr{M}_{\mathrm{bs}}$ for all elements $v\in\mathscr{M}$ such that $\{v\neq 0\}$ is a bounded subset of M. Lastly, for every $v,w\in\mathscr{M}$ we have the triangle inequality

$$|v+w|\leq|v|+|w| \quad \mathfrak{m}\text{-a.e.},$$

which shows that the map $|\cdot|\colon\mathscr{M}\to L^p(M)$ is continuous.

$\mathscr{M}$ is called *Hilbert module* if it is an L^2-normed L^∞-module and a Hilbert space [Gig18, Def. 1.2.20, Prop. 1.2.21]. Its pointwise norm $|\cdot|$ satisfies a pointwise $\mathfrak{m}$-a.e. parallelogram identity. In particular, it induces a *pointwise scalar product* $\langle\cdot,\cdot\rangle\colon\mathscr{M}^2\to L^1(M)$ which is $L^\infty(M)$-bilinear, $\mathfrak{m}$-a.e. nonnegative definite, local in both components, satisfies the pointwise $\mathfrak{m}$-a.e. Cauchy–Schwarz inequality, and reproduces the Hilbertian scalar product on $\mathscr{M}$ by integration w.r.t. $\mathfrak{m}$.

Let $\mathscr{M}$ and $\mathscr{N}$ be L^p-normed L^∞-modules, $p\in[1,\infty]$, such that $\mathscr{N}$ is a closed subspace of M. Then the quotient $\mathscr{M}/\mathscr{N}$ is an L^p-normed L^∞-module as well [Gig18, Prop. 1.2.14] with pointwise norm given by

$$|[v]| := \operatorname{essinf}\big\{|v+w| : w\in\mathscr{N}\big\}.$$

For instance, given $E \in \mathscr{B}^{\mathfrak{m}}(M)$, the L^∞-module $\mathscr{M}\big|_E$ consisting of all $v \in \mathscr{M}$ such that $\{v \neq 0\} \subset E$ can be canonically identified with $\mathscr{M}/\mathscr{M}\big|_{E^c}$.

Duality Let $\mathscr{M}$ and $\mathscr{N}$ be L^∞-normed modules. Slightly abusing notation, denote both pointwise norms by $|\cdot|$. A map $T\colon \mathscr{M} \to \mathscr{N}$ is called *module morphism* if it is a bounded linear map in the sense of functional analysis and

$$T(f\,v) = f\,T(v)$$

for every $v \in \mathscr{M}$ and every $f \in L^\infty(M)$. The set of all such module morphisms is written $\mathrm{Hom}(\mathscr{M};\mathscr{N})$ and is equipped with the usual operator norm $\|\cdot\|_{\mathscr{M};\mathscr{N}}$. We term $\mathscr{M}$ and $\mathscr{N}$ *isomorphic* (*as L^∞-modules*) if there exist $T \in \mathrm{Hom}(\mathscr{M};\mathscr{N})$ and $S \in \mathrm{Hom}(\mathscr{N};\mathscr{M})$ such that $T \circ S = \mathrm{Id}_{\mathscr{N}}$ and $S \circ T = \mathrm{Id}_{\mathscr{M}}$. Any such T is called *module isomorphism*. If in addition, such a T is a norm isometry, it is called *module isometric isomorphism*, while it is a *pointwise module isometric isomorphism* if it even preserves pointwise norms $\mathfrak{m}$-a.e., i.e. for every $v \in \mathscr{M}$,

$$|T(v)| = |v| \quad \mathfrak{m}\text{-a.e.}$$

The *dual module* to $\mathscr{M}$ is defined by

$$\mathscr{M}^* := \mathrm{Hom}(\mathscr{M};L^1(M))$$

and will be endowed with the usual operator norm. The pointwise pairing between $v \in \mathscr{M}$ and $L \in \mathscr{M}^*$ is denoted by $L(v) \in L^1(M)$. If $\mathscr{M}$ is L^p-normed, then $\mathscr{M}^*$ is an L^q-normed L^∞-normed module, where $p, q \in [1,\infty]$ with $1/p + 1/q = 1$ [Gig18, Prop. 1.2.14] with naturally defined multiplication and, by a slight abuse of notation, pointwise norm given by

$$|L| := \operatorname{esssup}\big\{|L(v)| : v \in \mathscr{M},\ |v| \leq 1\ \mathfrak{m}\text{-a.e.}\big\}. \tag{3.2.4}$$

By [Gig18, Cor. 1.2.16], if $p < \infty$,

$$|v| = \operatorname{esssup}\big\{|L(v)| : L \in \mathscr{M}^*,\ |L| \leq 1\ \mathfrak{m}\text{-a.e.}\big\}$$

for every $v \in \mathscr{M}$. Moreover, if $p < \infty$, in the sense of functional analysis $\mathscr{M}^*$ and the Banach space dual $\mathscr{M}'$ of $\mathscr{M}$ are isometrically isomorphic [Gig18, Prop. 1.2.13]. In this case, the natural pointwise pairing map $\mathscr{J}\colon \mathscr{M} \to \mathscr{M}^{**}$, where $\mathscr{M}^{**} := \mathrm{Hom}(\mathscr{M}^*;L^1(M))$, belongs to $\mathrm{Hom}(\mathscr{M};\mathscr{M}^{**})$ and constitutes a norm isometry [Gig18, Prop. 1.2.15]. We term $\mathscr{M}$ *reflexive* (*as L^∞-module*) if $\mathscr{J}$ is surjective. If $\mathscr{M}$ is L^p-normed for $p \in (1,\infty)$, this is equivalent to $\mathscr{M}$ being reflexive as Banach space [Gig18, Cor. 1.2.18], while for $p = 1$, the implication from "reflexive as Banach space" to "reflexive as L^∞-module" still holds [Gig18, Prop. 1.2.13, Prop. 1.2.17]. In particular, all Hilbert modules are reflexive in both senses.

If $\mathscr{M}$ is a Hilbert module, we have the following analogue of the Riesz representation theorem [Gig18, Thm. 1.2.24]. For $v \in \mathscr{M}$, let $L_v \in \mathscr{M}^*$ be given by

$$L_v(w) := \langle v, w\rangle.$$

Proposition 3.2.19. *Let $\mathscr{M}$ be a Hilbert module. Then the map which sends $v \in \mathscr{M}$ to $L_v \in \mathscr{M}^*$ is a pointwise module isometric isomorphism, and in particular a norm isometry. Moreover, for every $l \in \mathscr{M}'$ there exists a unique $v \in \mathscr{M}$ with*

$$l = \int_M \langle v, \cdot\rangle \,\mathrm{d}\mathfrak{m}.$$

L^0-modules Let $\mathscr{M}$ be an L^∞-module. Following [Gig18, Sec. 1.3] we now recall a natural concept of building a topological vector space $\mathscr{M}^0$ of "measurable elements of M without integrability restrictions" containing $\mathscr{M}$ with continuous inclusion as well as a (non-relabeled) extension of the pointwise norm $|\cdot|\colon \mathscr{M}^0 \to L^0(M)$ such that for every $v \in \mathscr{M}^0$, $v \in \mathscr{M}$ if and only if $|v| \in L^p(M)$.

Let $(B_i)_{i\in\mathbf{N}}$ a Borel partition of M such that $\mathfrak{m}[B_i] \in (0,\infty)$ for every $i \in \mathbf{N}$. Denote by $\mathscr{M}^0$ the completion of $\mathscr{M}$ w.r.t. the distance $\mathsf{d}_{\mathscr{M}^0}\colon \mathscr{M}^2 \to [0,\infty)$ with

$$\mathsf{d}_{\mathscr{M}^0}(v,w) := \sum_{i\in\mathbf{N}} \frac{2^{-i}}{\mathfrak{m}[B_i]} \int_{B_i} \min\{|v-w|, 1\}\,\mathrm{d}\mathfrak{m}. \tag{3.2.5}$$

We refer to $\mathscr{M}^0$ as the *L^0-module* associated to $\mathscr{M}$. The induced topology on $\mathscr{M}^0$ does not depend on the choice of $(B_i)_{i\in\mathbf{N}}$ [Gig18, p. 31]. Additionally, scalar and functional multiplication, and the pointwise norm $|\cdot|$ extend continuously to $\mathscr{M}^0$, so that all $\mathfrak{m}$-a.e. properties mentioned in Subsection 3.2.3 hold for general elements in $\mathscr{M}^0$ and $L^0(M)$ in place of $\mathscr{M}$ and $L^\infty(M)$. The pointwise pairing of $\mathscr{M}$ and $\mathscr{M}^*$ extends uniquely and continuously to a bilinear map on $\mathscr{M}^0 \times (\mathscr{M}^*)^0$ with values in $L^0(M)$ such that for every $v \in \mathscr{M}^0$ and every $L \in (\mathscr{M}^*)^0$,

$$|L(v)| \leq |L|\,|v| \quad \mathfrak{m}\text{-a.e.},$$

and we have the following characterization of elements in $(\mathscr{M}^*)^0$ [Gig18, Prop. 1.3.2].

Proposition 3.2.20. *Let $T\colon \mathscr{M}^0 \to L^0(M)$ be a linear map for which there exists $f \in L^0(M)$ such that for every $v \in \mathscr{M}$,*

$$|T(v)| \leq f\,|v| \quad \mathfrak{m}\text{-a.e.}$$

Then there exists a unique $L \in (\mathscr{M}^*)^0$ such that for every $v \in \mathscr{M}$,

$$L(v) = T(v) \quad \mathfrak{m}\text{-a.e.},$$

and we furthermore have

$$|L| \leq f \quad \mathfrak{m}\text{-a.e.}$$

Remark 3.2.21. To some extent, one can make sense of L^0-normed modules w.r.t. a *submodular outer measure* μ^* on M [DGP21, Def. 2.4]. A prominent example of such a μ^* is the *$\mathscr{E}$-capacity* $\mathrm{cap}_{\mathscr{E}}$ [DGP21, Def. 2.6, Prop. 2.8], which — towards the aim of defining *quasi-continuity* of vector fields over metric measure spaces — motivated the authors of [DGP21] to study this kind of modules. However, what lacks for such μ^* is a working definition of dual modules, and in particular Proposition 3.2.20 seems unavailable [DGP21, Rem. 3.3]. ■

Remark 3.2.22. The concept of L^0-modules is tightly linked to the one of *measurable fields of Hilbert spaces* [Ebe99]. See [Gig18, Rem. 1.4.12] and [HRT13, Ch. 2]. ■

Local dimension and dimensional decomposition Given an L^∞-module $\mathscr{M}$ and $E \in \mathscr{B}^{\mathfrak{m}}(M)$, we say that $v_1, \dots, v_n \in \mathscr{M}$, $n \in \mathbf{N}$, are *independent* (*on E*) if all functions $f_1, \dots, f_n \in L^\infty(M)$ obeying

$$f_1\,v_1 + \cdots + f_n\,v_n = 0 \quad \mathfrak{m}\text{-a.e.} \quad \text{on } E$$

vanish $\mathfrak{m}$-a.e. on E. This notion of local independence is well-behaved under passage to subsets and under module isomorphisms, see [Gig18, p. 34] for details. The *span* $\mathrm{span}_E\, \mathscr{V}$ of a subset $\mathscr{V} \subset \mathscr{M}$ on $E \subset M$ — briefly $\mathrm{span}\, \mathscr{V}$ if $E = M$ — is the space consisting of all $v \in \mathscr{M}\big|_E$ which possess the following property: there exists a disjoint partition $(E_k)_{k\in\mathbf{N}}$ of E in $\mathscr{B}^{\mathfrak{m}}(M)$ such that for every $k \in \mathbf{N}$, we can find $m_k \in \mathbf{N}$, $v_1^k, \dots, v_{m_k}^k \in \mathscr{V}$ and $f_1^k, \dots, f_{m_k}^k \in L^\infty(M)$ such that

$$1_{E_k}\, v = f_1^k\, v_1^k + \dots + f_{m_k}^k\, v_{m_k}^k.$$

Its closure $\mathrm{cl}_{\|\cdot\|_{\mathscr{M}}}\, \mathrm{span}_E\, \mathscr{V}$ is usually referred to as the space *generated* by $\mathscr{V}$ on E, or simply by $\mathscr{V}$ if $E = M$ [Gig18, Def. 1.4.2].

If $\mathscr{V}$ is a finite set, $\mathrm{span}_E\, \mathscr{V}$ is closed [Gig18, Prop. 1.4.6], a fact which gives additional strength to the following notions [Gig18, Def. 1.4.3].

Definition 3.2.23. *A family $\{v_1, \dots, v_n\} \subset \mathscr{M}$, $n \in \mathbf{N}$, is a said to be a* (local) basis (on E) *if $v_1, \dots, v_n$ are independent on E, and*

$$\mathrm{span}_E\, \{v_1, \dots, v_n\} = \mathscr{M}\big|_E.$$

We say that $\mathscr{M}$ has (local) dimension n (on E) *if there exists a local basis of $\mathscr{M}$ on E. We say that $\mathscr{M}$ has* infinite dimension *on E if it does not have finite dimension on E according to the previous sentence.*

For the well-posedness of this definition and its link to local independence, we refer to [Gig18, Prop. 1.4.4]. If $\mathscr{M}$ is L^p-normed, $p < \infty$, and has dimension $n \in \mathbf{N}$ on E, then $\mathscr{M}^*$ has dimension n on E as well [Gig18, Thm. 1.4.7.].

The following important structural result is due to [Gig18, Prop. 1.4.5].

Proposition 3.2.24. *For every L^∞-module $\mathscr{M}$, there exists a unique Borel partition $(E_n)_{n\in\mathbf{N}\cup\{\infty\}}$ of M such that*

a. *for every $n \in \mathbf{N}$ with $\mathfrak{m}[E_n] > 0$, $\mathscr{M}$ has dimension n on E_n, and*
b. *for every $E \in \mathscr{B}^{\mathfrak{m}}(M)$ with $E \subset E_\infty$, $\mathscr{M}$ has infinite dimension on E.*

Remark 3.2.25. Using Proposition 3.2.24, it is possible to establish a one-to-one-correspondence between separable Hilbert modules and *direct integrals* of separable Hilbert spaces [Gig18, Thm. 1.4.11, Rem. 1.4.12]. Albeit at a structural level, this provides the link of Subsection 3.2.4 to earlier axiomatizations of spaces of 1-forms and vector fields on Dirichlet spaces [BK19, CS03, Ebe99, HRT13, HT15, IRT12], and in view of the universal property of Theorem 3.2.32 below, we do not enter into details here and leave these to the interested reader. We only point out the remark at [Gig18, p. 42] that the interpretation of this link should be treated with some care. ■

Remark 3.2.26 (Hino index)**.** Unlike Proposition 3.2.24, in Dirichlet form theory there already exists a natural notion of "pointwise tangent space dimension" in terms of the (*pointwise*) *Hino index* introduced in [Hin10], which is quickly recorded now for our situation of $\mathscr{E}$ admitting a carré du champ (which is to say that $\mathfrak{m}$ is *minimal $\mathscr{E}$-dominant* [DSS20, Hin10]). For the modules under our consideration, these two notions of local dimension turn out to coincide, see Corollary 3.2.38 and also Subsection 3.3.3.

Let $(f_i)_{i\in\mathbf{N}}$ be a sequence in $\mathscr{F}$ whose linear span is dense in $\mathscr{F}$. The *pointwise index* of $\mathscr{E}$ [Hin10, Def. 2.9, Prop. 2.10] is the function $\mathrm{p}\colon M \to \mathbf{N}_0 \cup \{\infty\}$ given by

$$\mathrm{p} := \sup_{n\in\mathbf{N}} \mathrm{rank}\, \big[\Gamma(f_i, f_j)\big]_{i,j\in\{1,\dots,n\}}.$$

See [Hin10, Ch. 2] for a thorough discussion on the well-definedness of p. In particular, by [Hin10, Prop. 2.11] we know that $\mathrm{p} > 0$ $\mathfrak{m}$-a.e. unless $\mathcal{E}$ is trivial. The *index* $\mathrm{p}^* \in \mathbf{N} \cup \{\infty\}$ of $\mathcal{E}$ is then defined by

$$\mathrm{p}^* := \|\mathrm{p}\|_{L^\infty(M)}.$$

These definitions are in fact independent of the choice of minimal $\mathcal{E}$-dominant reference measure. By [Hin10, Prop. 2.10], we have the following result. (A probabilistic aspect of it not treated here is that p^* coincides with the *martingale dimension* [Hin08, Kus89] w.r.t. the Markov process on M associated to $\mathcal{E}$, see [Hin10, Thm. 3.4].) ■

Lemma 3.2.27. *For every $n \in \mathbf{N}$ and every $f_1, \dots, f_n \in \mathcal{F}$,*

$$\operatorname{rank}\big[\Gamma(f_i, f_j)\big]_{i,j\in\{1,\dots,n\}} \leq \mathrm{p} \leq \mathrm{p}^* \quad \mathfrak{m}\text{-a.e.}$$

Moreover, p *is the $\mathfrak{m}$-a.e. smallest function satisfying the first $\mathfrak{m}$-a.e. inequality for every $n \in \mathbf{N}$ and every $f_1, \dots, f_n \in \mathcal{F}$.*

Tensor products Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two *Hilbert* modules. Again, by a slight abuse of notation, we denote both pointwise scalar products by $\langle\cdot,\cdot\rangle$.

Let $\mathcal{M}_1^0 \odot \mathcal{M}_2^0$ be the "tensor product" consisting of all finite linear combinations of formal elements $v \otimes w$, $v \in \mathcal{M}_1^0$ and $w \in \mathcal{M}_2^0$, obtained by factorizing appropriate vector spaces [Gig18, Sec. 1.5]. It naturally comes with a multiplication $\cdot\colon L^0(M) \times (\mathcal{M}_1^0 \odot \mathcal{M}_2^0) \to L^0(M)$ defined through

$$f\,(v \otimes w) := (f\,v) \otimes w = v \otimes (f\,w)$$

and a pointwise scalar product $:\colon (\mathcal{M}_1^0 \odot \mathcal{M}_2^0)^2 \to L^0(M)$ given by

$$(v_1 \otimes w_1) : (v_2 \otimes w_2) := \langle v_1, v_2\rangle\,\langle w_1, w_2\rangle, \tag{3.2.6}$$

both extended to $\mathcal{M}_1^0 \odot \mathcal{M}_2^0$ by (bi-)linearity. Then $:$ is bilinear, $\mathfrak{m}$-a.e. nonnegative definite, symmetric, and local in both components [GP20a, Lem. 3.2.19].

The pointwise *Hilbert–Schmidt norm* $|\cdot|_{\mathrm{HS}}\colon \mathcal{M}_1^0 \odot \mathcal{M}_2^0 \to L^0(M)$ is given by

$$|A|_{\mathrm{HS}} := \sqrt{A : A}. \tag{3.2.7}$$

This map satisfies the $\mathfrak{m}$-a.e. triangle inequality and is 1-homogeneous w.r.t. multiplication with $L^0(M)$-functions [Gig18, p. 44].

Consequently, the map $\|\cdot\|_{\mathcal{M}_1\otimes\mathcal{M}_2}\colon \mathcal{M}_1^0 \odot \mathcal{M}_2^0 \to [0,\infty]$ defined through

$$\|A\|_{\mathcal{M}_1\otimes\mathcal{M}_2} := \big\||A|_{\mathrm{HS}}\big\|_{L^2(M)}$$

has all properties of a norm except that it might take the value ∞.

Definition 3.2.28. *The* tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$ *is the completion w.r.t. $\|\cdot\|_{\mathcal{M}_1\otimes\mathcal{M}_2}$ of the subspace that consists of all $A \in \mathcal{M}_1^0 \odot \mathcal{M}_2^0$ such that $\|A\|_{\mathcal{M}_1\otimes\mathcal{M}_2} < \infty$.*

Inductively, for $\mathcal{M} := \mathcal{M}_1$ and $k \in \mathbf{N}$, up to unique identification we set

$$\mathcal{M}^{\otimes k} := \mathcal{M}^{\otimes(k-1)} \otimes \mathcal{M} = \mathcal{M} \otimes \mathcal{M}^{\otimes(k-1)},$$

where we conventionally set $\mathcal{M}^{\otimes 0} := L^2(M)$ as well.

Through (3.2.6), $\mathscr{M}_1 \otimes \mathscr{M}_2$ naturally becomes a Hilbert module [Gig18, p. 45]. If $\mathscr{M}_1$ and $\mathscr{M}_2$ are separable, then so is $\mathscr{M}_1 \otimes \mathscr{M}_2$. Indeed, if $D_i \subset \mathscr{M}_i$ are countable dense subsets consisting of *bounded* elements, $i \in \{1, 2\}$, then the linear span of elements of the form $v \otimes w$, $v \in D_1$ and $w \in D_2$, is dense in $\mathscr{M}_1 \otimes \mathscr{M}_2$. Here, boundedness is essential as underlined by the next remark.

Remark 3.2.29. The space $\mathscr{M}_1 \otimes \mathscr{M}_2$ should not be confused with the tensor product $\mathscr{M}_1 \otimes_{\mathrm{H}} \mathscr{M}_2$ in the Hilbert space sense [KR83]. Indeed, in general these do not coincide [Gig18, Rem. 1.5.2]. For instance, for $v \in \mathscr{M}_1$ and $w \in \mathscr{M}_2$ we always have $v \otimes_{\mathrm{H}} w \in \mathscr{M}_1 \otimes_{\mathrm{H}} \mathscr{M}_2$ since the corresponding norm is

$$\|v \otimes_{\mathrm{H}} w\|_{\mathscr{M}_1 \otimes_{\mathrm{H}} \mathscr{M}_2} = \|v\|_{\mathscr{M}_1} \|w\|_{\mathscr{M}_2},$$

but according to (3.2.6), the norm

$$\|v \otimes w\|_{\mathscr{M}_1 \otimes \mathscr{M}_2} = \Big[\int_M |v|^2 \, |w|^2 \, \mathrm{d}\mathfrak{m}\Big]^{1/2}$$

in $\mathscr{M}_1^0 \odot \mathscr{M}_2^0$ might well be infinite unless, for instance, v or w is bounded.

More intuitively, we should think about $v \otimes w \in \mathscr{M}_1^0 \odot \mathscr{M}_2^0$ as section $x \mapsto v(x) \otimes w(x)$ over M, while $v \otimes_{\mathrm{H}} w \in \mathscr{M}_1 \otimes_{\mathrm{H}} \mathscr{M}_2$ is interpreted as section $(x, y) \mapsto v(x) \otimes w(y)$ over M^2. The latter point of view is not relevant in this chapter, but is crucial e.g. in obtaining a spectral representation of the heat kernel on 1-forms on compact $\mathrm{RCD}^*(K, N)$ spaces, $K \in \mathbf{R}$ and $N \in [1, \infty)$, in Chapter 4 below. ■

Lastly, we introduce the concept of *symmetric* and *antisymmetric parts* in the case $\mathscr{M} := \mathscr{M}_1 = \mathscr{M}_2$. Denote by $A^\top \in \mathscr{M}^{\otimes 2}$ the *transpose* of $A \in \mathscr{M}^{\otimes 2}$ as defined in [Gig18, Sec. 1.5]. For instance, for bounded $v, w \in \mathscr{M}$ we have

$$(v \otimes w)^\top = w \otimes v. \tag{3.2.8}$$

It is an involutive pointwise module isometric isomorphism. We shall call $A \in \mathscr{M}^{\otimes 2}$ *symmetric* if $A = A^\top$ and *antisymmetric* if $A = -A^\top$. We write $\mathscr{M}^{\otimes 2}_{\mathrm{sym}}$ and $\mathscr{M}^{\otimes 2}_{\mathrm{asym}}$ for the subspaces of symmetric and antisymmetric elements in $\mathscr{M}^{\otimes 2}$, respectively. These are closed and pointwise $\mathfrak{m}$-a.e. orthogonal w.r.t. $:$. As usual, for every $A \in \mathscr{M}^{\otimes 2}$ there exist a unique $A_{\mathrm{sym}} \in \mathscr{M}^{\otimes 2}_{\mathrm{sym}}$, the *symmetric part* of A, and a unique $A_{\mathrm{asym}} \in \mathscr{M}^{\otimes 2}_{\mathrm{asym}}$, the *antisymmetric part* of A, such that

$$A = A_{\mathrm{sym}} + A_{\mathrm{asym}}.$$

In particular, we have

$$\big|A\big|^2_{\mathrm{HS}} = \big|A_{\mathrm{sym}}\big|^2_{\mathrm{HS}} + \big|A_{\mathrm{asym}}\big|^2_{\mathrm{HS}} \quad \mathfrak{m}\text{-a.e.} \tag{3.2.9}$$

Next, we present a duality formula for symmetric parts crucially exploited later in Lemma 3.6.2. If $D \subset \mathscr{M}$ is a set of bounded elements generating $\mathscr{M}$ in the sense of the local dimension discussion from Subsection 3.2.3, then $\{v \otimes v : v \in D\}$ generates $\mathscr{M}^{\otimes 2}_{\mathrm{sym}}$, and after [Gig18, Prop. 1.4.9] for every $A \in \mathscr{M}^{\otimes 2}$ we have the duality formula

$$\begin{aligned} \big|A_{\mathrm{sym}}\big|^2_{\mathrm{HS}} = \operatorname{esssup}\Big\{2A : \sum_{j=1}^m v_j \otimes v_j - \Big|\sum_{j=1}^m v_j \otimes v_j\Big|^2_{\mathrm{HS}} : \\ m \in \mathbf{N},\ v_1, \dots, v_m \in D\Big\}. \end{aligned} \tag{3.2.10}$$

Traces Let $\mathscr{M}$ be a Hilbert module over M. In terms of local bases outlined in the local dimension paragraph in Subsection 3.2.3, it is possible to define the *trace* of an element $A \in \mathscr{M}^{\otimes 2}_{\mathrm{sym}}$.

As usual, Gram–Schmidt orthonormalization combined with [Gig18, Thm. 1.4.11] entails the following. Denoting by $(E_n)_{n\in\mathbf{N}\cup\{\infty\}}$ the dimensional decomposition of $\mathscr{M}$ according to Proposition 3.2.24, for every $n \in \mathbf{N}$ and every Borelian $E \subset E_n$ with $\mathfrak{m}[E] \in (0,\infty)$, there exists a basis $\{e_1^n,\dots,e_n^n\} \subset \mathscr{M}\big|_E$ of $\mathscr{M}$ on E with

$$\left\langle e_i^n, e_j^n\right\rangle = \delta_{ij} \quad \mathfrak{m}\text{-a.e.} \quad \text{on } E \tag{3.2.11}$$

for every $i,j \in \{1,\dots,n\}$. Moreover, for every Borelian $E \subset E_\infty$ with $\mathfrak{m}[E] \in (0,\infty)$ there exists a sequence $(e_i^\infty)_{i\in\mathbf{N}}$ in $\mathscr{M}\big|_E$ which generates $\mathscr{M}$ on E and satisfies (3.2.11) for every $i,j \in \mathbf{N}$ and $n := \infty$. Any such $\{e_1^n,\dots,e_n^n\}$, $n \in \mathbf{N}$, or $(e_i^\infty)_{i\in\mathbf{N}}$ is called a *pointwise orthonormal basis* of $\mathscr{M}$ on E. In particular, for every $E \subset E_n$ as above, $n \in \mathbf{N}\cup\{\infty\}$ and every $v \in \mathscr{M}$ we can write

$$1_E\, v = \sum_{i=1}^n \left\langle v, e_i^n\right\rangle e_i^n.$$

Lastly, if $E \subset E_n$ is such a Borel set, $n \in \mathbf{N}\cup\{\infty\}$, we define

$$1_E\ \operatorname{tr} A := \sum_{i=1}^n A : (e_i^n \otimes e_i^n). \tag{3.2.12}$$

This does not depend on the choice of the pointwise orthonormal basis.

Exterior products Let $\mathscr{M}$ be a Hilbert module and $k \in \mathbf{N}_0$. Set $\Lambda^0\mathscr{M}^0 := L^0(M)$ and, for $k \geq 1$, let $\Lambda^k\mathscr{M}^0$ be the "exterior product" constructed by suitably factorizing $(\mathscr{M}^0)^{\odot k}$ [Gig18, Sec. 1.5]. The representative of $v_1 \odot \cdots \odot v_k$, $v_1,\dots,v_k \in \mathscr{M}^0$, in $\Lambda^k\mathscr{M}^0$ is written $v_1 \wedge \cdots \wedge v_k$. $\Lambda^k\mathscr{M}^0$ naturally comes with a multiplication $\cdot\colon L^0(M)\times\Lambda^k\mathscr{M}^0 \to \Lambda^k\mathscr{M}^0$ via

$$f\,(v_1 \wedge \dots v_k) := (f\,v_1) \wedge \cdots \wedge v_k = \cdots = v_1 \wedge \cdots \wedge (f\,v_k)$$

and a pointwise scalar product $\langle\cdot,\cdot\rangle\colon (\Lambda^k\mathscr{M}^0)^2 \to L^0(M)$ defined by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle := \det\left[\langle v_i, w_j\rangle\right]_{i,j\in\{1,\dots,k\}} \tag{3.2.13}$$

up to a factor $k!$, both extended to $\Lambda^k\mathscr{M}^0$ by (bi-)linearity. Then $\langle\cdot,\cdot\rangle$ is bilinear, $\mathfrak{m}$-a.e. nonnegative definite, symmetric, and local in both components.

Remark 3.2.30. Given any $k, k' \in \mathbf{N}_0$, the map assigning to $v_1 \wedge \cdots \wedge v_k \in \Lambda^k\mathscr{M}^0$ and $w_1 \wedge \cdots \wedge w_{k'} \in \Lambda^{k'}\mathscr{M}^0$ the element $v_1 \wedge \cdots \wedge v_k \wedge w_1 \wedge \cdots \wedge w_{k'} \in \Lambda^{k+k'}\mathscr{M}^0$ can and will be uniquely extended by bilinearity and continuity to a bilinear map $\wedge\colon \Lambda^k\mathscr{M}^0 \times \Lambda^{k'}\mathscr{M}^0 \to \Lambda^{k+k'}\mathscr{M}^0$ termed *wedge product* [Gig18, p. 47]. If $k = 0$ or $k' = 0$, it simply corresponds to multiplication of elements of $\Lambda^k\mathscr{M}^0$ or $\Lambda^{k'}\mathscr{M}^0$, respectively, with functions in $L^0(M)$ according to (3.2.13). ■

By a slight abuse of notation, define the map $|\cdot|\colon \Lambda^k\mathscr{M}^0 \to L^0(M)$ by

$$|\omega| := \sqrt{\langle\omega,\omega\rangle}.$$

It obeys the $\mathfrak{m}$-a.e. triangle inequality and is homogeneous w.r.t. multiplication with $L^0(M)$-functions [Gig18, p. 47].

It follows that the map $\|\cdot\|_{\Lambda^k\mathscr{M}}\colon \Lambda^k\mathscr{M}^0 \to [0,\infty]$ defined by

$$\|\omega\|_{\Lambda^k\mathscr{M}} := \big\||\omega|\big\|_{L^2(M)}$$

has all properties of a norm except that $\|\omega\|_{\Lambda^k\mathscr{M}}$ might be infinite.

Definition 3.2.31. *The* (k-fold) exterior product $\Lambda^k\mathscr{M}$ *is defined as the completion w.r.t.* $\|\cdot\|_{\Lambda^k\mathscr{M}}$ *of the subspace consisting of all* $\omega \in \Lambda^k\mathscr{M}^0$ *such that* $\|\omega\|_{\Lambda^k\mathscr{M}} < \infty$.

The space $\Lambda^k\mathscr{M}$ naturally becomes a Hilbert module and, if $\mathscr{M}$ is separable, is separable as well [Gig18, p. 47].

3.2.4 First order differential structure

To shorten the presentation, we do not go into details about the first order differential structure of the given Dirichlet space and refer the interested reader instead to [Bra21] or to the references in Remark 3.2.41 or Remark 3.2.42. At least under our assumption of $\mathscr{E}$ admitting a carré du champ, the results in this subsection are completely analogous to the metric measure space results [Gig18, GP20a] to which we often refer for proof ideas. In [Bra21], we have put the first order treatises from Remark 3.2.42 into the framework of L^∞-modules and, as said, have considered more general reference measures, see Remark 3.2.36 below.

The following is due to [Bra21, Ch. 2], see also [GP20a, Thm. 4.1.1].

Theorem 3.2.32. *There exists a unique tuple* $(L^2(T^*M), \mathrm{d})$ *consisting of an* L^2*-normed* L^∞*-module over* M *with pointwise norm* $|\cdot|$ *as well as a continuous linear map* $\mathrm{d}\colon \mathscr{F}_{\mathrm{e}} \to L^2(T^*M)$ *such that*

(i) *for every* $f \in \mathscr{F}_{\mathrm{e}}$*, we have*

$$|\mathrm{d}f| = \Gamma(f)^{1/2} \quad \mathfrak{m}\text{-}a.e.,$$

(ii) $L^2(T^*M)$ *is generated in the sense of* L^∞*-modules by* $\mathrm{d}\,\mathscr{F}_{\mathrm{e}}$.

The uniqueness statement is intended up to unique isomorphism, that is, if $(\mathscr{M}, \mathscr{d})$ *is another tuple obeying the foregoing obstructions, there exists a unique module isomorphism* $\Phi\colon L^2(T^*M) \to \mathscr{M}$ *such that*

$$\Phi \circ \mathrm{d} = \mathscr{d}.$$

Definition 3.2.33. *In analogy with* [Gig18]*, the space* $L^2(T^*M)$ *is called* cotangent module. *Any element* ω *of* $L^2(T^*M)$ *is called* (differential) 1-form *or* covector field. *The map* d *is called* (exterior) differential.

Since $\mathscr{F}$ is separable, the cotangent module is a separable Hilbert module [Bra21, Lem. 2.7], see also [Gig18, Prop. 2.2.5]. Furthermore, as usual, we call a 1-form $\omega \in L^2(T^*M)$ *exact* if $\omega = \mathrm{d}f$ for some $f \in \mathscr{F}_{\mathrm{e}}$.

Remark 3.2.34. We point out for now that the notation $L^2(T^*M)$ is purely formal since we did and do not define any kind of cotangent bundle T^*M. It rather originates in the analogy of $L^2(T^*M)$ with the space of L^2-sections of the cotangent bundle T^*M in the smooth setting described in Remark 3.2.40 below. On the other hand, by the structural characterization of Hilbert modules as direct integral of measurable fields of certain Hilbert spaces $(\mathscr{H}_x)_{x\in M}$, see e.g. [HRT13, p. 4381] and Remark 3.2.42 below, one could think of a fictive cotangent bundle as something "a.e. defined". This point of view has been taken in the approaches [BK19, CS03, Ebe99, HRT13, HT15, IRT12]. ■

Remark 3.2.35. One can easily check that the "cotangent module" defined by replacing $\mathscr{F}_{\mathrm{e}}$ by any function class $\mathscr{A}$ satisfying $\mathscr{F} \subset \mathscr{A} \subset \mathscr{F}_{\mathrm{qloc}}$ in [Bra21, Ch. 2] agrees with the one from Theorem 3.2.32. (This inclusion holds for $\mathscr{F}_{\mathrm{e}}$ by the proof of [Kuw98, Thm. 4.1].) The differential $\mathrm{d}f$ can then be defined for every $f \in \mathscr{F}_{\mathrm{qloc}}$ for which $\Gamma(f) \in L^1(M)$ by locality, see Proposition 3.2.37 below. Here $\mathscr{F}_{\mathrm{qloc}}$ — in the notation of [Kuw98, Ch. 4], $\dot{\mathscr{F}}_{\mathrm{loc}}$ — is the space of all $\mathscr{E}$-quasi-local functions on M. This shows the compatibility with our approach and the one from [Gig18], see Remark 3.2.41. ■

Remark 3.2.36. It is possible to construct $L^2(T^*M)$ even without a carré du champ (w.r.t. $\mathfrak{m}$). The replacement for such a Γ-operator would then naturally be the density w.r.t. *any* given (minimal) $\mathscr{E}$-dominant measure (recall Remark 3.2.26), which always exist and can be defined in terms of energy measures for $\mathscr{E}$. Since the case without carré du champ will be irrelevant from Section 3.3 on, we leave the details, provided in [Bra21, Ch. 2], to the interested reader.

On the other hand, for a meaningful definition of tangent module, cf. Definition 3.2.43 below, which also makes sense of "$\mathfrak{m}$-a.e. defined vector fields", we need the carré du champ w.r.t. $\mathfrak{m}$ [Bra21, Sec. 3.1]. ■

As by Theorem 3.2.32 and the Hilbertian structure of $L^2(T^*M)$, the behavior of a given element of $L^2(T^*M)$ is completely determined by its interaction with differentials of functions in $\mathscr{F}_{\mathrm{e}}$, the usual calculus rules for the Γ-operator from Proposition 3.2.9 transfer to calculus rules for the differential d at the L^∞-module level as follows (also consult [Bra21, Prop. 2.11], [Gig18, Cor. 2.2.8] and [GP20a, Thm. 4.1.4]).

Proposition 3.2.37. *The following properties hold.*

(i) Locality. *For every $f, g \in \mathscr{F}_{\mathrm{e}}$,*

$$1_{\{f=g\}}\,\mathrm{d}f = 1_{\{f=g\}}\,\mathrm{d}g.$$

(ii) Chain rule. *For every $f \in \mathscr{F}_{\mathrm{e}}$ and every $\mathscr{L}^1$-negligible Borel set $C \subset \mathbf{R}$,*

$$1_{f^{-1}(C)}\,\mathrm{d}f = 0.$$

In particular, for every $\varphi \in \mathrm{Lip}(\mathbf{R})$,

$$\mathrm{d}(\varphi \circ f) = [\varphi' \circ f]\,\mathrm{d}f,$$

where the derivative $\varphi' \circ f$ is defined arbitrarily on the intersection of the set of non-differentiability points of φ with the image of f.

(iii) Leibniz rule. *For every $f, g \in \mathscr{F}_{\mathrm{eb}}(M)$,*

$$\mathrm{d}(f\,g) = f\,\mathrm{d}g + g\,\mathrm{d}f.$$

Further useful properties of the cotangent module which directly come from the definition are the following. Corollary 3.2.38 is an immediate consequence of the definition of the Hino index p (recall Remark 3.2.26) as well as the local dimension discussion from Subsection 3.2.3. Lemma 3.2.39 follows from L^2-lower semicontinuity of $\mathscr{E}$, see Proposition 3.2.5, [Bra21, Lem. 2.10] or [Gig18, Thm. 2.2.9].

Corollary 3.2.38. *Let $(E_n)_{n\in\mathbf{N}\cup\{\infty\}}$ be the dimensional decomposition of $L^2(T^*M)$, seen as an L^2-normed L^∞-module w.r.t. $\mathfrak{m}$. Denote by* p *the pointwise index from Remark 3.2.26. Then for every $n \in \mathbf{N}\cup\{\infty\}$,*

$$\mathrm{p} = n \quad \mathfrak{m}\text{-a.e.} \quad \text{on } E_n.$$

Lemma 3.2.39. *For every sequence $(f_n)_{n\in\mathbf{N}}$ in $\mathscr{F}_\mathrm{e}$ which converges to $f \in L^0(M)$ pointwise $\mathfrak{m}$-a.e. in such a way that the sequence $(\mathrm{d}f_n)_{n\in\mathbf{N}}$ converges to $\omega \in L^2(T^*M)$ in $L^2(T^*M)$, we have $f \in \mathscr{F}_\mathrm{e}$ as well as*

$$\mathrm{d}f = \omega. \tag{3.2.14}$$

In particular, if $f_n \in \mathscr{F}$ for every $n \in \mathbf{N}$, $f_n \rightharpoonup f$ in $L^2(M)$ and $\mathrm{d}f_n \rightharpoonup \omega$ in $L^2(T^*M)$ as $n \to \infty$ for some $f \in L^2(M)$ and $\omega \in L^2(T^*M)$, then $f \in \mathscr{F}$, and the identity (3.2.14) *holds accordingly.*

Remark 3.2.40 (Compatibility with the smooth case)**.** If M is a Riemannian manifold with boundary, in the setting of Example 3.2.13, $L^2(T^*M)$ coincides with the space of L^2-sections of the cotangent bundle T^*M w.r.t. $\mathfrak{m}$, and d is the usual $\mathfrak{m}$-a.e. defined differential for, say, boundedly supported Lipschitz functions on M. ■

Remark 3.2.41 (Compatibility with [Gig18])**.** By the $\mathfrak{m}$-a.e. equality between carré du champ and minimal weak upper gradient outlined in Example 3.2.14, see also Section 0.1, our approach is fully compatible with the one from [Gig18, Sec. 2.2] for *infinitesimally Hilbertian* metric measure spaces $(M, \mathsf{d}, \mathfrak{m})$.

Indeed, in the described setting we have $\mathscr{F} \subset \mathrm{S}^2(M) \subset \mathscr{F}_\mathrm{qloc}$, and the assertion follows from Remark 3.2.35. ■

Remark 3.2.42 (Compatibility with [BK19, CS03, Ebe99, HRT13, HT15, IRT12])**.** The space $\mathscr{H}$ of 1-forms constructed in [HRT13, Ch. 2] agrees with $L^2(T^*M)$. For $f \in \mathscr{F}\cap \mathrm{C}_0(M)$, the differential $\mathrm{d}f \in L^2(T^*M)$ corresponds to the element $f\otimes 1_M \in \mathscr{H}$ in [HRT13]. For instance, this follows by first proving that $\mathscr{H}$ is an L^2-normed L^∞-module w.r.t. an $\mathscr{E}$-dominant μ [HRT13, Ass. 2.1] (recall Remark 3.2.36) which is generated by $\mathscr{d}(\mathscr{F}\cap \mathrm{C}_0(M))$, $\mathscr{d} := \cdot \otimes 1_M$, and applying the universal property from Theorem 3.2.32 in the presence of a carré du champ, or [Bra21, Thm. 2.9] in the general case. This proof is already done somewhat implicitly in [HRT13, Ch. 2]. ■

Definition 3.2.43. *The* tangent module $(L^2(TM), \|\cdot\|_{L^2(TM)})$ *or simply $L^2(TM)$ is*

$$L^2(TM) := L^2(T^*M)^*,$$

the duality intended w.r.t. $\mathfrak{m}$ as usual, and it is endowed with the norm $\|\cdot\|_{L^2(TM)}$ induced by (3.2.4)*. The elements of $L^2(TM)$ will be called* vector fields.

As in Subsection 3.2.3, the pointwise pairing between $\omega \in L^2(T^*M)$ and $X \in L^2(TM)$ is denoted by $\omega(X) \in L^1(M)$, and, by a slight abuse of notation, $|X| \in L^2(M)$ denotes the pointwise norm of such an X. By the discussion after Theorem 3.2.32 and Proposition 3.2.19, $L^2(TM)$ is a separable Hilbert module.

Furthermore, in terms of the pointwise scalar product $\langle\cdot,\cdot\rangle$ on $L^2(T^*M)$ and $L^2(TM)$, respectively, Proposition 3.2.19 allows us to introduce the *musical isomorphisms* $\sharp\colon L^2(T^*M) \to L^2(TM)$ and $\flat := \sharp^{-1}$ defined by

$$\big\langle \omega^\sharp, X\big\rangle := \omega(X) =: \big\langle X^\flat, \omega\big\rangle \quad \mathfrak{m}\text{-a.e.} \tag{3.2.15}$$

Definition 3.2.44. *The* gradient *of a function* $f \in \mathscr{F}_{\mathrm{e}}$ *is defined by*

$$\nabla f := (\mathrm{d}f)^\sharp.$$

Observe from (3.2.15) that ∇f, where $f \in \mathscr{F}_{\mathrm{e}}$, is characterized as the unique element $X \in L^2(TM)$ which satisfies

$$\mathrm{d}f(X) = |\mathrm{d}f|^2 = |X|^2 \quad \mathfrak{m}\text{-a.e.}$$

This uniqueness may fail on "nonlinear" spaces — that we do not consider here — such as Finsler manifolds. Compare with [Gig18, Subsec. 2.3.1].

The gradient operator is clearly linear on $\mathscr{F}_{\mathrm{e}}$ and closed in the sense of Lemma 3.2.39. By (3.2.15) and Proposition 3.2.37, all calculus rules from Proposition 3.2.37 transfer accordingly to the gradient. Moreover, Theorem 3.2.32 and the definition of $\mathscr{F}_{\mathrm{e}}$ ensure that $L^2(TM)$ is generated, in the sense of L^∞-modules, by $\nabla\,\mathscr{F}_{\mathrm{e}}$ and by $\nabla\,\mathscr{F}$.

3.2.5 Divergences

Now we introduce and study two notions of *divergence* of suitable elements of $L^2(TM)$. The first in Definition 3.2.45 is an L^2-approach similar to [Gig18, Subsec. 2.3.3]. The second in Definition 3.2.46 axiomatizes a measure-valued divergence **div**. Both approaches are compatible in the sense of Lemma 3.2.50.

L^2-divergence The following is similar to [Gig18, Def. 2.3.11]. See also [HRT13, Ch. 3] for a similar construction for regular Dirichlet spaces. For instance, it can be used to prove a duality formula for $\mathscr{E}$, cf. e.g. [Bra21, Gig18].

Definition 3.2.45. *We define the space* $\mathscr{D}(\mathrm{div})$ *to consist of all* $X \in L^2(TM)$ *for which there exists a function* $f \in L^2(M)$ *such that for every* $h \in \mathscr{F}$,

$$-\int_M h\, f \,\mathrm{d}\mathfrak{m} = \int_M \mathrm{d}h(X)\,\mathrm{d}\mathfrak{m}.$$

In case of existence, f *is unique, called the* divergence *of* X *and denoted by* $\mathrm{div}\, X$.

The uniqueness comes from the density of $\mathscr{F}$ in $L^2(M)$. Note that div is a linear operator on $\mathscr{D}(\mathrm{div})$, which thus turns $\mathscr{D}(\mathrm{div})$ into a vector space.

By the integration by parts formula for Δ, we have $\nabla\,\mathscr{D}(\Delta) \subset \mathscr{D}(\mathrm{div})$ and

$$\mathrm{div}\,\nabla f = \Delta f \quad \mathfrak{m}\text{-a.e.} \tag{3.2.16}$$

for every $f \in \mathscr{D}(\Delta)$. Moreover, employing the Leibniz rule in Proposition 3.2.9, one easily can verify that for every $X \in \mathscr{D}(\mathrm{div})$ and every $f \in \mathscr{F}_{\mathrm{eb}}$ with $|\mathrm{d}f| \in L^\infty(M)$, we have $f\,X \in \mathscr{D}(\mathrm{div})$ and

$$\mathrm{div}(f\,X) = f\,\mathrm{div}\,X + \mathrm{d}f(X) \quad \mathfrak{m}\text{-a.e.} \tag{3.2.17}$$

Measure-valued divergence The next definition has partly been inspired by the work [BCM19, Def. 4.1].

Definition 3.2.46. *We define the space* $\mathscr{D}(\mathbf{div})$ *to consist of all* $X \in L^2(TM)$ *for which there exists* $\nu \in \mathfrak{M}^{\pm}_{\sigma\mathrm{R}}(M)_{\mathscr{E}}$ *such that for every* $h \in \mathscr{F}_{\mathrm{bc}}$*,*

$$-\int_M \widetilde{h}\,\mathrm{d}\nu = \int_M \mathrm{d}h(X)\,\mathrm{d}\mathfrak{m}. \tag{3.2.18}$$

In case of existence, ν *is unique, termed the* measure-valued divergence *of* X *and denoted by* $\mathbf{div}\,X$*.*

The uniqueness statement follows by density of $\mathscr{F}_{\mathrm{bc}}$ in $\mathscr{F}$ by quasi-regularity of $\mathscr{E}$. The divergence $\mathbf{div}$ is clearly a linear operator on

$$\mathscr{D}_{\mathrm{TV}}(\mathbf{div}) := \big\{X \in L^2(TM) : \|\mathbf{div}\,X\|_{\mathrm{TV}} < \infty\big\},$$

and the latter is a vector space.

Normal components and Gauß–Green's formula Before we proceed with various properties of the notions from Definition 3.2.45 and Definition 3.2.46, it is convenient to introduce some further notation.

Definition 3.2.47. *Given* $X \in \mathscr{D}(\mathbf{div})$*, its* divergence $\mathrm{div}_1\,X \in L^1_{\mathrm{loc}}(M)$ *and its* normal component $\mathbf{n}\,X \in \mathfrak{M}^{\pm}_{\sigma\mathrm{R}}(M)_{\mathscr{E}}$ *are defined by*

$$\begin{aligned} \mathrm{div}_1\,X &:= \frac{\mathrm{d}\mathbf{div}_{\ll}\,X}{\mathrm{d}\mathfrak{m}},\\ \mathbf{n}\,X &:= -\,\mathbf{div}_{\perp}\,X. \end{aligned}$$

We define $\mathscr{D}_{L^2}(\mathbf{div})$ *as the space of all* $X \in \mathscr{D}(\mathbf{div})$ *such that* $\mathrm{div}_1\,X \in L^2(M)$*.*

Definition 3.2.47 is justified in the smooth setting according to the subsequent Example 3.2.48 and Remark 3.2.49.

Example 3.2.48. Let M be a Riemannian manifold with boundary ∂M. Recall that we denote by n the outward pointing unit normal vector field at ∂M. Then for every smooth vector field X on M with compact support and every $h \in \mathrm{C}^\infty_{\mathrm{c}}(M)$, by Green's formula, see e.g. [Lee97, p. 44], we have

$$-\int_M \mathrm{d}h(X)\,\mathrm{d}\mathfrak{v} = \int_M h\,\mathrm{div}_{\mathfrak{v}}\,X\,\mathrm{d}\mathfrak{v} - \int_{\partial M} h\,\langle X, \mathsf{n}\rangle\,\mathrm{d}\mathfrak{s}.$$

Here, $\mathrm{div}_{\mathfrak{v}}\,X$ is the usual divergence w.r.t. $\mathfrak{v}$. Thus, since $\mathrm{C}^\infty_{\mathrm{c}}(M)$ is a core for $\mathscr{F}$ (recall Example 3.2.13), $X \in \mathscr{D}_{\mathrm{TV}}(\mathbf{div})$ with

$$\begin{aligned} \mathrm{div}\,X &= \mathrm{div}_{\mathfrak{v}}\,X \quad \mathfrak{v}\text{-a.e.},\\ \mathbf{n}\,X &= \langle X, \mathsf{n}\rangle\,\mathfrak{s}\\ &= \langle X^\perp, \mathsf{n}\rangle\,\mathfrak{s}. \end{aligned}$$

Hence $\mathbf{n}\,X$ is a measure which is supported on ∂M.

If $\mathfrak{v}$ is instead replaced by $\mathfrak{m} := \mathrm{e}^{-2w}\,\mathfrak{v}$, $w \in \mathrm{C}^2(M)$, then for every X as above, in terms of the metric tensor $\langle\cdot,\cdot\rangle$ we have $X \in \mathscr{D}_{\mathrm{TV}}(\mathbf{div})$ with

$$\begin{aligned} \mathrm{div}\,X &= \mathrm{div}_{\mathfrak{v}}\,X - 2\,\langle\nabla w, X\rangle \quad \mathfrak{m}\text{-a.e.},\\ \mathbf{n}\,X &= \mathrm{e}^{-2w}\,\langle X, \mathsf{n}\rangle\,\mathfrak{s}\\ &= \mathrm{e}^{-2w}\,\langle X^\perp, \mathsf{n}\rangle\,\mathfrak{s}. \end{aligned}$$

■

In fact, this smooth framework is the main motivation for the terminology of Definition 3.2.47, since in general we a priori do not know anything about the support of $\mathbf{n}\,X$, $X \in \mathscr{D}(\mathbf{div})$. As suggested by Example 3.2.48 and the following Remark 3.2.49, we may and will interpret $\mathbf{n}\,X$ as the normal component of X w.r.t. a fictive *outward pointing* unit normal vector field — or more generally, as the "normal" part of X w.r.t. a certain $\mathfrak{m}$-singular set. Examples which link our *intrinsic* approach to normal components with existing *extrinsic* ones from [BPS19, BCM19, Stu20] are discussed in Section 3.7 below.

As it turns out in Lemma 3.2.54 and Subsection 3.2.8, most vector fields of interest have vanishing normal component. Hence, normal components mostly do not appear in our subsequent treatise. In possible further applications it might still be useful to see the normal component of vector fields in $\mathscr{D}(\mathbf{div})$.

Remark 3.2.49 (Gauß–Green formula)**.** In terms of Definition 3.2.46, given any $X \in \mathscr{D}(\mathbf{div})$ and $h \in \mathscr{F}_{\mathrm{bc}}$, in analogy to Example 3.2.48 we have

$$-\int_M \mathrm{d}h(X)\,\mathrm{d}\mathfrak{m} = \int_M h \,\mathrm{div}\, X\,\mathrm{d}\mathfrak{m} - \int_M \widetilde{h}\;\mathrm{d}\mathbf{n}\,X. \qquad \blacksquare$$

Calculus rules The proof of the following simple result directly follows from the respective definitions and is thus omitted.

Lemma 3.2.50. *If* $X \in \mathscr{D}(\mathrm{div})$ *satisfies* $(\mathrm{div}\,X)^+ \in L^1(M)$ *or* $(\mathrm{div}\,X)^- \in L^1(M)$*, then* $X \in \mathscr{D}_{L^2}(\mathbf{div})$ *and*

$$\begin{aligned}\mathrm{div}_1\, X &= \mathrm{div}\, X \quad \mathfrak{m}\text{-}a.e.,\\ \mathbf{n}\,X &= 0.\end{aligned}$$

Conversely, if $X \in \mathscr{D}_{L^2}(\mathbf{div})$ *with* $\mathbf{n}\,X = 0$*, then* $X \in \mathscr{D}(\mathrm{div})$ *with*

$$\mathrm{div}\, X = \mathrm{div}_1\, X \quad \mathfrak{m}\text{-}a.e.$$

Remark 3.2.51. The additional integrability condition in the first part of Lemma 3.2.50 ensure that $\mathrm{div}\,X\,\mathfrak{m}$ is a well-defined signed Borel measure for $X \in \mathscr{D}(\mathrm{div})$. In fact, unlike (3.2.16) this small technical issue prevents us from saying that in general, if $f \in \mathscr{D}(\Delta)$ then $\nabla f \in \mathscr{D}_{L^2}(\mathbf{div})$ with $\mathrm{div}_1\,\nabla f = \Delta f$ $\mathfrak{m}$-a.e. and $\mathbf{n}\,\nabla f = 0$. This would correspond to the role of Δ as Neumann Laplacian.

However, by the integration by parts Definition 3.2.45 as well as Lemma 3.2.52 and Lemma 3.2.54 below there is still formal evidence in keeping this link in mind. In particular, we may and will interpret every $X \in \mathscr{D}(\mathrm{div})$ as having vanishing normal component although the natural assignment $\mathbf{div}\,X := \mathrm{div}\,X\,\mathfrak{m}$ — which entails $\mathbf{n}\,X = 0$ — might not be well-defined in $\mathfrak{M}^{\pm}_{\sigma}(M)_{\mathscr{E}}$. ■

Based upon Lemma 3.2.50, as long as confusion is excluded we make no further notational distinction and identify, for suitable $X \in L^2(TM)$,

$$\mathrm{div}\,X = \mathrm{div}_1\,X.$$

Lemma 3.2.52. *For every* $X \in \mathscr{D}(\mathbf{div})$ *and every* $f \in \mathscr{F}_{\mathrm{eb}}$ *such that* $|X| \in L^\infty(M)$ *or* $|\mathrm{d}f| \in L^\infty(M)$*, we have* $f\,X \in \mathscr{D}(\mathbf{div})$ *with*

$$\mathbf{div}(f\,X) = \widetilde{f}\,\mathbf{div}\,X + \mathrm{d}f(X)\,\mathfrak{m},$$

$$\begin{aligned}\operatorname{div}(f\,X) &= f\operatorname{div} X + \mathrm{d}f(X) \quad \mathfrak{m}\text{-}a.e.,\\ \mathbf{n}(f\,X) &= \widetilde{f}\,\mathbf{n}\,X.\end{aligned}$$

In particular, if $X \in \mathscr{D}_{L^2}(\mathbf{div})$ *then also* $f\,X \in \mathscr{D}_{L^2}(\mathbf{div})$.

Proof. The first identity, from which the second follows, is straightforward to deduce from Proposition 3.2.9. The last claim on $\mathscr{D}_{L^2}(\mathbf{div})$ is a direct consequence of these two identities. To prove the remaining claim $\mathbf{n}(f\,X) = \widetilde{f}\,\mathbf{n}\,X$, we compute

$$\begin{aligned}\mathbf{n}(f\,X) &= -\,\mathbf{div}_\perp(f\,X)\\ &= f\operatorname{div} X\,\mathfrak{m} - \widetilde{f}\,\mathbf{div}\,X\\ &= -\widetilde{f}\,\mathbf{div}_\perp X\\ &= \widetilde{f}\,\mathbf{n}\,X.\end{aligned}$$ □

Example 3.2.53. Every identity stated in Lemma 3.2.52 is fully justified in the smooth context of Example 3.2.48. In particular, for every smooth vector field X on M with compact support and every $f \in \mathrm{C}_\mathrm{c}^\infty(M)$,

$$\mathbf{n}(f\,X) = \langle f\,X, \mathsf{n}\rangle\,\mathfrak{s} = f\,\langle X, \mathsf{n}\rangle\,\mathfrak{s} = f\,\mathbf{n}\,X.$$ ■

Finally, we show that Lemma 3.2.50 is not void, i.e. that there will exist an L^2-dense set of vector fields — see Subsection 3.2.8 below — which satisfy both hypotheses of Lemma 3.2.50 even in a slightly stronger version.

Lemma 3.2.54. *Suppose that* $f \in \mathscr{D}(\Delta)$ *and* $g \in \mathscr{F}_\mathrm{b}$. *Moreover, suppose that* $|\nabla f| \in L^\infty(M)$ *or* $|\mathrm{d}g| \in L^\infty(M)$. *Then* $g\,\nabla f \in \mathscr{D}_{\mathrm{TV}}(\mathbf{div}) \cap \mathscr{D}(\operatorname{div})$ *with*

$$\begin{aligned}\operatorname{div}(g\,\nabla f) &= \mathrm{d}g(\nabla f) + g\,\Delta f \quad \mathfrak{m}\text{-}a.e.,\\ \mathbf{n}(g\,\nabla f) &= 0.\end{aligned}$$

Proof. By (3.2.17) and (3.2.16), we already know that $g\,\nabla f \in \mathscr{D}(\operatorname{div})$.

To show that $g\,\nabla f \in \mathscr{D}_{\mathrm{TV}}(\mathbf{div})$, note that under the given assumptions, the r.h.s. of the identity for $\operatorname{div}(g\,\nabla f)$ belongs to $L^1(M)$. In particular, by what we have already proved before, $\nu := \operatorname{div} X\,\mathfrak{m} \in \mathfrak{M}_\mathrm{f}^\pm(M)_\mathscr{E}$ satisfies the defining property (3.2.18) for $\mathbf{div}\,X$, yielding the claim. □

3.2.6 Basic notions of tamed spaces

From now on, we employ the following convention: given $f \in L^0(M)$, integrals of possibly degenerate terms such as $1/f$ are (consistently) understood as restricted on $\{f \neq 0\}$ without further notice.

In this subsection, we outline the theory of *tamed spaces* introduced in [ER+20] that will be needed below throughout.

Quasi-local distributions Given an $\mathscr{E}$-quasi-open set $G \subset M$, let $\mathscr{F}_G^{-1}$ denote the dual space of the closed [CF12, p. 84] subspace

$$\mathscr{F}_G := \big\{f \in \mathscr{F} : \widetilde{f} = 0\ \mathscr{E}\text{-q.e. on } G^\mathrm{c}\big\}$$

of $\mathscr{F}$. Note that if $G' \subset M$ is $\mathscr{E}$-quasi-open as well with $G \subset G'$, then $\mathscr{F}_G^{-1} \supset \mathscr{F}_{G'}^{-1}$. Let $\mathscr{F}_{\mathrm{qloc}}^{-1}$ denote the space of $\mathscr{E}$*-quasi-local distributions* on $\mathscr{F}$, i.e. the space of all objects

κ for which there exists an $\mathscr{E}$-quasi-open $\mathscr{E}$-nest $(G_n)_{n\in\mathbf{N}}$ such that $\kappa \in \bigcap_{n\in\mathbf{N}} \mathscr{F}_{G_n}^{-1}$. Every distribution $\kappa \in \mathscr{F}_{\mathrm{qloc}}^{-1}$ is uniquely associated with an *additive functional*, or briefly AF, $(\mathsf{a}_t^{\kappa,\cdot})_{t\geq 0}$ in the sense of [ER+20, Lem. 2.7, Lem. 2.9]. Here, uniqueness means up to $\mathfrak{m}$-equivalence of AF's [FOT11, p. 423]. (We refer to [FOT11, Ch. II] or [CF12, Ch. 4, App. A] for a concise overview over basic notions about AF's.) The AF $(\mathsf{a}_t^{\kappa,\cdot})_{t\geq 0}$ is independent of the chosen $\mathscr{E}$-nest [ER+20, Lem. 2.11]. All AF's will be understood as being zero beyond the explosion time ζ.

Remark 3.2.55. (The defining properties of) AF's are linked to the Markov process b associated with the Dirichlet form $\mathscr{E}$ [CF12, Def. A.3.1]. ■

Example 3.2.56. If $\kappa \in \mathscr{F}_{\mathrm{qloc}}^{-1}$ is induced in the evident way through a nearly Borel [CF12, Def. A.1.28] function $f \in L^2(M)$, by [ER+20, Rem. 2.8], for every $t \in [0, \zeta)$,

$$\mathsf{a}_t^{\kappa,x} = \frac{1}{2}\int_0^{2t} f(\mathsf{b}_s^x)\,\mathrm{d}s,$$

where b^x is Brownian motion starting in a given $x \in M$. ■

Extended Kato class The relevant distributions $\kappa \in \mathscr{F}_{\mathrm{qloc}}^{-1}$ in this chapter, see Remark 3.2.61, will be induced by signed measures in the *extended Kato class* $\mathbf{K}_{1-}(M)$, which is introduced now. In fact, Feynman–Kac semigroups and energy forms induced by more general distributions $\kappa \in \mathscr{F}_{\mathrm{qloc}}^{-1}$ are studied in [ER+20].

Given any Borelian $f\colon M \to \mathbf{R}$, set

$$\text{q-sup}\, f := \inf\Big\{\sup_{x\in(M')^{\mathrm{c}}} f(x) : M' \subset M \text{ is } \mathscr{E}\text{-polar}\Big\}.$$

Recall from [CF12, p. 84] that a measure $\nu \in \mathfrak{M}^+(M)$ is *$\mathscr{E}$-smooth* if it does not charge $\mathscr{E}$-polar sets and $\nu[F_n] < \infty$ for every $n \in \mathbf{N}$, for some $\mathscr{E}$-nest $(F_n)_{n\in\mathbf{N}}$ of closed sets. Every Radon measure charging no $\mathscr{E}$-polar set is $\mathscr{E}$-smooth, but the converse does not hold [DSS20, Ex. 2.33]. By the Revuz correspondence [CF12, Thm. A.3.5], any $\mathscr{E}$-smooth $\nu \in \mathfrak{M}^+(M)$ is uniquely associated to a positive continuous AF, or briefly PCAF, $(\mathsf{a}_t^{\nu,\cdot})_{t\geq 0}$ by the subsequent identity, valid for every nonnegative $f \in L_0(M)$:

$$\int_M f\,\mathrm{d}\nu = \lim_{t\to 0}\frac{1}{t}\int_M \mathbf{E}\Big[\int_0^t f(\mathsf{b}_{2s}^x)\,\mathrm{d}\mathsf{a}_s^\nu\Big]\,\mathrm{d}\mathfrak{m}(x).$$

Of course, the existence of a *compact* $\mathscr{E}$-nest, by quasi-regularity of $\mathscr{E}$, ensures that every σ-finite $\nu \in \mathfrak{M}^+(M)$ charging no $\mathscr{E}$-polar set is $\mathscr{E}$-smooth. Hence the following definition [ER+20, Def. 2.21] is meaningful.

Definition 3.2.57. *Given $\rho \geq 0$, the* ρ-Kato class $\mathbf{K}_\rho(M)$ *of M is defined to consist of all $\mu \in \mathfrak{M}_\sigma^\pm(M)$ which do not charge $\mathscr{E}$-polar sets with*

$$\lim_{t\to 0}\text{q-sup}\,\mathbf{E}\big[\mathsf{a}_t^{2|\mu|,\cdot}\big] \leq \rho.$$

$\mathbf{K}_0(M)$ *is called* Kato class *of M, while the* extended Kato class *of M is*

$$\mathbf{K}_{1-}(M) := \bigcup_{\rho\in[0,1)} \mathbf{K}_\rho(M).$$

Remark 3.2.58. By definition, we have $\mu \in \mathbf{K}_\rho(M)$ if and only if $|\mu| \in \mathbf{K}_\rho(M)$ if and only if $\mu^+, \mu^- \in \mathbf{K}_\rho(M)$, $\rho \geq 0$. ■

Remark 3.2.59. In the evident way by the assignment $\kappa := \mathrm{v}\,\mathfrak{m}$, the functional Kato class $\mathrm{K}(M)$ from Definition 2.1.2 can be naturally identified as subset of $\mathbf{K}_0(M)$. ■

The following important lemma is due to [ER+20, Cor. 2.25]. The immediate corollary that one can always choose $\rho' < 1$ for $\mu \in \mathbf{K}_{1-}(M)$ therein is used crucially in the sequel (and also in [ER+20]), see e.g. Corollary 3.3.12 and Lemma 3.6.8.

Lemma 3.2.60. *For every $\rho, \rho' \geq 0$ with $\rho < \rho'$ and every $\mu \in \mathbf{K}_\rho(M)$ there exists $\alpha' \in \mathbf{R}$ such that for every $f \in \mathscr{F}$,*

$$\int_M \widetilde{f}^2 \,\mathrm{d}\mu \leq \rho' \,\mathscr{E}(f) + \alpha' \int_M f^2 \,\mathrm{d}\mathfrak{m}.$$

In particular, w.r.t. a sequence $(G_n)_{n\in\mathbf{N}}$ of open subsets of M witnessing the σ-finiteness of $\nu := |\mu|$, by Cauchy–Schwarz's inequality every $\mu \in \mathbf{K}_\rho(M)$, $\rho \geq 0$, induces a (non-relabeled) element $\mu \in \mathscr{F}^{-1}_{\mathrm{qloc}}(M)$ by setting, for $f \in \bigcup_{n\in\mathbf{N}} \mathscr{F}_{G_n}$,

$$\langle \mu \,|\, f \rangle := \int_M \widetilde{f} \,\mathrm{d}\mu.$$

If $\mu \in \mathbf{K}_\rho(M)$ is nonnegative, then the AF's coming from the Revuz correspondence w.r.t. μ and from its property as inducing an element in $\mathscr{F}^{-1}_{\mathrm{qloc}}$ are $\mathfrak{m}$-equivalent. The key feature about general $\mu \in \mathbf{K}_{1-}(M)$ is that by Khasminskii's lemma [ER+20, Lem. 2.24] (recall also Lemma 2.1.4), the induced distribution is *moderate* [ER+20, Def. 2.13], i.e.

$$\sup_{t\in[0,1]} \ \text{q-sup}\,\mathbf{E}\big[\mathrm{e}^{-\mathsf{a}_t^{2\mu,\cdot}}\big] < \infty.$$

Feynman–Kac semigroup Let $q \in [1,2]$, and note that $q\kappa/2 \in \mathbf{K}_{1-}(M)$ for every $\kappa \in \mathbf{K}_{1-}(M)$. Given such κ, we define a family $(\mathsf{P}_t^{q\kappa})_{t\geq 0}$ of operators acting on nonnegative nearly Borel functions $f \in L_0(M)$ by

$$\mathsf{P}_t^{q\kappa} f := \mathbf{E}\big[\mathrm{e}^{-\mathsf{a}_t^{q\kappa,\cdot}} f(\mathsf{b}_{2t})\,1_{\{t<\zeta/2\}}\big].$$

It naturally extends to nearly Borelian $f \in L_0(M)$ for which the latter expectation for $|f|$ in place of f is finite, see [ER+20, Def. 2.10]. (For later convenience, we have to change the notation from [ER+20] a bit, also at later times, see Remark 3.2.62 below.) It is $\mathfrak{m}$-symmetric and maps $\mathfrak{m}$-equivalence classes to $\mathfrak{m}$-equivalence classes. Since $q\kappa/2$ is moderate, it extends to an exponentially bounded semigroup of linear operators on $L^p(M)$ for every $p \in [1,\infty]$ [ER+20, Lem. 2.11, Rem. 2.14]. That is, there exists a finite constant $C > 0$ such that for every $p \in [1,\infty]$ and every $t \geq 0$,

$$\big\|\mathsf{P}_t^{q\kappa}\big\|_{L^p(M);L^p(M)} \leq C\,\mathrm{e}^{Ct}.$$

The perturbed energy form One of the main results from [ER+20] is that for $\kappa \in \mathbf{K}_{1-}(M)$ — in fact, for more general $\kappa \in \mathscr{F}^{-1}_{\mathrm{qloc}}$, cf. [ER+20, Thm. 2.49] — $(\mathsf{P}_t^{q\kappa})_{t\geq 0}$ is properly associated to an energy form $\mathscr{E}^{q\kappa}$, $q \in [1,2]$. Indeed, by [ER+20, Thm. 2.47, Thm. 2.49, Cor. 2.51], the quadratic form

$$\mathscr{E}^{q\kappa}(f) := \mathscr{E}(f) + q\,\big\langle \kappa \,\big|\, f^2 \big\rangle \tag{3.2.19}$$

with finiteness domain $\mathscr{D}(\mathscr{E}^{q\kappa}) = \mathscr{F}$ is closed, lower semibounded and associated with $(\mathsf{P}_t^{q\kappa})_{t\geq 0}$ [FOT11, Thm. 1.3.1, Lem. 1.3.2].

The corresponding generator, henceforth termed $\Delta^{q\kappa}$ with domain $\mathscr{D}(\Delta^{q\kappa})$, is called *Schrödinger operator* with potential $q\kappa$.

Remark 3.2.61. One reason for considering perturbations of $\mathscr{E}$ by $\kappa \in \mathbf{K}_{1-}(M)$ is the following. By Lemma 3.2.60, the map $f \mapsto \langle \kappa^- \,|\, f^2 \rangle$ on $\mathscr{F}$ is form bounded w.r.t. $\mathscr{E}$, hence w.r.t. $\mathscr{E}^{q\kappa^+}$, with some form bound $\rho' < 1$. Hence, by [ER+20, Thm. 2.49], $\mathscr{E}^{q\kappa}$ is closed with domain $\mathscr{D}(\mathscr{E}^{q\kappa}) = \{f \in \mathscr{F} : \langle \kappa^+ \,|\, f^2 \rangle < \infty\}$. Again by Lemma 3.2.60, the latter is all of $\mathscr{F}$, which is technically required in the setting of Subsection 3.2.7, compare with [ER+20, Ch. 6]. See also Remark 3.2.66 below. ■

Remark 3.2.62. For our analytic and geometric purposes, we use differently scaled forms, operators and semigroups than [ER+20]. Let us list the relations of the main objects in [ER+20], on the l.h.s.'s, with our notation, on the respective r.h.s.'s:

$$\begin{aligned}
\mathcal{E} &= \mathscr{E}/2,\\
\mathsf{L} &= \Delta/2,\\
P_t &= \mathsf{P}_{t/2},\\
B_t &= \mathsf{b}_t \text{ [sic]},\\
\mathcal{E}^{q\kappa/2} &= \mathscr{E}^{q\kappa}/2\\
\mathsf{L}^{q\kappa/2} &= \Delta^{q\kappa}/2,\\
P_t^{q\kappa/2} &= \mathsf{P}_{t/2}^{q\kappa}.
\end{aligned}$$ ■

Tamed spaces

Definition 3.2.63. *Suppose that $q \in \{1, 2\}$, $\kappa \in \mathbf{K}_{1-}(M)$ and $N \in [1, \infty]$. We say that $(M, \mathscr{E}, \mathfrak{m})$ or simply M satisfies the q-*Bakry–Émery condition*, briefly* $\mathrm{BE}_q(\kappa, N)$*, if for every $f \in \mathscr{D}(\Delta)$ with $\Delta f \in \mathscr{F}$ and every nonnegative $\phi \in \mathscr{D}(\Delta^{q\kappa})$ with $\Delta^{q\kappa}\phi \in L^\infty(M)$, we have*

$$\begin{aligned}
&\frac{1}{q}\int_M \Delta^{q\kappa}\phi\,|\nabla f|^q\,\mathrm{d}\mathfrak{m} - \int_M \phi\,|\nabla f|^{q-1}\,\langle \nabla f, \nabla\Delta f\rangle\,\mathrm{d}\mathfrak{m}\\
&\qquad\geq \frac{1}{N}\int_M \phi\,|\nabla f|^{q-1}\,(\Delta f)^2\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

The latter term is understood as 0 if $N := \infty$.

Remark 3.2.64. Through a mollification argument using $(\mathsf{P}_t^{q\kappa})_{t\geq 0}$ [ER+20, Lem. 6.2], the class of elements $\phi \in \mathscr{D}(\Delta^{q\kappa})$ with $\Delta^{q\kappa}\phi \in L^\infty(M)$ is dense in $L^2(M)$. ■

Assumption 3.2.65. *Throughout the rest of the present chapter, assume that M satisfies the* $\mathrm{BE}_2(\kappa, N)$ *condition for given $\kappa \in \mathbf{K}_{1-}(M)$ and $N \in [1, \infty]$.*

For certain $k\colon M \to \mathbf{R}$ and $N \in [1, \infty]$, write $\mathrm{BE}_2(k, N)$ instead of $\mathrm{BE}_2(k\,\mathfrak{m}, N)$. Of course, Definition 3.2.63 [ER+20, Def. 3.1, Def. 3.5] generalizes the well-known Bakry–Émery condition for uniform lower Ricci bounds [AGS15, Bak85, BE85, EKS15, Gig15]. Variable lower Ricci bounds have been first studied by [BHS21, Stu15] in a synthetic context. Compare with Chapter 1 and in particular Definition 1.1.4 therein.

In the framework of Assumption 3.2.65, we say that $(M, \mathscr{E}, \mathfrak{m})$ or simply M is *tamed*. Although this is not the original definition of taming from [ER+20, Def. 3.2], for $\kappa \in \mathbf{K}_{1-}(M)$ they are in fact equivalent, see Subsection 3.2.7 below.

Remark 3.2.66. From the taming point of view, one might regard the implicit assumption that $\kappa^+ \in \mathbf{K}_{1-}(M)$ (recall Remark 3.2.58) as unnatural (see also the notion of

Kato decomposability in Definition 2.1.2). However, for the *qualitative* message of this chapter — the existence of a rich second order calculus — one can simply ignore κ^+ by setting it to zero. To obtain *quantitative* results in applications, to bypass the assumption $\kappa^+ \in \mathbf{K}_{1-}(M)$, a useful tool could be appropriate "cutoffs" and monotone approximations by elements in $\mathbf{K}_{1-}(M)$, see e.g. Lemma 1.2.1 above. ■

Example 3.2.67 (Manifolds)**.** Any compact Riemannian manifold M is tamed by

$$\kappa := \mathscr{k}\,\mathfrak{v} + \ell\,\mathfrak{s}$$

which in fact belongs to $\mathbf{K}_0(M)$ [ER+20, Thm. 4.4] (recall Example 3.2.13). More generally, reminiscent of Theorem 2.1.3 and Example 2.5.5, let M be a "regular" Lipschitz Riemannian manifold, in the sense of [BR21], that is quasi-isometric to a Riemannian manifold with uniformly lower bounded Ricci curvature. Suppose that the Ricci curvature of M, where defined, is bounded from below by a function $\mathscr{k} \in L^p(M, \Xi\,\mathfrak{v})$, $p > d/2$, where $\Xi\colon M \to \mathbf{R}$ is given by $\Xi(x) := \mathfrak{v}[B_1(x)]^{-1}$. Then M is tamed by $\kappa := \mathscr{k}\,\mathfrak{v} \in \mathbf{K}_0(M)$. ■

Example 3.2.68 (RCD spaces)**.** Every $\mathrm{RCD}(K,\infty)$ space $(M,\mathsf{d},\mathfrak{m})$, $K \in \mathbf{R}$, according to Example 3.2.14 is tamed by $\kappa := K\,\mathfrak{m} \in \mathbf{K}_0(M)$ [AGS14b]. ■

Example 3.2.69 (Almost smooth spaces)**.** Let $(M,\mathsf{d},\mathfrak{m})$ be a d-dimensional almost smooth metric measure space, $d \in \mathbf{N}$, in the sense of [Hon18b, Def. 3.1, Def. 3.16], examples of which include the gluing of two pointed, compact Riemannian manifolds (*not* necessarily of the same dimension, in which case the parameter d below is just the maximum of all considered dimensions) at their base points. Then, under few further assumptions, Honda proved that if the "generalized Ricci curvature" of M is bounded from below by $K(d-1)$, $K \in \mathbf{R}$, then the $\mathrm{BE}_2(K(d-1),d)$ condition holds for M [Hon18b, Thm. 3.7, Thm. 3.17]. The $\mathrm{RCD}^*(K(d-1),N)$ condition, however, does not hold in general [Hon18b, Rem. 3.9]. ■

Example 3.2.70 (Configuration spaces)**.** Further important nonsmooth examples are configuration spaces $\mathscr{Y}$ over Riemannian manifolds M [AKR98, EH15]. The Dirichlet form $\mathscr{E}^{\mathscr{Y}}$ on $\mathscr{Y}$ constructed in [AKR98] is quasi-regular and strongly local, cf. the proof of [AKR98, Thm. 6.1]. If $\mathrm{Ric} \geq K$ on M, $K \in \mathbf{R}$, then $(\mathscr{Y},\mathscr{E}^{\mathscr{Y}},\pi)$ is tamed by $\kappa := K\,\pi \in \mathbf{K}_0(M)$ as well by [EKS15, Thm. 4.7] and [ER+20, Thm. 3.6]. Here π is the Poisson (probability) measure on $\mathscr{Y}$, up to intensity.

A similar result even over more general spaces is announced in [DSS20]. ■

Spaces that are tamed by some measures in $\mathbf{K}_{1-}(M)$ or even $\mathbf{K}_0(M)$ may also have cusp-like singularities [ER+20, Thm. 4.6], have singular [ER+20, Thm. 2.36] or not semiconvex boundary [ER+20, Thm. 4.7] or be of Harnack-type [BR21, ER+20].

Intrinsically complete Dirichlet spaces An interesting, but not exhaustive, class of tamed spaces we sometimes consider is the one of intrinsically complete M as introduced in [ER+20, Def. 3.8]. Compare with Lemma 2.3.1.

Definition 3.2.71. *We call $(M,\mathscr{E},\mathfrak{m})$ or simply M* intrinsically complete *if there exists a sequence $(\phi_n)_{n\in\mathbf{N}}$ in $\mathscr{F}$ such that $\mathfrak{m}\big[\{\phi_n > 0\}\big] < \infty$, $0 \leq \phi_n \leq 1$ $\mathfrak{m}$-a.e. and $|\nabla\phi_n| \leq 1$ $\mathfrak{m}$-a.e. for every $n \in \mathbf{N}$ as well as $\phi_n \to 1_M$ and $|\nabla\phi_n| \to 0$ pointwise $\mathfrak{m}$-a.e. as $n \to \infty$.*

Intrinsically complete tamed spaces are *stochastically complete*, i.e.

$$\mathsf{P}_t 1_M = 1_M \quad \mathfrak{m}\text{-a.e.}$$

for every $t \geq 0$ [ER+20, Thm. 3.11]. In this chapter, intrinsically complete spaces provide a somewhat better version of Lemma 3.2.6, sometimes used to get rid of differentials in certain expressions. See e.g. Remark 3.4.7 and Remark 3.5.3. However, none of our results will severely rely on intrinsic completeness.

3.2.7 Self-improvement and singular Γ_2-calculus

In fact, under the above Assumption 3.2.65, the condition $\mathrm{BE}_2(\kappa, N)$ is *equivalent* to $\mathrm{BE}_1(\kappa, N)$ [ER+20, Thm. 6.9], $N \in [1, \infty]$, albeit the latter is a priori stronger [ER+20, Thm. 3.4, Prop. 3.7]. In particular, the heat flow $(\mathsf{P}_t)_{t\geq 0}$ satisfies the important contraction estimate

$$|\nabla \mathsf{P}_t f| \leq \mathsf{P}_t^\kappa |\nabla f| \quad \mathfrak{m}\text{-a.e.} \tag{3.2.20}$$

for every $f \in \mathscr{F}$ and every $t \geq 0$. See [Bak85, BE85, Sav14] for corresponding results for constant κ and Theorem 1.3.6 for the first nonconstant result in this direction.

We briefly recapitulate the singular Γ_2-calculus developed in [ER+20, Sav14], since the involved calculus objects, in particular the test ones from Subsection 3.2.7 below, are crucial in our treatise as well, see e.g. Theorem 3.3.11 and Theorem 3.6.9.

Test functions Define the set of *test functions* by

$$\mathrm{Test}(M) := \big\{ f \in \mathscr{D}(\Delta) \cap L^\infty(M) : |\nabla f| \in L^\infty(M),\ \Delta f \in \mathscr{F} \big\}$$

It is an algebra w.r.t. pointwise multiplication, and if $f \in \mathrm{Test}(M)^n$ and $\varphi \in \mathrm{C}^\infty(\mathbf{R}^n)$ with $\varphi(0) = 0$, $n \in \mathbf{N}$, then $\varphi \circ f \in \mathrm{Test}(M)$ as well [Sav14, Lem. 3.2].

Since $\mathrm{BE}_2(\kappa, N)$ implies $\mathrm{BE}_2(-\kappa^-, N)$ [ER+20, Prop. 6.7], a variant of the reverse Poincaré inequality states that for every $f \in L^2(M) \cap L^\infty(M)$ and every $t > 0$,

$$|\nabla \mathsf{P}_t f|^2 \leq \frac{1}{2t} \big\| \mathsf{P}_t^{-2\kappa^-} \big\|_{L^\infty(M);L^\infty(M)} \, \|f\|_{L^\infty(M)} \quad \mathfrak{m}\text{-a.e.},$$

and hence $\mathsf{P}_t f \in \mathrm{Test}(M)$ [ER+20, Cor. 6.8]. In particular, $\mathrm{Test}(M)$ is dense in $\mathscr{F}$.

We use the following two approximation results henceforth exploited at various instances. For convenience, we outline the proof of Lemma 3.2.72. Lemma 3.2.73, which yields a useful density result for the slightly smaller set

$$\mathrm{Test}_{L^\infty}(M) := \big\{ f \in \mathrm{Test}(M) : \Delta f \in L^\infty(M) \big\},$$

results from (3.2.20) and an approximation by a mollified heat flow [ER+20, Sav14].

Lemma 3.2.72. *For every $f \in \mathrm{Test}(M)$, there exist sequences $(g_n)_{n\in\mathbf{N}}$ and $(h_n)_{n\in\mathbf{N}}$ in $\mathrm{Test}(M)$ which are bounded in $L^\infty(M)$ with $g_n\, h_n \to f$ in $\mathscr{F}$ as $n \to \infty$.*

Proof. Given any $k, m \in \mathbf{N}$, we define $g_k := 2 \arctan(k\, f)/\pi \in \mathrm{Test}(M)$ and $h_m := (f^2 + 2^{-m})^{1/2} - 2^{-m/2} \in \mathrm{Test}(M)$. Then $g_k\, h_m \to g_k\, |f|$ in $L^2(M)$ as $m \to \infty$ for every $k \in \mathbf{N}$, and $g_k\, |f| \to f$ in $L^2(M)$ as $k \to \infty$. Using Proposition 3.2.37 and Lebesgue's theorem, it follows that $g_k\, h_m \to g_k\, |f|$ in $\mathscr{F}$ as $m \to \infty$ for every $k \in \mathbf{N}$, and $g_k\, |f| \to f$ in $\mathscr{F}$ as $k \to \infty$. We conclude by a diagonal argument. $\square$

Lemma 3.2.73. *For every $f \in \mathscr{F}$ such that $a \leq f \leq b$ $\mathfrak{m}$-a.e., $a, b \in [-\infty, \infty]$, there exists a sequence $(f_n)_{n\in\mathbf{N}}$ in $\mathrm{Test}_{L^\infty}(M)$ which converges to f in $\mathscr{F}$ such that $a \leq f_n \leq b$ $\mathfrak{m}$-a.e. for every $n \in \mathbf{N}$. If moreover $|\nabla f| \in L^\infty(M)$, then $(f_n)_{n\in\mathbf{N}}$ can be constructed such that $(|\nabla f_n|)_{n\in\mathbf{N}}$ is bounded in $L^\infty(M)$.*

Measure-valued Schrödinger operator A further regularity property of functions $f \in \mathrm{Test}(M)$ that will be crucial in defining the κ-Ricci measure in Subsection 3.6.1 is that their carré du champs $|\nabla f|^2$ have $\mathscr{F}$-regularity under $\mathrm{BE}_2(\kappa, N)$. In fact, $|\nabla f|^2$ even admits a measure-valued Schrödinger operator in the sense of Definition 3.2.74. This is recorded in Proposition 3.2.75 and is due to [ER⁺20, Lem. 6.4].

Definition 3.2.74. *We define $\mathscr{D}(\mathbf{\Delta}^{2\kappa})$ to consist of all $u \in \mathscr{F}$ for which there exists $\iota \in \mathfrak{M}^{\pm}_{\sigma\mathrm{R}}(M)_{\mathscr{E}}$ such that for every $h \in \mathscr{F}$, we have $\widetilde{h} \in L^1(M, \iota)$ and*

$$\int_M \widetilde{h}\,\mathrm{d}\iota = -\mathscr{E}^{2\kappa}(h, u).$$

In case of existence, ι is unique, denoted by $\mathbf{\Delta}^{2\kappa} u$ and shall be called the measure-valued Schrödinger operator *with potential 2κ.*

Proposition 3.2.75. *For every $f \in \mathrm{Test}(M)$ we have $|\nabla f|^2 \in \mathscr{F}$ and even $|\nabla f|^2 \in \mathscr{D}(\mathbf{\Delta}^{2\kappa})$. Moreover,*

$$\frac{1}{2}\mathbf{\Delta}^{2\kappa}|\nabla f|^2 - \big\langle \nabla f, \nabla \Delta f\big\rangle\,\mathfrak{m} \geq \frac{1}{N}\,(\Delta f)^2\,\mathfrak{m}.$$

An advantage of interpreting the Schrödinger operator associated to $\mathscr{E}^{2\kappa}$ as a measure is that the potential 2κ can be separated from $\mathbf{\Delta}^{2\kappa}$ to give the *measure-valued Laplacian* $\mathbf{\Delta} := \mathbf{\Delta}^{2\kappa} + 2\kappa$, which fits well with the divergence objects from Subsection 3.2.5, see the part about the measure-valued Laplacian in Subsection 3.6.2 below, and is a "drifted" version of the measure-valued Laplacian from Subsection 1.3.1 above. This is technically convenient in later defining the drift-free Ricci measure **Ric** without κ-dependency. However, for possible later extensions, e.g. when κ is not a (signed) measure, we decided not to separate the distribution κ from the other calculus objects under consideration until Subsection 3.6.2.

Singular $\mathbf{\Gamma}_2$-operator Given Proposition 3.2.75, following [ER⁺20, Sav14] we introduce the map $\mathbf{\Gamma}_2^{2\kappa}\colon \mathrm{Test}(M) \to \mathfrak{M}^{+}_{\sigma\mathrm{R}}(M)_{\mathscr{E}}$ by

$$\mathbf{\Gamma}_2^{2\kappa}(f) := \frac{1}{2}\mathbf{\Delta}^{2\kappa}|\nabla f|^2 - \big\langle \nabla f, \nabla \Delta f\big\rangle\,\mathfrak{m}. \tag{3.2.21}$$

According to Lebesgue's decomposition, we decompose

$$\mathbf{\Gamma}_2^{2\kappa}(f) = \mathbf{\Gamma}_2^{2\kappa}(f)_{\ll} + \mathbf{\Gamma}_2^{2\kappa}(f)_{\perp}$$

w.r.t. $\mathfrak{m}$, $f \in \mathrm{Test}(M)$. A consequence of Proposition 3.2.75 is that

$$\mathbf{\Gamma}_2^{2\kappa}(f)_{\perp} \geq 0$$

and, defining $\gamma_2^{2\kappa}(f) := \mathrm{d}\mathbf{\Gamma}_2^{2\kappa}(f)_{\ll}/\mathrm{d}\mathfrak{m} \in L^1(M)$,

$$\gamma_2^{2\kappa}(f) \geq \frac{1}{N}\,(\Delta f)^2 \quad \mathfrak{m}\text{-a.e.}$$

Further calculus rules of $\mathbf{\Gamma}_2^{2\kappa}$ are summarized in the next Lemma 3.2.76. To this aim, note that $\langle\nabla u, \nabla v\rangle \in \mathscr{F}$ for every $u, v \in \mathrm{Test}(M)$ by Proposition 3.2.75, whence it makes sense define the "pre-Hessian" $\mathrm{H}[\cdot]\colon \mathrm{Test}(M)^3 \to L^2(M)$ by

$$2\,\mathrm{H}[f](g_1, g_2) := \big\langle \nabla g_1, \nabla\langle\nabla f, \nabla g_2\rangle\big\rangle + \big\langle \nabla g_2, \nabla\langle\nabla f, \nabla g_1\rangle\big\rangle \\ - \big\langle \nabla f, \nabla\langle\nabla g_1, \nabla g_2\rangle\big\rangle. \tag{3.2.22}$$

Lemma 3.2.76. *Let $\alpha \in \mathbf{N}$, $q \in \mathrm{Test}(M)^\alpha$ and $\varphi \in \mathrm{C}^\infty(\mathbf{R}^\alpha)$ with $\varphi(0) = 0$. Moreover, given any $i, j \in \{1, \dots, \alpha\}$, set $\varphi_i := \partial_i\varphi$ and $\varphi_{ij} := \partial_i\partial_j\varphi$. Define $\mathbf{A}^{2\kappa}[\varphi\circ q] \in \mathfrak{M}_{\mathrm{f}}^{\pm}(M)_{\mathscr{E}}$ and $\mathrm{B}[\varphi\circ q], \mathrm{C}[\varphi\circ q], \mathrm{D}[\varphi\circ q] \in L^1(M)$ by*

$$\begin{aligned}
\mathbf{A}^{2\kappa}[\varphi\circ q] &:= \sum_{i,j=1}^{\alpha} \big[\varphi_i \circ \widetilde{q}\big]\,\big[\varphi_j \circ \widetilde{q}\big]\,\mathbf{\Gamma}_2^{2\kappa}(q_i, q_j),\\
\mathrm{B}[\varphi\circ q] &:= 2\sum_{i,j,k=1}^{\alpha} \big[\varphi_i \circ q\big]\,\big[\varphi_{jk} \circ q\big]\,\mathrm{H}[q_i](q_j, q_k),\\
\mathrm{C}[\varphi\circ q] &:= \sum_{i,j,k,l=1}^{\alpha} \big[\varphi_{ik} \circ q\big]\,\big[\varphi_{jl} \circ q\big]\,\langle\nabla q_i, \nabla q_j\rangle\,\langle\nabla q_k, \nabla q_l\rangle,\\
\mathrm{D}[\varphi\circ q] &:= \Big[\sum_{i=1}^{\alpha} \big[\varphi_i \circ q\big]\,\Delta q_i + \sum_{i,j=1}^{\alpha} \big[\varphi_{ij} \circ q\big]\,\langle\nabla q_i, \nabla q_j\rangle\Big]^2.
\end{aligned}$$

Then we have the identities

$$\begin{aligned}
\mathbf{\Gamma}_2^{2\kappa}(\varphi\circ q) &= \mathbf{A}^{2\kappa}[\varphi\circ q] + \big[\mathrm{B}[\varphi\circ q] + \mathrm{C}[\varphi\circ q]\big]\,\mathfrak{m},\\
\big[\Delta(\varphi\circ q)\big]^2\,\mathfrak{m} &= \mathrm{D}[\varphi\circ q]\,\mathfrak{m}.
\end{aligned}$$

Remark 3.2.77. Note that all measures in Lemma 3.2.76 are identified as *finite*. This technical point is shown in Proposition 3.2.79 below. Albeit the latter is an *a posteriori* consequence of $\mathrm{BE}_1(\kappa,\infty)$, see also Remark 3.2.80, the results of [ER+20, Lem. 6.5, Thm. 6.6, Thm. 6.9] — in particular Lemma 3.2.76 — are still deducible *a priori* from Assumption 3.2.65 by restriction of the identities from [ER+20, Lem. 6.5, Thm. 6.6] to subsets of finite measure, or interpreting the asserted identities in a weak sense. ■

Finiteness of total variations The final goal of this subsection is to prove in Proposition 3.2.79 that $\|\mathbf{\Delta}^{2\kappa}|\nabla f|^2\|_{\mathrm{TV}} < \infty$, $f \in \mathrm{Test}(M)$. Besides the technical Remark 3.2.77, this fact will be of decisive help in *continuously* extending the κ-Ricci measure beyond regular vector fields, see Theorem 3.6.9.

The following is a minor variant of [ER+20, Lem. 6.2] with potential κ instead of 2κ, proven in a completely analogous way.

Lemma 3.2.78. *Let $u \in L^2(M) \cap L^\infty(M)$ be nonnegative, and let $g \in L^2(M)$. Suppose that for every nonnegative $\phi \in \mathscr{D}(\Delta^\kappa) \cap L^\infty(M)$ with $\Delta^\kappa\phi \in L^\infty(M)$,*

$$\int_M u\,\Delta^\kappa\phi\,\mathrm{d}\mathfrak{m} \geq -\int_M g\,\phi\,\mathrm{d}\mathfrak{m}.$$

Then $u \in \mathscr{F}$ as well as

$$\mathscr{E}^\kappa(u) \leq \int_M u\,g\,\mathrm{d}\mathfrak{m}.$$

Moreover, there exists a unique measure $\boldsymbol{\sigma} \in \mathfrak{M}^+_\sigma(M)_{\mathscr{E}}$ such that for every $h \in \mathscr{F}$, we have $\widetilde{h} \in L^1(M, \boldsymbol{\sigma})$ and

$$\int_M \widetilde{h}\, \mathrm{d}\boldsymbol{\sigma} = -\mathscr{E}^\kappa(h,u) + \int_M h\, g\, \mathrm{d}\mathfrak{m}.$$

Proposition 3.2.79. *For every $f \in \mathrm{Test}(M)$, $|\nabla f|$ belongs to $\mathscr{F}_\mathrm{b}$, and the signed Borel measure $\boldsymbol{\Delta}^{2\kappa}|\nabla f|^2$ has finite total variation. Moreover,*

$$\begin{aligned} \boldsymbol{\Gamma}_2^{2\kappa}(f)[M] &= \int_M (\Delta f)^2\, \mathrm{d}\mathfrak{m} - \big\langle \kappa \,\big|\, |\nabla f|^2 \big\rangle, \\ \boldsymbol{\Delta}^{2\kappa}|\nabla f|^2[M] &= -2\,\big\langle \kappa \,\big|\, |\nabla f|^2 \big\rangle. \end{aligned} \tag{3.2.23}$$

Proof. By Proposition 3.2.75, we already know that $|\nabla f|^2 \in \mathscr{D}(\boldsymbol{\Delta}^{2\kappa})$. Now recall that by the self-improvement property of $\mathrm{BE}_2(\kappa, N)$, $(M, \mathscr{E}, \mathfrak{m})$ obeys $\mathrm{BE}_1(\kappa, \infty)$ according to Definition 3.2.63 [ER+20, Thm. 6.9]. By Lemma 3.2.78 applied to $u := |\nabla f| \in L^2(M) \cap L^\infty(M)$ and $g := -1_{\{|\nabla f|>0\}} \langle \nabla f, \nabla \Delta f\rangle\, |\nabla f|^{-1} \in L^2(M)$ as well as by (3.2.19), we obtain $|\nabla f| \in \mathscr{F}_\mathrm{b}$ and the unique existence of an element $\boldsymbol{\sigma} \in \mathfrak{M}^+_{\sigma\mathrm{R}}(M)_{\mathscr{E}}$ such that for every $h \in \mathscr{F}$, we have $\widetilde{h} \in L^1(M, \boldsymbol{\sigma})$ and

$$\int_M \widetilde{h}\, \mathrm{d}\boldsymbol{\sigma} = -\mathscr{E}^\kappa\big(h, |\nabla f|\big) - \int_{\{|\nabla f|>0\}} h\, \big\langle \nabla f, \nabla \Delta f \big\rangle\, |\nabla f|^{-1}\, \mathrm{d}\mathfrak{m}. \tag{3.2.24}$$

Inserting $h := \phi\, |\nabla f|$ for $\phi \in \mathscr{F}_\mathrm{b}$ in (3.2.24) yields

$$\int_M \widetilde{\phi}\, |\nabla f|_\sim\, \mathrm{d}\boldsymbol{\sigma} = -\mathscr{E}^\kappa\big(\phi\, |\nabla f|, |\nabla f|\big) - \int_M \phi\, \big\langle \nabla f, \nabla \Delta f \big\rangle\, \mathrm{d}\mathfrak{m}.$$

Hence for such ϕ and using the Definition 3.2.74 of $\boldsymbol{\Delta}^{2\kappa}$,

$$\begin{aligned} \int_M \widetilde{\phi}\, \mathrm{d}\boldsymbol{\Delta}^{2\kappa}|\nabla f|^2 &= -\mathscr{E}^{2\kappa}\big(\phi, |\nabla f|^2\big) \\ &= -2 \int_M |\nabla f|\, \big\langle \nabla \phi, \nabla |\nabla f| \big\rangle\, \mathrm{d}\mathfrak{m} - 2\, \big\langle \kappa \,\big|\, \phi\, |\nabla f|^2 \big\rangle \\ &= -2\, \mathscr{E}^\kappa\big(\phi\, |\nabla f|, |\nabla f|\big) + 2 \int_M \phi\, \big|\nabla |\nabla f|\big|^2\, \mathrm{d}\mathfrak{m} \\ &= 2 \int_M \widetilde{\phi}\, |\nabla f|_\sim\, \mathrm{d}\boldsymbol{\sigma} + 2 \int_M \phi\, \big\langle \nabla f, \nabla \Delta f \big\rangle\, \mathrm{d}\mathfrak{m} \\ &\qquad + 2 \int_M \phi\, \big|\nabla |\nabla f|\big|^2\, \mathrm{d}\mathfrak{m}. \end{aligned}$$

Since $\phi \in \mathscr{F}_\mathrm{b}$ is arbitrary, we get

$$\boldsymbol{\Delta}^{2\kappa}|\nabla f|^2 = 2\, |\nabla f|_\sim\, \boldsymbol{\sigma} + 2\, \big\langle \nabla f, \nabla \Delta f \big\rangle\, \mathfrak{m} + 2\, \big|\nabla |\nabla f|\big|^2\, \mathfrak{m}. \tag{3.2.25}$$

Indeed, the r.h.s. is well-defined since $\langle \nabla f, \nabla \Delta f\rangle\, \mathfrak{m}$ and $\big|\nabla|\nabla f|\big|^2\, \mathfrak{m}$ define (signed) Borel measures of finite total variation. Setting $h := |\nabla f|$ in (3.2.24) implies that $|\nabla f|_\sim\, \boldsymbol{\sigma}$ is finite as well, whence $\boldsymbol{\Delta}^{2\kappa}|\nabla f|^2$ is of finite total variation.

Finally, the second identity from (3.2.23) follows from combining (3.2.25) with (3.2.24) for $h := |\nabla f|$, which in turn gives the first identity by the definition (3.2.21). □

Remark 3.2.80. On $\mathrm{RCD}(K,\infty)$ spaces $(M,\mathsf{d},\mathfrak{m})$, $K \in \mathbf{R}$, according to Example 3.2.14 the argument for the finiteness of $\|\mathbf{\Delta}^{2K}|\nabla f|^2\|_{\mathrm{TV}}$ is more straightforward: it follows by conservativeness of the heat flow $(\mathsf{P}_t)_{t\geq 0}$ [AGS14a, Thm. 4.20] and does not require the detour over the $\mathrm{BE}_1(K,\infty)$ condition [Sav14, Lem. 2.6]. In the current chapter, conservativeness is *neither* assumed nor generally a *consequence* of Definition 3.2.63. ■

Remark 3.2.81 (Caveat)**.** The relation (3.2.24) suggests to derive that $|\nabla f| \in \mathscr{D}(\mathbf{\Delta}^\kappa)$, with $\mathbf{\Delta}^\kappa$ defined appropriately as in Definition 3.2.74, with

$$\mathbf{\Delta}^\kappa|\nabla f| = \boldsymbol{\sigma} + \big\langle \nabla f, \nabla\Delta f\big\rangle\,|\nabla f|^{-1}\,\mathfrak{m}.$$

However, it is not clear if the r.h.s. defines an element of $\mathfrak{M}^{\pm}_{\sigma\mathrm{R}}(M)$ since neither the summands on the r.h.s. typically define finite measures, nor we really know whether the last summand is a signed measure (it might take both the value ∞ and $-\infty$).

The situation changes when treating the L^1-Bochner inequality for test vector fields, see Theorem 3.6.21 below. ■

3.2.8 Lebesgue spaces and test objects

This subsection is a survey over the definition of the spaces $L^p(T^*M)$ and $L^p(TM)$, $p \in [1,\infty]$, of p-integrable (co-)tangent vector fields w.r.t. $\mathfrak{m}$.

The L^0-modules $L^0(T^*M)$ and $L^0(TM)$ Let $L^0(T^*M)$ and $L^0(TM)$ be the L^0-modules as in Subsection 3.2.3 associated to $L^2(T^*M)$ and $L^2(TM)$, i.e.

$$\begin{aligned}L^0(T^*M) &:= L^2(T^*M)^0,\\ L^0(TM) &:= L^2(TM)^0.\end{aligned}$$

The characterization of Cauchy sequences in these spaces [Gig18, p. 31] grants that the pointwise norms $|\cdot|\colon L^2(T^*M) \to L^2(M)$ and $|\cdot|\colon L^2(TM) \to L^2(M)$ as well as the musical isomorphisms $\flat\colon L^2(TM) \to L^2(T^*M)$ and $\sharp\colon L^2(T^*M) \to L^2(TM)$ uniquely extend to (non-relabeled) continuous maps $|\cdot|\colon L^0(T^*M) \to L^0(M)$, $|\cdot|\colon L^0(TM) \to L^0(M)$, $\flat\colon L^0(TM) \to L^0(T^*M)$ and $\sharp\colon L^0(T^*M) \to L^0(M)$. (And the latter two will restrict to pointwise isometric module isomorphisms between the respective L^p-spaces, $p \in [1,\infty]$, introduced below.)

The Lebesgue spaces $L^p(T^*M)$ and $L^p(TM)$ For $p \in [1,\infty]$, let $L^p(T^*M)$ and $L^p(TM)$ be the Banach spaces consisting of all $\omega \in L^0(T^*M)$ and $X \in L^0(TM)$ such that $|\omega| \in L^p(M)$ and $|X| \in L^p(M)$, respectively, endowed with the norms

$$\begin{aligned}\|\omega\|_{L^p(T^*M)} &:= \big\||\omega|\big\|_{L^p(M)},\\ \|X\|_{L^p(TM)} &:= \big\||X|\big\|_{L^p(M)}.\end{aligned}$$

Since by Theorem 3.2.32, $L^2(T^*M)$ is separable — and so is $L^2(TM)$ by Proposition 3.2.19 — one easily derives that if $p < \infty$, the spaces $L^p(T^*M)$ and $L^p(TM)$ are separable as well. Since $L^2(T^*M)$ and $L^2(TM)$ are reflexive as Hilbert spaces, by the duality discussion from Subsection 3.2.3 it follows that $L^p(T^*M)$ and $L^p(TM)$ are reflexive for every $p \in [1,\infty]$, and that for $q \in [1,\infty]$ such that $1/p + 1/q = 1$, in the sense of L^∞-modules we have the duality

$$L^p(T^*M)^* = L^q(TM).$$

Test and regular objects As in [Gig18, p. 102], using Lemma 3.2.73 we see that the linear span of all elements of the form $h\,\nabla g$, $g \in \mathrm{Test}_{L^\infty}(M)$ and $h \in \mathrm{Test}(M)$, is weakly* dense in $L^\infty(TM)$. This motivates to consider the subsequent subclasses of $L^2(TM)$ consisting of *test vector fields* or *regular vector fields*, respectively:

$$\begin{aligned}
\mathrm{Test}_{L^\infty}(TM) &:= \Big\{\sum_{i=1}^n g_i\,\nabla f_i : n \in \mathbf{N},\ f_i, g_i \in \mathrm{Test}_{L^\infty}(M)\Big\},\\
\mathrm{Test}(TM) &:= \Big\{\sum_{i=1}^n g_i\,\nabla f_i : n \in \mathbf{N},\ f_i, g_i \in \mathrm{Test}(M)\Big\},\\
\mathrm{Reg}(TM) &:= \Big\{\sum_{i=1}^n g_i\,\nabla f_i : n \in \mathbf{N},\ f_i \in \mathrm{Test}(M),\ g_i \in \mathrm{Test}(M) \cup \mathbf{R}\,1_M\Big\}.
\end{aligned}$$

Remark 3.2.82. These three classes play different roles in the sequel. $\mathrm{Test}_{L^\infty}(TM)$ is just needed for technical reasons when some second order L^∞-control is required. $\mathrm{Test}(TM)$ is usually the class of vector fields w.r.t. which certain objects are defined by testing against, while $\mathrm{Reg}(TM)$ is the typical class of vector fields for which such objects *are* defined. We make this distinction between $\mathrm{Test}(TM)$ and $\mathrm{Reg}(TM)$, which has not been done in [Gig18], for the reason that we want to include both vector fields with "regular" zeroth order part as well as pure gradient vector fields for differential objects such as the covariant derivative, see Theorem 3.4.3, or the exterior differential, see Theorem 3.5.5. However, under the usual closures that we take below, it is not clear if gradient vector fields belong to those w.r.t. test rather than regular objects. Compare with Remark 3.4.7. ■

Of course $\mathrm{Test}_{L^\infty}(TM) \subset \mathrm{Test}(TM), \mathrm{Test}_{L^\infty}(TM) \subset L^1(TM) \cap L^\infty(TM)$, while merely $\mathrm{Reg}(TM) \subset L^2(TM) \cap L^\infty(TM)$. By Lemma 3.2.54, we have $\mathrm{Test}(TM) \subset \mathscr{D}_{\mathrm{TV}}(\mathbf{div}) \cap \mathscr{D}(\mathrm{div})$ — as well as $\mathbf{n}\,X = 0$ for every $X \in \mathrm{Test}(TM)$ — while only $\mathrm{Reg}(TM) \subset \mathscr{D}(\mathrm{div})$. By Theorem 3.2.32 and Proposition 3.2.19, all classes are dense in $L^p(TM)$, $p \in [1,\infty)$, and weakly* dense in $L^\infty(TM)$. Furthermore, we set

$$\begin{aligned}
\mathrm{Test}_{L^\infty}(TM) &:= \mathrm{Test}_{L^\infty}(TM)^\flat,\\
\mathrm{Test}(T^*M) &:= \mathrm{Test}(TM)^\flat,\\
\mathrm{Reg}(T^*M) &:= \mathrm{Reg}(TM)^\flat.
\end{aligned}$$

Lebesgue spaces on tensor products Denote the two-fold tensor products of $L^2(T^*M)$ and $L^2(TM)$, respectively, in the sense outlined in Subsection 3.2.3 by

$$\begin{aligned}
L^2((T^*)^{\otimes 2}M) &:= L^2(T^*M)^{\otimes 2},\\
L^2(T^{\otimes 2}M) &:= L^2(TM)^{\otimes 2}.
\end{aligned}$$

By the tensor product discussion from Subsection 3.2.3, Theorem 3.2.32 and Proposition 3.2.19, both are separable Hilbert modules. They are pointwise isometrically module isomorphic: the respective pairing is initially defined by

$$(\omega_1 \otimes \omega_2)(X_1 \otimes X_2) := \omega_1(X_1)\,\omega_2(X_2) \quad \mathfrak{m}\text{-a.e.}$$

for $\omega_1, \omega_2 \in L^2(T^*M) \cap L^\infty(T^*M)$ and $X_1, X_2 \in L^2(T^*M) \cap L^\infty(T^*M)$, and is extended by linearity and continuity to $L^2((T^*)^{\otimes 2}M)$ and $L^2(T^{\otimes 2}M)$, respectively. By a slight abuse of notation, this pairing, with Proposition 3.2.19, induces the musical isomorphisms $\flat\colon L^2(T^{\otimes 2}M) \to L^2((T^*)^{\otimes 2}M)$ and $\sharp := \flat^{-1}$ given by

$$A^\sharp : T := A(T) =: A : T^\flat \quad \mathfrak{m}\text{-a.e.}$$

We let $L^p((T^*)^{\otimes 2}M)$ and $L^p(T^{\otimes 2}M)$, $p \in \{0\} \cup [1,\infty]$, be defined similarly to Subsection 3.2.8. For $p \in [1,\infty]$, these spaces naturally become Banach which, if $p < \infty$, are separable.

Lastly, we define the subsequent L^p-dense sets, $p \in [1,\infty]$, intended strongly if $p < \infty$ and weakly* if $p = \infty$, reminiscent of Subsection 3.2.3:

$$\begin{aligned}
\mathrm{Test}_{L^\infty}((T^*)^{\otimes 2}M) &:= \mathrm{Test}_{L^\infty}(T^*M)^{\odot 2},\\
\mathrm{Test}_{L^\infty}(T^{\otimes 2}M) &:= \mathrm{Test}_{L^\infty}(TM)^{\odot 2},\\
\mathrm{Test}((T^*)^{\otimes 2}M) &:= \mathrm{Test}(T^*M)^{\odot 2},\\
\mathrm{Test}(T^{\otimes 2}M) &:= \mathrm{Test}(TM)^{\odot 2},\\
\mathrm{Reg}((T^*)^{\otimes 2}M) &:= \mathrm{Reg}(T^*M)^{\odot 2},\\
\mathrm{Reg}(T^{\otimes 2}M) &:= \mathrm{Reg}(TM)^{\odot 2}.
\end{aligned}$$

Lebesgue spaces on exterior products Given any $k \in \mathbf{N}_0$, we set

$$\begin{aligned}
L^2(\Lambda^k T^*M) &:= \Lambda^k L^2(T^*M),\\
L^2(\Lambda^k TM) &:= \Lambda^k L^2(TM),
\end{aligned}$$

where the exterior products are intended as in Subsection 3.2.3. For $k \in \{0,1\}$, we employ the consistent interpretations

$$\begin{aligned}
L^2(\Lambda^1 T^*M) &:= L^2(T^*M),\\
L^2(\Lambda^1 TM) &:= L^2(TM),\\
L^2(\Lambda^0 T^*M) &:= L^2(\Lambda^0 TM) := L^2(M).
\end{aligned}$$

By Subsection 3.2.3, these are naturally Hilbert modules. Analogously to the previous paragraph, $L^2(\Lambda^k T^*M)$ and $L^2(\Lambda^k TM)$ are pointwise isometrically module isomorphic. For brevity, the induced pointwise pairing between $\omega \in L^2(\Lambda^k T^*M)$ and $X_1 \wedge \dots X_k \in L^2(\Lambda^k TM)$, $X_1, \dots, X_k \in L^2(TM) \cap L^\infty(TM)$, is written

$$\omega(X_1, \dots, X_k) := \omega(X_1 \wedge \dots \wedge X_k).$$

We let $L^p(\Lambda^k T^*M)$ and $L^p(\Lambda^k TM)$, $p \in \{0\} \cup [1,\infty]$, be defined similarly to $L^p(T^*M)$ and $L^p(TM)$, respectively. For $p \in [1,\infty]$, these spaces are Banach and, if $p < \infty$, additionally separable.

Denote the formal k-th exterior products, $k \in \mathbf{N}_0$, of the above test and regular classes through

$$\begin{aligned}
\mathrm{Test}_{L^\infty}(\Lambda^k T^*M) &:= \Big\{ \sum_{i=1}^n f_i^0 \,\mathrm{d} f_i^1 \wedge \dots \wedge \mathrm{d} f_i^k : n \in \mathbf{N},\ f_i^j \in \mathrm{Test}_{L^\infty}(M) \Big\},\\
\mathrm{Test}_{L^\infty}(\Lambda^k TM) &:= \Big\{ \sum_{i=1}^n f_i^0 \,\nabla f_i^1 \wedge \dots \wedge \nabla f_i^k : n \in \mathbf{N},\ f_i^j \in \mathrm{Test}_{L^\infty}(M) \Big\},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Test}(\Lambda^k T^*M) &:= \Big\{\sum_{i=1}^n f_i^0\, \mathrm{d}f_i^1 \wedge \cdots \wedge \mathrm{d}f_i^k : n \in \mathbf{N},\ f_i^j \in \mathrm{Test}(M)\Big\},\\
\mathrm{Test}(\Lambda^k TM) &:= \Big\{\sum_{i=1}^n f_i^0\, \nabla f_i^1 \wedge \cdots \wedge \nabla f_i^k : n \in \mathbf{N},\ f_i^j \in \mathrm{Test}(M)\Big\}\\
\mathrm{Reg}(\Lambda^k T^*M) &:= \Big\{\sum_{i=1}^n f_i^0\, \mathrm{d}f_i^1 \wedge \cdots \wedge \mathrm{d}f_i^k : n \in \mathbf{N},\ f_i^j \in \mathrm{Test}(M),\\
&\qquad\qquad f_i^0 \in \mathrm{Test}(M) \cup \mathbf{R}\,1_M\Big\},\\
\mathrm{Reg}(\Lambda^k TM) &:= \Big\{\sum_{i=1}^n f_i^0\, \nabla f_i^1 \wedge \cdots \wedge \nabla f_i^k : n \in \mathbf{N},\ f_i^j \in \mathrm{Test}(M),\\
&\qquad\qquad f_i^0 \in \mathrm{Test}(M) \cup \mathbf{R}\,1_M\Big\}.
\end{aligned}$$

We employ the evident interpretations for $k = 1$, while the respective spaces for $k = 0$ are identified with those spaces to which their generic elements's zeroth order terms belong to. These classes are dense in their respective L^p-spaces, $p \in [1, \infty]$ — strongly if $p < \infty$, and weakly* if $p = \infty$.

3.3 Hessian

3.3.1 The Sobolev space $\mathscr{D}(\mathrm{Hess})$

Now we define the key object of our second order differential structure, namely the *Hessian* of suitable functions $f \in \mathscr{F}$. We choose an integration by parts procedure as in [Gig18, Subsec. 3.3.1], motivated by the subsequent Riemannian example.

Example 3.3.1. Let M be a Riemannian manifold with boundary. The metric compatibility of ∇ allows us to rephrase the definition of the Hessian $\mathrm{Hess}\, f \in \Gamma((T^*)^{\otimes 2}M)$ of a function $f \in \mathrm{C}^\infty(M)$ pointwise as

$$\begin{aligned}
2\,\mathrm{Hess}\, f(\nabla g_1, \nabla g_2) &= 2\,\big\langle \nabla_{\nabla g_1} \nabla f, \nabla g_2\big\rangle\\
&= \big\langle \nabla g_1, \nabla\langle \nabla f, \nabla g_2\rangle\big\rangle + \big\langle \nabla g_2, \nabla\langle \nabla f, \nabla g_1\rangle\big\rangle\\
&\qquad - \big\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\big\rangle
\end{aligned}\tag{3.3.1}$$

for every $g_1, g_2 \in \mathrm{C}_\mathrm{c}^\infty(M)$, see e.g. [Pet06, p. 28]. The first equality ensures that $\mathrm{Hess}\, f$ is C^∞-linear in both components. Thus, since smooth gradient vector fields locally generate TM, the second equality characterizes the Hessian of f.

We now restrict our attention to those g_1 and g_2 whose derivatives constitute *Neumann vector fields*, i.e.

$$\langle \nabla g_1, \mathsf{n}\rangle = \langle \nabla g_2, \mathsf{n}\rangle = 0 \quad \text{on } \partial M, \tag{3.3.2}$$

e.g. to $g_1, g_2 \in \mathrm{C}_\mathrm{c}^\infty(M^\circ)$. Multiply (3.3.1) by a function $h \in \mathrm{C}_\mathrm{c}^\infty(M)$ and integrate (by parts). In this case, recall that $h\,\nabla g_1, h\,\nabla g_2 \in \mathscr{D}_{\mathrm{TV}}(\mathbf{div}) \cap \mathscr{D}(\mathrm{div})$ with

$$\begin{aligned}
\mathbf{div}(h\,\nabla g_1) &= \mathrm{div}_\mathfrak{v}(h\,\nabla g_1)\,\mathfrak{v},\\
\mathbf{div}(h\,\nabla g_2) &= \mathrm{div}_\mathfrak{v}(h\,\nabla g_2)\,\mathfrak{v}
\end{aligned}\tag{3.3.3}$$

by Example 3.2.48. The resulting integral identity reads

$$
\begin{aligned}
2\int_M h\,\mathrm{Hess} f(\nabla g_1,\nabla g_2)\,\mathrm{d}\mathfrak{v}
&= -\int_M \langle\nabla f,\nabla g_2\rangle\,\mathrm{div}_{\mathfrak{v}}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{v} - \int_M \langle\nabla f,\nabla g_1\rangle\,\mathrm{div}_{\mathfrak{v}}(h\,\nabla g_2)\,\mathrm{d}\mathfrak{v}\\
&\quad - \int_M h\,\big\langle\nabla f,\nabla\langle\nabla g_1,\nabla g_2\rangle\big\rangle\,\mathrm{d}\mathfrak{v}.
\end{aligned}
$$

In turn, this integral identity characterizes $\mathrm{Hess} f$ on M° by the arbitrariness of g_1, g_2 and h, and hence on M by the existence of a smooth extension to all of M. ■

Observe that on the r.h.s. of the previous integral identity, no second order expression in f is present. Moreover, as we have already noted in Lemma 3.2.54, $\mathrm{Test}(TM)$ is a large class of vector fields obeying a nonsmooth version of (3.3.2). Next, note that all volume integrals on the r.h.s. — with $\mathfrak{v}$ replaced by $\mathfrak{m}$ — are well-defined for every $f\in\mathscr{F}$ and $g_1,g_2,h\in\mathrm{Test}(M)$. Indeed, the first one exists since $\nabla g_2\in L^\infty(M)$, whence $\langle\nabla f,\nabla g_2\rangle\in L^2(M)$, and since $h\,\nabla g_1\in\mathscr{D}_{\mathrm{TV}}(\mathbf{div})\cap\mathscr{D}(\mathrm{div})$ with $\mathrm{div}(h\,\nabla g_1)\in L^2(M)$ by Lemma 3.2.54. An analogous argument applies for the second volume integral. The last one is well-defined since $\langle\nabla g_1,\nabla g_2\rangle\in\mathscr{F}$ thanks to Proposition 3.2.75 and polarization, and since $h\in L^\infty(M)$.

These observations lead to the following definition.

Definition 3.3.2. *We define the space* $\mathscr{D}(\mathrm{Hess})$ *to consist of all* $f\in\mathscr{F}$ *for which there exists* $A\in L^2((T^*)^{\otimes 2}M)$ *such that for every* $g_1,g_2,h\in\mathrm{Test}(M)$*,*

$$
\begin{aligned}
2\int_M h\,A(\nabla g_1,\nabla g_2)\,\mathrm{d}\mathfrak{m}
&= -\int_M \langle\nabla f,\nabla g_1\rangle\,\mathrm{div}(h\,\nabla g_2)\,\mathrm{d}\mathfrak{m} - \int_M \langle\nabla f,\nabla g_2\rangle\,\mathrm{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{m}\\
&\quad - \int_M h\,\big\langle\nabla f,\nabla\langle\nabla g_1,\nabla g_2\rangle\big\rangle\,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

If such an A *exists, it is unique, denoted by* $\mathrm{Hess} f$ *and termed the* Hessian *of* f*.*

Indeed, given any $f\in\mathscr{D}(\mathrm{Hess})$ there is at most one A as in Definition 3.3.2 by density of $\mathrm{Test}(T^{\otimes 2}M)$ in $L^2(T^{\otimes 2}M)$, since $\mathrm{Test}(M)$ is an algebra. In particular, $\mathscr{D}(\mathrm{Hess})$ is a vector space and Hess is a linear operator on it. Further elementary properties are collected in Theorem 3.3.3.

The space $\mathscr{D}(\mathrm{Hess})$ is endowed with the norm $\|\cdot\|_{\mathscr{D}(\mathrm{Hess})}$ given by

$$
\|f\|^2_{\mathscr{D}(\mathrm{Hess})} := \|f\|^2_{L^2(M)} + \|\mathrm{d}f\|^2_{L^2(T^*M)} + \|\mathrm{Hess} f\|^2_{L^2((T^*)^{\otimes 2}M)}.
$$

Furthermore we define the energy functional $\mathscr{E}_2\colon\mathscr{F}\to[0,\infty]$ by

$$
\mathscr{E}_2(f) := \begin{cases} \displaystyle\int_M \big|\mathrm{Hess} f\big|^2_{\mathrm{HS}}\,\mathrm{d}\mathfrak{m} & \text{if } f\in\mathscr{D}(\mathrm{Hess}),\\ \infty & \text{otherwise.}\end{cases}
$$

Theorem 3.3.3. *The space* $\mathscr{D}(\mathrm{Hess})$*, the Hessian* Hess *and the functional* $\mathscr{E}_2$ *have the following properties.*

(i) $\mathscr{D}(\mathrm{Hess})$ *is a separable Hilbert space w.r.t.* $\|\cdot\|_{\mathscr{D}(\mathrm{Hess})}$.

(ii) *The Hessian is a closed operator on* $\mathscr{D}(\mathrm{Hess})$, *i.e. the image of the map* $\mathrm{Id}\times\mathrm{Hess}\colon \mathscr{D}(\mathrm{Hess}) \to \mathscr{F}\times L^2((T^*)^{\otimes 2}M)$ *is a closed subspace of* $\mathscr{F}\times L^2((T^*)^{\otimes 2}M)$.

(iii) *For every* $f\in\mathscr{D}(\mathrm{Hess})$, *the tensor* $\mathrm{Hess}\, f$ *is symmetric, i.e.*

$$\mathrm{Hess}\, f = (\mathrm{Hess}\, f)^\top$$

according to the definition of the transpose from (3.2.8).

(iv) $\mathscr{E}_2$ *is* $\mathscr{F}$*-lower semicontinuous, and for every* $f\in\mathscr{F}$,

$$\begin{aligned}\mathscr{E}_2(f) = \sup\Big\{ &-\sum_{l=1}^r \int_M \langle\nabla f,\nabla g_l\rangle\,\mathrm{div}(h_l\, h_l'\,\nabla g_l')\,\mathrm{d}\mathfrak{m}\\ &-\sum_{l=1}^r \int_M \langle\nabla f,\nabla g_l'\rangle\,\mathrm{div}(h_l\, h_l'\,\nabla g_l)\,\mathrm{d}\mathfrak{m}\\ &-\sum_{l=1}^r \int_M h_l\, h_l'\,\big\langle\nabla f,\nabla\langle\nabla g_l,\nabla g_l'\rangle\big\rangle\,\mathrm{d}\mathfrak{m}\\ &-\int_M \Big|\sum_{l=1}^r h_l\, h_l'\,\nabla g_l\otimes\nabla g_l'\Big|^2\,\mathrm{d}\mathfrak{m} :\\ &\qquad\qquad r\in\mathbf{N},\ g_r, g_r', h_r, h_r'\in\mathrm{Test}(M)\Big\}.\end{aligned}$$

Proof. Item (ii) follows since the r.h.s. of the defining property of the Hessian in Definition 3.3.2 is continuous in f and A w.r.t. weak convergence in $\mathscr{F}$ and $L^2((T^*)^{\otimes 2}M)$, respectively, for fixed $g_1, g_2, h\in\mathrm{Test}(M)$.

The Hilbert space property of $\mathscr{D}(\mathrm{Hess})$ in (i) is a direct consequence of the completeness of $\mathscr{F}$, (ii) and since $\|\cdot\|_{\mathscr{D}(\mathrm{Hess})}$ trivially satisfies the parallelogram identity. Hence, we are left with the separability of $\mathscr{D}(\mathrm{Hess})$. Since $\mathscr{F}$ and $L^2((T^*)^{\otimes 2}M)$ are separable, their product is a separable Hilbert space w.r.t. the norm $\|\cdot\|$, where

$$\|(f,A)\|^2 := \big\|f\big\|_{\mathscr{F}}^2 + \big\|A\big\|_{L^2((T^*)^{\otimes 2}M)}^2.$$

In particular, $\mathrm{Id}\times\mathrm{Hess}\colon \mathscr{D}(\mathrm{Hess})\to\mathscr{F}\times L^2((T^*)^{\otimes 2}M)$ is a bijective isometry onto its image, whence the claim follows from (ii).

Concerning (iii), setting $h := h_1\, h_2$ with $h_1, h_2\in\mathrm{Test}(M)$, we easily see that for every $g_1, g_2\in\mathrm{Test}(M)$ the r.h.s. of the defining property of $\mathrm{Hess}\, f$, $f\in\mathscr{D}(\mathrm{Hess})$, in Definition 3.3.2 is symmetric in h_1 and h_2 as well as g_1 and g_2, respectively, and it is furthermore bilinear in $h_1\,\nabla g_1$ and $h_2\,\nabla g_2$. Hence, using (3.2.8) we deduce the symmetry of $\mathrm{Hess}\, f$, $f\in\mathscr{D}(\mathrm{Hess})$, on $\mathrm{Test}(T^{\otimes 2}M)$ and hence on all of $L^2(T^{\otimes 2}M)$ by a density argument.

The $\mathscr{F}$-lower semicontinuity of $\mathscr{E}_2$ in (iv) directly follows since bounded subsets of the Hilbert space $L^2((T^*)^{\otimes 2}M)$ are weakly relatively compact, combined with Mazur's lemma and (ii).

We finally turn to the duality formula in (iv).

Let us first prove the inequality "≥", for which we assume without restriction that $f\in\mathscr{D}(\mathrm{Hess})$. By duality of $\mathscr{D}(\mathrm{Hess})$ and its Hilbert space dual $\mathscr{D}(\mathrm{Hess})'$ as well as the density of $\mathrm{Test}(T^{\otimes 2}M)$ in $L^2(T^{\otimes 2}M)$,

$$\mathscr{E}_2(f) = \sup\Big\{2\sum_{i=1}^n \int_M \mathrm{Hess}\, f(X_i, X_i')\,\mathrm{d}\mathfrak{m}$$

$$- \int_M \Big| \sum_{i=1}^n X_i \otimes X_i' \Big|^2 \,\mathrm{d}\mathfrak{m} : n \in \mathbf{N},\ X_i, X_i' \in \mathrm{Test}(TM) \Big\}.$$

Let us write $X_i \otimes X_i' := h_{i1}\, h_{i1}'\, \nabla g_{i1} \otimes \nabla g_{i1}' + \dots + h_{im}\, h_{im}'\, \nabla g_{im} \otimes g_{im}'$ for certain elements $g_{ij}, g_{ij}', h_{ij}, h_{ij}' \in \mathrm{Test}(M)$, $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ with $n, m \in \mathbf{N}$. Then by Definition 3.3.2 and since $\mathrm{Test}(M)$ is an algebra,

$$\begin{aligned}
&2 \sum_{i=1}^n \int_M \mathrm{Hess}\, f(X_i, X_i')\,\mathrm{d}\mathfrak{m} \\
&\qquad = 2 \sum_{i=1}^n \sum_{j=1}^m \int_M h_{i1}\, h_{i1}'\, \mathrm{Hess}\, f(\nabla g_{i1}, \nabla g_{i1}')\,\mathrm{d}\mathfrak{m} \\
&\qquad = - \sum_{i=1}^n \sum_{j=1}^m \int_M \langle \nabla f, \nabla g_{ij}\rangle \,\mathrm{div}(h_{ij}\, h_{ij}'\, \nabla g_{ij}')\,\mathrm{d}\mathfrak{m} \\
&\qquad\qquad - \sum_{i=1}^n \sum_{j=1}^m \int_M \langle \nabla f, \nabla g_{ij}'\rangle \,\mathrm{div}(h_{ij}\, h_{ij}'\, \nabla g_{ij})\,\mathrm{d}\mathfrak{m} \\
&\qquad\qquad - \sum_{i=1}^n \sum_{j=1}^m \int_M h_{ij}\, h_{ij}' \,\big\langle \nabla f, \nabla \langle \nabla g_{ij}, \nabla g_{ij}'\rangle \big\rangle \,\mathrm{d}\mathfrak{m},
\end{aligned}$$

which terminates the proof of "$\geq$".

Turning to "$\leq$" in (iv), we may and will assume without loss of generality that the supremum, henceforth denoted by C, on the r.h.s. of the claimed formula is finite. Consider the operator $\Phi\colon \mathrm{Test}(T^{\otimes 2}M) \to \mathbf{R}$ given by

$$\begin{aligned}
&2\,\Phi \sum_{i=1}^n \sum_{j=1}^m h_{ij}\, h_{ij}'\, \nabla g_{ij} \otimes \nabla g_{ij}' \\
&\qquad := - \sum_{i=1}^n \sum_{j=1}^m \int_M \langle \nabla f, \nabla g_{ij}\rangle \,\mathrm{div}(h_{ij}\, h_{ij}'\, \nabla g_{ij}')\,\mathrm{d}\mathfrak{m} \\
&\qquad\qquad - \sum_{i=1}^n \sum_{j=1}^m \int_M \langle \nabla f, \nabla g_{ij}'\rangle \,\mathrm{div}(h_{ij}\, h_{ij}'\, \nabla g_{ij})\,\mathrm{d}\mathfrak{m} \\
&\qquad\qquad - \sum_{i=1}^n \sum_{j=1}^m \int_M h_{ij}\, h_{ij}' \,\big\langle \nabla f, \nabla \langle \nabla g_{ij}, \nabla g_{ij}'\rangle \big\rangle \,\mathrm{d}\mathfrak{m}.
\end{aligned} \tag{3.3.4}$$

The value of $\Phi(T)$ is independent of the particular way of writing $T \in \mathrm{Test}(T^{\otimes 2}M)$. Indeed, if $T = 0$ but $\Phi(T) \neq 0$, letting $\lambda \to \infty$ in the identity $\Phi(\lambda\, T)\ \mathrm{sgn}\,\Phi(T) = \Phi(T)\,\lambda\ \mathrm{sgn}\,\Phi(T)$ implied by (3.3.4) would contradict the assumption that $C < \infty$. The map Φ is thus well-defined, it is linear, and for every $T \in \mathrm{Test}(T^{\otimes 2}M)$,

$$2\,\Phi(T) \leq C + \big\|T\big\|^2_{L^2(T^{\otimes 2}M)}.$$

Replacing T by $\lambda\, T$ and optimizing over $\lambda \in \mathbf{R}$ gives

$$|\Phi(T)| \leq \sqrt{C}\, \|T\|_{L^2(T^{\otimes 2}M)} \tag{3.3.5}$$

for every $T \in \mathrm{Test}(T^{\otimes 2}M)$. Hence, Φ uniquely induces a (non-relabeled) element of the Hilbert space dual $L^2(T^{\otimes 2}M)'$ of $L^2(T^{\otimes 2}M)$. By Proposition 3.2.19, we find a unique element $A' \in L^2((T^*)^{\otimes 2}M)$ such that

$$\Phi(T) = \int_M A'(T)\,\mathrm{d}\mathfrak{m}$$

for every $T \in L^2(T^{\otimes 2}M)$. Now Lemma 3.2.72, Lemma 3.2.54 as well as the continuity of Φ allow us to replace the terms $h_{ij}\,h'_{ij}$ by arbitrary elements $k_{ij} \in \mathrm{Test}(M)$, still retaining the identity (3.3.4) with k_{ij} in place of $h_{ij}\,h'_{ij}$, $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m\}$. In particular, by Definition 3.3.2, we deduce that $f \in \mathscr{D}(\mathrm{Hess})$ and $A' = \mathrm{Hess}\, f$. By Proposition 3.2.19 again and (3.3.5), we obtain

$$\|\mathrm{Hess}\, f\|_{L^2((T^*)^{\otimes 2}M)} = \|\Phi\|_{L^2(T^{\otimes 2}M)'} \leq \sqrt{C},$$

which is precisely what was left prove. □

Remark 3.3.4. If $\mathscr{E}_2$ is extended to $L^2(M)$ by $\mathscr{E}_2(f) := \infty$ for $f \in L^2(M) \setminus \mathscr{F}$, it is unclear if the resulting functional is L^2-lower semicontinuous. To bypass this issue in applications, one might instead use that by Theorem 3.3.3, the functional $\mathscr{E}_2^\varepsilon\colon L^2(M) \to [0,\infty]$ given by

$$\mathscr{E}_2^\varepsilon(f) := \begin{cases} \varepsilon \displaystyle\int_M |\nabla f|^2 \,\mathrm{d}\mathfrak{m} + \int_M \big|\mathrm{Hess}\, f\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} & \text{if } f \in \mathscr{D}(\mathrm{Hess}),\\ \infty & \text{otherwise} \end{cases}$$

is L^2-lower semicontinuous for every $\varepsilon > 0$.

If M is, say, a compact Riemannian manifold without boundary, one can easily prove using the Bochner identity that the (nonpositive) generator associated with $\mathscr{E}_2^\varepsilon$ [FOT11, Thm. 1.3.1] is the Paneitz-type operator

$$-\Delta^2 + \varepsilon\,\Delta f + \mathrm{div}(\mathrm{Ric}^\flat\,\nabla\cdot). \qquad \blacksquare$$

Remark 3.3.5. In general, the Hessian is not the trace of the Laplacian in the sense of (3.2.12). This already happens on weighted Riemannian manifolds without boundary: of course, the associated Laplacian Δ is defined by partial integration w.r.t. the reference measure [Gri09, Sec. 3.6], while the definition of Hessian only depends on the metric tensor. See also the second part of Example 3.2.48. Examples of abstract spaces for which this *is* the case — and which currently enjoy high research interest [BNS20, DPG18, HZ20] — are *noncollapsed* $\mathrm{RCD}(K,N)$ spaces, $K \in \mathbf{R}$ and $N \in [1,\infty)$ [DPG18, Thm. 1.12]. See also Remark 3.3.16 below. ■

Remark 3.3.6. In line with Remark 3.3.5, although a priori $\mathfrak{m}$ plays a role in Definition 3.3.2, we expect the Hessian to only depend on conformal transformations of $\langle\cdot,\cdot\rangle$, but not on drift transformations of $\mathfrak{m}$. For instance, this is known on $\mathrm{RCD}^*(K,N)$ spaces, $K \in \mathbf{R}$ and $N \in [1,\infty)$, see e.g. [Han19, Prop. 3.11] or [HS21, Lem. 2.16], and it does not seem hard to adapt the arguments from [Han19] to more general settings. ■

Remark 3.3.7. As an alternative to Definition 3.3.2, one can define $\mathscr{D}(\mathrm{Hess})$ as the finiteness domain of the r.h.s. of the duality formula in (iv) in Theorem 3.3.11. The Hessian of $f \in \mathscr{D}(\mathrm{Hess})$ is then well-defined by the same duality arguments as in the proof of Theorem 3.3.11. ■

3.3.2 Existence of many functions in $\mathscr{D}(\mathrm{Hess})$

Up to now, we still do not know whether $\mathscr{D}(\mathrm{Hess})$ is nonempty. The ultimate goal of this subsection is to prove that $\mathrm{Test}(M) \subset \mathscr{D}(\mathrm{Hess})$ in Theorem 3.3.11, whence $\mathscr{D}(\mathrm{Hess})$ is even dense in $L^2(M)$.

The strategy is reliant on [Gig18, Subsec. 3.3.2], which has itself been inspired by the "self-improvement" works [Bak85, BE85], see [Gig18, Rem. 3.3.10] and also

[ER⁺20, Sav14, Stu18a]. The key technical part (not only for Theorem 3.3.11, but in fact for Theorem 3.6.9 below as well) is contained in Lemma 3.3.9, where — loosely speaking and up to introducing the relevant objects later — we show that

$$|\nabla X : T|^2 \leq \Big[\Delta^{2\kappa}\frac{|X|^2}{2} + \big\langle X, (\vec{\Delta} X^\flat)^\sharp\big\rangle - \big|(\nabla X)_{\mathrm{asym}}\big|_{\mathrm{HS}}^2\Big]\,\big|T\big|_{\mathrm{HS}}^2 \quad \mathfrak{m}\text{-a.e.}$$

for $X, T \in \mathrm{Test}(TM)$. Of course, neither we introduced the covariant derivative ∇, Definition 3.4.2 or the Hodge Laplacian $\vec{\Delta}$, Definition 3.5.21, yet, nor in general we have $|X|^2 \in \mathscr{D}(\Delta^{2\kappa})$ for $X \in \mathrm{Test}(TM)$. Reminiscent of Proposition 3.2.75 and [ER⁺20, Cor. 6.3], we instead rephrase the above inequality in terms of measures, and the involved objects ∇X and $\vec{\Delta} X^\flat$ therein as the "r.h.s.'s of the identities one would expect for ∇X and $\vec{\Delta} X^\flat$ for $X \in \mathrm{Test}(M)$", rigorously proven in Theorem 3.4.3 and Lemma 3.6.1 below. In particular, by optimization over $T \in \mathrm{Test}(TM)$,

$$\big|(\nabla X)_{\mathrm{sym}}\big|_{\mathrm{HS}}^2 \leq \Delta^{2\kappa}\frac{|X|^2}{2} + \big\langle X, (\vec{\Delta} X^\flat)^\sharp\big\rangle - \big|(\nabla X)_{\mathrm{asym}}\big|_{\mathrm{HS}}^2 \quad \mathfrak{m}\text{-a.e.},$$

which is the Bochner inequality for vector fields according to (3.2.9). For $X := \nabla f$, $f \in \mathrm{Test}(M)$, this essentially provides Theorem 3.3.11. Details about this inequality for general $X \in \mathrm{Reg}(TM)$, leading to Theorem 3.6.9, are due to Lemma 3.6.2.

We start with a technical preparation. Given $\mu, \nu \in \mathfrak{M}_{\mathrm{f}}^+(M)$, we define the Borel measure $\sqrt{\mu\,\nu} \in \mathfrak{M}_{\mathrm{f}}^+(M)$ as follows. Let $\iota \in \mathfrak{M}_{\mathrm{f}}^+(M)$ with $\mu \ll \iota$ and $\nu \ll \iota$ be arbitrary, denote the respective densities w.r.t. ι by $f, g \in L^1(M, \iota)$, and set

$$\sqrt{\mu\,\nu} := \sqrt{f\,g}\,\iota.$$

For instance, one can choose $\iota := |\mu| + |\nu|$ [Hal50, Thm. 30.A] — in fact, the previous definition is independent of the choice of ι, whence $\sqrt{\mu\,\nu}$ is well-defined.

The following important measure theoretic lemma is due to [Gig18, Lem. 3.3.6].

Lemma 3.3.8. *Let $\mu_1, \mu_2, \mu_3 \in \mathfrak{M}_{\mathrm{f}}^\pm(M)$ satisfy the inequality*

$$\lambda^2\,\mu_1 + 2\lambda\,\mu_2 + \mu_3 \geq 0$$

for every $\lambda \in \mathbf{R}$. Then the following properties hold.

(i) *The elements μ_1 and μ_3 are nonnegative, and*

$$|\mu_2| \leq \sqrt{\mu_1\,\mu_3}.$$

(ii) *We have $\mu_2 \ll \mu_1$, $\mu_2 \ll \mu_3$ and*

$$\|\mu_2\|_{\mathrm{TV}} \leq \sqrt{\|\mu_1\|_{\mathrm{TV}}\,\|\mu_3\|_{\mathrm{TV}}}.$$

(iii) *The $\mathfrak{m}$-singular parts $(\mu_1)_\perp$ and $(\mu_3)_\perp$ of μ_1 and μ_3 are nonnegative. Moreover, expressing the densities of the $\mathfrak{m}$-absolutely continuous parts of μ_i by $\rho_i := \mathrm{d}(\mu_i)_\ll/\mathrm{d}\mathfrak{m} \in L^1(M)$, $i \in \{1,2,3\}$, we have*

$$|\rho_2|^2 \leq \rho_1\,\rho_3 \quad \mathfrak{m}\text{-}a.e.$$

In the subsequent lemma, all terms where N' is infinite are interpreted as being zero. Similar proofs can be found in [Bra20, Gig18, Han18a].

Lemma 3.3.9. *Let $N' \in [N, \infty]$, $n, m \in \mathbf{N}$, $f, g \in \mathrm{Test}(M)^n$ and $h \in \mathrm{Test}(M)^m$. Define $\mu_1[f,g] \in \mathfrak{M}_{\mathrm{f}}^{\pm}(M)$ as*

$$\begin{aligned}\mu_1[f,g] &:= \sum_{i,i'=1}^n \widetilde{g}_i\,\widetilde{g}_{i'}\,\mathbf{\Gamma}_2^{2\kappa}(f_i, f_{i'}) + 2\sum_{i,i'=1}^n g_i\,\mathrm{H}[f_i](f_{i'}, g_{i'})\,\mathfrak{m}\\ &\qquad + \frac{1}{2}\sum_{i,i'=1}^n \big[\langle\nabla f_i, \nabla f_{i'}\rangle\,\langle\nabla g_i, \nabla g_{i'}\rangle + \langle\nabla f_i, \nabla g_{i'}\rangle\,\langle\nabla g_i, \nabla f_{i'}\rangle\big]\,\mathfrak{m}\\ &\qquad - \frac{1}{N'}\Big[\sum_{i=1}^n \big[g_i\,\Delta f_i + \langle\nabla f_i, \nabla g_i\rangle\big]\Big]^2\,\mathfrak{m}\end{aligned}$$

As in Lemma 3.3.8, we denote the density of the $\mathfrak{m}$-absolutely continuous part of $\mu_1[f,g]$ by $\rho_1[f,g] := \mathrm{d}\mu_1[f,g]_{\ll}/\mathrm{d}\mathfrak{m} \in L^1(M)$. Then the $\mathfrak{m}$-singular part $\mu_1[f,g]_\perp$ of $\mu_1[f,g]$ as well as $\rho_1[f,g]$ are nonnegative, and

$$\begin{aligned}&\Big[\sum_{i=1}^n\sum_{j=1}^m \big[\langle\nabla f_i, \nabla h_j\rangle\,\langle\nabla g_i, \nabla h_j\rangle + g_i\,\mathrm{H}[f_i](h_j, h_j)\big]\\ &\qquad\qquad - \frac{1}{N'}\sum_{i=1}^n\sum_{j=1}^m \big[g_i\,\Delta f_i + \langle\nabla f_i, \nabla g_i\rangle\big]\,|\nabla h_j|^2\Big]^2\\ &\qquad \le \rho_1[f,g]\Big[\sum_{j,j'=1}^m \langle\nabla h_j, \nabla h_{j'}\rangle^2 - \frac{1}{N'}\Big[\sum_{j=1}^m |\nabla h_j|^2\Big]^2\Big] \quad \mathfrak{m}\text{-a.e.}\end{aligned}$$

Proof. We define $\mu_2[f,g,h], \mu_3[h] \in \mathfrak{M}_{\mathrm{f}}^{\pm}(M)$ by

$$\begin{aligned}\mu_2[f,g,h] &:= \sum_{i=1}^n\sum_{j=1}^m \big[\langle\nabla f_i, \nabla h_j\rangle\,\langle\nabla g_i, \nabla h_j\rangle + g_i\,\mathrm{H}[f_i](h_j, h_j)\big]\,\mathfrak{m}\\ &\qquad - \frac{1}{N'}\sum_{i=1}^n\sum_{j=1}^m \big[g_i\,\Delta f_i + \langle\nabla f_i, \nabla g_i\rangle\big]\,|\nabla h_j|^2\,\mathfrak{m},\\ \mu_3[h] &:= \Big[\sum_{j,j'=1}^m \langle\nabla h_j, \nabla h_{j'}\rangle^2 - \frac{1}{N'}\Big[\sum_{j=1}^m |\nabla h_j|^2\Big]^2\Big]\,\mathfrak{m}.\end{aligned}$$

Both claims readily follow from Lemma 3.3.8 as soon as $\lambda^2\,\mu_1[f,g] + 2\lambda\,\mu_2[f,g,h] + \mu_3[h] \ge 0$ for every $\lambda \in \mathbf{R}$, which is what we concentrate on in the sequel.

Let $\lambda \in \mathbf{R}$ and pick $a, b \in \mathbf{R}^n$ as well as $c \in \mathbf{R}^m$. Define the function $\varphi \in \mathrm{C}^\infty(\mathbf{R}^{2n+m})$ through

$$\varphi(x,y,z) := \sum_{i=1}^n \big[\lambda\,x_i\,y_i + a_i\,x_i - b_i\,y_i\big] + \sum_{j=1}^m \big[(z_j - c_j)^2 - c_j^2\big].$$

For every $i \in \{1, \dots, n\}$ and every $j \in \{1, \dots, m\}$, those first and second partial derivatives of φ which, do not always vanish identically read

$$\begin{aligned}\varphi_i(x,y,z) &= \lambda\,y_i + a_i,\\ \varphi_{n+i}(x,y,z) &= \lambda\,x_i - b_i,\\ \varphi_{2n+j}(x,y,z) &= 2(z_j - c_j),\\ \varphi_{i,n+i}(x,y,z) &= \lambda,\\ \varphi_{n+i,i}(x,y,z) &= \lambda,\end{aligned}$$

$$\varphi_{2n+j,2n+j}(x,y,z) = 2.$$

For convenience, we write

$$\begin{aligned}
\mathbf{A}^{2\kappa}(\lambda,a,b,c) &:= \mathbf{A}^{2\kappa}[\varphi\circ q],\\
\mathrm{B}(\lambda,a,b,c) &:= \mathrm{B}[\varphi\circ q],\\
\mathrm{C}(\lambda,a,b,c) &:= \mathrm{C}[\varphi\circ q],\\
\mathrm{D}(\lambda,a,b,c) &:= \mathrm{D}[\varphi\circ q],
\end{aligned}$$

where the respective r.h.s.'s are defined as in Lemma 3.2.76 for $\alpha := 2n+m$ and $q := (f,g,h)$. Using the same Lemma 3.2.76, we compute

$$\begin{aligned}
\mathbf{A}^{2\kappa}(\lambda,a,b,c) &= \sum_{i,i'=1}^{n} (\lambda\,\widetilde{g}_i + a_i)\,(\lambda\,\widetilde{g}_{i'} + a_{i'})\,\mathbf{\Gamma}_2^{2\kappa}(f_i,f_{i'}) + \text{other terms},\\
\mathrm{B}(\lambda,a,b,c) &= 4\sum_{i,i'=1}^{n} (\lambda\,g_i + a_i)\,\lambda\,\mathrm{H}[f_i](f_{i'},g_{i'})\\
&\qquad + 4\sum_{i=1}^{n}\sum_{j=1}^{m}(\lambda\,g_i + a_i)\,\mathrm{H}[f_i](h_j,h_j) + \text{other terms},\\
\mathrm{C}(\lambda,a,b,c) &= 2\sum_{i,i'=1}^{n}\lambda^2\left[\langle\nabla f_i,\nabla f_{i'}\rangle\,\langle\nabla g_i,\nabla g_{i'}\rangle + \langle\nabla f_i,\nabla g_{i'}\rangle\,\langle\nabla g_i,\nabla f_{i'}\rangle\right]\\
&\qquad + 8\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda\,\langle\nabla f_i,\nabla h_j\rangle\,\langle\nabla g_i,\nabla h_j\rangle\\
&\qquad + 4\sum_{j,j'=1}^{m}\langle\nabla h_j,\nabla h_{j'}\rangle^2 + \text{other terms},\\
\mathrm{D}(\lambda,a,b,c) &= \sum_{i,i'=1}^{n}(\lambda\,g_i + a_i)\,(\lambda\,g_{i'} + a_i)\,\Delta f_i\,\Delta f_{i'}\\
&\qquad + 4\sum_{i,i'=1}^{n}\lambda\,(\lambda\,g_i + a_i)\,\Delta f_i\,\langle\nabla f_{i'},\nabla g_{i'}\rangle\\
&\qquad + 4\sum_{i,i'=1}^{n}\lambda^2\,\langle\nabla f_i,\nabla g_i\rangle\,\langle\nabla f_{i'},\nabla g_{i'}\rangle\\
&\qquad + 4\sum_{i=1}^{n}\sum_{j=1}^{m}(\lambda\,g_i + a_i)\,\Delta f_i\,|\nabla h_j|^2\\
&\qquad + 8\sum_{i=1}^{n}\sum_{j=1}^{m}\lambda\,\langle\nabla f_i,\nabla g_i\rangle\,|\nabla h_j|^2\\
&\qquad + 4\Big[\sum_{j=1}^{m}|\nabla h_j|^2\Big]^2 + \text{other terms}.
\end{aligned}$$

Here, every "other term" contains at least one factor of the form $\lambda\,\widetilde{f}_i - b_i$ or $\widetilde{h}_j - c_j$ for some $i\in\{1,\dots,n\}$ and $j\in\{1,\dots,m\}$.

By Lemma 3.2.76 and Proposition 3.2.75 with the nonnegativity of $\mathrm{D}(\lambda,a,b,c)$ as well as the trivial inequality $1/N \geq 1/N'$,

$$\mathbf{A}^{2\kappa}(\lambda,a,b,c) + \Big[\mathrm{B}(\lambda,a,b,c) + \mathrm{C}(\lambda,a,b,c) - \frac{1}{N'}\,\mathrm{D}(\lambda,a,b,c)\Big]\,\mathfrak{m} \geq 0.$$

By the arbitrariness of $a, b \in \mathbf{R}^n$ and $c \in \mathbf{R}^m$, for every Borel partition $(E_p)_{p\in\mathbf{N}}$ of M, every Borel set $F \subset M$ and all sequences $(a_k)_{k\in\mathbf{N}}$ and $(b_k)_{k\in\mathbf{N}}$ in $\mathbf{R}^n$ as well as $(c_k)_{k\in\mathbf{N}}$ in $\mathbf{R}^m$,

$$\begin{aligned} 1_F \sum_{k\in\mathbf{N}} 1_{E_k} \Big[\mathbf{A}^{2\kappa}(\lambda, a_k, b_k, c_k) + \Big[\mathrm{B}(\lambda, a_k, b_k, c_k) + \mathrm{C}(\lambda, a_k, b_k, c_k) \\ - \frac{1}{N'}\,\mathrm{D}(\lambda, a_k, b_k, c_k)\Big]\,\mathfrak{m}\Big] \geq 0. \end{aligned} \tag{3.3.6}$$

We now choose the involved quantities appropriately. Let $(F_k)_{k\in\mathbf{N}}$ be an $\mathscr{E}$-nest with the property that the restrictions of $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{h}$ to F_k are continuous for every $k \in \mathbf{N}$, and set $F := \bigcup_{k\in\mathbf{N}} F_k$. Since F^c is an $\mathscr{E}$-polar set and thus not seen by $\mathfrak{m}$ and $\mathbf{\Gamma}_2^{2\kappa}(f_i, f_{i'})$, $i, i' \in \{1, \dots, n\}$, its contribution to the subsequent manipulations is ignored. For $l \in \mathbf{N}$ we now take a Borel partition $(E_k^l)_{k\in\mathbf{N}}$ of M and sequences $(a_k^l)_{k\in\mathbf{N}}$ and $(b_k^l)_{k\in\mathbf{N}}$ in $\mathbf{R}^n$ as well as $(c_k^l)_{k\in\mathbf{N}}$ in $\mathbf{R}^m$ with

$$\sup_{k,l\in\mathbf{N}} \big[|a_k^l| + |b_k^l| + |c_k^l|\big] < \infty$$

in such a way that

$$\begin{aligned} \lim_{l\to\infty} \sum_{k\in\mathbf{N}} 1_{E_k^l}\, a_k^l &= \lambda\,\widetilde{g}, \\ \lim_{l\to\infty} \sum_{k\in\mathbf{N}} 1_{E_k^l}\, b_k^l &= \lambda\,\widetilde{f}, \\ \lim_{l\to\infty} \sum_{k\in\mathbf{N}} 1_{E_k^l}\, c_k^l &= \widetilde{h} \end{aligned}$$

pointwise on F. Thus, the l.h.s. of (3.3.6) with $(E_k)_{k\in\mathbf{N}}$, $(a_k)_{k\in\mathbf{N}}$, $(b_k)_{k\in\mathbf{N}}$ and $(c_k)_{k\in\mathbf{N}}$ replaced by $(E_k^l)_{k\in\mathbf{N}}$, $(a_k^l)_{k\in\mathbf{N}}$, $(b_k^l)_{k\in\mathbf{N}}$ and $(c_k^l)_{k\in\mathbf{N}}$, $l \in \mathbf{N}$, respectively, converges w.r.t. $\|\cdot\|_{\mathrm{TV}}$ as $l \to \infty$. In fact, in the limit as $l \to \infty$ every "other term" above becomes zero, and the prefactors $\lambda\,\widetilde{g}_i + (a_k^l)_i$ become $2\lambda\,\widetilde{g}_i$, $i \in \{1, \dots, n\}$. We finally obtain

$$\begin{aligned} &4\lambda^2 \sum_{i,i'=1}^n \widetilde{g}_i\,\widetilde{g}_{i'}\,\mathbf{\Gamma}_2^{2\kappa}(f_i, f_{i'}) \\ &\quad + 8\lambda^2 \sum_{i,i'=1}^n g_i\,\mathrm{H}[f_i](f_{i'}, g_{i'})\,\mathfrak{m} + 8\lambda \sum_{i=1}^n \sum_{j=1}^m g_i\,\mathrm{H}[f_i](h_j, h_j)\,\mathfrak{m} \\ &\quad + 2\lambda^2 \sum_{i,i'=1}^n \big[\langle\nabla f_i, \nabla f_{i'}\rangle\,\langle\nabla g_i, \nabla g_{i'}\rangle + \langle\nabla f_i, \nabla g_{i'}\rangle\,\langle\nabla g_i, \nabla f_{i'}\rangle\big]\,\mathfrak{m} \\ &\qquad + 8\lambda \sum_{i=1}^n \sum_{j=1}^m \langle\nabla f_i, \nabla h_j\rangle\,\langle\nabla g_i, \nabla h_j\rangle\,\mathfrak{m} \\ &\qquad + 4 \sum_{j,j'=1}^m \langle\nabla h_j, \nabla h_{j'}\rangle^2\,\mathfrak{m} \\ &\quad - \frac{4\lambda^2}{N'} \Big[\sum_{i=1}^n \big[g_i\,\Delta f_i + \langle\nabla f_i, \nabla g_i\rangle\big]\Big]^2\,\mathfrak{m} \\ &\qquad - \frac{8\lambda}{N'} \sum_{i=1}^n \sum_{j=1}^m \big[g_i\,\Delta f_i + \langle\nabla f_i, \nabla g_i\rangle\big]\,|\nabla h_j|^2\,\mathfrak{m} \\ &\qquad - \frac{4}{N'} \Big[\sum_{j=1}^m |\nabla h_j|^2\Big]^2 \geq 0. \end{aligned}$$

Dividing by 4 and sorting terms by the order of λ yields the claim. □

We note the following consequence of Lemma 3.3.9 that is used in Theorem 3.3.11 below as well, but becomes especially important in Subsection 3.3.3.

Remark 3.3.10. The nonnegativity of $\mu_3[h]$ from Lemma 3.3.9 can be translated into the following trace inequality, compare with [Bra20, Rem. 2.19] and the proof of [Han18a, Prop. 3.2]. With the pointwise trace defined as in (3.2.12), we have

$$\Big|\sum_{j=1}^m \nabla h_j \otimes \nabla h_j\Big|_{\mathrm{HS}}^2 \geq \frac{1}{N}\,\mathrm{tr}\Big[\sum_{j=1}^m \nabla h_j \otimes \nabla h_j\Big]^2 \quad \mathfrak{m}\text{-a.e.}$$

for every $m \in \mathbf{N}$ and every $h \in \mathrm{Test}(M)^m$. ■

Theorem 3.3.11. *Every $f \in \mathrm{Test}(M)$ belongs to $\mathscr{D}(\mathrm{Hess})$ and satisfies*

$$\mathrm{Hess}\, f(\nabla g_1, \nabla g_2) = \mathrm{H}[f](g_1, g_2) \quad \mathfrak{m}\text{-a.e.} \tag{3.3.7}$$

for every $g_1, g_2 \in \mathrm{Test}(M)$. Moreover, denoting by $\gamma_2^{2\kappa}(f) \in L^1(M)$ the density of the $\mathfrak{m}$-absolutely continuous part of $\mathbf{\Gamma}_2^{2\kappa}(f)$, we have

$$\big|\mathrm{Hess}\, f\big|_{\mathrm{HS}}^2 \leq \gamma_2^{2\kappa}(f) \quad \mathfrak{m}\text{-a.e.} \tag{3.3.8}$$

Proof. Recall that indeed $\gamma_2^{2\kappa}(f) \in L^1(M)$ by Proposition 3.2.79. Let $g, h_1, \dots, h_m \in \mathrm{Test}(M)$, $m \in \mathbf{N}$. Applying Lemma 3.3.9 for $N' := \infty$ and $n := 1$ then entails

$$\begin{aligned}
&\Big[\sum_{j=1}^m \big[\langle \nabla f, \nabla h_j\rangle\, \langle \nabla g, \nabla h_j\rangle + g\,\mathrm{H}[f](h_j, h_j)\big]\Big]^2\\
&\qquad \leq \Big[g^2\, \gamma_2^{2\kappa}(f) + 2g\,\mathrm{H}[f](f,g) + \frac{1}{2}\,|\nabla f|^2\, |\nabla g|^2 + \frac{1}{2}\,\langle \nabla f, \nabla g\rangle^2\Big]\\
&\qquad\qquad \times \sum_{j,j'=1}^m \langle \nabla h_j, \nabla h_{j'}\rangle^2\\
&\qquad = \Big[g^2\, \gamma_2^{2\kappa}(f) + g\,\big\langle \nabla|\nabla f|^2, \nabla g\big\rangle + \frac{1}{2}\,|\nabla f|^2\, |\nabla g|^2 + \frac{1}{2}\,\langle \nabla f, \nabla g\rangle^2\Big]\\
&\qquad\qquad \times \sum_{j,j'=1}^m \langle \nabla h_j, \nabla h_{j'}\rangle^2 \quad \mathfrak{m}\text{-a.e.}
\end{aligned}$$

In the last identity, we used the definition (3.2.22) of $\mathrm{H}[f](f,g)$. Using the first part of Lemma 3.2.73 and possibly passing to subsequences, this $\mathfrak{m}$-a.e. inequality extends to all $g \in \mathscr{F} \cap L^\infty(M)$. Thus, successively setting $g := g_n$, $n \in \mathbf{N}$, where $(g_n)_{n\in\mathbf{N}}$ is the sequence provided by Lemma 3.2.6, together with the locality of ∇ from Proposition 3.2.37, and by the definition (3.2.7) of the pointwise Hilbert–Schmidt norm of $L^2(T^{\otimes 2}M)$, we obtain

$$\Big|\sum_{j=1}^m \mathrm{H}[f](h_j, h_j)\Big| \leq \gamma_2^{2\kappa}(f)^{1/2}\,\Big|\sum_{j=1}^m \nabla h_j \otimes \nabla h_j\Big|_{\mathrm{HS}} \quad \mathfrak{m}\text{-a.e.} \tag{3.3.9}$$

This implies pointwise $\mathfrak{m}$-a.e. off-diagonal estimates as follows. Given any $m' \in \mathbf{N}$ and $h_j, h_j' \in \mathrm{Test}(M)$, $j \in \{1, \dots, m'\}$, since

$$\sum_{j=1}^{m'} \mathrm{H}[f](h_j, h_j') = \frac{1}{2}\sum_{j=1}^{m'} \big[\mathrm{H}[f](h_j + h_j', h_j + h_j') - \mathrm{H}[f](h_j, h_j) - \mathrm{H}[f](h_j', h_j')\big]$$

holds $\mathfrak{m}$-a.e., applying (3.3.9), using that

$$\begin{aligned}\frac{1}{2}\sum_{j=1}^{m'}&\big[\nabla(h_j+h'_j)\otimes\nabla(h_j+h'_j)-\nabla h_j\otimes\nabla h_j-\nabla h'_j\otimes\nabla h'_j\big]\\ &=\frac{1}{2}\sum_{j=1}^{m'}\big[\nabla h_j\otimes\nabla h'_j+\nabla h'_j\otimes\nabla h_j\big]\\ &=\Big[\sum_{j=1}^{m'}\nabla h_j\otimes\nabla h'_j\Big]_{\mathrm{sym}}\end{aligned}$$

and finally employing that $|T_{\mathrm{sym}}|_{\mathrm{HS}}\leq|T|_{\mathrm{HS}}$ for every $T\in L^2(T^{\otimes 2}M)$, we get

$$\begin{aligned}\Big|\sum_{j=1}^{m'}\mathrm{H}[f](h_j,h'_j)\Big|&\leq\gamma_2^{2\kappa}(f)^{1/2}\Big|\frac{1}{2}\sum_{j=1}^{m'}\big[\nabla h_j\otimes\nabla h'_j+\nabla h'_j\otimes\nabla h_j\big]\Big|_{\mathrm{HS}}\\ &\leq\gamma_2^{2\kappa}(f)^{1/2}\Big|\sum_{j=1}^{m'}\nabla h_j\otimes\nabla h'_j\Big|_{\mathrm{HS}}\quad\mathfrak{m}\text{-a.e.}\end{aligned}$$

We replace h_j by $a_j\,h_j$, $j\in\{1,\dots,m'\}$, for arbitrary $a_1,\dots,a_{m'}\in\mathbf{Q}$. This gives

$$\Big|\sum_{j=1}^{m'}a_j\,\mathrm{H}[f](h_j,h'_j)\Big|\leq\gamma_2^{2\kappa}(f)^{1/2}\Big|\sum_{j=1}^{m'}a_j\,\nabla h_j\otimes\nabla h'_j\Big|_{\mathrm{HS}}\quad\mathfrak{m}\text{-a.e.}\tag{3.3.10}$$

In fact, since $\mathbf{Q}$ is countable, we find an $\mathfrak{m}$-negligible Borel set $B\subset M$ on whose complement (3.3.10) holds pointwise for every $a_1,\dots,a_{m'}\in\mathbf{Q}$. Since both sides of (3.3.10) are continuous in $a_1,\dots,a_{m'}$, by density of $\mathbf{Q}$ in $\mathbf{R}$ we deduce that (3.3.10) holds pointwise on B^{c} for every $a_1,\dots,a_{m'}\in\mathbf{R}$. Therefore, given any $g_1,\dots,g_{m'}\in\mathrm{Test}(M)$, up to possibly removing a further $\mathfrak{m}$-negligible Borel set $C\subset M$, for every $x\in(B\cup C)^{\mathrm{c}}$ we may replace a_j by $g_j(x)$, $j\in\{1,\dots,m'\}$, in (3.3.10). This leads to

$$\Big|\sum_{j=1}^{m'}g_j\,\mathrm{H}[f](h_j,h'_j)\Big|^2\leq\gamma_2^{2\kappa}(f)^{1/2}\Big|\sum_{j=1}^{m'}g_j\,\nabla h_j\otimes\nabla h'_j\Big|_{\mathrm{HS}}\quad\mathfrak{m}\text{-a.e.}\tag{3.3.11}$$

We now define the operator $\Phi\colon\mathrm{Test}(T^{\otimes 2}M)\to L^0(M)$ by

$$\Phi\sum_{j=1}^{m'}g_j\,g'_j\,\nabla h_j\otimes\nabla h'_j:=\sum_{j=1}^{m'}g_j\,g'_j\,\mathrm{H}[f](h_j,h'_j).\tag{3.3.12}$$

From (3.3.11) and the algebra property of $\mathrm{Test}(M)$, it follows that Φ is well-defined, i.e. the value of $\Phi(T)$ does not depend on the specific way of representing a given element $T\in\mathrm{Test}(T^{\otimes 2}M)$. Moreover, the map Φ is clearly linear, and for every $g\in\mathrm{Test}(M)$ and every $T\in\mathrm{Test}(T^{\otimes 2}M)$,

$$\Phi(g\,T)=g\,\Phi(T).\tag{3.3.13}$$

Since the $\mathfrak{m}$-singular part $\mathbf{\Gamma}_2^{2\kappa}(f)_\perp$ of $\mathbf{\Gamma}_2^{2\kappa}(f)$ is nonnegative, by (3.2.23) we get

$$\int_M\gamma_2^{2\kappa}(f)\,\mathrm{d}\mathfrak{m}\leq\mathbf{\Gamma}_2^{2\kappa}(f)[M]=\int_M(\Delta f)^2\,\mathrm{d}\mathfrak{m}-\big\langle\kappa\,\big|\,|\nabla f|^2\big\rangle.\tag{3.3.14}$$

After integrating (3.3.11) and employing Cauchy–Schwarz's inequality,

$$\|\Phi(T)\|_{L^1(M)} \leq \Big[\int_M (\Delta f)^2 \,\mathrm{d}\mathfrak{m} - \big\langle \kappa \,\big|\, |\nabla f|^2 \big\rangle\Big]^{1/2} \|T\|_{L^2(T^{\otimes 2}M)}$$

holds for every $T \in \mathrm{Test}(T^{\otimes 2}M)$. Thus, by density of $\mathrm{Test}(T^{\otimes 2}M)$ in $L^2(T^{\otimes 2}M)$ and (3.3.13), Φ uniquely extends to a (non-relabeled) continuous, L^∞-linear map from $L^2(T^{\otimes 2}M)$ into $L^1(M)$, whence $\Phi \in L^2((T^*)^{\otimes 2}M)$ by definition of the latter space.

To check that $f \in \mathscr{D}(\mathrm{Hess})$ and $\Phi = \mathrm{Hess}\, f$, first note that by the continuity of Φ and Lemma 3.2.72, we can replace $g_j\, g_j'$ by arbitrary $k_j \in \mathrm{Test}(M)$, $j \in \{1, \dots, m'\}$, still retaining the identity (3.3.12). Therefore, slightly changing the notation in (3.3.12), let $g_1, g_2, h \in \mathrm{Test}(M)$ and use (3.3.13), the definition (3.2.22) of $\mathrm{H}[f]$ and Lemma 3.2.54 to derive that

$$\begin{aligned}
2\int_M & h\,\Phi(\nabla g_1, \nabla g_2)\,\mathrm{d}\mathfrak{m}\\
&= \int_M h\,\big\langle \nabla g_1, \nabla\langle \nabla f, \nabla g_2\rangle\big\rangle\,\mathrm{d}\mathfrak{m} + \int_M h\,\big\langle \nabla g_1, \nabla\langle \nabla f, \nabla g_2\rangle\big\rangle\,\mathrm{d}\mathfrak{m}\\
&\qquad - \int_M h\,\big\langle \nabla g_1, \nabla\langle \nabla f, \nabla g_2\rangle\big\rangle\,\mathrm{d}\mathfrak{m}\\
&= -\int_M \langle \nabla f, \nabla g_2\rangle\,\mathrm{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{m} - \int_M \langle \nabla f, \nabla g_2\rangle\,\mathrm{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{m}\\
&\qquad - \int_M h\,\big\langle \nabla g_1, \nabla\langle \nabla f, \nabla g_2\rangle\big\rangle\,\mathrm{d}\mathfrak{m},
\end{aligned}$$

which is the desired assertion $f \in \mathscr{D}(\mathrm{Hess})$ and $\Phi = \mathrm{Hess}\, f$.

The same argument gives (3.3.7), while the inequality (3.3.8) is due to (3.3.11), the density of $\mathrm{Test}(T^{\otimes 2}M)$ in $L^2(T^{\otimes 2}M)$ as well as the definition (3.2.7) of the pointwise Hilbert–Schmidt norm. □

Theorem 3.3.11 implies the following qualitative result. A quantitative version of it, as directly deduced in [Gig18, Cor. 3.3.9] from [Gig18, Thm. 3.3.8], is however not yet available only with the information collected so far. See Remark 3.3.13 below.

Corollary 3.3.12. *Every $f \in \mathscr{D}(\Delta)$ belongs to the closure of* $\mathrm{Test}(M)$ *in* $\mathscr{D}(\mathrm{Hess})$*, and in particular to* $\mathscr{D}(\mathrm{Hess})$. *More precisely, let $\rho' \in (0,1)$ and $\alpha' \in \mathbf{R}$ be as in Lemma 3.2.60 for $\mu := \kappa^-$. Then for every $f \in \mathscr{D}(\Delta)$, we have $f \in \mathscr{D}(\mathrm{Hess})$ with*

$$\int_M \big|\mathrm{Hess}\, f\big|_{\mathrm{HS}}^2\,\mathrm{d}\mathfrak{m} \leq \frac{1}{1-\rho'}\int_M (\Delta f)^2\,\mathrm{d}\mathfrak{m} + \frac{\alpha'}{1-\rho'}\int_M |\nabla f|^2\,\mathrm{d}\mathfrak{m}.$$

Proof. Since $(M, \mathscr{E}, \mathfrak{m})$ satisfies $\mathrm{BE}_1(\kappa, \infty)$ by [ER+20, Thm. 6.9], it also trivially obeys $\mathrm{BE}_1(-\kappa^-, \infty)$, see also [ER+20, Prop. 6.7]. As in the proof of Proposition 3.2.79,

$$\mathscr{E}^{-\kappa^-}\big(|\nabla f|\big) \leq \int_M (\Delta f)^2\,\mathrm{d}\mathfrak{m}$$

holds for every $f \in \mathrm{Test}(M)$. Hence, using (3.2.19) we estimate

$$\begin{aligned}
\big\langle \kappa^- \,\big|\, |\nabla f|^2\big\rangle &\leq \rho'\,\mathscr{E}\big(|\nabla f|\big) + \alpha'\int_M |\nabla f|^2\,\mathrm{d}\mathfrak{m}\\
&= \rho'\,\mathscr{E}^{-\kappa^-}\big(|\nabla f|\big) + \rho'\,\big\langle \kappa^- \,\big|\, |\nabla f|^2\big\rangle + \alpha'\int_M |\nabla f|^2\,\mathrm{d}\mathfrak{m}
\end{aligned}$$

$$\leq \rho' \int_M (\Delta f)^2 \, \mathrm{d}\mathfrak{m} + \rho' \left\langle \kappa^- \,\middle|\, |\nabla f|^2 \right\rangle + \alpha' \int_M |\nabla f|^2 \, \mathrm{d}\mathfrak{m}.$$

The claim for $f \in \mathrm{Test}(M)$ now follows easily. We already know from Theorem 3.3.11 that $f \in \mathscr{D}(\mathrm{Hess})$. Integrating (3.3.8) and using (3.3.14) thus yields

$$\begin{aligned}\int_M \left|\mathrm{Hess}\, f\right|_{\mathrm{HS}}^2 \, \mathrm{d}\mathfrak{m} &\leq \int_M (\Delta f)^2 \, \mathrm{d}\mathfrak{m} - \left\langle \kappa \,\middle|\, |\nabla f|^2 \right\rangle \\ &\leq \int_M (\Delta f)^2 \, \mathrm{d}\mathfrak{m} + \left\langle \kappa^- \,\middle|\, |\nabla f|^2 \right\rangle \\ &\leq \frac{1}{1-\rho'} \int_M (\Delta f)^2 \, \mathrm{d}\mathfrak{m} + \frac{\alpha'}{1-\rho'} \int_M |\nabla f|^2 \, \mathrm{d}\mathfrak{m}.\end{aligned} \tag{3.3.15}$$

Finally, given $f \in \mathscr{D}(\Delta)$, let $f_n := \max\{\min\{f, n\}, -n\} \in L^2(M) \cap L^\infty(M)$, $n \in \mathbf{N}$. Note that $\mathsf{P}_t f_n \in \mathrm{Test}(M)$ for every $t > 0$ and every $n \in \mathbf{N}$, and that $\mathsf{P}_t f_n \to \mathsf{P}_t f$ in $\mathscr{F}$ as well as, thanks to Lemma 3.2.12, $\Delta \mathsf{P}_t f_n \to \Delta \mathsf{P}_t f$ in $L^2(M)$ as $n \to \infty$. Moreover $\mathsf{P}_t f \to f$ in $\mathscr{F}$ as well as $\Delta \mathsf{P}_t f = \mathsf{P}_t \Delta f \to \Delta f$ in $L^2(M)$ as $t \to 0$. These observations imply that f belongs to the closure of $\mathrm{Test}(M)$ in $\mathscr{D}(\mathrm{Hess})$, whence $f \in \mathscr{D}(\mathrm{Hess})$ by Theorem 3.3.3, and the claimed inequality, with unchanged constants, is clearly stable under this approximation procedure. □

Remark 3.3.13. The subtle reason why we still cannot deduce (3.3.15) for general $f \in \mathscr{D}(\Delta)$ is that we neither know whether the r.h.s. of (3.3.15) makes sense — which essentially requires $|\nabla f| \in \mathscr{F}$ — nor, in the notation of the previous proof, whether $\left\langle \kappa \,\middle|\, |\nabla \mathsf{P}_t f_n|^2 \right\rangle \to \left\langle \kappa \,\middle|\, |\nabla f|^2 \right\rangle$ as $n \to \infty$ and $t \to 0$. (Neither we know if $\mathscr{E}\big(|\nabla \mathsf{P}_t f_n|\big) \to \mathscr{E}\big(|\nabla f|\big)$ as $n \to \infty$ and $t \to 0$.) Both points are trivial in the more restrictive $\mathrm{RCD}(K, \infty)$ case from [Gig18, Cor. 3.3.9], $K \in \mathbf{R}$. In our setting, solely Lemma 3.2.60 does not seem sufficient to argue similarly. Instead, both points will follow from Lemma 3.4.13 and Lemma 3.6.2, see Corollary 3.4.14 and Corollary 3.6.3. ■

3.3.3 Structural consequences of Lemma 3.3.9

Solely in this subsection, we assume that $\mathrm{BE}_2(\kappa, N)$ holds for $N < \infty$. In this case, we derive a nontrivial upper bound on the local dimension of $L^2(TM)$ — and hence of $L^2(T^*M)$ by Proposition 3.2.19 — in Proposition 3.3.14. Our proof follows [Han18a, Prop. 3.2]. Reminiscent of Remark 3.2.26 and Corollary 3.2.38, as a byproduct we obtain an upper bound on the Hino index of M. The key point is the trace inequality derived in Remark 3.3.10.

Let $(E_n)_{n \in \mathbf{N} \cup \{\infty\}}$ be the dimensional decomposition of $L^2(TM)$ from Proposition 3.2.24. The *maximal essential local dimension* of $L^2(TM)$ is the quantity

$$\dim_{L^\infty, \max} L^2(TM) := \sup\big\{n \in \mathbf{N} \cup \{\infty\} : \mathfrak{m}[E_n] > 0\big\}.$$

Proposition 3.3.14. *We have*

$$\dim_{L^\infty, \max} L^2(TM) \leq \lfloor N \rfloor.$$

Moreover, if N is an integer, then for every $f \in \mathrm{Test}(M)$,

$$\mathrm{tr}\, \mathrm{Hess}\, f = \Delta f \quad \mathfrak{m}\text{-a.e.} \quad \text{on } E_N.$$

Proof. Suppose to the contrapositive that $\mathfrak{m}[E_n] > 0$ for some $n \in \mathbf{N} \cup \{\infty\}$ with $n > N$. Let $m \in \mathbf{N}$ be a finite number satisfying $N < m \leq n$. Let $B \subset E_m$ be a given Borel set of finite, but positive $\mathfrak{m}$-measure. By Theorem 3.2.32 and [Gig18, Thm. 1.4.11], there exist vectors $V_1, \dots, V_m \in L^2(TM)$ such that $1_{B^c} V_i = 0$ and $\langle V_i, V_j \rangle = \delta_{ij}$ $\mathfrak{m}$-a.e. in B for every $i, j \in \{1, \dots, m\}$ which generate $L^2(TM)$ on B. Recall that by Remark 3.3.10, for every $h \in \mathrm{Test}(M)^m$,

$$\Big|\sum_{j=1}^m \nabla h_j \otimes \nabla h_j\Big|_{\mathrm{HS}}^2 \geq \frac{1}{N} \operatorname{tr}\Big[\sum_{j=1}^m \nabla h_j \otimes \nabla h_j\Big]^2 \quad \mathfrak{m}\text{-a.e.} \tag{3.3.16}$$

By a similar argument as for Theorem 3.3.11, we can replace ∇h_j by $f_j \nabla h_j$, $j \in \{1, \dots, m\}$, for arbitrary $f_1, \dots, f_m \in L^\infty(M)$, still retaining (3.3.16). Again by Theorem 3.2.32 and Subsection 3.2.8, the linear span of such vector fields generates $L^2(TM)$ on B. We can thus further replace $f_j \nabla h_j$ by $1_B V_j$, $j \in \{1, \dots, m\}$, and (3.3.16) translates into

$$m = \Big|\sum_{j=1}^m V_j \otimes V_j\Big|_{\mathrm{HS}}^2 \geq \frac{1}{N} \operatorname{tr}\Big[\sum_{j=1}^m V_j \otimes V_j\Big]^2 = \frac{m^2}{N} \quad \mathfrak{m}\text{-a.e.} \quad \text{on } B. \tag{3.3.17}$$

This is in contradiction with the assumption $N < m$.

The second claim is only nontrivial if $\mathfrak{m}[E_N] > 0$. In this case, retain the notation of the previous part and observe that under our given assumptions, equality occurs in (3.3.17) for m replaced by N. Using Lemma 3.3.9, (3.3.7) and similar arguments as for Theorem 3.3.11 to get rid of the term containing $g \in \mathrm{Test}(M)$ and from above to pass from ∇h_j to V_j, $j \in \{1, \dots, N\}$, we get

$$\begin{aligned}&\Big|\sum_{j=1}^N \operatorname{Hess} f(V_j, V_j) - (\Delta f)^2\Big|^2 \\ &\qquad = \Big|\sum_{j=1}^N \operatorname{Hess} f(V_j, V_j) - \frac{1}{N}(\Delta f)^2 \sum_{j=1}^N |V_j|^2\Big|^2 = 0 \quad \mathfrak{m}\text{-a.e.} \quad \text{on } B.\end{aligned}$$

This provides the assertion by the arbitrariness of B. □

Corollary 3.3.15. *For every $k \in \mathbf{N}$ with $k > \lfloor N \rfloor$,*

$$L^2(\Lambda^k T^* M) = \{0\}.$$

Proposition 3.3.14 opens the door for considering an N-Ricci tensor on $\mathrm{BE}_2(\kappa, N)$ spaces with $N < \infty$, see the discussion about the dimension-dependent Ricci tensor in Subsection 3.6.2 below.

Remark 3.3.16. It is an interesting task to carry out a detailed study of sufficient and necessary conditions for the constancy of the local dimension of $L^2(TM)$ as well as its maximality. Natural questions in this respect are the following.

a. Under which hypotheses does there exist $d \in \{1, \dots, \lfloor N \rfloor\}$ such that

$$\mathfrak{m}[E_d^c] = 0? \tag{3.3.18}$$

b. If (3.3.18) holds for some $d \in \{1, \dots, \lfloor N \rfloor\}$, which conclusions can be drawn for the space $(M, \mathscr{E}, \mathfrak{m})$? Does it satisfy $\mathrm{BE}_2(\kappa, d)$?

c. What happens if N is an integer and $d = N$ in (3.3.18)?

In [Hon18b], a general class of examples which obey $\mathrm{BE}_2(K,N)$, $K \in \mathbf{R}$ and $N \in [1,\infty)$, but do not have constant local dimension has been pointed out. The latter already happens, for instance, for metric measure spaces obtained by gluing together two compact pointed Riemannian manifolds at their base points. The point is that such a space does not satisfy the Sobolev-to-Lipschitz property, hence cannot be $\mathrm{RCD}(K',\infty)$ for any $K' \in \mathbf{R}$ [AGS14a, Thm. 6.2].

On the other hand, some existing results in the framework of $\mathrm{RCD}(K,N)$ spaces $(M,\mathsf{d},\mathfrak{m})$, $K \in \mathbf{R}$ and $N \in [1,\infty)$, are worth mentioning.

Originating in [MN19], a. has completely been solved in [BS20, Thm. 0.1]. (This result from [BS20] uses optimal transport tools. See [Han18a] for the connections of the latter to [Gig18].) However, it is still unknown what the corresponding value of d really is, e.g. whether it generally coincides with the Hausdorff dimension of (M,d).

Questions b. and c. are still subject to high research interest. Results in this direction have been initiated in [DPG18]. Indeed, every $\mathrm{RCD}(K,N)$ space with integer N and $\mathfrak{m}[E_N^{\mathrm{c}}] = 0$ is weakly noncollapsed [DPG18, Def. 1.10] by [DPG18, Thm. 1.12, Rem. 1.13]. In fact, it is conjectured in [DPG18, Rem. 1.11] that every such weakly noncollapsed $\mathrm{RCD}(K,N)$ space is *noncollapsed* [DPG18, Def. 1.1], i.e. $\mathfrak{m}$ is a constant multiple of $\mathscr{H}^N$. This conjecture is true if M is compact [HZ20, Cor. 1.3]. Lastly, we mention for completeness that the second question in b. is true for general $\mathrm{RCD}(K,N)$ spaces [DPG18, Rem. 1.13]. ■

Since to our knowledge, the result from [BS20] in Remark 3.3.16 has not yet been considered in the context of Hino indices, let us phrase it separately according to our compatibility result in Corollary 3.2.38 to bring it to a broader audience.

Corollary 3.3.17. *Let $(M,\mathscr{E},\mathfrak{m})$ be the Dirichlet space induced by an $\mathrm{RCD}(K,N)$ space $(M,\mathsf{d},\mathfrak{m})$, $K \in \mathbf{R}$ and $N \in [1,\infty)$. Then there exists $d \in \{1,\dots,\lfloor N\rfloor\}$ such that the pointwise Hino index of $\mathscr{E}$ is $\mathfrak{m}$-a.e. constantly equal to d.*

3.3.4 Calculus rules

This subsection contains calculus rules for the Hessian as well as preparatory material, such as different function spaces, required to develop them. We shortly comment on the two major challenges in establishing these.

First, the calculus rules in the third paragraph below — which easily hold for test functions — do not transfer to arbitrary elements in $\mathscr{D}(\mathrm{Hess})$ in general, at least not by approximation. In fact, in general $\mathrm{Test}(M)$ is even *not* dense in $\mathscr{D}(\mathrm{Hess})$: on a compact smooth Riemannian manifold M with boundary, any nonconstant, affine $f\colon M \to \mathbf{R}$ belongs to $\mathscr{D}(\mathrm{Hess})$, but as a possible limit of elements of $\mathrm{Test}(M)$ in $\mathscr{D}(\mathrm{Hess})$ it would necessarily have $\mathfrak{s}$-a.e. vanishing normal derivative at ∂M by Lemma 3.2.54, Example 3.2.48 and the trace theorem from Proposition 3.2.1 for $k := p := 2$ and $E := M \times \mathbf{R}$, see also [Sch95, p. 32]. Hence, in Definition 3.3.27 we consider the closure $\mathscr{D}_{\mathrm{reg}}(\mathrm{Hess})$ of $\mathrm{Test}(M)$ in $\mathscr{D}(\mathrm{Hess})$. Many calculus rules will "only" hold for this class.

Second, to prove a product rule for the Hessian, Proposition 3.3.30, similarly to Definition 3.3.2 above one would try to define the space $W^{2,1}(M)$ of all $f \in L^1(M)$ having a gradient $\nabla f \in L^1(TM)$ and a Hessian $\mathrm{Hess}\, f \in L^1((T^*)^{\otimes 2}M)$. However, we refrain from doing so, since in this case the term $\big\langle \nabla f, \nabla\langle\nabla g_1,\nabla g_2\rangle\big\rangle$, $g_1, g_2 \in \mathrm{Test}(M)$, appearing in the defining property of $\mathrm{Hess}\, f$ in Definition 3.3.2 cannot be guaranteed to have any integrability by the lack of L^∞-bounds on $\nabla\langle\nabla g_1, \nabla g_2\rangle$. (In view of Proposition 3.4.11 and Lemma 3.4.13, this would amount to the strong requirement of bounded Hessians of g_1 and g_2. Compare with [GP20b, Rem. 4.10, Rem. 4.11, Rem. 4.13].) A similar issue arises for the Hessian chain rule in Proposition 3.3.31. Since the requirement $\nabla f \in L^2(M)$ thus cannot be dropped, we are forced to work with the space $\mathscr{D}_{2,2,1}(\mathrm{Hess})$ introduced in Definition 3.3.28.

First order spaces

Definition 3.3.18. *We define the space $\mathscr{G}$ to consist of all $f \in L^1(M)$ for which there exists $\eta \in L^1(T^*M)$ such that*

$$\int_M \eta(\nabla g)\, h \,\mathrm{d}\mathfrak{m} = -\int_M f \,\mathrm{div}(h\,\nabla g)\,\mathrm{d}\mathfrak{m}$$

for every $g \in \mathrm{Test}_{L^\infty}(M)$ and every $h \in \mathrm{Test}(M)$. If such an η exists, it is unique, denoted by $\mathrm{d}_1 f$ and termed the differential *of f.*

The defining equality makes sense by Lemma 3.2.54. The uniqueness statement follows from weak* density of $\mathrm{Test}(TM)$ in $L^\infty(TM)$ as discussed in Subsection 3.2.8. Therefore, d_1 becomes a linear operator on $\mathscr{G}$, which turns the latter into a real vector space. Since both sides of the defining property for d_1 in Definition 3.3.18 are strongly L^1-continuous in f and η, respectively, for fixed $g \in \mathrm{Test}_{L^\infty}(M)$ and $h \in \mathrm{Test}(M)$, $\mathscr{G}$ is a Banach space w.r.t. the norm $\|\cdot\|_{\mathscr{G}}$ given by

$$\|f\|_{\mathscr{G}} := \|f\|_{L^1(M)} + \|\mathrm{d}_1 f\|_{L^1(T^*M)}.$$

Products of test functions belong to $\mathscr{G}$ by Lemma 3.2.54. In fact, by Lemma 3.2.72, $\mathscr{G}\cap\mathscr{F}$ and $\mathscr{G}\cap\mathrm{Test}(M)$ are dense in $\mathscr{F}$. The subsequent definition — that we introduce since we do not know if $\mathscr{G}\cap\mathrm{Test}(M)$ is dense in $\mathscr{G}$ — is thus non-void.

Definition 3.3.19. *We define the space $\mathscr{G}_{\mathrm{reg}} \subset \mathscr{G}$ by*

$$\mathscr{G}_{\mathrm{reg}} := \mathrm{cl}_{\|\cdot\|_{\mathscr{G}}}\big[\mathscr{G}\cap\mathrm{Test}(M)\big].$$

One checks through Lemma 3.2.72 and Lemma 3.2.73 that $\mathscr{G}_{\mathrm{reg}}$ is dense in $L^1(M)$.

A priori, the differential from Definition 3.3.18 could differ from the differential d on $\mathscr{F}_{\mathrm{e}}$ from Definition 3.2.33. However, these objects agree on the intersection $\mathscr{G}\cap\mathscr{F}_{\mathrm{e}}$. In particular, in what follows we simply write $\mathrm{d}f$ in place of $\mathrm{d}_1 f$ for $f \in \mathscr{G}$ although the axiomatic difference should always be kept in mind.

Lemma 3.3.20. *If $f \in \mathscr{G}\cap\mathscr{F}_{\mathrm{e}}$, then*

$$\mathrm{d}_1 f = \mathrm{d}f.$$

Proof. Given any $n \in \mathbf{N}$, the function $f_n := \max\{\min\{f, n\}, -n\} \in \mathscr{F}_{\mathrm{e}}$ belongs to $L^1(M)\cap L^\infty(M)$ and thus to $\mathscr{F}$ by Proposition 3.2.5. Let $g \in \mathrm{Test}_{L^\infty}(M)$ and $h \in \mathrm{Test}(M)$ be arbitrary. Observe that $f_n \to f$ in $L^1(M)$ and $\mathrm{d}f_n = 1_{\{|f|<n\}}\,\mathrm{d}f \to \mathrm{d}f$ in $L^2(T^*M)$ as $n \to \infty$. Since $h\,\nabla g \in \mathscr{D}_{\mathrm{TV}}(\mathbf{div})\cap\mathscr{D}(\mathrm{div})$ with $\mathbf{n}(h\,\nabla g) = 0$ and $\mathrm{div}(h\,\nabla g) = \langle\nabla h, \nabla g\rangle + h\,\Delta g \in L^\infty(M)$ by Lemma 3.2.54,

$$\int_M \mathrm{d}_1 f(\nabla g)\, h\,\mathrm{d}\mathfrak{m} = -\int_M f\,\mathrm{div}(h\,\nabla g)\,\mathrm{d}\mathfrak{m}$$

$$
\begin{aligned}
&= -\lim_{n\to\infty} \int_M f_n \operatorname{div}(h\,\nabla g)\,\mathrm{d}\mathfrak{m}\\
&= \lim_{n\to\infty} \int_{\{|f|<n\}} \mathrm{d}f(\nabla g)\,h\,\mathrm{d}\mathfrak{m}\\
&= \int_M \mathrm{d}f(\nabla g)\,h\,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

The statement follows from the weak* density of $\mathrm{Test}_{L^\infty}(TM)$ in $L^\infty(TM)$. □

Occasionally we adopt the dual perspective of $\mathrm{d}f$ for a given $f \in \mathscr{G}$ similarly to Definition 3.2.44 — more precisely, under the compatibility granted by Lemma 3.3.20 and keeping in mind Subsection 3.2.8, define $\nabla f \in L^1(TM)$ by

$$\nabla f := (\mathrm{d}f)^\sharp.$$

Lemma 3.3.21. *For every $f \in \mathscr{G}_{\mathrm{reg}}$ and every $g \in \mathscr{F}$ such that $\mathrm{d}g \in L^\infty(T^*M)$ and $\Delta g \in L^\infty(M)$, we have*

$$\int_M f\,\Delta g\,\mathrm{d}\mathfrak{m} = -\int_M \mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m}.$$

Proof. Given a sequence $(f_n)_{n\in\mathbf{N}}$ in $\mathrm{Test}(M)\cap\mathscr{G}$ converging f in $\mathscr{G}$, since $f_n \in \mathscr{F}$ for every $n\in\mathbf{N}$, by Lemma 3.3.20 we have

$$
\begin{aligned}
\int_M f\,\Delta g\,\mathrm{d}\mathfrak{m} &= \lim_{n\to\infty}\int_M f_n\,\Delta g\,\mathrm{d}\mathfrak{m}\\
&= -\lim_{n\to\infty}\int_M \mathrm{d}f_n(\nabla g)\,\mathrm{d}\mathfrak{m}\\
&= -\int_M \mathrm{d}f(\nabla g)\,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

□

Remark 3.3.22. It is unclear to us whether the integration by parts formula from Lemma 3.3.21 holds if merely $f \in \mathscr{G}$. The subtle point is that in general, we are not able to get rid of the zeroth order term h in Definition 3.3.18 — Lemma 3.2.6 does not give global first order controls. Of course, what still rescues this important identity (see e.g. Proposition 3.4.24 below) for $\mathscr{G}_{\mathrm{reg}}$-functions is its validity for the approximating test functions, and Lemma 3.3.20.

By the same reason, in our setting we lack an analogue of [Gig18, Prop. 3.3.18], see also [GP20a, Le. 6.2.26], which grants $\mathscr{F}$-regularity of $f\in\mathscr{G}\cap L^2(M)$ as soon as $\mathrm{d}f \in L^2(T^*M)$. Compare with Remark 3.5.3 and also Remark 3.6.34 below.

Both statements are clearly true if M is intrinsically complete. ■

In line with Remark 3.3.22, the following proposition readily follows from corresponding properties of test functions from Proposition 3.2.37, Lemma 3.3.20 and an argument as for [Gig18, Thm. 2.2.6].

Proposition 3.3.23. *The following hold for the space $\mathscr{G}_{\mathrm{reg}}$ and the differential* d.

(i) Locality. *For every $f, g \in \mathscr{G}_{\mathrm{reg}}$,*

$$1_{\{f=g\}}\,\mathrm{d}f = 1_{\{f=g\}}\,\mathrm{d}g.$$

(ii) Chain rule. *For every $f \in \mathscr{G}_{\mathrm{reg}}$ and every $\mathscr{L}^1$-negligible Borel set $C \subset \mathbf{R}$,*

$$1_{f^{-1}(C)}\,\mathrm{d}f = 0.$$

In particular, for every $\varphi \in \mathrm{Lip}(\mathbf{R})$, we have $\varphi \circ f \in \mathscr{G}_{\mathrm{reg}}$ with

$$\mathrm{d}(\varphi \circ f) = \big[\varphi' \circ f\big]\,\mathrm{d}f,$$

where the derivative $\varphi' \circ f$ is defined arbitrarily on the intersection of the set of non-differentiability points of φ with the image of f.

(iii) Leibniz rule. *For every $f, g \in \mathscr{G}_{\mathrm{reg}} \cap L^\infty(M)$, $f\,g \in \mathscr{G}_{\mathrm{reg}}$ with*

$$\mathrm{d}(f\,g) = f\,\mathrm{d}g + g\,\mathrm{d}f.$$

Remark 3.3.24. We do not know if Proposition 3.3.23 holds for functions merely in $\mathscr{G}$. E.g., unlike the identification result Lemma 3.3.20 it is unclear whether the gradient estimate from (3.2.20) holds for every $f \in \mathscr{G}$ or whether P_t maps $\mathscr{G}$ into $\mathscr{G}$, $t > 0$. ■

Lemma 3.3.25. *If $f \in \mathscr{F}_{\mathrm{e}} \cap L^1(M)$ with $\mathrm{d}f \in L^1(T^*M)$, then $f \in \mathscr{G}_{\mathrm{reg}}$.*

Proof. Define $f_n \in \mathscr{F}_{\mathrm{e}} \cap L^1(M) \cap L^\infty(M)$ by $f_n := \max\{\min\{f, n\}, -n\}$, $n \in \mathbf{N}$. By Proposition 3.2.37, we have $\mathrm{d}f_n = 1_{\{|f|<n\}}\,\mathrm{d}f \in L^1(T^*M)$, and therefore $\mathrm{d}f_n \to \mathrm{d}f$ in $L^1(T^*M)$ as $n \to \infty$.

We claim that $f_n \in \mathscr{G}_{\mathrm{reg}}$ for every $n \in \mathbf{N}$, which then readily yields the conclusion of the lemma. Indeed, given $n \in \mathbf{N}$ and $t > 0$, we have $\mathsf{P}_t f_n \in \mathrm{Test}(M) \cap L^1(M)$, and in fact $\mathsf{P}_t f_n \in \mathscr{G}$ by Lemma 3.2.54. Since $\mathsf{P}_t f_n \to f_n$ in $\mathscr{F}$ as $t \to 0$, by Lebesgue's theorem and the assumption that $f \in L^1(M)$ it follows that $\mathsf{P}_t f_n \to f_n$ in $L^1(M)$ as $t \to 0$. Moreover, since $\mathrm{d}\mathsf{P}_t f_n \to \mathrm{d}f_n$ in $L^2(T^*M)$ as $t \to 0$ and since $(\mathrm{d}\mathsf{P}_t f_n)_{t\in[0,1]}$ is bounded in $L^1(T^*M)$ thanks to (3.2.20) and exponential boundedness of $(\mathsf{P}_t^\kappa)_{t\geq 0}$ in $L^1(M)$ [ER+20, Rem. 2.14], applying Lebesgue's theorem again we obtain that $\mathrm{d}\mathsf{P}_t f_n \to \mathrm{d}f_n$ in $L^1(T^*M)$ as $t \to 0$. □

Lemma 3.3.26. *For every $f, g \in \mathscr{F}$ we have $f\,g \in \mathscr{G}_{\mathrm{reg}}$ with*

$$\mathrm{d}(f\,g) = f\,\mathrm{d}g + g\,\mathrm{d}f.$$

Proof. Again we define $f_n, g_n \in \mathscr{F} \cap L^\infty(M)$ by $f_n := \max\{\min\{f, n\}, -n\}$ and $g_n := \max\{\min\{g, n\}, -n\}$, $n \in \mathbf{N}$. By Proposition 3.2.37, we have $f_n\,g_n \in \mathscr{F}_{\mathrm{e}} \cap L^1(M)$ with the identities

$$\mathrm{d}(f_n\,g_n) = f_n\,\mathrm{d}g_n + g_n\,\mathrm{d}f_n = f_n\,1_{\{|g|<n\}}\,\mathrm{d}g + g_n\,1_{\{|f|<n\}}\,\mathrm{d}f$$

whence $f_n\,g_n \in \mathscr{G}_{\mathrm{reg}}$ for every $n \in \mathbf{N}$ by Lemma 3.3.25. Since $f_n\,g_n \to f\,g$ in $L^1(M)$ and $f_n\,1_{\{|g|<n\}}\,\mathrm{d}g + g_n\,1_{\{|f|<n\}}\,\mathrm{d}f \to f\,\mathrm{d}g + g\,\mathrm{d}f$ in $L^1(T^*M)$ as $n \to \infty$, respectively, the conclusion follows. □

Second order spaces

Definition 3.3.27. *We define the space $\mathscr{D}_{\mathrm{reg}}(\mathrm{Hess}) \subset \mathscr{D}(\mathrm{Hess})$ as*

$$\mathscr{D}_{\mathrm{reg}}(\mathrm{Hess}) := \mathrm{cl}_{\|\cdot\|_{\mathscr{D}(\mathrm{Hess})}}\,\mathrm{Test}(M).$$

By Corollary 3.3.12, we have

$$\mathscr{D}_{\mathrm{reg}}(\mathrm{Hess}) = \mathrm{cl}_{\|\cdot\|_{\mathscr{D}(\mathrm{Hess})}}\mathscr{D}(\Delta).$$

This space plays an important role in Proposition 3.3.32, but also in later discussions on calculus rules for the covariant derivative, see Subsection 3.4.2.

As indicated in the beginning of Subsection 3.3.4, we consider the space $\mathscr{D}_{2,2,1}(\mathrm{Hess})$ of $\mathscr{F}$-functions with an L^1-Hessian, which seems to be the correct framework for Proposition 3.3.30 and Proposition 3.3.31 below. (The numbers in the subscript denote the degree of integrability of f, $\mathrm{d}f$ and $\mathrm{Hess}\, f$, respectively, $f \in \mathscr{D}_{2,2,1}(\mathrm{Hess})$.) As after Example 3.3.1, one argues that the defining property is well-defined.

Definition 3.3.28. *We define the space* $\mathscr{D}_{2,2,1}(\mathrm{Hess})$ *to consist of all* $f \in \mathscr{F}$ *for which there exists* $A \in L^1((T^*)^{\otimes 2}M)$ *such that for every* $g_1, g_2, h \in \mathrm{Test}(M)$*,*

$$\begin{aligned}
2\int_M h\, A(\nabla g_1, \nabla g_2)\, \mathrm{d}\mathfrak{m}
&= -\int_M \langle \nabla f, \nabla g_1\rangle\, \mathrm{div}(h\, \nabla g_2)\, \mathrm{d}\mathfrak{m} - \int_M \langle \nabla f, \nabla g_2\rangle\, \mathrm{div}(h\, \nabla g_1)\, \mathrm{d}\mathfrak{m}\\
&\quad - \int_M h\, \big\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\big\rangle\, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

In case of existence, A is unique, denoted by $\mathrm{Hess}_1 f$ *and termed the* Hessian *of f.*

By the weak* density of $\mathrm{Test}(T^{\otimes 2}M)$ in $L^\infty(T^{\otimes 2}M)$, the uniqueness statement in Definition 3.3.28 is indeed true given that such an element A exists. Furthermore, if $f \in \mathscr{D}_{2,2,1}(\mathrm{Hess}) \cap \mathscr{D}(\mathrm{Hess})$ then of course

$$\mathrm{Hess}_1 f = \mathrm{Hess}\, f.$$

We shall thus simply write Hess in place of Hess_1 also for functions in $\mathscr{D}_{2,2,1}(\mathrm{Hess})$ without further notice. $\mathscr{D}_{2,2,1}(\mathrm{Hess})$ is a real vector space, and Hess is a linear operator on it.

The space $\mathscr{D}_{2,2,1}(\mathrm{Hess})$ is endowed with the norm $\|\cdot\|_{\mathscr{D}_{2,2,1}(\mathrm{Hess})}$ given by

$$\|f\|_{\mathscr{D}_{2,2,1}(\mathrm{Hess})} := \|f\|_{L^2(M)} + \|\mathrm{d}f\|_{L^2(T^*M)} + \|\mathrm{Hess}\, f\|_{L^1((T^*)^{\otimes 2}M)}.$$

Since both sides of the defining property in Definition 3.3.28 are continuous in f and A w.r.t. convergence in $\mathscr{F}$ and $L^1((T^*)^{\otimes 2}M)$, respectively, this norm turns $\mathscr{D}_{2,2,1}(\mathrm{Hess})$ into a Banach space. It is also separable, since the map $\mathrm{Id}\times\mathrm{d}\times\mathrm{Hess}\colon \mathscr{D}_{2,2,1}(\mathrm{Hess}) \to L^2(M)\times L^2(T^*M)\times L^1((T^*)^{\otimes 2}M)$ is an isometry onto its image, where the latter space is endowed with the usual product norm

$$\|(f,\omega,A)\| := \|f\|_{L^2(M)} + \|\omega\|_{L^2(T^*M)} + \|A\|_{L^1((T^*)^{\otimes 2}M)}$$

which is separable by Theorem 3.2.32 and the discussions in Subsection 3.2.8.

Calculus rules for the Hessian After having introduced the relevant spaces, we finally proceed with the calculus rules for functions in $\mathscr{D}_{2,2,1}(\mathrm{Hess})$.

The next lemma will be technically useful. For $h \in \mathrm{Test}(M)$, the asserted equality is precisely the defining property of $\mathrm{Hess}\, f$ stated in Definition 3.3.2. The general case $h \in \mathscr{F} \cap L^\infty(M)$ follows by replacing h by $\mathsf{P}_t h \in \mathrm{Test}(M)$, $t > 0$, and letting $t \to 0$ with the aid of Lemma 3.2.54.

Lemma 3.3.29. *For every $f \in \mathscr{D}(\mathrm{Hess})$, every $g_1, g_2 \in \mathrm{Test}(M)$ and every $h \in \mathscr{F} \cap L^\infty(M)$,*

$$\begin{aligned}2\int_M h \,\mathrm{Hess}\, f(\nabla g_1, \nabla g_2)\,\mathrm{d}\mathfrak{m}\\ &= -\int_M \langle \nabla f, \nabla g_1\rangle \,\mathrm{div}(h\,\nabla g_2)\,\mathrm{d}\mathfrak{m} - \int_M \langle \nabla f, \nabla g_2\rangle \,\mathrm{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{m}\\ &\quad - \int_M h\,\big\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\big\rangle\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Proposition 3.3.30 (Product rule)**.** *If $f, g \in \mathscr{D}(\mathrm{Hess}) \cap L^\infty(M)$, we have $f\,g \in \mathscr{D}_{2,2,1}(\mathrm{Hess})$ with*

$$\mathrm{Hess}(f\,g) = g\,\mathrm{Hess}\, f + f\,\mathrm{Hess}\, g + \mathrm{d}f \otimes \mathrm{d}g + \mathrm{d}g \otimes \mathrm{d}f.$$

Proof. Note that $f\,g \in \mathscr{F}$, and that the r.h.s. of the claimed identity for $\mathrm{Hess}(f\,g)$ defines an element in $L^1((T^*)^{\otimes 2}M)$. Now, given any $g_1, g_2, h \in \mathrm{Test}(M)$, by Proposition 3.2.37 and Lemma 3.2.54 it follows that

$$\begin{aligned}&-\big\langle \nabla(f\,g), \nabla g_1\big\rangle\,\mathrm{div}(h\,\nabla g_2)\\ &\quad= -f\,\langle \nabla g, \nabla g_1\rangle\,\mathrm{div}(h\,\nabla g_2) - g\,\langle \nabla f, \nabla g_1\rangle\,\mathrm{div}(h\,\nabla g_2)\\ &\quad= -\langle \nabla g, \nabla g_1\rangle\,\mathrm{div}(f\,h\,\nabla g_2) + h\,\langle \nabla g, \nabla g_1\rangle\,\langle \nabla f, \nabla g_2\rangle\\ &\qquad - \langle \nabla f, \nabla g_1\rangle\,\mathrm{div}(g\,h\,\nabla g_2) + h\,\langle \nabla f, \nabla g_1\rangle\,\langle \nabla g, \nabla g_2\rangle \quad \mathfrak{m}\text{-a.e.},\end{aligned}$$

and analogously

$$\begin{aligned}&-\big\langle \nabla(f\,g), \nabla g_2\big\rangle\,\mathrm{div}(h\,\nabla g_1)\\ &\quad= -\langle \nabla g, \nabla g_2\rangle\,\mathrm{div}(f\,h\,\nabla g_1) + h\,\langle \nabla g, \nabla g_2\rangle\,\langle \nabla f, \nabla g_1\rangle\\ &\qquad - \langle \nabla f, \nabla g_2\rangle\,\mathrm{div}(g\,h\,\nabla g_1) + h\,\langle \nabla f, \nabla g_2\rangle\,\langle \nabla g, \nabla g_1\rangle \quad \mathfrak{m}\text{-a.e.},\end{aligned}$$

while finally

$$\begin{aligned}&-h\,\big\langle \nabla(f\,g), \nabla\langle \nabla g_1, \nabla g_2\rangle\big\rangle\\ &\quad= -h\,f\,\big\langle \nabla g, \nabla\langle \nabla g_1, \nabla g_2\rangle\big\rangle - h\,g\,\big\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\big\rangle \quad \mathfrak{m}\text{-a.e.}\end{aligned}$$

Adding up these three identities and using Lemma 3.3.29 for f with $g\,h$ in place of h and for g with $f\,h$ in place of h and by (3.2.6), the claim readily follows. □

Proposition 3.3.31 (Chain rule)**.** *Let $f \in \mathscr{D}(\mathrm{Hess})$, and let $\varphi \in \mathrm{C}^1(\mathbf{R})$ such that $\varphi' \in \mathrm{Lip}_\mathrm{b}(\mathbf{R})$. If $\mathfrak{m}[M] = \infty$, we also assume $\varphi(0) = 0$. Then $\varphi \circ f \in \mathscr{D}_{2,2,1}(\mathrm{Hess})$ as well as*

$$\mathrm{Hess}(\varphi \circ f) = \big[\varphi'' \circ f\big]\,\mathrm{d}f \otimes \mathrm{d}f + \big[\varphi' \circ f\big]\,\mathrm{Hess}\, f,$$

where $\varphi'' \circ f$ is defined arbitrarily on the intersection of the set of non-differentiability points of φ'.

Proof. By Proposition 3.2.37 and the boundedness of φ', we have $\varphi \circ f \in \mathscr{F}$. The r.h.s. of the claimed identity defines an object in $L^1(M)$. Now, given any $g_1, g_2, h \in \mathrm{Test}(M)$, as in the proof of Proposition 3.3.30 we have

$$-\big\langle \nabla(\varphi \circ f), \nabla g_1\big\rangle\,\mathrm{div}(h\,\nabla g_2)$$

$$
\begin{aligned}
&= -\big[\varphi' \circ f\big]\,\langle \nabla f, \nabla g_1\rangle\,\mathrm{div}(h\,\nabla g_2)\\
&= -\langle \nabla f, \nabla g_1\rangle\,\mathrm{div}\big(\big[\varphi' \circ f\big]\,h\,\nabla g_2\big)\\
&\qquad + h\,\big[\varphi'' \circ f\big]\,\langle \nabla f, \nabla g_1\rangle\,\langle \nabla f, \nabla g_2\rangle \quad \mathfrak{m}\text{-a.e.},
\end{aligned}
$$

and analogously

$$
\begin{aligned}
&-\big\langle \nabla(\varphi\circ f), \nabla g_2\big\rangle\,\mathrm{div}(h\,\nabla g_1)\\
&\qquad = -\langle \nabla f, \nabla g_2\rangle\,\mathrm{div}\big(\big[\varphi' \circ f\big]\,h\,\nabla g_1\big)\\
&\qquad\qquad + h\,\big[\varphi'' \circ f\big]\,\langle \nabla f, \nabla g_2\rangle\,\langle \nabla f, \nabla g_1\rangle \quad \mathfrak{m}\text{-a.e.},
\end{aligned}
$$

while finally

$$
-h\,\big\langle \nabla(\varphi\circ f), \nabla\langle \nabla g_1, \nabla g_2\rangle\big\rangle = -h\,\big[\varphi' \circ f\big]\,\big\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\big\rangle \quad \mathfrak{m}\text{-a.e.}
$$

Adding up these three identities and using Lemma 3.3.29 for f with $h\,[\varphi' \circ f]$ in place of h and by (3.2.6), the claim readily follows. □

Proposition 3.3.32 (Product rule for gradients)**.** *The following properties hold for every $f \in \mathscr{D}(\mathrm{Hess})$ and every $g \in \mathscr{D}_{\mathrm{reg}}(\mathrm{Hess})$.*

(i) *We have $\langle \nabla f, \nabla g\rangle \in \mathscr{G}$ with*

$$
\mathrm{d}\langle \nabla f, \nabla g\rangle = \mathrm{Hess}\, f(\nabla g, \cdot) + \mathrm{Hess}\, g(\nabla f, \cdot).
$$

(ii) *For every $g_1, g_2 \in \mathscr{D}_{\mathrm{reg}}(\mathrm{Hess})$,*

$$
\begin{aligned}
2\,\mathrm{Hess}\, f(\nabla g_1, \nabla g_2) &= \big\langle \nabla g_1, \nabla\langle \nabla f, \nabla g_2\rangle\big\rangle + \big\langle \nabla g_2, \nabla\langle \nabla f, \nabla g_1\rangle\big\rangle\\
&\qquad - \big\langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\big\rangle \quad \mathfrak{m}\text{-a.e.}
\end{aligned}
$$

(iii) *If $f \in \mathscr{D}_{\mathrm{reg}}(\mathrm{Hess})$ as well, then $\langle \nabla f, \nabla g\rangle \in \mathscr{G}_{\mathrm{reg}}$.*

Proof. We first treat item (i) under the additional assumption that $g \in \mathrm{Test}(M)$. Let $g' \in \mathrm{Test}_{L^\infty}(M)$ and $h \in \mathrm{Test}(M)$. Let $(f_n)_{n\in\mathbf{N}}$ be a sequence in $\mathrm{Test}(M)$ which converges to f in $\mathscr{F}$. Then by Definition 3.3.2,

$$
\begin{aligned}
&\int_M h\,\big[\mathrm{Hess}\, f(\nabla g, \nabla g') + \mathrm{Hess}\, g(\nabla f, \nabla g')\big]\,\mathrm{d}\mathfrak{m}\\
&\quad = \lim_{n\to\infty}\int_M h\,\big[\mathrm{Hess}\, f(\nabla g, \nabla g') + \mathrm{Hess}\, g(\nabla f_n, \nabla g')\big]\,\mathrm{d}\mathfrak{m}\\
&\quad = -\frac{1}{2}\int_M \langle \nabla f, \nabla g\rangle\,\mathrm{div}(h\,\nabla g')\,\mathrm{d}\mathfrak{m} - \frac{1}{2}\int_M \langle \nabla f, \nabla g'\rangle\,\mathrm{div}(h\,\nabla g)\,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad - \frac{1}{2}\int_M h\,\big\langle \nabla f, \nabla\langle \nabla g, \nabla g'\rangle\big\rangle\,\mathrm{d}\mathfrak{m}\\
&\qquad - \frac{1}{2}\lim_{n\to\infty}\int_M \langle \nabla g, \nabla f_n\rangle\,\mathrm{div}(h\,\nabla g')\,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad - \frac{1}{2}\lim_{n\to\infty}\int_M \langle \nabla g, \nabla g'\rangle\,\mathrm{div}(h\,\nabla f_n)\,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad - \frac{1}{2}\lim_{n\to\infty}\int_M h\,\big\langle \nabla g, \nabla\langle \nabla f_n, \nabla g'\rangle\big\rangle\,\mathrm{d}\mathfrak{m}
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_M \langle\nabla f,\nabla g\rangle\,\mathrm{div}(h\,\nabla g')\,\mathrm{d}\mathfrak{m} - \frac{1}{2}\int_M \langle\nabla f,\nabla g'\rangle\,\mathrm{div}(h\,\nabla g)\,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad + \frac{1}{2}\lim_{n\to\infty}\int_M \langle\nabla f_n,\nabla g'\rangle\,\mathrm{div}(h\,\nabla g)\,\mathrm{d}\mathfrak{m}\\
&\qquad - \frac{1}{2}\int_M h\,\big\langle\nabla f,\nabla\langle\nabla g,\nabla g'\rangle\big\rangle\,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad + \frac{1}{2}\lim_{n\to\infty}\int_M h\,\big\langle\nabla\langle\nabla g,\nabla g'\rangle,\nabla f_n\big\rangle\,\mathrm{d}\mathfrak{m}\\
&= -\int_M \langle\nabla f,\nabla g\rangle\,\mathrm{div}(h\,\nabla g')\,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

In the second last step, we used Lemma 3.2.54 and the fact that $\langle\nabla g,\nabla g'\rangle\in\mathcal{F}$ as well as $\langle\nabla f_n,\nabla g'\rangle\in\mathcal{F}$ for every $n\in\mathbf{N}$ provided by Proposition 3.2.75. Since $\mathrm{Hess}\,f(\nabla g,\cdot)+\mathrm{Hess}\,g(\nabla f,\cdot)\in L^1(T^*M)$, the claim follows from the definition of $\mathcal{G}$ in Definition 3.3.18.

To cover the case of general $g\in\mathcal{D}_{\mathrm{reg}}(\mathrm{Hess})$, simply observe that given any $f\in\mathcal{D}(\mathrm{Hess})$, $g'\in\mathrm{Test}_{L^\infty}(M)$ and $h\in\mathrm{Test}(M)$, both sides of the above computation are continuous in g w.r.t. $\|\cdot\|_{\mathcal{D}(\mathrm{Hess})}$.

Point (ii) easily follows by expressing each summand in the r.h.s. of the claimed identity in terms of the formula from (i), and the symmetry of the Hessian known from Theorem 3.3.3.

We finally turn to (iii). If $f,g\in\mathrm{Test}(M)$, then $\langle\nabla f,\nabla g\rangle\in\mathcal{F}\cap L^1(M)$ thanks to Proposition 3.2.75, while $\mathrm{d}\langle\nabla f,\nabla g\rangle\in L^1(T^*M)$ by (i). The $\mathcal{G}_{\mathrm{reg}}$-regularity of $\langle\nabla f,\nabla g\rangle$ thus follows from Lemma 3.3.25. For general $f,g\in\mathcal{D}_{\mathrm{reg}}(\mathrm{Hess})$, let $(f_n)_{n\in\mathbf{N}}$ and $(g_n)_{n\in\mathbf{N}}$ be two sequences in $\mathrm{Test}(M)$ such that $f_n\to f$ and $g_n\to g$ in $\mathcal{D}(\mathrm{Hess})$ as $n\to\infty$, respectively. Then clearly $\langle\nabla f_n,\nabla g_n\rangle\to\langle\nabla f,\nabla g\rangle$ in $L^1(M)$ as $n\to\infty$, while $\mathrm{Hess}\,f_n(\nabla g_n,\cdot)+\mathrm{Hess}\,g_n(\nabla f_n,\cdot)\to\mathrm{Hess}\,f(\nabla g,\cdot)+\mathrm{Hess}\,g(\nabla f,\cdot)$ in $L^1(T^*M)$ as $n\to\infty$. By the $\mathcal{G}_{\mathrm{reg}}$-regularity of $\langle\nabla f_n,\nabla g_n\rangle$ for every $n\in\mathbf{N}$ already shown above, the proof is terminated. □

Combining the last two items of Proposition 3.3.32 with Proposition 3.3.23 thus entails the following locality property.

Lemma 3.3.33 (Locality of the Hessian). *For every $f,g\in\mathcal{D}_{\mathrm{reg}}(\mathrm{Hess})$,*

$$
1_{\{f=g\}}\,\mathrm{Hess}\,f = 1_{\{f=g\}}\,\mathrm{Hess}\,g.
$$

3.4 Covariant derivative

3.4.1 The Sobolev space $W^{1,2}(TM)$

In a similar kind as in Example 3.3.1, the smooth context motivates how we should define a nonsmooth covariant derivative acting on vector fields, having now the notion of Hessian at our disposal.

Example 3.4.1. Let M be a Riemannian manifold with boundary. The covariant derivative ∇X of $X\in\Gamma(TM)$ is uniquely defined by the pointwise relation

$$
\begin{aligned}
\big\langle\nabla_{\nabla g_1}X,\nabla g_2\big\rangle &= \nabla X:(\nabla g_1\otimes\nabla g_2)\\
&= \big\langle\nabla\langle X,\nabla g_2\rangle,\nabla g_1\big\rangle - \mathrm{Hess}\,g_2(\nabla g_1,X)
\end{aligned}
\tag{3.4.1}
$$

for every $g_1, g_2 \in \mathrm{C}_\mathrm{c}^\infty(M)$, see e.g. [GP20a, Thm. 6.2.1]. That is, ∇ is tensorial in its first and derivative in its second component, torsion-free and metrically compatible if and only if the second equality in (3.4.1) holds for every g_1 and g_2 as above. This yields an alternative definition of the Levi-Civita connection ∇ in place of Koszul's formula, see e.g. [Pet06, p. 25], which does not use Lie brackets. (Indeed, in our setting it is not even clear how to define the Lie bracket without covariant derivatives.)

As in Example 3.3.1, ∇ is still uniquely determined on $\Gamma(TM)$ by the validity of (3.4.1) for every $g_1, g_2 \in \mathrm{C}_\mathrm{c}^\infty(TM)$ for which

$$\langle \nabla g_1, \mathsf{n}\rangle = \langle \nabla g_2, \mathsf{n}\rangle = 0 \quad \text{on } \partial M.$$

For such g_1 and g_2, we integrate (3.4.1) against $h\,\mathfrak{m}$ for a given $h \in \mathrm{C}_\mathrm{c}^\infty(M)$ to obtain, using again the regularity discussion around (3.3.3),

$$\begin{aligned}\int_M h\,\nabla X : (\nabla g_1 \otimes \nabla g_2)\,\mathrm{d}\mathfrak{v} \\ = -\int_M \langle X, \nabla g_2\rangle \operatorname{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{v} - \int_M h \operatorname{Hess} g_2(X, \nabla g_1)\,\mathrm{d}\mathfrak{v}.\end{aligned} \tag{3.4.2}$$

This identity characterizes ∇X by the existence of a smooth extension to ∂M. ■

In our setting, the r.h.s. of (3.4.2) — with $\mathfrak{v}$ replaced by $\mathfrak{m}$ — still makes sense for $g_1, g_2, h \in \mathrm{Test}(M)$ and is compatible with the smooth case (in the sense that no boundary terms show up in the nonsmooth terminology of Definition 3.2.47), which is argued as in the paragraph after Example 3.3.1. Indeed, for $X \in L^2(TM)$ we have $\langle X, \nabla g_2\rangle \in L^2(M)$ as well as $\operatorname{div}(h\,\nabla g_1) = \langle \nabla h, \nabla g_1\rangle + h\,\Delta g_1 \in L^2(M)$ by Lemma 3.2.54, while the second integral on the r.h.s. of (3.4.2) is trivially well-defined.

This motivates the subsequent definition.

Definition 3.4.2. *The space $W^{1,2}(TM)$ is defined to consist of all $X \in L^2(TM)$ for which there exists $T \in L^2(T^{\otimes 2}M)$ such that for every $g_1, g_2, h \in \mathrm{Test}(M)$,*

$$\begin{aligned}\int_M h\,T : (\nabla g_1 \otimes \nabla g_2)\,\mathrm{d}\mathfrak{m} \\ = -\int_M \langle X, \nabla g_2\rangle \operatorname{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{m} - \int_M h \operatorname{Hess} g_2(X, \nabla g_1)\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

In case of existence, the element T is unique, denoted by ∇X and termed the covariant derivative *of X.*

Arguing as for the Hessian after Definition 3.3.2, the uniqueness statement in Definition 3.4.2 is derived. In particular, $W^{1,2}(TM)$ constitutes a vector space and the covariant derivative ∇ is a linear operator on it. Further properties can be consulted in Theorem 3.4.3 below.

The space $W^{1,2}(TM)$ is endowed with the norm $\|\cdot\|_{W^{1,2}(TM)}$ given by

$$\|X\|_{W^{1,2}(TM)}^2 := \|X\|_{L^2(TM)}^2 + \|\nabla X\|_{L^2(T^{\otimes 2}M)}^2.$$

We also define the *covariant* functional $\mathscr{E}_{\mathrm{cov}}\colon L^2(TM) \to [0,\infty]$ by

$$\mathscr{E}_{\mathrm{cov}}(X) := \begin{cases} \displaystyle\int_M |\nabla X|_{\mathrm{HS}}^2\,\mathrm{d}\mathfrak{m} & \text{if } X \in W^{1,2}(TM),\\ \infty & \text{otherwise.}\end{cases} \tag{3.4.3}$$

Theorem 3.4.3. *The space $W^{1,2}(TM)$, the covariant derivative ∇ and the functional $\mathscr{E}_{\mathrm{cov}}$ possess the following properties.*

(i) *$W^{1,2}(TM)$ is a separable Hilbert space w.r.t. $\|\cdot\|_{W^{1,2}(TM)}$.*
(ii) *The covariant derivative ∇ is a closed operator. That is, the image of the map $\mathrm{Id}\times\nabla\colon W^{1,2}(TM)\to L^2(TM)\times L^2(T^{\otimes 2}M)$ is a closed subspace of $L^2(TM)\times L^2(T^{\otimes 2}M)$.*
(iii) *For every $f\in\mathrm{Test}(M)$ and every $g\in\mathrm{Test}(M)\cup\mathbf{R}\,1_M$, we have $g\,\nabla f\in W^{1,2}(TM)$ with*

$$\nabla(g\,\nabla f) = \nabla g\otimes\nabla f + g\,(\mathrm{Hess}\,f)^\sharp,$$

with the interpretation $\nabla 1_M := 0$. In particular $\mathrm{Reg}(TM)\subset W^{1,2}(TM)$, and $W^{1,2}(TM)$ is dense in $L^2(TM)$.
(iv) *The functional $\mathscr{E}_{\mathrm{cov}}$ is lower semicontinuous, and for every $X\in L^2(TM)$ we have the duality formula*

$$\begin{aligned}\mathscr{E}_{\mathrm{cov}}(X) = \sup\Big\{&-2\sum_{i=1}^n\int_M\langle X,Z_i\rangle\,\mathrm{div}\,Y_i\,\mathrm{d}\mathfrak{m} - 2\sum_{i=1}^n\int_M\nabla Z_i : (Y_i\otimes X)\,\mathrm{d}\mathfrak{m}\\ &-\int_M\Big|\sum_{i=1}^n Y_i\otimes Z_i\Big|^2\,\mathrm{d}\mathfrak{m} : n\in\mathbf{N},\ Y_i,Z_i\in\mathrm{Test}(TM)\Big\}.\end{aligned}$$

Proof. Item (ii) is addressed by observing that given any $g_1,g_2,h\in\mathrm{Test}(M)$, both sides of the defining property of ∇ in Definition 3.4.2 are continuous in X and T w.r.t. weak convergence in $L^2(TM)$ and $L^2(T^{\otimes 2}M)$, respectively.

For (i), it is clear from (ii) and the trivial parallelogram identity of $\|\cdot\|_{W^{1,2}(TM)}$ that $W^{1,2}(TM)$ is a Hilbert space. To prove its separability, we endow $L^2(TM)\times L^2(T^{\otimes 2}M)$ with the product norm $\|\cdot\|$ given by

$$\|(X,T)\|^2 := \big\|X\big\|^2_{L^2(TM)} + \big\|T\big\|^2_{L^2(T^{\otimes 2}M)}.$$

Recall from Theorem 3.2.32, Proposition 3.2.19 and the discussions in Subsection 3.2.3 and Subsection 3.2.8 that $L^2(TM)$ and $L^2(T^{\otimes 2}M)$ are separable Hilbert spaces, and so is their Cartesian product. As $\mathrm{Id}\times\nabla\colon W^{1,2}(TM)\to L^2(TM)\times L^2(T^{\otimes 2}M)$ is a bijective isometry onto its image, the proof of (i) is completed.

Item (iii) for $g := 1_M$ follows from Proposition 3.3.32. Indeed, given any $g_1,g_2,h\in\mathrm{Test}(M)$, by definition of $\mathscr{G}$ we have

$$\begin{aligned}&\int_M h\,\mathrm{Hess}\,f(\nabla g_1,\nabla g_2)\,\mathrm{d}\mathfrak{m}\\ &\qquad = \int_M h\,\big\langle\nabla\langle\nabla f,\nabla g_2\rangle,\nabla g_1\big\rangle\,\mathrm{d}\mathfrak{m} - \int_M h\,\mathrm{Hess}\,g_2(\nabla f,\nabla g_2)\,\mathrm{d}\mathfrak{m}\\ &\qquad = -\int_M\langle\nabla f,\nabla g_2\rangle\,\mathrm{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{m} - \int_M h\,\mathrm{Hess}\,g_2(\nabla f,\nabla g_2)\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

The argument for $g\in\mathrm{Test}(M)$ follows similar lines. To prove it, let $g_1,g_2,h\in\mathrm{Test}(M)$. Recall from Proposition 3.2.75 that $\langle\nabla f,\nabla g_2\rangle\in\mathscr{F}$. Since additionally $\langle\nabla f,\nabla g_2\rangle\in L^\infty(M)$, we also have $g\,\langle\nabla f,\nabla g_2\rangle\in\mathscr{F}$ by the Leibniz rule for the gradient — compare with Proposition 3.2.37 — which, also taking into account Proposition 3.3.32 and Lemma 3.3.20, yields

$$\mathrm{d}\big[g\,\langle\nabla f,\nabla g_2\rangle\big] = \langle\nabla f,\nabla g_2\rangle\,\mathrm{d}g + g\,\mathrm{Hess}\,f(\nabla g_2,\cdot) + g\,\mathrm{Hess}\,g_2(\nabla f,\cdot).$$

This entails that

$$\begin{aligned}
&-\int_M g\,\langle\nabla f,\nabla g_2\rangle\,\mathrm{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{m} - \int_M h\,g\,\mathrm{Hess}\,g_2(\nabla f,\nabla g_1)\,\mathrm{d}\mathfrak{m}\\
&\qquad = \int_M \mathrm{d}\big[g\,\langle\nabla f,\nabla g_2\rangle\big](h\,\nabla g_1)\,\mathrm{d}\mathfrak{m} - \int_M h\,g\,\mathrm{Hess}\,g_2(\nabla f,\nabla g_1)\,\mathrm{d}\mathfrak{m}\\
&\qquad = \int_M h\,\big[\langle\nabla g,\nabla g_1\rangle\,\langle\nabla f,\nabla g_2\rangle + g\,\mathrm{Hess}\,f(\nabla g_1,\nabla g_2)\big]\,\mathrm{d}\mathfrak{m},
\end{aligned}$$

which is the claim by the definition (3.2.6) of the pointwise tensor product.

Concerning (iv), the lower semicontinuity of $\mathcal{E}_{\mathrm{cov}}$ follows from the fact that bounded sets in the Hilbert space $L^2(T^{\otimes 2}M)$ are weakly relatively compact, and from the closedness of ∇ that has been shown in (ii).

The proof of the duality formula for $\mathcal{E}_{\mathrm{cov}}$ is analogous to the one for the duality formula for $\mathcal{E}_2$ in Theorem 3.3.3. Details are left to the reader. □

Remark 3.4.4. Similarly to Remark 3.3.6 and motivated by the $\mathrm{RCD}^*(K,N)$ result [Han19, Prop. 3.12], $K \in \mathbf{R}$ and $N \in [1,\infty)$, in practice the notion of covariant derivative from Definition 3.4.2 should only depend on conformal transformations of $\langle\cdot,\cdot\rangle$, but be independent of drift transformations of $\mathfrak{m}$. ■

3.4.2 Calculus rules

In this subsection, we proceed in showing less elementary — still expected — calculus rules for the covariant derivative. Among these, we especially regard Proposition 3.4.11 and Lemma 3.4.13 as keys for the functionality of our second order axiomatization which also potentially involves "boundary contributions".

Some auxiliary spaces of vector fields As in Subsection 3.3.4, we introduce two Sobolev spaces of vector fields we use in the sequel. In fact, the space $H^{1,2}(TM)$ which is introduced next plays a dominant role later, see e.g. Subsection 3.4.3.

Definition 3.4.5. *We define the space* $H^{1,2}(TM) \subset W^{1,2}(TM)$ *as*

$$H^{1,2}(TM) := \mathrm{cl}_{\|\cdot\|_{W^{1,2}(TM)}}\mathrm{Reg}(TM).$$

By Theorem 3.4.3, we see that $\nabla\,\mathcal{D}_{\mathrm{reg}}(\mathrm{Hess}) \subset H^{1,2}(TM)$.

$H^{1,2}(TM)$ is in general a strict subset of $W^{1,2}(TM)$. For instance, on compact Riemannian manifolds with boundary, this follows as in the beginning of Subsection 3.3.4 by (3.2.16), Lemma 3.2.54, Example 3.2.48 and Proposition 3.2.1 for $E := TM$.

Remark 3.4.6. For noncollapsed mGH-limits of Riemannian manifolds without boundary under uniform Ricci and diameter bounds, it is proved in [Hon17, Prop. 4.5] that $H^{1,2}(TM) = W^{1,2}(TM)$. This does not conflict with our above argument involving the presence of a boundary, since spaces "with boundary" are known not to appear as noncollapsed Ricci limits [CC97, Thm. 6.1]. ■

Remark 3.4.7. Besides technical reasons, Definition 3.4.5 has the advantage that it includes both gradient vector fields of test functions as well as general elements of $\mathrm{Test}(TM)$ in the calculus rules below. See also Section 3.6.

The "$H^{1,2}$-space" of vector fields introduced in [Gig18, Def. 3.4.3] is rather given by $\mathrm{cl}_{\|\cdot\|_{W^{1,2}(TM)}}\mathrm{Test}(TM)$, which — in our generality — is a priori *smaller* than $H^{1,2}(TM)$.

The converse inclusion seems more subtle: as similarly encountered in Remark 3.3.22, the issue is that we do not really know how to pass from zeroth-order factors belonging to $\mathrm{Test}(M)$ to constant ones in a topology which also takes into account "derivatives" of vector fields in our generality. However, since "$\supset$" holds e.g. if M is intrinsically complete by Theorem 3.4.3, Definition 3.4.5 and the approach from [Gig18, Def. 3.4.3] give rise to the same space on $\mathrm{RCD}(K,\infty)$ spaces, $K \in \mathbf{R}$. ■

As in Definition 3.3.28, in view of Lemma 3.4.9 we introduce a space intermediate between $W^{1,2}(TM)$ and "$W^{1,1}(TM)$", a suitable space (that we do not define) of all $X \in L^1(TM)$ with covariant derivative ∇X in $L^1(TM)$. The problem arising in defining the latter, compare with Definition 3.4.2, is that we do not know if there exists a large enough class of $g_2 \in \mathrm{Test}(M)$ such that $\mathrm{Hess}\, g_2 \in L^\infty((T^*)^{\otimes 2}M)$.

Definition 3.4.8. *The space $W^{(2,1)}(TM)$ is defined to consist of all $X \in L^2(TM)$ for which there exists $T \in L^1(T^{\otimes 2}M)$ such that for every $g_1, g_2, h \in \mathrm{Test}(M)$,*

$$\begin{aligned}\int_M & h\,T : (\nabla g_1 \otimes \nabla g_2)\,\mathrm{d}\mathfrak{m}\\ &= -\int_M \langle X, \nabla g_2\rangle\,\mathrm{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{m} - \int_M h\,\mathrm{Hess}\, g_2(X, \nabla g_1)\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

In case of existence, the element T is unique, denoted by $\nabla_1 X$ and called the covariant derivative *of X.*

Indeed, the uniqueness follows from weak* density of $\mathrm{Test}(T^{\otimes 2}M)$ in $L^\infty(T^{\otimes 2}M)$. In particular, $W^{(2,1)}(TM)$ is clearly a vector space, and ∇_1 is a linear operator on it. By definition, for $X \in W^{(2,1)}(TM) \cap W^{1,2}(TM)$ we furthermore have

$$\nabla_1 X = \nabla X,$$

whence we subsequently write ∇ in place of ∇_1 as long as confusion is excluded.

We endow $W^{(2,1)}(TM)$ with the norm $\|\cdot\|_{W^{(2,1)}(TM)}$ given by

$$\|X\|_{W^{(2,1)}(TM)} := \|X\|_{L^2(TM)} + \|\nabla X\|_{L^1(T^{\otimes 2}M)}.$$

Since both sides of the defining property in Definition 3.4.8 are continuous w.r.t. weak convergence in X and T, respectively, this norm turns $W^{(2,1)}(TM)$ into a Banach space. Moreover, since the map $\mathrm{Id} \times \nabla \colon W^{(2,1)}(TM) \to L^2(TM) \times L^1(T^{\otimes 2}M)$ is a bijective isometry onto its image — where $L^2(TM) \times L^1(T^{\otimes 2}M)$ is endowed with the usual product norm — Theorem 3.2.32, Proposition 3.2.19 and the tensor product discussion from Subsection 3.2.8 show that $W^{(2,1)}(TM)$ is separable.

Calculus rules for the covariant derivative

Lemma 3.4.9 (Leibniz rule)**.** *Let $X \in W^{1,2}(TM)$ and $f \in \mathscr{F} \cap L^\infty(M)$. Then $f\,X \in W^{(2,1)}(TM)$ and*

$$\nabla(f\,X) = \nabla f \otimes X + f\,\nabla X.$$

Proof. We first assume that $f \in \mathrm{Test}(M)$. Given any $g_1, g_2, h \in \mathrm{Test}(M)$, we have $h\,f \in \mathrm{Test}(M)$ since $\mathrm{Test}(M)$ is an algebra. By Lemma 3.2.54 and the Leibniz rule for the gradient, Proposition 3.2.37, we obtain

$$\mathrm{div}(h\,f\,\nabla g_1) = h\,\langle \nabla f, \nabla g_1\rangle + f\,\mathrm{div}(h\,\nabla g_1) \quad \mathfrak{m}\text{-a.e.}$$

Hence, by definition of ∇X from Definition 3.4.2 and Theorem 3.3.11,

$$\begin{aligned}\int_M h\,(f\,\nabla X) : (\nabla g_1 \otimes \nabla g_2)\,\mathrm{d}\mathfrak{m}\\ &= -\int_M \langle X, \nabla g_2\rangle\,\mathrm{div}(h\,f\,\nabla g_1)\,\mathrm{d}\mathfrak{m} - \int_M h\,f\,\mathrm{Hess}\,g_2(X,\nabla g_1)\,\mathrm{d}\mathfrak{m}\\ &= -\int_M h\,\langle \nabla f, \nabla g_1\rangle\,\langle X, \nabla g_2\rangle\,\mathrm{d}\mathfrak{m} - \int_M \langle f\,X, \nabla g_2\rangle\,\mathrm{div}(h\,\nabla g_1)\,\mathrm{d}\mathfrak{m}\\ &\qquad - \int_M h\,\mathrm{Hess}\,g_2(f\,X, \nabla g_1)\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Rearranging terms according to Definition 3.4.8 yields the claim.

For general $f \in \mathscr{F} \cap L^\infty(M)$, note that on the one hand $\mathsf{P}_t f \in \mathrm{Test}(M)$ and $\|\mathsf{P}_t f\|_{L^\infty(M)} \le \|f\|_{L^\infty(M)}$ holds for every $t > 0$, and on the other hand $\mathsf{P}_t f \to f$ in $\mathscr{F}$ as $t \to 0$. Thus, we easily see that $\mathsf{P}_t f\,X \to f\,X$ in $L^2(TM)$ as well as $\nabla(\mathsf{P}_t f\,X) = \nabla\mathsf{P}_t f \otimes X + \mathsf{P}_t f\,\nabla X \to f \otimes X + f\,\nabla X$ in $L^1(T^{\otimes 2}M)$ as $t \to 0$, and this suffices to conclude the proof. □

Remark 3.4.10. Elementary approximation arguments, also using Lemma 3.2.73 for (ii), entail the following two $H^{1,2}$-variants of Lemma 3.4.9.

(i) If $f \in \mathrm{Test}(M)$ and $X \in H^{1,2}(TM)$, then $f\,X \in H^{1,2}(TM)$ and

$$\nabla(f\,X) = \nabla f \otimes X + f\,\nabla X.$$

(ii) The same conclusion as in (i) holds if merely $f \in \mathscr{F} \cap L^\infty(M)$ and $X \in H^{1,2}(TM) \cap L^\infty(TM)$. ■

In view of the important *metric compatibility* of ∇ now discussed in Proposition 3.4.11 as well as its consequences below, we shall first make sense of *directional derivatives* of a given $X \in W^{1,2}(TM)$ in the direction of $Z \in L^0(TM)$. There exists a unique vector field $\nabla_Z X \in L^0(TM)$ which satisfies

$$\langle \nabla_Z X, Y\rangle = \nabla X : (Z \otimes Y) \quad \mathfrak{m}\text{-a.e.} \tag{3.4.4}$$

for every $Y \in L^0(TM)$. Indeed, the r.h.s. is a priori well-defined for every $Y, Z \in L^0(TM)$ for which $Z \otimes Y \in L^2(T^{\otimes 2}M)$, but this definition can and will be uniquely extended by continuity to a bilinear map $\nabla X : (\cdot \otimes \cdot) \colon L^0(TM)^2 \to L^0(M)$, and the appropriate existence of $\nabla_Z X$ is then due to Proposition 3.2.20. In particular,

$$|\nabla_Z X| \le |\nabla X|\,|Z| \quad \mathfrak{m}\text{-a.e.} \tag{3.4.5}$$

Proposition 3.4.11 (Metric compatibility)**.** *Let $X \in W^{1,2}(TM)$ and $Y \in H^{1,2}(TM)$. Then $\langle X, Y\rangle \in \mathscr{G}$, and for every $Z \in L^0(TM)$,*

$$\mathrm{d}\langle X, Y\rangle(Z) = \big\langle \nabla_Z X, Y\big\rangle + \big\langle X, \nabla_Z Y\big\rangle \quad \mathfrak{m}\text{-a.e.}$$

Moreover, if $X \in H^{1,2}(TM)$ as well, then $\langle X, Y\rangle \in \mathscr{G}_{\mathrm{reg}}$.

Proof. We first prove the claim for $Y = \nabla f$ for some $f \in \mathrm{Test}(M)$. In this case, given any $g, h \in \mathrm{Test}(M)$, by definition of ∇X,

$$\int_M h\,\nabla X : (\nabla g \otimes \nabla f)\,\mathrm{d}\mathfrak{m}$$

$$= -\int_M \langle X, \nabla f\rangle \operatorname{div}(h\,\nabla g)\,\mathrm{d}\mathfrak{m} - \int_M h \operatorname{Hess} f(X, \nabla g)\,\mathrm{d}\mathfrak{m}.$$

Rearranging terms and recalling Definition 3.3.18 as well as Theorem 3.4.3, we see that $\langle X, \nabla f\rangle \in L^1(M)$ belongs to $\mathscr{G}$ and

$$\mathrm{d}\langle X, \nabla f\rangle = \nabla X : (\cdot \otimes f) + (\operatorname{Hess} f)^\sharp : (\cdot \otimes \nabla f).$$

It follows from (3.4.4) that for every $Z \in \mathrm{Test}(TM)$,

$$\mathrm{d}\langle X, \nabla f\rangle(Z) = \big\langle \nabla_Z X, \nabla f\big\rangle + \big\langle \nabla_Z \nabla f, X\big\rangle \quad \mathfrak{m}\text{-a.e.}$$

By L^∞-linear extension and the density of $\mathrm{Test}(TM)$ in $L^2(TM)$, this identity holds in fact for every $Z \in L^2(TM)$, and therefore extends to arbitrary $Z \in L^0(TM)$ by construction of the latter space in Subsection 3.2.3.

The case $Y := g\,\nabla f$, $f, g \in \mathrm{Test}(M)$, follows by the trivial $\mathfrak{m}$-a.e. identity $\langle X, Y\rangle = \langle g\,X, \nabla f\rangle$, the previously derived identity and Lemma 3.4.9.

By linearity of the covariant derivative and Theorem 3.4.3, the foregoing discussion thus readily covers the case of general $Y \in \mathrm{Reg}(TM)$.

Now, given any $Y \in H^{1,2}(TM)$, a sequence $(Y_n)_{n\in\mathbf{N}}$ in $\mathrm{Reg}(TM)$ such that $Y_n \to Y$ in $W^{1,2}(TM)$ as $n \to \infty$ and $Z \in L^2(TM) \cap L^\infty(TM)$, we have $\langle X, Y_n\rangle \to \langle X, Y\rangle$ and $\langle \nabla_Z X, Y_n\rangle + \langle \nabla_Z Y_n, X\rangle \to \langle \nabla_Z X, Y\rangle + \langle \nabla_Z Y, X\rangle$ in $L^1(M)$ as $n \to \infty$. Passing to the limit in the definition of $\mathscr{G}$, it straightforwardly follows that $\langle X, Y\rangle \in \mathscr{G}$. The claimed identity for $\mathrm{d}\langle X, Y\rangle(Z)$ follows after passing to suitable subsequences, and it extends to arbitrary $Z \in L^0(TM)$ as in the first step of the current proof.

To prove the last claim, suppose first that $X, Y \in \mathrm{Reg}(M)$, in which case $\langle X, Y\rangle \in \mathscr{G} \cap \mathscr{F}$ by Theorem 3.4.3, what we already proved, Proposition 3.2.75 and Proposition 3.2.37. By the duality between $L^1(TM)$ and $L^\infty(T^*M)$ as L^∞-modules, see Subsection 3.2.8, and Remark 3.4.12 below, we deduce that $\mathrm{d}\langle X, Y\rangle \in L^1(TM)$, whence $\langle X, Y\rangle \in \mathscr{G}_{\mathrm{reg}}$ thanks to Lemma 3.3.20 and Lemma 3.3.25.

Lastly, given arbitrary $X, Y \in H^{1,2}(TM)$, let $(X_n)_{n\in\mathbf{N}}$ and $(Y_n)_{n\in\mathbf{N}}$ be sequences in $\mathrm{Reg}(TM)$ that converge to X and Y in $W^{1,2}(TM)$, respectively. Then clearly $\langle X_n, Y_n\rangle \to \langle X, Y\rangle$ in $L^1(M)$ as $n \to \infty$, while Remark 3.4.12 below ensures that $\mathrm{d}\langle X_n, Y_n\rangle \to \mathrm{d}\langle X, Y\rangle$ in $L^1(T^*M)$ as $n \to \infty$, which is the claim. □

Remark 3.4.12. By duality and (3.4.5), it follows in particular from Proposition 3.4.11 that for every $X \in W^{1,2}(TM)$ and every $Y \in H^{1,2}(TM)$,

$$|\mathrm{d}\langle X, Y\rangle| \le |\nabla X|_{\mathrm{HS}}\,|Y| + |\nabla Y|_{\mathrm{HS}}\,|X| \quad \mathfrak{m}\text{-a.e.}$$ ■

The following lemma is a version of what is known as *Kato's inequality* (for the Bochner Laplacian) in the smooth case [HSU80, Ch. 2], inspired by the work [Kat72]. See also [DGP21, Lem. 3.5].

Lemma 3.4.13 (Kato's inequality)**.** *For every $X \in H^{1,2}(TM)$, $|X| \in \mathscr{F}$ and*

$$\big|\nabla|X|\big| \le \big|\nabla X\big|_{\mathrm{HS}} \quad \mathfrak{m}\text{-a.e.}$$

In particular, if $X \in H^{1,2}(TM) \cap L^\infty(TM)$ then $|X|^2 \in \mathscr{F}$.

Proof. We initially prove the frist claim for $X \in \mathrm{Reg}(TM)$. Define $\varphi_n \in \mathrm{Lip}([0,\infty))$ by $\varphi_n(t) := (t + 1/n^2)^{1/2} - 1/n$, $n \in \mathbf{N}$. By polarization, Proposition 3.2.75 and the Leibniz rule stated in Proposition 3.2.37, we have $|X|^2 \in \mathscr{F}$, and thus $\varphi_n \circ |X|^2 \in \mathscr{F}$

for every $n \in \mathbf{N}$. Moreover, $\varphi_n \circ |X|^2 \to |X|$ pointwise $\mathfrak{m}$-a.e. and in $L^2(M)$ as $n \to \infty$, as well as $\varphi_n'(t)^2\, t \leq 1/4$ for every $t \geq 0$ and every $n \in \mathbf{N}$. Hence by Remark 3.4.12 and — since also $|X|^2 \in \mathscr{G}_{\mathrm{reg}}$ by Proposition 3.4.11 — Lemma 3.3.20,

$$\begin{aligned}\big|\nabla(\varphi_n \circ |X|^2)\big|^2 &= \big|\varphi_n' \circ |X|^2\big|^2 \,\big|\nabla |X|^2\big|^2\\ &\leq 4\,\big|\varphi_n' \circ |X|^2\big|^2\, |X|^2\, \big|\nabla X\big|_{\mathrm{HS}}^2 \\ &\leq \big|\nabla X\big|_{\mathrm{HS}}^2 \quad \mathfrak{m}\text{-a.e.}\end{aligned} \tag{3.4.6}$$

Therefore $|X| \in \mathscr{F}$ by Proposition 3.2.5, and $(\varphi_n \circ |X|^2)_{n\in\mathbf{N}}$ converges $\mathscr{F}$-weakly to $|X|$. By Mazur's lemma, suitable convex combinations of elements of $(\varphi_n \circ |X|^2)_{n\in\mathbf{N}}$ converge $\mathscr{F}$-strongly to $|X|$. The convexity of the carré du champ, see e.g. [AGS15, p. 249], and (3.4.6) imply the claimed $\mathfrak{m}$-a.e. upper bound on $\big|\nabla |X|\big|$.

For general $X \in H^{1,2}(TM)$, let $(X_n)_{n\in\mathbf{N}}$ be a sequence in $\mathrm{Reg}(TM)$ such that $X_n \to X$ in $H^{1,2}(TM)$ and $|X_n| \to |X|$ both pointwise $\mathfrak{m}$-a.e. and in $L^2(M)$ as $n \to \infty$. By what we proved above, $(|X_n|)_{n\in\mathbf{N}}$ is bounded in $\mathscr{F}$, whence $|X| \in \mathscr{F}$ again by Proposition 3.2.5. Still by what we already proved, every L^2-weak limit of subsequences of $(\big|\nabla |X_n|\big|)_{n\in\mathbf{N}}$ is clearly no larger than $|\nabla X|_{\mathrm{HS}}$ $\mathfrak{m}$-a.e., whence we obtain $\big|\nabla |X|\big| \leq |\nabla X|_{\mathrm{HS}}$ $\mathfrak{m}$-a.e. by Mazur's lemma. (See also (2.10) in [AGS15].)

If $X \in H^{1,2}(TM) \cap L^\infty(TM)$, the $\mathscr{F}$-regularity of $|X|^2$ follows from the one of $|X|$ and the chain rule in Proposition 3.2.37. □

This Lemma 3.4.13 has numerous important consequences. First, the $\mathscr{F}$-regularity asserted therein makes it possible in Section 3.6 to pair the function $|X| \in \mathscr{F}$, $X \in H^{1,2}(TM)$, with the given distribution $\kappa \in \mathscr{F}^{-1}_{\mathrm{qloc}}(M)$. Second, it yields the improved semigroup comparison for the covariant heat flow in Theorem 3.4.26. Third, it cancels out the covariant term appearing in the definition of the Ricci curvature, Theorem 3.6.9, leading to a vector q-Bochner inequality for $q \in [1,2]$, Theorem 3.6.21. Under additional assumptions, the latter again implies improved semigroup comparison results in Theorem 3.6.33, this time for the contravariant heat flow. Fourth, it is regarded as the key technical tool in showing that every $X \in H^{1,2}(TM)$ has a "quasi-continuous representative" similar to [DGP21, Thm. 3.14]. This latter topic is not addressed here, but the arguments of [DGP21] do not seem hard to adapt to our setting.

Corollary 3.4.14. *The real-valued function $X \mapsto \langle \kappa \,|\, |X|^2\rangle$ defined on $H^{1,2}(TM)$ is $H^{1,2}$-continuous. More precisely, let $\rho' \in (0,1)$ and $\alpha' \in \mathbf{R}$ satisfy Lemma 3.2.60 for every $\mu \in \{\kappa^+, \kappa^-, |\kappa|\}$. Then for every $X, Y \in H^{1,2}(TM)$,*

$$\big|\big\langle \mu \,\big|\, |X|^2\big\rangle^{1/2} - \big\langle \mu \,\big|\, |Y|^2\big\rangle^{1/2}\big|^2 \leq \rho'\, \mathscr{E}_{\mathrm{cov}}(X-Y) + \alpha'\, \big\|X-Y\big\|^2_{L^2(TM)}.$$

Proof. Since $\kappa = \kappa^+ - \kappa^-$, it suffices to prove the last statement. Given $X, Y \in H^{1,2}(TM)$, by Lemma 3.4.13 we have $|X|, |Y|, |X-Y| \in \mathscr{F}$. In particular, the expressions $\big\langle \mu \,\big|\, |X|^2\big\rangle$ and $\big\langle \mu \,\big|\, |Y|^2\big\rangle$ make sense. Since

$$\begin{aligned}&\big|\big\langle \mu \,\big|\, |X|^2\big\rangle^{1/2} - \big\langle \mu \,\big|\, |Y|^2\big\rangle^{1/2}\big|^2\\ &\qquad\leq \big\langle \mu \,\big|\, \big||X| - |Y|\big|^2\big\rangle\\ &\qquad\leq \big\langle \mu \,\big|\, |X-Y|^2\big\rangle\\ &\qquad\leq \rho'\, \mathscr{E}\big(|X-Y|\big) + \alpha'\, \big\|X-Y\big\|^2_{L^2(TM)}\\ &\qquad\leq \rho'\, \mathscr{E}_{\mathrm{cov}}(X-Y) + \alpha'\, \big\|X-Y\big\|^2_{L^2(TM)}\end{aligned}$$

by Lemma 3.2.60 and again thanks to Lemma 3.4.13, the claim readily follows. □

Remark 3.4.15. We should not expect the function in Corollary 3.4.14 to admit a continuous extension to $L^2(TM)$ in general. The reason is once again the case of compact Riemannian manifolds M with boundary and, say, with nonnegative Ricci curvature. In this case, $\kappa := \ell\,\mathfrak{s} \in \mathbf{K}_0(M)$ by [ER+20, Lem. 2.33, Thm. 4.4], where $\ell\colon \partial M \to \mathbf{R}$ designates the lowest eigenvalue function of the second fundamental form $\mathbb{I}$ at ∂M. The pairing $\big\langle \kappa \,\big|\, |X|^2 \big\rangle$ then simply does not make sense for general vector fields in $L^2(TM)$, which are only defined up to $\mathfrak{v}$-negligible sets (unlike the $H^{1,2}$-case, where the well-definedness of $\big\langle \kappa \,\big|\, |X|^2 \big\rangle$ comes from the trace theorem). ■

Lemma 3.4.16 (Triviality of the torsion tensor)**.** *Suppose that* $f \in \mathscr{D}_{\mathrm{reg}}(\mathrm{Hess})$ *and* $X, Y \in W^{1,2}(TM)$*. Then* $\langle X, \nabla f\rangle, \langle Y, \nabla f\rangle \in \mathscr{G}$ *and*

$$\big\langle X, \nabla\langle Y, \nabla f\rangle\big\rangle - \big\langle Y, \nabla\langle X, \nabla f\rangle\big\rangle = \mathrm{d}f(\nabla_X Y - \nabla_Y X) \quad \mathfrak{m}\text{-}a.e.$$

Proof. Proposition 3.4.11 shows that $\langle Y, \nabla f\rangle, \langle X, \nabla f\rangle \in \mathscr{G}$ with

$$\begin{aligned}
\big\langle X, \nabla\langle Y, \nabla f\rangle\big\rangle &= \nabla Y : (X \otimes \nabla f) + \mathrm{Hess}\, f(X, Y)\\
&= \mathrm{d}f(\nabla_X Y) + \mathrm{Hess}\, f(X, Y) \quad \mathfrak{m}\text{-a.e.},\\
\big\langle Y, \nabla\langle X, \nabla f\rangle\big\rangle &= \nabla X : (Y \otimes \nabla f) + \mathrm{Hess}\, f(Y, X)\\
&= \mathrm{d}f(\nabla_Y X) + \mathrm{Hess}\, f(X, Y) \quad \mathfrak{m}\text{-a.e.}
\end{aligned}$$

In the last step, we used the symmetry of Hess f from Theorem 3.3.3. Subtracting the two previous identities gives the assertion. □

Remark 3.4.17. In the setting of Lemma 3.4.16, a more familiar way — compared to classical Riemannian geometry — of writing the stated identity is

$$X(Y\,f) - Y(X\,f) = \mathrm{d}f(\nabla_X Y - \nabla_Y X) \quad \mathfrak{m}\text{-a.e.},$$

defining $Xf := \langle X, \nabla f\rangle$, $Yf := \langle Y, \nabla f\rangle$, and accordingly $X(Y\,f)$ and $Y(X\,f)$. ■

Since $\mathrm{d}\,\mathscr{F}$ generates $L^2(T^*M)$ by Theorem 3.2.32 and the definition of the extended domain $\mathscr{F}_{\mathrm{e}}$, it follows by Lemma 3.2.73 that $\mathrm{d}\,\mathscr{D}_{\mathrm{reg}}(\mathrm{Hess})$ generates $L^2(T^*M)$ as well, in the sense of L^∞-modules. Hence, by Lemma 3.4.16, given $X, Y \in W^{1,2}(TM)$, $\nabla_X Y - \nabla_Y X$ is the unique vector field $Z \in L^1(TM)$ such that

$$X(Y\,f) - Y(X\,f) = \mathrm{d}f(Z) \quad \mathfrak{m}\text{-a.e.}$$

for every $f \in \mathscr{D}_{\mathrm{reg}}(\mathrm{Hess})$. This motivates the following definition.

Definition 3.4.18. *The* Lie bracket $[X, Y] \in L^1(TM)$ *of two given vector fields* $X, Y \in W^{1,2}(TM)$ *is defined by*

$$[X, Y] := \nabla_X Y - \nabla_Y X.$$

We terminate with the following locality property. It directly follows from the second part of Proposition 3.4.11 as well as Proposition 3.3.23.

Lemma 3.4.19 (Locality of the covariant derivative)**.** *If* $X, Y \in H^{1,2}(TM)$*, then*

$$1_{\{X=Y\}}\, \nabla X = 1_{\{X=Y\}}\, \nabla Y.$$

3.4.3 Heat flow on vector fields

Now we introduce and study the canonical heat flow $(\mathsf{T}_t)_{t\geq 0}$ on L^2-vector fields. First, after defining its generator in Definition 3.4.20, following well-known lines [Bre73] we collect elementary properties of $(\mathsf{T}_t)_{t\geq 0}$ in Theorem 3.4.23. Then we prove the important semigroup comparison result Theorem 3.4.26 between $(\mathsf{T}_t)_{t\geq 0}$ and $(\mathsf{P}_t)_{t\geq 0}$, using Lemma 3.4.13.

Bochner Laplacian In fact, we have a preliminary choice to make, i.e. either to define the Bochner Laplacian $\square$ on $W^{1,2}(TM)$ or on the *strictly smaller* space $H^{1,2}(TM)$. We choose the latter one since the calculus rules from Subsection 3.4.2 are more powerful, in particular in view of Proposition 3.4.24 below. Also, no ambiguity occurs for the background boundary conditions, see Example 3.4.22.

Definition 3.4.20. *We define $\mathscr{D}(\square)$ to consist of all $X \in H^{1,2}(TM)$ for which there exists $Z \in L^2(TM)$ such that for every $Y \in H^{1,2}(TM)$,*

$$\int_M \langle Y, Z\rangle \,\mathrm{d}\mathfrak{m} = -\int_M \nabla Y : \nabla X \,\mathrm{d}\mathfrak{m}.$$

In case of existence, Z is uniquely determined, denoted by $\square X$ and termed the Bochner Laplacian (*or* connection Laplacian *or* horizontal Laplacian) *of X.*

Observe that $\mathscr{D}(\square)$ is a vector space, and that $\square\colon \mathscr{D}(\square) \to L^2(TM)$ is a linear operator. Both is easy to see from the linearity of the covariant derivative.

We modify the functional from (3.4.3) with domain $W^{1,2}(TM)$ by introducing the "augmented" covariant energy functional $\widetilde{\mathscr{E}}_{\mathrm{cov}}\colon L^2(TM) \to [0,\infty]$ with

$$\widetilde{\mathscr{E}}_{\mathrm{cov}}(X) := \begin{cases} \displaystyle\int_M \big|\nabla X\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} & \text{if } X \in H^{1,2}(TM),\\ \infty & \text{otherwise.}\end{cases}$$

Clearly, its (non-relabeled) polarization $\widetilde{\mathscr{E}}_{\mathrm{cov}}\colon H^{1,2}(TM)^2 \to \mathbf{R}$ is a closed, symmetric form, and $\square$ is the nonpositive, self-adjoint generator uniquely associated to it according to [FOT11, Thm. 1.3.1].

A first elementary consequence of this discussion, Lemma 3.4.13 and Rayleigh's theorem is the following inequality between the spectral bottoms of Δ and $\square$.

Corollary 3.4.21. *We have*

$$\inf \sigma(-\Delta) \leq \inf \sigma(-\square).$$

Example 3.4.22. Let M be a Riemannian manifold with boundary. Recall that every element of $H^{1,2}(TM)$ has $\mathfrak{s}$-a.e. vanishing normal component at ∂M by the local version of Proposition 3.2.1. In particular, $\square$ coincides with the self-adjoint realization in $L^2(TM)$ of the restriction of the usual (non-relabeled) Bochner Laplacian $\square$ to the class of compactly supported elements $X \in \Gamma(TM)$ satisfying the following mixed boundary conditions on ∂M, see (3.2.1):

$$\begin{aligned} X^\perp &= 0,\\ (\nabla_{\mathsf{n}} X)^\parallel &= 0. \end{aligned} \tag{3.4.7}$$

Indeed, for *any* compactly supported $X, Y \in \Gamma(TM)$,

$$\int_M \langle \Box X, Y\rangle \,\mathrm{d}\mathfrak{v} - \int_M \langle X, \Box Y\rangle \,\mathrm{d}\mathfrak{v} = \int_{\partial M} \langle \nabla_{\mathsf{n}} X, Y\rangle \,\mathrm{d}\mathfrak{s} - \int_{\partial M} \langle X, \nabla_{\mathsf{n}} Y\rangle \,\mathrm{d}\mathfrak{s}$$

according to the computations carried out in [Cha10, Ch. 2]. The last two integrals vanish under (3.4.7). Moreover, as remarked in [Cha10] this suffices to recover the defining integration by parts formula from Definition 3.4.20.

Let us remark for completeness that in [Cha10], the boundary conditions that are *dual* to (3.4.7) have been considered. See also [Sch95, Prop. 1.2.6]. ■

Heat flow and its elementary properties Analogously to the functional heat flow, we may and will define the *heat flow* on vector fields as the semigroup $(\mathsf{T}_t)_{t\geq 0}$ of bounded, linear and self-adjoint operators on $L^2(TM)$ by

$$\mathsf{T}_t := \mathrm{e}^{\Box t}.$$

Following e.g. [Bre73] or [Gig18, Subsec. 3.4.4], the subsequent elementary properties of $(\mathsf{T}_t)_{t\geq 0}$ are readily established.

Theorem 3.4.23. *The following properties of $(\mathsf{T}_t)_{t\geq 0}$ hold for every $X \in L^2(TM)$ and every $t > 0$.*

(i) *The curve $t \mapsto \mathsf{T}_t X$ belongs to $\mathrm{C}^1((0,\infty); L^2(TM))$ with*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{T}_t X = \Box \mathsf{T}_t X.$$

(ii) *If $X \in \mathscr{D}(\Box)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{T}_t X = \mathsf{T}_t \Box X.$$

In particular, we have the identity

$$\Box\, \mathsf{T}_t = \mathsf{T}_t\, \Box \quad \text{on } \mathscr{D}(\Box).$$

(iii) *For every $s \in [0, t]$,*

$$\|\mathsf{T}_t X\|_{L^2(TM)} \leq \|\mathsf{T}_s X\|_{L^2(TM)}.$$

(iv) *The function $t \mapsto \widetilde{\mathscr{E}}_{\mathrm{cov}}(\mathsf{T}_t X)$ belongs to $\mathrm{C}^1((0,\infty))$, is nonincreasing, and its derivative satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathscr{E}}_{\mathrm{cov}}(\mathsf{T}_t X) = -2\int_M \big|\Box \mathsf{T}_t X\big|^2 \,\mathrm{d}\mathfrak{m}.$$

(v) *If $X \in H^{1,2}(TM)$, the map $t \mapsto \mathsf{T}_t X$ is contiuous on $[0,\infty)$ w.r.t. strong convergence in $H^{1,2}(TM)$.*

(vi) *We have*

$$\widetilde{\mathscr{E}}_{\mathrm{cov}}(\mathsf{T}_t X) \leq \frac{1}{2t}\,\big\|X\big\|^2_{L^2(TM)},$$
$$\big\|\Box \mathsf{T}_t X\big\|^2_{L^2(TM)} \leq \frac{1}{2t^2}\,\big\|X\big\|^2_{L^2(TM)}.$$

Functional inequalities and L^p-properties The calculus rules from Subsection 3.4.2 allow us to derive useful functional inequalities of $(\mathsf{T}_t)_{t\geq 0}$ w.r.t. $(\mathsf{P}_t)_{t\geq 0}$. The main result, essentially coming from Proposition 3.4.11 and Lemma 3.4.13, is the L^1-estimate from Theorem 3.4.26. L^p-consequences of it, $p \in [1,\infty]$, are stated in Corollary 3.4.28.

In fact, the latter requires the following L^2-version of it *in advance* for technical reasons, see Remark 3.4.27 — in particular, Proposition 3.4.24 does *not* follow from Theorem 3.4.26 just by Jensen's inequality for $(\mathsf{P}_t)_{t\geq 0}$.

Proposition 3.4.24. *For every $X \in L^2(TM)$ and every $t \geq 0$,*

$$|\mathsf{T}_t X|^2 \leq \mathsf{P}_t\big(|X|^2\big) \quad \mathfrak{m}\text{-a.e.}$$

Proof. We only prove the nontrivial part in which $t > 0$. Let $\phi \in \mathrm{Test}_{L^\infty}(M)$ be nonnegative. Define the function $F\colon [0,t] \to \mathbf{R}$ by

$$F(s) := \int_M \phi\, \mathsf{P}_{t-s}\big(|\mathsf{T}_s X|^2\big)\,\mathrm{d}\mathfrak{m} = \int_M \mathsf{P}_{t-s}\phi\, |\mathsf{T}_s X|^2\,\mathrm{d}\mathfrak{m}.$$

Of course, F is well-defined. Moreover, $s \mapsto \mathsf{P}_{t-s}\phi$ is Lipschitz continuous as a map from $[0,t)$ into $L^\infty(M,\mathfrak{m})$ since its derivative at a given $s \in [0,t)$ is equal to $-\Delta\mathsf{P}_{t-s}\phi = -\mathsf{P}_{t-s}\Delta\phi$. Furthermore, since $s \mapsto \mathsf{T}_s X$ is continuous as a map from $[0,t]$ into $L^2(TM)$ and locally absolutely continuous as a map from $(0,t]$ into $L^2(TM)$, the L^1-valued map $s \mapsto |\mathsf{T}_s X|^2$ is continuous on $[0,t]$ and locally absolutely continuous on $(0,t]$. All in all, this discussion implies that F is continuous on $[0,t]$ and locally absolutely continuous on $(0,t)$. By exchanging differentiation and integration, for $\mathscr{L}^1$-a.e. $s \in (0,t)$ we thus get

$$F'(s) = -\int_M \Delta\mathsf{P}_{t-s}\phi\, |\mathsf{T}_s X|^2\,\mathrm{d}\mathfrak{m} + 2\int_M \mathsf{P}_{t-s}f\,\langle \mathsf{T}_s X, \square\mathsf{T}_s X\rangle\,\mathrm{d}\mathfrak{m}.$$

Observe that $\mathsf{P}_{t-s}\phi \in \mathrm{Test}(M)$ with $\Delta\mathsf{P}_{t-s}\phi = \mathsf{P}_{t-s}\Delta\phi \in L^\infty(M)$ as well as $|\mathsf{T}_s X|^2 \in \mathscr{G}_{\mathrm{reg}}$ for every $s \in (0,t)$ by Proposition 3.4.11. By Lemma 3.3.21 and Remark 3.4.10, for $\mathscr{L}^1$-a.e. $s \in (0,t)$ we have

$$\begin{aligned}
F'(s) &= \int_M \big\langle \nabla\mathsf{P}_{t-s}\phi, \nabla|\mathsf{T}_s X|^2\big\rangle\,\mathrm{d}\mathfrak{m} - 2\int_M \big\langle \nabla(\mathsf{P}_{t-s}\phi\,\mathsf{T}_s X), \nabla\mathsf{T}_s X\big\rangle\,\mathrm{d}\mathfrak{m} \qquad (3.4.8)\\
&= \int_M \big\langle \nabla\mathsf{P}_{t-s}\phi, \nabla|\mathsf{T}_s X|^2\big\rangle\,\mathrm{d}\mathfrak{m} - 2\int_M [\nabla\mathsf{P}_{t-s}\phi \otimes \mathsf{T}_s X] : \nabla\mathsf{T}_s X\,\mathrm{d}\mathfrak{m}\\
&\qquad - 2\int_M \mathsf{P}_{t-s}\phi\,\big|\nabla\mathsf{T}_s X\big|_{\mathrm{HS}}^2\,\mathrm{d}\mathfrak{m}\\
&\leq \int_M \big\langle \nabla\mathsf{P}_{t-s}\phi, \nabla|\mathsf{T}_s X|^2\big\rangle\,\mathrm{d}\mathfrak{m} - 2\int_M [\nabla\mathsf{P}_{t-s}\phi \otimes \mathsf{T}_s X] : \nabla\mathsf{T}_s X\,\mathrm{d}\mathfrak{m} = 0.
\end{aligned}$$

In the last equality, we used Proposition 3.4.11. Therefore,

$$\int_M \phi\,|\mathsf{T}_t X|^2\,\mathrm{d}\mathfrak{m} = F(t) \leq F(0) = \int_M \phi\,\mathsf{P}_t\big(|X|^2\big)\,\mathrm{d}\mathfrak{m}.$$

This proves the claim thanks to the arbitrariness of ϕ by Lemma 3.2.73. □

In applications, the following corollary of Proposition 3.4.24 could be useful.

Corollary 3.4.25. *For every $X \in \mathscr{D}(\Box)$, there exists a sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\Box) \cap L^\infty(TM)$ which converges to X in $H^{1,2}(TM)$ such that in addition, $\Box X_n \to \Box X$ in $L^2(TM)$ as $n \to \infty$. If $X \in L^\infty(TM)$ in addition, this sequence can be constructed to be bounded in $L^\infty(TM)$.*

Proof. Define $X_k := 1_{\{|X|\leq k\}}\, X \in L^2(TM) \cap L^\infty(TM)$, $k \in \mathbf{N}$, and, given any $t > 0$, consider the element $X_{t,k} := \mathsf{T}_t X_k$ which, thanks to Proposition 3.4.24, belongs to $\mathscr{D}(\Box) \cap L^\infty(TM)$. By Theorem 3.4.23, we have $X_{t,k} \to \mathsf{T}_t X$ in $H^{1,2}(TM)$ and $\Box X_{t,k} \to \Box\mathsf{T}_t X$ in $L^2(TM)$ as $k \to \infty$ for every $t > 0$. Furthermore, $\mathsf{T}_t X \to X$ in $H^{1,2}(TM)$ and $\Box\mathsf{T}_t X = \mathsf{T}_t\Box X \to \Box X$ in $L^2(TM)$ as $t \to 0$ again by Theorem 3.4.23. The claim follows by a diagonal argument. □

The following improvement of Proposition 3.4.24 is an instance of the correspondence between *form domination* and *semigroup domination* [HSU77, Ouh99, Shi97, Sim77]. It extends analogous results for Riemannian manifolds without boundary [HSU77, HSU80].

Theorem 3.4.26. *For every $X \in L^2(TM)$ and every $t \geq 0$,*

$$|\mathsf{T}_t X| \leq \mathsf{P}_t|X| \quad \mathfrak{m}\text{-a.e.}$$

Proof. Again, we restrict ourselves to $t > 0$. By the L^2-continuity of both sides of the claimed inequality in X, it is sufficient to prove the latter for $X \in \mathrm{Test}(M)$. Given any $\varepsilon > 0$, define the function $\varphi_\varepsilon \in \mathrm{C}^\infty([0,\infty))\cap\mathrm{Lip}([0,\infty))$ by $\varphi_\varepsilon(r) := (r+\varepsilon)^{1/2} - \varepsilon^{1/2}$. Moreover, let $\phi \in \mathrm{Test}(M)$ be nonnegative with $\Delta\phi \in L^\infty(M)$. As in the proof of Proposition 3.4.24, one argues that the function $F_\varepsilon\colon [0,t] \to \mathbf{R}$ with

$$F_\varepsilon(s) := \int_M \phi\, \mathsf{P}_{t-s}\big(\varphi_\varepsilon \circ |\mathsf{T}_s X|^2\big)\,\mathrm{d}\mathfrak{m} = \int_M \mathsf{P}_{t-s}\phi\,\big[\varphi_\varepsilon \circ |\mathsf{T}_s X|^2\big]\,\mathrm{d}\mathfrak{m}$$

is continuous on $[0,t]$, locally absolutely continuous on $(0,t)$, and in differentiating it, integration and differentiation can be switched at $\mathscr{L}^1$-a.e. $s \in (0,t)$, yielding

$$\begin{aligned} F_\varepsilon'(s) = &-\int_M \Delta\mathsf{P}_{t-s}\phi\,\big[\varphi_\varepsilon \circ |\mathsf{T}_s X|^2\big]\,\mathrm{d}\mathfrak{m}\\ &\qquad + 2\int_M \mathsf{P}_{t-s}\phi\,\big[\varphi_\varepsilon' \circ |\mathsf{T}_s X|^2\big]\,\langle \mathsf{T}_s X, \Box\mathsf{T}_s X\rangle\,\mathrm{d}\mathfrak{m}. \end{aligned}$$

By Proposition 3.4.24, we have $\mathsf{T}_s X \in L^\infty(TM)$ and hence $|\mathsf{T}_s X|^2 \in \mathscr{F}$ for every $s \in (0,t)$ by Lemma 3.4.13. In particular $\varphi_\varepsilon \circ |\mathsf{T}_s X|^2, \varphi_\varepsilon' \circ |\mathsf{T}_s X|^2 \in \mathscr{F} \cap L^\infty(M)$, and hence by Proposition 3.2.37 and Remark 3.4.10,

$$\begin{aligned} F_\varepsilon'(s) &= \int_M \big\langle \Delta\mathsf{P}_{t-s}\phi, \nabla\big[\varphi_\varepsilon \circ |\mathsf{T}_s X|^2\big]\big\rangle\,\mathrm{d}\mathfrak{m}\\ &\qquad - 2\int_M \nabla\big[\mathsf{P}_{t-s}\phi\,\big[\varphi_\varepsilon' \circ |\mathsf{T}_s X|^2\big]\,\mathsf{T}_s X\big] : \nabla\mathsf{T}_s X\,\mathrm{d}\mathfrak{m}\\ &= \int_M \big[\varphi_\varepsilon' \circ |\mathsf{T}_s X|^2\big]\,\big\langle \nabla\mathsf{P}_{t-s}\phi, \nabla|\mathsf{T}_s X|^2\big\rangle\,\mathrm{d}\mathfrak{m}\\ &\qquad - 2\int_M \mathsf{P}_{t-s}\phi\,\big[\varphi_\varepsilon'' \circ |\mathsf{T}_s X|^2\big]\,\nabla|\mathsf{T}_s X|^2 \otimes \mathsf{T}_s X : \nabla\mathsf{T}_s X\,\mathrm{d}\mathfrak{m}\\ &\qquad - 2\int_M \big[\varphi_\varepsilon' \circ |\mathsf{T}_s X|^2\big]\,\nabla\mathsf{P}_{t-s}\phi \otimes \mathsf{T}_s X : \nabla\mathsf{T}_s X\,\mathrm{d}\mathfrak{m} \end{aligned}$$

$$
\begin{aligned}
&\quad - 2\int_M \mathsf{P}_{t-s}\phi\, \big[\varphi_\varepsilon' \circ |\mathsf{T}_s X|^2\big]\, \big|\nabla \mathsf{T}_s X\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m}\\
&= -2\int_M \mathsf{P}_{t-s}\phi\, \big[\varphi_\varepsilon'' \circ |\mathsf{T}_s X|^2\big]\, \nabla|\mathsf{T}_s X|^2 \otimes \mathsf{T}_s X : \nabla \mathsf{T}_s X \,\mathrm{d}\mathfrak{m}\\
&\quad - 2\int_M \mathsf{P}_{t-s}\phi\, \big[\varphi_\varepsilon' \circ |\mathsf{T}_s X|^2\big]\, \big|\nabla \mathsf{T}_s X\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m}.
\end{aligned}
$$

In the last step, we used Proposition 3.4.11 to cancel out two integrals. Lastly, one easily verifies that $-2r\,\varphi_\varepsilon''(r) \leq \varphi_\varepsilon'(r)$ for every $r \geq 0$, and that $-\varphi_\varepsilon''$ is nonnegative. Taking Lemma 3.4.13 into account, we thus get

$$
\begin{aligned}
&- 2\,\big[\varphi_\varepsilon'' \circ |\mathsf{T}_s X|^2\big]\, \nabla|\mathsf{T}_s X|^2 \otimes \mathsf{T}_s X : \nabla \mathsf{T}_s X\\
&\qquad = -4\,\big[\varphi_\varepsilon'' \circ |\mathsf{T}_s X|^2\big]\, |\mathsf{T}_s X|\, \nabla|\mathsf{T}_s X| \otimes \mathsf{T}_s X : \nabla \mathsf{T}_s X\\
&\qquad \leq -4\,\big[\varphi_\varepsilon'' \circ |\mathsf{T}_s X|^2\big]\, |\mathsf{T}_s X|^2\, \big|\nabla|\mathsf{T}_s X|\big|\, |\nabla \mathsf{T}_s X|_{\mathrm{HS}}\\
&\qquad \leq 2\,\big[\varphi_\varepsilon' \circ |\mathsf{T}_s X|^2\big]\, \big|\nabla \mathsf{T}_s X\big|_{\mathrm{HS}}^2 \quad \mathfrak{m}\text{-a.e.}
\end{aligned}
$$

This shows that $F'(s) \leq 0$ for $\mathscr{L}^1$-a.e. $s \in (0,t)$, whence

$$
\int_M \phi\,\big[\varphi_\varepsilon \circ |\mathsf{T}_t X|^2\big]\,\mathrm{d}\mathfrak{m} = F_\varepsilon(t) \leq F_\varepsilon(0) = \int_M \phi\, \mathsf{P}_t(\varphi_\varepsilon \circ |X|^2)\,\mathrm{d}\mathfrak{m}
$$

for every $\varepsilon > 0$. Sending $\varepsilon \to 0$ with the aid of Lebesgue's theorem and using the arbitrariness of ϕ via Lemma 3.2.73 gives the desired assertion. □

Note that the only essential tool to prove Proposition 3.4.24 and Theorem 3.4.26 is the metric compatibility of ∇ from Proposition 3.4.11. In particular, no curvature shows up in both statements.

Remark 3.4.27. In the notation of the proof of Theorem 3.4.26, Lemma 3.4.13 only guarantees that $|\mathsf{T}_s X| \in \mathscr{F}$, $s \in (0,t)$. However, the required regularity $|\mathsf{T}_s X|^2 \in \mathscr{F}$ is unclear without any a priori information about L^∞-L^∞-regularizing properties of $(\mathsf{T}_t)_{t\geq 0}$, which is precisely provided by Proposition 3.4.24. In turn, the proof of the latter only needs $\mathscr{G}_{\mathrm{reg}}$-regularity of $|\mathsf{T}_s X|^2$, which is true for any $X \in L^2(TM)$ by Proposition 3.4.11. To integrate by parts in (3.4.8), this missing $\mathscr{F}$-regularity is compensated by Lemma 3.3.21, which is one key feature of the space $\mathscr{G}_{\mathrm{reg}}$ (recall Remark 3.3.22 as well). ■

Corollary 3.4.28. *The heat flow $(\mathsf{T}_t)_{t\geq 0}$ uniquely extends to a semigroup of bounded linear operators on $L^p(TM)$ for every $p \in [1,\infty]$ such that, for every $X \in L^p(TM)$ and every $t \geq 0$,*

$$
|\mathsf{T}_t X|^p \leq \mathsf{P}_t\big(|X|^p\big) \quad \mathfrak{m}\text{-a.e.},
$$

and in particular

$$
\|\mathsf{T}_t\|_{L^p(TM),L^p(TM)} \leq 1.
$$

It is strongly continuous on $L^p(TM)$ if $p < \infty$ and weakly continuous on $L^\infty(TM)$.*

3.5 Exterior derivative

Throughout this section, let us fix $k \in \mathbf{N}_0$.

3.5.1 The Sobolev space $\mathscr{D}(\mathbf{d}^k)$

We now give a meaning to the exterior derivative acting on suitable k-forms, i.e. elements of $L^2(\Lambda^k T^*M)$ (recall Subsection 3.2.3).

Definition and basic properties Before the motivating smooth Example 3.5.1, a notational comment is in order. Given $\omega \in L^0(\Lambda^k T^*M)$ and $X_0, \dots, X_k, Y \in L^0(TM)$, we shall use the standard abbreviations

$$\begin{aligned}\omega(\widehat{X}_i) &:= \omega(X_0, \dots, \widehat{X}_i, \dots, X_k)\\ &:= \omega(X_0 \wedge \dots \wedge X_{i-1} \wedge X_{i+1} \wedge \dots \wedge X_k),\\ \omega(Y, \widehat{X}_i, \widehat{X}_j) &:= \omega(Y, X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k)\\ &:= \omega(Y \wedge X_0 \wedge \dots \wedge X_{i-1} \wedge X_{i+1} \wedge \dots \wedge X_{j-1} \wedge X_{j+1} \wedge \dots \wedge X_k).\end{aligned}$$

Example 3.5.1. On a Riemannian manifold M with boundary, the exterior derivative $\mathrm{d}\colon \Gamma(\Lambda^k T^*M) \to \Gamma(\Lambda^{k+1}T^*M)$ is defined by three axioms [Lee18, Thm. 9.12]. It can be shown [Pet06, Sec. A.2] that the unique such d satisfies the following pointwise, chart-free representation for any $\omega \in \Gamma(\Lambda^k T^*M)$ and any $X_0, \dots, X_k \in \Gamma_c(TM)$:

$$\begin{aligned}\mathrm{d}\omega(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^i\, \mathrm{d}\big[\omega(\widehat{X}_i)\big](X_i)\\ &\qquad + \sum_{i=0}^k \sum_{j=i+1}^k (-1)^{i+j}\, \omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j).\end{aligned} \tag{3.5.1}$$

By the discussion from Subsection 3.2.1, the map d is still uniquely determined on $\Gamma(\Lambda^k T^*M)$ by this identity when restricting to those $X_0, \dots, X_k$ for which

$$\langle X_0, \mathsf{n}\rangle = \dots = \langle X_k, \mathsf{n}\rangle = 0 \quad \text{on } \partial M.$$

In this case, integrating (3.5.1) leads to

$$\begin{aligned}\int_M \mathrm{d}\omega(X_0, \dots, X_k)\,\mathrm{d}\mathfrak{v} &= \int_M \sum_{i=0}^k (-1)^i\, \omega(\widehat{X}_i)\,\mathrm{div}\, X_i \,\mathrm{d}\mathfrak{v}\\ &\qquad + \int_M \sum_{i=0}^k \sum_{j=i+1}^k (-1)^{i+j}\, \omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j)\,\mathrm{d}\mathfrak{v}\end{aligned}$$

after integration by parts in conjunction with Example 3.2.48. Given this integral identity for every compactly supported $X_0, \dots, X_k \in \Gamma(\Lambda^k T^*M)$ with vanishing normal parts at ∂M as above, the differential $\mathrm{d}\omega \in \Gamma(\Lambda^{k+1}T^*M)$ of $\omega \in \Gamma(\Lambda^k T^*M)$ is of course still uniquely determined. ■

Now note that the r.h.s. of the last integral identity — with $\mathfrak{v}$ replaced by $\mathfrak{m}$ — is meaningful for arbitrary $\omega \in L^2(\Lambda^k T^*M)$ and $X_0, \dots, X_k \in \mathrm{Test}(TM)$. Indeed, $\omega(\widehat{X}_i) \in L^2(M)$ since $X_0, \dots, X_k \in L^\infty(TM)$, and $X_0, \dots, X_k \in \mathscr{D}_{\mathrm{TV}}(\mathbf{div}) \cap \mathscr{D}(\mathrm{div})$ with $\mathrm{div}\, X_i \in L^2(M)$ and $\mathbf{n}\, X_i = 0$ by Lemma 3.2.54, $i \in \{0, \dots, k\}$. Moreover, by (3.4.5) the Lie bracket $[X_i, X_j]$ belongs to $L^2(TM)$, whence $\omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j) \in L^2(M)$, $i \in \{0, \dots, k\}$ and $j \in \{i+1, \dots, k\}$.

These considerations motivate the subsequent definition. (We only make explicit the degree k in the name of the space, but not in the differential object itself.)

Definition 3.5.2. *We define* $\mathscr{D}(\mathrm{d}^k)$ *to consist of all* $\omega \in L^2(\Lambda^k T^*M)$ *for which there exists* $\eta \in L^2(\Lambda^{k+1}T^*M)$ *such that for every* $X_0, \dots, X_k \in \mathrm{Test}(TM)$*,*

$$\begin{aligned}\int_M \eta(X_0, \dots, X_k)\,\mathrm{d}\mathfrak{m} &= \int_M \sum_{i=0}^k (-1)^{i+1}\,\omega(\widehat{X}_i)\,\mathrm{div}\,X_i\,\mathrm{d}\mathfrak{m}\\ &\quad + \int_M \sum_{i=0}^k \sum_{j=i+1}^k (-1)^{i+j}\,\omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j)\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

In case of existence, the element η *is unique, denoted by* $\mathrm{d}\omega$ *and termed the* exterior derivative (*or* exterior differential) *of* ω*.*

The uniqueness follows by density of $\mathrm{Test}(\Lambda^{k+1}T^*M)$ in $L^2(\Lambda^{k+1}T^*M)$ as discussed in Subsection 3.2.8. It is then clear that $\mathscr{D}(\mathrm{d}^k)$ is a real vector space and that d is a linear operator on it.

We always endow $\mathscr{D}(\mathrm{d}^k)$ with the norm $\|\cdot\|_{\mathscr{D}(\mathrm{d}^k)}$ given by

$$\|\omega\|^2_{\mathscr{D}(\mathrm{d}^k)} := \|\omega\|^2_{L^2(\Lambda^k T^*M)} + \|\mathrm{d}\omega\|^2_{L^2(\Lambda^{k+1}T^*M)}.$$

We introduce the functional $\mathscr{E}_\mathrm{d}\colon L^2(\Lambda^k T^*M) \to [0, \infty]$ with

$$\mathscr{E}_\mathrm{d}(\omega) := \begin{cases} \displaystyle\int_M |\mathrm{d}\omega|^2\,\mathrm{d}\mathfrak{m} & \text{if } \omega \in \mathscr{D}(\mathrm{d}^k),\\ \infty & \text{otherwise.}\end{cases}$$

We do not make explicit the dependency of $\mathscr{E}_\mathrm{d}$ on the degree k. It will always be clear from the context which one is intended.

Remark 3.5.3. By Lemma 3.2.54 it is easy to see that $\mathscr{F}$ is contained in $\mathscr{D}(\mathrm{d}^0)$, and that $\mathrm{d}\omega$ is simply the exterior differential from Definition 3.2.33, $\omega \in \mathscr{F}$. The reverse inclusion, however, seems more subtle, but at least holds true if M is intrinsically complete as in Definition 3.2.71. Compare with Remark 3.3.22, [Gig18, p. 136] and (the proof of) [Gig18, Prop. 3.3.13]. ■

Remark 3.5.4. Similarly to Remark 3.3.6 and Remark 3.4.4, motivated by its axiomatization in Riemannian geometry we expect the differential d to neither depend on conformal transformations of $\langle\cdot,\cdot\rangle$, nor on drift transformations of $\mathfrak{m}$. ■

The next theorem collects basic properties of the above notions. It is proven in a similar fashion as Theorem 3.3.3 and Theorem 3.4.3.

Theorem 3.5.5. *The space* $\mathscr{D}(\mathrm{d}^k)$*, the exterior derivative* d *and the functional* $\mathscr{E}_\mathrm{d}$ *satisfy the following properties.*

(i) $\mathscr{D}(\mathrm{d}^k)$ *is a separable Hilbert space w.r.t.* $\|\cdot\|_{\mathscr{D}(\mathrm{d}^k)}$.
(ii) *The exterior differential is a closed operator. That is, the image of the map* $\mathrm{Id}\times\mathrm{d}\colon \mathscr{D}(\mathrm{d}^k) \to L^2(\Lambda^k T^*M)\times L^2(\Lambda^{k+1}T^*M)$ *is a closed subspace of* $L^2(\Lambda^k T^*M)\times L^2(\Lambda^{k+1}T^*M)$.
(iii) *The functional* $\mathscr{E}_\mathrm{d}$ *is* L^2*-lower semicontinuous, and for every* $\omega \in L^2(\Lambda^k T^*M)$ *we have the duality formula*

$$\mathscr{E}_\mathrm{d}(\omega) = \sup\Big\{2\sum_{l=1}^n \int_M \sum_{i=0}^k (-1)^{i+1}\,\omega(\widehat{X}_i^l)\,\mathrm{div}\,X_i^l\,\mathrm{d}\mathfrak{m}$$

$$+ 2 \sum_{l=1}^{n} \int_M \sum_{i=0}^{k} \sum_{j=i+1}^{k} (-1)^{i+j}\, \omega([X_i^l, X_j^l], \widehat{X}_i^l, \widehat{X}_j^l)\, \mathrm{d}\mathfrak{m}$$
$$- \int_M \Big| \sum_{j=1}^{n} X_0^l \wedge \cdots \wedge X_k^l \Big|^2 \mathrm{d}\mathfrak{m} : n \in \mathbf{N},\ X_i^l \in \mathrm{Test}(TM) \Big\}.$$

(iv) *For every* $f_0 \in \mathrm{Test}(M) \cup \mathbf{R}\, 1_M$ *and every* $f_1, \dots, f_k \in \mathrm{Test}(M)$ *we have* $f_0\, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \in \mathscr{D}(\mathrm{d}^k)$ *with*

$$\mathrm{d}(f_0\, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = \mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k,$$

with the usual interpretation $\mathrm{d}1_M := 0$. *In particular* $\mathrm{Reg}(\Lambda^k T^*M) \subset \mathscr{D}(\mathrm{d}^k)$, *and* $\mathscr{D}(\mathrm{d}^k)$ *is dense in* $L^2(\Lambda^k T^*M)$.

Proof. The items (i), (ii) and (iii) follow completely analogous lines as the proofs of corresponding statements in Theorem 3.3.3 and Theorem 3.4.3. We omit the details.

We turn to (iv). We concentrate on the proof of the claimed formula, from which the last two statements then readily follow by linearity of d. First observe that the r.h.s. of the claimed identity belongs to $L^2(\Lambda^{k+1}T^*M)$. By definition (3.2.6) of the pointwise scalar product in $L^2(\Lambda^k T^*M)$ and Proposition 3.4.24, we have $\omega(X_1, \dots, X_k) \in \mathscr{F} \cap L^\infty(M)$, where $\omega := \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$. Direct computations using the Definition 3.4.18 of the Lie bracket and Theorem 3.4.3 yield

$$\sum_{i=0}^{k} (-1)^i\, \mathrm{d}\big[\omega(\widehat{X}_i)\big](X_i) = - \sum_{i=0}^{k} \sum_{j=i+1}^{k} (-1)^{i+j}\, \omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j) \quad \mathfrak{m}\text{-a.e.}$$

For $f_0 \in \mathrm{Test}(M)$, it thus follows from Lemma 3.2.54 that

$$\begin{aligned}
&\int_M \sum_{i=0}^{k} (-1)^{i+1} f_0\, \omega(\widehat{X}_i)\, \mathrm{div}\, X_i \, \mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \sum_{i=0}^{k} \sum_{j=i+1}^{k} (-1)^{i+j} f_0\, \omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j)\, \mathrm{d}\mathfrak{m} \\
&= \int_M \sum_{i=0}^{k} (-1)^i\, \mathrm{d}f_0(X_i)\, \omega(\widehat{X}_i)\, \mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \sum_{i=0}^{k} (-1)^i f_0\, \mathrm{d}\big[\omega(\widehat{X}_i)\big](X_i)\, \mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \sum_{i=0}^{k} \sum_{j=i+1}^{k} (-1)^{i+j} f_0\, \omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j)\, \mathrm{d}\mathfrak{m} \\
&= \int_M (\mathrm{d}f_0 \wedge \omega)(X_0, \dots, X_k)\, \mathrm{d}\mathfrak{m},
\end{aligned}$$

which shows the first claimed identity. The same computation can be done for $f_0 \in \mathbf{R}\, 1_M$ with the formal interpretation $\mathrm{d}f_0 := 0$. □

Remark 3.5.6. For arbitrary, not necessarily tamed Dirichlet spaces, the spaces $L^2(\Lambda^k T^*M)$ and $L^2(\Lambda^{k+1}T^*M)$ from the exterior product part of Subsection 3.2.3 make sense. One is then tempted to define the exterior derivative of $f_0\, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$ for appropriate $f_0, \dots, f_k \in \mathscr{F}_\mathrm{e}$ simply as $\mathrm{d}f_0 \wedge \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \in L^2(\Lambda^{k+1}T^*M)$. However, it is in general not clear if d defined in that way is closable. In our approach, this is clear from Definition 3.5.2 by integration by parts, for which it has been crucial to know the existence of a large class of vector fields whose Lie bracket is well-defined. In our approach, this is precisely $\mathrm{Test}(TM)$, whose nontriviality — in fact, density in $L^2(TM)$ — is a consequence of the (extended Kato condition on the) lower Ricci bound κ, see Subsection 3.2.7 and Subsection 3.2.8. ■

Calculus rules We proceed with further calculus rules for d. In view of Proposition 3.5.8 below, the following preliminary lemma is required.

Lemma 3.5.7. *Suppose that $\omega \in \mathscr{D}(\mathrm{d}^k)$, and that $f \in \mathscr{F} \cap L^\infty(M)$. Then for every $X_0, \dots, X_k \in \mathrm{Test}(TM)$,*

$$\begin{aligned}
&\int_M f\, \mathrm{d}\omega(X_0, \dots, X_k)\, \mathrm{d}\mathfrak{m} \\
&\qquad = \int_M \sum_{i=0}^k (-1)^{i+1}\, \omega(\widehat{X}_i)\, \mathrm{div}(f\, X_i)\, \mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \sum_{i=0}^k \sum_{j=i+1}^k (-1)^{i+j}\, f\, \omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j)\, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

Proof. We first prove the claim for $f \in \mathrm{Test}(M)$. As $f\, X_0 \in \mathrm{Test}(M)$, by definition of the Lie bracket and Lemma 3.4.9 we have

$$\begin{aligned}
[f\, X_0, X_j] &= \nabla_{f\, X_0} X_j - \nabla_{X_j}(f\, X_0) \\
&= f\, \nabla_{X_0} X_j - \mathrm{d}f(X_j)\, X_0 - f\, \nabla_{X_j} X_0 \\
&= f\, [X_0, X_j] - \mathrm{d}f(X_j)\, X_0
\end{aligned}$$

for every $j \in \{1, \dots, k\}$. Hence, by Lemma 3.2.52,

$$\begin{aligned}
&\int_M f\, \mathrm{d}\omega(X_0, \dots, X_k)\, \mathrm{d}\mathfrak{m} \\
&\qquad = \int_M \mathrm{d}\omega(f\, X_0, X_1, \dots, X_k)\, \mathrm{d}\mathfrak{m} \\
&\qquad = -\int_M \omega(\widehat{X}_0)\, \mathrm{d}f(X_0)\, \mathrm{d}\mathfrak{m} + \int_M \sum_{i=0}^k (-1)^{i+1} f\, \omega(\widehat{X}_i)\, \mathrm{div}\, X_i\, \mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \sum_{i=0}^k \sum_{j=i+1}^k f\, \omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j)\, \mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \sum_{j=1}^k (-1)^{j+1}\, \omega(\widehat{X}_j)\, \mathrm{d}f(X_j)\, \mathrm{d}\mathfrak{m}.
\end{aligned}$$

Since $\mathrm{div}(f\, X_i) = \mathrm{d}f(X_i) + f\, \mathrm{div}\, X_i$ $\mathfrak{m}$-a.e. by Lemma 3.2.52, we are done.

The claim for general $f \in \mathscr{F} \cap L^\infty(M)$ follows by the approximation result from Lemma 3.2.73 together with Lemma 3.2.52. □

Proposition 3.5.8 (Leibniz rule)**.** *Let $\omega \in \mathscr{D}(\mathrm{d}^k)$ and, for some $k' \in \mathbf{N}_0$, suppose that $\omega' \in \mathrm{Reg}(\Lambda^{k'} T^*M)$. Then $\omega \wedge \omega' \in \mathscr{D}(\mathrm{d}^{k+k'})$ with*

$$\mathrm{d}(\omega \wedge \omega') = \mathrm{d}\omega \wedge \omega' + (-1)^k\, \omega \wedge \mathrm{d}\omega'.$$

Proof. We proceed by induction on k' and start with $k' = 0$. In this case, ω' is simply an element $f \in \mathrm{Test}(M)$. Given any $X_0, \dots, X_k \in \mathrm{Test}(TM)$, by Lemma 3.2.52 and Lemma 3.5.7 we obtain

$$\begin{aligned}
&\int_M \sum_{i=0}^{k} (-1)^{i+1} f\, \omega(\widehat{X}_i) \operatorname{div} X_i \,\mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \sum_{i=0}^{k} \sum_{j=i+1}^{k} (-1)^{i+j} f\, \omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j) \,\mathrm{d}\mathfrak{m} \\
&\quad = \int_M \sum_{i=0}^{k} (-1)^{i+1} \omega(\widehat{X}_i) \operatorname{div}(f\, X_i) \,\mathrm{d}\mathfrak{m} + \int_M \sum_{i=0}^{k} (-1)^{i} \omega(\widehat{X}_i)\, \mathrm{d}f(X_i) \,\mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \sum_{i=0}^{k} \sum_{j=i+1}^{k} (-1)^{i+j} f\, \omega([X_i, X_j], \widehat{X}_i, \widehat{X}_j) \,\mathrm{d}\mathfrak{m} \\
&\quad = \int_M f\, \mathrm{d}\omega(X_0, \dots, X_k) \,\mathrm{d}\mathfrak{m} + \int_M (\mathrm{d}f \wedge \omega)(X_0, \dots, X_k) \,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

In the last equality, we used the definition (3.2.13) of the pointwise scalar product in $L^2(\Lambda^k T^* M)$. Therefore, we obtain that $f\, \omega \in \mathscr{D}(\mathrm{d}^k)$ with

$$\mathrm{d}(f\, \omega) = f\, \mathrm{d}\omega + \mathrm{d}f \wedge \omega = f\, \mathrm{d}\omega + (-1)^k\, \omega \wedge \mathrm{d}f,$$

which is precisely the claim for $k' = 0$.

Before we proceed with the induction step, we show the claim under the assumption that $\omega' := \mathrm{d}f$ for some $f \in \mathrm{Test}(M)$, in which case we more precisely claim that $\omega \wedge \mathrm{d}f \in \mathscr{D}(\mathrm{d}^{k+1})$ with

$$\mathrm{d}(\omega \wedge \mathrm{d}f) = \mathrm{d}\omega \wedge \mathrm{d}f, \tag{3.5.2}$$

keeping in mind that $\mathrm{d}(\mathrm{d}f) = 0$ by Theorem 3.5.5. To this aim, let $X_0, \dots, X_{k+1} \in \mathrm{Test}(TM)$. By definition (3.2.13) of the pointwise scalar product and Lemma 3.5.7 — which can be applied since $\mathrm{d}f(X_i) \in \mathscr{F} \cap L^\infty(M)$ by Proposition 3.4.11 — we get

$$\begin{aligned}
&\int_M (\mathrm{d}\omega \wedge \mathrm{d}f)(X_0, \dots, X_{k+1}) \,\mathrm{d}\mathfrak{m} \\
&\quad = \int_M \sum_{i=0}^{k+1} (-1)^{i+k+1} \mathrm{d}\omega(\widehat{X}_i)\, \mathrm{d}f(X_i) \,\mathrm{d}\mathfrak{m} \\
&\quad = \int_M \sum_{i=0}^{k+1} \sum_{\substack{j=0,\\ j\neq i}}^{k+1} a_{ij}\, \omega(\widehat{X}_i, \widehat{X}_j) \operatorname{div}\big[\mathrm{d}f(X_i)\, X_j\big] \,\mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \sum_{i=0}^{k+1} \sum_{\substack{j=0,\\ j\neq i}}^{k+1} \sum_{\substack{j'=j+1,\\ j'\neq i}}^{k+1} b_{ijj'}\, \omega([X_j, X_{j'}], \widehat{X}_i, \widehat{X}_j, \widehat{X}_{j'})\, \mathrm{d}f(X_i) \,\mathrm{d}\mathfrak{m},
\end{aligned}$$

where, for $i, j, j' \in \{0, \dots, k+1\}$,

$$a_{ij} := \begin{cases} (-1)^{i+j+k+1} & \text{if } j \leq i, \\ (-1)^{i+j+k} & \text{otherwise,} \end{cases}$$

$$b_{ijj'} := \begin{cases} (-1)^{i+j+j'+k} & \text{if } j < i < j', \\ (-1)^{i+j+j'+k+1} & \text{otherwise.} \end{cases}$$

Then (3.5.2) directly follows since, by Lemma 3.4.16,

$$\begin{aligned}&\operatorname{div}\big[\mathrm{d}f(X_i)\,X_j\big] - \operatorname{div}\big[\mathrm{d}f(X_j)\,X_i\big]\\&\qquad = \mathrm{d}f(X_i)\operatorname{div}X_j - \mathrm{d}f(X_j)\operatorname{div}X_i - \mathrm{d}f([X_i,X_j])\quad \mathfrak{m}\text{-a.e.}\end{aligned}$$

Now we are ready to perform the induction step. Given the assertion for $k'-1$ with $k'\in\mathbf{N}$, by linearity it suffices to consider the case $\omega' := f_0\,\mathrm{d}f_1\wedge\dots\wedge\mathrm{d}f_{k'}$, where $f_0\in\mathrm{Test}(TM)\cup\mathbf{R}\,1_M$ and $f_1,\dots,f_{k'}\in\mathrm{Test}(M)$. By Theorem 3.5.5 we have $f_0\,\omega\in\mathscr{D}(\mathrm{d}^k)$. Writing $\omega'':=\mathrm{d}f_2\wedge\dots\wedge\mathrm{d}f_{k'}$ thus yields

$$\begin{aligned}\mathrm{d}(\omega\wedge\omega') &= \mathrm{d}\big[(\omega\wedge f_0\,\mathrm{d}f_1)\wedge\omega''\big]\\&= \mathrm{d}\big[(f_0\,\omega)\wedge\mathrm{d}f_1\big]\wedge\omega''\\&= \big[f_0\,\mathrm{d}\omega + \mathrm{d}f_0\wedge\omega\big]\wedge\mathrm{d}f_1\wedge\omega''\\&= \mathrm{d}\omega\wedge\omega' + (-1)^k\,\omega\wedge\mathrm{d}\omega'\end{aligned}$$

where we used the induction hypothesis in the second identity and Theorem 3.5.5 in the last two equalities. □

Remark 3.5.9. In Proposition 3.5.8, by evident integrability issues we cannot go really beyond the assumption $\omega\in\mathrm{Reg}(\Lambda^{k'}T^*M)$, not even to the space $\mathscr{D}_{\mathrm{reg}}(\mathrm{d}^k)$ introduced in the subsequent Definition 3.5.10. ■

Definition 3.5.10. *We define the space* $\mathscr{D}_{\mathrm{reg}}(\mathrm{d}^k)\subset\mathscr{D}(\mathrm{d}^k)$ *by*

$$\mathscr{D}_{\mathrm{reg}}(\mathrm{d}^k) := \mathrm{cl}_{\|\cdot\|_{\mathscr{D}(\mathrm{d})}}\mathrm{Reg}(\Lambda^k T^*M).$$

This definition is non-void thanks to Theorem 3.5.5 — in fact, $\mathscr{D}_{\mathrm{reg}}(\mathrm{d}^k)$ is a dense subspace of $L^2(\Lambda^k T^*M)$. As in Remark 3.4.7 we see that $\mathscr{D}_{\mathrm{reg}}(\mathrm{d}^k)$ coincides with the closure of $\mathrm{Test}(\Lambda^k T^*M)$ in $\mathscr{D}(\mathrm{d}^k)$ if M is intrinsically complete, but might be larger in general. In line with this observation, we also do not know if $\mathscr{D}(\mathrm{d}^0)=\mathscr{F}$ unless constant functions belong to $\mathscr{F}$.

Proposition 3.5.11. *For every* $\omega\in\mathscr{D}_{\mathrm{reg}}(\mathrm{d}^k)$*, we have* $\mathrm{d}\omega\in\mathscr{D}_{\mathrm{reg}}(\mathrm{d}^{k+1})$ *with*

$$\mathrm{d}(\mathrm{d}\omega) = 0.$$

Proof. The statement for $\omega\in\mathrm{Reg}(\Lambda^k T^*M)$ follows from Theorem 3.5.5 above. By definition of $\mathscr{D}(\mathrm{d}^k)$ and the closedness of d again by Theorem 3.5.5, the claim extends to arbitrary $\omega\in\mathscr{D}(\mathrm{d}^k)$. □

Remark 3.5.12. A locality property for d such as

$$1_{\{\omega=0\}}\,\mathrm{d}\omega = 0$$

for general $\omega\in\mathscr{D}(\mathrm{d}^k)$ seems hard to obtain from our axiomatization. Compare with a similar remark at [Gig18, p. 140]. ■

3.5.2 Nonsmooth de Rham cohomology and Hodge theorem

Motivated by Proposition 3.5.11, the goal of this subsection is to make sense of a nonsmooth de Rham complex. The link with the smooth setting is outlined in Remark 3.5.25.

Given $k \in \mathbf{N}_0$ we exceptionally designate by d^k the exterior differential defined on $\mathscr{D}(\mathrm{d}^k)$, which is well-defined by Proposition 3.5.11. Define the spaces $\mathrm{C}_k(M)$ and $\mathrm{E}_k(M)$ of *closed* and *exact* k-forms by

$$\begin{aligned}\mathrm{C}_k(M) &:= \ker \mathrm{d}^k\\ &= \big\{\omega \in \mathscr{D}(\mathrm{d}^k) : \mathrm{d}\omega = 0\big\},\\ \mathrm{E}_k(M) &:= \operatorname{im} \mathrm{d}^{k-1}\\ &= \big\{\omega \in \mathscr{D}(\mathrm{d}^k) : \omega = \mathrm{d}\omega' \text{ for some } \omega' \in \mathscr{D}(\mathrm{d}^{k-1})\big\}.\end{aligned}$$

By Theorem 3.5.5, we know that $\mathrm{C}_k(M)$ is a closed subspace of $L^2(\Lambda^k T^*M)$, but not if the same is true for $\mathrm{E}_k(M)$. Since $\mathrm{E}_k(M) \subset \mathrm{C}_k(M)$ and $\mathrm{C}_k(M)$ is L^2-closed, the L^2-closure of $\mathrm{E}_k(M)$ is contained in $\mathrm{C}_k(M)$ as well, hence the following definition is meaningful and non-void.

Definition 3.5.13. *The k-th* de Rham cohomology group *of M is defined by*

$$H^k_{\mathrm{dR}}(M) := \mathrm{C}_k(M) \,\big/\, \mathrm{cl}_{\|\cdot\|_{L^2(\Lambda^k T^*M)}} \mathrm{E}_k(M).$$

In view of the Hodge Theorem 3.5.23, we first need to make sense of the Hodge Laplacian and of harmonic k-forms. We start with the following. (Again, we only make explicit the degree k in the denotation of the space, but not of the differential object itself.)

Definition 3.5.14. *Given any $k \geq 1$, the space $\mathscr{D}(\delta^k)$ is defined to consist of all $\omega \in L^2(\Lambda^k T^*M)$ for which there exists $\rho \in L^2(\Lambda^{k-1}T^*M)$ such that for every $\eta \in$ $\mathrm{Test}(\Lambda^{k-1}T^*M)$, we have the identity*

$$\int_M \langle \rho, \eta\rangle \,\mathrm{d}\mathfrak{m} = \int_M \langle \omega, \mathrm{d}\eta\rangle \,\mathrm{d}\mathfrak{m}.$$

If it exists, ρ is unique, denoted by $\delta\omega$ and called the codifferential *of ω. We simply define $\mathscr{D}(\delta^0) := L^2(M)$ and $\delta := 0$ on this space.*

By the density of $\mathrm{Test}(\Lambda^{k-1}T^*M)$ in $L^2(\Lambda^{k-1}T^*M)$, the uniqueness statement is indeed true. Furthermore, δ is a closed operator, i.e. the image of the assignment $\mathrm{Id} \times \delta \colon \mathscr{D}(\delta^k) \to L^2(\Lambda^k T^*M) \times L^2(\Lambda^{k-1}T^*M)$ is closed in $L^2(\Lambda^k T^*M) \times L^2(\Lambda^{k-1}T^*M)$. Lastly, by comparison of Definition 3.5.14 with Definition 3.2.45 we have $\mathscr{D}(\delta^1) = \mathscr{D}(\mathrm{div})^\flat$, and for every $\omega \in \mathscr{D}(\delta^1)$,

$$\delta\omega = -\operatorname{div}\omega^\sharp \quad \mathfrak{m}\text{-a.e.} \tag{3.5.3}$$

The next result shows that $\mathscr{D}(\delta^k)$ is nonempty — in fact, it is dense in $L^2(\Lambda^k T^*M)$. There and in the sequel, for appropriate $f_1, \dots, f_k \in \mathscr{F}_{\mathrm{e}}$ and $i, j \in \{1, \dots, k\}$ with $i < j$, we use the abbreviations

$$\begin{aligned}\{\widehat{\mathrm{d}f_i}\} &:= \mathrm{d}f_1 \wedge \dots \wedge \widehat{\mathrm{d}f_i} \wedge \dots \wedge \mathrm{d}f_k\\ \{\widehat{\mathrm{d}f_i}, \widehat{\mathrm{d}f_j}\} &:= \mathrm{d}f_1 \wedge \dots \wedge \widehat{\mathrm{d}f_i} \wedge \dots \wedge \widehat{\mathrm{d}f_j} \wedge \dots \wedge \mathrm{d}f_k.\end{aligned}$$

Lemma 3.5.15. *For every* $f_0 \in \mathrm{Test}(M) \cup \mathbf{R}\,1_M$ *and every* $f_1, \dots, f_k \in \mathrm{Test}(M)$*, we have* $f_0\,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \in \mathscr{D}(\delta^k)$ *with*

$$\delta(f_0\,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = \sum_{i=1}^{k} (-1)^i \left[f_0\,\Delta f_i + \langle \mathrm{d}f_0, \mathrm{d}f_i\rangle\right] \{\widehat{\mathrm{d}f}_i\}$$
$$+ \sum_{i=1}^{k} \sum_{j=i+1}^{k} (-1)^{i+j}\, f_0\, [\nabla f_i, \nabla f_j]^\flat \wedge \{\widehat{\mathrm{d}f}_i, \widehat{\mathrm{d}f}_j\},$$

with the usual interpretation $\mathrm{d}1_M := 0$.

Proof. We abbreviate $\omega := f_0\,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$. By linearity, it clearly suffices to consider the defining property from Definition 3.5.14 for $\eta := g_1\,\mathrm{d}g_2 \wedge \cdots \wedge \mathrm{d}g_k$, $g_1, \dots, g_k \in \mathrm{Test}(M)$. Write $\mathfrak{S}_k$ for the set of permutations of $\{1, \dots, k\}$. Then by (3.2.13) and the Leibniz formula,

$$\int_M \langle \omega, \mathrm{d}\eta\rangle\,\mathrm{d}\mathfrak{m} = \int_M \sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}\sigma\, f_0 \left\langle \nabla g_1, \nabla f_{\sigma(1)}\right\rangle \prod_{i=2}^{k} \left\langle \nabla g_i, \nabla f_{\sigma(i)}\right\rangle \mathrm{d}\mathfrak{m}.$$

Since $\nabla g_1 \in \mathscr{D}(\mathrm{div})$, regardless of whether $f_0 \in \mathrm{Test}(M)$ or $f_0 \in \mathbf{R}\,1_M$ — and with appropriate interpretation $\nabla f_0 := 0$ in the latter case — integration by parts and then using Lemma 3.2.54 and Proposition 3.4.11 yields

$$\begin{aligned}
\int_M \langle \omega, \mathrm{d}\eta\rangle\,\mathrm{d}\mathfrak{m} &= -\int_M \sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}\sigma\, g_1 \operatorname{div}\Big[f_0\,\nabla f_{\sigma(1)} \prod_{i=2}^{k} \left\langle \nabla g_i, \nabla f_{\sigma(i)}\right\rangle\Big]\,\mathrm{d}\mathfrak{m} \\
&= -\int_M \sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}\sigma\, g_1 \left\langle \nabla f_0, \nabla f_{\sigma(1)}\right\rangle \prod_{i=2}^{k} \left\langle \nabla g_i, \nabla f_{\sigma(i)}\right\rangle \mathrm{d}\mathfrak{m} \\
&\quad - \int_M \sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}\sigma\, g_1\, f_0\, \Delta f_{\sigma(1)} \prod_{i=2}^{k} \left\langle \nabla g_i\, \nabla f_{\sigma(i)}\right\rangle \mathrm{d}\mathfrak{m} \\
&\quad - \int_M \sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}\sigma\, g_1\, f_0 \sum_{i=2}^{k} \Big[\operatorname{Hess} g_i(\nabla f_{\sigma(1)}, \nabla f_{\sigma(i)}) \\
&\qquad\qquad \times \prod_{\substack{j=2,\\ j\neq i}}^{k} \left\langle \nabla g_j, \nabla f_{\sigma(j)}\right\rangle\Big]\,\mathrm{d}\mathfrak{m} \\
&\quad - \int_M \sum_{\sigma\in\mathfrak{S}_k} \operatorname{sgn}\sigma\, g_1\, f_0 \sum_{i=2}^{k} \Big[\operatorname{Hess} f_{\sigma(i)}(\nabla f_{\sigma(1)}, \nabla g_i) \\
&\qquad\qquad \times \prod_{\substack{j=2,\\ j\neq i}}^{k} \left\langle \nabla g_j, \nabla f_{\sigma(j)}\right\rangle\Big]\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

The second last integral vanishes identically, which follows by symmetry of the Hessian and by comparing a given $\sigma \in \mathfrak{S}_k$ with the permutation that swaps $\sigma(1)$ and $\sigma(i)$ in σ, $i \in \{2, \dots, k\}$. The claim follows from the combinatorial formulas

$$\sum_{\substack{\sigma\in\mathfrak{S}_k,\\ \sigma(1)=I}} \operatorname{sgn}\sigma \prod_{j=1}^{k} \left\langle \nabla g_j, \nabla f_{\sigma(j)}\right\rangle = (-1)^{I+1} \left\langle \mathrm{d}g_2 \wedge \cdots \wedge \mathrm{d}g_k, \{\widehat{\mathrm{d}f}_I\}\right\rangle$$

$$\sum_{\substack{\sigma\in\mathfrak{S}_k,\\ \sigma(1)=I,\\ \sigma(J)=K}} \operatorname{sgn}\sigma \prod_{\substack{j=1,\\ j\neq J}}^{k} \big\langle \nabla g_i, \nabla f_{\sigma(i)}\big\rangle = a_{IJK}\,\big\langle \{\widehat{\mathrm{d}g_J}\}, \{\widehat{\mathrm{d}f_I}, \widehat{\mathrm{d}f_K}\}\big\rangle$$

for every $I, K \in \{1, \dots, k\}$ with $I \neq K$ and every $J \in \{2, \dots, k\}$, where

$$a_{IJK} := \begin{cases} (-1)^{1+I+J+K} & \text{if } I < K,\\ (-1)^{I+J+K} & \text{otherwise,}\end{cases}$$

from the identity

$$\operatorname{Hess} f_K(\nabla f_I, \cdot) - \operatorname{Hess} f_I(\nabla f_K, \cdot) = [\nabla f_I, \nabla f_K]^\flat$$

stemming from the definition of the Lie bracket and Theorem 3.3.3, and

$$\begin{aligned}\sum_{J=2}^{k} (-1)^J \,&\big\langle [\nabla f_I, \nabla f_K], \nabla g_J\big\rangle \big\langle \{\widehat{\mathrm{d}g_J}\}, \{\widehat{\mathrm{d}f_I}, \widehat{\mathrm{d}f_K}\}\big\rangle\\ &= \big\langle \mathrm{d}g_2 \wedge \cdots \wedge \mathrm{d}g_k, [\nabla f_I, \nabla f_K]^\flat \wedge \{\widehat{\mathrm{d}f_I}, \widehat{\mathrm{d}f_K}\}\big\rangle. \qquad \square\end{aligned}$$

Remark 3.5.16. By approximation, using Theorem 3.5.5 and Lemma 3.5.15, one readily proves the following. Let $f \in \mathscr{F}_{\mathrm{eb}}$ and $\omega \in H^{1,2}(T^*M)$. Assume moreover that $\mathrm{d}f \in L^\infty(T^*M)$ or that $\omega \in L^\infty(T^*M)$. Then $f\,\omega \in H^{1,2}(T^*M)$ with

$$\begin{aligned}\mathrm{d}(f\,\omega) &= f\,\mathrm{d}\omega + \mathrm{d}f \wedge \omega,\\ \delta(f\,\omega) &= f\,\delta\omega - \langle \mathrm{d}f, \omega\rangle. \qquad \blacksquare\end{aligned}$$

Definition 3.5.17. *We define the space* $W^{1,2}(\Lambda^k T^*M)$ *by*

$$W^{1,2}(\Lambda^k T^*M) := \mathscr{D}(\mathrm{d}^k) \cap \mathscr{D}(\delta^k).$$

By Theorem 3.5.5 and Lemma 3.5.15, we already know that $W^{1,2}(\Lambda^k T^*M)$ is a dense subspace of $L^2(\Lambda^k T^*M)$.

We endow $W^{1,2}(\Lambda^k T^*M)$ with the norm $\|\cdot\|_{W^{1,2}(\Lambda^k T^*M)}$ given by

$$\|\omega\|^2_{W^{1,2}(\Lambda^k T^*M)} := \|\omega\|^2_{L^2(\Lambda^k T^*M)} + \|\mathrm{d}\omega\|^2_{L^2(\Lambda^{k+1} T^*M)} + \|\delta\omega\|^2_{L^2(\Lambda^{k-1} T^*M)}$$

and we define the *contravariant* functional $\mathscr{E}_{\mathrm{con}}\colon L^2(\Lambda^k T^*M) \to [0, \infty]$ by

$$\mathscr{E}_{\mathrm{con}}(\omega) := \begin{cases} \dfrac{1}{2}\displaystyle\int_M \big[|\mathrm{d}\omega|^2 + |\delta\omega|^2\big]\,\mathrm{d}\mathfrak{m} & \text{if } \omega \in W^{1,2}(\Lambda^k T^*M),\\ \infty & \text{otherwise.}\end{cases}$$

Arguing as for Theorem 3.3.3, Theorem 3.4.3 and Theorem 3.5.5, $W^{1,2}(\Lambda^k T^*M)$ becomes a separable Hilbert space w.r.t. $\|\cdot\|_{W^{1,2}(\Lambda^k T^*M)}$. Moreover, the functional $\mathscr{E}_{\mathrm{con}}$ is clearly L^2-lower semicontinuous.

Again by Theorem 3.5.5 and Lemma 3.5.15, we have $\mathrm{Reg}(\Lambda^k T^*M) \subset W^{1,2}(\Lambda^k T^*M)$, so that the following definition makes sense.

Definition 3.5.18. *The space* $H^{1,2}(\Lambda^k T^*M) \subset W^{1,2}(\Lambda^k T^*M)$ *is defined by*

$$H^{1,2}(\Lambda^k T^*M) := \mathrm{cl}_{\|\cdot\|_{W^{1,2}(\Lambda^k T^*M)}} \mathrm{Reg}(\Lambda^k T^*M).$$

Remark 3.5.19 (Absolute boundary conditions)**.** We adopt the interpretation that ω and $\mathrm{d}\omega$ have "vanishing normal components" for any given $\omega \in H^{1,2}(\Lambda^k T^*M)$. For instance, on a compact Riemannian manifold M with boundary, by (3.2.2) every $\omega \in \mathrm{Reg}(\Lambda^k T^*M)$ satisfies the *absolute boundary conditions* [Sch95, Sec. 2.6]

$$\begin{aligned} \mathrm{n}\,\omega &= 0,\\ \mathrm{n}\,\mathrm{d}\omega &= 0 \end{aligned} \tag{3.5.4}$$

$\mathfrak{s}$-a.e. at ∂M. By Gaffney's inequality — see Remark 3.5.20 below — Proposition 3.5.11 and Lemma 3.5.15, both ω and $\mathrm{d}\omega$ belong to the corresponding $W^{1,2}$-Sobolev spaces over $\mathbf{F} := \Lambda^k T^*M$ and $\mathbf{F} := \Lambda^{k+1}T^*M$ induced by the Levi-Civita connection ∇ as introduced in Subsection 3.2.1, respectively, and these inclusions are continuous. Hence, the trace theorem can be applied, and (3.5.4) passes to the limit in the definition of $H^{1,2}(\Lambda^k T^*M)$. In particular, (3.5.4) holds $\mathfrak{s}$-a.e. for every $\omega \in H^{1,2}(\Lambda^k T^*M)$. ■

Remark 3.5.20 (Gaffney's inequality)**.** In general, $H^{1,2}(\Lambda^k T^*M)$ does not coincide with $W^{1,2}(\Lambda^k T^*M)$. On a compact Riemannian manifold M with boundary, our usual argument using Proposition 3.2.1 gives the claim, up to an important technical detail to be fixed before. (For notational simplicity, we restrict ourselves to the case $k = 1$. In the general case, $H^{1,2}(TM)$ has to be replaced by $W^{1,2}(\mathbf{F})$ defined according to Subsection 3.2.1 with $\mathbf{F} := \Lambda^k T^*M$.)

In general, d and δ are continuous w.r.t. convergence in $H^{1,2}(TM)^\flat$ [sic] by [Sch95, p. 62], whence $H^{1,2}(T^*M) \subset H^{1,2}(TM)^\flat$ with continuous inclusion. However, to apply the trace theorem to infer that all elements of $H^{1,2}(T^*M)$ have vanishing normal component at ∂M, the reverse inclusion is required (compare with the foregoing Remark 3.5.19 and with Lemma 3.6.8 below). The latter is a classical result by Gaffney, see [Sch95, Cor. 2.1.6] for a proof: there exists a finite constant $C > 0$ such that for every $\omega \in H^{1,2}(TM)^\flat$ with $\mathrm{n}\,\omega = 0$ $\mathfrak{s}$-a.e. on ∂M, we have

$$\big\|\omega^\sharp\big\|^2_{W^{1,2}(TM)} \leq C\,\big\|\omega\big\|^2_{W^{1,2}(T^*M)}. \qquad \blacksquare$$

Definition 3.5.21. *The space $\mathscr{D}(\vec{\Delta}_k)$ is defined to consist of all $\omega \in H^{1,2}(\Lambda^k T^*M)$ for which there exists $\alpha \in L^2(\Lambda^k T^*M)$ such that for every $\eta \in H^{1,2}(\Lambda^k T^*M)$,*

$$\int_M \langle \alpha, \eta\rangle\,\mathrm{d}\mathfrak{m} = \int_M \big[\langle \mathrm{d}\omega, \mathrm{d}\eta\rangle + \langle \delta\omega, \delta\eta\rangle\big]\,\mathrm{d}\mathfrak{m}.$$

In case of existence, the element α is unique, denoted by $\vec{\Delta}_k\omega$ and termed the Hodge Laplacian *of ω. Moreover, the space $\mathscr{H}(\Lambda^k T^*M)$ of* harmonic k-forms *is defined as the space of all $\omega \in H^{1,2}(\Lambda^k T^*M)$ with* $\mathrm{d}\omega = 0$ *and* $\delta\omega = 0$.

For the most important case $k = 1$, we shall write $\vec{\Delta}$ instead of $\vec{\Delta}_k$. By the integration by parts formula for the Neumann Laplacian Δ,

$$\vec{\Delta}_0 = -\Delta.$$

Moreover, the Hodge Laplacian $\vec{\Delta}_k$ is a closed operator, which can be e.g. seen by identifying $\vec{\Delta}_k\omega$, $\omega \in \mathscr{D}(\vec{\Delta}_k)$, with the only element in the subdifferential of $\widetilde{\mathscr{E}}_{\mathrm{con}}(\omega)$, where the functional $\widetilde{\mathscr{E}}_{\mathrm{con}}$ is defined in (3.6.14) below [Gig18, p. 145]. In particular, $\mathscr{H}(\Lambda^k T^*M)$ is a closed subspace of $L^2(\Lambda^k T^*M)$.

Remark 3.5.22. Our definition of harmonic k-forms follows [Sch95, Def. 2.2.1]. Of course, in the framework of Definition 3.5.21, $\omega \in \mathscr{H}(\Lambda^k T^*M)$ if and only if $\omega \in \mathscr{D}(\vec{\Delta}_k)$ with $\vec{\Delta}_k\omega = 0$, see e.g. [Gig18, p. 145]. This, however, is a somewhat implicit consequence of the interpretation of any $\omega \in \mathscr{H}(\Lambda^k T^*M)$ as obeying absolute boundary conditions (recall Remark 3.5.19). Compare with [Sch95, Prop. 1.2.6, Prop. 2.1.2, Cor. 2.1.4]. In general, the vanishing of $\vec{\Delta}_k\omega$ is a weaker condition than asking for $\mathrm{d}\omega$ and $\delta\omega$ to vanish identically for appropriate $\omega \in L^2(\Lambda^k T^*M)$ [Sch95, p. 68]. ■

We can now state and prove the following variant of Hodge's theorem.

Theorem 3.5.23 (Hodge theorem)**.** *The map $\omega \mapsto [\omega]$ from $\mathscr{H}(\Lambda^k T^*M)$ into $H^k_{\mathrm{dR}}(M)$ is an isomorphism of Hilbert spaces.*

Proof. Set $H := \mathrm{C}_k(M)$ and $V := \mathrm{E}_k(M)$. Thanks to Theorem 3.5.5, H is a Hilbert space w.r.t. $\|\cdot\|_H := \|\cdot\|_{L^2(\Lambda^k T^*M)}$. Moreover, V is a subspace of H. Lastly, by Definition 3.5.14 it is elementary to see that

$$V^\perp = \mathscr{H}(\Lambda^k T^*M),$$

where we intend the orthogonal complement w.r.t. the usual scalar product in $L^2(\Lambda^k T^*M)$. Therefore, Theorem 3.5.23 follows since by basic Hilbert space theory [KR83, Thm. 2.2.3], the map sending any $\omega \in V^\perp$ to $\omega + \mathrm{cl}_{\|\cdot\|_H} V \in H/\mathrm{cl}_{\|\cdot\|_H} V$ is a Hilbert space isomorphism. □

Remark 3.5.24. It is unclear if $W^{1,2}(\Lambda^k T^*M) \setminus H^{1,2}(\Lambda^k T^*M)$ contains elements with vanishing differential and codifferential, hence our choice of the domain of definition of $\vec{\Delta}_k$ and of harmonic k-forms. Compare with [Gig18, Rem. 3.5.16]. ■

Remark 3.5.25. Theorem 3.5.23 is a variant of the Hodge theorem on compact manifolds M with boundary [Sch95, Thm. 2.6.1]. Quite interestingly, in contrast to our setting — where boundary conditions somewhat come as a byproduct of our class of "smooth k-forms" — no boundary conditions are needed to build the corresponding de Rham cohomology group $H^k_{\mathrm{dR}}(M)$ [Sch95, p. 103]. *Still*, the latter is isomorphic to the space of harmonic *Neumann fields* [Sch95, Def. 2.2.1] which by definition satisfy absolute boundary conditions. This follows from the *Hodge–Morrey decomposition* [Sch95, Thm. 2.4.2] combined with the *Friedrichs decomposition* [Sch95, Thm. 2.4.8].

It is worth mentioning that in this setting, $\dim \mathscr{H}(\Lambda^k T^*M)$ coincides with the k-th Betti number of M [Sch95, p. 68]. ■

3.6 Curvature measures

We are now in a position to introduce various concepts of *curvature*. In Subsection 3.6.1, we prove our main result, Theorem 3.6.9, where the *κ-Ricci measure* $\mathbf{Ric}^\kappa$ is made sense of, among others in terms of $\mathbf{\Delta}^{2\kappa}$. In Subsection 3.6.2, we separate κ from $\mathbf{\Delta}^{2\kappa}$, which induces the *measure-valued Laplacian* $\mathbf{\Delta}$ fully compatible with the definition of the measure-valued divergence **div**, see Definition 3.6.11. In turn, this allows us to define a Ricci curvature on M.

3.6.1 κ-Ricci measure

Our main Theorem 3.6.9 requires some technical preliminary ingredients that are subsequently discussed.

Preliminary preparations

Lemma 3.6.1. *For every $g \in \mathrm{Test}(M) \cup \mathbf{R}\,1_M$ and every $f \in \mathrm{Test}(M)$, we have $g\,\mathrm{d}f \in \mathscr{D}(\vec{\Delta})$ with*

$$\vec{\Delta}(g\,\mathrm{d}f) = -g\,\mathrm{d}\Delta f - \Delta g\,\mathrm{d}f - 2\,\mathrm{Hess}\,f(\nabla g, \cdot),$$

with the usual interpretations $\nabla 1_M := 0$ and $\Delta 1_M := 0$. More generally, for every $X \in \mathrm{Reg}(TM)$ and every $h \in \mathrm{Test}(M) \cup \mathbf{R}\,1_M$, we have $h\,X^\flat \in \mathscr{D}(\vec{\Delta})$ with

$$\vec{\Delta}(h\,X) = h\,\vec{\Delta}X - \Delta h\,X - 2\,\nabla_{\nabla h} X.$$

Proof. The respective r.h.s.'s of the claimed identities belong to $L^2(T^*M)$, which grants their meaningfulness.

To prove the first, we claim that for every $f', g' \in \mathrm{Test}(M)$,

$$\begin{aligned}\int_M \big\langle \mathrm{d}(g\,\mathrm{d}f), \mathrm{d}(g'\,\mathrm{d}f')\big\rangle\,\mathrm{d}\mathfrak{m} &+ \int_M \delta(g\,\mathrm{d}f)\,\delta(g'\,\mathrm{d}f')\,\mathrm{d}\mathfrak{m}\\ &= -\int_M \big\langle g\,\mathrm{d}\Delta f + \Delta g\,\mathrm{d}f + 2\,\mathrm{Hess}\,f(\nabla g, \cdot), g'\,\mathrm{d}f'\big\rangle\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

By linearity of both sides in $g'\,\mathrm{d}f'$ and density, the first identity for $\vec{\Delta}(g\,\mathrm{d}f)$ then readily follows. Indeed, since $\delta(g\,\mathrm{d}f) = -\langle \nabla g, \nabla f\rangle - g\,\Delta f$ $\mathfrak{m}$-a.e. by (3.5.3) — with appropriate interpretation if $g \in \mathbf{R}\,1_M$ according to (3.2.16) — $\delta(g\,\mathrm{d}f)$ belongs to $\mathscr{F}$ by Proposition 3.4.11, and we have

$$\begin{aligned}&\int_M \delta(g\,\mathrm{d}f)\,\delta(g'\,\mathrm{d}f')\,\mathrm{d}\mathfrak{m}\\ &\qquad = -\int_M \big\langle \mathrm{d}\big[\langle \nabla g, \nabla f\rangle + g\,\Delta f\big], g'\,\mathrm{d}f'\big\rangle\,\mathrm{d}\mathfrak{m}\\ &\qquad = -\int_M \big\langle \mathrm{Hess}\,g(\nabla f, \cdot) + \mathrm{Hess}\,f(\nabla g, \cdot) + \mathrm{d}g\,\Delta f + g\,\mathrm{d}\Delta f, g'\,\mathrm{d}f'\big\rangle\,\mathrm{d}\mathfrak{m},\end{aligned}$$

with $\mathrm{Hess}\,g := 0$ whenever $g \in \mathbf{R}\,1_M$. On the other hand, by Theorem 3.5.5 we have $\mathrm{d}(g\,\mathrm{d}f) = \mathrm{d}g \wedge \mathrm{d}f$, which belongs to $\mathscr{D}(\delta^2)$ by Lemma 3.5.15 with

$$\begin{aligned}&\int_M \big\langle \mathrm{d}(g\,\mathrm{d}f), \mathrm{d}(g'\,\mathrm{d}f')\big\rangle\,\mathrm{d}\mathfrak{m}\\ &\qquad = \int_M \big\langle \delta(\mathrm{d}g \wedge \mathrm{d}f), g'\,\mathrm{d}f'\big\rangle\,\mathrm{d}\mathfrak{m}\\ &\qquad = \int_M \big\langle \Delta f\,\mathrm{d}g - \Delta g\,\mathrm{d}f - [\nabla g, \nabla f]^\flat, g'\,\mathrm{d}f'\big\rangle\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Adding up these two identities yields the claim since

$$[\nabla g, \nabla f]^\flat = \mathrm{Hess}\,f(\nabla g, \cdot) - \mathrm{Hess}\,g(\nabla f, \cdot).$$

Concerning the second claim, from what we already proved, the linearity of $\vec{\Delta}$ and since $\mathrm{Test}(M) \cup \mathbf{R}\,1_M$ is an algebra, it follows that $h\,X^\flat \in \mathscr{D}(\vec{\Delta})$. Again by linearity, it thus suffices to consider the case $X := g\,\nabla f$, $g \in \mathrm{Test}(M) \cup \mathbf{R}\,1_M$ and $f \in \mathrm{Test}(M)$. Indeed, we have

$$\vec{\Delta}(h\,g\,\mathrm{d}f) = -h\,g\,\mathrm{d}f - \Delta(h\,g)\,\mathrm{d}f - 2\,\mathrm{Hess}\,f(\nabla(h\,g), \cdot)$$

$$
\begin{aligned}
&= -h\, g\, \mathrm{d}\Delta f - g\, \Delta h\, \mathrm{d}f - 2\langle \nabla h, \nabla g\rangle\, \mathrm{d}f - h\, \Delta g\, \mathrm{d}f\\
&\qquad - 2\, g\, \mathrm{Hess}\, f(\nabla h, \cdot) - 2\, h\, \mathrm{Hess}\, f(\nabla g, \cdot)\\
&= h\, \vec{\Delta}(g\, \mathrm{d}f) - \Delta h\, (g\, \mathrm{d}f) - 2\, \big[\nabla_{\nabla h}(g\, \nabla f)\big]^\sharp.
\end{aligned}
$$

In the last identity, we used the identity

$$\nabla(g\, \nabla f) = \nabla g \otimes \nabla f + g\, (\mathrm{Hess}\, f)^\sharp$$

inherited from Theorem 3.3.3. □

Recall from Theorem 3.2.32 that $\nabla\, \mathrm{Test}(M)$, a set consisting of $\mathfrak{m}$-essentially bounded elements, generates $L^2(TM)$ in the sense of L^∞-modules. Hence, for every $A \in L^2(T^{\otimes 2}M)$, by (3.2.10) its symmetric part obeys the duality formula

$$
\begin{aligned}
\big|A_{\mathrm{sym}}\big|_{\mathrm{HS}}^2 = \operatorname{esssup}\Big\{ 2A : \sum_{j=1}^m \nabla h_j \otimes \nabla h_j - \Big|\sum_{j=1}^m \nabla h_j \otimes \nabla h_j\Big|_{\mathrm{HS}}^2 :\\
m \in \mathbf{N},\ h_1, \dots, h_m \in \mathrm{Test}(M)\Big\}.
\end{aligned}
\tag{3.6.1}
$$

This is crucial in the next Lemma 3.6.2. Therein, the divergence of $X \in \mathrm{Reg}(TM)$ is understood in the sense of Definition 3.2.45. Recall that if $X \in \mathrm{Test}(TM)$, this is the same as interpreting it according to Definition 3.2.46 by Lemma 3.2.50.

Lemma 3.6.2. *For every $X \in \mathrm{Reg}(TM)$, we have $|X|^2 \in \mathscr{D}(\mathbf{\Delta}^{2\kappa})$ with*

$$\mathbf{\Delta}^{2\kappa} \frac{|X|^2}{2} \geq \Big[\big|\nabla X\big|_{\mathrm{HS}}^2 - \big\langle X, (\vec{\Delta} X^\flat)^\sharp \big\rangle\Big]\, \mathfrak{m}.$$

Proof. The Leibniz rule for $\mathbf{\Delta}^{2\kappa}$ [ER+20, Cor. 6.3] together with polarization and the linearity of $\mathbf{\Delta}^{2\kappa}$ on $\mathscr{F}$, recall in particular Proposition 3.2.75, ensure that $|X|^2 \in \mathscr{F}$, and in fact $|X|^2 \in \mathscr{D}(\mathbf{\Delta}^{2\kappa})$.

Write $X := g_1\, \nabla f_1 + \cdots + g_n\, \nabla f_n$ for certain $g_i \in \mathrm{Test}(M) \cup \mathbf{R}\, 1_M$ and $f_i \in \mathrm{Test}(M)$, $i \in \{1, \dots, n\}$. Retaining the notation from Lemma 3.3.9 for $N' := \infty$, we first claim the identity

$$\rho_1[f, g] = \mathbf{\Delta}^{2\kappa} \frac{|X|^2}{2} + \Big[\big\langle X, (\vec{\Delta} X^\flat)^\sharp \big\rangle - \big|(\nabla X)_{\mathrm{asym}}\big|_{\mathrm{HS}}^2\Big]\, \mathfrak{m}. \tag{3.6.2}$$

Again by the Leibniz rule for $\mathbf{\Delta}^{2\kappa}$ from [ER+20, Cor. 6.3] and Proposition 3.3.32,

$$
\begin{aligned}
\mathbf{\Delta}^{2\kappa} \frac{|X|^2}{2} &= \frac{1}{2} \sum_{i,i'=1}^n \widetilde{g}_i\, \widetilde{g}_{i'}\, \mathbf{\Delta}^{2\kappa} \langle \nabla f_i, \nabla f_{i'}\rangle + \sum_{i,i'=1}^n \big\langle \nabla[g_i\, g_{i'}], \nabla \langle \nabla f_i, \nabla f_{i'}\rangle \big\rangle\, \mathfrak{m}\\
&\qquad + \frac{1}{2} \sum_{i,i'=1}^n \langle \nabla f_i, \nabla f_{i'}\rangle\, \Delta[g_i\, g_{i'}]\, \mathfrak{m}\\
&= \frac{1}{2} \sum_{i,i'=1}^n \widetilde{g}_i\, \widetilde{g}_{i'}\, \mathbf{\Delta}^{2\kappa} \langle \nabla f_i, \nabla f_{i'}\rangle\\
&\qquad + \sum_{i,i'=1}^n \big[\mathrm{Hess}\, f_i(\nabla f_{i'}, \nabla[g_i\, g_{i'}]) + \mathrm{Hess}\, f_{i'}(\nabla f_i, \nabla[g_i\, g_{i'}])\big]\, \mathfrak{m}\\
&\qquad + \sum_{i,i'=1}^n \big[g_{i'}\, \Delta g_i\, \langle \nabla f_i, \nabla f_{i'}\rangle + \langle \nabla g_i, \nabla g_{i'}\rangle\, \langle \nabla f_i, \nabla f_{i'}\rangle\big]\, \mathfrak{m}
\end{aligned}
$$

$$= \frac{1}{2} \sum_{i,i'=1}^{n} \widetilde{g_i}\,\widetilde{g_{i'}}\,\mathbf{\Delta}^{2\kappa}\langle\nabla f_i, \nabla f_{i'}\rangle$$
$$+ 2 \sum_{i,i'=1}^{n} \big[g_i \operatorname{Hess} f_i(\nabla f_{i'}, \nabla g_{i'}) + g_i \operatorname{Hess} f_{i'}(\nabla f_i, \nabla g_{i'})\big]\,\mathfrak{m}$$
$$+ \sum_{i,i'=1}^{n} \big[g_{i'}\,\Delta g_i\,\langle\nabla f_i, \nabla f_{i'}\rangle + \langle\nabla g_i, \nabla g_{i'}\rangle\,\langle\nabla f_i, \nabla f_{i'}\rangle\big]\,\mathfrak{m}.$$

Furthermore, by Lemma 3.6.1 we obtain

$$\vec{\Delta}X^\flat = -\sum_{i=1}^{n} \big[g_i\,\mathrm{d}\Delta f_i + \Delta g_i\,\mathrm{d}f_i + 2\operatorname{Hess} f_i(\nabla g_i, \cdot)\big],$$

which entails

$$\big\langle X, (\vec{\Delta}X^\flat)^\sharp\big\rangle = -\sum_{i,i'=1}^{n} \big[g_{i'}\,\Delta g_i\,\langle\nabla f_i, \nabla f_{i'}\rangle + 2\,g_{i'}\operatorname{Hess} f_i(\nabla g_i, \nabla f_{i'})\big]$$
$$-\sum_{i,i'=1}^{n} \big[g_i\,g_{i'}\,\langle\nabla\Delta f_i, \nabla f_{i'}\rangle\big] \quad \mathfrak{m}\text{-a.e.}$$

Next, by item (iii) in Theorem 3.4.3, the symmetry of the Hessian ensured by Theorem 3.3.3, and the definition of the anti-symmetric part of an element in $L^2(T^{\otimes 2}M)$,

$$(\nabla X)_{\mathrm{asym}} = \frac{1}{2}\sum_{i=1}^{n} \big[\nabla g_i \otimes \nabla f_i - \nabla f_i \otimes \nabla g_i\big]$$

from which we get

$$\big|(\nabla X)_{\mathrm{asym}}\big|_{\mathrm{HS}}^2 = \frac{1}{2}\sum_{i,i'=1}^{n} \big[\langle\nabla f_i, \nabla f_{i'}\rangle\,\langle\nabla g_i, \nabla g_{i'}\rangle - \langle\nabla f_i, \nabla g_{i'}\rangle\,\langle\nabla g_i, \nabla f_{i'}\rangle\big] \quad \mathfrak{m}\text{-a.e.}$$

Lastly, the definition (3.2.21) of the measure-valued Γ_2-operator yields

$$\mathbf{\Gamma}_2^{2\kappa}(f_i, f_{i'}) = \frac{1}{2}\,\mathbf{\Delta}^{2\kappa}\langle\nabla f_i, \nabla f_{i'}\rangle - \frac{1}{2}\big[\langle\nabla\Delta f_i, \nabla f_{i'}\rangle + \langle\nabla\Delta f_{i'}, \nabla f_i\rangle\big]\,\mathfrak{m}$$

for every $i, i' \in \{1, \dots, n\}$. Patching terms together straightforwardly leads to (3.6.2).

Therefore, w.r.t. $\mathfrak{m}$ we have

$$\Big[\mathbf{\Delta}^{2\kappa}\frac{|X|^2}{2}\Big]_\perp = \rho_1[f, g]_\perp \geq 0$$

thanks to Lemma 3.3.9. To finally prove the claimed inequality, it thus suffices to show that, setting $\delta^{2\kappa}|X|^2/2 := \mathrm{d}(\mathbf{\Delta}^{2\kappa}|X|^2/2)_\ll/\mathrm{d}\mathfrak{m}$,

$$\delta^{2\kappa}\frac{|X|^2}{2} \geq \big|\nabla X\big|_{\mathrm{HS}}^2 - \big\langle X, (\vec{\Delta}X^\flat)^\sharp\big\rangle \quad \mathfrak{m}\text{-a.e.} \tag{3.6.3}$$

Indeed, having (3.6.2) at our disposal, Lemma 3.3.9 implies that for every $m \in \mathbf{N}$ and for every $h_1, \dots, h_m \in \mathrm{Test}(M)$,

$$\Big|\nabla X : \sum_{j=1}^{m} \nabla h_j \otimes \nabla h_j\Big| \leq \Big[\delta^{2\kappa}\frac{|X|^2}{2} + \big\langle X, (\vec{\Delta}X^\flat)^\sharp\big\rangle - \big|(\nabla X)_{\mathrm{asym}}\big|_{\mathrm{HS}}^2\Big]^{1/2}$$

$$\times \Big|\sum_{j=1}^m \nabla h_j \otimes \nabla h_j\Big|_{\mathrm{HS}} \quad \mathfrak{m}\text{-a.e.}$$

Applying Young's inequality at the r.h.s. and optimizing over $h_1, \dots, h_m \in \mathrm{Test}(M)$ according to (3.6.1) yields

$$\big|(\nabla X)_{\mathrm{sym}}\big|_{\mathrm{HS}}^2 \leq \delta^{2\kappa}\,\frac{|X|^2}{2} + \big\langle X, (\vec{\Delta} X^\flat)^\sharp\big\rangle - \big|(\nabla X)_{\mathrm{asym}}\big|_{\mathrm{HS}}^2 \quad \mathfrak{m}\text{-a.e.}$$

which is the remaining claim (3.6.3) by the decomposition (3.2.9). □

The following consequence of Lemma 3.6.2 is not strictly needed in Theorem 3.6.9, but gives an idea about the reasoning for Lemma 3.6.8 below.

Corollary 3.6.3. *For every $f \in \mathscr{D}(\Delta)$, we have $|\nabla f| \in \mathscr{F}$ and*

$$\mathscr{E}_2(f) \leq \int_M (\Delta f)^2 \,\mathrm{d}\mathfrak{m} - \big\langle \kappa \,\big|\, |\nabla f|^2\big\rangle.$$

Proof. Since $\mathscr{D}(\Delta) \subset \mathscr{D}_{\mathrm{reg}}(\mathrm{Hess})$ by Corollary 3.3.12 and $\nabla\,\mathscr{D}_{\mathrm{reg}}(\mathrm{Hess}) \subset H^{1,2}(TM)$ by Theorem 3.4.3, Lemma 3.4.13 implies that $|\nabla f| \in \mathscr{F}$ whenever $f \in \mathscr{D}(\Delta)$. In particular, $\mathscr{E}_2(f)$ is finite and the pairing $\big\langle \kappa \,\big|\, |\nabla f|^2\big\rangle$ is well-defined for every such f.

The claimed estimate for $f \in \mathrm{Test}(M)$ follows by integrating Lemma 3.6.2 for $X := \nabla f \in \mathrm{Reg}(TM)$ over all of M, and then using (3.2.16) and (3.2.23). In the more general case $f \in \mathscr{D}(\Delta)$, let $(f_k)_{k\in\mathbf{N}}$ be a sequence in $\mathrm{Test}(M)$ as constructed in the proof of Corollary 3.3.12, i.e. which satisfies $\mathscr{E}_2(f_k) \to \mathscr{E}_2(f)$, $\Delta f_k \to \Delta f$ in $L^2(M)$ and $\nabla f_k \to \nabla f$ in $L^2(TM)$ as $k \to \infty$. The first and the third convergence, with Theorem 3.4.3, imply that $\nabla f_k \to \nabla f$ in $H^{1,2}(TM)$ as $k \to \infty$. Hence

$$\lim_{k\to\infty} \big\langle \kappa \,\big|\, |\nabla f_k|\big\rangle = \big\langle \kappa \,\big|\, |\nabla f|\big\rangle$$

by Corollary 3.4.14. The conclusion follows easily. □

For Lemma 3.6.8, but also for Theorem 3.6.9 and Theorem 3.6.21 below, we shall need the subsequent Lemma 3.6.4. (Recall Subsection 3.2.6 for the well-definedness of $\mathscr{E}^{q\kappa}$ for $\kappa \in \mathbf{K}_{1-}(M)$ and $q \in [1,2]$.) Corollary 3.6.6 is then deduced along the same lines as Proposition 3.2.79 after setting $q = 1$ and letting $\varepsilon \to 0$ in Lemma 3.6.4.

Lemma 3.6.4. *Let $X \in \mathrm{Reg}(TM)$, and let $\phi \in \mathscr{D}(\Delta^{q\kappa}) \cap L^\infty(M)$ be nonnegative with $\Delta^{q\kappa}\phi \in L^\infty(M)$. Given any $\varepsilon > 0$, we define $\varphi_\varepsilon \in \mathrm{C}^\infty([0,\infty))$ by $\varphi_\varepsilon(r) := (r+\varepsilon)^{q/2} - \varepsilon^{q/2}$. Then for every $q \in [1,2]$,*

$$\begin{aligned}
&\int_M \big[\varphi_\varepsilon \circ |X|^2\big]\,\Delta^{q\kappa}\phi \,\mathrm{d}\mathfrak{m}\\
&\qquad \geq -2\int_M \phi\,\big[\varphi_\varepsilon' \circ |X|^2\big]\,\big\langle X, (\vec{\Delta} X^\flat)^\sharp\big\rangle \,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad + 2\,\big\langle \kappa \,\big|\, \phi\,|X|^2\,\varphi_\varepsilon' \circ |X|^2\big\rangle - q\,\big\langle \kappa \,\big|\, \phi\,\varphi_\varepsilon \circ |X|^2\big\rangle.
\end{aligned}$$

Proof. According to our choice of q, we have $2r\,\varphi_\varepsilon''(r) \geq -\varphi_\varepsilon'(r)$ for every $r \geq 0$. Recall from Lemma 3.4.13 and Proposition 3.2.9 that $\phi\,\varphi_\varepsilon' \circ |X|^2 \in \mathscr{F}_{\mathrm{b}}$. Therefore, by (3.2.19), Proposition 3.2.9, Lemma 3.6.2 and Lemma 3.4.13 we get

$$\frac{1}{2}\int_M \big[\varphi_\varepsilon \circ |X|^2\big]\,\Delta^{q\kappa}\phi \,\mathrm{d}\mathfrak{m} - \big\langle \kappa \,\big|\, \phi\,|X|^2\,\varphi_\varepsilon' \circ |X|^2\big\rangle$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_M \big[\varphi_\varepsilon' \circ |X|^2\big]\,\big\langle \nabla\phi, \nabla|X|^2\big\rangle \,\mathrm{d}\mathfrak{m}\\
&\qquad - \big\langle \kappa \,\big|\, \phi\,\big[q(\varphi_\varepsilon \circ |X|^2)/2 + |X|^2\,\varphi_\varepsilon' \circ |X|^2\big]\big\rangle\\
&= -\frac{1}{2}\,\mathscr{E}^{2\kappa}\big(\phi\,\varphi_\varepsilon' \circ |X|^2, |X|^2\big) + \frac{1}{2}\int_M \phi\,\big[\varphi_\varepsilon'' \circ |X|^2\big]\,\big|\nabla|X|^2\big|^2 \,\mathrm{d}\mathfrak{m}\\
&\qquad - \frac{q}{2}\,\big\langle \kappa \,\big|\, \phi\,\varphi_\varepsilon \circ |X|^2\big\rangle\\
&\geq \int_M \phi\,\big[\varphi_\varepsilon' \circ |X|^2\big]\,\big|\nabla X\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} - \int_M \phi\,\big[\varphi_\varepsilon' \circ |X|^2\big]\,\big\langle X, (\vec{\Delta} X^\flat)^\sharp\big\rangle \,\mathrm{d}\mathfrak{m}\\
&\qquad + 2\int_M \phi\,\big[\varphi'' \circ |X|^2\big]\,|X|^2\,\big|\nabla|X|\big|^2 \,\mathrm{d}\mathfrak{m} - \frac{q}{2}\,\big\langle \kappa \,\big|\, \phi\,\varphi_\varepsilon \circ |X|^2\big\rangle\\
&\geq -\int_M \phi\,\big[\varphi_\varepsilon' \circ |X|^2\big]\,\big\langle X, (\vec{\Delta} X^\flat)^\sharp\big\rangle \,\mathrm{d}\mathfrak{m} - \frac{q}{2}\,\big\langle \kappa \,\big|\, \phi\,\varphi_\varepsilon \circ |X|^2\big\rangle.
\end{aligned}
$$

Multiplying this inequality by 2 terminates the proof. □

Remark 3.6.5. Let Assumption 3.6.29 below hold. Then, after partial integration of the Hodge Laplacian term and approximation, the conclusion of Lemma 3.6.4, with κ replaced by κ_n, $n \in \mathbf{N}$, holds under the more general hypothesis $q = 2$, $\varepsilon = 0$, $X \in \mathscr{D}(\vec{\Delta})^\sharp$ and $\phi \in \mathrm{Test}_{L^\infty}(M)$. See also Lemma 3.6.32 below. ■

Corollary 3.6.6. *For every $X \in \mathrm{Reg}(TM)$,*

$$\mathbf{\Delta}^{2\kappa}|X|^2[M] = -2\,\big\langle \kappa \,\big|\, |X|^2\big\rangle.$$

We will have to identify 1-forms in $H^{1,2}(T^*M)$ with their vector field counterparts in while additionally retaining their respective first order regularities in Theorem 3.6.9. This is discussed now. In some sense, Lemma 3.6.8 can be seen as an analogue of Gaffney's inequality in Remark 3.5.20 under curvature lower bounds.

Definition 3.6.7. *We define the space $H^{1,2}_\sharp(TM)$ as the image of $H^{1,2}(T^*M)$ under the map $\sharp$, endowed with the norm*

$$\|X\|_{H^{1,2}_\sharp(TM)} := \|X^\flat\|_{H^{1,2}(T^*M)}.$$

Lemma 3.6.8. *$H^{1,2}_\sharp(TM)$ is a subspace of $H^{1,2}(TM)$. The aforementioned natural inclusion is continuous. Additionally, for every $X \in H^{1,2}_\sharp(TM)$,*

$$\mathscr{E}_{\mathrm{cov}}(X) \leq \mathscr{E}_{\mathrm{con}}(X^\flat) - \big\langle \kappa \,\big|\, |X|^2\big\rangle.$$

Proof. Let $\rho' \in [0,1)$ and $\alpha' \in \mathbf{R}$ be as in Lemma 3.2.60 for $\mu := \kappa^-$.

Clearly $\mathrm{Reg}(TM) = \mathrm{Reg}(T^*M)^\sharp \subset H^{1,2}_\sharp(TM)$ as well as $\mathrm{Reg}(TM) \subset H^{1,2}(TM)$ by definition of the respective spaces. Moreover, the claimed inequality for $X \in \mathrm{Reg}(TM)$ follows after evaluating the inequality in Lemma 3.6.2 at M and using Lemma 3.6.1 and Corollary 3.6.6.

To prove that $H^{1,2}_\sharp(T^*M) \subset H^{1,2}(TM)$ with continuous inclusion, we again first study what happens for $X \in \mathrm{Reg}(TM)$. By the previous step, Lemma 3.2.60 for $\mu := \kappa^-$ together with Lemma 3.4.13, and (3.5.3),

$$\mathscr{E}_{\mathrm{cov}}(X) \leq -\big\langle \kappa \,\big|\, |X|^2\big\rangle + \mathscr{E}_{\mathrm{con}}(X^\flat) \tag{3.6.4}$$

$$\begin{aligned} &\leq \left\langle \kappa^- \,\middle|\, |X|^2 \right\rangle + \mathscr{E}_{\mathrm{con}}(X^\flat) \\ &\leq \rho' \, \mathscr{E}_{\mathrm{cov}}(X) + \alpha' \, \|X\|^2_{L^2(TM)} + \mathscr{E}_{\mathrm{con}}(X^\flat). \end{aligned}$$

Rearranging yields

$$\mathscr{E}_{\mathrm{cov}}(X) \leq \frac{\alpha'}{1-\rho'} \, \|X\|^2_{L^2(TM)} + \frac{1}{1-\rho'} \, \mathscr{E}_{\mathrm{con}}(X^\flat). \tag{3.6.5}$$

This inequality extends to arbitrary $X \in H^{1,2}_\sharp(T^*M)$ by applying it to all members of a sequence $(X_n)_{n\in\mathbf{N}}$ in $\mathrm{Reg}(TM)$ such that $X_n \to X$ in $H^{1,2}_\sharp(TM)$ as $n \to \infty$ and using the L^2-lower semicontinuity of $\mathscr{E}_{\mathrm{cov}}$ from Theorem 3.4.3. In particular, as the r.h.s. of (3.6.5) is continuous in $H^{1,2}_\sharp(TM)$, both $H^{1,2}_\sharp(TM) \subset H^{1,2}(TM)$ and the continuity of this inclusion follow. In particular, by Corollary 3.4.14 the inequality (3.6.4) is stable under this limit procedure. □

Main result We now come to the main result of this chapter.

Theorem 3.6.9. *There exists a unique continuous mapping* $\mathbf{Ric}^\kappa \colon H^{1,2}_\sharp(TM)^2 \to \mathfrak{M}^\pm_{\mathrm{f}}(M)_{\mathscr{E}}$ *satisfying the identity*

$$\begin{aligned} \mathbf{Ric}^\kappa(X,Y) = \mathbf{\Delta}^{2\kappa} \frac{\langle X, Y\rangle}{2} + \Big[\frac{1}{2} \big\langle X, (\vec{\Delta} Y^\flat)^\sharp \big\rangle \\ + \frac{1}{2} \big\langle Y, (\vec{\Delta} X^\flat)^\sharp \big\rangle - \nabla X : \nabla Y \Big] \, \mathfrak{m} \end{aligned} \tag{3.6.6}$$

for every $X, Y \in \mathrm{Reg}(TM)$. *The map* $\mathbf{Ric}^\kappa$ *is symmetric and* $\mathbf{R}$*-bilinear. Furthermore, for every* $X, Y \in H^{1,2}_\sharp(TM)$*, it obeys*

$$\begin{aligned} \mathbf{Ric}^\kappa(X,X) &\geq 0, \\ \mathbf{Ric}^\kappa(X,Y)[M] &= \int_M \big[\langle \mathrm{d}X^\flat, \mathrm{d}Y^\flat \rangle + \delta X^\flat \, \delta Y^\flat \big] \, \mathrm{d}\mathfrak{m} \\ &\qquad - \int_M \nabla X : \nabla Y \, \mathrm{d}\mathfrak{m} - \big\langle \kappa \,\big|\, \langle X, Y \rangle \big\rangle, \\ \big\| \mathbf{Ric}^\kappa(X,Y) \big\|^2_{\mathrm{TV}} &\leq \big[\mathscr{E}_{\mathrm{con}}(X^\flat) + \big\langle \kappa \,\big|\, |X|^2 \big\rangle \big] \big[\mathscr{E}_{\mathrm{con}}(Y^\flat) + \big\langle \kappa \,\big|\, |Y|^2 \big\rangle \big]. \end{aligned} \tag{3.6.7}$$

Proof. Given any $X, Y \in \mathrm{Reg}(TM)$, we define $\mathbf{Ric}^\kappa(X,Y) \in \mathfrak{M}^\pm_{\mathrm{f}}(M)_{\mathscr{E}}$ by (3.6.6). This assignment is well-defined since $\langle X, Y \rangle \in \mathscr{D}(\mathbf{\Delta}^{2\kappa})$ by Lemma 3.6.2 and gives a nonnegative element. The map $\mathbf{Ric}^\kappa \colon \mathrm{Reg}(TM)^2 \to \mathfrak{M}^\pm_{\mathrm{f}}(M)_{\mathscr{E}}$ defined in that way is clearly symmetric and $\mathbf{R}$-bilinear.

Let X and Y as above. Then by Lemma 3.6.1 and Corollary 3.6.6,

$$\begin{aligned} \|\mathbf{Ric}^\kappa(X,X)\|_{\mathrm{TV}} &= \mathbf{Ric}^\kappa(X,X)[M] \\ &= \mathscr{E}_{\mathrm{con}}(X^\flat) - \mathscr{E}_{\mathrm{cov}}(X) - \big\langle \kappa \,\big|\, |X|^2 \big\rangle \\ &\leq \mathscr{E}_{\mathrm{con}}(X^\flat) - \big\langle \kappa \,\big|\, |X|^2 \big\rangle. \end{aligned}$$

By symmetry and $\mathbf{R}$-bilinearity of $\mathbf{Ric}^\kappa$, for every $\lambda \in \mathbf{R}$ we obtain

$$\begin{aligned} \lambda^2 \, \mathbf{Ric}^\kappa(X,X) + 2\lambda \, \mathbf{Ric}^\kappa(X,Y) + \mathbf{Ric}^\kappa(Y,Y) \\ = \mathbf{Ric}^\kappa(\lambda X + Y, \lambda X + Y) \geq 0. \end{aligned}$$

Lemma 3.3.8 thus implies that

$$\|\mathbf{Ric}^\kappa(X,Y)\|_{\mathrm{TV}} \leq \big[\mathscr{E}_{\mathrm{con}}(X^\flat) - \big\langle \kappa \,\big|\, |X|^2\big\rangle\big]^{1/2} \big[\mathscr{E}_{\mathrm{con}}(Y^\flat) - \big\langle \kappa \,\big|\, |Y|^2\big\rangle\big]^{1/2}.$$

Since the r.h.s. is jointly continuous in X and Y w.r.t. convergence in $H^{1,2}_\sharp(TM)$ by Lemma 3.6.8 and Corollary 3.4.14, the above map $\mathbf{Ric}^\kappa$ extends continuously and uniquely to a (non-relabeled) map $\mathbf{Ric}^\kappa\colon H^{1,2}_\sharp(TM)^2 \to \mathfrak{M}^\pm_{\mathrm{f}}(M)$.

In particular, the last inequality of (3.6.7) directly comes as a byproduct of the previous argument, while the first inequality of (3.6.7) follows by continuously extending Lemma 3.6.2 to any $X \in H^{1,2}_\sharp(TM)$. The second identity for $X \in \mathrm{Reg}(TM)$ follows from Lemma 3.6.1 and Corollary 3.6.6 after integration by parts. Since both sides are jointly $H^{1,2}_\sharp$-continuous in X and Y by Lemma 3.6.8 and Corollary 3.4.14, this identity easily extends to all $X, Y \in H^{1,2}_\sharp(TM)$. □

Remark 3.6.10. In the abstract setting of diffusion operators, Sturm [Stu18a] introduced a "pointwise", possibly dimension-dependent Ricci tensor. Although the smoothness assumptions are not really justified in our setting [Gig18, Rem. 3.6.6], it would be interesting to study the behavior of $\mathbf{Ric}^\kappa$ under conformal and drift transformations of $\langle\cdot,\cdot\rangle$ and $\mathfrak{m}$, respectively, as done in the abstract framework of [Stu18a]. ■

3.6.2 Ricci curvature from the κ-Ricci measure

Next, we separate κ from $\mathbf{Ric}^\kappa$ in Theorem 3.6.9 to define the *Ricci curvature*.

Measure-valued Laplacian

Definition 3.6.11. *We define $\mathscr{D}(\mathbf{\Delta})$ to consist of all $f \in \mathscr{F}$ for which $\nabla f \in \mathscr{D}_{L^2}(\mathbf{div})$, in which case we define the* measure-valued Laplacian *of f by*

$$\mathbf{\Delta} f := \mathbf{div}\,\nabla f.$$

Lemma 3.6.12. *The signed Borel measure $\widetilde{u}^2\,\kappa$ has finite total variation for every $u \in \mathscr{F}$. In particular, if $u^2 \in \mathscr{D}(\mathbf{\Delta}^{2\kappa})$, we have $u^2 \in \mathscr{D}(\mathbf{\Delta})$ with*

$$\mathbf{\Delta}(u^2) = \mathbf{\Delta}^{2\kappa}(u^2) + 2\,\widetilde{u}^2\,\kappa.$$

In particular $\mathbf{\Delta}|\nabla f|^2$ has finite total variation for every $f \in \mathrm{Test}(M)$.

Proof. The first claim follows by observing that $\kappa \in \mathbf{K}_{1-}(M)$ if and only if $|\kappa| \in \mathbf{K}_{1-}(M)$, whence $\widetilde{u}^2\,|\kappa|$ is a finite measure by [ER+20, Cor. 2.25].

The remaining claims then follow by straightforward computations using Definition 3.2.74, (3.2.19) and Proposition 3.2.79. □

Dimension-free Ricci curvature Keeping in mind Theorem 3.6.9, we define the map $\mathbf{Ric}\colon H^{1,2}_\sharp(TM)^2 \to \mathfrak{M}^\pm_{\mathrm{f}}(M)_{\mathscr{E}}$ by

$$\mathbf{Ric}(X,Y) := \mathbf{Ric}^\kappa(X,Y) + \langle X, Y\rangle_\sim\,\kappa.$$

This map is well-defined, symmetric and $\mathbf{R}$-bilinear — indeed, by polarization, Lemma 3.6.8 and Lemma 3.4.13, $\langle X,Y\rangle_\sim\,\kappa \in \mathfrak{M}^\pm_{\mathrm{f}}(M)_{\mathscr{E}}$ for every $X, Y \in H^{1,2}_\sharp(TM)$. $\mathbf{Ric}$ is also jointly continuous, which follows from polarization, Corollary 3.4.14 and

Lemma 3.6.8 again. Lastly, by Theorem 3.6.9, Lemma 3.6.2 and Lemma 3.6.12, for every $X, Y \in \mathrm{Reg}(TM)$,

$$\begin{aligned}\mathbf{Ric}(X, Y) = \mathbf{\Delta}\frac{\langle X, Y\rangle}{2} + \Big[\frac{1}{2}\big\langle X, (\vec{\Delta}Y^\flat)^\sharp\big\rangle \\ + \frac{1}{2}\big\langle X, (\vec{\Delta}Y^\flat)^\sharp\big\rangle - \nabla X : \nabla Y\Big]\,\mathfrak{m}.\end{aligned} \tag{3.6.8}$$

Of course, (3.6.8) is defined to recover the familiar *vector Bochner formula*

$$\mathbf{\Delta}\frac{|X|^2}{2} + \big\langle X, (\vec{\Delta}X^\flat)^\sharp\big\rangle\,\mathfrak{m} = \mathbf{Ric}(X, X) + \big|\nabla X\big|_{\mathrm{HS}}^2\,\mathfrak{m}$$

for $X \in \mathrm{Reg}(TM)$, which, setting $X := \nabla f$, $f \in \mathrm{Test}(M)$ and using Lemma 3.6.1 as well as Theorem 3.4.3, in turn reduces to the *Bochner identity*

$$\mathbf{\Delta}\frac{|\nabla f|^2}{2} - \big\langle \nabla f, \nabla\Delta f\big\rangle\,\mathfrak{m} = \mathbf{Ric}(\nabla f, \nabla f) + \big|\mathrm{Hess}\, f\big|_{\mathrm{HS}}^2\,\mathfrak{m}.$$

In general, $\mathbf{Ric}(X, X)$ is no longer nonnegative, but it is bounded from below by κ in the following sense, which is a direct consequence of (3.6.8), (3.6.7) as well as the respective $H^{1,2}_\sharp$-continuity of both sides of the resulting inequality.

Proposition 3.6.13. *For every $X \in H^{1,2}_\sharp(TM)$, the map* $\mathbf{Ric}$ *defined above obeys*

$$\mathbf{Ric}(X, X) \geq |X|^2_\sim\,\kappa.$$

It is unclear whether (3.6.8) or (3.6.6) hold for general $X, Y \in H^{1,2}_\sharp(TM)$, since we do not know whether $\langle X, Y\rangle \in \mathscr{D}(\mathbf{\Delta})$ or $\langle X, Y\rangle \in \mathscr{D}(\mathbf{\Delta}^{2\kappa})$ in the respective situation. Still, (3.6.8) makes sense in the subsequent weak form. See also Remark 3.6.17 below.

Remark 3.6.14. In [Han20, Thm. 2.4], it is shown that on any smooth, connected Riemannian manifold M with boundary (with measure $\mathfrak{m} := \mathrm{e}^{-2w}\,\mathfrak{v}$, $w \in \mathrm{C}^2(M)$, so that $\mathfrak{s} = \mathrm{e}^{-2w}\,\mathscr{H}^{d-1}\big|_{\partial M}$), with $\mathbf{Ric}$ the Ricci measure in the sense of [Gig18, Thm. 3.6.7] — and hence of Theorem 3.6.9 — we have

$$\mathbf{Ric}(\nabla f, \nabla f) = \mathrm{Ric}(\nabla f, \nabla f)\,\mathfrak{m} + \mathbb{I}(\nabla f^{\parallel}, \nabla f^{\parallel})\,\mathfrak{s}$$

for every $f \in \mathrm{C}_\mathrm{c}^\infty(M)$ with $\mathrm{d}f(\mathsf{n}) = 0$ on ∂M. Therefore,

$$\mathbf{Ric}_\perp(\nabla f, \nabla f) = \mathbb{I}(\nabla f^{\parallel}, \nabla f^{\parallel})\,\mathfrak{s}. \qquad \blacksquare$$

Remark 3.6.15 (Convexity of RCD spaces)**.** On an $\mathrm{RCD}(K, \infty)$ space, $K \in \mathbf{R}$, [Gig18, Thm. 3.6.7] implies that every such space is *intrinsically convex* in the sense that

$$\mathbf{Ric}_\perp(X, X) \geq 0 \tag{3.6.9}$$

for every $X \in H^{1,2}_\sharp(TM)$. In other words, every such space is necessarily *convex* in the sense that (3.6.9) holds for, say, every $X \in \mathrm{Reg}(TM)$. (This notion of convexity is frequently used the smooth setting, see e.g. [Wan14, Def. 1.2.2].) Of course, by Remark 3.6.14 this is not true any more for general tamed spaces.

The former fact seems natural from various perspectives.

First, we know from [AGS14b, Thm. 6.18] that *geodesic convexity* of a subset Y of M is a *sufficient* condition for it to naturally become again $\mathrm{RCD}(K, \infty)$ as soon as $\mathfrak{m}[\partial Y] = 0$ and $\mathfrak{m}[Y] > 0$. Through (3.6.9) and [Han20] we have thus provided a nonsmooth analogue of the fact that every, say, compact, geodesically convex Riemannian manifold with boundary has nonnegative second fundamental form [Pet06, Lem. 61]. (The converse, of course, does not hold in general, e.g. for disks on the cylinder $\mathbf{S}^1 \times \mathbf{R}$ with diameter larger than π.)

Second, recent results [Stu20] and examples [Wan14] show that on nonconvex domains — even if the boundary has small concavity — show that in general, one cannot expect uniform lower Ricci bounds solely described by relative entropies. ■

Lemma 3.6.16. *For every $X, Y \in H^{1,2}_\sharp(TM)$ and every $f \in \mathrm{Test}(M)$,*

$$\begin{aligned}\int_M \widetilde{f}\,\mathrm{d}\mathbf{Ric}(X,Y) &= \int_M \big[\big\langle \mathrm{d}X^\flat, \mathrm{d}(f\,Y^\flat)\big\rangle + \delta X^\flat\,\delta(f\,Y^\flat) - \nabla X : \nabla(f\,Y)\big]\,\mathrm{d}\mathfrak{m}\\ &= \int_M \big[\mathrm{Hess}\,f(X,Y) + \mathrm{d}f(X\,\mathrm{div}\,Y + Y\,\mathrm{div}\,X)\big]\,\mathrm{d}\mathfrak{m}\\ &\qquad + \int_M f\,\big[\big\langle \mathrm{d}X^\flat, \mathrm{d}Y^\flat\big\rangle + \delta X^\flat\,\delta Y^\flat - \nabla X : \nabla Y\big]\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Proof. For a given $f \in \mathrm{Test}(M)$, all terms are continuous in X and Y w.r.t. convergence in $H^{1,2}_\sharp(TM)$. Hence, without restriction, we may and will assume in the sequel that $X, Y \in \mathrm{Reg}(TM)$.

Under these assumptions, some lengthy computations carried out in the proof of [Bra21, Lem. 8.14] show that

$$\begin{aligned}&\int_M \mathrm{d}X^\flat(\nabla f, Y)\,\mathrm{d}\mathfrak{m}\\ &\qquad = \int_M \big[-\langle X, Y\rangle\,\Delta f + \langle X, \nabla f\rangle\,\mathrm{div}\,Y - \big\langle X, [\nabla f, Y]\big\rangle\big]\,\mathrm{d}\mathfrak{m}.\end{aligned} \tag{3.6.10}$$

With this in hand, we initially prove the second equality. First, using Theorem 3.5.5, (3.6.10), the definition of the Lie bracket and Theorem 3.4.3,

$$\begin{aligned}&\int_M \big\langle \mathrm{d}X^\flat, \mathrm{d}(f\,Y^\flat)\big\rangle\,\mathrm{d}\mathfrak{m}\\ &\qquad = \int_M \big[\mathrm{d}X^\flat(\nabla f, Y) + f\,\big\langle \mathrm{d}X^\flat, \mathrm{d}Y^\flat\big\rangle\big]\,\mathrm{d}\mathfrak{m}\\ &\qquad = \int_M \big[-\langle X, Y\rangle\,\Delta f + \langle X, \nabla f\rangle\,\mathrm{div}\,Y - \big\langle X, [\nabla f, Y]\big\rangle\big]\,\mathrm{d}\mathfrak{m}\\ &\qquad\qquad + \int_M f\,\big\langle \mathrm{d}X^\flat, \mathrm{d}Y^\flat\big\rangle\,\mathrm{d}\mathfrak{m}\\ &\qquad = \int_M \big[-\langle X, Y\rangle\,\Delta f + \langle X, \nabla f\rangle\,\mathrm{div}\,Y - \nabla Y : (\nabla f \otimes X)\big]\,\mathrm{d}\mathfrak{m}\\ &\qquad\qquad + \int_M \big[\mathrm{Hess}\,f(X,Y) + f\,\big\langle \mathrm{d}X^\flat, \mathrm{d}Y^\flat\big\rangle\big]\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Second, by (3.5.3), Lemma 3.2.52 and Lemma 3.2.54,

$$\begin{aligned}\int_M \delta X^\flat\,\delta(f\,Y^\flat)\,\mathrm{d}\mathfrak{m} &= \int_M \mathrm{div}\,X\ \mathrm{div}(f\,Y)\,\mathrm{d}\mathfrak{m}\\ &= \int_M \big[\mathrm{div}\,X\,\langle \nabla f, Y\rangle + f\,\mathrm{div}\,X\ \mathrm{div}\,Y\big]\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Third, by Lemma 3.4.9,

$$-\int_M \nabla X : \nabla(f\,Y)\,\mathrm{d}\mathfrak{m} = -\int_M \big[\nabla X : (\nabla f \otimes Y) - f\,\nabla X : \nabla Y\big]\,\mathrm{d}\mathfrak{m}.$$

Adding up these three identities and employing that

$$\int_M \big[-\langle X, Y\rangle\,\Delta f - \nabla Y : (\nabla f \otimes X) - \nabla X : (\nabla f \otimes Y)\big]\,\mathrm{d}\mathfrak{m} = 0$$

thanks to (3.2.16), Lemma 3.4.13 and Proposition 3.4.11, the second claimed equality in the lemma is shown.

To prove the first identity, denote the r.h.s. of it by $\mathrm{A}(X,Y)$. By what we have proved above, it follows that $\mathrm{A}(X,Y) = \mathrm{A}(Y,X)$. Since $f\,X, f\,Y \in H^{1,2}_\sharp(TM)$ by Theorem 3.5.5, by Lemma 3.6.12 and (3.6.8) we get

$$\begin{aligned}
\int_M \widetilde{f}\,\mathrm{d}\mathbf{Ric}(X,Y) &= -\frac{1}{2}\int_M \big[\big\langle \nabla f, \nabla\langle X,Y\rangle\big\rangle + 2f\,\nabla X : \nabla Y\big]\,\mathrm{d}\mathfrak{m}\\
&\qquad + \frac{1}{2}\int_M \big[\big\langle \mathrm{d}(f\,X^\flat), \mathrm{d}Y^\flat\big\rangle + \delta(f\,X^\flat)\,\delta Y^\flat\big]\,\mathrm{d}\mathfrak{m}\\
&\qquad + \frac{1}{2}\int_M \big[\big\langle \mathrm{d}X^\flat, \mathrm{d}(f\,Y^\flat)\big\rangle + \delta X^\flat\,\delta(f\,Y^\flat)\big]\,\mathrm{d}\mathfrak{m}\\
&= -\frac{1}{2}\int_M \big[\nabla(f\,X) : \nabla Y + \nabla X : \nabla(f\,Y)\big]\,\mathrm{d}\mathfrak{m}\\
&\qquad + \frac{1}{2}\int_M \big[\big\langle \mathrm{d}(f\,X^\flat), \mathrm{d}Y^\flat\big\rangle + \delta(f\,X^\flat)\,\delta Y^\flat\big]\,\mathrm{d}\mathfrak{m}\\
&\qquad + \frac{1}{2}\int_M \big[\big\langle \mathrm{d}X^\flat, \mathrm{d}(f\,Y^\flat)\big\rangle + \delta X^\flat\,\delta(f\,Y^\flat)\big]\,\mathrm{d}\mathfrak{m}\\
&= \frac{1}{2}\big[\mathrm{A}(X,Y) + \mathrm{A}(Y,X)\big] = \mathrm{A}(X,Y).
\end{aligned}$$

In the second step, we used Proposition 3.4.11 and then Lemma 3.4.9. □

Remark 3.6.17 (Weitzenböck identity)**.** If $X \in \mathscr{D}(\vec{\Delta})^\sharp \cap \mathscr{D}(\square)$ in Lemma 3.6.16, from the latter we could deduce that

$$\int_M \widetilde{f}\,\mathrm{d}\mathbf{Ric}(X,Y) = \int_M f\,\big\langle Y, (\vec{\Delta}X^\flat)^\sharp + \square X\big\rangle\,\mathrm{d}\mathfrak{m}$$

for every $Y \in H^{1,2}_\sharp(TM)$, which is strongly reminiscent of the Weitzenböck formula [Pet06, Cor. 21]. In other words, $\mathbf{Ric}(X,Y) \ll \mathfrak{m}$ and

$$\mathrm{ric}(X,Y) = \big\langle Y, (\vec{\Delta}X^\flat)^\sharp + \square X\big\rangle \quad \mathfrak{m}\text{-a.e.}, \tag{3.6.11}$$

which, if $Y \in \mathscr{D}(\vec{\Delta})^\sharp \cap \mathscr{D}(\square)$ as well, especially implies the *pointwise* symmetry

$$\big\langle Y, (\vec{\Delta}X^\flat)^\sharp + \square X\big\rangle = \big\langle (\vec{\Delta}Y^\flat)^\sharp + \square Y, X\big\rangle \quad \mathfrak{m}\text{-a.e.}$$

The identity (3.6.11) plays a crucial role in deriving the Feynman–Kac formula for the semigroup on differential 1-forms on Riemannian manifolds, with or without boundary, as well as Bismut–Elworthy–Li formulas for $\mathrm{d}\mathsf{P}_t f$, $f \in L^2(M) \cap L^\infty(M)$ and $t > 0$ [Bis84a, EL94b, Hsu02a, Wan14]. Still, we do not know whether $\mathscr{D}(\vec{\Delta})^\sharp \cap \mathscr{D}(\square) \neq \{0\}$.

Let us remark that for $X \in H^{1,2}_\sharp(TM)$ to belong to both $\mathscr{D}(\vec{\Delta})^\sharp$ and $\mathscr{D}(\square)$, from (3.6.11) one would necessarily have $\mathbf{Ric}_\perp(X,\cdot) = 0$. On a compact Riemannian manifold M with boundary, this is underlined by comparison of the boundary conditions for $\vec{\Delta}$ and $\square$. Indeed, let $X \in \Gamma(TM)$. By (3.4.7), recall that $X \in \mathscr{D}(\square)$ means that

$$\begin{aligned} X^\perp &= 0,\\ (\nabla_{\mathsf{n}} X)^\parallel &= 0 \end{aligned}$$

on ∂M according to (3.2.1). On the other hand, $X^\flat \in \mathscr{D}(\vec{\Delta})$ entails absolute boundary conditions as in Remark 3.5.19 for $X^\flat$ which, by [Hsu02a, Lem. 4.1], are equivalent to

$$\begin{aligned} X^\perp &= 0,\\ (\nabla_{\mathsf{n}} X)^\parallel - \mathbb{I}(X^\parallel,\cdot) &= 0 \end{aligned}$$

at ∂M. Hence, we must have $\mathbb{I}(X^\parallel,\cdot) = 0$ on ∂M. ∎

Lemma 3.6.18. *For every $X, Y \in H^{1,2}_\sharp(TM)$ and every $f \in \mathrm{Test}(M)$,*

$$\mathbf{Ric}(f\,X, Y) = \widetilde{f}\,\mathbf{Ric}(X,Y).$$

Proof. By the $H^{1,2}_\sharp$-continuity of both sides in X and Y (recall Theorem 3.5.5 and Lemma 3.5.15), we may and will assume without restriction that $X, Y \in \mathrm{Reg}(TM)$. Owing to Lemma 3.2.73, it moreover suffices to prove that for every $g \in \mathrm{Test}(M)$,

$$\int_M \widetilde{g}\,\mathrm{d}\mathbf{Ric}(f\,X,Y) = \int_M \widetilde{g}\,\widetilde{f}\,\mathrm{d}\mathbf{Ric}(X,Y). \tag{3.6.12}$$

As in the proof of Lemma 3.6.16 above, by (3.6.8) and Lemma 3.6.1 we have

$$\begin{aligned} \int_M \widetilde{g}\,\mathrm{d}\mathbf{Ric}(f\,X,Y) &= \frac{1}{2}\int_M \big[\Delta g\, f\,\langle X,Y\rangle + g\,f\,\big\langle X, (\vec{\Delta}Y^\flat)^\sharp\big\rangle\big]\,\mathrm{d}\mathfrak{m}\\ &\quad + \frac{1}{2}\int_M \big[g\,\big\langle Y, (\vec{\Delta}(f\,Y^\flat))^\sharp\big\rangle - 2g\,\nabla(f\,X) : \nabla Y\big]\,\mathrm{d}\mathfrak{m},\\ \int_M \widetilde{g}\,\widetilde{f}\,\mathrm{d}\mathbf{Ric}(X,Y) &= \frac{1}{2}\int_M \big[\Delta(g\,f)\,\langle X,Y\rangle + g\,f\,\big\langle X, (\vec{\Delta}Y^\flat)^\sharp\big\rangle\big]\,\mathrm{d}\mathfrak{m}\\ &\quad + \frac{1}{2}\int_M \big[g\,f\,\big\langle Y, (\vec{\Delta}X^\flat)^\sharp\big\rangle - 2g\,f\,\nabla X : \nabla Y\big]\,\mathrm{d}\mathfrak{m}. \end{aligned}$$

Using Lemma 3.2.11 and Lemma 3.4.9 yields

$$\begin{aligned} \Delta(g\,f) &= g\,\Delta f + 2\,\langle\nabla g, \nabla f\rangle + f\,\Delta g &&\mathfrak{m}\text{-a.e.},\\ \nabla(f\,X) : \nabla Y &= f\,\nabla X : \nabla Y + (\nabla f \otimes X) : \nabla Y &&\mathfrak{m}\text{-a.e.}, \end{aligned}$$

while Lemma 3.6.1 ensures that

$$(\vec{\Delta}(f\,X))^\sharp = f\,(\vec{\Delta}X^\flat)^\sharp - \Delta f\,X - 2\,\nabla_{\nabla f} X.$$

Lastly, since $\langle X,Y\rangle \in \mathscr{F}_{\mathrm{b}}$ by Lemma 3.4.13, $\langle X,Y\rangle\,\nabla f \in \mathscr{D}_{\mathrm{TV}}(\mathbf{div}) \cap \mathscr{D}(\mathrm{div})$ with $\mathbf{n}(\langle X,Y\rangle\,\nabla f) = 0$ by Lemma 3.2.54. Together with Proposition 3.4.11, this yields

$$\int_M \langle\nabla g, \nabla f\rangle\,\langle X,Y\rangle\,\mathrm{d}\mathfrak{m} = -\int_M g\,\mathrm{div}\big(\langle X,Y\rangle\,\nabla f\big)\,\mathrm{d}\mathfrak{m}$$

$$= -\int_M \big[g\,\Delta f\,\langle X,Y\rangle + \nabla X : (\nabla f \otimes Y)\big]\,\mathrm{d}\mathfrak{m}$$
$$+ \int_M \nabla Y : (\nabla f \otimes X)\,\mathrm{d}\mathfrak{m}.$$

Using the last four identities, a term-by-term comparison in the above identities for both sides of (3.6.12) precisely yield (3.6.12). □

Dimension-dependent Ricci tensor Following the $\mathrm{RCD}^*(K,N)$-treatise [Han18a], $K \in \mathbf{R}$, and motivated by [BE85], we shortly outline the definition of an N-Ricci tensor on $\mathrm{BE}_2(\kappa,N)$ spaces for $N \in [1,\infty)$. The latter condition is assumed to hold for $(M,\mathscr{E},\mathfrak{m})$ throughout this subsection. Details are left to the reader.

Keeping in mind Proposition 3.3.14, let $(E_n)_{n\in\mathbf{N}\cup\{\infty\}}$ be the dimensional decomposition of $L^2(TM)$ and define the function $\dim_{\mathrm{loc}}\colon M \to \{1,\dots,\lfloor N\rfloor\}$ by

$$\dim_{\mathrm{loc}} := 1_{E_1} + 2\,1_{E_2} + \dots + \lfloor N\rfloor\,1_{\lfloor N\rfloor}.$$

Arguing as for [Han18a, Prop. 4.1], we see that for every $X \in H^{1,2}_\sharp(TM)$, if $N \in \mathbf{N}$ then

$$\operatorname{tr}\nabla X = \operatorname{div} X \quad \mathfrak{m}\text{-a.e.} \quad \text{on } E_N$$

according to the definition (3.2.12) of the trace of a generic $A \in L^2(T^{\otimes 2}M)$. The function $\mathrm{R}_N : H^{1,2}_\sharp(TM)^2 \to \mathbf{R}$ defined by

$$\mathrm{R}_N(X,Y) := \begin{cases} \dfrac{\big[\operatorname{tr}\nabla X - \operatorname{div} X\big]\,\big[\operatorname{tr}\nabla Y - \operatorname{div} Y\big]}{N - \dim_{\mathrm{loc}}} & \text{if } \dim_{\mathrm{loc}} < N,\\ 0 & \text{otherwise}\end{cases}$$

is thus well-defined. In fact, the assignment $(X,Y) \mapsto \mathrm{R}_N(X,Y)\,\mathfrak{m}$ is continuous as a map from $H^{1,2}_\sharp(TM)^2$ into $\mathfrak{M}_{\mathrm{f}}^{\pm}(M)_{\mathscr{E}}$ thanks to Lemma 3.6.8.

Definition 3.6.19. *The map* $\mathbf{Ric}_N : H^{1,2}_\sharp(TM)^2 \to \mathfrak{M}_{\mathrm{f}}^{\pm}(M)_{\mathscr{E}}$ *given by*

$$\mathbf{Ric}_N(X,Y) := \mathbf{Ric}(X,Y) - \mathrm{R}_N(X,Y)\,\mathfrak{m}$$

is henceforth called N-Ricci tensor *of* $(M,\mathscr{E},\mathfrak{m})$.

Theorem 3.6.20. *For every* $X \in H^{1,2}_\sharp(TM)$,

$$\mathbf{Ric}_N(X,X) \geq |X|^2_\sim\,\kappa,$$
$$\mathbf{\Delta}\frac{|X|^2}{2} + \big\langle X, (\vec{\Delta}X^\flat)^\sharp\big\rangle \geq \mathbf{Ric}_N(X,X) + \frac{1}{N}\,|\operatorname{div} X|^2\,\mathfrak{m}.$$

Proof. We only outline the main differences to the proof of [Han18a, Thm. 4.3]. Set

$$\mathbf{Ric}^\kappa_N(X,Y) := \mathbf{Ric}^\kappa(X,Y) - \mathrm{R}_N(X,Y)\,\mathfrak{m}$$

for $X,Y \in H^{1,2}_\sharp(TM)$. By Lemma 3.6.2 and absolute continuity of $\mathrm{R}_N(X,Y)$ w.r.t. $\mathfrak{m}$, we see that $\mathbf{Ric}^\kappa_N(X,X)_\perp \geq 0$, understood w.r.t. $\mathfrak{m}$, for every such X. The nonnegativity of $\mathbf{Ric}^\kappa_N(X,X)_\ll$ for $X := \nabla f$, $f \in \mathrm{Test}(M)$, is argued similarly to [Han18a, Thm. 3.3] up to replacing $\mathbf{\Gamma}_2(f)$ therein by $\mathbf{\Gamma}_2^{2\kappa}(f)$, compare with Lemma 3.3.9. By Lemma 3.6.18 and the definition of $\mathbf{Ric}$, it follows that $\mathbf{Ric}^\kappa$ is both $\mathbf{R}$- and Test-bilinear. The same is true for R_N by Lemma 3.4.9 and the definition (3.2.12) of the trace. Proceeding now as in the proof of [Han18a, Thm. 4.3] implies the nonnegativity of $\mathbf{Ric}^\kappa_N(X,X)$ for every $X \in \mathrm{Reg}(TM)$, and hence for every $X \in H^{1,2}_\sharp(TM)$ by continuity. We conclude the first inequality from the definition of $\mathbf{Ric}$ again. The argument for the second is the same as in [Han18a]. □

3.6.3 Vector Bochner inequality

The subsequent vector q-Bochner inequality is a direct consequence from Lemma 3.6.4, $q \in [1,2]$. For $\mathrm{RCD}(K,\infty)$ or $\mathrm{RCD}^*(K,N)$ spaces, $K \in \mathbf{R}$ and $N \in [1,\infty)$, it is due to [Bra20, Thm. 3.13] for $q = 1$, see the beginning of Chapter 4. Note that the assumption of Theorem 3.6.21 is satisfied if $X \in \mathrm{Test}(TM)$ by Lemma 3.6.1.

Let $\mathscr{D}(\mathbf{\Delta}^{q\kappa})$ be defined w.r.t. the closed form $\mathscr{E}^{q\kappa}$ as in Definition 3.2.74, $q \in [1,2]$.

Theorem 3.6.21. *Suppose that $X \in \mathrm{Reg}(TM)$ satisfies $\vec{\Delta} X^\flat \in L^1(T^*M)$. Then for every $q \in [1,2]$, we have $|X|^q \in \mathscr{D}(\mathbf{\Delta}^{q\kappa})$ and*

$$\mathbf{\Delta}^{q\kappa}\frac{|X|^q}{q} + |X|^{q-2}\left\langle X, (\vec{\Delta} X^\flat)^\sharp\right\rangle \mathfrak{m} \geq 0.$$

Proof. Letting $\varepsilon \to 0$ in Lemma 3.6.4 with Lebesgue's theorem yields

$$\int_M \frac{|X|^q}{q}\,\Delta^{q\kappa}\phi\,\mathrm{d}\mathfrak{m} \geq -\int_M \phi\,|X|^{q-2}\left\langle X, (\vec{\Delta} X^\flat)^\sharp\right\rangle \mathrm{d}\mathfrak{m}$$

for every $\phi \in \mathscr{D}(\Delta^{q\kappa}) \cap L^\infty(M)$ with $\Delta^{q\kappa}\phi \in L^\infty(M)$. Since the function on the r.h.s. which involves X belongs to $L^1(M) \cap L^2(M)$ — and here is where we use that $\vec{\Delta} X^\flat \in L^1(M)$ — a variant of [ER⁺20, Lem. 6.2] for $\mathscr{E}^{q\kappa}$ in place of $\mathscr{E}^{2\kappa}$ implies that $|X|^q \in \mathscr{D}(\mathbf{\Delta}^{q\kappa})$ with the desired inequality. □

Remark 3.6.22. If $\kappa \in \mathbf{K}_0(M)$, the quadratic form $\mathscr{E}^{q\kappa}$ is well-defined and closed even for $q \in [2,\infty)$, and Theorem 3.6.21 can be deduced along the same lines for this range of q even without the assumption that $\vec{\Delta} X^\flat \in L^1(T^*M)$. Compare e.g. with the functional treatise from Section 1.3. ■

3.6.4 Heat flow on 1-forms

A slightly more restrictive variant of Theorem 3.6.21 yields functional inequalities for the heat flow $(\mathsf{H}_t)_{t\geq 0}$ on 1-forms, see Theorem 3.6.33. The latter is shortly introduced before, along with its basic properties. A thorough study of $(\mathsf{H}_t)_{t\geq 0}$ on $\mathrm{RCD}(K,\infty)$ spaces, $K \in \mathbf{R}$, will be pursued in Chapter 4, following [Bra20], in the RCD framework for which Assumption 3.6.29 below clearly holds.

Heat flow and its elementary properties Analogously to the functional heat flow and Subsection 3.4.3, we define the *heat flow* on 1-forms as the semigroup $(\mathsf{H}_t)_{t\geq 0}$ of bounded, linear and self-adjoint operators on $L^2(T^*M)$ by

$$\mathsf{H}_t := \mathrm{e}^{-\vec{\Delta} t}. \tag{3.6.13}$$

It is associated [FOT11, Thm. 1.3.1] to the functional $\widetilde{\mathscr{E}}_{\mathrm{con}}\colon L^2(T^*M) \to [0,\infty]$ with

$$\widetilde{\mathscr{E}}_{\mathrm{con}}(\omega) := \begin{cases} \displaystyle\int_M \left[|\mathrm{d}\omega|^2 + |\delta\omega|^2\right] \mathrm{d}\mathfrak{m} & \text{if } \omega \in H^{1,2}(T^*M),\\ \infty & \text{otherwise.}\end{cases} \tag{3.6.14}$$

Theorem 3.6.23. *The subsequent properties of $(\mathsf{H}_t)_{t\geq 0}$ hold for every $\omega \in L^2(T^*M)$ and every $t > 0$.*

(i) *The curve* $t \mapsto \mathsf{H}_t\omega$ *belongs to* $\mathrm{C}^1((0,\infty); L^2(T^*M))$ *with*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}_t\omega = -\vec{\Delta}\mathsf{H}_t\omega.$$

(ii) *If* $\omega \in \mathscr{D}(\vec{\Delta})$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}_t\omega = -\mathsf{H}_t\vec{\Delta}\omega.$$

In particular, we have the identity

$$\vec{\Delta}\,\mathsf{H}_t = \mathsf{H}_t\,\vec{\Delta} \quad \text{on } \mathscr{D}(\vec{\Delta}).$$

(iii) *For every* $s \in [0,t]$,

$$\|\mathsf{H}_t\omega\|_{L^2(T^*M)} \leq \|\mathsf{H}_s\omega\|_{L^2(T^*M)}.$$

(iv) *The function* $t \mapsto \widetilde{\mathscr{E}}_{\mathrm{con}}(\mathsf{H}_t\omega)$ *belongs to* $\mathrm{C}^1((0,\infty))$, *is nonincreasing, and its derivative satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathscr{E}}_{\mathrm{con}}(\mathsf{H}_t\omega) = -2\int_M \big|\vec{\Delta}\mathsf{H}_t\omega\big|^2\,\mathrm{d}\mathfrak{m}.$$

(v) *If* $\omega \in H^{1,2}(T^*M)$, *the map* $t \mapsto \mathsf{H}_t\omega$ *is continuous on* $[0,\infty)$ *w.r.t. strong convergence in* $H^{1,2}(T^*M)$.

(vi) *We have*

$$\widetilde{\mathscr{E}}_{\mathrm{con}}(\mathsf{H}_t\omega) \leq \frac{1}{2t}\,\|\omega\|^2_{L^2(TM)},$$
$$\big\|\vec{\Delta}\mathsf{H}_t\omega\big\|^2_{L^2(T^*M)} \leq \frac{1}{2t^2}\,\|\omega\|^2_{L^2(T^*M)}.$$

Via the closedness of d from Lemma 3.2.39 together with Lemma 3.6.1, the following Lemma 3.6.24 is verified. The spectral bottom inequality from Corollary 3.6.28 follows from the second identity of (3.6.7), Lemma 3.4.13, (3.2.19) and Rayleigh's theorem.

Lemma 3.6.24. *For every* $f \in \mathscr{F}$ *and every* $t > 0$, $\mathrm{d}\mathsf{P}_t f \in \mathscr{D}(\vec{\Delta})$ *and*

$$\mathsf{H}_t\mathrm{d}f = \mathrm{d}\mathsf{P}_t f.$$

Remark 3.6.25. It is part of the statement of Lemma 3.6.24 that $\mathrm{d}\mathsf{P}_t f \in H^{1,2}(T^*M)$. Indeed, if $f \in \mathscr{F}_\mathrm{b}$, we even have $\mathrm{d}\mathsf{P}_t f \in \mathrm{Reg}(T^*M)$. Using that by Theorem 3.5.5 and Lemma 3.5.15, $\mathrm{d}(\mathrm{d}\mathsf{P}_t f) = 0$ and $\delta(\mathrm{d}\mathsf{P}_t f) = -\Delta\mathsf{P}_t f$ for such f, the claim for general elements of $\mathscr{F}$ easily follows by truncation and Lemma 3.2.12. ■

Remark 3.6.26. Analogously to (3.6.13), it is possible to define the heat flow $(\mathsf{H}_t^k)_{t\geq 0}$ in $L^2(\Lambda^k T^*M)$ with generator $-\vec{\Delta}_k$ for any $k \in \mathbf{N}$. However, it is not clear if the commutation relation from Lemma 3.6.24 holds between $(\mathsf{H}_t^k)_{t\geq 0}$ and $(\mathsf{H}_t^{k-1})_{t\geq 0}$ for $k \geq 2$. Compare with [Bra20, Rem. 3.4]. ■

Corollary 3.6.27. *If* $\omega \in \mathscr{D}(\delta)$ *and* $t > 0$, *then* $\mathsf{H}_t\omega \in \mathscr{D}(\delta)$ *with*

$$\delta\mathsf{H}_t\omega = \mathsf{P}_t\delta\omega.$$

Corollary 3.6.28. *We have*

$$\inf\sigma(-\Delta^\kappa) \leq \inf\sigma(\vec{\Delta}).$$

Functional inequalities and L^p-properties Unlike the results for $(\mathsf{H}_t)_{t\geq 0}$ from [Bra20] for $\mathrm{RCD}(K,\infty)$ spaces, $K \in \mathbf{R}$, see Chapter 4 below, a collateral effect of the singular potential κ is that we do not know how the domains $\mathscr{D}(\Delta)$ and $\mathscr{D}(\Delta^{2\kappa})$ are related. Compare with Remark 3.6.34 below. We thus restrict ourselves to the following assumption throughout this paragraph.

Assumption 3.6.29. *In the framework of Assumption 3.2.65, there exists $\mathscr{k} \in L^1_{\mathrm{loc}}(M)$ in the functional extended Kato class of M [ER⁺20, Def. 2.20] which is uniformly bounded from below by some $K \in \mathbf{R}$, such that*

$$\kappa = \mathscr{k}\,\mathfrak{m}.$$

We define the sequence $(\kappa_n)_{n\in\mathbf{N}}$ in $\mathbf{K}_{1-}(M)$ by

$$\kappa_n := \mathscr{k}_n\,\mathfrak{m}$$

with $\mathscr{k}_n := \min\{n, \mathscr{k}\} \in L^\infty(M)$, $n \in \mathbf{N}$. (Similar cutoff arguments have been used in Chapter 1 already.) Observe that $(M, \mathscr{E}, \mathfrak{m})$ obeys $\mathrm{BE}_2(\kappa_n, N)$ for every $n \in \mathbf{N}$. A priori, the Schrödinger operator $\Delta^{2\kappa_n}$ is the *form sum* [Far75, p. 19] of Δ and $-2\mathscr{k}_n$, the latter being viewed as self-adjoint [Far75, Thm. 1.7] multiplication operator on $L^2(M)$ with domain $\mathscr{D}(-2\kappa_n) := \{f \in L^2(M) : \mathscr{k}_n f \in L^2(M)\} = L^2(M)$, $n \in \mathbf{N}$. In fact [Far75, Prop. 3.1], $\Delta^{2\kappa_n}$ is an *operator sum*, i.e. $f \in \mathscr{D}(\Delta)$ if and only if $f \in \mathscr{D}(\Delta^{2\kappa_n})$ for every $n \in \mathbf{N}$, and for such f,

$$\Delta^{2\kappa_n} f = \Delta f - 2\mathscr{k}_n f \quad \mathfrak{m}\text{-a.e.} \tag{3.6.15}$$

Proposition 3.6.30. *For every $\omega \in L^2(T^*M)$ and every $t \geq 0$,*

$$|\mathsf{H}_t\omega|^2 \leq \mathsf{P}_t^{2\kappa}\big(|\omega|^2\big) \quad \mathfrak{m}\text{-}a.e.$$

Proof. We only concentrate on the nontrivial part $t > 0$. Let $\phi \in \mathrm{Test}_{L^\infty}(M)$ be nonnegative, and define $F\colon [0,t] \to \mathbf{R}$ by

$$F(s) := \int_M \phi\,\mathsf{P}_{t-s}^{2\kappa_n}\big(|\mathsf{H}_s\omega|^2\big)\,\mathrm{d}\mathfrak{m} = \int_M \mathsf{P}_{t-s}^{2\kappa_n}\phi\,|\mathsf{H}_s\omega|^2\,\mathrm{d}\mathfrak{m},$$

where $n \in \mathbf{N}$. As in the proof of Proposition 3.4.24, we argue that F is locally absolutely continuous on $(0,t)$, and that for $\mathscr{L}^1$-a.e. $s \in (0,t)$,

$$F'(s) = -\int_M \Delta^{2\kappa_n}\mathsf{P}_{t-s}^{2\kappa_n}\phi\,|\mathsf{H}_s\omega|^2\,\mathrm{d}\mathfrak{m} - 2\int_M \mathsf{P}_{t-s}^{2\kappa_n}\phi\,\big\langle \mathsf{H}_s\omega, \vec{\Delta}\mathsf{H}_s\omega\big\rangle\,\mathrm{d}\mathfrak{m}.$$

Given such an $s \in (0,t)$, using a mollified version of $(\mathsf{P}_t)_{t\geq 0}$ [Sav14, p. 1648] we construct a sequence $(f_i)_{i\in\mathbf{N}}$ of nonnegative functions in $\mathrm{Test}_{L^\infty}(M)$ such that $(f_i)_{i\in\mathbf{N}}$ and $(\Delta f_i)_{i\in\mathbf{N}}$ are bounded in $L^\infty(M)$, and $f_i \to \mathsf{P}_{t-s}^{2\kappa_n}\phi$ as well as $\Delta f_i \to \Delta\mathsf{P}_{t-s}^{2\kappa_n}\phi$ pointwise $\mathfrak{m}$-a.e. as $i \to \infty$. By Lebesgue's theorem, (3.6.15), Remark 3.6.5 and (3.5.3), we obtain that

$$\begin{aligned} F'(s) = -\lim_{i\to\infty}\int_M \Delta^{2\kappa_n} f_i\,|\mathsf{H}_s\omega|^2\,\mathrm{d}\mathfrak{m}& \\ -\lim_{i\to\infty} 2\int_M f_i\,\big\langle \mathsf{H}_s\omega, \vec{\Delta}\mathsf{H}_s\omega\big\rangle\,\mathrm{d}\mathfrak{m} &\leq 0. \end{aligned}$$

Integrating this inequality from 0 to t, employing the arbitrariness of ϕ and letting $n \to \infty$ via Levi's theorem readily provides the claimed inequality. □

As for Corollary 3.4.25, we have the following consequence of Proposition 3.6.30. A similar argument as for Lemma 3.6.32, providing an extension of Lemma 3.6.4 beyond regular vector fields which is needed for the proof of Theorem 3.6.33, is due to [Bra20].

Corollary 3.6.31. *For every $\omega \in \mathscr{D}(\vec{\Delta})$, there exists a sequence $(\omega_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\vec{\Delta}) \cap L^\infty(T^*M)$ which converges to ω in $H^{1,2}(T^*M)$ such that in addition, $\vec{\Delta}\omega_n \to \vec{\Delta}\omega$ in $L^2(T^*M)$ as $n \to \infty$. If $\omega \in L^\infty(T^*M)$, this sequence can be constructed to be uniformly bounded in $L^\infty(T^*M)$.*

Lemma 3.6.32. *For every $n \in \mathbf{N}$, the conclusion from Lemma 3.6.4 holds for κ replaced by κ_n as well as $q = 1$, $\varepsilon > 0$, $X \in \mathscr{D}(\vec{\Delta})^\sharp \cap L^\infty(TM)$ and $\phi \in \mathscr{D}(\Delta) \cap L^\infty(M)$.*

Proof. We shortly outline the argument. Let $\psi \in \mathscr{F}_{\mathrm{b}}$ be nonnegative, and let $(X_i)_{i\in\mathbf{N}}$ and $(\psi_j)_{j\in\mathbf{N}}$ be sequences in $\mathrm{Reg}(TM)$ and $\mathrm{Test}(M)$ converging to X and ψ in $H^{1,2}_\sharp(TM)$ and $\mathscr{F}$, respectively. By Lemma 3.2.73, we may and will assume that ψ_j is nonnegative for every $j \in \mathbf{N}$. Integrating Lemma 3.6.2, for κ replaced by κ_n, $n \in \mathbf{N}$, against ψ_j and using that $|X_i|^2 \in \mathscr{F}_{\mathrm{b}}$ by Lemma 3.4.13, for every $i, j \in \mathbf{N}$,

$$\begin{aligned} &-\frac{1}{2}\int_M \big\langle \nabla\psi_j, \nabla|X_i|^2\big\rangle \,\mathrm{d}\mathfrak{m} - \int_M \mathscr{k}_n\,\psi_j\,\frac{|X_i|^2}{2}\,\mathrm{d}\mathfrak{m} \\ &\qquad \geq \int_M \psi_j\,\big|\nabla X_i\big|_{\mathrm{HS}}^2\,\mathrm{d}\mathfrak{m} - \int_M \psi_j\,\big\langle X_i, (\vec{\Delta}X_i^\flat)^\sharp\big\rangle\,\mathrm{d}\mathfrak{m}. \end{aligned}$$

Integrating by parts the last term, using Proposition 3.4.11 for the first, Lemma 3.6.8 for the third and Theorem 3.5.5 and Lemma 3.5.15 for the last term, and finally integrating by parts back the last term we send $i \to \infty$. This yields the previous inequality for X_i replaced by X. Employing Lemma 3.6.8 and Proposition 3.4.11 again for the first term together with $X \in L^\infty(TM)$ and $\nabla\psi_j \to \nabla\psi$ in $L^2(TM)$ as $j \to \infty$, the above estimate still holds for ψ_j replaced by ψ. Lastly, we insert $\psi := \phi\,[\varphi_\varepsilon' \circ |X|^2]$, $\varepsilon > 0$, where φ_ε is defined as in Lemma 3.6.4 for $q = 1$. The term containing $\varphi_\varepsilon'' \circ |X|^2$ coming from the Leibniz rule in the first integral cancels out with the third integral thanks to Lemma 3.4.13, and elementary further computations entail the claim. □

Theorem 3.6.33 is known as *Hess–Schrader–Uhlenbrock inequality* [HSU77, HSU80, Sim77] in the case when M is a Riemannian manifold. A similar, analytic access to the latter on compact M (with possibly *convex* boundary) is due to [Ouh99, Shi97, Shi00]. On general compact M, it has been derived in [Hsu02a] using probabilistic methods. In the noncompact case without boundary, one can appeal to both analytic [Gün17a] or stochastic [DT01, Li92] methods. Moreover, recently, L^p-properties of $(\mathsf{H}_t)_{t\geq 0}$ and related heat kernel estimates on Riemannian manifolds have been studied in [MO20] under Kato curvature conditions.

Theorem 3.6.33 (Hess–Schrader–Uhlenbrock inequality)**.** *For every $\omega \in L^2(T^*M)$ and every $t \geq 0$,*

$$|\mathsf{H}_t\omega| \leq \mathsf{P}_t^\kappa|\omega| \quad \mathfrak{m}\text{-}a.e.$$

Proof. Let $(\omega_l)_{l\in\mathbf{N}}$ be a sequence in $L^1(T^*M) \cap L^\infty(T^*M)$ which is obtained by appropriately cutting off and truncating the given ω. (In this case, truncation means multiplication with an indicator function of $\{|\omega| \leq R\}$, $R > 0$.) Moreover, given any $\varepsilon > 0$, define $\varphi_\varepsilon \in \mathrm{C}^\infty([0,\infty))$ by $\varphi_\varepsilon(r) := (r+\varepsilon)^{1/2} - \varepsilon^{1/2}$. For a nonnegative $\phi \in \mathrm{Test}_{L^\infty}(M)$, given any $l \in \mathbf{N}$, consider the function $F_\varepsilon\colon [0,t] \to \mathbf{R}$ with

$$F_\varepsilon(s) := \int_M \phi\, \mathsf{P}^{\kappa_n}_{t-s}\big(\varphi_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big)\,\mathrm{d}\mathfrak{m} = \int_M \mathsf{P}^{\kappa_n}_{t-s}\phi\,\big[\varphi_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big]\,\mathrm{d}\mathfrak{m}.$$

As for Proposition 3.6.30, the function F_ε is readily verified to be continuous on $[0,t]$, locally absolutely continuous on $(0,t)$, and integration and differentiation can be swapped in computing its derivative $F'_\varepsilon(s)$ at $\mathscr{L}^1$-a.e. $s \in (0,t)$.

Given such an $s \in (0,t)$, consider a sequence $(f_i)_{i\in\mathbf{N}}$ of nonnegative functions in $\mathrm{Test}_{L^\infty}(M)$ associated to $\mathsf{P}^{\kappa_n}_{t-s}\phi$ as in the proof of Proposition 3.6.30. Then, according to Lemma 3.6.32 — since $\mathsf{H}_s\omega_l \in L^\infty(T^*M)$ thanks to Proposition 3.6.30 — (3.2.16) and Lebesgue's theorem,

$$\begin{aligned}
F'_\varepsilon(s) &= -\int_M \Delta^{\kappa_n}\mathsf{P}^{\kappa_n}_{t-s}\phi\,\big[\varphi_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big]\,\mathrm{d}\mathfrak{m}\\
&\qquad - 2\int_M \mathsf{P}^{\kappa_n}_{t-s}\phi\,\big[\varphi'_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big]\,\big\langle \mathsf{H}_s\omega_l, \vec{\Delta}\mathsf{H}_s\omega_l\big\rangle\,\mathrm{d}\mathfrak{m}\\
&= -\lim_{i\to\infty}\int_M \Delta^{\kappa_n} f_i\,\big[\varphi_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big]\,\mathrm{d}\mathfrak{m}\\
&\qquad - \lim_{i\to\infty} 2\int_M f_i\,\big[\varphi'_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big]\,\big\langle \mathsf{H}_s\omega_l, \vec{\Delta}\mathsf{H}_s\omega_l\big\rangle\,\mathrm{d}\mathfrak{m}\\
&\leq -\lim_{i\to\infty} 2\,\big\langle \kappa_n \,\big|\, f_i\,|\mathsf{H}_s\omega_l|^2\,\varphi_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big\rangle + \lim_{i\to\infty}\big\langle \kappa_n \,\big|\, f_i\,\varphi_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big\rangle\\
&= -2\,\big\langle \kappa_n \,\big|\, \mathsf{P}^{\kappa_n}_{t-s}\phi\,|\mathsf{H}_s\omega_l|^2\,\varphi'_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big\rangle + \big\langle \kappa_n \,\big|\, \mathsf{P}^{\kappa_n}_{t-s}\phi\,\varphi_\varepsilon \circ |\mathsf{H}_s\omega_l|^2\big\rangle.
\end{aligned}$$

Integrating this inequality from 0 to t, sending $\varepsilon \to 0$ with the aid of Lebesgue's theorem and employing the arbitrariness of ϕ imply that, for every $l, n \in \mathbf{N}$,

$$|\mathsf{H}_t\omega_l| \leq \mathsf{P}^{\kappa_n}_t|\omega_l| \quad \mathfrak{m}\text{-a.e.}$$

Sending $l \to \infty$ and $n \to \infty$ using Levi's theorem terminates the proof. □

Remark 3.6.34. Technical issues prevent us from proving Proposition 3.6.30 or Theorem 3.6.33 beyond Assumption 3.6.29. The key reason is that integrated versions or inequalities derived from (3.6.6) and (3.6.7) are hard to obtain beyond $X \in \mathrm{Reg}(TM)$ or integrands both belonging to $\mathrm{Test}(M)$ and $\mathscr{D}(\Delta^{2\kappa})$.

It is outlined in Remark 3.6.5 how to obtain more general versions under Assumption 3.6.29. The key obstacle, however, lies in dealing with the behavior of the term in (3.6.6) containing the Hodge Laplacian or, in other words, to obtain an analogue to Lemma 3.6.32. In general, L^∞-bounds for derivatives of $\mathsf{P}^{2\kappa}_{t-s}\phi$ or $\mathsf{P}^{\kappa}_{t-s}\phi$, $s \in (0,t)$, lack for sufficiently many nonnegative $\phi \in L^2(M)$, but being able to integrate by parts this term essentially requires e.g. $\mathsf{P}^{2\kappa}_{t-s}\phi \in L^\infty(M)$ and $\mathrm{d}\mathsf{P}^{2\kappa}_{t-s}\phi \in L^\infty(T^*M)$. (A related question is whether and when not only $\mathsf{P}^{2\kappa}_{t-s}\phi, \Delta^{2\kappa}\mathsf{P}^{2\kappa}_{t-s}\phi \in L^\infty(M)$ — which can always be achieved by [ER+20, Sec. 6.1] — but also $\mathrm{d}\mathsf{P}^{2\kappa}_{t-s}\phi \in L^\infty(T^*M)$ holds.) Compare with Proposition 3.5.8, Remark 3.5.9, (3.5.3) and Lemma 3.2.52. We also cannot leave the Hodge Laplacian term as it is because we do not know if $\mathrm{Reg}(T^*M)$ is dense in $\mathscr{D}(\vec{\Delta})$ w.r.t. the induced graph norm. Under Assumption 3.6.29, these deductions could still be done thanks to the explicit relation (3.6.15) between $\mathscr{D}(\Delta^{2\kappa})$ and $\mathscr{D}(\Delta)$. ■

Remark 3.6.35. If we know that, given $\omega \in L^2(T^*M)$, there exists a sequence $(\omega_n)_{n\in\mathbf{N}}$ in $L^2(T^*M)$ that L^2-converges to ω such that $\mathsf{H}_t\omega_n \in L^\infty(T^*M)$ for every $t > 0$ and every $n \in \mathbf{N}$, then Theorem 3.6.33 then Theorem 3.6.33 can be deduced by the same arguments as above for more general $\kappa \in \mathbf{K}_{1-}(M)$. In particular, since $\mathsf{H}_t\,\mathrm{d}f_n \in L^\infty(T^*M)$ for every $t > 0$ and every $n \in \mathbf{N}$, $f_n := \max\{\min\{f,n\},-n\}$, by Lemma 3.6.24, Theorem 3.6.33 recovers the gradient estimate [ER+20, Thm. 6.9] for any $f \in \mathscr{F}$ by (3.5.3).

On Riemannian manifolds with not necessarily convex boundary, such a sequence can be constructed under further geometric assumptions [AL17, Thm. 5.2]. (The latter result, in fact, implies Theorem 3.6.33 by Gronwall's inequality.) ■

3.7 Extrinsic approaches

Lastly, we compare some recent *extrinsic* approaches [BPS19, BCM19, Stu20] to boundary objects on RCD spaces with our intrinsic notions. More precisely, we outline some links to our notions of divergences and normal components.

3.7.1 Sets of finite perimeter

Let $(M,\mathsf{d},\mathfrak{m})$ be a locally compact $\mathrm{RCD}(K,\infty)$ space, $K \in \mathbf{R}$, with induced Dirichlet space $(M,\mathscr{E},\mathfrak{m})$, cf. Example 3.2.14.

Identification of the measure-valued divergence Following [BCM19, Def. 4.1], let $\mathscr{DM}^p(M)$, $p \in [1,\infty]$, be the space of all $X \in L^p(TM)$ such that there exists $\mathit{div}\,X \in \mathfrak{M}_{\mathrm{f}}^{\pm}(M)_{\mathscr{E}}$ such that for every $h \in \mathrm{Lip}_{\mathrm{bs}}(M)$,

$$-\int_M h\,\mathrm{d}\mathit{div}\,X = \int_M \mathrm{d}h(X)\,\mathrm{d}\mathfrak{m}.$$

The density of $\mathrm{Lip}_{\mathrm{bs}}(M)$ in $W^{1,2}(M)$ [AGS14a] and [BCM19, Prop. 4.6] yield the following.

Lemma 3.7.1. *Every $X \in \mathscr{DM}^2(M)$ belongs to $\mathscr{D}(\mathbf{div})$, and*

$$\mathit{div}\,X = \mathbf{div}\,X.$$

Gauß–Green formula For boundary objects to really appear, we use the Gauß–Green formulas for appropriate subsets $E \subset M$ obtained in [BCM19].

We say that a Borel set $E \subset M$ has *finite perimeter* [BCM19, Def. 3.3] if $1_E \in \mathrm{BV}(M)$. It is associated with a Radon measure $|\mathrm{D}1_E| \in \mathfrak{M}_{\mathrm{fR}}^{+}(M)$ [BCM19, Thm. 3.4] which is supported on ∂E and, if $\mathfrak{m}[\partial E] = 0$, in particular singular to $\mathfrak{m}$ [BCM19, Rem. 3.5]. Here, the class of functions of *bounded variation* $\mathrm{BV}(M) \subset L^1(M)$ can be defined in various ways [ADM14, BCM19, Mir03] which all lead to the same spaces and objects in a large generality [ADM14, Thm. 1.1]. (In particular, we require sets of finite perimeter to have finite $\mathfrak{m}$-measure, although this is not strictly needed [BPS19, Def. 1.1, Def. 1.2].)

We now make the following assumptions on E.

a. E satisfies the obstructions from Remark 3.2.15.
b. The inclusion $W^{1,2}(M)\big|_E \subset W^{1,2}(E)$ from (3.2.3) is dense.
c. E or E^{c} is a set of finite perimeter.

Item a. guarantees that $(E, \mathsf{d}_E, \mathfrak{m}_E)$ induces a quasi-regular, strongly local Dirichlet space $(E, \mathscr{E}_E, \mathfrak{m}_E)$, hence a tangent module $L^2(TE)$ w.r.t. $\mathfrak{m}_E$. By (3.2.3), we can identify $L^2(TM)\big|_E$ with $L^2(TE)$. Given any $X \in L^2(TM)$, denote by $X_E \in L^2(TE)$ the image of $1_E\, X$ under this identification. Of course, b. is satisfied if E has the extension property $W^{1,2}(M)\big|_E = W^{1,2}(E)$, and if $\mathsf{d}_E \leq C\,\mathsf{d}$ on E^2 for some finite $C > 1$. For a different variant of this condition b., see Subsection 3.7.2 below.

Proposition 3.7.2. *For every $X \in \mathrm{Test}(TM)$, there exists a unique $\langle X, \upsilon_E\rangle_{\partial E} \in L^\infty(\partial E, |\mathrm{D}1_E|)$ such that for every $h \in W^{1,2}(E)$,*

$$-\int_E \mathrm{d}h(X)\,\mathrm{d}\mathfrak{m} = \int_E h\,\frac{\mathrm{d}\mathit{div}\,X}{\mathrm{d}\mathfrak{m}}\,\mathrm{d}\mathfrak{m} + \int_{\partial E} \widetilde{h}\,\big\langle X, \upsilon_E\big\rangle_{\partial E}\,\mathrm{d}|\mathrm{D}1_E|. \tag{3.7.1}$$

In particular, we have $X_E \in \mathscr{D}_{\mathrm{TV}}(\mathbf{div}_E)$ with

$$\begin{aligned} \mathrm{div}_E\, X_E &= \frac{\mathrm{d}\mathit{div}\,X}{\mathrm{d}\mathfrak{m}} \quad \mathfrak{m}_E\text{-a.e.},\\ \mathbf{n}_E\, X_E &= -\big\langle X, \upsilon_E\big\rangle_{\partial E}\,|\mathrm{D}1_E|. \end{aligned}$$

Proof. The last statement follows from (3.7.1), whence we concentrate on (3.7.1). By Lemma 3.2.54 and Lemma 3.7.1, we have $|\mathit{div}\,X| \ll \mathfrak{m}$. (And furthermore, $\mathit{div}\,X$ has finite total variation.) Hence, under c., [BCM19, Prop. 6.11, Thm. 6.13] implies that there exists a unique function $\langle X, \upsilon_E\rangle_{\partial E} \in L^\infty(\partial E, |\mathrm{D}1_E|)$ such that (3.7.1) holds for every $h \in \mathrm{Lip}_{\mathrm{bs}}(M)$. By b., $\mathscr{E}$-quasi-uniform approximation [CF12, Thm. 1.3.3] and [BCM19, Prop. 4.6], the latter extends to arbitrary $h \in W^{1,2}(E)$. □

Remark 3.7.3. The notation $\langle X, \upsilon_E\rangle_{\partial E}$, $X \in \mathrm{Test}(TM)$, in Proposition 3.7.2 is purely formal, in the sense that the authors of [BCM19] neither consider any "tangent module" with scalar product $\langle\cdot,\cdot\rangle_{\partial E}$ over ∂E, nor define a unit normal vector field υ_E. ■

Example 3.7.4. Another version of the Gauß–Green formula on $\mathrm{RCD}(K, N)$ spaces $(M, \mathsf{d}, \mathfrak{m})$, $K \in \mathbf{R}$ and $N \in [1, \infty)$, has been obtained in [BPS19, Thm. 2.2]. Retain the assumptions a., b. and c. on $E \subset M$. Then there exists a unique $\upsilon_E \in L^2_E(TM)$, the tangent module over ∂E [BPS19, Thm. 2.1], with $|\upsilon_E| = 1$ $|\mathrm{D}1_E|$-a.e. on ∂E such that for every $X \in H^{1,2}(TM) \cap \mathscr{D}(\mathrm{div}) \cap L^\infty(TM)$,

$$\int_E \mathrm{div}\,X\,\mathrm{d}\mathfrak{m} = -\int_{\partial E} \big\langle \mathrm{tr}_E(X), \upsilon_E\big\rangle\,\mathrm{d}|\mathrm{D}1_E|.$$

Here $\mathrm{tr}_E\colon H^{1,2}(TM) \cap L^\infty(TM) \to L^2_E(TM)$ is the *trace operator* over ∂E.

Replacing X by $h\,X$, where $h \in W^{1,2}(M) \cap L^\infty(M)$ has bounded support — recall Lemma 3.2.52 and Remark 3.4.10 — and using (3.2.17) as well as the arbitrariness of h we obtain that $X_E \in \mathscr{D}(\mathbf{div}_E)$ with

$$\begin{aligned} \mathrm{div}_E\, X_E &= \mathrm{div}\,X \quad \mathfrak{m}_E\text{-a.e.},\\ \mathbf{n}_E\, X_E &= -\big\langle X, \upsilon_E\big\rangle\,|\mathrm{d}1_E|. \end{aligned}$$

■

3.7.2 Regular semiconvex subsets

Consider the canonical Dirichlet space induced by an $\mathrm{RCD}(k, N)$ metric measure space $(M, \mathsf{d}, \mathfrak{m})$, see Example 3.2.14, where $k\colon M \to \mathbf{R}$ is continuous and lower bounded as well as $N \in [2, \infty)$ [Stu20, Def. 3.1, Def. 3.3, Thm. 3.4]. Let $E \subset M$ be as in Remark 3.2.15 with $\mathfrak{m}[E] < \infty$.

For a function f on M or E, denote by f_n its truncation $\max\{\min\{f, n\}, -n\}$ at the levels n and $-n$, $n \in \mathbf{N}$. Following [Stu20, Def. 2.1] we set

$$\begin{aligned} W^{1,1+}(M) := \Big\{ f \in L^1(M) : f_n \in \mathscr{F} \text{ for every } n \in \mathbf{N}, \\ \sup_{n\in\mathbf{N}} \big\| |f_n| + |\mathrm{d} f_n| \big\|_{L^1(M)} < \infty \Big\}. \end{aligned}$$

Let $W^{1,1+}(E)$ be defined analogously w.r.t. $W^{1,2}(E)$ and $|\mathrm{d}\cdot|_E$. We assume that E has *regular boundary* [Stu20, p. 1702], i.e. $v \in \mathscr{D}(\Delta)$ with $v, \Delta v \in \mathrm{C}(M) \cap L^\infty(M)$, where $v := \mathsf{d}(\cdot, E) - \mathsf{d}(\cdot, E^c)$ is the signed distance function from ∂E, and

$$W^{1,1+}(M)\big|_E = W^{1,1+}(E).$$

Then thanks to [Stu20, Lem. 6.10], there exists a nonnegative $\sigma \in \mathfrak{M}_\mathrm{f}^+(M)_\mathscr{E}$ supported on ∂E such that for every $h \in W^{1,2}_\mathrm{b}(M)$,

$$\int_{\partial E} \widetilde{h}\,\mathrm{d}\sigma = \int_E \mathrm{d}v(\nabla h)\,\mathrm{d}\mathfrak{m} + \int_M \Delta v\, h\,\mathrm{d}\mathfrak{m}. \tag{3.7.2}$$

Lemma 3.7.5. *In the notation of the Gauß–Green part in Subsection 3.2.5, the vector field $(\nabla v)_E \in L^2(TE)$ belongs to $\mathscr{D}_{L^2}(\mathbf{div}_E)$ with*

$$\begin{aligned} \mathrm{div}_E(\nabla v)_E &= \Delta v \quad \mathfrak{m}_E\text{-a.e.},\\ \mathbf{n}_E(\nabla v)_E &= \sigma. \end{aligned}$$

Proof. Since $\mathfrak{m}[E] < \infty$, any given $h \in W^{1,2}_\mathrm{bc}(E)$ belongs to $W^{1,1+}(E)$, and hence to $W^{1,1+}(M)\big|_E$ by regularity of ∂E. Thus, there exists $\overline{h} \in W^{1,1+}(M)$ such that $\overline{h} = h$ $\mathfrak{m}$-a.e. on E. In particular, $\overline{h}_n \in W^{1,2}_\mathrm{b}(M)$ for every $n \in \mathbf{N}$. Since $\overline{h}_n = h$ $\mathfrak{m}$-a.e. on E for large enough $n \in \mathbf{N}$, the claim follows from (3.7.2). □

Remark 3.7.6. If the integration by parts formula as in [Stu20, Lem. 6.11] holds — which, in fact, uniquely characterizes σ — a similar argument as for Lemma 3.7.5 yields that for every $f \in \mathscr{D}(\Delta)$ with $\nabla f \in L^\infty(TM)$, we have $(\nabla f)_E \in \mathscr{D}(\mathbf{div}_E)$ with

$$\begin{aligned} \mathrm{div}_E(\nabla f)_E &= \Delta f \quad \mathfrak{m}_E\text{-a.e.},\\ \mathbf{n}_E(\nabla f)_E &= \langle \nabla f, \nabla v\rangle_\sim\, \sigma. \end{aligned}$$

The latter is well-defined thanks to Lemma 3.4.13 and Corollary 3.3.12. ■

Proposition 3.7.7. *Given any $X \in \mathrm{Reg}(M)$, define $X^\perp \in L^2(TM)$ by*

$$X^\perp := \langle X, \nabla v\rangle\, \nabla v.$$

Then $X^\perp_E \in \mathscr{D}(\mathbf{div}_E)$ with

$$\begin{aligned} \mathrm{div}_E\, X^\perp_E &= \langle X, \nabla v\rangle\, \Delta v + \nabla X : (\nabla v \otimes \nabla v) \quad \mathfrak{m}_E\text{-a.e.},\\ \mathbf{n}_E\, X^\perp_E &= \langle X, \nabla v\rangle_\sim\, \sigma. \end{aligned}$$

Proof. Observe that $\langle X, \nabla v\rangle \in W^{1,2}_\mathrm{b}(M)$, so that both the statements make sense. The claimed formulas follow from Lemma 3.7.5 as well as Lemma 3.2.52 while noting that by Proposition 3.4.11 and since $|\nabla v| = 1$ $\mathfrak{m}$-a.e.,

$$\begin{aligned} \big\langle \nabla\langle X, \nabla v\rangle, \nabla v\big\rangle &= \nabla X : (\nabla v \otimes \nabla v) + \mathrm{Hess}\, v(X, \nabla v)\\ &= \nabla X : (\nabla v \otimes \nabla v) + \mathrm{d}|\nabla v|^2(X)/2\\ &= \nabla X : (\nabla v \otimes \nabla v) \quad \mathfrak{m}\text{-a.e.} \end{aligned}$$ □

Remark 3.7.8. We do not know if the pointwise defined second fundamental form from [Stu20, Rem. 5.13] is related to $\mathbf{Ric}_E$. By Proposition 3.7.7 these notions coincide if $X \in \mathrm{Reg}(TM)$ obeys $|X|^2 \in \mathscr{D}(\mathrm{Hess})$, but we do not know if many of such X can be found in general. At least, the taming condition for $(E, \mathscr{E}_E, \mathfrak{m}_E)$ is already provided once one can verify that the taming distribution

$$\kappa := -k^- \, \mathfrak{m}_E - \ell^- \, \sigma,$$

for some appropriate $\ell \in \mathrm{C}(M)$, according to [Stu20, Thm. 6.14] and [ER⁺20, Prop. 2.16] belongs to $\mathbf{K}_{1-}(E)$. See [BR21, ER⁺20] for examples in this direction. ■

Chapter Four

Heat flow on 1-forms under lower Ricci bounds. Functional inequalities, spectral theory, and heat kernel

This chapter is based on the author's work [Bra20], from which large parts are taken over *verbatim*.

In this final chapter, we fix again an $\mathrm{RCD}(K,\infty)$ space $(M,\mathsf{d},\mathfrak{m})$ according to Section 0.1, $K \in \mathbf{R}$. Occasionally, we will assume the more restrictive $\mathrm{RCD}^*(K,N)$ or $\mathrm{RCD}(K,N)$ condition, $N \in [1,\infty)$. Required details on these notions which are not yet contained in Section 0.1 or Section 1.2 are summarized in Section 4.2 below.

We retain the notations from Chapter 3 except the more standard ones for the (extended) Cheeger energy domains $W^{1,2}(M) := \mathscr{F}$ and $\mathrm{S}^2(M) := \mathscr{F}_{\mathrm{e}}$, recall Remark 3.2.4. Readers who want to read this chapter as independently of Chapter 3 as possible are recommended to familiarize themselves with Subsection 3.2.4, Subsection 3.2.5 and Subsection 3.2.8 for notational purposes, with the calculus rules from Theorem 3.3.3, Theorem 3.4.3 and Theorem 3.5.5, and with the heat flow on 1-forms $(\mathsf{H}_t)_{t\geq 0}$ introduced in Subsection 3.6.4.

4.1 Main results

In [Bra20], we proved the RCD versions of Hess-Schrader–Uhlenbrock's inequality, Theorem 3.6.33, as well as the 1-vector Bochner inequality, Theorem 3.6.21, which in turn lead to the named improvements in the tamed framework of Chapter 3 above. For both results, we already observed in [Bra20] the importance of Kato's inequality from Lemma 3.4.13 in its RCD version from [DGP21, Lem. 3.5] (which has been used therein to obtain different results). In this chapter, we now provide several applications especially from Theorem 3.6.33 and Lemma 3.4.13 for RCD spaces. We discuss further functional inequalities for $(\mathsf{H}_t)_{t\geq 0}$ and spectral properties of its generator, the (negative) Hodge Laplacian $-\vec{\Delta}$. The final outcome of our discussion is an appropriate definition and the construction of a *heat kernel* for $(\mathsf{H}_t)_{t\geq 0}$ in the nonsmooth setting.

In the mentioned $\mathrm{RCD}(K,\infty)$ framework, $K \in \mathbf{R}$, the previously mentioned Hess–Schrader–Uhlenbrock inequality is stated again below for convenience. It is worth to note that its L^2-counterpart — i.e. the RCD version of Proposition 3.6.30 — is due to [Gig18, Prop. 3.6.10]. However, the latter frequently turns out to be too weak to deduce the results presented in this Chapter 4 (in particular in view of Section 4.5), whence the need of the following improved Theorem 4.1.1.

Theorem 4.1.1. *For every $\omega \in L^2(T^*M)$ and every $t \geq 0$,*

$$|\mathsf{H}_t\omega| \leq \mathrm{e}^{-Kt}\,\mathsf{P}_t|\omega| \quad \mathfrak{m}\text{-}a.e.$$

Motivation and possible further extensions Our motivation to study the heat flow $(\mathsf{H}_t)_{t\geq 0}$ in more detail than in [Gig18] comes from different directions. First, we provide a further contribution to the large diversity of works generalizing important "classical smooth" statements to nonsmooth spaces. Second, we believe that RCD spaces (or more general spaces with "lower Ricci bounds" [BHS21, ER+20, Stu20]) with the tensor language of [Gig15, Gig18] and Chapter 3 are the correct framework to develop nonsmooth notions of stochastic differential geometry, e.g. (damped) stochastic parallel transports, a project which, as mentioned above, currently lacks in the nonsmooth setting and which we attack in the future. Therein, in establishing Bismut–Elworthy–Li-type derivative formulas for the functional heat flow $(\mathsf{P}_t)_{t\geq 0}$ ([Bis84a, EL94b], see also Chapter 2) — which in turn are expected to provide further regularity information about it — a good understanding of its 1-form counterpart $(\mathsf{H}_t)_{t\geq 0}$ is essential [DT01]. Lastly, in smooth contexts, heat kernel methods for 1-forms are useful in many important applications, all of which could lead to RCD analogues that, however, are not addressed in this thesis. Exemplary, let us quote

- a deeper understanding of the Hodge theorem (see [Gig18] for the RCD result and Theorem 3.5.23 below) by the study of the heat kernel [MR51],
- a proof variant of index theorems in Riemannian geometry [Bis84b, Cha84, Hsu02a], along with introducing a working notion of nonsmooth Dirac operators on RCD spaces,
- the study of boundedness of the Riesz transform, see e.g. [CS08, Cou13, CDS20, Dev14, MO20] and the references therein, and
- the study of its short-time asymptotics playing dominant roles in theoretical physics and quantum gravity [Avr00, MP49, Ros97].

Logarithmic Sobolev inequalities The first consequence of Theorem 4.1.1 we discuss are *logarithmic Sobolev inequalities* for $(\mathsf{H}_t)_{t\geq 0}$. Such inequalities for functions and their connections to the functional heat flow $(\mathsf{P}_t)_{t\geq 0}$, initiated in [Gro75], have been an active field of research in past decades. For an overview over the vast literature on this subject, see [BGL14, Dav89]. In a similar manner, in this chapter we relate logarithmic Sobolev inequalities for 1-forms to certain further integral properties of $(\mathsf{H}_t)_{t\geq 0}$ described below. There is some ambiguity in formulating the former depending on whether one regards 1-forms as vector fields or really as contravariant objects. For brevity, we only outline Definition 4.3.4, where we say that a sufficiently regular vector field X over M obeys the 2-logarithmic Sobolev inequality $\mathrm{LSI}_2(\beta,\chi)$ with parameters $\beta > 0$ and $\chi \in \mathbf{R}$ if

$$\int_M |X|^2 \log|X|\,\mathrm{d}\mathfrak{m} \leq \beta\,\big\|\nabla X\big\|^2_{L^2(T^{\otimes 2}M)} + \chi\,\big\|X\big\|^2_{L^2(TM)} + \big\|X\big\|^2_{L^2(TM)}\log\|X\|_{L^2(TM)}.$$

The advantage of this form is that it follows from logarithmic Sobolev inequalities for functions, known to hold in various cases [CM17, Vil09], via Kato's inequality Lemma 3.4.13, see Lemma 4.3.8. It also implies its contravariant pendant from Definition 4.3.5 for arbitrary exponents, see Proposition 4.3.10.

The integral properties of $(\mathsf{H}_t)_{t\geq 0}$ to be derived are the following. We call $(\mathsf{H}_t)_{t\geq 0}$

- *hypercontractive* if there exist $T \in (0, \infty]$ and a strictly increasing C^1-function $p\colon [0, T) \to (1, \infty)$ such that H_t is bounded from $L^{p(0)}(T^*M)$ to $L^{p(t)}(T^*M)$ for every $t \in (0, T)$, and
- *ultracontractive* if there exist $p_0 \in (1, \infty)$ and $T > 0$ such that H_T is bounded from $L^{p_0}(T^*M)$ to $L^\infty(T^*M)$.

In great generality, in Theorem 4.3.12 we study when certain logarithmic Sobolev inequalities imply hyper- or ultracontractivity of $(\mathsf{H}_t)_{t\geq 0}$. We also treat a partial converse in Theorem 4.3.16.

Read in concrete applications, according to all these discussions and the known functional examples from [CM17, Vil09], we deduce the following hypercontractivity. (According to [CM17, Vil09], if $K > 0$ in either case, then the constant β in Theorem 4.1.2 can be chosen to be $(N-1)/KN$ or $1/K$, respectively.)

Theorem 4.1.2. *On any compact* $\mathrm{RCD}^*(K, N)$ *space with* $N \in (1, \infty)$ *or, for* $K > 0$, *any* $\mathrm{RCD}(K, \infty)$ *space, there exists a constant* $\beta > 0$ *such that for every* $p_0 \in (1, \infty)$, H_t *is bounded from* $L^{p_0}(T^*M)$ *to* $L^{p(t)}(T^*M)$ *with operator norm no larger than* e^{-Kt} *for every* $t > 0$, *where the function* $p\colon [0, \infty) \to (1, \infty)$ *is given by*

$$p(t) := 1 + (p_0 - 1)\,\mathrm{e}^{2t/\beta}.$$

Many of our arguments for Theorem 4.1.2 are inspired by the functional treatise [Dav89]. In the case of non-weighted Riemannian manifolds, logarithmic Sobolev inequalities for 1-forms have been studied with similar results in [Cha07].

Spectral behavior of Hodge's Laplacian As indicated in Corollary 3.6.28, Kato's inequality Lemma 3.4.13 also connects the spectra of the Hodge and the (negative) functional Laplacian. The study of the former is our goal in Section 4.4.

The following is first shown in full generality in Theorem 4.4.3 and Corollary 4.4.4.

Theorem 4.1.3. *If a positive real number belongs to the spectrum of* $-\Delta$, *then it is also contained in the spectrum of* $\vec{\Delta}$. *Similar inclusions hold between the respective point and essential spectra. In particular,*

$$\inf \sigma(-\Delta + K) \leq \inf \sigma(\vec{\Delta}) \leq \inf \sigma(\vec{\Delta}) \setminus \{0\} \leq \inf \sigma(-\Delta) \setminus \{0\}.$$

The stated spectral inclusions are known in the non-weighted Riemannian setting by [CL19]. Our proof of the former adopts a similar strategy, relying on a suitable variant of Weyl's criterion. The first stated spectral gap inequality follows by basic spectral theory and is well-known in the smooth setting. See e.g. [Gün17a] for a more general smooth treatise and further references.

On compact $\mathrm{RCD}^*(K, N)$ spaces, as in the case of functions, the spectrum of $\vec{\Delta}$ can be characterized much better. A key tool towards an explicit understanding of it in this case is the following Rellich-type compact embedding theorem, Theorem 4.4.8.

Theorem 4.1.4. *If* $(M, \mathsf{d}, \mathfrak{m})$ *is a compact* $\mathrm{RCD}^*(K, N)$ *space, the formal operator* $\vec{\Delta}^{-1}$ *is compact.*

For Ricci limit spaces, i.e. noncollapsed mGH-limits of sequences of non-weighted Riemannian manifolds with uniformly lower bounded Ricci curvatures, Theorem 4.1.4 is due to [Hon17, Hon18a]. In the very recent work [HZ20], Theorem 4.1.4 has been proven independently in a different way using so-called δ-splitting maps.

The proof of Theorem 4.1.4 uses several powerful properties of $(\mathsf{H}_t)_{t\geq 0}$ on compact $\mathrm{RCD}(K,N)$ spaces. Using that $(\mathsf{P}_t)_{t\geq 0}$ admits a heat kernel which obeys Gaussian bounds [JLZ16, Stu95, Tam19], together with Theorem 4.1.1 and Bishop–Gromov's inequality, we see in Theorem 4.3.3 that the heat operator H_t maps $L^p(T^*M)$ boundedly into $L^\infty(T^*M)$ for every $t > 0$ and every $p \in [1,\infty]$. In particular, H_t is a Hilbert–Schmidt operator on $L^2(T^*M)$, and Theorem 4.1.4 as well as expected properties of the spectrum of $\vec{\Delta}$ stated in Theorem 4.4.12 are then deduced by abstract functional analysis. We also establish L^∞-estimates on eigenforms of $\vec{\Delta}$, with an explicit growth rate for positive eigenvalues. See Corollary 4.4.13 and Proposition 4.4.14.

The last part of Section 4.4, especially Theorem 4.4.18, is devoted to the proof of the independence of the L^p-spectrum of $\vec{\Delta}$ on $p \in [1,\infty]$, provided $(M,\mathsf{d},\mathfrak{m})$ is an $\mathrm{RCD}^*(K,N)$ space satisfying, for every $\varepsilon > 0$, the volume growth condition

$$\sup_{x\in M}\int_M \mathrm{e}^{-\varepsilon\,\mathsf{d}(x,y)}\,\mathfrak{m}[B_1(x)]^{-1/2}\,\mathfrak{m}[B_1(y)]^{-1/2}\,\mathrm{d}\mathfrak{m}(y) < \infty.$$

On non-weighted Riemannian manifolds, this is shown in [Cha05]. Our proof, based on a perturbation argument, Theorem 4.1.1 and functional heat kernel bounds, is inspired by similar results for the functional Laplacian [HV86, HV87, SC92, Stu93]. See also [CF12, DL+10, KS14, Tak07, TT09] for further works in this direction for Markov processes and Feynman–Kac semigroups.

Heat kernel Up to now no general result ensuring the *existence* of a heat kernel for $(\mathsf{H}_t)_{t\geq 0}$ was known in the setting of [Gig18]. Outside the scope of noncompact, even weighted Riemannian manifolds [Gün17a, Pat71, Ros97], there are only few metric measure constructions under restrictive structural (existence of a *continuous* covector bundle with constant fiber dimensions) and volume doubling assumptions [CS08, Sik04]. Our axiomatization and existence proof of a heat kernel for $(\mathsf{H}_t)_{t\geq 0}$ on $\mathrm{RCD}(K,\infty)$ spaces is hoped to push forward research in the above areas on such spaces. Our general study applies to non-locally compact or non-doubling, possibly infinite-dimensional RCD spaces.

Let us motivate our axiomatization via the heat kernel p of $(\mathsf{P}_t)_{t\geq 0}$ from [AGS14b], cf. Section 4.2 for details. Slightly abusing notation, it induces a map $\mathsf{p}\colon (0,\infty)\times L^0(M)^2 \to L^0(M^2)$ sending $t > 0$ and $(g,f) \in L^0(M)^2$ to the $\mathfrak{m}^{\otimes 2}$-measurable function given by $\mathsf{p}_t[g,f](x,y) := \mathsf{p}_t(x,y)\,g(x)\,f(y)$ such that for a sufficiently large class of functions $f,g \in L^0(M)$, we have $\mathsf{p}_t[g,f] \in L^1(M^2)$ as well as

$$g\,\mathsf{P}_t f = \int_M \mathsf{p}_t[g,f](\cdot,y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.}$$

Let us turn to 1-forms. Recall that a heat kernel for $(\mathsf{H}_t)_{t\geq 0}$ in the smooth, possibly weighted setting is a jointly smooth map $\mathsf{h}\colon (0,\infty)\times M^2 \to (T^*M)^* \boxtimes T^*M$ — i.e. for every $t > 0$ and every $(x,y) \in M^2$, $\mathsf{h}_t(x,y)$ is a homomorphism mapping T^*_yM to T^*_xM — satisfying, for every $\omega \in L^2(T^*M)$,

$$\mathsf{H}_t\omega = \int_M \mathsf{h}_t(\cdot,y)\,\omega(y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.} \tag{4.1.1}$$

The heat kernel for 1-forms has first been constructed on compact spaces by [Pat71] using the so-called *parametrix construction*. See also [Gün17a, Ros97]. Since $\mathrm{RCD}(K,\infty)$ spaces a priori do neither come with any covector bundle nor with a smooth structure, the fiberwise notion (4.1.1) is be replaced by "testing the identity (4.1.1) pointwise against sufficiently many 1-forms". Motivated by our functional considerations, we understand a mapping $\mathsf{h}\colon (0,\infty)\times L^0(T^*M)^2 \to L^0(M^2)$ to be a *heat kernel* for $(\mathsf{H}_t)_{t\geq 0}$ if, for every $t>0$, h_t is L^0-bilinear, and for all sufficiently regular 1-forms $\omega,\eta\in L^0(T^*M)$, we have $\mathsf{h}_t[\eta,\omega]\in L^1(M^2)$ with the identity

$$\langle \eta, \mathsf{H}_t\omega\rangle = \int_M \mathsf{h}_t[\eta,\omega](\cdot,y)\,\mathrm{d}\mathfrak{m}(y)\quad \mathfrak{m}\text{-a.e.}$$

Theorem 4.1.5. *The heat kernel for* $(\mathsf{H}_t)_{t\geq 0}$ *in the indicated sense exists and is unique.*

The proof strategy for this result, see Theorem 4.5.5 for the precise formulation, is the following. Motivated by similar functional results [Stu95, SC10], a crucial tool to obtain integral kernels for certain operators is a *Dunford–Pettis*-type theorem [DP40, DS58], a very general L^∞-module version of which we prove in Theorem 4.5.3. Boiled down to the 1-form setting, it states that any linear operator which is bounded from $L^1(T^*M)$ to $L^\infty(T^*M)$ in the Banach sense admits an integral kernel, the concept of which is similar to the axiomatization of the 1-form heat kernel. Now for $t>0$, the heat operator H_t is not bounded from $L^1(T^*M)$ to $L^\infty(T^*M)$ in this generality. But by [Tam19], given any $\varepsilon>0$ there exist constants $C_1, C_2>0$ with

$$\mathsf{p}_t(x,y)\leq \mathfrak{m}\big[B_{\sqrt{t}}(x)\big]^{-1/2}\,\mathfrak{m}\big[B_{\sqrt{t}}(x)\big]^{-1/2}\,\exp\Big[C_1\big(1+C_2\,t\big)-\frac{\mathsf{d}^2(x,y)}{(4+\varepsilon)t}\Big]$$

for every $t>0$ and $\mathfrak{m}^\otimes$-a.e. $(x,y)\in M^2$. By Theorem 4.1.1, the perturbed operator

$$\mathsf{A}_t := \phi_t\,\mathsf{H}_t\,\phi_t,$$

where $\phi_t(x) := \mathfrak{m}[B_{\sqrt{t}}(x)]^{1/2}$, is thus bounded from $L^1(T^*M)$ to $L^\infty(T^*M)$ and therefore admits an integral kernel — formally multiplying A_t by ϕ_t^{-1} from both sides then yields the desired integral kernel h_t for H_t. Note that for this argument, it is essential that $(\mathsf{P}_t)_{t\geq 0}$ has a heat kernel. (This explains best our restriction to uniform lower Ricci bounds, a more general result is not available up to now.)

Having existence of h at our disposal, further properties of h such as symmetry, Hess–Schrader–Uhlenbrock's inequality for the "pointwise operator norm" $|\mathsf{h}_t|_\otimes$ of h_t,

$$|\mathsf{h}_t|_\otimes(x,y)\leq \mathrm{e}^{-Kt}\,\mathsf{p}_t(x,y)$$

for $\mathfrak{m}^{\otimes 2}$-a.e. $(x,y)\in M^2$ holds for every $t>0$, and Chapman–Kolmogorov's formula are stated in Theorem 4.5.7 below.

Two further results are then finally given on the class of $\mathrm{RCD}(K,N)$ spaces. In Theorem 4.5.11, for every $t>0$ we first prove the trace inequality

$$\operatorname{tr}\mathsf{H}_t\leq (\dim_{\mathsf{d},\mathfrak{m}} M)\,\mathrm{e}^{-Kt}\,\operatorname{tr}\mathsf{P}_t.$$

Here, $\dim_{\mathsf{d},\mathfrak{m}} M$, a positive integer not larger than N, is the *essential dimension* of $(M,\mathsf{d},\mathfrak{m})$ in the sense of [BS20, MN19]. This generalizes similar results on possibly weighted Riemannian manifolds [Gün17a, HSU80, Ros88]. Furthermore, our spectral analysis for $\vec{\Delta}$ from Theorem 4.1.4 entails a spectral resolution identity for h_t in Theorem 4.5.13 as soon as M is also compact.

4.2 Preliminaries

Hilbert–Schmidt operators and Hilbert space tensor products We start with a technical addendum to the L^∞-module discussion from Subsection 3.2.3 which will be needed in Subsection 4.5.3. Let $\mathscr{M}$ be a Hilbert module over M with pointwise scalar product $\langle\cdot,\cdot\rangle$. We call a linear operator $\mathsf{S}\colon \mathscr{M}\to\mathscr{M}$ a *Hilbert–Schmidt operator* if for some, or equivalently *any*, countable orthonormal bases $(v_i)_{i\in\mathbf{N}}$ and $(w_{i'})_{i'\in\mathbf{N}}$ of $\mathscr{M}$,

$$\big\|\mathsf{S}\big\|_{\mathrm{HS}}^2 := \sum_{i,i'\in\mathbf{N}} \Big[\int_M \langle \mathsf{S}v_i, w_{i'}\rangle\,\mathrm{d}\mathfrak{m}\Big]^2 < \infty.$$

The two-fold *Hilbert space tensor product* $\mathscr{M}^{\otimes_{\mathrm{H}}2}$ of $\mathscr{M}$, see e.g. Remark 3.2.29 or [Gig18, KR83] for the details, is isometrically isomorphic to the space of all Hilbert–Schmidt operators from $\mathscr{M}'$ to $\mathscr{M}$, endowed with the norm $\|\cdot\|_{\mathrm{HS}}$. Up to isomorphism, it is characterized by the following universal property [KR83, Thm. 2.6.4]. Given a real Hilbert space H, a bilinear $\mathsf{G}\colon\mathscr{M}^2\to H$ is termed *weakly Hilbert–Schmidt* if for some, or equivalently *any*, countable orthonormal bases $(v_i)_{i\in\mathbf{N}}$ and $(w_{i'})_{i'\in\mathbf{N}}$ as above,

$$\big\|\mathsf{G}\big\|_{\mathrm{wHS}}^2 := \sup\Big\{\big\|h\big\|_H^{-2}\sum_{i,i'\in\mathbf{N}}\big(\mathsf{G}(v_i,w_{i'})\,\big|\,h\big)_H^2 : h\in H\setminus\{0\}\Big\} < \infty.$$

Theorem 4.2.1. *The mapping* $\mathsf{e}\colon\mathscr{M}^2\to\mathscr{M}^{\otimes_{\mathrm{H}}2}$ *defined by*

$$\mathsf{e}(\eta,\omega) := \eta\otimes_{\mathrm{H}}\omega$$

is weakly Hilbert–Schmidt. Moreover, given any real Hilbert space H, for every weakly Hilbert–Schmidt mapping $\mathsf{G}\colon\mathscr{M}^2\to H$*, there exists a unique bounded operator* $\mathsf{T}\colon\mathscr{M}^{\otimes_{\mathrm{H}}2}\to H$ *which satisfies the identities*

$$\begin{aligned}\mathsf{G} &= \mathsf{T}\circ\mathsf{e},\\ \|\mathsf{G}\|_{\mathrm{wHS}} &= \|\mathsf{T}\|_{\mathscr{M}^{\otimes_{\mathrm{H}}2};H}.\end{aligned}$$

Furthermore, it is not difficult to prove the subsequent result which, in fact, is true for the Hilbert space tensor product of *any* two Hilbert modules over M, mostly applied to $\mathscr{M} := L^2(T^*M)$ in Subsection 4.5.3. (Recall that the language of Subsection 3.2.3 from [Gig18] works for any measure space. From the Dirichlet space perspective, in a standard way one can endow the product of two infinitesimally Hilbertian metric measure spaces with a Dirichlet structure, see e.g. [AGS15, Sec. 5.1].)

Lemma 4.2.2. *The Hilbert space tensor product $\mathscr{M}^{\otimes_{\mathrm{H}}2}$ has a natural structure of a Hilbert module over the product space $(M^2,\mathsf{d}^2,\mathfrak{m}^{\otimes 2})$ such that the multiplication $\cdot\colon L^\infty(M^2)\times\mathscr{M}^{\otimes_{\mathrm{H}}2}\to\mathscr{M}^{\otimes_{\mathrm{H}}2}$ and the pointwise norm $|\cdot|\colon\mathscr{M}^{\otimes_{\mathrm{H}}2}\to L^2(M^2)$ satisfy*

$$\begin{aligned}\big[f(\mathrm{pr}_1)\,g(\mathrm{pr}_2)\big]\,(v_1\otimes_{\mathrm{H}}v_2) &= (f\,v_1)\otimes_{\mathrm{H}}(g\,v_2),\\ |v_1\otimes_{\mathrm{H}}v_2| &= |v_1|(\mathrm{pr}_1)\,|w_2|(\mathrm{pr}_2)\end{aligned}$$

for every $v_1,v_2\in\mathscr{M}$ and every $f,g\in L^\infty(M)$.

Heat kernel on RCD(K,∞) spaces By [AGS14b, Sec. 6.1] and the identification of the heat flows on $L^2(M)$ and $\mathscr{P}_2(M)$ outlined in Section 1.2, the former flow $(\mathsf{P}_t)_{t\geq 0}$

admits a *heat kernel*, i.e. for every $t > 0$ there exists a symmetric $\mathfrak{m}^{\otimes 2}$-measurable map $\mathsf{p}_t : M^2 \to (0,\infty)$ such that $\int_M \mathsf{p}_t(\cdot, y)\,\mathrm{d}\mathfrak{m}(y) = 1$ and, for every $f \in L^2(M)$,

$$\mathsf{P}_t f = \int_M \mathsf{p}_t(\cdot, y)\, f(y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.}$$

The following result from [Tam19] yields a Gaussian upper bound for p_t, $t > 0$.

Theorem 4.2.3. *For every $\varepsilon > 0$, there exist constants $C_1 > 0$, depending only on ε, and $C_2 \geq 0$, depending only on K, such that for every $t > 0$ and $\mathfrak{m}^{\otimes 2}$-a.e. $(x, y) \in M^2$,*

$$\mathsf{p}_t(x, y) \leq \mathfrak{m}\big[B_{\sqrt{t}}(x)\big]^{-1/2}\,\mathfrak{m}\big[B_{\sqrt{t}}(y)\big]^{-1/2} \exp\Big[C_1\big(1 + C_2 t\big) - \frac{\mathsf{d}^2(x, y)}{(4+\varepsilon)t}\Big].$$

If $K \geq 0$, the constant C_2 can be chosen equal to zero.

RCD*(K, N) and RCD(K, N) spaces We recall useful properties of $\mathrm{RCD}(K, \infty)$ spaces admitting a synthetic notion of "upper dimension bound" $N \in [1, \infty)$ in addition. Two a priori different such notions exist and have been introduced following [LV09, Stu06b]: $\mathrm{RCD}^*(K, N)$ spaces [BS10, EKS15] and $\mathrm{RCD}(K, N)$ spaces [Gig15]. We do not detail their definitions and refer the reader to the cited references for details. Here, we only collect basic properties of these which are needed in this Chapter 4.

Every $\mathrm{RCD}(K, N)$ space is an $\mathrm{RCD}^*(K, N)$ space, and these conditions (even without infinitesimal Hilbertianity) coincide if $\mathfrak{m}[M] < \infty$ [CM21, Thm. 1.1]. It is conjectured that this holds for $\mathfrak{m}$-essentially nonbranching [RS14, p. 832] infinite metric measure spaces as well. In fact, we will need the stronger $\mathrm{RCD}(K, N)$ property only at one particular place, namely in Subsection 4.5.3 when speaking about "spaces with constant dimension", compare with Remark 3.3.16. All other results presented in the sequel hold for more general $\mathrm{RCD}^*(K, N)$ spaces. Moreover, from the Lagrangian viewpoint it is technically a bit more challenging to define $\mathrm{RCD}^*(K, 1)$ and $\mathrm{RCD}(K, 1)$ spaces, $K \in \mathbf{R}$ — here, these conditions consistently have to be read as "$\mathrm{RCD}^*(K, 1+\varepsilon)$ holds for every $\varepsilon > 0$", and similarly for the second case.

The first property is the following corollary of *Bishop–Gromov's inequality*, cf. [EKS15, Prop. 3.6] or [Stu06b, Thm. 2.3]. For every $D > 0$, for every $r, R \in (0, D)$ with $r < R$, there exists a constant $C < \infty$ depending only on K, N and D such that

$$\frac{\mathfrak{m}[B_R(x)]}{\mathfrak{m}[B_r(x)]} \leq C\left(\frac{R}{r}\right)^N \tag{4.2.1}$$

for every $x \in M$. In particular, since $B_1(y) \subset B_{1+\mathsf{d}(x,y)}(x)$, for every $y \in M$,

$$\mathfrak{m}[B_1(y)] \leq C\,\mathrm{e}^{N\,\mathsf{d}(x,y)}\,\mathfrak{m}[B_1(x)]. \tag{4.2.2}$$

A further consequence of (4.2.1) is that $\mathfrak{m}$ is *locally doubling*, that is, for every $x \in M$ and $r \in (0, D)$ as above we have

$$\mathfrak{m}[B_{2r}(x)] \leq 2^N\, C\,\mathfrak{m}[B_r(x)]. \tag{4.2.3}$$

In turn, this condition implies local compactness of M [Stu06b, Cor. 2.4]. In particular, every finite diameter $\mathrm{RCD}^*(K, N)$ space is necessarily compact — this is in particular the case when $K > 0$ [Stu06b, Cor. 2.6].

Since $\mathrm{RCD}^*(K, N)$ spaces also satisfy local $(1,1)$- and $(2,2)$-Poincaré inequalities [EKS15, Raj12a], the general study from [Stu95, Stu96] yields the existence of a locally

Hölder continuous representative of the heat kernel p on $(0,\infty)\times M^2$. By [JLZ16, Thm. 1.2], for every $\varepsilon > 0$, there exist constants $C_3, C_4 > 1$ depending only on K, N and ε, such that for every $x, y \in M$ and every $t > 0$,

$$\mathsf{p}_t(x,y) \leq C_3\,\mathfrak{m}\big[B_{\sqrt{t}}(x)\big]^{-1}\,\exp\Big[C_4 t - \frac{\mathsf{d}^2(x,y)}{(4+\varepsilon)t}\Big]. \tag{4.2.4}$$

4.3 Improved integral estimates for the heat flow

4.3.1 Basic L^p-properties and L^p-L^∞-regularization

From Theorem 3.6.33 above and a standard procedure, the following is immediate by approximation. It is worth to emphasize that the restriction of H_t to $L^\infty(T^*M)$ is defined as the Banach space adjoint of the restriction of H_t to $L^1(T^*M)$ for every $t \geq 0$. See also Subsection 4.4.3.

Theorem 4.3.1. *For every $p \in [1,\infty]$, $(\mathsf{H}_t)_{t\geq 0}$ then extends to a semigroup of bounded linear operators from $L^p(T^*M)$ into $L^p(T^*M)$, strongly continuous if $p < \infty$ and weakly* continuous if $p = \infty$, which satisfies, for every $t \geq 0$,*

$$\|\mathsf{H}_t\|_{L^p(T^*M);L^p(T^*M)} \leq \mathrm{e}^{-Kt}.$$

Remark 4.3.2. This result implies the L^p-contractivity of $(\mathsf{H}_t)_{t\geq 0}$ for every $p \in [1,\infty]$ under nonnegative lower Ricci bounds. However, we should not expect contractivity in larger generality, not even on Riemannian manifolds [Str83, Str86]. ■

On compact $\mathrm{RCD}^*(K,N)$ spaces, $N \in [1,\infty)$, the heat operator H_t is not only bounded from $L^p(T^*M)$ to $L^p(T^*M)$, but also from $L^p(T^*M)$ to $L^\infty(T^*M)$ for every $t > 0$ and every $p \in [1,\infty]$. This is the content of the following result which will be crucial in Subsection 4.4.2 and Subsection 4.5.3.

Theorem 4.3.3 (L^p-L^∞-regularization)**.** *Let $(M,\mathsf{d},\mathfrak{m})$ be a compact* $\mathrm{RCD}^*(K,N)$ *space, $K \in \mathbf{R}$ and $N \in [1,\infty)$. Furthermore, let $t > 0$ and $p \in [1,\infty]$. Then H_t is bounded from $L^p(T^*M)$ to $L^\infty(T^*M)$.*

Proof. By Hölder's inequality, it suffices to prove boundedness of H_t from $L^1(T^*M)$ to $L^\infty(T^*M)$. Let $\omega \in L^1(T^*M)\cap L^2(T^*M)$ with $\|\omega\|_{L^1(T^*M)} \leq 1$ be arbitrary; the consideration of such 1-forms is enough by the density of $L^1(T^*M)\cap L^2(T^*M)$ in $L^1(T^*M)$. By (4.2.2), there exist $z \in M$ and a constant $C > 0$ such that $\mathfrak{m}\big[B_{\sqrt{t}}(\cdot)\big]^{-1} \leq C\,\mathfrak{m}\big[B_{\sqrt{t}}(z)\big]^{-1}$ on M. The conclusion follows by observing that by Theorem 4.1.1 and (4.2.4), there exist constants $C_3, C_4 > 1$ depending only on K and N such that

$$\begin{aligned}|\mathsf{H}_t\omega| &\leq \mathrm{e}^{-Kt}\int_M \mathsf{p}_t(\cdot,y)\,|\omega|(y)\,\mathrm{d}\mathfrak{m}(y)\\ &\leq C_3\,\mathrm{e}^{-(K-C_4)t}\,\mathfrak{m}\big[B_{\sqrt{t}}(\cdot)\big]^{-1}\\ &\leq C\,C_3\,\mathrm{e}^{-(K-C_4)t}\,\mathfrak{m}\big[B_{\sqrt{t}}(z)\big]^{-1}\quad \mathfrak{m}\text{-a.e.}\end{aligned}$$

□

4.3.2 Logarithmic Sobolev inequalities

We come to an important class of functional inequalities, namely *logarithmic Sobolev inequalities* for 1-forms and their relation to integral-type inequalities for $(\mathsf{H}_t)_{t\geq 0}$. More precisely, following [Cha07, Dav89] we show that the former imply, for certain $t > 0$ and every $p_0 \in [1,\infty)$, the boundedness of H_t from $L^{p_0}(T^*M)$ into $L^{p(t)}(T^*M)$, where p is a real-valued function with $p(0) = p_0$. This property is called *hypercontractivity*. Under more restrictive assumptions, for some finite $T > 0$ it is even possible to prove the boundedness of H_T from $L^{p_0}(T^*M)$ to $L^\infty(T^*M)$, a property termed *ultracontractivity*. See Theorem 4.3.12 (compare with Theorem 4.3.3 above).

A certain reverse implication also holds, see Theorem 4.3.16.

Definition 4.3.4. *Let $\beta > 0$ and $\chi \in \mathbf{R}$. We say that a vector field $X \in H^{1,2}(TM) \cap L^1(TM) \cap L^\infty(TM)$ satisfies the* 2-logarithmic Sobolev inequality *with constants β and χ, briefly* $\mathrm{LSI}_2(\beta,\chi)$*, if*

$$\int_M |X|^2 \log|X| \,\mathrm{d}\mathfrak{m} \leq \beta \left\|\nabla X\right\|^2_{L^2(TM)} + \chi \left\|X\right\|^2_{L^2(TM)} + \left\|X\right\|^2_{L^2(TM)} \log \|X\|_{L^2(TM)}.$$

Definition 4.3.5. *Let $\varepsilon > 0$, $\gamma \in \mathbf{R}$ and $p \in [1,\infty)$. We say that a 1-form $\omega \in \mathscr{D}(\vec{\Delta}) \cap L^1(T^*M) \cap L^\infty(T^*M)$ obeys the* form p-logarithmic Sobolev inequality *with constants ε and γ, briefly* $\mathrm{fLSI}_p(\varepsilon,\gamma)$*, if $\vec{\Delta}\omega \in L^p(T^*M)$ and, with the convention $0^0 := 0$, we have*

$$\int_M |\omega|^p \log|\omega| \,\mathrm{d}\mathfrak{m} \leq \varepsilon \int_M |\omega|^{p-2} \langle \omega, \vec{\Delta}\omega\rangle \,\mathrm{d}\mathfrak{m} + \gamma \left\|\omega\right\|^p_{L^p(T^*M)} + \left\|\omega\right\|^p_{L^p(T^*M)} \log\|\omega\|_{L^p(T^*M)}.$$

Remark 4.3.6. On Riemannian manifolds, a similar definition as Definition 4.3.5 has been given and considered in [Cha07, Def. 2.1] for the more restrictive case $p \in (2,\infty)$. The definition of a 2-logarithmic Sobolev inequality therein, on the other hand, is similar to Definition 4.3.4. ■

Remark 4.3.7. We do not discuss the case of 1-logarithmic Sobolev inequalities since it is not clear, even having an appropriate version of such an inequality at our disposal, that for $\omega \in L^1(T^*M) \cap L^\infty(T^*M)$, the function $|\omega| \log|\omega|$ is integrable.

On the other hand, the integrability of $|\omega|^p \log|\omega|$ for $p \in (1,\infty)$ is clear by local boundedness of the function $r \mapsto r^\delta \log r$ on $[0,\infty)$ for every $\delta > 0$. ■

Later, special interest will be devoted to the class

$$V_{1,\infty} := \bigcup_{t>0} \mathsf{H}_t\big(L^1(T^*M) \cap L^\infty(T^*M)\big)$$

By Theorem 3.6.23 and Theorem 4.1.1, $V_{1,\infty}$ is contained in $\mathscr{D}(\vec{\Delta})$ as well as in $L^p(T^*M)$ for every $p \in [1,\infty]$, is invariant under the action of H_t for every $t > 0$, and it is strongly dense in the latter space if $p < \infty$. Additionally, since the infinitesimal generator of the restriction of $(\mathsf{H}_t)_{t\geq 0}$ onto $L^p(T^*M)$ applied to any $\omega \in V_{1,\infty}$ coincides with $\vec{\Delta}$, we have $\vec{\Delta}\omega \in L^p(T^*M)$ for every $p \in [1,\infty]$. (See Subsection 4.3.1 and Subsection 4.4.3 for the correct interpretation in the case $p = \infty$.)

Relations between different logarithmic Sobolev inequalities In view of Proposition 4.3.10, we first focus on the 2-logarithmic Sobolev inequality. In particular, we show how to derive it from its functional counterpart in the next Lemma 4.3.8.

Given any $\beta > 0$, following (1.2) of [CM17] (replacing α by $1/\beta$ therein), a nonnegative $f \in \mathrm{Lip}(M) \cap L^1(M)$ is said to satisfy the *functional* 2*-logarithmic Sobolev inequality* with constant β provided

$$2\int_M f \log f \,\mathrm{d}\mathfrak{m} - 2\int_M f \,\mathrm{d}\mathfrak{m} \log \int_M f \,\mathrm{d}\mathfrak{m} \leq \beta \int_M \frac{|\nabla f|^2}{f} \,\mathrm{d}\mathfrak{m}. \tag{4.3.1}$$

Lemma 4.3.8. *Let $\beta > 0$, and suppose that every nonnegative $f \in \mathrm{Lip}(M) \cap L^1(M)$ obeys the functional 2-logarithmic Sobolev inequality with constant β. Then every $X \in H^{1,2}(TM) \cap L^1(TM) \cap L^\infty(TM)$ satisfies $\mathrm{LSI}_2(\beta, 0)$.*

Proof. Let $R > 1$ and $z \in M$, and let $\psi_R \in \mathrm{Lip}_{\mathrm{bs}}(M)$ be a cutoff function with $\psi_R(M) = [0,1]$, identically equal to 1 on $B_R(z)$ and identically equal to 0 on $B_{R+1}(z)^{\mathrm{c}}$.

Given any $X \in H^{1,2}(TM) \cap L^1(TM) \cap L^\infty(TM)$, observe that $\mathsf{P}_{1/n}|X| \in \mathrm{Test}(M) \cap \mathrm{Lip}(M) \cap L^1(M)$ for every $n \in \mathbf{N}$, where we identify $\mathsf{P}_{1/n}|X|$ with its Lipschitz $\mathfrak{m}$-a.e. representative by the Sobolev-to-Lipschitz property of $(M, \mathsf{d}, \mathfrak{m})$ (recall Section 0.1). We may and will assume that the sequence $(g_n)_{n\in\mathbf{N}}$ in $\mathrm{Lip}_{\mathrm{bs}}(M) \cap L^1(M)$, where $g_n := \psi_R\, \mathsf{P}_{1/n}|X|$, converges to $\psi_R\,|X|$ pointwise $\mathfrak{m}$-a.e. and strongly in $W^{1,2}(M)$. Setting $f_n := g_n^2$, $n \in \mathbf{N}$, entails

$$|\nabla f_n|^2 = 4\, f_n\, |\nabla g_n|^2 \quad \mathfrak{m}\text{-a.e.}$$

Lebesgue's theorem as well as (4.3.1) applied to f_n for every $n \in \mathbf{N}$ yield

$$\begin{aligned}
&2\int_M \psi_R^2\,|X|^2 \log\big(\psi_R^2\,|X|^2\big) \,\mathrm{d}\mathfrak{m} - 2\int_M \psi_R^2\,|X|^2 \,\mathrm{d}\mathfrak{m} \log \int_M \psi_R^2\,|X|^2 \,\mathrm{d}\mathfrak{m} \\
&\qquad = \lim_{n\to\infty} \Big[2\int_M f_n \log f_n \,\mathrm{d}\mathfrak{m} - 2\int_M f_n \,\mathrm{d}\mathfrak{m} \log \int_M f_n \,\mathrm{d}\mathfrak{m}\Big] \\
&\qquad \leq \lim_{n\to\infty} \beta \int_M \frac{|\nabla f_n|^2}{f_n} \,\mathrm{d}\mathfrak{m} \\
&\qquad = \lim_{n\to\infty} 4\beta \int_M |\nabla g_n|^2 \,\mathrm{d}\mathfrak{m} = 4\beta \int_M \big|\nabla(\psi_R\,|X|)\big|^2 \,\mathrm{d}\mathfrak{m},
\end{aligned}$$

and the claim follows by letting $R \to \infty$, employing Lebesgue's theorem and Kato's inequality from Lemma 3.4.13. □

Example 4.3.9. By [Vil09, Thm. 30.21] if $(M, \mathsf{d}, \mathfrak{m})$ is an $\mathrm{RCD}(K, \infty)$ space with $K > 0$, or [CM17, Thm. 1.9] in the case when $(M, \mathsf{d}, \mathfrak{m})$ is a compact $\mathrm{RCD}^*(K, N)$ space, $K \in \mathbf{R}$ and $N \in (1, \infty)$, the hypothesis of Lemma 4.3.8 is known to be satisfied for some finite $\beta > 0$. If $K > 0$ in the respective cases, the constant β can explicitly be chosen to be $1/K$ and $(N-1)/KN$. ■

Proposition 4.3.10. *Let $\beta > 0$ and $\chi \in \mathbf{R}$. Define the functions $\varepsilon, \gamma \in \mathrm{C}((1,\infty))$ by*

$$\begin{aligned}
\varepsilon(p) &:= \frac{\beta p}{2(p-1)}, \\
\gamma(p) &:= \frac{2\chi}{p} - \frac{K\beta p}{2(p-1)}.
\end{aligned}$$

Assume that every $X \in H^{1,2}(T^*M)^\sharp \cap L^1(TM) \cap L^\infty(TM)$ obeys $\mathrm{LSI}_2(\beta, \chi)$ according to Definition 4.3.4. Then every $\omega \in V_{1,\infty}$ obeys $\mathrm{fLSI}_p(\varepsilon(p), \gamma(p))$ for every $p \in (1, \infty)$.

Proof. The claim for $p = 2$ follows from the Gaffney-type inequality, Lemma 3.6.8. Thus we concentrate on the case $p \in (1, 2) \cup (2, \infty)$.

Given any $\tau > 0$, the function $\varphi_\tau \in \mathrm{C}^\infty([0, \infty))$ given by $\varphi_\tau(r) := (r + \tau)^{p/2-1}$ obeys the following inequalities for every $r \geq 0$:

$$0 \leq \frac{p}{p-2}\, \varphi_\tau'(r)\, r \leq \varphi_\tau(r) + \varphi_\tau'(r)\, r. \tag{4.3.2}$$

Approximating φ_τ by functions in $\mathrm{Lip}_{\mathrm{bs}}([0, \infty))$, we have $\varphi_\tau \circ |\omega| \in \mathscr{F}_{\mathrm{eb}}$ whence, by Remark 3.5.16 and Lemma 3.4.13, we obtain $[\varphi_\tau \circ |\omega|]\, \omega^\sharp \in H^{1,2}(T^*M)^\sharp \cap L^1(TM) \cap L^\infty(TM)$ for every $\tau > 0$. By $\mathrm{LSI}_2(\beta, \chi)$ applied to $X := [\varphi_\tau \circ |\omega|]\, \omega^\sharp$ and letting $\tau \downarrow 0$, employing Lebesgue's theorem and Lemma 3.6.8, we infer that

$$\begin{aligned}
&\int_M |\omega|^p \log|\omega| \,\mathrm{d}\mathfrak{m} - \frac{2\chi}{p} \big\|\omega\big\|_{L^p(T^*M)}^p - \big\|\omega\big\|_{L^p(T^*M)}^p \log\|\omega\|_{L^p}\\
&\qquad\leq \liminf_{\tau\downarrow 0} \Big[\frac{2}{p}\int_M \big|[\varphi_\tau \circ |\omega|]\,\omega\big|^2 \log\big|[\varphi_\tau\circ|\omega|]\,\omega\big| \,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad - \frac{2\chi}{p}\big\|[\varphi_\tau\circ|\omega|]\,\omega\big\|_{L^2(T^*M)}^2\\
&\qquad\qquad - \frac{2}{p}\big\|[\varphi_\tau\circ|\omega|]\,\omega\big\|_{L^2(T^*M)}^2 \log\big\|[\varphi_\tau\circ|\omega|]\,\omega\big\|_{L^2(T^*M)}\Big]\\
&\qquad\leq \liminf_{\tau\downarrow 0} \frac{2\beta}{p}\int_M \big|\nabla\big[[\varphi_\tau\circ|\omega|]\,\omega^\sharp\big]\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m}.
\end{aligned}\tag{4.3.3}$$

It remains to estimate the last limit in (4.3.3). Recal that by (3.2.13),

$$\big|\mathrm{d}|\omega| \wedge \omega\big|^2 = \big|\mathrm{d}|\omega|\big|^2\,|\omega|^2 - \langle \mathrm{d}|\omega|, \omega\rangle^2.$$

Therefore, for every $\tau > 0$, we observe by Lemma 3.6.8 and Remark 3.5.16, taking into account that $[\varphi_\tau^2 \circ |\omega|]\,\omega \in H^{1,2}(T^*M)$ as well, and finally integration by parts that

$$\begin{aligned}
&\int_M \big|\nabla\big[[\varphi_\tau\circ|\omega|]\,\omega^\sharp\big]\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} + K\,\big\|[\varphi_\tau\circ|\omega|]\,\omega\big\|_{L^2(T^*M)}^2 \qquad (4.3.4)\\
&\qquad\leq \int_M \Big[\big|\mathrm{d}\big[[\varphi_\tau\circ|\omega|]\,\omega\big]\big|^2 + \big|\delta\big[[\varphi_\tau\circ|\omega|]\,\omega\big]\big|^2\Big] \,\mathrm{d}\mathfrak{m}\\
&\qquad= \int_M \Big[[\varphi_\tau^2\circ|\omega|]\,|\mathrm{d}\omega|^2 + 2\,[\varphi_\tau\circ|\omega|]\,[\varphi_\tau'\circ|\omega|]\,\langle \mathrm{d}|\omega|\wedge\omega, \mathrm{d}\omega\rangle\Big] \,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad + \int_M \Big[[(\varphi_\tau')^2\circ|\omega|]\,\big|\mathrm{d}|\omega|\wedge\omega\big|^2 + [\varphi_\tau^2\circ|\omega|]\,|\delta\omega|^2\Big] \,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad - \int_M \Big[2\,[\varphi_\tau\circ|\omega|]\,[\varphi_\tau'\circ|\omega|]\,\delta\omega\,\langle \mathrm{d}|\omega|, \omega\rangle\\
&\qquad\qquad\qquad - [(\varphi_\tau')^2\circ|\omega|]\,\langle \mathrm{d}|\omega|, \omega\rangle^2\Big] \,\mathrm{d}\mathfrak{m}\\
&\qquad= \int_M \Big[\big\langle \mathrm{d}\big[[\varphi_\tau^2\circ|\omega|]\,\omega\big], \mathrm{d}\omega\big\rangle + \delta\big[[\varphi_\tau^2\circ|\omega|]\,\omega\big]\,\delta\omega\Big] \,\mathrm{d}\mathfrak{m}\\
&\qquad\qquad + \int_M [(\varphi_\tau')^2\circ|\omega|]\,|\omega|^2\,\big|\mathrm{d}|\omega|\big|^2 \,\mathrm{d}\mathfrak{m}\\
&\qquad= \int_M [\varphi_\tau^2\circ|\omega|]\,\langle \omega, \vec{\Delta}\omega\rangle \,\mathrm{d}\mathfrak{m} + \int_M [(\varphi_\tau')^2\circ|\omega|]\,|\omega|^2\,\big|\mathrm{d}|\omega|\big|^2 \,\mathrm{d}\mathfrak{m}. \qquad (4.3.5)
\end{aligned}$$

Thanks to (4.3.2), the Leibniz rule, the chain rule and Lemma 3.4.13, we have

$$[(\varphi'_\tau)^2 \circ |\omega|]\, |\omega|^2 \left|\mathrm{d}|\omega|\right|^2 \leq \frac{(p-2)^2}{p^2} \left|\nabla\big[[\varphi_\tau \circ |\omega|]\, \omega^\sharp\big]\right|_{\mathrm{HS}}^2 \quad \mathfrak{m}\text{-a.e.}$$

Rearranging the estimate resulting from this bound with the inequality between (4.3.4) and (4.3.5) and then sending $\tau \downarrow 0$ yields

$$\begin{aligned}\liminf_{\tau\downarrow 0} \frac{2\beta}{p} &\int_M \left|\nabla\big[[\varphi_\tau \circ |\omega|]\, \omega^\sharp\big]\right|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} \\ &\leq \varepsilon(p) \int_M |\omega|^{p-2} \langle \omega, \vec{\Delta}\omega\rangle \,\mathrm{d}\mathfrak{m} - \frac{K\beta p}{2(p-1)} \|\omega\|_{L^p(T^*M)}^p.\end{aligned}$$

From (4.3.3), this readily provides the claim. □

Remark 4.3.11. Let $\beta > 0$ and $\chi \in \mathbf{R}$, and define $\varepsilon, \gamma \in \mathrm{C}((1,\infty))$ by

$$\begin{aligned}\varepsilon(p) &:= \frac{\beta p}{2(p-1)},\\ \gamma(p) &:= \frac{2\chi}{p} - \frac{K\beta(p-2)^2}{2p(p-1)}.\end{aligned}$$

With a slight modification of the proof of Proposition 4.3.10, it is possible to show that if every element in $\mathscr{D}(\vec{\Delta}) \cap L^1(T^*M) \cap L^\infty(T^*M)$ obeys $\mathrm{fLSI}_2(\beta,\chi)$, then every $\omega \in V_{1,\infty}$ satisfies $\mathrm{fLSI}_p(\varepsilon(p),\gamma(p))$ for every $p \in (2,\infty)$. Up to changing the involved constants, this assumption is weaker compared to the one of Proposition 4.3.10. ■

From logarithmic Sobolev inequalitites to hyper- and ultracontractivity

Theorem 4.3.12. *Let $p_0 \in (1,\infty)$. Let $\varepsilon \in \mathrm{C}([p_0,\infty))$ be a positive function, and let $\gamma \in \mathrm{C}([p_0,\infty))$. Suppose that the integrals*

$$\begin{aligned}T &:= \int_{p_0}^\infty \frac{\varepsilon(r)}{r}\,\mathrm{d}r,\\ C &:= \int_{p_0}^\infty \frac{\gamma(r)}{r}\,\mathrm{d}r\end{aligned}$$

exist with values in $(0,\infty]$ and $(-\infty,\infty]$, respectively. Define $p \in \mathrm{C}^1([0,T))$ and $A \in \mathrm{C}^1([0,\infty))$ through the relations

$$\begin{aligned}\int_{p_0}^{p(t)} \frac{\varepsilon(r)}{r}\,\mathrm{d}r &:= t,\\ A(t) &:= \int_0^t \frac{\gamma(p(r))}{\varepsilon(p(r))}\,\mathrm{d}r.\end{aligned}$$

Assume $\mathrm{fLSI}_p(\varepsilon(p),\gamma(p))$ for every $\omega \in V_{1,\infty}$ and every $p \in [p_0,\infty)$. Then the following properties hold.

(i) Hypercontractivity. *For every $t \in [0,T)$,*

$$\|\mathsf{H}_t\|_{L^{p_0}(T^*M);L^{p(t)}(T^*M)} \leq \mathrm{e}^{A(t)}.$$

(ii) Ultracontractivity. *If* $T < \infty$ *and* $C < \infty$,

$$\|\mathsf{H}_T\|_{L^{p_0}(T^*M);L^\infty(T^*M)} \leq \mathrm{e}^C.$$

Proof. First observe that $p(0) = p_0$, that $A(0) = 0$, and that p is strictly increasing with $p(t) \to \infty$ as $t \to T$. Moreover, $A(t) \to C$ as $t \to T$ thanks to the relations

$$\begin{aligned} p' &= \frac{p}{\varepsilon(p)},\\ A' &= \frac{\gamma(p)}{\varepsilon(p)} = \frac{\gamma(p)\, p'}{p}. \end{aligned} \tag{4.3.6}$$

Owing to (i), given $\omega \in V_{1,\infty} \setminus \{0\}$ we assume that $\mathsf{H}_t\omega \neq 0$ for every $t \in [0,T)$, which is always true at least for small times. Otherwise, the following computations are performed until the heat flow dies out. Consider the function $F \in \mathrm{C}^1([0,T))$ with

$$F(t) := \mathrm{e}^{-A(t)}\, \|\mathsf{H}_t\omega\|_{L^{p(t)}(T^*M)}.$$

By Theorem 3.6.23, $t \mapsto |\mathsf{H}_t\omega|^2$ is continuously differentiable on $[0,\infty)$ in $L^2(T^*M)$ with derivative $-2\,\langle \mathsf{H}_t\omega, \vec{\Delta}\mathsf{H}_t\omega\rangle \in L^2(T^*M)$ for every $t \geq 0$. In particular,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathsf{H}_t\omega|^{p(t)} = |\mathsf{H}_t\omega|^{p(t)}\,\big[p'(t)\log|\mathsf{H}_t\omega| - p(t)\,|\mathsf{H}_t\omega|^{-2}\,\langle \mathsf{H}_t\omega, \vec{\Delta}\mathsf{H}_t\omega\rangle\big] \quad \mathfrak{m}\text{-a.e.},$$

and the assertion on the regularity of F indeed follows by the integrability assumptions on ω, C^1-regularity of p, Theorem 4.1.1 and arguing as in Remark 4.3.7.

Moreover, for every $t \in [0,T)$, from $\mathrm{fLSI}_{p(t)}(\varepsilon(p(t)), \gamma(p(t)))$ applied to $\mathsf{H}_t\omega \in V_{1,\infty}$ as well as (4.3.6),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\log F(t) &= -A'(t) + \big\|\mathsf{H}_t\omega\big\|_{L^{p(t)}(T^*M)}^{-1}\,\frac{\mathrm{d}}{\mathrm{d}t}\|\mathsf{H}_t\omega\|_{L^{p(t)}(T^*M)}\\ &= -A'(t) - \frac{p'(t)}{p(t)}\log\big\|\mathsf{H}_t\omega\big\|_{L^{p(t)}(T^*M)}^{p(t)}\\ &\qquad + \frac{1}{p(t)}\,\big\|\mathsf{H}_t\omega\big\|_{L^{p(t)}(T^*M)}^{-p(t)}\,\frac{\mathrm{d}}{\mathrm{d}t}\big\|\mathsf{H}_t\omega\big\|_{L^{p(t)}(T^*M)}^{p(t)}\\ &= -\frac{\gamma(p(t))}{\varepsilon(p(t))} - \frac{1}{\varepsilon(p(t))}\log\big\|\mathsf{H}_t\omega\big\|_{L^{p(t)}(T^*M)}^{p(t)}\\ &\qquad + \frac{1}{\varepsilon(p(t))}\,\big\|\mathsf{H}_t\omega\big\|_{L^{p(t)}(T^*M)}^{-p(t)}\int_M |\mathsf{H}_t\omega|^{p(t)}\log|\mathsf{H}_t\omega|\,\mathrm{d}\mathfrak{m}\\ &\qquad - \big\|\mathsf{H}_t\omega\big\|_{L^{p(t)}(T^*M)}^{-p(t)}\int_M |\mathsf{H}_t\omega|^{p(t)-2}\,\langle \mathsf{H}_t\omega, \vec{\Delta}\mathsf{H}_t\omega\rangle\,\mathrm{d}\mathfrak{m} \leq 0. \end{aligned}$$

Hence, F is nonincreasing, yielding (i) by the density of $V_{1,\infty}$ in $L^{p_0}(T^*M)$.

Concerning (ii), invoking the strict increasingness of p and Hölder's inequality, for every $s,t \in [0,T)$ with $s < t$ and every bounded Borelian $B \subset M$ with $\mathfrak{m}[B] > 0$,

$$\begin{aligned} \|1_B\,\mathsf{H}_t\omega\|_{L^{p(s)}(T^*M)} &\leq \mathfrak{m}[B]^{1-p(s)/p(t)}\,\|\mathsf{H}_t\omega\|_{L^{p(t)}(T^*M)}\\ &\leq \mathfrak{m}[B]^{1-p(s)/p(t)}\,\mathrm{e}^{A(t)-A(s)}\,\|\mathsf{H}_s\omega\|_{L^{p(s)}(T^*M)}\\ &\leq \mathfrak{m}[B]^{1-p(s)/p(t)}\,\mathrm{e}^{A(t)}\,\|\omega\|_{L^{p_0}(T^*M)}. \end{aligned}$$

The claim follows by letting $t \to T$ and $s \to T$ in such a way that $p(s)/p(t) \to 1$ and afterwards using the arbitrariness of B as well as the density of $V_{1,\infty}$ in $L^{p_0}(T^*M)$. □

Example 4.3.13. Given any $\beta > 0$, the functions $\varepsilon, \gamma \in \mathrm{C}((1,\infty))$ with

$$\varepsilon(p) := \frac{\beta p}{2(p-1)},$$
$$\gamma(p) := -\frac{K\beta p}{2(p-1)}.$$

are the coefficients in Proposition 4.3.10 arising from the setup of Lemma 4.3.8 and Example 4.3.9. Retaining the notation from Theorem 4.3.12, subject to these coefficients and any $p_0 \in (1,\infty)$, the value T is always infinite, while C takes the values $-\infty$, 0 or ∞ depending on whether $K > 0$, $K = 0$ or $K < 0$. Moreover, the functions p and A from Theorem 4.3.12 read

$$p(t) = 1 + (p_0 - 1)\,\mathrm{e}^{2t/\beta},$$
$$A(t) = \int_{p_0}^{p(t)} \frac{\gamma(s)}{s}\,\mathrm{d}s = -Kt. \qquad \blacksquare$$

Corollary 4.3.14. *In the setting of Example 4.3.13, given any $p_0 \in (1,\infty)$, for every $t \geq 0$, we have*

$$\|\mathsf{H}_t\|_{L^{p_0}(T^*M);L^{p(t)}(T^*M)} \leq \mathrm{e}^{-Kt}.$$

Corollary 4.3.15. *In the setting of Example 4.3.13, let $\omega \in \mathscr{D}(\vec{\Delta})$ be an eigenform for $\vec{\Delta}$ with eigenvalue $\lambda \geq 0$, i.e. $\vec{\Delta}\omega = \lambda\,\omega$. Then $\omega \in L^q(T^*M)$ for every $q \in (2,\infty)$ with*

$$\|\omega\|_{L^q(T^*M)} \leq (q-1)^{(\lambda-K)\beta/2}\,\|\omega\|_{L^2(T^*M)}.$$

Proof. Note that $\mathsf{H}_t\omega = \mathrm{e}^{-\lambda t}\,\omega$ for every $t \geq 0$. We apply Corollary 4.3.14 to the initial value $p_0 := 2$. Given any $q \in (1,\infty)$, since $p(t) = q$ if and only if $t = \log(q-1)\beta/2$, for this value of t we have

$$\begin{aligned}\|\omega\|_{L^q(T^*M)} &= \mathrm{e}^{\lambda t}\,\|\mathsf{H}_t\omega\|_{L^{p(t)}(T^*M)}\\ &\leq \mathrm{e}^{(\lambda-K)t}\,\|\omega\|_{L^{p_0}(T^*M)}\\ &= (q-1)^{(\lambda-K)\beta/2}\,\|\omega\|_{L^2(T^*M)}.\end{aligned} \qquad \square$$

From ultracontractivity to logarithmic Sobolev inequalities

Theorem 4.3.16. *Let $T \in (0,\infty]$ as well as $c \in \mathrm{C}((0,T))$. Suppose that*

$$\|\mathsf{H}_t\|_{L^2(T^*M),L^\infty(T^*M)} \leq \mathrm{e}^{c(t)}$$

holds for every $t \in (0,T)$. Then every $\omega \in \mathscr{D}(\vec{\Delta}) \cap L^1(T^*M) \cap L^\infty(T^*M)$ satisfies $\mathrm{fLSI}_2(\varepsilon, c(\varepsilon))$ *for every $\varepsilon \in (0,T)$.*

Proof. Let $U := \big\{\xi \in \mathbf{C} : \Re\xi \in [0,1]\big\}$. Given any $\varepsilon \in (0,T)$ and $\xi \in U$, consider

$$\mathsf{S}_\xi := \mathrm{e}^{-\varepsilon\xi\vec{\Delta}}$$

as a linear operator acting on $L^2(T^*M) + \mathrm{i}\,L^2(T^*M)$. For every $\eta, \rho \in L^2(T^*M) + \mathrm{i}\,L^2(T^*M)$, the canonical bilinear form in $L^2(T^*M) + \mathrm{i}\,L^2(T^*M)$ induced by S_ξ evaluated at (η, ρ) is continuous in U, and its restriction to the interior of U is holomorphic. For every $\eta \in L^2(T^*M) + \mathrm{i}\, L^2(T^*M)$ and every $\vartheta \in \mathbf{R}$, we have

$$\begin{aligned} \|\mathsf{S}_{\mathrm{i}\vartheta}\eta\|_{L^2(T^*M)} &\leq \|\eta\|_{L^2(T^*M)}, \\ \|\mathsf{S}_{1+\mathrm{i}\vartheta}\eta\|_{L^\infty(T^*M)} &\leq \mathrm{e}^{-c(\varepsilon)}\, \|\mathsf{S}_{\mathrm{i}\vartheta}\eta\|_{L^2(T^*M)} \leq \mathrm{e}^{-c(\varepsilon)}\, \|\eta\|_{L^2(T^*M)}. \end{aligned}$$

Given any $\omega \in \mathscr{D}(\vec{\Delta}) \cap L^1(T^*M) \cap L^\infty(T^*M)$ such that $\|\omega\|_{L^2(T^*M)} = 1$, for every $\tau \in (0,1)$, via Stein's interpolation theorem we infer

$$\|\mathsf{H}_{\varepsilon\tau}\omega\|_{L^{2/(1-\tau)}(T^*M)} = \|\mathsf{S}_\tau\omega\|_{L^{2/(1-\tau)}(T^*M)} \leq \mathrm{e}^{-c(\varepsilon)\tau}. \tag{4.3.7}$$

Define $p \in \mathrm{C}^1([0,\varepsilon))$ by $p(t) := 2\varepsilon/(\varepsilon - t)$. Setting $\tau := t/\varepsilon$ in (4.3.7) translates into

$$\big\|\mathsf{H}_t\omega\big\|_{L^{p(t)}(T^*M)}^{p(t)} \leq \mathrm{e}^{-c(\varepsilon)\,p(t)\,t/\varepsilon},$$

and the claim follows after differentiating both sides at 0 via

$$\int_M |\omega|^2 \left[\frac{2}{\varepsilon}\log|\omega| - 2\,|\omega|^{-2}\,\langle\omega, \vec{\Delta}\omega\rangle\right] \mathrm{d}\mathfrak{m} \leq \frac{2c(\varepsilon)}{\varepsilon}. \qquad \square$$

Example 4.3.17. If P_t is bounded from $L^2(M)$ to $L^\infty(M)$, then so is H_t by means of Theorem 4.1.1. Compare this with (the proof of) Theorem 4.3.3.

See [Cha07, Ch. 4] for an application to certain Gaussian upper bounds for the heat kernel on 1-forms in the non-weighted smooth setting. ■

4.4 Spectral properties of the Hodge Laplacian

Next, we study properties of the *spectrum* $\sigma(\vec{\Delta})$ of $\vec{\Delta}$, fixing first some notation. We denote the *resolvent set* of $\vec{\Delta}$ by $\rho(\vec{\Delta}) := \mathbf{C} \setminus \sigma(\vec{\Delta})$. The *point spectrum* of $\vec{\Delta}$ is denoted by $\sigma_{\mathrm{p}}(\vec{\Delta})$, and the *essential spectrum* of $\vec{\Delta}$ will be termed $\sigma_{\mathrm{e}}(\vec{\Delta})$.

Since $\vec{\Delta}$ is self-adjoint and nonnegative, we immediately have

$$\sigma(\vec{\Delta}) \subset [0, \infty).$$

Eigenspaces w.r.t. different eigenvalues are mutually orthogonal in $L^2(T^*M)$.

4.4.1 Inclusion of spectra

In this subsection, we show that, except the critical value 0, the spectrum of the negative functional Laplacian $-\Delta$ is contained in $\sigma(\vec{\Delta})$. Similar inclusions hold between the respective point and essential spectra. See Theorem 4.4.3. Our proof follows the smooth treatise for [CL19, Cor. 4.4, Cor. 4.5].

We shall need the subsequent characterization of points in the (essential) spectrum of $\vec{\Delta}$. See [CL19, Prop. 2.5] and the references therein for a more general statement.

Lemma 4.4.1. *For every $\lambda > 0$, we have $\lambda \in \sigma(\vec{\Delta})$ if and only if there exist $\alpha < 0$ and a sequence $(\omega_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\vec{\Delta})$ such that*

a. $\|\omega_n\|_{L^2(T^*M)} = 1$ *for every $n \in \mathbf{N}$, and*
b. *for every $j \in \{1, 2\}$, one has*

$$\lim_{n\to\infty} \int_M \big\langle (\vec{\Delta} - \alpha)^{-j}\omega_n, \vec{\Delta}\omega_n - \lambda\,\omega_n\big\rangle\,\mathrm{d}\mathfrak{m} = 0.$$

Moreover, a number $\lambda > 0$ belongs to the essential spectrum of $\vec{\Delta}$ if and only if some sequence $(\omega_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\vec{\Delta})$ satisfies the previous conditions a. *and* b. *as well as*

c. $\omega_n \rightharpoonup 0$ *in* $L^2(T^*M)$ *as* $n \to \infty$.

Lemma 4.4.2. *Let $\alpha < 0$. Then for every $f \in W^{1,2}(M)$, we have*

$$(\vec{\Delta} - \alpha)^{-1}\mathrm{d}f = \mathrm{d}(-\Delta - \alpha)^{-1} f,$$

while for every $\omega \in \mathscr{D}(\delta)$, we have

$$(-\Delta - \alpha)^{-1}\delta\omega = \delta(\vec{\Delta} - \alpha)^{-1}\omega.$$

Proof. Any $\alpha < 0$ belongs to $\rho(\vec{\Delta})$ and $\rho(-\Delta)$, thus $\vec{\Delta} - \alpha$ and $-\Delta - \alpha$ are invertible with bounded inverse. Furthermore, the second identity follows from the first by definition of δ and the self-adjointness of $(\vec{\Delta} - \alpha)^{-1}$, since α is real — we thus concentrate on the proof of the first equality.

Given any $f \in W^{1,2}(M)$, let $u \in \mathscr{D}(\Delta)$ be the unique solution to the equation $-\Delta u - \alpha\, u = f$ on M. By Lemma 3.6.24, for every $t > 0$ we have $\mathrm{d}\mathsf{P}_t u \in \mathscr{D}(\vec{\Delta})$ and

$$\vec{\Delta}\mathrm{d}\mathsf{P}_t u - \alpha\, \mathrm{d}\mathsf{P}_t u = -\mathrm{d}(\Delta \mathsf{P}_t u + \alpha\, \mathsf{P}_t u) = \mathrm{d}\mathsf{P}_t f.$$

Therefore $\mathsf{H}_t \mathrm{d}u = (\vec{\Delta} - \alpha)^{-1}\mathsf{H}_t \mathrm{d}f$ again by Lemma 3.6.24, and the claim follows by letting $t \to 0$ in the latter identity. □

Theorem 4.4.3. *We have the inclusions*

$$\begin{aligned}\sigma_{\mathrm{p}}(-\Delta) &\subset \sigma_{\mathrm{p}}(\vec{\Delta}),\\ \sigma(-\Delta) \setminus \{0\} &\subset \sigma(\vec{\Delta}),\\ \sigma_{\mathrm{e}}(-\Delta) \setminus \{0\} &\subset \sigma_{\mathrm{e}}(\vec{\Delta}).\end{aligned}$$

Proof. The first inclusion is elementary, since for every $\lambda \in \sigma_{\mathrm{p}}(-\Delta)$ and its corresponding eigenfunction $f \in \mathscr{D}(\Delta)$, by Lemma 3.6.24, $\mathrm{d}\mathsf{P}_1 f \in \mathscr{D}(\vec{\Delta})$ and

$$\vec{\Delta}\mathrm{d}\mathsf{P}_1 f = -\mathrm{d}\Delta\mathsf{P}_1 f = \lambda\, \mathrm{d}\mathsf{P}_1 f.$$

To prove the second inclusion, let $\lambda \in \sigma(-\Delta) \setminus \{0\}$. By Weyl's criterion [HS96, Thm. 5.10] applied to $-\Delta$, for every $n \in \mathbf{N}$ there exists $g_n \in \mathscr{D}(\Delta)$ with

$$\begin{aligned}\|g_n\|_{L^2(M)} &= 1,\\ \|\Delta g_n + \lambda\, g_n\|_{L^2(M)} &\leq 2^{-n}.\end{aligned}$$

Moreover, for every $n \in \mathbf{N}$ there exists $t_n > 0$ such that $f_n := \mathsf{P}_{t_n} g_n \in \mathscr{D}(\Delta)$ obeys

$$\begin{aligned}1/\sqrt{2} \leq\ \|f_n\|_{L^2(M)} &\leq 1,\\ \|\Delta f_n + \lambda f_n\|_{L^2(M)} &\leq 2^{-n}.\end{aligned} \tag{4.4.1}$$

Provided that $2^{-n} \leq \lambda/4$, from (4.4.1) we get

$$\begin{aligned}\int_M |\mathrm{d}f_n|^2\, \mathrm{d}\mathfrak{m} &= -\int_M f_n\, \Delta f_n\, \mathrm{d}\mathfrak{m}\\ &\geq -\|\Delta f_n + \lambda\, f_n\|_{L^2(M)} + \lambda\, \big\|f_n\big\|^2_{L^2(M)} \geq \frac{\lambda}{4} > 0.\end{aligned} \tag{4.4.2}$$

Possibly relabeling $(f_n)_{n\in\mathbf{N}}$, we assume that $(\|\mathrm{d}f_n\|^2_{L^2(T^*M)})_{n\in\mathbf{N}}$ is uniformly bounded from below by $\lambda/4$. In particular, for $j \in \{1,2\}$ it follows from Lemma 3.6.24, Lemma 4.4.2, contractivity of $(-\Delta+1)^{-j}$ in $L^2(M)$, (4.4.1) and (4.4.2) that

$$\begin{aligned}
&\Big|\int_M \big\langle (\vec{\Delta}+1)^{-j}\mathrm{d}f_n, (\vec{\Delta}-\lambda)\mathrm{d}f_n\big\rangle \,\mathrm{d}\mathfrak{m}\Big| \\
&\qquad = \Big|\int_M \big[\delta(\vec{\Delta}+1)^{-j}\mathrm{d}f_n\big]\,(\Delta f_n + \lambda\, f_n)\,\mathrm{d}\mathfrak{m}\Big| \\
&\qquad = \Big|\int_M \big[(-\Delta+1)^{-j}\Delta f_n\big]\,(\Delta f_n + \lambda\, f_n)\,\mathrm{d}\mathfrak{m}\Big| \\
&\qquad \leq 2^{-n}\,\|\Delta f_n\|_{L^2(M)} \leq 2^{-n}\,(\lambda + 2^{-n}) \leq \frac{4(\lambda+1)}{\lambda}\,2^{-n}\,\|\mathrm{d}f_n\|^2_{L^2(T^*M)}.
\end{aligned}$$

In particular, the sequence $(\omega_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\vec{\Delta})$ — by Lemma 3.6.24 — given by

$$\omega_n := \|\mathrm{d}f_n\|^{-1}_{L^2(T^*M)}\,\mathrm{d}f_n, \tag{4.4.3}$$

obeys a. and b. from Lemma 4.4.1, whence $\lambda \in \sigma(\vec{\Delta})$.

Turning to the last inclusion, if $\lambda \in \sigma_{\mathrm{e}}(-\Delta)$, then the sequence $(f_n)_{n\in\mathbf{N}}$ from the previous step can be constructed to satisfy $f_n \rightharpoonup 0$ in $L^2(M)$ as $n \to \infty$ in addition to (4.4.1) above [HS96, Thm. 7.2]. Therefore, for every $\eta \in \mathrm{Test}(T^*M)$ we obtain for the sequence $(\omega_n)_{n\in\mathbf{N}}$ defined in (4.4.3) that

$$\lim_{n\to\infty}\Big|\int_M \langle\omega_n,\eta\rangle\,\mathrm{d}\mathfrak{m}\Big| \leq \lim_{n\to\infty}\frac{2}{\sqrt{\lambda}}\Big|\int_M f_n\,\delta\eta\,\mathrm{d}\mathfrak{m}\Big| = 0.$$

Since $\|\omega_n\|_{L^2(T^*M)} = 1$ for every $n\in\mathbf{N}$, this provides c. in Lemma 4.4.1. □

From Corollary 3.6.28, we directly deduce the following.

Corollary 4.4.4. *Under the assumptions of Theorem 4.4.3, we have*

$$\inf\sigma(-\Delta+K) \leq \inf\sigma(\vec{\Delta}) \leq \inf\sigma(\vec{\Delta})\setminus\{0\} \leq \inf\sigma(-\Delta)\setminus\{0\}.$$

Remark 4.4.5. The same proof shows that the statements of Theorem 4.4.3 and Corollary 4.4.4 are still valid in the tamed space setting from Chapter 3. ■

4.4.2 The spectrum in the compact case

Much more about $\sigma(\vec{\Delta})$ can be said if $(M,\mathsf{d},\mathfrak{m})$ is a compact $\mathrm{RCD}^*(K,N)$ space, $K\in\mathbf{R}$ and $N\in[1,\infty)$. In this framework, adopted in this subsection, we prove that $\sigma(\vec{\Delta})$ is discrete and only consists of eigenvalues, see Theorem 4.4.12. A closely related result is that the natural inclusion of $H^{1,2}(T^*M)$ into $L^2(T^*M)$ is compact, Theorem 4.4.8. In turn, by abstract functional analysis, this follows if H_t is a Hilbert–Schmidt operator on $L^2(T^*M)$ for every $t>0$, which is the content of Corollary 4.4.7.

Afterwards, we establish the boundedness of eigenforms for $\vec{\Delta}$ with an explicit growth rate for their L^∞-norms for positive eigenvalues, see Corollary 4.4.13 and Proposition 4.4.14. The entire discussion in this subsection heavily relies on the L^2-L^∞-regularization property of $(\mathsf{H}_t)_{t\geq 0}$ from Theorem 4.3.3.

The elementary proof of the subsequent lemma is given in [Bra20], see also the references therein. Corollary 4.4.7 then follows from Theorem 4.3.3.

Lemma 4.4.6. *Suppose that* S *is a linear operator which maps* $L^2(T^*M)$ *boundedly into* $L^\infty(T^*M)$*. Then* S *is a Hilbert–Schmidt operator.*

Corollary 4.4.7. *For every* $t > 0$, H_t *is a Hilbert–Schmidt operator on* $L^2(T^*M)$*.*

In particular, H_t is compact on $L^2(T^*M)$ for every $t > 0$. Employing standard functional analytic results, see e.g. [LSW10, Cor. 1.5], this entails the following crucial Rellich-type theorem. (The reader is also invited to consult [HZ20, Thm. 6.1], where the same result has independently been proven by different means via δ-splitting maps.) It implies in particular that the space $\mathscr{H}(T^*M)$ of harmonic 1-forms (recall Definition 3.5.21) is a closed subspace of $L^2(T^*M)$, hence the L^2-orthogonal projection T onto $\mathscr{H}(T^*M)$ is well-defined. Lemma 4.4.9 is then easily argued by contradiction.

Theorem 4.4.8 (Compactness of $\vec{\Delta}^{-1}$)**.** *The natural inclusion of* $H^{1,2}(T^*M)$ *into* $L^2(T^*M)$ *is compact.*

Lemma 4.4.9. *There exists a constant* $C < \infty$ *such that for every* $\omega \in L^2(T^*M)$*,*

$$\big\|\omega - \mathsf{T}\omega\big\|^2_{L^2(T^*M)} \leq C\,\mathscr{E}_{\mathrm{con}}(\omega). \tag{4.4.4}$$

Remark 4.4.10. Lemma 4.4.9 can be seen as a qualitative global Poincaré inequality for $\vec{\Delta}$. In contrast to logarithmic Sobolev inequalities, we did not derive local or global Poincaré inequalities for 1-forms from the corresponding functional estimates. Combining Lemma 3.4.13 with [Raj12a, Thm. 1.1, Thm. 1.2] or [Vil09, Thm. 30.24], this would be possible to some extent. Paying the price of a less explicit constant, the point is however that the terms $\int_{B_r(x)} f\,\mathrm{d}\mathfrak{m}$ or $\int_M f\,\mathrm{d}\mathfrak{m}$ if $\mathfrak{m}[M] < \infty$, respectively, appearing in the functional versions are the L^2-orthogonal projections of f onto the space of harmonic functions on the respective L^2-spaces, while their 1-form counterparts $\int_{B_r(x)} |\omega|\,\mathrm{d}\mathfrak{m}$ or $\int_M |\omega|\,\mathrm{d}\mathfrak{m}$ appearing in the derived estimates arguing as for Lemma 4.3.8 would clearly lack this interpretation. ∎

Corollary 4.4.11. *Let* $C > 0$ *be any constant for which* (4.4.4) *holds, and let* $\lambda \in \sigma(\vec{\Delta}) \setminus \{0\}$*. Then* λ *is an eigenvalue of* $\vec{\Delta}$ *and satisfies the inequality*

$$\lambda \geq 1/C.$$

Proof. Let $\lambda \in \sigma(\vec{\Delta}) \setminus \{0\}$. As in the proof of Theorem 4.4.3, by Weyl's criterion there exists a sequence $(\omega_n)_{n\in\mathbf{N}}$ in $\mathscr{D}(\vec{\Delta})$ such that, for every $n \in \mathbf{N}$,

$$\begin{aligned} \|\omega_n\|_{L^2(T^*M)} &= 1,\\ \|\vec{\Delta}\omega_n - \lambda\,\omega_n\|_{L^2(T^*M)} &\leq 2^{-n}. \end{aligned} \tag{4.4.5}$$

To prove that λ is an eigenvalue, observe that $(\|\omega_n\|_{H^{1,2}(T^*M)})_{n\in\mathbf{N}}$ is uniformly bounded by (4.4.5). According to Theorem 4.4.8, a non-relabeled subsequence of $(\omega_n)_{n\in\mathbf{N}}$ converges weakly in $H^{1,2}(T^*M)$ and strongly in $L^2(T^*M)$ to some $\omega \in H^{1,2}(T^*M)$ with $\|\omega\|_{L^2(T^*M)} = 1$. Given any $\rho \in \mathrm{Test}(T^*M)$, since

$$\begin{aligned} \lambda\int_M \langle\rho,\omega\rangle\,\mathrm{d}\mathfrak{m} &= \lim_{n\to\infty}\int_M \langle\rho,\vec{\Delta}\omega_n\rangle\,\mathrm{d}\mathfrak{m}\\ &= \lim_{n\to\infty}\int_M \big[\langle\mathrm{d}\rho,\mathrm{d}\omega_n\rangle + \delta\rho\,\delta\omega_n\big]\,\mathrm{d}\mathfrak{m}\\ &= \int_M \big[\langle\mathrm{d}\rho,\mathrm{d}\omega\rangle + \delta\rho\,\delta\omega\big]\,\mathrm{d}\mathfrak{m}, \end{aligned}$$

we also obtain that $\omega \in \mathscr{D}(\vec{\Delta})$ with $\vec{\Delta}\omega = \lambda\,\omega$, which is the claim.

The bound $\lambda \geq 1/C$ then follows by inserting ω into (4.4.4), recalling that eigenspaces w.r.t. different eigenvalues are orthonormal in $L^2(T^*M)$. □

In view of Corollary 4.4.11, given $\lambda \geq 0$ we denote the *eigenspace* of $\vec{\Delta}$ w.r.t. λ by

$$\mathrm{E}_\lambda(\vec{\Delta}) := \big\{\omega \in \mathscr{D}(\vec{\Delta}) : \vec{\Delta}\omega = \lambda\,\omega\big\}.$$

The proofs of the following basic results are standard once having Theorem 4.4.8 as well as Corollary 4.4.11 at our disposal. We refer to [Hon17, Thm. 4.3] as well as [Sak96, Lem. VI.3.6, Prop. VI.3.8] for similar statements in the smooth setting and for comprehensive proofs.

Theorem 4.4.12. *The spectrum $\sigma(\vec{\Delta})$ has the following properties.*

(i) Finite dimensionality. *For every $\lambda \geq 0$, the real vector space dimension of $\mathrm{E}_\lambda(\vec{\Delta})$ is finite.*
(ii) Discreteness, unboundedness. *The spectrum $\sigma(\vec{\Delta})$ is discrete (i.e. for every $\lambda \in \sigma(\vec{\Delta})$ there exists some $r > 0$ such that $(\lambda - r, \lambda + r) \cap \sigma(\vec{\Delta}) = \{\lambda\}$), consists only of eigenvalues, and is unbounded.*
(iii) Variational principle. *Let $(\lambda_i)_{i\in\mathbf{N}}$ be an increasing enumeration of the eigenvalues of $\vec{\Delta}$ counted with multiplicities. Let $\mathbf{S}$ denote the unit sphere in $L^2(T^*M)$. Then for every $i \in \mathbf{N}$,*

$$\lambda_i = \inf\Big\{\sup_{\omega\in E\cap\mathbf{S}} \mathscr{E}_{\mathrm{con}}(\omega) : E \subset H^{1,2}(T^*M) \textit{ subspace with } \dim E = i\Big\}.$$

(iv) Orthonormal eigenbasis. *The direct sum*

$$\mathrm{E}(\vec{\Delta}) := \bigoplus_{\lambda\in\sigma(\vec{\Delta})} \mathrm{E}_\lambda(\vec{\Delta})$$

is dense both in $H^{1,2}(T^*M)$ and $L^2(T^*M)$, endowed with their respective norms. In particular, there exists a countable orthonormal basis $(\omega_i)_{i\in\mathbf{N}}$ of $L^2(T^*M)$ such that, for every $i \in \mathbf{N}$, we have $\omega_i \in \mathrm{E}_{\lambda_i}(\vec{\Delta})$ for some $\lambda_i \in \sigma(\vec{\Delta})$.

Further, since $\omega = \mathsf{H}_1\omega$ for every $\omega \in \mathscr{H}(T^*M)$, Theorem 4.3.3 immediately provides Corollary 4.4.13 below. An argument as in the proof of Theorem 4.3.3 with a finer estimation then yields Proposition 4.4.14. For similar statements, see [AH+21, Prop. 7.1] for functions — whose proof is adopted in our approach — and [Hon17, Prop. 4.14] for arbitrary tensor fields in the Ricci limit framework.

Corollary 4.4.13. *We have $\mathscr{H}(T^*M) \subset L^\infty(T^*M)$.*

Proposition 4.4.14. *Let the constant $D > 0$ obey $\operatorname{diam} M \leq D$. Assume that $\omega \in \mathscr{D}(\vec{\Delta})$ is an eigenform with eigenvalue $\lambda \in [D^{-2}, \infty)$ and $\|\omega\|_{L^2(T^*M)} = 1$. Then there exists a constant $C < \infty$ depending only on K, N and D such that*

$$\|\omega\|_{L^\infty(T^*M)} \leq C\,\lambda^{N/4}.$$

Proof. Since $\omega \in \mathrm{E}_\lambda(\vec{\Delta})$, it follows that $\mathsf{H}_t\omega = \mathrm{e}^{-\lambda t}\omega$ for every $t \geq 0$. Thus, for $t \in (0, D^2]$ to be determined later, Theorem 4.1.1 and then (4.2.4) for $\varepsilon := 1$ yield the existence of constants $C_1, C_2 < \infty$ depending only on K and N such that

$$|\omega| \leq \mathrm{e}^{(\lambda+K^-)t}\int_M \mathsf{p}_t(\cdot, y)\,|\omega|(y)\,\mathrm{d}\mathfrak{m}(y)$$

$$\leq \mathrm{e}^{(\lambda+K^-)t}\Big[\int_M \mathsf{p}_t^2(\cdot,y)\,\mathrm{d}\mathfrak{m}(y)\Big]^{1/2}$$
$$\leq C_1\,\mathrm{e}^{(\lambda+K^-+C_2)t}\,\mathfrak{m}\big[B_{\sqrt{t}}(\cdot)\big]^{-1}\Big[\int_M \mathrm{e}^{-2\mathsf{d}^2(\cdot,y)/5t}\,\mathrm{d}\mathfrak{m}(y)\Big]^{1/2}\quad \mathfrak{m}\text{-a.e.}$$

Arguing exactly as in the proof of [AH+21, Prop. 7.1], using (4.2.1), under the given assumptions we find a constant $C < \infty$ depending only on K, N and D such that

$$\mathfrak{m}\big[B_{\sqrt{t}}(\cdot)\big]^{-1}\Big[\int_M \mathrm{e}^{-2\mathsf{d}^2(\cdot,y)/5t}\,\mathrm{d}\mathfrak{m}(y)\Big]^{1/2}\leq C\Big(\frac{D}{\sqrt{t}}\Big)^{N/2}\quad \mathfrak{m}\text{-a.e.}$$

The choice of $t := 1/\lambda$ gives the desired estimate. □

4.4.3 Independence of the L^p-spectrum on p

In this subsection, let us fix an $\mathrm{RCD}^*(K,N)$ space $(M,\mathsf{d},\mathfrak{m})$, where $K\in\mathbf{R}$ and $N\in[1,\infty)$. Under a volume growth assumption stated in Definition 4.4.15, following [Cha05, HV86, Stu93] we show that the L^p-spectrum of $\vec{\Delta}$ is independent of $p\in[1,\infty]$, see Theorem 4.4.18 below.

To keep the presentation clear, in this subsection we denote by $\vec{\Delta}_2 := \vec{\Delta}$ the Hodge Laplacian acting on $L^2(T^*M)$ and by $(\mathsf{H}_{2,t})_{t\geq 0}$ the associated semigroup $\mathsf{H}_{2,t} := \mathsf{H}_t$. Recalling Theorem 4.3.1, by $(\mathsf{H}_{p,t})_{t\geq 0}$ we denote the extension of $(\mathsf{H}_{2,t})_{t\geq 0}$ to $L^p(T^*M)$ for every $p\in[1,\infty)$. Let $\vec{\Delta}_p$ be the infinitesimal generator of $(\mathsf{H}_{p,t})_{t\geq 0}$. We also define $\mathsf{H}_{\infty,t}$ and $\vec{\Delta}_\infty$ on $L^\infty(T^*M)$ as the adjoints of $\mathsf{H}_{1,t}$ and $\vec{\Delta}_1$, respectively.

Given any $p\in[1,\infty]$ and $n\in\mathbf{N}$, by [Yos80, Thm. IX.4.1, Cor. IX.4.1] we know that, for every $\xi\in\rho(\vec{\Delta}_p)$ with $\Re\xi < K^-$, we have

$$(\vec{\Delta}_p-\xi)^{-n} = \frac{1}{(n-1)!}\int_0^\infty \mathrm{e}^{\xi t}\,t^{n-1}\,\mathsf{H}_{p,t}\,\mathrm{d}t. \tag{4.4.6}$$

Definition 4.4.15. *We say that the reference measure* $\mathfrak{m}$ *on* M *is* uniformly subexponentially integrable *if for every* $\varepsilon > 0$*, we have*

$$\sup_{x\in M}\int_M \mathrm{e}^{-\varepsilon\mathsf{d}(x,y)}\,\mathfrak{m}[B_1(x)]^{-1/2}\,\mathfrak{m}[B_1(y)]^{-1/2}\,\mathrm{d}\mathfrak{m}(y)<\infty.$$

Remark 4.4.16. By an analogous argument to [Stu93, Prop. 1], $\mathfrak{m}$ is uniformly subexponentially integrable if for every $\varepsilon > 0$, there exists $C<\infty$ such that for every $x\in M$ and every $r>0$, we have

$$\mathfrak{m}[B_r(x)]\leq C\,\mathrm{e}^{\varepsilon r}\,\mathfrak{m}[B_1(x)].$$ ■

Example 4.4.17. If $(M,\mathsf{d},\mathfrak{m})$ is globally doubling (4.2.3), then $\mathfrak{m}$ is uniformly subexponentially integrable. Indeed, by a well-known iteration argument starting from (4.2.3), the sufficient condition from the previous Remark 4.4.16 follows from the existence of finite constants $\alpha,\beta>0$ such that

$$\mathfrak{m}[B_r(x)]\leq \beta\,r^\alpha\,\mathfrak{m}[B_1(x)]$$

holds for every $x\in M$ and every $r\geq 1$.

Since $\mathrm{RCD}^*(K,N)$ spaces with a nonnegative lower Ricci bound are globally doubling [Stu06b, Cor. 2.4], uniform subexponential integrability of $\mathfrak{m}$ is granted in this case, i.e. as soon as $K\geq 0$. ■

Theorem 4.4.18. *Assume that* $\mathfrak{m}$ *is uniformly subexponentially integrable. Then the spectrum* $\sigma(\vec{\Delta}_p)$ *of the operator* $\vec{\Delta}_p$ *acting on* $L^p(T^*M)$ *is equal to* $\sigma(\vec{\Delta}_2)$ *for every* $p \in [1,\infty]$*. Furthermore, for every* $p, q \in [1,\infty]$*, every isolated eigenvalue of* $\vec{\Delta}_p$ *with finite algebraic multiplicity is also an isolated eigenvalue of* $\vec{\Delta}_q$ *with the same algebraic multiplicity.*

The key point of the proof of Theorem 4.4.18 is a perturbation argument whose core we outsource into Lemma 4.4.20, Lemma 4.4.21 and Corollary 4.4.22 below. Before that, we quickly fix some notation.

We define the measurable function $\phi_1 \colon M \to \mathbf{R}$ by

$$\phi_1(x) := \mathfrak{m}[B_1(x)]^{1/2}.$$

Given any $\varepsilon > 0$, we consider the class

$$\Gamma_\varepsilon := \big\{\psi \in W^{1,2}(M) \cap \mathrm{C_b}(M) : |\mathrm{d}\psi| \leq \varepsilon \ \mathfrak{m}\text{-a.e.}\big\}$$

and recall from [AGS15, Thm. 4.17] that, for every $x, y \in M$,

$$\varepsilon\, \mathsf{d}(x,y) = \sup\big\{\psi(x) - \psi(y) : \psi \in \Gamma_\varepsilon\big\}. \tag{4.4.7}$$

Lastly, given $\psi \in \Gamma_\varepsilon$, by $\mathrm{e}^{\psi}\, \vec{\Delta}_2\, \mathrm{e}^{-\psi}$ we intend the linear, densely defined operator on $L^2(T^*M)$ given by setting, for arbitrary $\omega, \eta \in \mathrm{Test}(T^*M)$,

$$\begin{aligned}
&\int_M \big\langle \eta, (\mathrm{e}^{\psi}\, \vec{\Delta}_2\, \mathrm{e}^{-\psi})\omega \big\rangle \,\mathrm{d}\mathfrak{m} \\
&\qquad := \int_M \langle \eta, \vec{\Delta}_2 \omega\rangle \,\mathrm{d}\mathfrak{m} - \int_M |\nabla\psi|^2 \,\langle \eta, \omega\rangle \,\mathrm{d}\mathfrak{m} \\
&\qquad\qquad + \int_M \big[\nabla\omega^\sharp(\nabla\psi, \eta^\sharp) - \nabla\eta^\sharp(\nabla\psi, \omega^\sharp)\big] \,\mathrm{d}\mathfrak{m}.
\end{aligned} \tag{4.4.8}$$

Remark 4.4.19. Observe that if ψ is sufficiently regular, say, $\psi \in \Gamma_\varepsilon \cap \mathrm{Test}(M)$ with $\Delta\psi \in L^\infty(M)$, then $\mathrm{e}^{\psi}\, \vec{\Delta}_2\, \mathrm{e}^{-\psi}\omega$ is pointwise well-defined on any $\omega \in \mathrm{Test}(T^*M)$ as composition of the multiplication operators e^{ψ} and $\mathrm{e}^{-\psi}$ as well as the Hodge Laplacian $\vec{\Delta}_2$ in the indicated order by Lemma 3.6.1. In this case, (4.4.8) follows by a straightforward computation using Lemma 3.2.11 and Proposition 3.4.11.

The class of functions in $\Gamma_\varepsilon \cap \mathrm{Test}(M)$ with bounded Laplacian is dense in Γ_ε w.r.t. strong convergence in $W^{1,2}(M)$, see Lemma 3.2.73. ■

Lemma 4.4.20. *For every compact* $V \subset \rho(\vec{\Delta}_2)$*, there exist* $\varepsilon \in (0,1)$ *and a constant* $C < \infty$ *such that for every* $\xi \in V$ *and every* $\psi \in \Gamma_\varepsilon$*,* $\xi \in \rho(\mathrm{e}^{\psi}\, \vec{\Delta}_2\, \mathrm{e}^{-\psi})$ *with*

$$\big\|(\mathrm{e}^{\psi}\, \vec{\Delta}_2\, \mathrm{e}^{-\psi} - \xi)^{-1}\big\|_{L^2(T^*M);L^2(T^*M)} \leq C. \tag{4.4.9}$$

Proof. Given any $\varepsilon \in (0,1)$ to be determined later and $\psi \in \Gamma_\varepsilon$, the operator

$$\mathsf{T}_\psi := \mathrm{e}^{\psi}\, \vec{\Delta}_2\, \mathrm{e}^{-\psi} - \vec{\Delta}_2$$

is well-defined on $\mathrm{Test}(T^*M) + \mathrm{i}\,\mathrm{Test}(T^*M)$ and therefore a densely defined linear operator on the complexified vector space $L^2(T^*M) + \mathrm{i}\,L^2(T^*M)$. (4.4.8) yields

$$\int_M \langle \mathsf{T}_\psi\omega, \overline{\omega}\rangle \,\mathrm{d}\mathfrak{m} = -\int_M |\nabla\psi|^2\, |\omega|^2 \,\mathrm{d}\mathfrak{m} + 2\mathrm{i}\int_M \Im\big(\nabla\omega^\sharp(\nabla\psi, \overline{\omega}^\sharp)\big) \,\mathrm{d}\mathfrak{m}$$

for every $\omega \in \mathrm{Test}(T^*M) + \mathrm{i}\,\mathrm{Test}(T^*M)$. Young's inequality and Lemma 3.6.8 thus give

$$\begin{aligned}\Big|\int_M \langle \mathsf{T}_\psi \omega, \overline{\omega}\rangle \,\mathrm{d}\mathfrak{m}\Big| &\leq \varepsilon^2 \,\big\|\omega\big\|^2_{L^2(T^*M)} + 2\varepsilon \int_M |\nabla\omega^\sharp|_{\mathrm{HS}}\, |\omega| \,\mathrm{d}\mathfrak{m}\\ &\leq (\varepsilon+1)\,\varepsilon\,\big\|\omega\big\|^2_{L^2(T^*M)} + \varepsilon \int_M \big|\nabla\omega^\sharp\big|^2_{\mathrm{HS}} \,\mathrm{d}\mathfrak{m}\\ &\leq (\varepsilon+1-K)\,\varepsilon\,\big\|\omega\big\|^2_{L^2(T^*M)} + \varepsilon \int_M \langle \vec{\Delta}_2\omega, \omega\rangle \,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Given any compact $V \subset \rho(\vec{\Delta}_2)$, there exists $\varepsilon \in (0,1)$ such that for every $\xi \in V$,

$$\begin{aligned}&2\,\big\|\big((\varepsilon+1-K)\,\varepsilon + \varepsilon\,\vec{\Delta}_2\big)\big(\vec{\Delta}_2 - \xi\big)^{-1}\big\|_{L^2(T^*M);L^2(T^*M)}\\ &\qquad \leq 2\varepsilon\,\big(3+|K|+|\xi|\big)\,\big\|\big(\vec{\Delta}_2-\xi\big)^{-1}\big\|_{L^2(T^*M);L^2(T^*M)} + 2\varepsilon < 1.\end{aligned}$$

From the above form boundedness of T_ψ by $\vec{\Delta}_2$, which is uniform in $\psi \in \Gamma_\varepsilon$, and under the previous choice of ε, [Kat95, Thm. VI.3.9] both gives $\xi \in \rho(\mathrm{e}^{\psi}\,\vec{\Delta}_2\,\mathrm{e}^{-\psi})$ for every $\psi \in \Gamma_\varepsilon$ and provides us with the existence of a finite constant C such that (4.4.9) holds uniformly in $\xi \in V$ and $\psi \in \Gamma_\varepsilon$. □

Recall that every real $\alpha < 0$ belongs to $\rho(\vec{\Delta}_2)$. Therefore, using (4.2.2) in the first case, the operators $(\vec{\Delta}_2 - \alpha)^{-1/2}\,\mathrm{e}^{-\psi}\,\phi_1$ and $(\vec{\Delta}_2-\alpha)^{-1/2}\,\mathrm{e}^{-\psi}$ are well-defined on $L^\infty_{\mathrm{bs}}(T^*M)$, hence densely defined on $L^p(T^*M)$ for every $p \in [1,\infty)$.

Lemma 4.4.21. *There exists $\alpha < 0$ such that for every $\varepsilon \in (0,1)$, there exist an even $n \in \mathbf{N}$ and a constant $C < \infty$ such that for every $\psi \in \Gamma_\varepsilon$, we have*

$$\begin{aligned}\big\|\mathrm{e}^{\psi}\,(\vec{\Delta}_2-\alpha)^{-n/2}\,\mathrm{e}^{-\psi}\,\phi_1\big\|_{L^1(T^*M);L^2(T^*M)} &\leq C,\\ \big\|\phi_1\,\mathrm{e}^{\psi}\,(\vec{\Delta}_2-\alpha)^{-n/2}\,\mathrm{e}^{-\psi}\big\|_{L^2(T^*M);L^\infty(T^*M)} &\leq C.\end{aligned}$$

Proof. Fix any real $\alpha < 0$ to be determined later, an arbitrary $\beta \in \mathbf{R}$ as well as an even $n \in \mathbf{N}$ with $n \geq \lfloor N+4\rfloor$. Employing the formula (4.4.6) and then Theorem 4.1.1, (4.2.4) as well as (4.2.1), there exist constants $c, C_1, C_2, C_3 < \infty$ with

$$\begin{aligned}\big|(\vec{\Delta}_2-\alpha)^{-n/2}\eta\big| &\leq c\int_0^\infty \mathrm{e}^{\alpha t}\,t^{n/2-1}\,|\mathsf{H}_{2,t}\eta|\,\mathrm{d}t\\ &\leq c\int_0^\infty \mathrm{e}^{(\alpha-K)t}\,t^{n/2-1}\int_M \mathsf{p}_t(\cdot,y)\,|\eta|(y)\,\mathrm{d}\mathfrak{m}(y)\,\mathrm{d}t\\ &\leq c\,C_1\int_0^\infty \Big[\mathrm{e}^{(\alpha-K+C_2)t}\,t^{n/2-1}\,\mathfrak{m}\big[B_{\sqrt{t}}(\cdot)\big]^{-1}\\ &\qquad\qquad \times \int_M \mathrm{e}^{-\mathsf{d}^2(\cdot,y)/5t}\,|\eta|(y)\,\mathrm{d}\mathfrak{m}(y)\Big]\,\mathrm{d}t\\ &\leq c\,C_1\,C_3\int_0^\infty \Big[\mathrm{e}^{(\alpha-K+C_2)t}\,t^{n/2-1}\max\big\{t^{-N/2},1\big\}\,\phi_1^{-2}\\ &\qquad\qquad \int_M \mathrm{e}^{-\mathsf{d}^2(x,y)/5t}\,|\eta|(y)\,\mathrm{d}\mathfrak{m}(y)\Big]\,\mathrm{d}t\\ &\leq c\,C_1\,C_3\,\Big[\int_0^\infty \mathrm{e}^{(\alpha-K+C_2+5\beta^2/4)t}\,t^{n/2-1}\,\max\big\{t^{-N/2},1\big\}\,\mathrm{d}t\Big]\,\phi_1^{-2}\\ &\qquad\qquad \times\int_M \mathrm{e}^{-\beta\mathsf{d}(\cdot,y)}\,|\eta|(y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.}\end{aligned}$$

for every $\eta \in L^2(T^*M)$. In the last inequality, we also used that

$$\frac{5t}{4}\,\beta^2 - \beta\,\mathsf{d}(x,y) + \frac{\mathsf{d}^2(x,y)}{5t} \geq 0.$$

Setting $\alpha := \min\{-1 - K + C_2 + 5\beta^2/4, -1\}$ gives the existence of some $C_4 < \infty$ with

$$\big|(\vec{\Delta}_2 - \alpha)^{-n/2}\eta\big| \leq C_4\,\phi_1^{-2}\int_M \mathrm{e}^{-\beta\mathsf{d}(\cdot,y)}\,|\eta|(y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.} \tag{4.4.10}$$

The next step is to use (4.4.10) subject to a particular choice of $\beta \in \mathbf{R}$ to be determined later. Let $\varepsilon \in (0,1)$ and $\psi \in \Gamma_\varepsilon$ be arbitrary. Since $\mathrm{e}^{\psi}\,(\vec{\Delta}_2 - \alpha)^{-n/2}\,\mathrm{e}^{-\psi}\,\phi_1$ is the formal adjoint of $\phi_1\,\mathrm{e}^{\psi}\,(\vec{\Delta}_2 - \alpha)^{-n/2}\,\mathrm{e}^{-\psi}$, the first estimate will actually follow from the second inequality. To prove the latter, for every $\omega \in L^\infty_{\mathrm{bs}}(T^*M)$, inserting $\eta := \mathrm{e}^{-\psi}\,\omega$ into (4.4.10) for arbitrary $\beta > \varepsilon$ and using (4.4.7) yields

$$\begin{aligned}
&\big|\phi_1\,\mathrm{e}^{\psi}\,(\vec{\Delta}_2 - \alpha)^{-n/2}\,\mathrm{e}^{-\psi}\,\omega\big| \\
&\qquad \leq C_4\,\phi_1^{-1}\,\mathrm{e}^{\psi}\int_M \mathrm{e}^{-\beta\mathsf{d}(\cdot,y)}\,\mathrm{e}^{-\psi(y)}\,|\omega|(y)\,\mathrm{d}\mathfrak{m}(y) \\
&\qquad \leq C_4\left[\phi_1^{-2}\int_M \mathrm{e}^{-2(\beta-\varepsilon)\mathsf{d}(\cdot,y)}\,\mathrm{d}\mathfrak{m}(y)\right]^{1/2}\|\omega\|_{L^2(T^*M)} \\
&\qquad \leq C_4\Big[\sum_{j\in\mathbf{N}} \mathrm{e}^{-2(\beta-\varepsilon)(j-1)}\,\mathfrak{m}[B_j(\cdot)]\,\mathfrak{m}[B_1(\cdot)]^{-1}\Big]^{1/2}\|\omega\|_{L^2(T^*M)} \quad \mathfrak{m}\text{-a.e.}
\end{aligned}$$

By (4.2.1), the last sum is uniformly bounded uniformly on M and in $\varepsilon \in (0,1)$ as soon as $\beta > 0$ is chosen large enough. The previous inequality for arbitrary $\omega \in L^2(T^*M)$ follows by density of $L^\infty_{\mathrm{bs}}(T^*M)$, after possibly passing to pointwise $\mathfrak{m}$-a.e. convergent subsequences. □

Corollary 4.4.22. *For every compact $V \subset \rho(\vec{\Delta}_2)$, there exist $\varepsilon \in (0,1)$, an even $n \in \mathbf{N}$ and a constant $C < \infty$ such that for every $\xi \in V$, one has*

$$\big\|(\vec{\Delta}_2 - \xi)^{-n}\big\|_{L^\infty(T^*M),L^\infty(T^*M)} \leq C\,\sup_{x\in M}\int_M \mathrm{e}^{-\varepsilon\mathsf{d}(x,y)}\,\phi_1^{-1}(x)\,\phi_1^{-1}(y)\,\mathrm{d}\mathfrak{m}(y).$$

Proof. Let $V \subset \rho(\vec{\Delta}_2)$ be compact, and let $\varepsilon \in (0,1)$ be as provided by Lemma 4.4.20 and $n \in \mathbf{N}$ be as provided by Lemma 4.4.21. For every $\xi \in V$ and every $\psi \in \Gamma_\varepsilon$, the first revolvent identity gives

$$\begin{aligned}
&\phi_1\,\mathrm{e}^{\psi}\,(\vec{\Delta}_2 - \xi)^{-n}\,\mathrm{e}^{-\psi}\,\phi_1 \\
&\qquad = \sum_{j=0}^{n}\left[\binom{n}{j}(\xi-\alpha)^j\,\big(\phi_1\,\mathrm{e}^{\psi}\,(\vec{\Delta}_2-\alpha)^{-n/2}\,\mathrm{e}^{-\psi}\big)\,\big(\mathrm{e}^{\psi}\,(\vec{\Delta}_2-\xi)^{-1}\,\mathrm{e}^{-\psi}\big)^j\right. \\
&\qquad\qquad\qquad \left.\big(\mathrm{e}^{\psi}\,(\vec{\Delta}_2-\alpha)^{-n/2}\,\mathrm{e}^{-\psi}\,\phi_1\big)\right].
\end{aligned}$$

By Lemma 4.4.20 and Lemma 4.4.21 we find a constant $C < \infty$ such that for every $\xi \in V$ and $\psi \in \Gamma_\varepsilon$,

$$\big\|\phi_1\,\mathrm{e}^{\psi}\,(\vec{\Delta}_2 - \xi)^{-n}\,\mathrm{e}^{-\psi}\,\phi_1\big\|_{L^1(T^*M),L^\infty(T^*M)} \leq C.$$

Hence, $\phi_1\,\mathrm{e}^{\psi}\,(\vec{\Delta}_2-\xi)^{-n}\,\mathrm{e}^{-\psi}\,\phi_1$ is representable as an integral operator in the sense of Theorem 4.5.3 — in particular, for every $\eta\in L^\infty_{\mathrm{bs}}(T^*M)$, we obtain

$$\big|\phi_1\,\mathrm{e}^{\psi}\,(\vec{\Delta}_2-\xi)^{-n}\,\mathrm{e}^{-\psi}\,\phi_1\eta\big|\le C\int_M|\eta|(y)\,\mathrm{d}\mathfrak{m}(y)\quad\mathfrak{m}\text{-a.e.}$$

Setting $\eta:=\phi_1^{-1}\,\mathrm{e}^{\psi}\,\omega$, where $\omega\in L^\infty_{\mathrm{bs}}(T^*M)$ is arbitrary,

$$\big|(\vec{\Delta}_2-\xi)^{-n}\omega\big|\le C\int_M\mathrm{e}^{-\psi}\,\mathrm{e}^{\psi(y)}\,\phi_1^{-1}\,\phi_1^{-1}(y)\,|\omega|(y)\,\mathrm{d}\mathfrak{m}(y)\quad\mathfrak{m}\text{-a.e.}$$

By the arbitrariness of $\psi\in\Gamma_\varepsilon$ and (4.4.7), we obtain

$$\big|(\vec{\Delta}_2-\xi)^{-n}\omega\big|\le C\int_M\mathrm{e}^{-\varepsilon\mathsf{d}(\cdot,y)}\,\phi_1^{-1}\,\phi_1^{-1}(y)\,|\omega|(y)\,\mathrm{d}\mathfrak{m}(y)\quad\mathfrak{m}\text{-a.e.}$$

The latter estimate is indeed true for every $\omega\in L^\infty(T^*M)$ by an elementary cutoff argument, which establishes the desired assertion. □

Proof of Theorem 4.4.18. Fix an arbitrary $p\in[1,2)\cup(2,\infty]$. We concentrate on the inclusion $\sigma(\vec{\Delta}_p)\subset\sigma(\vec{\Delta}_2)$. The inclusion $\sigma(\vec{\Delta}_p)\supset\sigma(\vec{\Delta}_2)$ follows as for [Cha05, Prop. 9], and the argument for the isolated eigenvalues is the same as in (the references given in) the proof of [HV86, Prop. 2.2].

Let $V\subset\rho(\vec{\Delta}_2)$ be compact with $V\cap(-\infty,0)\neq\emptyset$. Let $n\in\mathbf{N}$ be as in Corollary 4.4.22. Since $\mathfrak{m}$ is uniformly subexponentially integrable, by Corollary 4.4.22 and taking adjoints, we see that $(\vec{\Delta}_2-\xi)^{-n}$ is bounded from $L^p(T^*M)$ into $L^p(T^*M)$ for $p\in\{1,\infty\}$. By Riesz–Thorin's interpolation theorem, $(\vec{\Delta}_2-\xi)^{-n}$ is actually bounded from $L^p(T^*M)$ into $L^p(T^*M)$ for every $p\in[1,\infty]$.

By (4.4.6) and since $\mathsf{H}_{2,t}=\mathsf{H}_{p,t}$ on $L^2(T^*M)\cap L^p(T^*M)$ for every $t\ge 0$, we get

$$(\vec{\Delta}_2-\xi)^{-n}=(\vec{\Delta}_p-\xi)^{-n}\quad\text{on }L^2(T^*M)\cap L^p(T^*M)\tag{4.4.11}$$

for every $\xi\in\rho(\vec{\Delta}_2)\cap(-\infty,0)$. Since $V\cap(-\infty,0)\neq\emptyset$ and the map $\xi\mapsto(\vec{\Delta}_2-\xi)^{-n}$ is analytic on $\rho(\vec{\Delta}_2)$, (4.4.11) holds for every $\xi\in V$. Thus, $(\vec{\Delta}_p-\xi)^{-n}$ extends to a bounded linear operator from $L^p(T^*M)$ into $L^p(T^*M)$ for every $\xi\in V$, with

$$(\vec{\Delta}_p-\xi)^{-n}=(\vec{\Delta}_2-\xi)^{-n}\quad\text{on }L^p(T^*M).$$

It follows that $V\subset\rho(\vec{\Delta}_p)$. Taking complements, we deduce the claimed inclusion. □

Example 4.4.23. Let $N\in\mathbf{N}$ and $K<0$. The N-dimensional hyperbolic space $\mathbf{H}^N_K$ with constant sectional curvature K is an $\mathrm{RCD}^*(K(N-1),N)$ space when endowed with its Riemannian distance and Riemannian volume measure. In this situation, it is due to [Cha05, Thm. 14] that the set $\sigma(\vec{\Delta}_p)$ *does* depend on $p\in[1,\infty]$. ■

4.5 Heat kernel

4.5.1 Dunford–Pettis' theorem

A crucial step in proving the existence of a heat kernel is a Dunford–Pettis-type theorem for co- or contravariant objects, see Theorem 4.5.3 below. See [Cha05, Lem. 11] for a smooth analogue obtained via computations in local coordinates. Our Theorem 4.5.3 significantly enlarges the scope of the latter to the language of L^1-normed L^∞-modules from Subsection 3.2.3 (and although in this Chapter 4, we work in the metric measure space setting, the discussion from this subsection is valid on any measure space over which suitable L^∞-modules are placed).

The following definition provides the setting for our understanding of integral operators on an L^p-normed L^∞-module $\mathscr{M}$, $p \in [1,\infty]$.

Definition 4.5.1. *Given any $p, r \in [1,\infty]$, let $\mathscr{M}$ be a separable L^p-normed L^∞-module, and $\mathscr{N}$ be a separable L^r-normed L^∞-module. Let $\mathscr{M}^0$ and $\mathscr{N}^0$ be their corresponding L^0-modules as introduced in Subsection 3.2.3. We denote by $\mathscr{N}^0 \boxtimes \mathscr{M}^0$ the space of all L^0-bilinear maps $\mathsf{a}\colon \mathscr{N}^0 \times \mathscr{M}^0 \to L^0(M^2)$. In the case $p = r$ and $\mathscr{M} = \mathscr{N}$, we briefly write $(\mathscr{M}^0)^{\boxtimes 2} := \mathscr{M}^0 \boxtimes \mathscr{M}^0$. Lastly, for $\mathsf{a} \in \mathscr{N}^0 \boxtimes \mathscr{M}^0$, we define the $\mathfrak{m}^{\otimes 2}$-measurable function $|\mathsf{a}|_\otimes \colon M^2 \to [0,\infty]$ by*

$$|\mathsf{a}|_\otimes(x,y) := \operatorname{esssup}\big\{|\mathsf{a}[s,v]|(x,y) : s \in \mathscr{N}^0,\ v \in \mathscr{M}^0,\ |s|,|v| \leq 1\ \mathfrak{m}\text{-}a.e.\big\}.$$

A crucial ingredient for Theorem 4.5.3 is the subsequent result from [DP40, Thm. 2.2.5]. Its advantage compared to the more general result [DS58, Thm. VI.8.6] — providing a similar statement with the Banach dual of *any* separable Banach space as target domain — is described in Remark 4.5.4.

Proposition 4.5.2. *Assume that $\mathsf{B}\colon L^1(M) \to L^\infty(M)$ is a linear and bounded map. Then there exists an $\mathfrak{m}^{\otimes 2}$-measurable kernel $\mathsf{b}\colon M^2 \to \mathbf{R}$ such that*

$$\|\mathsf{b}\|_{L^\infty(M^2)} = \|\mathsf{B}\|_{L^1(M),L^\infty(M)}$$

and, for every $g \in L^1(M)$,

$$\mathsf{B}g = \int_M \mathsf{b}(\cdot,y)\, g(y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-}a.e.$$

Such a kernel is unique in the sense that if $\widetilde{\mathsf{b}}\colon M^2 \to \mathbf{R}$ is another $\mathfrak{m}^{\otimes 2}$-measurable kernel fulfilling the foregoing obstructions, then $\widetilde{\mathsf{b}}$ does $\mathfrak{m}^{\otimes 2}$-a.e. coincide with b.

For convenience reasons, we will sometimes write the pairing $\mathsf{L}(v) \in L^1(M)$ of $\mathsf{L} \in \mathscr{M}^*$ and $v \in \mathscr{M}$ as $\langle v \,|\, \mathsf{L}\rangle$ or $v(\mathsf{L})$.

Theorem 4.5.3 (Dunford–Pettis theorem for L^∞-modules)**.** *Let $\mathscr{M}$ and $\mathscr{N}$ be separable L^1-normed L^∞-modules defined over M. Suppose that $\mathsf{A}\colon \mathscr{M} \to \mathscr{N}^*$ is a linear map with $\|\mathsf{A}\|_{\mathscr{M};\mathscr{N}^*} < \infty$. Then there exists $\mathsf{a} \in \mathscr{N}^0 \boxtimes \mathscr{M}^0$ such that*

$$\big\||\mathsf{a}|_\otimes\big\|_{L^\infty(M^2)} = \|\mathsf{A}\|_{\mathscr{M};\mathscr{N}^*}$$

and, for every $v \in \mathscr{M}$ and every $s \in \mathscr{N}$, we have $\mathsf{a}[s,v] \in L^1(M^2)$ with

$$\langle s \,|\, \mathsf{A}v\rangle = \int_M \mathsf{a}[s,v](\cdot,y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-}a.e.$$

The element a is unique in the sense that for any $\widetilde{\mathsf{a}} \in \mathscr{N}^0 \boxtimes \mathscr{M}^0$ satisfying the foregoing obstructions, $\mathsf{a}[s,v] = \widetilde{\mathsf{a}}[s,v]$ holds $\mathfrak{m}^{\otimes 2}$-a.e. for every $v \in \mathscr{M}^0$ and every $s \in \mathscr{N}^0$.

Proof. **Step 1.** *Integral kernel for the pointwise pairing with A.* Let $\mathscr{D}_\infty$, $\mathscr{M}_\infty$ and $\mathscr{N}_\infty$ be countable dense subsets in $L^0(M)$, $\mathscr{M}$ and $\mathscr{N}$ made of $\mathfrak{m}$-essentially bounded elements. We may and will assume that $1_M, 0 \in \mathscr{D}_\infty$. Let $\mathscr{D}_\diamond$ be the smallest algebra of functions in $L^0(M)$ w.r.t. pointwise multiplication which contains $\mathscr{D}_\infty$. Furthermore, let $\mathscr{M}_\diamond$ and $\mathscr{N}_\diamond$ be the sets of all finite linear combinations of elements of the form $f\,v$ and $f\,s$, respectively, where $f \in \mathscr{D}_\diamond$, $v \in \mathscr{M}_\infty$ and $s \in \mathscr{N}_\infty$. The classes $\mathscr{D}_\diamond$, $\mathscr{M}_\diamond$ and $\mathscr{N}_\diamond$ are all countable, consist of $\mathfrak{m}$-essentially bounded elements, and are dense in $L^0(M)$, $\mathscr{M}^0$ and $\mathscr{N}^0$, respectively.

Given any $v \in \mathscr{M}_\diamond$ and any $s \in \mathscr{N}_\diamond$, define $\mathsf{B}[s,v]\colon L^1(M) \to L^\infty(M)$ by

$$\mathsf{B}[s,v]g := \big\langle s \,\big|\, \mathsf{A}(g\,v)\big\rangle.$$

The map $\mathsf{B}[s,v]$ is clearly linear, and it is well-defined and bounded since

$$\begin{aligned}\big\|\mathsf{B}[s,v]g\big\|_{L^\infty(M)} &\leq \big\|\mathsf{A}(g\,v)\big\|_{\mathscr{N}^*}\,\big\||s|\big\|_{L^\infty(M)} \leq \|\mathsf{A}\|_{\mathscr{M},\mathscr{N}^*}\,\|g\,v\|_{\mathscr{M}}\,\big\||s|\big\|_{L^\infty(M)}\\ &\leq \|\mathsf{A}\|_{\mathscr{M},\mathscr{N}^*}\,\|g\|_{L^1(M)}\,\big\||v|\big\|_{L^\infty(M)}\,\big\||s|\big\|_{L^\infty(M)}.\end{aligned}$$

By Proposition 4.5.2, there exists a kernel $\mathsf{b}[s,v] \in L^\infty(M^2)$, $\mathfrak{m}^{\otimes 2}$-a.e. uniquely determined in a proper way, such that for every $g \in L^1(M)$,

$$\mathsf{B}[s,v]g = \int_M \mathsf{b}[s,v](\cdot,y)\,g(y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.} \tag{4.5.1}$$

Step 2. *Properties of the obtained integral kernel.* An immediate property coming from the fact that $\mathsf{B}[s,c\,v]g = \mathsf{B}[s,v](c\,g)$ and $\mathsf{B}[d\,s,v]g = d\,\mathsf{B}[s,v]g$ for every $g \in L^1(M)$ and every $c,d \in L^\infty(M)$, and the $\mathfrak{m}^{\otimes 2}$-a.e. uniqueness of the induced integral kernel is the following bilinearity. For every $c,d \in \mathscr{D}_\diamond$, every $v,v' \in \mathscr{M}_\diamond$ and every $s,s' \in \mathscr{N}_\diamond$, we have

$$\begin{aligned}\mathsf{b}[d\,s+s', c\,v+v'] &= d(\mathrm{pr}_1)\,c(\mathrm{pr}_2)\,\mathsf{b}[s,v] + d(\mathrm{pr}_1)\,\mathsf{b}[s,v']\\ &\quad + c(\mathrm{pr}_2)\,\mathsf{b}[s',v] + \mathsf{b}[s',v'] \quad \mathfrak{m}^{\otimes 2}\text{-a.e.}\end{aligned} \tag{4.5.2}$$

Next, for every $v \in \mathscr{N}_\diamond$ and every $s \in \mathscr{N}_\diamond$, we claim that

$$|\mathsf{b}[s,v]| \leq \|\mathsf{A}\|_{\mathscr{M};\mathscr{N}^*}\,|s|(\mathrm{pr}_1)\,|v|(\mathrm{pr}_2) \quad \mathfrak{m}^{\otimes 2}\text{-a.e.} \tag{4.5.3}$$

Indeed, let $g,h \in L^1(M)$ be nonnegative. Multiplying both sides of the identity (4.5.1) with h and integrating w.r.t. $\mathfrak{m}$ yields

$$\begin{aligned}\int_{M^2} \mathsf{b}[s,v](x,y)\,g(y)\,h(x)\,\mathrm{d}\mathfrak{m}^{\otimes 2}(x,y) &= \int_M \mathsf{B}[s,v]g(x)\,h(x)\,\mathrm{d}\mathfrak{m}(x)\\ &\leq \int_M |\mathsf{A}(g\,v)|(x)\,|s|(x)\,h(x)\,\mathrm{d}\mathfrak{m}(x)\\ &\leq \int_{M^2} \|\mathsf{A}\|_{\mathscr{M},\mathscr{N}^*}\,|v|(y)\,|s|(x)\,g(y)\,h(x)\,\mathrm{d}\mathfrak{m}^{\otimes 2}(x,y).\end{aligned} \tag{4.5.4}$$

Changing the sign in (4.5.4), the claim follows by the arbitrariness of g and h.

Step 3. *Definition of* a. Since the topology of $\mathscr{M}^0$ and $\mathscr{N}^0$ is intrinsic, in the sense indicated in the L^0-module part from Subsection 3.2.3, we consider the distances $\mathsf{d}_{\mathscr{M}^0}$ and $\mathsf{d}_{\mathscr{N}^0}$ as defined w.r.t. a fixed partition $(E_j)_{j\in\mathbf{N}}$ of M into Borel subsets of finite and positive $\mathfrak{m}$-measure. Then $(E_j^2)_{j\in\mathbf{N}}$ is a partition of M^2 into Borel sets of finite and positive $\mathfrak{m}^{\otimes 2}$-measure w.r.t. which we define the distance $\mathsf{d}_{L^0(M^2)}$ on $L^0(M^2)$ analogously to (3.2.5).

If $v \in \mathscr{M}_\diamond$ and $s \in \mathscr{N}_\diamond$, we define

$$\mathsf{a}[s,v] := \mathsf{b}[s,v].$$

Next, let any $v \in \mathscr{M}^0$ and $s \in \mathscr{N}^0$ satisfy $|v|, |s| \in L^\infty(M)$. By density of $\mathscr{M}_\diamond$ in $\mathscr{M}$ and by the definition of $\mathscr{M}^0$, there exists a sequence $(v_n)_{n\in\mathbf{N}}$ in $\mathscr{N}_\diamond$ converging to v w.r.t. $\mathsf{d}_{\mathscr{M}^0}$. We extract a sequence $(s_i)_{i\in\mathbf{N}}$ in $\mathscr{N}_\diamond$ converging to s w.r.t. $\mathsf{d}_{\mathscr{N}^0}$. Define

$$C := \max\big\{\|\mathsf{A}\|_{\mathscr{M},\mathscr{N}^*} + 1, \||v|\|_{L^\infty} + 1, \||s|\|_{L^\infty} + 1\big\}.$$

Given any $\varepsilon > 0$, select $L \in \mathbf{N}$ such that, for every $n, n', i, i' \geq L$,

$$\max\big\{\mathsf{d}_{\mathscr{M}^0}(v_n, v_{n'}),\ \mathsf{d}_{\mathscr{M}^0}(v, v_{n'}),\ \mathsf{d}_{\mathscr{N}^0}(s_i, s_{i'}),\ \mathsf{d}_{\mathscr{N}^0}(s_i, s)\big\} \leq \frac{\varepsilon}{6C^2}.$$

Using the elementary facts that

$$\begin{aligned}\min\{a+b, 1\} &\leq \min\{a, 1\} + \min\{b, 1\},\\ \min\{ab, 1\} &\leq \min\{a, 1\} + \min\{b, 1\}\end{aligned}$$

for every $a, b \in [0, \infty)$ as well as (4.5.2) and (4.5.3) thus yields

$$\begin{aligned}
&\sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]^2} \int_{E_j^2} \min\big\{|\mathsf{a}[s_i, v_n] - \mathsf{a}[s_{i'}, v_{n'}]|, 1\big\}\,\mathrm{d}\mathfrak{m}^{\otimes 2}\\
&\quad\leq \sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]^2} \int_{E_j^2} \min\big\{|\mathsf{a}[s_i, v_n - v_{n'}]|, 1\big\}\,\mathrm{d}\mathfrak{m}^{\otimes 2}\\
&\qquad\quad + \sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]^2} \int_{E_j^2} \min\big\{|\mathsf{a}[s_i - s_{i'}, v_{n'}]|, 1\big\}\,\mathrm{d}\mathfrak{m}^{\otimes 2}\\
&\quad\leq C \sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]^2} \int_{E_j^2} \min\big\{|s_i|(\mathrm{pr}_1)\,|v_n - v_n'|(\mathrm{pr}_1), 1\big\}\,\mathrm{d}\mathfrak{m}^{\otimes 2}\\
&\qquad\quad + C \sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]^2} \int_{E_j^2} \min\big\{|s_i - s_{i'}|(\mathrm{pr}_1)\,|v_{n'}|(\mathrm{pr}_2), 1\big\}\,\mathrm{d}\mathfrak{m}^{\otimes 2}\\
&\quad\leq C \sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]^2} \int_{E_j^2} \min\big\{|s_i - s|(\mathrm{pr}_1)\,|v_n - v_n'|(\mathrm{pr}_2), 1\big\}\,\mathrm{d}\mathfrak{m}^{\otimes 2}\\
&\qquad\quad + C^2 \sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]} \int_{E_j} \min\big\{|v_n - v_n'|, 1\big\}\,\mathrm{d}\mathfrak{m}\\
&\qquad\quad + C \sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]^2} \int_{E_j^2} \min\big\{|s_i - s_{i'}|(\mathrm{pr}_1)\,|v_{n'} - v|(\mathrm{pr}_2), 1\big\}\,\mathrm{d}\mathfrak{m}^{\otimes 2}\\
&\qquad\quad + C^2 \sum_{j\in\mathbf{N}} \frac{2^{-j}}{\mathfrak{m}[E_j]} \int_{E_j} \min\big\{|s_i - s_{i'}|, 1\big\}\,\mathrm{d}\mathfrak{m} \leq \varepsilon.
\end{aligned}$$

Thus $(\mathsf{b}[s_i, v_n])_{n,i\in\mathbf{N}}$ is a Cauchy sequence in $L^0(M^2)$ — we define $\mathsf{a}[s, v]$ as its $\mathfrak{m}^{\otimes 2}$-a.e. unique limit in $L^0(M^2)$. A similar argument shows that this definition of $\mathsf{a}[s, v]$ is independent of the particularly chosen approximating sequences in $\mathscr{M}_\diamond$ and $\mathscr{N}_\diamond$, respectively. Moreover, the identities (4.5.2), for arbitrary $c, d \in L^\infty(M)$, and (4.5.3) remain true for b replaced by a.

Lastly, for arbitrary $v \in \mathscr{M}^0$ and $s \in \mathscr{N}^0$, the sequences $(v_n)_{n\in\mathbf{N}}$ and $(s_i)_{i\in\mathbf{N}}$ given by $v_n := [1_{[0,n]} \circ |v|]\, v$ and $s_i := [1_{[0,i]} \circ |s|]\, s$ converge to v and s in $\mathscr{M}^0$ and $\mathscr{N}^0$, respectively. Indeed, observe that $|v - v_n| \to 0$ pointwise $\mathfrak{m}$-a.e. as $n \to \infty$ and $|s - s_i| \to 0$ pointwise $\mathfrak{m}$-a.e. as $i \to \infty$, and the claim follows since pointwise

$\mathfrak{m}$-a.e. convergent sequences converge in measure on finite measure spaces. By (4.5.2) and (4.5.3), $(\mathsf{b}[s_i, v_n])_{n,i\in\mathbf{N}}$ is a Cauchy sequence in $L^0(M^2)$ — we again define $\mathsf{a}[s, v]$ as its $\mathfrak{m}^{\otimes 2}$-a.e. unique limit. Once again, it is easily seen that this definition does not depend on the chosen approximating sequences for v and s, respectively, and that (4.5.2), for arbitrary $c, d \in L^0(M)$, and (4.5.3) hold for a instead of b.

Step 4. *Properties of* a. According to the previous Step 3, we already know that $\mathsf{a} \in \mathscr{N}^0 \boxtimes \mathscr{M}^0$. Moreover, from (4.5.3) for a, it already follows that $\mathsf{a}[s, v] \in L^1(M^2)$ for every $v \in \mathscr{M}$ and every $s \in \mathscr{N}$, and that

$$\big\| |\mathsf{a}|_\otimes \big\|_{L^\infty(M^2)} \leq \|\mathsf{A}\|_{\mathscr{M};\mathscr{N}^*}.$$

To show the claimed integral identity, let $z \in M$. For any sequences $(v_n)_{n\in\mathbf{N}}$ in $\mathscr{M}_\diamond$ and $(s_i)_{i\in\mathbf{N}}$ in $\mathscr{N}_\diamond$ converging to v and s in $\mathscr{M}$ and $\mathscr{N}$, respectively, from (4.5.1) we get, for every $n, i, k \in \mathbf{N}$,

$$\big\langle s_i \,\big|\, A(1_{B_k(z)}\, v_n) \big\rangle = \int_{B_k(z)} \mathsf{a}[s_i, v_n](\cdot, y)\, \mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.}$$

Letting $k \to \infty$ with the continuity of A and then $n \to \infty$ and $i \to \infty$, employing that $\mathsf{a}[s_i, v_n] \to \mathsf{a}[s, v]$ in $L^1(M^2)$ by virtue of (4.5.3), the desired claim is deduced.

From this, the inequality

$$\big\| |\mathsf{a}|_\otimes \big\|_{L^\infty(M^2)} \geq \|\mathsf{A}\|_{\mathscr{M};\mathscr{N}^*}$$

follows by observing that for every $v \in \mathscr{M}$, by definition of the pointwise norm in $\mathscr{N}^*$,

$$\begin{aligned} |\mathsf{A}v| &\leq \int_M \operatorname{esssup}\big\{\mathsf{a}[s, v] : s \in \mathscr{N}^0,\ |s| \leq 1\ \mathfrak{m}\text{-a.e.}\big\}\, \mathrm{d}\mathfrak{m} \\ &\leq \big\| |\mathsf{a}|_\otimes \big\|_{L^\infty(M^2)}\, \|v\|_{\mathscr{M}} \quad \mathfrak{m}\text{-a.e.} \end{aligned}$$

The uniqueness statement is clear by L^0-bilinearity of all considered mappings. □

Remark 4.5.4. In the setting of Theorem 4.5.3, the general result [DS58, Thm. VI.8.6] would provide a map a on M, $\mathfrak{m}$-essentially uniquely determined in a proper way, such that $a(y)\colon \mathscr{M} \to \mathscr{N}^*$ is linear for $\mathfrak{m}$-a.e. $y \in M$ and, for every $v \in \mathscr{M}$ and every s,

$$\int_M \big\langle s \,\big|\, \mathsf{A}v \big\rangle \, \mathrm{d}\mathfrak{m} = \int_M \int_M \big\langle s \,\big|\, a(y)v \big\rangle(x)\, \mathrm{d}\mathfrak{m}(x)\, \mathrm{d}\mathfrak{m}(y).$$

However, it is not clear that the map $(x, y) \mapsto \langle s \,|a(y)v\rangle(x)$ is $\mathfrak{m}^{\otimes 2}$-measurable — a property which is implicitly used at many places in the proof of Theorem 4.5.3. Even in functional treatises, this is considered as a delicate detail [GH14, Ch. 3] and explains why we chose the formulation of Definition 4.5.1 with target space $L^0(M^2)$. ■

4.5.2 Explicit construction as integral kernel

We are now in a position to state our main result, valid for any $\mathrm{RCD}(K, \infty)$ space $(M, \mathsf{d}, \mathfrak{m})$, $K \in \mathbf{R}$. On weighted Riemannian manifolds with not necessarily uniform lower Ricci bounds, a version of it has been proven in [Gün17a, Thm. XI.1] using Lebesgue's differentiation theorem and thus local compactness of the underlying space, an assumption we do not make. See also [Gri09] for a thorough functional treatment.

Theorem 4.5.5. *There exists a mapping* $\mathsf{h}\colon (0,\infty) \to L^0(T^*M)^{\boxtimes 2}$ *such that for every* $p,q \in [1,\infty]$ *with* $1/p + 1/q = 1$*, if* $\omega \in L^p(T^*M)$ *and* $\eta \in L^q(T^*M)$*, and every* $t > 0$ *we have* $\mathsf{h}_t[\eta,\omega] \in L^1(M^2)$ *with*

$$\langle \eta, \mathsf{H}_t\omega\rangle = \int_M \mathsf{h}_t[\eta,\omega](\cdot,y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.}$$

The previous mapping h *is uniquely determined in the sense that for every mapping* $\widetilde{\mathsf{h}}\colon (0,\infty) \to L^0(T^*M)^{\boxtimes 2}$ *satisfying the foregoing obstructions, for every* $\omega,\eta \in L^0(T^*M)$ *and every* $t > 0$*, the identity* $\mathsf{h}_t[\eta,\omega] = \widetilde{\mathsf{h}}_t[\eta,\omega]$ *holds* $\mathfrak{m}^{\otimes 2}$*-a.e.*

Proof. **Step 1.** *Kernel for a perturbation of* H_t. Let $t > 0$. Let us define the weight function $\phi_t : M \to \mathbf{R}$, locally bounded by the volume growth property (recall Section 1.2), and the operator $\mathsf{A}_t : L^1_{\mathrm{bs}}(T^*M) \to L^0(T^*M)$ as

$$\begin{aligned}\phi_t(x) &:= \mathfrak{m}\big[B_{\sqrt{t}}(x)\big]^{1/2},\\ \mathsf{A}_t &:= \phi_t\,\mathsf{H}_t\,\phi_t.\end{aligned}$$

By Theorem 4.1.1 and the functional heat kernel bound from Theorem 4.2.3, there exist constants $C_1, C_2 < \infty$ such that for every $\omega \in L^1_{\mathrm{bs}}(T^*M)$,

$$\begin{aligned}|\mathsf{A}_t\omega| &\leq \mathrm{e}^{-Kt}\int_M \phi_t\,\mathsf{p}_t(\cdot,y)\,\phi_t(y)\,|\omega|(y)\,\mathrm{d}\mathfrak{m}(y)\\ &\leq \mathrm{e}^{-Kt}\,\mathrm{e}^{C_1(1+C_2t)}\,\|\omega\|_{L^1(T^*M)} \quad \mathfrak{m}\text{-a.e.}\end{aligned}$$

Therefore, A_t uniquely extends to a bounded and linear operator from $L^1(T^*M)$ into $L^\infty(T^*M)$, whose extension we still denote by A_t. Theorem 4.5.3 thus provides us with some element $\mathsf{a}_t \in L^0(T^*M)^{\boxtimes 2}$, uniquely determined in a proper way, such that for every $\omega,\eta \in L^1(T^*M)$, we have

$$\langle \eta, \mathsf{A}_t\omega\rangle = \int_M \mathsf{a}_t[\eta,\omega](\cdot,y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.}$$

In fact, arguing as in the proof of Theorem 4.5.3,

$$|\mathsf{a}_t|_\otimes \leq \mathrm{e}^{-Kt}\,\phi_t(\mathrm{pr}_1)\,\phi_t(\mathrm{pr}_2)\,\mathsf{p}_t \quad \mathfrak{m}^{\otimes 2}\text{-a.e.} \tag{4.5.5}$$

Step 2. *Removing the weights.* Given the element a_t extracted in the previous step and any $\varepsilon,\iota > 0$, we define $\mathsf{h}_t^{\varepsilon,\iota} \in L^0(T^*M)^{\boxtimes 2}$ through

$$\mathsf{h}_t^{\varepsilon,\iota}[\eta,\omega] := \mathsf{a}_t\Big[\frac{1}{\phi_t+\varepsilon}\,\eta, \frac{1}{\phi_t+\iota}\,\omega\Big].$$

It is clear from (4.5.5) and Theorem 4.2.3 that $\mathsf{h}_t^{\varepsilon,\iota}[\eta,\omega] \in L^1(M^2)$ for every $\omega,\eta \in L^1(T^*M)$. Moreover, if in addition $\omega \in L^1_{\mathrm{bs}}(T^*M)$, then

$$\Big\langle \eta, \frac{\phi_t}{\phi_t+\varepsilon}\,\mathsf{H}_t\Big[\frac{\phi_t}{\phi_t+\iota}\,\omega\Big]\Big\rangle = \int_M \mathsf{h}_t^{\varepsilon,\iota}[\eta,\omega](\cdot,y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.} \tag{4.5.6}$$

Next, observe that for every $\omega,\eta \in L^0(T^*M)$ and every $\varepsilon',\iota' > 0$, by (4.5.5),

$$\big|\mathsf{h}_t^{\varepsilon,\iota}[\eta,\omega] - \mathsf{h}_t^{\varepsilon',\iota'}[\eta,\omega]\big|$$

$$
\begin{aligned}
&\leq \Big|\mathsf{a}_t\Big[\frac{1}{\phi_t+\varepsilon}\,\eta, \frac{1}{\phi_t+\iota}\,\omega - \frac{1}{\phi_t+\iota'}\,\omega\Big]\Big|\\
&\qquad + \Big|\mathsf{a}_t\Big[\frac{1}{\phi_t+\varepsilon}\,\eta - \frac{1}{\phi_t+\varepsilon'}\,\eta, \frac{1}{\phi_t+\iota'}\,\omega\Big]\Big|\\
&\leq \mathrm{e}^{-Kt}\,|\eta|(\mathrm{pr}_1)\,|\omega|(\mathrm{pr}_2)\,\mathsf{p}_t \times \Big|\frac{\phi_t(\mathrm{pr}_2)}{\phi_t(\mathrm{pr}_2)+\iota} - \frac{\phi_t(\mathrm{pr}_2)}{\phi_t(\mathrm{pr}_2)+\iota'}\Big|\\
&\qquad + \Big[\mathrm{e}^{-Kt}\,|\eta|(\mathrm{pr}_1)\,|\omega|(\mathrm{pr}_2)\,\mathsf{p}_t\\
&\qquad\qquad \times \Big|\frac{\phi_t(\mathrm{pr}_1)}{\phi_t(\mathrm{pr}_1)+\varepsilon} - \frac{\phi_t(\mathrm{pr}_1)}{\phi_t(\mathrm{pr}_1)+\varepsilon'}\Big|\Big] \quad \mathfrak{m}^{\otimes 2}\text{-a.e.}
\end{aligned}
$$

Thus, independently of the choice of sequences $(\varepsilon_n)_{n\in\mathbf{N}}$ and $(\iota_n)_{n\in\mathbf{N}}$ in $(0,\infty)$ converging to 0 in place of ε and ι, the two-parameter family $(\mathsf{h}_t^{\varepsilon,\iota}[\eta,\omega])_{\varepsilon,\iota>0}$ has a unique limit in $L^0(M^2)$ — we define $\mathsf{h}_t[\eta,\omega]$ to be this limit and denote by $\mathsf{h}_t \in L^0(T^*M)^{\boxtimes 2}$ the induced element according to Definition 4.5.1.

Step 3. *Properties of* h. Turning to the claimed integral representation of H_t, given any $\omega \in L^p(T^*M)$ and $\eta \in L^q(T^*M)$ where $p,q \in [1,\infty]$ are dual to each other, we integrate (4.5.6) and let $\varepsilon,\iota \to 0$. On the one hand, by Hölder's inequality, Theorem 4.3.1 and Lebesgue's theorem,

$$\lim_{\varepsilon,\iota\to 0}\int_M \Big\langle \eta, \frac{\phi_t}{\phi_t+\varepsilon}\,\mathsf{H}_t\Big[\frac{\phi_t}{\phi_t+\iota}\,\omega\Big]\Big\rangle\,\mathrm{d}\mathfrak{m} = \int_M \langle\eta,\mathsf{H}_t\omega\rangle\,\mathrm{d}\mathfrak{m}.$$

On the other hand, from the construction in the previous step,

$$|\mathsf{h}_t|_\otimes \leq \mathrm{e}^{-Kt}\,\mathsf{p}_t \quad \mathfrak{m}^{\otimes 2}\text{-a.e.}$$

so that a further application of Lebesgue's theorem to (4.5.6) entails

$$\int_M \langle\eta,\mathsf{H}_t\omega\rangle\,\mathrm{d}\mathfrak{m} = \int_{M^2} \mathsf{h}_t[\eta,\omega]\,\mathrm{d}\mathfrak{m}^{\otimes 2}.$$

Replacing η by $f\,\eta$, $f \in L^\infty(M)$, finally gives the claimed pointwise $\mathfrak{m}$-a.e. equality.

The uniqueness statement is as clear as in the proof of Theorem 4.5.3. □

Definition 4.5.6. *We call the mapping* h *provided through Theorem 4.5.5 the* 1-form heat kernel *of* $(M,\mathsf{d},\mathfrak{m})$.

It is straightforward to check the following result using the symmetry and the semigroup property of $(\mathsf{H}_t)_{t\geq 0}$, the Chapman–Kolmogorov formula for the functional heat kernel [AGS14b, Thm. 6.1] as well as Theorem 4.1.1. Theorem 4.5.9 then follows from Theorem 4.2.3 and (4.2.4). (A help in understanding h and its properties could be the formal interpretation "$\mathsf{h}_t[\eta,\omega](x,y) = \mathsf{h}_t(x,y)\,\eta(x)\,\omega(y)$".)

Theorem 4.5.7. *For every $\omega,\eta \in L^0(T^*M)$ and every $s,t > 0$, the* 1*-form heat kernel* h *from Definition 4.5.6 obeys the following relations at $\mathfrak{m}^{\otimes 2}$-a.e. $(x,y) \in M^2$.*

(i) Symmetry. *We have*

$$\mathsf{h}_t[\eta,\omega](x,y) = \mathsf{h}_t[\omega,\eta](y,x)$$

(ii) Pointwise Hess–Schrader–Uhlenbrock inequality. *We have*

$$|\mathsf{h}_t|_\otimes(x,y) \leq \mathrm{e}^{-Kt}\,\mathsf{p}_t(x,y)$$

(iii) Chapman–Kolmogorov equation. *We have*

$$\int_M \mathsf{h}_{t+s}[\eta,\omega](\cdot,y)\,\mathrm{d}\mathfrak{m}(y) = \int_M \mathsf{h}_s[\eta,\mathsf{H}_t\omega](\cdot,y)\,\mathrm{d}\mathfrak{m}(y) \quad \mathfrak{m}\text{-a.e.}$$

Remark 4.5.8. $(\mathsf{H}_t)_{t\geq 0}$ does not localize, whence we cannot state the Chapman–Kolmogorov formula from (iii) in Theorem 4.5.7 as a pointwise $\mathfrak{m}^{\otimes 2}$-a.e. equality. ■

Theorem 4.5.9. *Given any $\varepsilon > 0$, there exist constants $C_1 > 0$, depending only on ε, and $C_2 \geq 0$, depending only on K, with $C_2 := 0$ if $K \geq 0$, such that for $\mathfrak{m}^{\otimes 2}$-a.e. $(x,y) \in M^2$,*

$$|\mathsf{h}_t|_\otimes(x,y) \leq \mathfrak{m}\big[B_{\sqrt{t}}(x)\big]^{-1/2}\,\mathfrak{m}\big[B_{\sqrt{t}}(y)\big]^{-1/2} \exp\Big[C_1\big(1+(C_2-K)t\big) - \frac{\mathsf{d}^2(x,y)}{(4+\varepsilon)t}\Big].$$

In particular, if $(M,\mathsf{d},\mathfrak{m})$ obeys the $\mathrm{RCD}^*(K,N)$ condition, $N \in [1,\infty)$, there exist constants $C_3, C_4 > 1$ depending only on ε, K and N such that at $\mathfrak{m}^{\otimes 2}$-a.e. $(x,y) \in M^2$,

$$|\mathsf{h}_t|_\otimes(x,y) \leq C_3\,\mathfrak{m}\big[B_{\sqrt{t}}(y)\big]^{-1} \exp\Big[(C_4-K)t - \frac{\mathsf{d}^2(x,y)}{(4+\varepsilon)t}\Big].$$

Remark 4.5.10. Following the arguments for Theorem 4.5.5, we deduce from Theorem 3.4.26 that on any $\mathrm{RCD}(K,\infty)$ space, $K \in \mathbf{R}$, the heat flow on vector fields $(\mathsf{T}_t)_{t\geq 0}$ from Subsection 3.4.3 has a heat kernel in the above sense as well. The pointwise operator norm of the latter is $\mathfrak{m}^{\otimes 2}$-a.e. no larger than the heat kernel of $(\mathsf{P}_t)_{t\geq 0}$, compare with Theorem 4.5.7, and the estimates from Theorem 4.5.9 hold for the vector field heat kernel with K replaced by 0. ■

4.5.3 Trace inequality and spectral resolution

Let $(M,\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space, $N \in [1,\infty)$. First, relying on Theorem 4.5.7 and Remark 4.5.8, we prove a trace inequality between H_t and P_t in Theorem 4.5.11, $t > 0$. In the smooth case, the corresponding bound is classical and has important applications to mathematical physics, see e.g. [Gün17a, Cor. XI.8], (1.1) in [HSU80] or [Ros88, Thm. 3.5]. A key feature of the $\mathrm{RCD}(K,N)$ framework is that, thanks to [BS20, GP16, Han18a, MN19], there exists precisely one $n \in \mathbf{N}$ such that $\mathfrak{m}[E_n] > 0$ within the dimensional decomposition $(E_n)_{n\in\mathbf{N}}$ of $L^2(T^*M)$ — actually, n is equal to the *essential dimension* $\dim_{\mathsf{d},\mathfrak{m}} M \in \{1,\dots,\lfloor N\rfloor\}$ of $(M,\mathsf{d},\mathfrak{m})$. See [BS20, MN19] for comprehensive accounts on the latter from the structure theoretic point of view.

Through Theorem 4.4.12, if M is additionally compact, we also prove a spectral resolution identity for h_t, $t > 0$, in Theorem 4.5.13. More precisely, we show that h_t can be viewed as an element in the two-fold Hilbert space tensor product $L^2(T^*M)^{\otimes_{\mathrm{H}} 2}$ of $L^2(T^*M)$ indicated in Section 4.2.

Theorem 4.5.11. *Intending the traces in the usual Hilbert space sense, for every $t > 0$,*

$$\operatorname{tr}\mathsf{H}_t \leq (\dim_{\mathsf{d},\mathfrak{m}} M)\,\mathrm{e}^{-Kt}\,\operatorname{tr}\mathsf{P}_t.$$

Proof. Abbreviate $d := \dim_{\mathsf{d},\mathfrak{m}} M$ and let $B \subset E_d$ be any bounded Borel set with $\mathfrak{m}[B] \in (0,\infty)$. Let $e_1,\dots,e_d \in L^2(T^*M)$ be local orthonormal basis vectors of $L^2(T^*M)$ on B as in Subsection 3.2.3, i.e. $\langle e_i,e_j\rangle = \delta_{ij}$ $\mathfrak{m}$-a.e. on B and $1_{B^c}\,e_i = 0$,

$i, j \in \{1, \dots, d\}$, an. We set $\omega_k := \mathfrak{m}[B]^{-1/2}\, e_k$ with $k \in \{1, \dots, d\}$ and complete this set of 1-forms to a countable orthonormal basis $(\omega_k)_{k \in \mathbf{N}}$ of $L^2(T^*M)$, for which we may and will assume that $1_B\, \langle \omega_k, e_i \rangle = 0$ for every $k > d$ and every $i \in \{1, \dots, d\}$. Indeed, for every $k \in \mathbf{N}$ with $k > d$, $1_B\, \omega_k$ can be written as linear combination of $\omega_1, \dots, \omega_d$.

Given any $\alpha \in \mathbf{N}$ with $\alpha > d$, by Theorem 4.5.7 and Cauchy–Schwarz's inequality,

$$\begin{aligned}
&\sum_{k=1}^{\alpha} \int_M \langle \mathsf{H}_t \omega_k, \omega_k \rangle \,\mathrm{d}\mathfrak{m} \\
&\quad = \sum_{k=1}^{\alpha} \sum_{i,i'=1}^{d} \int_{B^2} \langle \omega_k, e_i \rangle(\mathrm{pr}_1)\, \langle \omega_k, e_{i'} \rangle(\mathrm{pr}_2)\, \mathsf{h}_t[e_i, e_{i'}] \,\mathrm{d}\mathfrak{m}^{\otimes 2} \\
&\quad = \mathfrak{m}[B]^{-1} \sum_{i=1}^{d} \int_{B^2} \mathsf{h}_t[e_i, e_i] \,\mathrm{d}\mathfrak{m}^{\otimes 2} \\
&\quad = \mathfrak{m}[B]^{-1} \sum_{i=1}^{d} \int_{B\times M} \mathsf{h}_{t/2}[e_i, \mathsf{H}_{t/2} e_i] \,\mathrm{d}\mathfrak{m}^{\otimes 2} \\
&\quad \leq \mathfrak{m}[B]^{-1} \sum_{i=1}^{d} \int_{B\times M} |\mathsf{h}_{t/2}|_\otimes\, |\mathsf{H}_{t/2} e_i|(\mathrm{pr}_2) \,\mathrm{d}\mathfrak{m}^{\otimes 2} \\
&\quad \leq \mathfrak{m}[B]^{-1}\, d \int_{B\times M\times B} |\mathsf{h}_{t/2}|_\otimes(\mathrm{pr}_1, \mathrm{pr}_3)\, |\mathsf{h}_{t/2}|_\otimes(\mathrm{pr}_2, \mathrm{pr}_3) \,\mathrm{d}\mathfrak{m}^{\otimes 3} \qquad (4.5.7) \\
&\quad \leq d \int_{B^2} |\mathsf{h}_{t/2}|_\otimes^2 \,\mathrm{d}\mathfrak{m}^{\otimes 2} \\
&\quad \leq d\, \mathrm{e}^{-Kt} \int_{M^2} \mathsf{p}_{t/2}^2 \,\mathrm{d}\mathfrak{m}^{\otimes 2} = d\, \mathrm{e}^{-Kt}\, \mathrm{tr}\, \mathsf{P}_t.
\end{aligned}$$

In (4.5.7), we used Theorem 4.5.5 together with duality for the pointwise norm of $\mathsf{H}_{t/2} e_i$. The last identity straighfollows from the self-adjointness of the functional heat flow in $L^2(M)$. The claim follows by letting $\alpha \to \infty$. □

Remark 4.5.12. With a similar proof as for Theorem 4.5.11, if $(M, \mathsf{d}, \mathfrak{m})$ obeys the more general $\mathrm{RCD}^*(K, N)$ condition, $K \in \mathbf{R}$ and $N \in [1, \infty)$, taking Proposition 3.3.14 or [Han18a, Prop. 3.2] into account, we still have

$$\mathrm{tr}\, \mathsf{H}_t \leq \lfloor N \rfloor\, \mathrm{e}^{-Kt}\, \mathrm{tr}\, \mathsf{P}_t. \qquad \blacksquare$$

Now, we assume compactness of M. Let $(\omega_i)_{i \in \mathbf{N}}$ be any orthonormal basis of $L^2(T^*M)$ consisting of eigenforms for $\vec{\Delta}$ provided by Theorem 4.4.12, i.e. $\omega_i \in \mathrm{E}_{\lambda_i}(\vec{\Delta})$ for some $\lambda_i \in \sigma(\vec{\Delta})$ and every $i \in \mathbf{N}$.

Theorem 4.5.13. *Under the foregoing assumptions, for every $t > 0$ there exists a unique $\mathsf{g}_t \in L^2(T^*M)^{\otimes_{\mathrm{H}} 2}$ such that, for every $\omega, \eta \in L^2(T^*M)$,*

$$\langle \mathsf{g}_t, \eta \otimes_{\mathrm{H}} \omega \rangle = \mathsf{h}_t[\eta, \omega] \quad \mathfrak{m}^{\otimes 2}\text{-a.e.}$$

Moreover, w.r.t. strong convergence in $L^2(T^*M)^{\otimes_{\mathrm{H}} 2}$, g_t admits the representation

$$\mathsf{g}_t = \sum_{i \in \mathbf{N}} \mathrm{e}^{-\lambda_i t}\, \omega_i \otimes_{\mathrm{H}} \omega_i.$$

Proof. By virtue of (4.2.2) and Theorem 4.5.9, we have $|\mathsf{h}_t|_\otimes \in L^\infty(M^2)$. Hence, the bilinear map $\mathsf{G}_t : L^2(T^*M)^2 \to \mathbf{R}$ is well-defined, where

$$\mathsf{G}_t(\eta, \omega) := \int_{M^2} \mathsf{h}_t[\eta, \omega] \,\mathrm{d}\mathfrak{m}^{\otimes 2}.$$

Moreover, G_t is weakly Hilbert–Schmidt by Theorem 4.5.5 and Corollary 4.4.7. Therefore, by Theorem 4.2.1 there exists a unique bounded $\mathsf{T}_t : L^2(T^*M)^{\otimes_{\mathrm{H}} 2} \to \mathbf{R}$ such that $\mathsf{G}_t(\eta, \omega) = \mathsf{T}_t(\eta \otimes_{\mathrm{H}} \omega)$ for every $\omega, \eta \in L^2(T^*M)$. Recalling Lemma 4.2.2, by Proposition 3.2.19 there exists a unique element $\mathsf{g}_t \in L^2(T^*M)^{\otimes_{\mathrm{H}} 2}$ such that for every $\omega, \eta \in L^2(T^*M)$, we have

$$\int_{M^2} \langle \mathsf{g}_t, \eta \otimes_{\mathrm{H}} \omega \rangle \,\mathrm{d}\mathfrak{m}^{\otimes 2} = \mathsf{T}_t(\eta \otimes_{\mathrm{H}} \omega) = \mathsf{G}_t(\eta, \omega) = \int_{M^2} \mathsf{h}_t[\eta, \omega] \,\mathrm{d}\mathfrak{m}^{\otimes 2}.$$

Replacing ω and η by $f\,\omega$ and $g\,\eta$ for arbitrary $f, g \in L^\infty(M)$, respectively, provides the claimed $\mathfrak{m}^{\otimes 2}$-a.e. valid identity $\langle \mathsf{g}_t, \eta \otimes_{\mathrm{H}} \omega \rangle = \mathsf{h}_t[\eta, \omega]$.

It remains to prove the series representation of g_t. Since $(\omega_i \otimes_{\mathrm{H}} \omega_j)_{i,j \in \mathbf{N}}$ is an orthonormal basis of $L^2(T^*M)^{\otimes_{\mathrm{H}} 2}$, this simply follows by writing

$$\mathsf{g}_t = \sum_{i,j \in \mathbf{N}} c_{ij}(t)\, \omega_i \otimes_{\mathrm{H}} \omega_j$$

w.r.t. strong convergence in $L^2(T^*M)^{\otimes_{\mathrm{H}} 2}$, where the coefficients are given by

$$c_{ij}(t) = \int_{M^2} \langle \mathsf{g}_t, \omega_i \otimes_{\mathrm{H}} \omega_j \rangle \,\mathrm{d}\mathfrak{m}^{\otimes 2} = \int_M \langle \omega_i, \mathsf{H}_t \omega_j \rangle \,\mathrm{d}\mathfrak{m} = \mathrm{e}^{-\lambda_j t}\, \delta_{ij}. \qquad \square$$

Remark 4.5.14. By Theorem 4.5.13 and [Ros97, Prop. 3.1], our notion of the 1-form heat kernel from Definition 4.5.6 is fully compatible with the so-called *parametrix approach* to it on compact, non-weighted Riemannian manifolds [Pat71, Ch. 4]. More precisely, denoting the smooth heat kernel by $\mathsf{h} : (0, \infty) \times M^2 \to (T^*M)^* \boxtimes T^*M$ as in Section 4.1 by a slight abuse of notation, for all smooth 1-forms ω and η, we have $\mathsf{g}_t(\eta \otimes_{\mathrm{H}} \omega)(x, y) = \langle \eta(x), \mathsf{h}_t(x, y)\, \omega(y) \rangle$ for $\mathfrak{m}^{\otimes 2}$-a.e. $(x, y) \in M^2$. ■

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