A variational approach to Gibbs measures on function spaces

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Deutsche Zusammenfassung

In dieser Arbeit behandeln wir sogenannte Gibbs Maße auf Funktionenräumen. Diese sind heuristisch durch den Ausdruck

$$e^{-S(\phi)}\mathrm{d}\mu$$

gegeben. Dabei ist

$$S(\phi)=\lambda\int_{\Lambda}V(\phi)\mathrm{d}x$$

wobei $V\colon\mathbb{R}\to\mathbb{R}$ eine von unten beschränkte analytische Funktion ist und $\Lambda=\mathbb{R}^d,\mathbb{T}^d$ mit $d=2$, 3 und μ ein Gausssches Maß mit Kovarianz $(m^2-\Delta)$, $m^2>0$. Wir beschränken uns hier auf die Fälle $V(\phi)=\phi^4$ bekannt als Φ_d^4 Modell und $V(\phi)=\sin(\beta\phi)$ mit $\beta^2<4\pi$ bekannt als Sine-Gordon Modell. Das Hauptproblem bei der Konstruktion dieser Objekte ist, dass der Träger des Maßes μ Distributionen mit negativer Regularität beinhaltet und es deshalb nicht klar ist wie die Nichtlinearität $V(\phi)$ zu interpretieren ist. Um diese Schwierigkeit zu umgehen werden in der Literatur diese Objekte mittels Approximation konstruiert, dabei nähert man μ durch Gausssche Maße μ_T an, welche in regulären Räumen getragen sind. Zudem ersetzt man das Potential V durch ein "*renormiertes*" Potential V_T, dies ist nötig um im Limes ein nichttriviales Objekt zu erhalten. In unserer Arbeit setzen wir zu diesem Zweck die folgende Formel ein die ursprünglich von Boué und Dupuis bewiesen und in der Theorie der Großen Abweichungen eingesetzt wurde. Diese Formel lautet:

$$-\log\int e^{-f(\phi)-\lambda\int V_T(\phi)}\mathrm{d}\mu^T=\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[f(W_T+I_T(u))+\lambda\int_{\Lambda}V_T(W_T+I_T(u))+\frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+\times\Lambda)}\right]. \quad (1)$$

Hierbei ist W_T ein Gausscher Prozess mit $\mathrm{Law}(W_T)=\mu_T$, $\mathbb{H}_a$ ist ein Raum von Prozessen die bezüglich W_T adaptiert sind und I_T ist eine lineare Abbildung $L^2(\mathbb{R}_+\times\mathbb{R}^2)\to C([0,\infty],H^1)$.

In Kapitel 2 konstruieren wir mithilfe dieser Formel das Φ_3^4 Maß auf $\mathbb{T}^3$. Es gehört seit längerer Zeit zum „Volksglauben“ der mathematischen Physik, dass Φ_3^4 singulär bezüglich μ ist. Aus diesem Grund gab es unseres Wissens nach bisher keine Beschreibung diese Maßes in der Literatur welche nicht Bezug auf ein Approximationsverfahren nimmt. In Kapitel 2 sind wir fähig eine solche Beschreibung für die Laplacetransformation zu geben indem wir mittels Γ-Konvergenz den Limes in Gleichung (1) nehmen.

In Kapitel 3 setzen wir unsere Untersuchung des Φ_3^4 Maßes fort. Wir geben einen Beweis der Singularität von Φ_3^4 bezüglich μ. Weiterhin konstruieren wir ein Hilfsmaß ν sodass einerseits Φ_3^4 bezüglich ν absolut stetig ist, anderseits ν relativ einfach zu konstruieren und zu analysieren ist.

In Kapitel 4 beschäftigen wir uns mit dem Sine-Gordon Modell, diesmal auf $\mathbb{R}^2$. Anders als in den vorherigen Kapiteln geht die Hauptschwierkeit hier vom unendlichen Volumen von $\mathbb{R}^2$. Mithilfe von Sätzen aus der Stochastischen Kontrolle und der Polchinki Gleichung studieren wir die Abhängigkeit des Minimierers auf der rechten Seite von (1) von f. Dadurch können wir erneut eine Beschreibung der Laplacetransformation des Sine-Gordon Modells geben. Außerdem geben wir einen neuen Beweis der Osterwalder Schrader Axiome für Sine-Gordon.

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Table of contents

Chapter 1

Introduction

In this thesis we will consider measures on spaces of the space of Schwartz distributions $\mathscr{S}'(\Lambda)$ where $\Lambda = \mathbb{T}^d, \mathbb{R}^d$, of the form

$$e^{-S(\phi)}\mathrm{d}\phi, \tag{1.1}$$

with $S(\phi)$ being an action-functional of the form

$$S(\phi) = \lambda \int_\Lambda V(\phi) + m^2 \int \phi^2 + \int |\nabla \phi|^2, \quad V \in C^\infty(\mathbb{R}, \mathbb{R}),$$

and $\mathrm{d}\phi$ being a formal notation for the nonexistent Lebesgue measure in infinite dimensions.We will call such measures Gibbs measures. To give a meaning to eq. (1.1) we first observe that for $\lambda = 0$ we obtain the quadratic action

$$S_{\text{free}} = m^2 \int \phi^2 + \int |\nabla \phi|^2.$$

In this case the eq. (1.1) can be interpreted as the gaussian measure μ with covariance

$$\int \langle f, \phi \rangle \langle g, \phi \rangle \mathrm{d}\mu = \langle f, (m^2 - \Delta)^{-1} g \rangle,$$

this is known as the Gaussian Free Field(GFF). The Gaussian Free Field is known to be supported in Besov-Hölder spaces of regularity $(2-d)/2-\delta$ for any $\delta > 0$. This means that for $d \geqslant 2$ its samples are genuine distributions and not functions.

Now let us turn to models where $\lambda \neq 0$. The case $V(\phi) = \phi^4$ is known as the Φ^4_d model,the $V(\phi) = \cos(\beta\phi)$ case is known as the *Sine-Gordon* model and $V(\phi) = \exp(\beta\phi)$ is known as the Høegh-Krohn model. If we want to make sense of the Gibbs measures corresponding to these actions naively we would write

$$e^{-\lambda \int_\Lambda V(\phi(x))}\mathrm{d}\mu. \tag{1.2}$$

However this only works in $d = 1$, since for $d \geqslant 2$ we would be required to make sense of $\lambda \int_\Lambda V(\phi(x))$ for ϕ a distribution. At first sight this seems to be impossible, however we will show that one can leverage the properties of μ to make sense of eq. (1.2). The standard playbook to achieve this is the following:

- Approximate μ with Gaussian measures μ^T supported in more regular spaces.
- Replace the potential V by an adjusted ("*renormalized*") potential V_T, for example for the ϕ^4 model one replaces

  $$\phi^4 \to \phi^4 - a_T \phi^2 - b_T$$

  with $a_T, b_T \to \infty$ while for Sine Gordon

  $$\cos(\beta\phi) \to \alpha_T \cos(\beta\phi)$$

  with $\alpha_T \to \infty$.
- Take the weak limit of

  $$e^{-\lambda \int_\Lambda V_T(\phi(x))}\mathrm{d}\mu^T.$$

The second step is known as renormalization, it is necessary to obtain a nontrivial limit. Let us briefly comment on the role of the constants λ, β. For finite volume, that is $\Lambda = \mathbb{T}^d$, the size of λ is not particularly important for the analysis. However for $\Lambda = \mathbb{R}^d$ the models behave different depending on the size of λ. For example for the ϕ^4 model the correlation functions that is quantities which are formally written as

$$C(z, y) = \int \phi(z)\phi(y)e^{-\lambda \int_\Lambda V(\phi(x))\mathrm{d}x}\mathrm{d}\mu - \int \phi(y)e^{-\lambda \int_\Lambda V(\phi(x))\mathrm{d}x}\mathrm{d}\mu \int \phi(z)e^{-\lambda \int_\Lambda V(\phi(x))\mathrm{d}x}\mathrm{d}\mu$$

decay exponentially in $|y - z|$ for small λ while they decay only algebraically for large λ, see [69, 59, 41]. In total we can say that λ is not very influential on small scales but influential on large scales.

On the role of β on the other hand is more indicative of the small scale behavior of the model. The limit $T \to \infty$ outlined above can only be taken for $\beta^2 < 8\pi$. For $\beta^2 > 4\pi$ the resulting measure is expected to become singular with respect to the Gaussian Free Field and becomes increasingly difficult to construct as β^2 approaches 8π [89].

Up to this point we hope that we have conveyed to the reader a (admittedly not very precise) picture of what some Gibbs measures on function spaces look like, but so far we have not explained why they should be studied, which shall be our concern for the rest of this section. The most well known application is the use of *Euclidean Quantum Field Theories* (EQFTs) in *Constructive Quantum Field Theory* (CQFT): Euclidean Quantum Field theories are a special class of Gibbs measures on $\mathscr{S}'(\mathbb{R}^d)$ whose correlation function satisfy certain properties known as the *Osterwalder-Schrader Axioms*. Once one has an EQFT, the Osterwalder-Schrader reconstruction theorem then asserts that one can obtain from it a relativistic QFT in the sense of Wightman, which is one the aims of CQFT. In the next subsection we will attempt to explain these notions in more detail.

Before we move on to the next section, let us give a brief (and very incomplete overview) of the literature. Nelson [99, 100] investigated the relationship of Random Fields with Quantum Field Theories and studied in this context the Gaussian free field and the Φ_2^4 model. Nelson's analysis required the random field to satisfy a Markov property, which, in general, can be very tricky to prove, since as already outlined EQFT's are usually constructed by approximation with more regular measures and the Markov Property is hard to carry over to the limit. This problem was solved by Osterwalder and Schrader [104] who discovered that the Markov property can be replaced by the weaker *Reflection Positivity* (see section 1.1.2 and also [51]). Many works on $\Phi_{2,3}^4$, the Sine-Gordon as well as the exponential interaction, constructing these models in both finite and infinite volume and giving proofs of the *Osterwalder-Schrader axioms* (see Section 1.1.2), soon followed: [6, 66, 60, 61, 53, 107, 18, 31] is only and incomplete list. This development culminated in the works of Feldman and Osterwalder [54] and Magnen and Sénéor [92] where the authors gave a complete proof of the Osterwalder-Schrader axioms for Φ_3^4 for small λ . Even though the Markov property was shown for Sine-Gordon (at least for small β) and Φ_2^4 [8, 7] to our knowledge it remains open for Φ_3^4.

There have also been some results on dimensions $d \geqslant 4$: Aizenman [1] and Fröhlich [58] provided proofs of the triviality of Φ_d^4 with $d \geqslant 5$, in the sense that a large class of approximations converge to Gaussian measures. These results were recently extended by Aizenman and Duminil-Copin to Φ_4^4 [2], which is a case of substantial physical interest since physical space time is 4-dimensional. Later the $\Phi_{2,3}^4$, Sine-Gordon models were revisited using Renormalization Group methods [28, 33, 34, 16, 35, 36]. In recent years these models or more generally CQFT have again received substantial attention due to their connections with Singular Stochastic PDE's (see Section 1.2.1), whose understanding saw rapid progress after the pioneering works of Da Prato-Debussche [43], Hairer [77] and Gubinelli-Imkeller-Perkowski [73]. Another fascinating development is the connection with Liouville Quantum Gravity, and Conformal Field Theory [45, 86, 87, 74]. In this connection the Liouville measure, which is the Høegh-Krohn model with $m^2 = 0$ plays a vital role. We shall not further discuss this here, instead we refer to the nice reviews [85, 110].

1.1. Relativistic and Euclidean Quantum Field Theories

In this section we will give an overview of Quantum Field Theories in the sense of Wightman, and their relationship with Wightman functions and Euclidean Quantum Field Theories. We will follow [112] the contribution of Kazhdan to [46] and [113] for the Wightman axioms, and [67, 51] for the Osterwalder Schrader axioms. We can consider Quantum Field Theory as an attempt to reconcile Quantum Mechanics and Relativity Theory into a single framework. Heuristically Qantum Mechanics would entail that the states of our system should be described as vectors in a (separable) Hilbert space, with observables being described by operators. Relativity theory should imply that the system is invariant under an action of the Poincaré-Group and that observables commute on "space like" (see Definition 1.5 below) separated regions. This is made precise in the axioms detailed in the following subsection.

1.1.1. Wightman Axioms

In this section we discuss the Wightman axioms for Quantum Field Theory. We restrict ourselves to spinless bosonic theories.

Definition 1.1. *Throughout this subsection we will denote for* $x, y \in \mathbb{R}^d$ *by* $x \cdot y = x^1 y^1 - x^2 y^2 - \ldots - x^d y^d$ *the Lorenz scalar product,and by* $x^2 = x \cdot x$*. The Lorentz group* $\mathscr{L}$ *is the group of linear transformations that leave the Lorentz scalar product invariant. The Poincaré group* $\mathscr{P}$ *consists of transformations* $\{a, A\}$ *with*

$$\{a, A\}x = Ax + a$$

with $A \in \mathscr{L}$ *and* $a \in \mathbb{R}^d$ *and the group law defined by the composition.*

Definition 1.2. (Wightman Data) *Wightman's description of a quantum field theory begins with the following data:*

- *A separable Hilbert space* $\mathcal{H}$
- *A unitary representation* $U: \mathscr{P} \to \mathrm{Aut}(\mathcal{H})$
- *A dense subspace* $\mathcal{D} \subseteq \mathcal{H}$ *such that for any* $p \in \mathscr{P}$*,* $U(p)\mathcal{D} \subseteq \mathcal{D}$ *and a unique vector* $\Omega \in \mathcal{D}$ *such that* $U(p)\Omega = \Omega$ *for any* $p \in \mathscr{P}$*.*
- *A linear map* $\phi: \mathscr{S}(\mathbb{R}^d, \mathbb{C}) \to \mathrm{Op}(\mathcal{H})$*, where* $\mathrm{Op}(\mathcal{H})$ *is the space of unbounded operators on* $\mathcal{H}$ *called the field map, such that for any* $f \in \mathscr{S}(\mathbb{R}^d, \mathbb{C})$ $\mathcal{D} \subseteq \mathrm{Dom}(\phi(f))$*, and for any* $x \in \mathcal{D}$ $\phi(f)x \in \mathcal{D}$*.*

Axiom 1.3. (Spectral Condition) *Since operators of the form* $U(a, 1)$ *with* $a \in \mathbb{R}^d$*, and* 1 *denoting the identity matrix form a unitary subgroup of* $\mathrm{Aut}(\mathcal{H})$ *they can be written as*

$$U(a, 1) = e^{i\sum_{i=1}^{d} a_i P_i}.$$

with generators P_i*. We assume that the* $\mathrm{Spec}\,(P) \subseteq V_+$*, where we have written* $P = (P_i)_{i=1}^d$ *and* $\mathrm{spec}(P) = \prod_{i=1}^d \mathrm{spec}(P_i)$ *while*

$$V_+ = \{p \in \mathbb{R}_+ \times \mathbb{R}^{d-1}: p^2 \geqslant 0\}.$$

where we recall that $p^2 = p \cdot p$ *is the Lorentz scalar product.*

As in Quantum Mechanics the spectrum of P should be thought of as describing the energy of the system, so Axiom 1.3 is essentially a requirement for the energy to be positive. This condition also implies that we can define by spectral calculus the mass operator $M = \sqrt{P_1^2 - \sum_{i=2}^d P_i^2}$. If $\mathrm{Spec}\, M \subseteq \{0\} \cup [m, \infty)$ for some $m > 0$ we say that the theory has a mass gap.

Axiom 1.4. (Poincare Covariance) *The field map* ϕ *satisfies the following transformation rule:*

$$U(a, A)\phi(f)U(a, A)^{-1} = \phi(\{a, A\}f).$$

DEFINITION 1.5. *We say that* $f, g \in \mathscr{S}(\mathbb{R}^d, \mathbb{C})$ *are space like separated if for any* $x, y \in \mathbb{R} \times \mathbb{R}^{d-1}$ *such that* $(x-y)^2 \geqslant 0$ *it holds that* $f(x)g(y) = 0$.

AXIOM 1.6. (CAUSALITY) *If* $f, g \in \mathscr{S}(\mathbb{R}^d, \mathbb{C})$ *are space like separated*

$$[\phi(f), \phi(g)] = 0,$$

where $[\cdot, \cdot]$ *as usual denotes the commutator of two operators.*

AXIOM 1.7. (CYCLYCITY) *The set*

$$\left\{h\colon h = \prod_{i=1}^{N} \phi(f_i)\Omega, \text{for some } f_i \in \mathscr{S}(\mathbb{R}^d, \mathbb{C}), N \in \mathbb{N}\right\}$$

is dense in $\mathcal{H}$.

If Wightman data satisfies these axioms, we will say that it is a Quantum Field Theory. For a Quantum Field Theory we can consider the Wightman functions given by

$$\tilde{\mathcal{W}}_n(f_1, \ldots, f_n) = \langle \Omega, \phi(f_1) \ldots \phi(f_n)\Omega \rangle. \tag{1.3}$$

The Schwartz nuclear theorem (see for instance Theorem 2 on page 158 in [119]) implies the existence of a tempered distributions $\mathcal{W}_n \in \mathscr{S}'(\mathbb{R}^{nd}, \mathbb{C})$ such that

$$\tilde{\mathcal{W}}_n(f_1, \ldots, f_n) = \langle \mathcal{W}_n, f_1 \otimes \ldots \otimes f_n \rangle_{L^2(\mathbb{R}^{nd})}$$

It is natural to ask what kind of properties the $\mathcal{W}_n$ obey and under which conditions the Quantum Field Theory can be reconstructed from the Wightman functions. The answer to those questions is the content of the following two propositions:

PROPOSITION 1.8. *Let* $\tilde{\mathcal{W}}, \mathcal{W}_n$ *be constructed as above. Then they satisfy the following properties:*

- $\mathcal{W}_n$ *is invariant under* P *for all* $n \in \mathbb{N}$. *In particular* $\mathcal{W}_n$ *is translation invariant, hence*

  $$\mathcal{W}_n(x_1, \ldots, x_n) = W_n(\xi_1, \ldots, \xi_{n-1})$$

  where $\xi_j = x_j - x_{j+1}$, *for unique* $W_n \in \mathscr{S}'(\mathbb{R}^{(n-1)d}, \mathbb{C})$.

- *Let* $\hat{W}_n$ *be the Fourier transform of* W_n. *Then*

  $$\operatorname{supp} \hat{W}_n \subseteq (V_+)^{n-1}$$

  where V_+ *is defined in Axiom 1.3.*

- $\tilde{\mathcal{W}}_n(f_1^*, \ldots, f_n^*) = \overline{\tilde{\mathcal{W}}_n(f_1, \ldots, f_n)}$ *where* $f^*(x) = \overline{f(x)}$ *and* $\bar{\cdot}$ *denotes complex conjugation.*

- *Let* π *be a permutation of* $\{1, \ldots, n\}$ *and assume that* $(x_i - x_j)^2 \geqslant 0$ *for any* $i, j \in \{1, \ldots, n\}$. *Then*

  $$\mathcal{W}(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \mathcal{W}(x_1, \ldots, x_n).$$

- *Let* $f_n \in \mathscr{S}(\mathbb{R}^{nd}, \mathbb{C})$ $n \leqslant M$ *for some* $M \in \mathbb{N}$. *Then*

  $$\sum_{j,k \leqslant M} \tilde{\mathcal{W}}_{j+k}(f_j^* \otimes f_k) \geqslant 0$$

- *Let* $a \in \mathbb{R}^d$ *such that* $|a| = 1$. *Then as* $\lambda \to \infty$

  $$\mathcal{W}_n(x_1, \ldots x_j, x_{j+1} + \lambda a, x_n + \lambda a) \to \mathcal{W}_j(x_1, \ldots, x_j)\mathcal{W}_{n-j}(x_{j+1}, \ldots, x_n)$$

  where the convergence is in $\mathscr{S}'(\mathbb{R}^{nd}, \mathbb{C})$.

Remark 1.9. The last property of the Wightman functions is known as the cluster decomposition property or simply clustering. It is related to the vacuum vector Ω in Definition 1.2 being unique. In principle one does not have to require Ω to be unique in which case the cluster decomposition property does not hold however we will not go into this here.

The above theorem gives necessary conditions for the $\mathcal{W}_n$ to be coming from a Quantum Field Theory. It turns out these conditions are also sufficient.

Proposition 1.10. *Let $\mathcal{W}_n \in \mathscr{S}'(\mathbb{R}^{nd}, \mathbb{C})$ be a family of tempered distributions such that it satisfies all the properties from Proposition 1.8. Then there exists a Quantum Field Theory such that eq.(1.3) holds.*

For proof of Proposition 1.8 and Proposition 1.10 see [112] Section 3.3 and Section 3.4.

1.1.2. Osterwalder-Schrader Axioms

In view of the preceding section it is enough to construct the vacuum expectation values $\mathcal{W}_n$ to construct a Quantum Field Theory. Then the question becomes how to construct vacuum expectation values. Wightman functions are difficult to construct directly so instead we opt to construct their cousins: Schwinger functions. Heuristically Schwinger functions are Wightman functions formally evaluated in *complex Euclidean points*: for a point $x \in \mathbb{R}^{nd}$, $x = ((\tau_1, y_1), \ldots, (\tau_n, y_n))$ with $\tau_i \in \mathbb{R}, y_i \in \mathbb{R}^{d-1}$ we write the corresponding complex Euclidean point as $z(x) = (((i\tau_1, y_1), \ldots, (i\tau_n, y_n))) \in (\mathbb{C} \times \mathbb{R}^{d-1})^n$ and so if one can construct an extension of $\mathcal{W}_n$ to $(\mathbb{C} \times \mathbb{R}^{d-1})^n$ we can think of S_n as defined by

$$S_n(x) = \mathcal{W}_n(z(x)).$$

"In practice" often the reverse procedure is applied, the Schwinger functions are constructed first and then the Wightman functions are recovered by analytic continuation.

Note that

$$(z(x))^2 = |x|^2$$

where by $|x|^2$ we have denoted the euclidean norm on $\mathbb{R}^d$, since

$$(z(x))^2 = \sum_{i=1}^{n} -(i\tau_i)^2 + |y_i|^2 = \sum_{i=1}^{n} \tau_i^2 + |y_i|^2 = |x|^2.$$

so (at least heuristically) if the Wightman functions are invariant under the action of the Poincare group, the Schwinger functions should be invariant under the actions of the *Euclidean group*, which is one of the reasons they are easier to construct. In fact, in many cases Schwinger functions can be constructed as moments of a random field, as shall be described below. However as our primary interest in this section is to construct a Quantum Field Theory let us turn to how one can recover the Wightman functions (and subsequently the QFT) from Schwinger functions. The condition under which this is possible are known as the Osterwalder Schrader axioms.

In the following let $\{S_n \in \mathscr{S}'(\mathbb{R}^{nd})\}_{n \in \mathbb{N}}$ be a family of distributions.

Axiom 1.11. (Regularity) *$S_0 = 1$ and there exists a Schwartz semi-norm $\|\cdot\|_s$ such that*

$$|S_n(f_1 \otimes \ldots \otimes f_n)| \leqslant n! \prod_{i=1}^{n} \|f_i\|_s$$

Axiom 1.12. (Euclidean Invariance) *Let the Euclidean group with $G = (R, a)$ $R \in O(d), a \in \mathbb{R}^d$ acts on functions by*

$$(Gf)(x) = f(Rx - a).$$

Then

$$S_n(Gf_1 \otimes \ldots \otimes Gf_n) = S_n(f_1 \otimes \ldots \otimes f_n).$$

Axiom 1.13. (Reflection Positivity) *Let $\mathbb{R}_+^{nd}$ be the set*

$$\{x \in \mathbb{R}^{nd} \colon x = (x_1, \ldots., x_n) \text{ and } x_i = (\tau_i, y_i) \text{ with } \tau_i > 0 \text{ and } y_i \in \mathbb{R}^{d-1}\}$$

Furthermore define the reflection $\Theta(x) = \Theta((\tau, y)) = (-\tau, y)$ and its action on a function $f \in \mathscr{S}'(\mathbb{R}^{nd})$ by

$$\Theta f(x_1, \ldots, x_n) = f(\Theta x_1, \ldots., \Theta x_n).$$

Now we require that for all finite families $\{f_n \in \mathscr{S}(\mathbb{R}^{nd})\}_{n \leqslant M}$ *such that* $\operatorname{supp} f_n \subseteq \mathbb{R}_+^{nd}$ *we have*

$$\sum_{i,j=1}^{M} S_{i+j}(\Theta f_i \otimes f_j) \geqslant 0.$$

AXIOM 1.14. (SYMMETRY) *Let* π *be a permutation of* $\{1, ..., n\}$. *Then*

$$S_n(f_{\pi(1)} \otimes ... \otimes f_{\pi(n)}) = S_n(f_1 \otimes ... \otimes f_n)$$

AXIOM 1.15. (CLUSTERING) *For any* $j, n \in \mathbb{N}$, $1 \leqslant j \leqslant n$ *and* $a \in \mathbb{R}^d$ *such that* $|a| = 1$ *as* $\lambda \to \infty$

$$S_n(f_1 \otimes ... \otimes f_j \otimes f_{j+1}(\cdot + \lambda a) \otimes ... \otimes f_n(\cdot + \lambda a)) \to S_j(f_1 \otimes ... \otimes f_j) S_{n-j}(f_{j+1} \otimes ... \otimes f_n).$$

If furthermore there exists $m > 0$ *such that for any family of* $f_j \in C_c^\infty(\mathbb{R}^d)$ *there exists a constant* $C > 0$ *such that*

$$|S_n(f_1 \otimes ... \otimes f_j \otimes f_{j+1}(\cdot + \lambda a) \otimes ... \otimes f_n(\cdot + \lambda a)) - S_j(f_1 \otimes ... \otimes f_j) S_{n-j}(f_{j+1} \otimes ... \otimes f_n)| \leqslant C e^{-m|\lambda|}$$

We say that the clustering is exponential.

These conditions are sufficient to construct a Quantum Field Theory:

THEOREM 1.16. *Assume that* $\{S_n \in \mathscr{S}'(\mathbb{R}^{nd})\}_{n \in \mathbb{N}}$ *satisfies the Osterwalder-Schrader Axioms. Then there exists a unique corresponding set of Wightman functions satisfying the properties described in Proposition 1.8. By Proposition 1.10 there also exists a corresponding Quantum Field Theory.*

For a proof see [51].

Remark 1.17. Let us briefly sketch out how the Osterwalder Schrader Axioms relate the properties of the corresponding Wightman-QFT.

- Axiom 1.11 is a technical condition which makes the proof of Theorem 1.16.
- Euclidean Invariance is equivalent to Poincare of the corresponding Wightman theory.
- Axiom 1.13 enables one to build a Hilbert space for the corresponding Wightman theory.
- Axiom 1.14 is related to the Causality (Axiom 1.6).
- Axiom 1.15 is related to uniqueness of the Vacuum vector Ω in Definition 1.2.

Remark 1.18. If a set of Schwinger functions satisfies exponential clustering the corresponding Quantum Field Theory is known to have a mass gap: see [111].

We have already mentioned that Schwinger functions can be constructed as moments of a random field. Indeed, we can modify the definition of Reflection Positivity for measures:

DEFINITION 1.19. *Let* ν *be a measure on* $\mathscr{S}'(\mathbb{R}^d)$. *Let* $\mathcal{A}_+ \subseteq L^2(\nu, \mathbb{C})$ *be the set of of functionals which depend only on* $f|_{\mathbb{R}_+^d}$, *and* Θ *be the reflection as in Axiom 1.13. We say* ν *is reflection positive if for any* $A \in \mathcal{A}_+$

$$\int \overline{A(\phi)}(A(\Theta\phi))\nu(\mathrm{d}\phi) \geqslant 0.$$

Remark 1.20. It is not hard to see that Reflection Positivity is stable under weak convergence: If $\nu_n \to \nu$ is a sequence of weakly convergent reflection positive measures than also the limit ν is reflection positive.

Then we have the following proposition:

PROPOSITION 1.21. *Let ν be a reflection positive measure on $\mathscr{S}'(\mathbb{R}^d)$. Furthermore assume that there exists a Banach space $B \subseteq \mathscr{S}'(\mathbb{R}^d)$ equipped with the norm $\|\cdot\|_B$ such that for some $\alpha > 0$*

$$\int e^{\alpha \|\phi\|_B} \nu(\mathrm{d}\phi) < \infty \tag{1.4}$$

and that ν is invariant under the action of the Euclidean group, meaning that for any $F \in L^1(\nu)$

$$\int F(\phi)\mathrm{d}\nu = \int F(G\phi)\mathrm{d}\nu$$

for any $G = (R, a)$ $R \in O(d), a \in \mathbb{R}^d$. Then the moments of μ

$$S_n(f_1 \otimes \ldots \otimes f_n) := \int \langle f_1, \phi\rangle \ldots \langle f_n, \phi\rangle \nu(\mathrm{d}\phi)$$

satisfy Axioms 1.11-1.14.

The proof of the proposition is straightforward and we omit it. Note that Proposition 1.21 does not cover clustering, it has to be verified separately.

DEFINITION 1.22. *We say that a measure ν on $\mathscr{S}'(\mathbb{R}^d)$ satisfies the Osterwalder-Schrader axioms if it satisfies the assumptions of Proposition 1.21.*

Let us now provide an example where the assumptions of Proposition 1.21 are satisfied: The Gaussian Free Field (GFF), which is the Gaussian measure with covariance $(m^2 - \Delta)^{-1}$. We have already met it the introduction. Fernique's theorem implies that it satisfies eq. (1.4). Furthermore the GFF is invariant under the action of the Euclidean group since its covariance operator is. We will now prove that it is reflection positive.

LEMMA 1.23. *Let μ be a Gaussian measure on $\mathscr{S}'(\mathbb{R}^d)$ with covariance operator C. Then μ is reflection positive if for any $f \in S(\mathbb{R}^d)$*

$$\langle Cf, \Theta f\rangle_{L^2(\mathbb{R}^d)} \geqslant 0.$$

Proof. By density it is enough to show that

$$\int \overline{A(\phi)}(A(\Theta\phi))\nu(\mathrm{d}\phi) \geqslant 0 \tag{1.5}$$

for $A(\phi) = \sum_{k=1}^m c_k e^{i\langle \phi, f_k\rangle}$ with $f_k \in C_c^\infty(\mathbb{R}_+^d)$. Then by the formula for characteristic functions of Gaussian measures (see [83]) (1.5) reduces to

$$\begin{aligned} &\sum_{k,j=1}^m c_k c_j \int e^{i\langle \phi, f_k - \Theta f_j\rangle} \mathrm{d}\mu \\ =& \sum_{k,j=1}^m c_k c_j e^{-\langle f_k - \Theta f_j, C(f_k - \Theta f_j)\rangle/2} \\ =& \sum_{k,j=1}^m c_k c_j e^{-\langle f_k, C f_k\rangle/2} e^{-\langle f_j, C f_j\rangle/2} e^{\langle f_j, C\Theta f_k\rangle} \end{aligned}$$

Now if we denote by $M = (M_{jk})_{j,k \leqslant m}$, $M_{jk} = e^{\langle f_j, C\Theta f_k\rangle}$ and $v = (e^{-\langle f_1, Cf_1\rangle/2} c_1, \ldots, e^{-\langle f_m, Cf_m\rangle/2} c_m) \in \mathbb{R}^m$ and viewing M as an $m \times m$ matrix our computation becomes:

$$\int A(\phi)(A(\Theta\phi))\nu(\mathrm{d}\phi) = \langle Mv, v\rangle_{\mathbb{R}^m}.$$

It remains to prove that M is positive semi-definite. By the Schur-Hadamard product theorem this follows from $N_{i,j} = \langle f_i, C\Theta f_j\rangle$ being positive semi-definite, which in turn follows from the assumption. □

We now have the following lemma showing that the covariance of the Free Field is indeed reflection positive:

LEMMA 1.24. *Let* $C = (m^2 - \Delta)^{-1}$ $m \geqslant 0$. *Then for any* $f \in L^2(\mathbb{R}^d)$

$$\langle Cf, \Theta f\rangle_{L^2(\mathbb{R}^d)} \geqslant 0. \tag{1.6}$$

Proof. Here we follow [80]. For $p = (\xi, q) \in \mathbb{R}^d$ with $\xi \in \mathbb{R}, q \in \mathbb{R}^{d-1}$

$$\begin{aligned} \mathscr{F}f(p) &= \int e^{-ipx} f(x)\mathrm{d}x \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^{d-1}} e^{-i(qy+\tau\xi)} f(\tau, y)\mathrm{d}y\mathrm{d}\tau \end{aligned}$$

and we can continue this analytically in standard fashion in τ to the lower half plane in the variable ξ. By abuse of notation we denote with $\mathcal{F}f$ also the analytic continuation. In particular we obtain

$$(\mathscr{F}f)(-iE, q) = \int_{\mathbb{R}_+}\int_{\mathbb{R}^{d-1}} e^{-iqy-E\tau} f(\tau, y)\mathrm{d}y\mathrm{d}\tau$$

While for Θf

$$\begin{aligned} \overline{\mathscr{F}\Theta f(p)} &= \int e^{ipx} \Theta f(x)\mathrm{d}x \\ &= \int_{\mathbb{R}_+}\int_{\mathbb{R}^{d-1}} e^{iqy-i\tau\xi} f(\tau, y)\mathrm{d}y\mathrm{d}\tau \end{aligned}$$

and again this can be analytically continued to the lower half plane and we get

$$\overline{\mathscr{F}\Theta f(-iE, q)} = \int_{\mathbb{R}_+}\int_{\mathbb{R}^{d-1}} e^{iqy-iE\tau} f(\tau, y)\mathrm{d}y\delta\tau.$$

Note that $\overline{\mathscr{F}\Theta f(-iE, q)} = \overline{\mathscr{F}f(-iE, q)}$.

After this preliminary analysis we can rewrite

$$\begin{aligned} \langle Cf, \Theta f\rangle_{L^2(\mathbb{R}^d)} &= \int \mathscr{F}f(p)\overline{\mathscr{F}\Theta f(p)} \frac{1}{|p|^2 + m^2}\mathrm{d}p \\ &= \int \mathscr{F}f(\xi, q)\overline{\mathscr{F}\Theta f(\xi, q)} \frac{1}{|q|^2 + m^2 + \xi^2}\mathrm{d}q\mathrm{d}\xi \end{aligned}$$

We now want to use the contour integral argument to compute the ξ integral. Indeed the poles of in ξ are

$$\xi_\pm = \pm iw = \pm i(|q|^2 + m^2)$$

The Residue of

$$\mathscr{F}f(\xi, q)\overline{\mathscr{F}\Theta f(\xi, q)} \frac{1}{|q|^2 + m^2 + \xi^2} = \mathscr{F}f(\xi, q)\overline{\mathscr{F}\Theta f(\xi, q)} \frac{1}{(\xi - \xi_+)(\xi - \xi_-)}$$

at ξ_- is

$$\mathscr{F}f(iw_-, q)\overline{\mathscr{F}\Theta f(iw_-, q)} \frac{1}{i(w_+ - w_-)}$$

so by the contour argument

$$\begin{aligned} &\int \mathscr{F}f(\xi, q)\overline{\mathscr{F}\Theta f(\xi, q)} \frac{1}{|q|^2 + m^2 + \xi^2}\mathrm{d}q\mathrm{d}\xi \\ &= 2\pi i \int \mathscr{F}f(iw_-, q)\overline{\mathscr{F}\Theta f(iw_-, q)} \frac{1}{i(w_+ - w_-)}\mathrm{d}q \\ &= 2\pi \int |\mathscr{F}f(iw_-, q)|^2 \frac{1}{2(|q|^2 + m^2)}\mathrm{d}q \\ &\geqslant 0 \end{aligned}$$

□

Remark 1.25. It is also possible to show that the covariance operator $(m^2 - \Delta)^{-s}$ for $0 < s < 1$ is reflection positive [71]. However for $s > 1$ that is not the case.

So we have seen that the GFF satisfies the assumptions of Proposition 1.21. We will now show that its moments satisfy exponential clustering, thus satisfying all Osterwalder Schrader axioms.

Lemma 1.26. *Let μ be the Gaussian measure with covariance $(m^2-\Delta)^{-1}$ with $m^2>0$. Then its moments satisfy exponential clustering.*

Proof. We want to show that

$$\int \prod_{i=1}^{k} \langle f_i, \phi\rangle \prod_{i=k+1}^{n} \langle f_i(\cdot+a), \phi\rangle \mathrm{d}\mu - \int \prod_{i=1}^{k} \langle f_i, \phi\rangle \mathrm{d}\mu \int \prod_{i=k+1}^{n} \langle f_i, \phi\rangle \mathrm{d}\mu$$
$$\leqslant Ce^{-m|\lambda|}$$

if $f_i \in C_c^{\infty}$

Denote by P_n the set of partitions of $\{1, \ldots, n\}$ into pairs. Then by Wick's theorem (see for example Theorem 1.28 in [82]), defining by $\tilde{f}_i = f_i$ if $i \leqslant k$ and $\tilde{f}_i = f_i(\cdot+a)$ for $i > k$ we have

$$\int \prod_{i=1}^{n} \langle \tilde{f}_i, \phi\rangle \mathrm{d}\mu = \sum_{p\in P_n} \prod_{\{i,j\}\in p} \int \langle \tilde{f}_i, \phi\rangle \langle \tilde{f}_j, \phi\rangle \mathrm{d}\mu = \sum_{p\in P_n} \prod_{\{i,j\}\in p} \langle \tilde{f}_i, (m^2-\Delta)^{-1} \tilde{f}_j\rangle.$$

Now denote by P_n^k the set of partitions of $\{1, \ldots, n\}$ into pairs such that each pair is contained in either $\{1, \ldots, k\}$ or $\{k+1, \ldots, n\}$. Then it is not hard to see using that μ is translation invariant that

$$\int \prod_{i=1}^{n} \langle \tilde{f}_i, \phi\rangle \mathrm{d}\mu - \int \prod_{i=1}^{k} \langle f_i, \phi\rangle \mathrm{d}\mu \int \prod_{i=k+1}^{n} \langle f_i, \phi\rangle \mathrm{d}\mu$$
$$= \sum_{p\in P_n\setminus P_n^k} \prod_{\{i,j\}\in p} \langle \tilde{f}_i, (m^2-\Delta)^{-1} \tilde{f}_j\rangle.$$

Now assume that $p \in P_n \setminus P_n^k$. Then there exists at least one pair $\{i, j\} \in p$ such that $i \leqslant k$ and $j > k$. For this pair

$$\langle \tilde{f}_i, (m^2-\Delta)^{-1} \tilde{f}_j\rangle = \langle f_i, (m^2-\Delta)^{-1} f_j(\cdot+a)\rangle \leqslant Ce^{-m|a|},$$

which implies the statement. □

1.2. Connections with PDE's

In this section we will outline some connections of Gibbs measures on function spaces of the form $e^{-\int V(\phi)}\mu(\mathrm{d}\phi)$ with partial differential equations. We choose to focus here on two ways of connecting to PDE's.

1.2.1. Stochastic Quantization

Gibbs measures in the continuum have dynamical counterparts, formally given by the stochastic PDE's

$$\begin{aligned} \partial_t u(t,x) + (m^2-\Delta) u(t,x) + V'(u(t,x)) &= \xi(t,x) \\ u(0) &= u_0 \end{aligned}$$

with ξ is a space time white noise on $\mathbb{R}_+ \times \Lambda$, which has covariance

$$\mathbb{E}[\langle \xi, f\rangle_{L^2_+(\mathbb{R}\times\Lambda)} \langle \xi, f\rangle_{L^2_+(\mathbb{R}\times\Lambda)}] = \langle f, g\rangle_{L^2_+(\Lambda)}.$$

One expects that the Gibbs measure is the equilibrium for this Stochastic Partial Differential Equation, that is one can find, stationary solutions u such that formally for any $t \in \mathbb{R}_+$

$$u(t,\cdot) \sim \text{“}\exp\left(\int_{\Lambda} \phi (m^2-\Delta)\phi - \int_{\Lambda} V(\phi)\right) \mathrm{d}\phi\text{”}$$

and in some cases one even has convergence of the Law of $u(t,x)$ to the Gibbs measure as $t \to \infty$ for a large class of initial conditions u_0.

Let us focus on the case $V(u) = \frac{\lambda}{4} u^4$ and $d = 2$. Then we end up with the equation

$$\begin{aligned}\partial_t u + (m^2 - \Delta) u - u^3 &= \xi\\ u(0, x) &= u_0(x)\end{aligned}$$

It is known that in this case ξ has spacial regularity $-2 - \delta$ for any $\delta > 0$, so we cannot expect u to have spacial regularity any better that $-\delta$, and because of this we run into the problem of making sense of the non linearity u^3. The remedy for this is very similar to the one for the measure described above: We approximate the noise with a sequence of smooth noises ξ_ε, and replace the third power by the *Wick ordered* (see chapter 3 in [82]) third power $u^3 \rightarrow u^3 - C_\varepsilon u = \llbracket u_\varepsilon^3 \rrbracket$, where the notation $\llbracket \cdot \rrbracket$ is defined by

$$\begin{aligned}\llbracket u_\varepsilon^3 \rrbracket &:= u_\varepsilon^3 - 3\mathbb{E}[\eta_\varepsilon^2(0, 0)] u_\varepsilon,\\ \llbracket u_\varepsilon^2 \rrbracket &:= u_\varepsilon^2 - \mathbb{E}[\eta_\varepsilon^2(0, 0)]\end{aligned}$$

(note that $\mathbb{E}[\eta_\varepsilon^2(t, x)]$ is does not depend on t, x) and we have denoted by η_ε the stationary solution to

$$\partial_t \eta_\varepsilon + (m^2 - \Delta) \eta_\varepsilon = \xi_\varepsilon .$$

We see that the solution to

$$\begin{aligned}\partial_t u_\varepsilon + (m^2 - \Delta) u_\varepsilon - \llbracket u_\varepsilon^3 \rrbracket &= \xi_\varepsilon\\ u_\varepsilon(0, x) &= u_0(x)\end{aligned}$$

satisfies $u_\varepsilon = \eta_\varepsilon + v_\varepsilon$ where v solves

$$\begin{aligned}\partial_t v_\varepsilon + (m^2 - \Delta) v_\varepsilon - \llbracket (\eta_\varepsilon + v_\varepsilon)^3 \rrbracket &= 0\\ v_\varepsilon(0, x) &= u_0(x) - \eta_\varepsilon(x)\end{aligned}$$

Now

$$\llbracket (\eta_\varepsilon + v_\varepsilon)^3 \rrbracket = \sum_{i=1}^{3} \llbracket \eta_\varepsilon^i \rrbracket (v_\varepsilon)^{3-i}$$

and using probabilistic arguments one can show that as $\varepsilon \rightarrow 0$ $\llbracket \eta_\varepsilon^i \rrbracket$, converges to a random distribution of regularity $-\delta$ for any $\delta > 0$. By a contraction argument one can then make sense of the limiting equation for v_ε. The idea to consider the equation for $v_\varepsilon = u_\varepsilon - \eta_\varepsilon$ instead of the equation for u directly is known as the *Da Prato-Debussche trick* [43].

For $d = 3$ the Da Prato-Debussche trick is not enough, since one no longer has that $\llbracket \eta_\varepsilon^i \rrbracket$ converges in a space of regularity $-\delta$ (instead converging in a space of regularity $-i/2$), and one has to look for a more complicated ansatz for u and one needs more tools. Their development was achieved by Hairer [77] and Gubinelli Imkeller and Perkowski [73] which was applied to Φ_3^4 in [40], see also [78] for the parallel development for Sine-Gordon. In the approach of [73, 40] one takes further specified v as

$$\partial_t v_\varepsilon + (m^2 - \Delta) v_\varepsilon = \llbracket \eta_\varepsilon^3 \rrbracket + \llbracket \eta_\varepsilon^2 \rrbracket \succ v_\varepsilon + w_\varepsilon$$

where w_ε is another remainder term and $\succ$ is a *para-product*; it is a bilinear operation on functions and has the property that the function $f \succ g$ behaves at large frequencies like f, so in particular the regularity of the para-product is dictated by f, see appendix A for details. If one then adds additional renormalization constants beyond Wick ordering, one can solve the equation also in three dimensions. The goal of the program known as *Stochastic Quantization* is to use these equations to obtain control over the associated invariant measures (which we are also interested in). These has been achieved for Φ_3^4 in finite volume [9] and infinite volume in [71]. In [115] exponential convergence of the dynamical Φ_2^4 model in finite volume to equilibrium was proven, that is starting from any initial data the law of the solution at fixed time will converge to the Φ_2^4 measure. We note also that the development described here took place on $\mathbb{T}^{2,3}$, except for [71]. In infinite volume further work is necessary to handle the divergence of the noise see [72, 71, 95, 94, 96].

Recently another class of Stochastic SPDE's was shown to exhibit a connection to euclidean quantum field theories. These are elliptic PDE's on $\mathbb{R}^2 \times \mathbb{R}^d$ which formally look like

$$-\Delta u + m^2 u + V'(u) = \xi$$

where ξ is a white noise on $\mathbb{R}^2\times\mathbb{R}^d$. Then one expects that for any $z\in\mathbb{R}^2$ $u(z,\cdot)$ is distributed according to

$$u(z,\cdot)\sim\text{"}\exp\left(\int_\Lambda \phi(m^2-\Delta)\phi-\int_\Lambda V(\phi)\right)\mathrm{d}\phi\text{"}$$

see [4, 11].

1.2.2. Random Data dispersive equations

Gibbs measures in infinite dimensions can also be useful as *invariant measures* for certain dispersive PDE's , this direction of research has recently received some renewed interest as part of the more general program of studying dispersive PDE's with randomized initial data [38, 39], see also the review [116]. In [23] Bourgain considered considered the invariance of Φ_2^4 on $\mathbb{T}^2$ with respect to the flow of nonlinear dispersive equations, for example the nonlinear (cubic) Schrödinger equation

$$i\partial_t u+\Delta u=\llbracket |u|^2 u\rrbracket \tag{1.7}$$

where the Wick ordering is defined similarly as above. This invariance is nontrivial to interpret since this equation is known to be well posed for initial data with regularity 0 but not below, on the other hand the Φ_2^4 measure is known to be supported on spaces of regularity just below 0. Indeed one of the main contributions of Bourgain is that he was able to show that the flow of (1.7) is well defined almost surely with respect to the Φ_2^4 measure. In this sense one could interpret Bourgain's result as improving the properties of the equation in a probabilistic setting. Bourgain's argument consists of the following steps

1. Approximating (1.7) with $i\partial_t u^N+\Delta u^N=P_N|u|^2u-c_N u$ where P_N is a projection on functions with support in a ball of radius N in the frequency space, and c_N is again a diverging renormalization constant
2. Constructing invariant measures for the approximate equations, which is simpler since they are Gaussian outside of a finite dimensional space
3. Proving that the solutions of the approximate equations converge to solutions of (1.7) for small times (this is done via Bourgain's trick which is very similar to the Da Prato Debussche trick described above).
4. Using invariance of approximate measures under the approximate equations to piece local solutions together to obtain global solutions (this step is known as Bourgain's globalization argument).

Since then this program has been carried out for various other dispersive PDE's in finite [21, 102, 101] and infinite volume [22]. Step 3 of this construction is usually carried out for initial data distributed according to the free field, which then yields the statement for the full measure, since in finite volume Φ_2^4 is absolutely continuous with respect for the free field. Let us mention a recent development in this area which is closely related to Chapter 3 of this thesis. In a recent series of papers [26, 27] Bringmann carried out this program for the wave equation with Hartree nonlinearity on $\mathbb{T}^3$:

$$-\partial_{tt}^2 u-u-\Delta u=\llbracket (V*u^2)u\rrbracket,$$

where $V(x)=c_\beta|x|^{-(3-\beta)}$ with $3>\beta>0$. For $\beta<1$ the appropriate invariant measures μ_H^β are not absolutely continuous with respect to the free field. However relying on the approach developed in Chapter 3 Bringmann was able to construct reference measures ν_H^β such that $\mu_H^\beta\ll\nu_H^\beta$ and such that $\nu_H^\beta=\mathrm{Law}(W+I)$ where W is distributed according to the free field and I is a random function with positive regularity. Using this approach he was able to reduce the local theory to constructing the flow for short times starting from a random function distributed according to the Gaussian Free Field. Another crucial contribution of [27] is a modification of step 4 to equations whose invariant measures are not absolutely continuous with respect to the free field. Let us also mention that closely related to the invariance is the *quasi-invariance* of Gaussian measures under the flow of dispersive equations. In this branch of research one wants to construct Gaussian measures for dispersive PDE's which are quasi-invariant, that is the push forward under the flow of the measure is absolutely continuous with respect to the Gaussian measure. This is often done by finding a Gibbs measure which is equivalent to the Gaussian measure and invariant under the flow, see [103, 76].

1.3. The Sine-Gordon Tranformation of the Yukawa gas

As a last application we explain the relation of the Yukawa gas with the Sine-Gordon model. We follow here the nice exposition of [89]. The two dimensional Yukawa gas in a domain $\mathcal{O} \subseteq \mathbb{R}^2$ is a model describing a gas of charged particles interacting via the Yukawa potential given by

$$K_m(x,y) = \int_0^\infty \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t} - mt} \mathrm{d}t,$$

note that K_m is the kernel of $(m^2 - \Delta)^{-1}$. For the purpose of this section we will take $\mathcal{O}$ bounded. Then the partition function of the Yukawa gas is (at least formally) given by

$$Z^Y_{\alpha,\beta,\lambda} = \sum_{n=0}^\infty \frac{\lambda^n}{2^n n!} \sum_{(\tau_i)_{i=1}^n \in \{-1,1\}^n} \int_{\mathcal{O}^n} \exp\left(-\beta^2 \sum_{1 \leqslant i<j \leqslant n} \tau_i \tau_j K_m(x_i,x_j)\right) \prod_{i=1}^n \mathrm{d}x_i.$$

On the other hand let W be a random variable on a probability space $\mathbb{P}$ mapping into the Besov-Hölder space $\mathscr{C}^{-\varepsilon}(\mathcal{O})$ for some small $\varepsilon > 0$, distributed according to the Gaussian free field, that is

$$\mathbb{E}[\langle W, f\rangle_{L^2(\mathcal{O})} \langle W, g\rangle_{L^2(\mathcal{O})}] = \int_{\mathbb{R}^2} K_m(x,y) f(x) g(y) \mathrm{d}y \mathrm{d}x.$$

Let us consider the expectation

$$\mathbb{E}\left[\exp\left(-\lambda e^{\frac{\beta^2}{2} K_m(0)} \int_{\mathcal{O}} \cos(\beta W(x)) \mathrm{d}x\right)\right].$$

Obviously this makes no sense a priory since $K_m(0)$ is ∞ and W is only a distribution so $\cos(\beta W)$ has no meaning. However we can approximate it by

$$\mathbb{E}\left[\exp\left(-\lambda e^{\frac{\beta^2}{2} K^T_m(0)} \int_{\mathcal{O}} \cos(\beta W_T(x)) \mathrm{d}x\right)\right], \tag{1.8}$$

where

$$K^T_m(x,y) = \int_0^T \frac{1}{4\pi t} e^{-\frac{|x-y|^2}{4t} - mt} \mathrm{d}t.$$

Let $W_T(x)$ be a random field with covariance

$$\mathbb{E}[\langle W_T, f\rangle_{L^2(\mathcal{O})} \langle W_T, g\rangle_{L^2(\mathcal{O})}] = \int_{\mathbb{R}^2} K^T_m(x,y) f(x) g(y) \mathrm{d}y \mathrm{d}x.$$

Now (1.8) is well defined since $K^T_m(0) < \infty$ and consequently $W_T \in L^2(\mathcal{O})$ almost surely. Furthermore we can compute that for any $(\tau_i)_{i=1}^n \in \{-1,1\}^n$ with k positive elements and $n-k$ negative ones, we get

$$\begin{aligned} &\mathbb{E}\left[\left(\int_{\mathcal{O}} e^{i\beta W_T(x) + \frac{\beta^2}{2} K^T_m(0)} \mathrm{d}x\right)^k \left(\int_{\mathcal{O}} e^{-i\beta W_T(y) + \frac{\beta^2}{2} K^T_m(0)} \mathrm{d}y\right)^{n-k}\right] \\ = &\int_{\mathcal{O}^n} \exp\left(-\beta^2 \sum_{1 \leqslant i<j \leqslant n} \tau_i \tau_j K^T_m(x_i,x_j)\right) \prod_{i=1}^n \mathrm{d}x_i. \end{aligned}$$

and so expanding $\mathbb{E}\left[\exp\left(-\lambda e^{\frac{\beta^2}{2} K^T_m(0)} \int_{\mathcal{O}} \cos(\beta W_T(x)) \mathrm{d}x\right)\right]$ in a series we obtain

$$\begin{aligned} &\mathbb{E}\left[\exp\left(-\lambda e^{\frac{\beta^2}{2} K^T_m(0)} \int_{\mathcal{O}} \cos(\beta W_T(x)) \mathrm{d}x\right)\right] \\ = &\mathbb{E}\left[\exp\left(-\frac{\lambda}{2}\left(\int_{\mathcal{O}} e^{i\beta W_T(x) + \frac{\beta^2}{2} K^T_m(0)} \mathrm{d}x + \int_{\mathcal{O}} e^{-i\beta W_T(y) + \frac{\beta^2}{2} K^T_m(0)} \mathrm{d}y\right)\right)\right] \\ = &\sum_{n=0} \sum_{k=0}^n \frac{\lambda^n}{2^n n!} \binom{n}{k} \mathbb{E}\left[\left(\int_{\mathcal{O}} e^{i\beta W_T(x) + \frac{\beta^2}{2} K^T_m(0)} \mathrm{d}x\right)^k \left(\int_{\mathcal{O}} e^{-i\beta W_T(y) + \frac{\beta^2}{2} K^T_m(0)} \mathrm{d}y\right)^{n-k}\right] \\ = &\sum_{n=0}^\infty \frac{\lambda^n}{2^n n!} \sum_{(\tau_i)_{i=1}^n \in \{-1,1\}^n} \int_{\mathcal{O}^n} \exp\left(-\beta^2 \sum_{1 \leqslant i<j \leqslant n} \tau_i \tau_j K^T_m(x_i,x_j)\right) \prod_{i=1}^n \mathrm{d}x_i. \end{aligned}$$

So sending $T\to\infty$ we obtain that (at least formally) we have

$$Z^Y_{\alpha,\beta,\lambda}=\mathbb{E}\left[\exp\left(-\lambda e^{\frac{\beta^2}{2}K_m(0)}\int_{\mathcal{O}}\cos(\beta W(x))\mathrm{d}x\right)\right].$$

Remark 1.27. Let us draw the readers attention to the following detail: Due to the irregularity of the Gaussian Free Field one would have

$$\cos(\beta W_T)\to 0$$

in a suitable sense as $T\to\infty$. However

$$e^{\frac{\beta^2}{2}K_m^T(0)}\cos(\beta W_T)$$

will converge to a (nontrivial) well defined random distribution as $T\to\infty$. Usually this renormalization has to be put in somewhat artificially and lacks an obvious physical meaning, however here it comes out as a the correct quantity from a physically sensible computation.

1.4. Contributions

The main theme of this thesis is the study of continuum Gibbs measures through variational/stochastic control techniques. The objects we study will always (at least formally) be of the form (1.2). We can take advantage of this by using the a formula first established by Boué and Dupuis [20] in the context of large deviations and later generalized by Üstünel [117]. Before stating the formula let us introduce a way to regularize the gaussian free field which will be convenient for us: Below we will take W_T to be a gaussian process on a probability space $\mathbb{P}$ such that

- W_T is a continuous martingale in T with values in the space of Schwartz functions $\mathscr{S}(\Lambda)$.
- W_T is smooth almost surely for $T<\infty$.
- $\mathrm{Law}(W_T)\to\mu$ where μ is the gaussian free field.

We will always construct W as stochastic integral of a cylindrical Brownian motion on $L^2(\Lambda)$ with $\Lambda=\mathbb{T}^d$ for $d=2,3$ or $\Lambda=\mathbb{R}^2$. At this point it will be convenient to change notation and denote by

$$V_T(\phi)=\int_\Lambda\phi^4-a_T\int_\Lambda\phi^2+b_T\quad\text{or}\quad V_T(\phi)=\alpha^T\int\sin(\beta\phi)$$

with suitably chosen $a_T,b_T,\alpha^T\to\infty$. Then we we will be able to express the approximate measures as defined by $\exp(-V_T(\phi))\mu(\mathrm{d}\phi)$ as

$$\nu^T(A)=\mathbb{P}^T(W_T\in A)$$

with

$$\mathrm{d}\mathbb{P}^T=\exp(-V_T(W_T))\mathrm{d}\mathbb{P}$$

Theorem 1.28. *Assume that $\mathcal{V}\colon C^\infty\to\mathbb{R}$ such that $\mathbb{E}[|\mathcal{V}(W_T)|^2]+\mathbb{E}[\exp(-2\mathcal{V}(W_T))]<\infty$, then*

$$-\log\mathbb{E}[\exp(-\mathcal{V}(W_T))]=\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[\mathcal{V}(W_T+I_T(u))+\frac{1}{2}\int_0^T\|u_t\|^2_{L^2(\Lambda)}\mathrm{d}t\right].\qquad(1.9)$$

Here

- *$\mathbb{H}_a$ is the space of processes adapted to the filtration generated by $(W_t)_{t\in\mathbb{R}_+}$ such that $\int_0^\infty\|u_t\|^2_{L^2(\Lambda)}\mathrm{d}t<\infty$ $\mathbb{P}$-almost surely,*
- *I is a bounded linear map from $L^2([0,\infty)\times\Lambda)\to C([0,\infty],H^1(\Lambda))$.*

All of the subsequent chapters will make heavy use of this formula of this formula in somewhat different ways. We will give a proof of a modified version, in chapter 2, relying on Girsanov's transform, and another proof in chapter 4 based on stochastic control theory.

1.4.1. Chapter 2

Our aim in Chapter 2, which closely follows the paper [13], will be to construct the $\Phi^4_{2,3}$ measure on the finite domain $\Lambda = \mathbb{T}^2, \mathbb{T}^3$. Inspired by section 1.2.1 let us introduce the Wick ordering

$$\begin{aligned} [\![W_T^4]\!] &:= W_T^4 - 6\mathbb{E}[W_T^2]\, W_T^2 + 3\mathbb{E}[W_T^2]^2, \\ [\![W_T^3]\!] &:= W_T^3 - 3\mathbb{E}[W_T^2] W_T, \\ [\![W_T^2]\!] &:= W_T^2 - \mathbb{E}[W_T^2]. \end{aligned}$$

One can prove that $[\![W_T^i]\!]$ are martingales in T. For $d=2$ we can choose $V_T(W_T) = \lambda \int_\Lambda [\![W_T^4]\!]$ and apply eq. (1.9) to obtain

$$\begin{aligned} & \mathcal{W}_T(f) \\ := & -\log \mathbb{E}[\exp(-f(W_T) - V_T(W_T))] \\ = & \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[f(W_T + I_T(u)) + \lambda \int [\![W_T^4]\!] + \lambda \int [\![W_T^3]\!] I_T(u) + \lambda \int [\![W_T^2]\!] (I_T(u))^2 \right. \\ & \left. \lambda \int W_T (I_T(u))^3 + \lambda \int (I_T(u))^4 + \frac{1}{2}\int_0^T \|u\|^2_{L^2(\Lambda)} \right] \end{aligned} \tag{1.10}$$

Our objective is to get lower and upper bounds on the right hand side of (1.10) to obtain tightness for the sequence of measures ν_T. Note that the terms $\int_\Lambda (I_T(u))^4 + \frac{1}{2}\int_0^T \|u_t\|^2_{L^2(\Lambda)} \mathrm{d}t$ are "good" in the sense that they are positive, and we will use them to bound the other terms which do not have a sign. If f has linear growth on the space Hölder space of regularity $\mathscr{C}^{-\delta}(\Lambda)$, that is $|f(\varphi)| \leqslant C(1 + \|f\|_{\mathscr{C}^{-\delta}(\Lambda)})$, we can prove, using that $\sup_{T\in\mathbb{R}_+} \mathbb{E}\big[\|[\![W_T^i]\!]\|^p_{\mathscr{C}^{-\delta}(\Lambda)}\big] < \infty$, that

$$\begin{aligned} & \mathbb{E}\left[f(W_T + I_T(u)) + \lambda \int [\![W_T^4]\!] + \lambda \int [\![W_T^3]\!] I_T(u) + \lambda \int [\![W_T^2]\!] (I_T(u))^2 + \lambda \int W_T (I_T(u))^3 + \right. \\ & \left. \lambda \int (I_T(u))^4 + \frac{1}{2}\int_0^T \|u\|^2_{L^2(\Lambda)} \right] \\ \geqslant & -C + \frac{1}{2}\mathbb{E}\left[\lambda \int_\Lambda (I_T(u))^4 + \frac{1}{2}\int_0^T \|u\|^2_{L^2(\Lambda)} \right] \\ \geqslant & -C. \end{aligned}$$

Where we have used that $\mathbb{E}[\int_\Lambda [\![W_T^4]\!]] = 0$ since $[\![W_T^4]\!]$ is a martingale. On the other hand by choosing $u=0$ we obtain an upper bound to get

$$-C \leqslant \log \mathbb{E}[\exp(-f(W_T) - V_T(W_T))] \leqslant C$$

which is the desired goal. For Φ^4_3 the situation is significantly more complicated since we no longer have $\sup_T \mathbb{E}[\|[\![W_T^i]\!]\|_{\mathscr{C}^{-\delta}(\Lambda)}] < \infty$. Instead one can only prove

$$\sup_T \mathbb{E}\big[\|W_T\|^p_{\mathscr{C}^{-1/2-\delta}(\Lambda)}\big] < \infty, \quad \sup_T \mathbb{E}\big[\|[\![W_T^2]\!]\|^p_{\mathscr{C}^{-1-\delta}(\Lambda)}\big] < \infty, \quad \sup_T \frac{1}{\log(T)} \mathbb{E}\big[\|[\![W_T^3]\!]\|^p_{\mathscr{C}^{-1-\delta}(\Lambda)}\big] < \infty.$$

so one can no longer bound the terms

$$\lambda \int [\![W_T^3]\!] I_T(u), \quad \lambda \int [\![W_T^2]\!] (I_T(u))^2 \tag{1.11}$$

by the good terms since $\int_0^T \|u\|^2_{L^2(\Lambda)}$ only controls the H^1 norm of $I_T(u)$ uniformly in T. For this reason we need to introduce additional renormalization constants and choose

$$V_T(W_T) = \int [\![W_T^4]\!] - \gamma_T \int [\![W_T^2]\!] - \delta_T.$$

After this we have to make the correct ansatz for u so that $\frac{1}{2}\int_0^T \|u\|^2_{L^2(\Lambda)}$ cancels the divergent terms (1.11). This ansatz will be of the form

$$u_t = -J_t [\![W_t^3]\!] - J_t [\![W_t^2]\!] \succ \theta_t I_t(u) + l_t(u) \tag{1.12}$$

where J_t is a Fourier multiplier localized in frequency in an annulus of radius t, θ_t is a Fourier multiplier localizing the function in a ball of radius $t/2$ and $l_t(u)$ is a remainder term defined by this formula. Finally $''\succ''$ is a para-product already mentioned in Section 1.2.1 (see Appendix A for a precise definition). Performing this "change of variables" and choosing γ_T, δ_T suitably we can then obtain

$$-C \leqslant \mathcal{W}_T(f) \leqslant C$$

also in the $d=3$ case. It is also known that Φ_3^4 is singular with respect to the free field even in finite volume ([3] and chapter 3). For this reason to our knowledge an "explicit" description(that is one not making reference to a limiting procedure) has been lacking so far in the literature. In this chapter we will obtain such a description in the form of a variational formula for the Laplace transform of Φ_3^4 by passing to the limit $T \to \infty$ in (1.9) , after having made the change of variables (1.12) using Γ-convergence. To be more precise we will obtain the following statement

THEOREM 1.29. *Let $d=3$ and take a small $\kappa > 0$. There exist renormalization constants γ_T, δ_T (which depend polynomially on λ) such that the limit*

$$\mathcal{W}(f) := \lim_{T \to \infty} \mathcal{W}_T(f),$$

exists for every $f \in C(\mathscr{C}^{-1/2-\kappa}; \mathbb{R})$ with linear growth. Moreover the functional $\mathcal{W}(f)$ has the variational form

$$\mathcal{W}(f) = \inf_{u \in \mathbb{H}_a^{-1/2-\kappa}} \mathbb{E}\left[f(W_\infty + I_\infty(u)) + \Psi_\infty(u) + \lambda \|I_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|l(u)\|_{L^2([0,\infty)\times\Lambda)}^2 \right]$$

where $\Psi_\infty(u)$ a nice polynomial (non-random) functional of (W, u),independent of f, and $\mathbb{H}_a^{-1/2-\kappa}$ is the space of predictable processes (wrt. the Brownian filtration) in $L^2(\mathbb{R}_+; H^{-1/2-\kappa})$,with H^α being the Sobolev space defined by the norm $\|f\|_{H^\alpha}^2 = \langle f, (1-\Delta)^{2\alpha} f\rangle_{L^2(\Lambda)}$. In particular the measures ν^T converge to a unique limit ν^∞.

1.4.2. Chapter 3

In this chapter, which closely follows the preprint [14] we continue our study of the Φ_3^4 measure. As already mentioned a significant difficulty in the study of Φ_3^4 is that it is singular with respect to the Gaussian Free Field. In this chapter we will construct a "replacement" for the free field, that is a measure $\mathbb{Q}^v$ with respect to which the Φ_3^4 measure is absolutely continuous. $\mathbb{Q}^v$ is obtained from $\mathbb{P}$ by a Girsanov transform. Indeed the time t which for us parametrizes the scale provides a filtration $\mathscr{F}_T$ (the one generated by the cylindrical Brownian motion on $L^2(\Lambda)$). We will then take $\mathbb{Q}^u$ to satisfy

$$\left.\frac{\mathrm{d}\mathbb{Q}^u}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_T} = \exp\left(L_T^u - \frac{1}{2}\langle L^u \rangle_T \right), \qquad L_t^v = \int_0^t \langle u_s, \mathrm{d}W_s \rangle_{L^2(\Lambda)} \tag{1.13}$$

and v is a specific drift which is designed to cancel the divergences in the densities for Φ_3^4. Indeed the considerations from Chapter 2 suggest to take v satisfying the equation

$$u = \Xi(W^u, u) = \Xi(W - I(u), u)$$

with

$$\Xi_s(W, u) := -\lambda J_s \mathbb{W}_s^3 - \lambda J_s(\mathbb{W}_s^2 \succ \theta_s I_s(u)).$$

Actually we will have to make some technical modifications to Ξ to simplify the proof construction of $\mathbb{Q}^v$ and the proof of absolute continuity but we will not go into this here. We will then consider the sequence of densities

$$D_T := \frac{1}{Z_T} e^{-V_T(W_T)} \left(\frac{\mathrm{d}\mathbb{Q}^u}{\mathrm{d}\mathbb{P}} \right)^{-1}$$

and see that we can prove localized L^p bound on D_T. That is we will be able to show that

$$\sup_T \mathbb{E}_{\mathbb{Q}^u}\left[|D_T|^p \mathbb{1}_{\{\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}} \leqslant K\}} \right] < \infty.$$

which will imply absolute continuity of any accumulation point of the family $\mathbb{P}^T$,which is given by

$$\mathrm{d}\mathbb{P}^T = \exp(-V_T(W_T))\mathrm{d}\mathbb{P} \quad \text{with} \quad V_T(W_T) = \int W_T^4 - a_T \int W_T^2 - b_T,$$

with respect to $\mathbb{Q}^u$. As an application of our result we will provide a proof that Φ_3^4 is singular with respect to the free field by constructing an event which has probability 0 $\mathbb{P}$ and probability 1 under $\mathbb{Q}^u$. More precisely we will show that $S \subseteq \mathscr{C}^{-1/2-\varepsilon}(\Lambda)$ defined by

$$S := \left\{ f \in \mathscr{C}^{-1/2-\varepsilon}(\Lambda) \colon \frac{1}{T_n^{1/2+\delta}} \int_\Lambda [\![(\theta_{T_n} f)^4]\!] \to 0 \right\}$$

for some suitable sequence $T_n \to \infty$, and θ_T being a family of Fourier multipliers localized in a ball of radius T, satisfies $\mathbb{P}(W_\infty \in S) = 1$ and $\mathbb{Q}^u(W_\infty \in S) = 0$. Since Φ_3^4 is absolutely continuous with respect to $\mathbb{Q}^u$ this will prove singularity.

1.4.3. Chapter 4

In this chapter we study the Sine-Gordon model on $\mathbb{R}^2$. Recall that this is formally defined by

$$e^{-\lambda \int \cos(\beta\phi)} \mu(\mathrm{d}\phi)$$

with $\mu(\mathrm{d}\phi)$ now being the Gaussian free field on $\mathbb{R}^2$. In the previous chapters we have dealt with Φ_3^4 on a finite domain, where the difficulty arose from having to make sense of the non-linearity applied to an irregular distribution. In this chapter we will encounter the additional complication of "infrared divergence" that is of having to make sense of $\int \cos(\beta\phi)$ on the whole space $\mathbb{R}^2$ without $\cos(\beta\phi)$ having any decay property at infinity. After introducing a regularization and a spacial cutoff ρ we end up with the measure

$$\nu_{\mathrm{SG}}^{\rho,T}(A) = \mathbb{P}^{T,\rho}(W_T \in A)$$

with

$$\mathrm{d}\mathbb{P}^{T,\rho} = \frac{1}{Z_{T,\rho}} \exp\left(-\alpha(T) \int \rho \cos(\beta W_T) \right) \mathrm{d}\mathbb{P}$$

where W_T is constructed in analogy with Chapters 2 and 3, $\alpha(T)$ is a diverging renormalization constant introduced to prevent the limit from becoming trivial (see Section 4.2.2 and recall Section 1.3), $\rho \in C_c^\infty(\mathbb{R}^2)$ is a spacial cutoff, and $Z_{T,\rho}$ is a normalization constant turning $\mathbb{P}^{T,\rho}$ into a probability measure. We will be looking to take the limits $\rho \to 1, T \to \infty$. Using the Boué-Dupuis formula we will have

$$\begin{aligned} & -\log \int e^{-f(\phi)} \nu_{\mathrm{SG}}^{\rho,T}(\mathrm{d}\phi) \\ = & \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[f(W_T + I_T(u)) + \lambda\alpha(T) \int \rho \cos(\beta W_T + \beta I_T(u)) + \frac{1}{2} \int_0^T \|u\|_{L^2(\mathbb{R}^2)}^2 \right] \\ & - \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[\lambda\alpha(T) \int \rho \cos(\beta W_T + \beta I_T(u)) + \frac{1}{2} \int_0^T \|u\|_{L^2(\mathbb{R}^2)}^2 \right] \end{aligned}$$

Denoting u^f a minimizer of the functional

$$F^{T,\rho,f}(u) = \mathbb{E}\left[f(W_T + I_T(u)) + \lambda\alpha(T) \int \rho \cos(\beta W_T + \beta I_T(u)) + \frac{1}{2} \int_0^T \|u\|_{L^2(\mathbb{R}^2)}^2 \right],$$

we will see that our main goal will be to control the dependence of u^f on f and ρ. In fact we will show that $u^f - u$ decays exponentially fast outside of the support of f or more precisely

$$\int_0^T \int \exp(\gamma x) |u_t^f(x) - u_t^0(x)|^2 \mathrm{d}x \tag{1.14}$$

Using this and similar estimates we will be able to prove the following results

Theorem 1.30. *$\nu_{\mathrm{SG}}^{T,\rho}$ converges as $T \to \infty, \rho \to 1$ weakly to a measure ν_{SG} on $\mathscr{S}'(\mathbb{R}^2)$. Furthermore μ_{SG} satisfies*

$$-\log \int e^{-f(\varphi)} \nu_{\mathrm{SG}}(\mathrm{d}\varphi) = \inf_{u \in \mathbb{D}^f} \mathbb{E}\left[f(W_{0,\infty} + I_{0,\infty}(u) + I_{0,\infty}(u^\infty)) + \Psi(u) + \frac{1}{2}\int_0^\infty \|u_t\|_{L^2}^2 \mathrm{d}t \right]$$

where

- *$u^\infty \in L^\infty(\mathbb{P}, L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)) \cap L^2(\mathbb{P}, L^2(\mathbb{R}_+, L^2(\langle x \rangle^{-n})))$ where by $L^2(\langle x \rangle^{-n})$ we denote the space equipped with the norm $\|f\|_{L^2(\langle x \rangle^{-n})}^2 = \int_{\mathbb{R}^2} \langle x \rangle^{-n} f(x) \mathrm{d}x . u^\infty$ does not depend on f.*
- *I is a linear map improving regularity by 1 similarly as above*
- *$\Psi(u)$ is a functional of u which also depends on u^∞ and W, it will be specified below*
- *$\mathbb{D}^f$ is a subspace of $\mathbb{H}_a$ containing drifts with exponential decay in space, it will also be specified below, similarly to eq. (1.14)*

We will also obtain a description of the Sine-Gordon measure as a random shift of a gaussian measure, similarly to how we described the drift measure in Chapter 3 but this time for the full Sine-Gordon measure.

Theorem 1.31. *There exists a random variable $I \in L^\infty(\mathbb{P}, W^{1,\infty}(\mathbb{R}^2))$ such that*

$$\nu_{\mathrm{SG}} = \mathrm{Law}_{\mathbb{P}}(W_\infty + I).$$

Furthermore the Law of the pair (W_∞, I) is invariant under the action of the Euclidean group.

Remark 1.32. We remark here that in comparison with Chapter 3 the description of the random variable I is more complicated in this case since the equation it solves involves the value function of the the control problem which is a somewhat “implicit” object (see Proposition 4.13 in Section 4.2 for details).

We will also be able to prove that the Sine-Gordon measure satisfies the OS-Axioms described in section 1.1.2.

Theorem 1.33. *ν_{SG} satisfies the Osterwalder-Schrader axioms according to Definition 1.22. Furthermore the clustering is exponential and ν_{SG} is non-Gaussian.*

1.4.4. Large Deviations

In Chapter 2 and Chapter 4 we will also discuss *Large Deviation Principles* for the considered measures, that is the Φ_3^4 measure on $\mathbb{T}^3$ and the Sine-Gordon measure on $\mathbb{R}^2$, in the so called semi-classical limit. The theory of large deviations has seen many nice expositions see [50, 55, 47]. For a sequence of probability measures ν_n on a metric space S it codifies how fast probabilities of “unlikely” events go to 0. More precisely as $n \to \infty$ one looks for an estimate of the form

$$\nu_n(A) \approx \exp\left(-n \inf_{x \in A} I(A) \right),$$

where $I: S \to \mathbb{R}_+ \cup \{\infty\}$ is called a rate function. Rigorously we have the following definition:

Definition 1.34. *A sequence of measures ν_n on a polish space S satisfies the Large deviations principle with rate function I if for any closed set $A \subset S$*

$$\limsup_{n \to \infty} -\frac{1}{n} \log \nu_n(A) \geqslant \inf_{x \in A} I(x)$$

and for any open set $B \subset S$

$$\liminf_{n \to \infty} -\frac{1}{n} \log \nu_n(A) \leqslant \inf_{x \in A} I(x)$$

Large deviations are equivalent to what is known as the Laplace principle, which is a description of the limit of the Laplace transforms of ν_n.

DEFINITION 1.35. *A sequence of measures ν_n on a polish S satisfies the Laplace principle with rate function I if for any continuous bounded function $f: S \to \mathbb{R}$*

$$\lim_{n\to\infty} -\frac{1}{n}\log\int e^{-nf(x)}\mathrm{d}\nu_n(x) = \inf_{x\in S}\{f(x)+I(x)\}.$$

It is well known that the Large deviations principle and the Laplace principle are equivalent, see for instance [50], Section 1.2.

To look at the semi-classical limit we have to introduce Planck's constant into the measure. We can consider the measures formally given by

$$\nu^\hbar = \frac{1}{Z_\hbar}e^{-\frac{1}{\hbar}\int\lambda\phi^4+\frac{1}{2}m^2\varphi(x)^2+\frac{1}{2}|\nabla\varphi(x)|^2\mathrm{d}x}\mathrm{d}\varphi.$$

where again $Z_\hbar$ is a normalization constant.One can give a meaning to these measures in the way that has been shown for ν in the previous discussion. More rigorously this should be defined should be defined by the limit of the measures

$$\int g(\phi)\nu_T^\hbar(\mathrm{d}\phi) = \frac{\mathbb{E}\Big[g(\hbar^{1/2}W_T)\,e^{-\frac{1}{\varepsilon}V_T^\hbar(\hbar^{1/2}W_T)}\Big]}{Z_\hbar^T}.$$

In Chapter 2 we will consider the Φ_3^4 model on a finite domain, which again has to be appropriately renormalized (see Section 2.7 for details). Proving that the $T\to\infty$ limit exists can be done in exactly the same way as for the $\hbar=1$ case treated before, and we can also obtain a variational formula for the Laplace transform. Taking $\hbar\to 0$ in the variational formula we will obtain the following theorem.

THEOREM 1.36. *The sequence of measures $\nu_T^\hbar$ converges weakly on $\mathscr{C}^{-1/2-\varepsilon}$, for any $\varepsilon>0$, to a unique limit $\nu^\hbar$ as $T\to\infty$. Furthermore $\nu^\hbar$ satisfies a Laplace principle with rate function*

$$I(\psi) = \lambda\int\psi^4 + m^2\int\psi^2 + \int|\nabla\psi|^2 \tag{1.15}$$

as $\hbar\to 0$.

In Chapter 4 we can obtain a similar situation for Sine-Gordon where we can define $\mu_{\mathrm{SG},\hbar}$ as the limit of

$$\nu_\hbar^{\rho,T}(\mathrm{d}\varphi) = \frac{1}{Z_\hbar^{\rho,T}}e^{-\frac{\lambda}{\hbar}\alpha(T)\int\rho\cos(\hbar^{1/2}\beta\varphi)}\mu(\mathrm{d}\varphi),$$

as $T\to\infty$, $\rho\to 1$. We then have the following theorem.

THEOREM 1.37. *The measures $\nu_\hbar^{\rho,T}$ converge weakly on $H^{-1}(\langle x\rangle^{-n})$ for n large enough to a limiting measure $\nu_{\mathrm{SG},\hbar}$. $\nu_{\mathrm{SG},\hbar}$ satisfies a Large deviation principle with rate function*

$$I(\varphi) = \lambda\int(\cos(\varphi)-1) + \frac{1}{2}m^2\int\varphi^2 + \frac{1}{2}\int|\nabla\varphi(x)|^2$$

as $\hbar\to 0$.

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Chapter 2

A Variational Approach to Φ_3^4

2.1. Introduction

The Φ_d^4 Gibbs measure on the d-dimensional torus $\Lambda = \Lambda_L = \mathbb{T}_L^d = (\mathbb{R}/(2\pi L\mathbb{Z}))^d$ is the probability measure ν obtained as the weak limit for $T \to \infty$ of the family $(\nu_T)_{T>0}$ given by

$$\nu_T(\mathrm{d}\phi) = \frac{\exp[-V_T(\phi_T)]}{\mathscr{Z}_T}\vartheta(\mathrm{d}\phi), \tag{2.1}$$

where

$$V_T(\varphi) := \lambda \int_\Lambda (|\varphi(\xi)|^4 - a_T|\varphi(\xi)|^2 + b_T)\mathrm{d}\xi, \qquad \mathscr{Z}_T := \int e^{-V_T(\phi_T)}\vartheta(\mathrm{d}\phi).$$

Here $\lambda \geqslant 0$ is a fixed constant, Δ is the Laplacian on Λ, ϑ is the centered Gaussian measure with covariance $(1-\Delta)^{-1}$, $\mathscr{Z}_T$ is a normalization factor, a_T, b_T given constants and $\phi_T = \rho_T * \phi$ with ρ_T some appropriate smooth and compactly supported cutoff function such that $\rho_T \to \delta$ as $T \to \infty$. In comparison with the introduction have set $m^2 = 1$ here, since we are on finite volume the mass does not really play a role. The measures ϑ and ν_T are realized as probability measures on $\mathscr{S}'(\Lambda)$, the space of tempered distributions on Λ. They are supported on the Hölder–Besov space $\mathscr{C}^{(2-d)/2-\kappa}(\Lambda)$ for all small $\kappa > 0$. The existence of the limit ν is conditioned on the choice of a suitable sequence of *renormalization constants* $(a_T, b_T)_{T>0}$. The constant b_T is not necessary, but is useful to decouple the behavior of the numerator from that of the denominator in eq. (2.1).

The aim of this paper is to give a proof of convergence using a variational formula for the partition function $\mathscr{Z}_T$ and for the generating function of the measure ν_T. As a byproduct we obtain also a variational description for the generating function of the limiting measure ν via Γ-convergence of the variational problem. Let us remark that, to our knowledge, it is the first time that such explicit description of the unregulated Φ_3^4 measure is available.

Our work can be seen as an alternative realization of Wilson's [118] and Polchinski's [108] continuous renormalization group (RG) method. This method has been made rigorous by Brydges, Slade et al. [30, 28, 29] and as such witnesses a lot of progress and successes [33, 34, 16, 35, 36]. The key idea is the nonperturbative study of a certain infinite dimensional Hamilton–Jacobi–Bellman equation [32] describing the effective, scale dependent, action of the theory. Here we avoid the analysis involved in the direct study of the PDE by going to the equivalent stochastic control formulation, well established and understood in finite dimensions [56]. The time parameter of the evolution corresponds to an increasing amount of small scale fluctuations of the Euclidean field and our main tool is a variational representation formula, introduced by Boué and Dupuis [20], for the logarithm of the partition function interpreted as the value function of the control problem. See also the related papers of Üstünel [117] and Zhang [120] where extensions and further results on the variational formula are obtained. The variational formula has been used by Lehec [91] to prove some Gaussian functional inequalities, following the work of Borell [19]. In this representation we can avoid the analysis of an infinite dimensional second order operator and concentrate more on path-wise properties of the Euclidean interacting fields. We are able to leverage techniques developed for singular SPDEs, in particular the para-controlled calculus developed in [73], to perform the renormalization of various non-linear quantities and show uniform bounds in the $T \to \infty$ limit.

Define the normalized free energy $\mathcal{W}_T$ for the cutoff Φ_3^4 measure, as the functional

$$\mathcal{W}_T(f) := -\frac{1}{|\Lambda|}\log \int_{\mathscr{S}'(\Lambda)} \exp[-|\Lambda| f(\phi) - V_T(\phi_T)]\vartheta(\mathrm{d}\phi), \tag{2.2}$$

for all $f \in C(\mathscr{S}'(\Lambda); \mathbb{R})$. The main result of the paper is the following

THEOREM 2.1. *Let $d=3$ and take a small $\kappa>0$. There exist renormalization constants a_T, b_T (which depend polynomially on λ) such that the limit*

$$\mathcal{W}(f) := \lim_{T\to\infty} \mathcal{W}_T(f),$$

exists for every $f \in C(\mathscr{C}^{-1/2-\kappa}; \mathbb{R})$ with linear growth. Moreover the functional $\mathcal{W}(f)$ has the variational form

$$\mathcal{W}(f) = \inf_{u\in\mathbb{H}_a^{-1/2-\kappa}} \mathbb{E}\left[f(W_\infty + Z_\infty(u)) + \Psi_\infty(u) + \lambda\|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|l(u)\|^2_{L^2([0,\infty)\times\Lambda)}\right]$$

where

- *$\mathbb{E}$ denotes expectations on the Wiener space of a cylindrical Brownian motion $(X_t)_{t\geqslant 0}$ on $L^2(\Lambda)$ with law $\mathbb{P}$;*
- *$(W_t)_{t\geqslant 0}$ is a Gaussian martingale process adapted to $(X_t)_{t\geqslant 0}$ and such that $\mathrm{Law}_{\mathbb{P}}(W_t) = \mathrm{Law}_{\vartheta}(\phi_t)$;*
- *$\mathbb{H}_a^{-1/2-\kappa}$ is the space of predictable processes (wrt. the Brownian filtration) in $L^2(\mathbb{R}_+; H^{-1/2-\kappa})$;*
- *$(Z_t(u), l_t(u))_{t\geqslant 0}$ are explicit (non-random) functions of $u\in\mathbb{H}_a^{-1/2-\kappa}$ and W;*
- *$\Psi_\infty(u)$ a nice polynomial (non-random) functional of (W, u), independent of f.*

See Section 2.4 and in particular Lemma 2.22 and Theorem 2.23 for precise definitions of the various objects and a more detailed statement of this result. With respect to the notations in Lemma 2.22, observe that

$$f(W_\infty + Z_\infty(u)) + \Psi_\infty(u) = \Phi_\infty(\mathbb{W}, Z(u), K(u)).$$

Theorem 2.1 implies directly the convergence of $(\nu_T)_T$ to a limit measure ν on $\mathscr{S}'(\Lambda)$. Taking f in the linear dual of $\mathscr{C}^{-1/2-\kappa}$ it also gives the following formula for the Laplace transform of ν:

$$\int_{\mathscr{S}'(\Lambda)} \exp(-f(\phi))\nu(\mathrm{d}\phi) = \exp(-|\Lambda|(\mathcal{W}(f/|\Lambda|) - \mathcal{W}(0))). \tag{2.3}$$

To our knowledge this is the first such explicit description (i.e. without making reference of the limiting procedure). The difficulty is linked to the conjectured singularity of the Φ_3^4 measure with respect to the reference Gaussian measure. Another possible approach to an explicit description goes via integration by parts (IBP) formulas, see [10] for an early proof and a discussion of this approach. More recently [71] gives a self–contained proof of the IBP formula for any accumulation point of the Φ_3^4 in the full space. However is still not clear how to use these formulas directly to obtain uniqueness of the measure and/or other properties (either on the torus or on the more difficult situation of the full space). Therefore, while our approach here is limited to the finite volume situation, it could be used to prove additional results, like large deviations or weak universality very much like in the case of SPDEs, see e.g. [79, 63].

The parameter L, which determines the size of the spatial domain $\Lambda = \Lambda_L$, will be kept fixed all along the paper and we will not attempt here to obtain the infinite volume limit $L\to\infty$. For this reason we will avoid to explicitly show the dependence of $\mathcal{W}_T$ with Λ. However some care will be taken to obtain estimates uniform in the volume $|\Lambda|$.

An easy consequence of the estimates needed to establish the main theorem is the following corollary (well known in the literature, see e.g. [18]):

COROLLARY 2.2. *There exists functions $E_+(\lambda), E_-(\lambda)$ not depending on $|\Lambda|$, such that*

$$\lim_{\lambda\to 0+} \frac{E_\pm(\lambda)}{\lambda^3} = 0,$$

and, for any $\lambda>0$,

$$E_-(\lambda) \leqslant \mathcal{W}_T(0) \leqslant E_+(\lambda).$$

A similar statement for $d=2$ will be sketched below in order to introduce some of the ideas on which the $d=3$ proof is based.

The construction of the $\Phi^4_{2,3}$ measure in finite volume is a basic problem of constructive quantum field theory to which many works have been devoted, especially in the $d=2$ case. It is not our aim to provide here a comprehensive review of this literature. As far as the $d=3$ case is concerned, let us just mention some of the results that, to different extent, prove the existence of the limit as the ultraviolet (small scale) regularization is removed. After the early work by Glimm and Jaffe [65, 64], in part performed in the Hamiltonian formalism, all the subsequent research has been formulated in the Euclidean setting: i.e. as the problem of construction and study of the probability measures ν on a space of distributions. Feldman [53], Park [107], Feldman and Osterwalder [54], Magnén and Sénéor [92], Benfatto et al. [18], Brydges, Fröhlich and Sokal [31] and Brydges, Dimock and Hurd [28] obtained the main results we are aware of. Recent advances in the analysis of singular SPDEs put forward by the invention of regularity structures by M. Hairer [77] and related approaches [73, 40, 105], or even RG–inspired ones [84], have allowed to pursue the stochastic quantization program to a point where now it can be used to prove directly the existence of the finite volume Φ^4_3 measure in two different ways [97, 9]. Uniqueness by these methods requires additional efforts but seems at reach. Some results on the existence of the infinite volume measure [71] and dynamics [72] have been obtained recently. For an overview of the status of the constructive program wrt. the analysis of the $\Phi^4_{2,3}$ models the reader can consult the introduction to [9] and [71].

This chapter is organized as follows. In Section 2.2 we set up our main tool, the Boué–Dupuis variational formula of Theorem 2.4. Then, as a warm-up exercise, we use the formula to show bounds and existence of the Φ^4_2 measure in Section 2.3. We then pass to the more involved situation of three dimensions in Section 2.4 where we introduce the renormalized variational problem. In Section 2.5 we establish uniform bounds for this new problem and in Section 2.6 we prove Theorem 2.1. Section 2.8 and Section 2.9 are concerned with some details of the analytic and probabilistic estimates needed throughout the paper. Appendix A gathers background material on functional spaces, paraproducts and related functional analytic background material.

Convention. Let us fix some notations and objects.

- For $a\in\mathbb{R}^d$ let $\langle a\rangle := (1+|a|^2)^{1/2}$.
- The various constants appearing in the estimates will be understood uniform in $|\Lambda|$, unless otherwise stated.
- The constant $\kappa>0$ represents a small positive number which can be different from line to line.
- Denote with $\mathscr{S}(\Lambda)$ the space of Schwartz functions on Λ and with $\mathscr{S}'(\Lambda)$ the dual space of tempered distributions. The notation $\hat{f}$ or $\mathscr{F}f$ stands for the space Fourier transform of f and we will write $g(\mathrm{D})$ to denote the Fourier multiplier operator with symbol $g\colon\mathbb{R}^n\to\mathbb{R}$, i.e. $\mathscr{F}(g(\mathrm{D})f)=g\mathscr{F}f$.
- In order to easily keep track of the volume dependence of various objects we normalize the Lebesgue measure on Λ to have unit mass. We denote the normalized integral and measure by
$$⨍ f := \frac{1}{|\Lambda|}\int_\Lambda f, \quad đx = \frac{1}{|\Lambda|}\mathrm{d}x,$$
where $|\Lambda|$ is the volume of Λ. Norms in all the related functional spaces (Lebesgue, Sobolev and Besov spaces) are understood similarly normalized unless stated otherwise. This normalization of the functional spaces is used not because it is the most convenient one but because it is the one relevant to obtain uniform estimates in the volume of the variational functional. For example, another normalization of H^1 norm would no longer be controlled by the L^2 norm of the drift appearing in Theorem 2.4 below uniformly in $|\Lambda|$. Note that that with our choice of normalization the Sobolev embedding no longer holds uniformly in $|\Lambda|$. This is the reason why we carefully avoid to use it in the estimates of Section 2.8.

The reader is referred to Appendix A for an overview of the functional spaces and the additional notations used in the paper.

2.2. A stochastic control problem

We begin by constructing a probability space $(\Omega, \mathscr{B}, \mathbb{P})$ endowed with a process $(W_t)_{t\in[0,\infty]}$ belonging to $C([0,\infty], \mathscr{C}^{(2-d)/2-\kappa}(\Lambda))$ and such that $\mathrm{Law}_\vartheta(\phi_T) = \mathrm{Law}_\mathbb{P}(W_T)$ for all $T \geqslant 0$ and $\mathrm{Law}_\mathbb{P}(W_\infty) = \vartheta$, the Gaussian free field with covariance $(1-\Delta)^{-1}$.

Fix $\alpha < -d/2$ and let $\Omega := C(\mathbb{R}_+; H^{-\alpha})$, $(X_t)_{t\geqslant 0}$ the canonical process on Ω and $\mathscr{B}$ the Borel σ–algebra of Ω. On $(\Omega, \mathscr{B})$ consider the probability measure $\mathbb{P}$ which makes the canonical process X a cylindrical Brownian motion in $L^2(\Lambda)$. In the following $\mathbb{E}$ without any qualifiers will denote expectations wrt. $\mathbb{P}$ and $\mathbb{E}_\mathbb{Q}$ will denote expectations wrt. some other measure $\mathbb{Q}$. On the probability space $(\Omega, \mathscr{B}, \mathbb{P})$ there exists a collection $(B_t^n)_{n\in(L^{-1}\mathbb{Z})^d}$ of complex (2-dimensional) Brownian motions, such that $\overline{B_t^n} = B_t^{-n}$, B_t^n, B_t^m independent for $m \neq \pm n$ and $X_t = |\Lambda|^{-1/2}\sum_{n\in(L^{-1}\mathbb{Z})^d} e^{i\langle n,\cdot\rangle} B_t^n$. Note that X has a.s. trajectories in $C(\mathbb{R}_+, \mathscr{C}^{-d/2-\varepsilon}(\Lambda))$ for any $\varepsilon > 0$ by standard arguments.

Fix some $\rho \in C_c^\infty(\mathbb{R}_+, \mathbb{R}_+)$, decreasing such that $\rho(s) = 1$ for any $s \leqslant 1/2$ and $\rho(s) = 0$ for any $s \geqslant 1$. For $x \in \mathbb{R}^d$, set, $\rho_t(x) := \rho(\langle x\rangle/t)$ and

$$\sigma_t(x) := \left(\frac{\mathrm{d}}{\mathrm{d}t}(\rho_t^2(x))\right)^{1/2} = (2\dot{\rho}_t(x)\rho_t(x))^{1/2} = (-2(\langle x\rangle/t)\rho(\langle x\rangle/t)\rho'(\langle x\rangle/t))^{1/2}/t^{1/2},$$

where $\dot{\rho}_t$ is the partial derivative of ρ_t with respect to t. Consider the process $(W_t)_{t\geqslant 0}$ defined by

$$W_t := \frac{1}{|\Lambda|^{1/2}} \sum_{n\in(L^{-1}\mathbb{Z})^d} \int_0^t \frac{\sigma_s(n)}{\langle n\rangle} e^{i\langle n,\cdot\rangle} \mathrm{d}B_s^n, \qquad t \geqslant 0. \tag{2.4}$$

It is a centered Gaussian process with covariance

$$\begin{aligned}
\mathbb{E}[\langle W_t, \varphi\rangle \langle W_s, \psi\rangle] &= \frac{1}{|\Lambda|} \sum_{n,m\in(L^{-1}\mathbb{Z})^d} \mathbb{E}\left[\int_0^t \frac{\sigma_u(n)}{\langle n\rangle} \mathrm{d}B_u^n\, \hat{\varphi}(n) \overline{\int_0^s \frac{\sigma_u(m)}{\langle m\rangle} \mathrm{d}B_s^m\, \hat{\psi}(m)}\right] \\
&= \frac{1}{|\Lambda|} \sum_{n\in(L^{-1}\mathbb{Z})^d} \frac{\rho_{\min(s,t)}^2(n)}{\langle n\rangle^2} \hat{\varphi}(n)\overline{\hat{\psi}(n)},
\end{aligned}$$

for any $\varphi, \psi \in \mathscr{S}(\Lambda)$ and $t, s \geqslant 0$, by Fubini theorem and Itô isometry. By dominated convergence $\lim_{t\to\infty}\mathbb{E}[\langle W_t, \varphi\rangle\langle W_t, \psi\rangle] = |\Lambda|^{-1}\sum_{n\in(L^{-1}\mathbb{Z})^d} \langle n\rangle^{-2}\hat{\varphi}(n)\overline{\hat{\psi}(n)}$ for any $\varphi, \psi \in L^2(\Lambda)$.

Note that up to any finite time T the r.v. W_T has a bounded spectral support and the stopped process $W_t^T = W_{t\wedge T}$ for any fixed $T > 0$, is in $C(\mathbb{R}_+, W^{k,2}(\Lambda))$ for any $k \in \mathbb{N}$. Furthermore $(W_t^T)_t$ only depends on a finite subset of the Brownian motions $(B^n)_n$. Denote

$$W_t = \int_0^t J_s \mathrm{d}X_s, \qquad t \geqslant 0, \tag{2.5}$$

with $J_s := \langle \mathrm{D}\rangle^{-1}\sigma_s(\mathrm{D})$. Observe that W_t has a distribution given by the push-forward $(\rho_t(\mathrm{D}))_*\vartheta$ of ϑ through $\rho_t(\mathrm{D})$. We write the measure ν_T in (2.1) in terms of expectations over $\mathbb{P}$ as

$$\int g(\phi)\nu_T(\mathrm{d}\phi) = \frac{\mathbb{E}[g(W_T)\, e^{-V_T(W_T)}]}{\mathscr{Z}_T}, \tag{2.6}$$

for any bounded measurable $g: \mathscr{S}'(\Lambda) \to \mathbb{R}$.

For fixed T the polynomial appearing in the expression for $V_T(W_T)$ is bounded below (since $\lambda > 0$) and $\mathscr{Z}_T$ is well defined and also bounded away from zero (this follows easily from Jensen's inequality). However as $T \to \infty$ we tend to loose both these properties due to the fact that we will be obliged to take $a_T \to +\infty$ to renormalize the non–linear terms. To obtain uniform upper and lower bounds we need a more detailed analysis and we proceed as follows.

Denote by $\mathbb{H}_a$ the space of progressively measurable processes which are $\mathbb{P}$–almost surely in $\mathcal{H} := L^2(\mathbb{R}_+ \times \Lambda)$. We say that an element v of $\mathbb{H}_a$ is a *drift*. Below we will need also (generalized) drifts belonging to $\mathcal{H}^\alpha := L^2(\mathbb{R}_+; H^\alpha(\Lambda))$ for some $\alpha \in \mathbb{R}$, we denote the corresponding space with $\mathbb{H}_a^\alpha$. Consider the measure $\mathbb{Q}_T$ on $(\Omega, \mathscr{B})$ whose Radon–Nykodim derivative wrt. $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = \frac{e^{-V_T(W_T)}}{\mathscr{Z}_T}.$$

Since W_T depends on finitely many Brownian motions $(B^n)_n$, it is well known [109, 57] that any $\mathbb{P}$–absolutely continuous probability can be expressed via Girsanov transform. In particular, by the Brownian martingale representation theorem there exists a drift $u^T \in \mathbb{H}_a$ such that

$$\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^\infty u_s^T \mathrm{d}X_s - \frac{|\Lambda|}{2}\int_0^\infty \|u_s^T\|_{L^2}^2 \mathrm{d}s\right),$$

(recall that we normalized the $L^2(\Lambda)$ norm) and the entropy of $\mathbb{Q}_T$ wrt. $\mathbb{P}$ is given by

$$H(\mathbb{Q}_T|\mathbb{P}) = \mathbb{E}_{\mathbb{Q}_T}\left[\log \frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}}\right] = \frac{|\Lambda|}{2}\mathbb{E}_{\mathbb{Q}_T}\left[\int_0^\infty \|u_s^T\|_{L^2}^2 \mathrm{d}s\right].$$

Here equality holds also if one of the two quantities is $+\infty$. By Girsanov theorem, the canonical process X is a semimartingale under $\mathbb{Q}_T$ with decomposition

$$X_t = \tilde{X}_t + \int_0^t u_s^T \mathrm{d}s, \qquad t \geqslant 0,$$

where $(\tilde{X}_t)_t$ is a cylindrical $\mathbb{Q}_T$–Brownian motion in $L^2(\Lambda)$. Under $\mathbb{Q}_T$ the process $(W_t)_t$ has the semimartingale decomposition $W_t = \tilde{W}_t + U_t$ with

$$\tilde{W}_t := \int_0^t J_s \mathrm{d}\tilde{X}_s, \quad \text{and} \quad U_t = I_t(u^T),$$

where for any drift $v \in \mathbb{H}_a$ we define

$$I_t(v) := \int_0^t J_s v_s \mathrm{d}s.$$

The integral in the density can be restricted to $[0, T]$ since $u_t^T = 0$ if $t > T$. Now

$$-\log \mathscr{Z}_T = -\log\left[e^{-V_T(W_T)}\left(\frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}}\right)^{-1}\right] = V_T(W_T) + \int_0^\infty u_s^T \mathrm{d}X_s - \frac{|\Lambda|}{2}\int_0^\infty \|u_s^T\|^2 \mathrm{d}s, \tag{2.7}$$

and taking expectation of (2.7) wrt $\mathbb{Q}_T$ we get

$$-\log \mathscr{Z}_T = \mathbb{E}_{\mathbb{Q}_T}\left[V_T(\tilde{W}_T + I_T(u^T)) + \frac{|\Lambda|}{2}\int_0^\infty \|u_s^T\|^2 \mathrm{d}s\right]. \tag{2.8}$$

For any $v \in \mathbb{H}_a$ define the measure $\mathbb{Q}^v$ by

$$\frac{\mathrm{d}\mathbb{Q}^v}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^\infty v_s \mathrm{d}X_s - \frac{|\Lambda|}{2}\int_0^\infty \|v_s\|^2 \mathrm{d}s\right).$$

Denote with $\mathbb{H}_c \subseteq \mathbb{H}_a$ the set of drifts $v \in \mathbb{H}_a$ for which $\mathbb{Q}^v(\Omega) = 1$, in particular $u^T \in \mathbb{H}_c$. By Jensen's inequality and Girsanov transformation we have

$$\begin{aligned} -\log \mathscr{Z}_T = -\log \mathbb{E}_{\mathbb{P}}[e^{-V_T(W_T)}] &= -\log \mathbb{E}^v\left[e^{-V_T(W_T) - \int_0^\infty v_s \mathrm{d}X_s + \frac{|\Lambda|}{2}\int_0^\infty \|v_s\|^2 \mathrm{d}s}\right] \\ &\leqslant \mathbb{E}^v\left[V_T(W_T) + \int_0^\infty v_s \mathrm{d}X_s - \frac{|\Lambda|}{2}\int_0^\infty \|v_s\|^2 \mathrm{d}s\right], \end{aligned}$$

for all $v \in \mathbb{H}_c$, where $\mathbb{E}^v := \mathbb{E}_{\mathbb{Q}^v}$. We conclude that

$$-\log \mathscr{Z}_T \leqslant \mathbb{E}^v\left[V_T(W_T^v + I_T(v)) + \frac{|\Lambda|}{2}\int_0^\infty \|v_s\|^2 \mathrm{d}s\right], \tag{2.9}$$

where $W = W_T^v + I_T(v)$ and $\mathrm{Law}_{\mathbb{Q}^v}(W^v) = \mathrm{Law}_{\mathbb{P}}(W)$. The bound is saturated when $v = u^T$. We record this result in the following lemma which is a precursor of our main tool to obtain bounds on the partition function and related objects.

Lemma 2.3. *For any $f \in C(\mathscr{C}^{-1/2-\kappa}; \mathbb{R})$ with linear growth, the following variational formula for the free energy holds:*

$$\mathcal{W}_T(f) = -\frac{1}{|\Lambda|}\log \mathbb{E}\left[e^{-V_T^f(W_T)}\right] = \min_{v \in \mathbb{H}_c} \mathbb{E}^v\left[\frac{1}{|\Lambda|} V_T^f(W_T^v + I_T(v)) + \frac{1}{2}\int_0^\infty \|v_s\|_{L^2}^2 \mathrm{d}s\right].$$

where $V_T^f := |\Lambda| f + V_T$.

This formula is nice and easy to prove but somewhat inconvenient for certain manipulations since the space $\mathbb{H}_c$ is indirectly defined and the reference measure $\mathbb{E}^v$ and the process W^v depend on the drift v. A more straightforward formula has been found by Boué–Dupuis [20] which involves the fixed canonical measure $\mathbb{P}$ and a general adapted drift $u \in \mathbb{H}_a$. This formula will be our main tool in the following.

Theorem 2.4. *For any $f \in C(\mathscr{C}^{-1/2-\kappa}; \mathbb{R})$ with linear growth the Boué–Dupuis (BD) variational formula for the free energy holds:*

$$\mathcal{W}_T(f) = -\frac{1}{|\Lambda|}\log \mathbb{E}\left[e^{-V_T^f(W_T)}\right] = \inf_{v \in \mathbb{H}_a} \mathbb{E}\left[\frac{1}{|\Lambda|} V_T^f(W_T + I_T(v)) + \frac{1}{2}\int_0^\infty \|v_s\|_{L^2}^2 \mathrm{d}s\right].$$

where the expectation is taken wrt to the measure $\mathbb{P}$ on Ω.

Proof. The original proof can be found in Boué–Dupuis [20] for functionals bounded from above. In our setting the formula can be proved using the result of Üstünel [117] by observing that $V_T^f(Y_T)$ is a *tame* functional, according to his definitions. Namely, for some $p, q \geqslant 1$ such that $1/p + 1/q = 1$ we have

$$\mathbb{E}[|V_T^f(W_T)|^p] + \mathbb{E}[e^{-qV_T^f(W_T)}] < +\infty. \qquad \square$$

Remark 2.5. Some observations on these variational formulas.

a) They originates directly from the variational formula for the free energy of a statistical mechanical systems: V_T^f playing the role of the internal energy and the quadratic term playing the role of the entropy.

b) The infimum might not be attained in Theorem 2.4 (see e.g. Theorem 8 in [117]) while it is attained in Lemma 2.3.

c) The drift generated by absolutely continuous perturbations of the Wiener measure has been introduced and studied by Föllmer [57].

d) They are a non–Markovian and infinite dimensional extension of the well known stochastic control problem representation of the Hamilton–Jacobi–Bellman equation in finite dimensions [56].

e) The BD formula is easier to use than the formula in Lemma 2.3 since the probability do not depend on the drift v. Going from one formulation to the other requires proving that certain SDEs with functional drift admits strong solutions and that one is able to approximate unbounded functionals V_T by bounded ones. See Üstünel [117] and Lehec [91] for a streamlined proof of the BD formula and for applications to functional inequalities on Gaussian measures. For example, from this formula it is not difficult to prove integrability of functionals which are Lipschitz in the Cameron–Martin directions.

The next lemma provides a deterministic regularity result for $I(v)$ which will be useful below. In particular, it says that the drift v generates shifts of the Gaussian free field in directions which belong to H^1 uniformly in the scale parameter up to ∞. The space H^1 is the Cameron–Martin space of the free field [82].

Lemma 2.6. *Let $\alpha \in \mathbb{R}$. For any $v \in L^2([0,\infty), H^\alpha)$ we have*

$$\sup_{0 \leqslant t \leqslant T} \|I_t(v)\|_{H^{\alpha+1}}^2 + \sup_{0 \leqslant s < t \leqslant T} \frac{\|I_t(v) - I_s(v)\|_{H^{\alpha+1}}^2}{1 \wedge (t-s)} \lesssim \int_0^T \|v_r\|_{H^\alpha}^2 \mathrm{d}r,$$

and

$$\sup_{0 \leqslant t \leqslant T} \|I_t(v)\|_{H^{\alpha+1}}^2 \leqslant \int_0^T \|v_r\|_{H^\alpha}^2 \mathrm{d}r.$$

Proof. Using the fact that $\sigma_s(\mathrm{D})$ is diagonal in Fourier space, and denoting with $(e_k)_{k \in (L^{-1}\mathbb{Z})^d}$ the basis of trigonometric polynomials, we have

$$\begin{aligned}
\left\| \int_r^t \sigma_s(\mathrm{D}) v_s \mathrm{d}s \right\|_{H^\alpha}^2 &= \frac{1}{|\Lambda|} \sum_{k \in (L^{-1}\mathbb{Z})^d} \langle k \rangle^{2\alpha} \left| \int_r^t \langle \sigma_s(\mathrm{D}) e_k, v_s \rangle \mathrm{d}s \right|^2 \\
&\leqslant \frac{1}{|\Lambda|} \sum_{k \in (L^{-1}\mathbb{Z})^d} \langle k \rangle^{2\alpha} \left(\int_r^t |\langle \sigma_s(\mathrm{D}) e_k, e_k \rangle|^2 \mathrm{d}s \right) \left(\int_r^t |\langle e_k, v_s \rangle|^2 \mathrm{d}s \right) \\
&\leqslant \int_r^t \|v_s\|_{H^\alpha}^2 \mathrm{d}s \sup_k \int_r^t \langle e_k, \sigma_s(\mathrm{D})^2 e_k \rangle \mathrm{d}s \\
&\leqslant \int_r^t \|v_s\|_{H^\alpha}^2 \mathrm{d}s \sup_k \langle e_k, \rho_t^2(\mathrm{D}) e_k \rangle \leqslant \int_0^T \|v_s\|_{H^\alpha}^2 \mathrm{d}s.
\end{aligned}$$

Which is the second statement. On the other hand $\sigma_s(\mathrm{D})$ is a smooth Fourier multiplier and using Proposition A.7 we have the estimate $\|\sigma_s(\mathrm{D}) f\|_{H^\alpha} \lesssim \|f\|_{H^\alpha} / \langle s \rangle^{1/2}$ uniformly in $s \geqslant 0$, therefore, for all $0 \leqslant r \leqslant t \leqslant T$, we have

$$\begin{aligned}
\left\| \int_r^t \sigma_s(\mathrm{D}) v_s \mathrm{d}s \right\|_{H^\alpha}^2 &\leqslant \left(\int_r^t \|\sigma_s(\mathrm{D}) v_s\|_{H^\alpha} \mathrm{d}s \right)^2 \leqslant (t-r) \int_r^t \|\sigma_s(\mathrm{D}) v_s\|_{H^\alpha}^2 \mathrm{d}s \\
&\lesssim (t-r) \int_0^T \|v_s\|_{H^\alpha}^2 \mathrm{d}s.
\end{aligned}$$

We conclude that

$$\| I_t(v) - I_r(v) \|_{H^{\alpha+1}}^2 \lesssim \left\| \int_r^t \sigma_s(\mathrm{D}) v_s \mathrm{d}s \right\|_{H^\alpha}^2 \leqslant [1 \wedge (t-r)] \int_0^T \|v_s\|_{H^\alpha}^2 \mathrm{d}s.$$

□

Notation 2.7. *In the estimates below the symbol $E(\lambda)$ will denote a generic positive deterministic quantity, not depending on $|\Lambda|$ and such that $E(\lambda)/\lambda^3 \to 0$ as $\lambda \to 0$. Moreover the symbol Q_T will denote a generic random variable measurable wrt. $\sigma((W_t)_{t \in [0,T]})$ and belonging to $L^p(\mathbb{P})$ uniformly in T and $|\Lambda|$ for any $1 \leqslant p < \infty$.*

2.3. Two dimensions

As a warm up consider here the case $d = 2$ setting $f = 0$ for simplicity. From Theorem 2.4 we see that the relevant quantity to bound is of the form

$$F_T(u) := \mathbb{E}\left[\frac{1}{|\Lambda|} V_T(W_T + I_T(u)) + \frac{1}{2} \|u\|_{\mathcal{H}}^2 \right], \tag{2.10}$$

for $u \in \mathbb{H}_a$. From now on we leave implicit the integration variable over the spatial domain Λ and let $Z_t = I_t(u)$ for brevity. Choosing

$$a_T = 6\mathbb{E}[W_T(0)^2], \qquad b_T = 3\mathbb{E}[W_T(0)^2]^2, \tag{2.11}$$

we have

$$\frac{1}{|\Lambda|} V_T(W_T + Z_T) = \lambda \fint [\![W_T^4]\!] + 4\lambda \fint [\![W_T^3]\!] Z_T + 6\lambda \fint [\![W_T^2]\!] Z_T^2 + 4\lambda \fint W_T Z_T^3 + \lambda \fint Z_T^4,$$

where

$$\begin{aligned}
[\![W_T^4]\!] &:= W_T^4 - 6\mathbb{E}[W_T^2] W_T^2 + 3\mathbb{E}[W_T^2]^2, \\
[\![W_T^3]\!] &:= W_T^3 - 3\mathbb{E}[W_T^2] W_T, \\
[\![W_T^2]\!] &:= W_T^2 - \mathbb{E}[W_T^2],
\end{aligned}$$

denote the Wick powers of the Gaussian r.v. W_T [82]. These polynomials, when seen as stochastic processes in T, are $\mathbb{P}$–martingales wrt. the filtration of $(W_t)_t$. In particular they have an expression as iterated stochastic integrals wrt. the Brownian motions $(B^n_t)_{t,n}$ introduced in eq. (2.4). Using Theorem 2.4 with $u=0$ we readily have an upper bound for the free energy:

$$-\frac{1}{|\Lambda|}\log \mathscr{Z}_T \leqslant \lambda\mathbb{E}\left[\fint [\![W_T^4]\!]\right]=0.$$

For a lower bound we need to estimate from below the average under $\mathbb{P}$ of the variational expression

$$\lambda\fint [\![W_T^4]\!]+4\lambda\fint [\![W_T^3]\!]Z_T+6\lambda\fint [\![W_T^2]\!]Z_T^2+4\lambda\fint W_T Z_T^3+\lambda\fint Z_T^4+\frac{1}{2}\|u\|_{\mathcal{H}}^2.$$

The strategy we adopt is to bound *path-wise*, and for a generic drift u, the contributions

$$\Phi_T(Z):=\underbrace{4\lambda\fint [\![W_T^3]\!]Z_T}_{\text{I}}+\underbrace{6\lambda\fint [\![W_T^2]\!]Z_T^2}_{\text{II}}+\underbrace{4\lambda\fint W_T Z_T^3}_{\text{III}},$$

in term of quantities involving only the Wick powers of W which we can control in expectation and the last two *positive terms*

$$\frac{1}{2}\|u\|_{\mathcal{H}}^2+\lambda\fint Z_T^4.$$

Any residual positive contribution depending on u can be dropped in the lower bound making the dependence on the drift disappear. To control term I we see that by duality and Young's inequality, for any $\delta>0$,

$$\left|4\lambda\fint [\![W_T^3]\!]Z_T\right|\leqslant 4\lambda\|[\![W_T^3]\!]\|_{H^{-1}}\|Z_T\|_{H^1}\leqslant C(\delta,d)\lambda^2\|[\![W_T^3]\!]\|_{H^{-1}}^2+\delta\int_0^T\|u_s\|_{L^2}^2\mathrm{d}s. \tag{2.12}$$

For the term II the following fractional Leibniz rule is of help:

PROPOSITION 2.8. *Let $1<p<\infty$ and $p_1,p_2,p_1',p_2'>1$ such that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_1'}+\frac{1}{p_2'}=\frac{1}{p}$. Then for every $s,\alpha\geqslant 0$ there exists a constant C such that*

$$\|\langle \mathrm{D}\rangle^s(fg)\|_{L^p}\leqslant C\|\langle \mathrm{D}\rangle^{s+\alpha}f\|_{L^{p_2}}\|\langle \mathrm{D}\rangle^{-\alpha}g\|_{L^{p_1}}+C\|\langle \mathrm{D}\rangle^{s+\alpha}g\|_{L^{p_1'}}\|\langle \mathrm{D}\rangle^{-\alpha}f\|_{L^{p_2'}}.$$

Proof. See [75]. □

Using Proposition 2.8 we get, for any $\delta>0$, $1\geqslant\varepsilon>0$,

$$\begin{aligned}\left|6\lambda\fint [\![W_T^2]\!]Z_T^2\right| &\lesssim \lambda\|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}}\|Z_T^2\|_{W^{\varepsilon,\frac{5}{4}}}\\ &\lesssim \lambda\|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}}\|Z_T\|_{W^{\varepsilon,2}}\|Z_T\|_{L^{\frac{10}{3}}}\\ &\lesssim \lambda\|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}}\|Z_T\|_{W^{1,2}}\|Z_T\|_{L^4}\\ &\leqslant \frac{C^2\lambda^3}{2\delta}\|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}}^4+\frac{\delta}{4}\|Z_T\|_{W^{1,2}}^2+\frac{\delta\lambda}{4}\|Z_T\|_{L^4}^4.\end{aligned} \tag{2.13}$$

In order to bound the term III we observe the following:

LEMMA 2.9. *For any $\varepsilon>0$ there exists a $1\leqslant p<\infty$, and $K<\infty$ such that for any $f\in W^{-1/2-\varepsilon,p}$ and $g\in W^{1,2}\cap L^4$*

$$\lambda\left|\fint fg^3\right|\leqslant E(\lambda)\|f\|_{W^{-1/2-\varepsilon,p}}^K+\delta(\|g\|_{W^{1-\varepsilon,2}}^2+\lambda\|g\|_{L^4}^4).$$

Proof. By duality $|\fint fg^3|\leqslant\|f\|_{W^{-1/2-\varepsilon,p}}\|g^3\|_{W^{1/2+\varepsilon,p'}}$. Applying again Proposition 2.8 and Proposition A.8 of the appendix, we get

$$\begin{aligned}\|g^3\|_{W^{1/2+\varepsilon,14/13}} &\lesssim \|\langle \mathrm{D}\rangle^{1/2+\delta}g^3\|_{L^{14/13}}\lesssim\|\langle \mathrm{D}\rangle^{5/8}g\|_{L^{14/6}}\|g\|_{L^4}^2\\ &\lesssim \|g\|_{H^{7/8}}^{5/7}\|g\|_{L^4}^{17/7}.\end{aligned}$$

So

$$\lambda\left|\fint fg^3\right| \lesssim \lambda\|f\|_{W^{-1/2-\varepsilon,14}}\|g\|_{H^{7/8}}^{5/7}\|g\|_{L^4}^{17/7}$$
$$\lesssim \lambda^{11}\|f\|_{W^{-1/2-\varepsilon,14}}^{28}+\delta(\|g\|_{H^{7/8}}^2+\lambda\|g\|_{L^4}^4).$$

□

Using Lemma 2.9 we deduce

$$\left|4\lambda\fint W_T Z_T^3\right| \leqslant E(\lambda)\|W_T\|_{W^{-1/2-\varepsilon,p}}^K+\delta\,(\|Z_T\|_{W^{1-\varepsilon,2}}^2+\lambda\|Z_T\|_{L^4}^4). \tag{2.14}$$

Remark 2.10. This estimate is not optimal for $d=2$. Indeed in this case $(W_T)_T$ stays bounded in $W^{-\varepsilon,p}$ for any large p and it would have been enough to estimate Z_T^3 in $W^{\varepsilon,p'}$. The stronger estimate will be useful below for $d=3$ since there we will only have $W_T\in W^{-1/2-\varepsilon,p}$.

Using eqs. (2.12), (2.13) and (2.14) we obtain, for δ small enough,

$$|\Phi_T(Z)|\leqslant Q_T+\delta\left[\frac{1}{2}\|u\|_{\mathcal{H}}^2+\lambda\fint Z_T^4\right], \tag{2.15}$$

where

$$Q_T=O(\lambda^2)[1+\|[\![W_T^3]\!]\|_{H^{-1}}^2+\|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}}^4+\|W_T\|_{W^{-1/2-\varepsilon,p}}^K].$$

Therefore

$$F_T(u)\geqslant-\mathbb{E}[Q_T]+(1-\delta)\left[\frac{1}{2}\|u\|_{\mathcal{H}}^2+\lambda\fint Z_T^4\right]\geqslant-\mathbb{E}[Q_T].$$

This last average do not depends anymore on the drift and we are only left to show that

$$\sup_T\mathbb{E}[Q_T]<\infty.$$

However, it is well known that the Wick powers of the two dimensional Gaussian free field are distributions belonging to $L^a(\Omega,W^{-\varepsilon,b})$ for any $a\geqslant1$ and $b\geqslant1$ and hypercontractivity plus an easy argument gives the uniform boundedness of the above averages, see e.g. [98]. We have established:

Theorem 2.11. *For any $\lambda>0$ we have*

$$\sup_T\frac{1}{|\Lambda|}|\log\mathscr{Z}_T|\lesssim O(\lambda^2),$$

where the constant in the r.h.s. is independent of Λ.

Remark 2.12. Observe that the argument above remains valid upon replacing λ with λp with $p\geqslant1$. This implies that $e^{-V_T(Y_T)}$ is in all the L^p spaces wrt. the measure $\mathbb{P}$ uniformly in T and for any $p\geqslant1$.

2.4. Three dimensions

In three dimensions the strategy we used above fails. Indeed here the Wick products are less regular: $[\![W_T^2]\!]\in\mathscr{C}^{-1-\kappa}$ uniformly in T for any small $\kappa>0$ and $[\![W_T^3]\!]$ does not even converge to a well-defined random distribution. This implies that there is no straightforward approach to control the terms

$$\fint[\![W_T^3]\!]Z_T,\qquad\text{and}\qquad\fint[\![W_T^2]\!]Z_T^2, \tag{2.16}$$

like we did in Section 2.3. The only apriori estimate on the regularity of $Z_T=I_T(u)$ is in H^1, coming from Lemma 2.6 and the quadratic term in the variational functional $F_T(u)$. It is also well known that in three dimensions there are further divergences beyond the Wick ordering which have to be subtracted in order for the limiting measure to be non-trivial. For these reasons in the energy V_T we introduce further scale dependent renormalization constants γ_T,δ_T to have

$$\frac{1}{|\Lambda|}V_T^f(Y_T)=f(Y_T)+\fint(\lambda[\![Y_T^4]\!]-\lambda^2\gamma_T[\![Y_T^2]\!]-\delta_T). \tag{2.17}$$

where we Wick products $[\![Y_T^4]\!]$, $[\![Y_T^2]\!]$ are taken with respect to the variance of W_T (as opposed to Y_T).

Repeating the computation from Section 2.3 we arrive at

$$\begin{aligned} F_T(u) &= \mathbb{E}\left[f(W_T+Z_T)+\lambda\fint \mathbb{W}_T^3 Z_T+\frac{\lambda}{2}\fint \mathbb{W}_T^2 Z_T^2+4\lambda\fint W_T Z_T^3\right] \\ &\quad -\mathbb{E}\left[2\lambda^2\gamma_T\fint W_T Z_T+\lambda^2\gamma_T\fint Z_T^2+\delta_T\right]+\mathbb{E}\left[\lambda\fint Z_T^4+\frac{1}{2}\|u\|_{\mathcal{H}}^2\right]. \end{aligned} \tag{2.18}$$

where we introduced the convenient notations

$$\mathbb{W}_t^3 := 4[\![W_t^3]\!], \qquad \mathbb{W}_t^2 := 12[\![W_t^2]\!], \qquad t \geqslant 0,$$

and we recall that f is a fixed function belonging to $C(\mathscr{C}^{-1/2-\kappa};\mathbb{R})$ with linear growth.

As already observed, this form of the functional is not very useful in the limit $T\to\infty$ since some of the terms, taken individually, are not expected to behave well. We will perform a change of variables in the variational functional in order to obtain some explicit cancellations which will leave only quantities well behaved as $T\to\infty$. The main drawback is that the functional will have a less compact and canonical form.

Some care has to be taken in order for the resulting quantities to be still controlled by the coercive terms. We need to introduce a regularization which make compatible Fourier cutoffs with L^4 estimates. To introduce such a regularization fix smooth functions $\tilde{\theta}$, η: $\mathbb{R}^d\to\mathbb{R}_+$ such that $\tilde{\theta}(\xi)=1$ if $|\xi|\leqslant 1/4$ and $\tilde{\theta}(\xi)=0$ if $|\xi|\geqslant 1/3$, $\eta(\xi)=1$ if $|\xi|\leqslant 1$ and $\eta(\xi)=0$ if $|\xi|\geqslant 2$. Set $\tilde{\theta}_t(\xi):=\tilde{\theta}(\xi/t)$, then define

$$\theta_t(\xi)=(1-\eta(\xi))\tilde{\theta}_t(\xi)+\zeta(t)\eta(\xi)\tilde{\theta}_t(\xi)$$

where $\zeta(t)$: $\mathbb{R}_+\to\mathbb{R}$ is a smooth function such that $\zeta(t)=0$ for $t\leqslant 10$ and $\zeta(t)=1$ for $t\geqslant 11$.

$$\begin{aligned} \theta_t(\xi)\sigma_s(\xi) &= 0 \text{ for } s\geqslant t, \\ \theta_t(\xi) &= 1 \text{ for } |\xi|\leqslant ct \text{ for some } c>0 \text{ provided that } t\geqslant 11. \end{aligned} \tag{2.19}$$

By the Mihlin-Hörmander theorem we deduce that the operator $\theta_t=\theta_t(\mathrm{D})$ is bounded on L^p for any $1<p<\infty$, see Proposition A.7. In the following, for any $f\in C([0,\infty],\mathscr{S}'(\Lambda))$ we define $f_t^\flat:=\theta_t f_t$ then

$$Z_t^\flat=\theta_t Z_t=\int_0^t \theta_t\langle\mathrm{D}\rangle^{-1}\sigma_s(\mathrm{D})u_s\,\mathrm{d}s=\int_0^T \theta_t\langle\mathrm{D}\rangle^{-1}\sigma_s(\mathrm{D})u_s\,\mathrm{d}s=\theta_t Z_T.$$

In this way we have $\|Z_t^\flat\|_{L^p}\lesssim\|Z_T\|_{L^p}$ for all $t\leqslant T$. In the sequel we will always assume $T\geqslant 11$.

The renormalized functional will depend on some specific renormalized combinations of the martingales $([\![\mathbb{W}_t^k]\!])_{t,k}$. Therefore it will be also convenient to introduce a collective notation for all the stochastic objects appearing in the functionals and specify the topologies in which they are expected to be well behaved. Let

$$\mathbb{W} := (\mathbb{W}^1, \mathbb{W}^2, \mathbb{W}^{\langle 3\rangle}, \mathbb{W}^{[3]\circ 1}, \mathbb{W}^{2\diamond[3]}, \mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}),$$

with $\mathbb{W}^1 := W$,

$$\mathbb{W}_t^{\langle 3\rangle} := J_t\mathbb{W}_t^3, \quad \mathbb{W}_t^{[3]} := \int_0^t J_s\mathbb{W}_s^{\langle 3\rangle}\mathrm{d}s, \quad \mathbb{W}_t^{[3]\circ 1} := \mathbb{W}_t^1\circ\mathbb{W}_t^{[3]},$$

$$\mathbb{W}_t^{2\diamond[3]} := \mathbb{W}_t^2\circ\mathbb{W}_t^{[3]}+2\gamma_t\mathbb{W}_t^1, \quad \mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle} := (J_t\mathbb{W}_t^2)\circ(J_t\mathbb{W}_t^2)+2\dot{\gamma}_t.$$

where $\circ$ denotes the resonant product (see Definition A.9 in Appendix A). We do not need to include $\mathbb{W}^{[3]}$ in the data since it can be obtained as a function of $\mathbb{W}^{\langle 3\rangle}$ thanks to the bound

$$\begin{aligned} \|\mathbb{W}_t^{[3]}-\mathbb{W}_s^{[3]}\|_{\mathscr{C}^{1/2-2\kappa}} &\leqslant \int_s^t \|J_r\mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{1/2-2\kappa}}\mathrm{d}r \leqslant \left[\int_0^T \|J_r\mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{1/2-2\kappa}}^2\mathrm{d}r\right]^{1/2}|t-s|^{1/2} \\ &\leqslant \left[\int_0^T \|\mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{-1/2-\kappa}}^2\frac{\mathrm{d}r}{\langle r\rangle^{1+2\kappa}}\right]^{1/2}|t-s|^{1/2} \lesssim \sup_{r\in[0,T]}\|\mathbb{W}_r^{\langle 3\rangle}\|_{\mathscr{C}^{-1/2-\kappa}}^2|t-s|^{1/2}, \end{aligned}$$

valid for all $0 \leqslant s \leqslant t \leqslant T$ which shows that the deterministic linear map $\mathbb{W}^{\langle 3\rangle} \mapsto \mathbb{W}^{[3]}$ is continuous from $C([0,\infty],\mathscr{C}^{-1/2-\kappa})$ to $C^{1/2}([0,\infty],\mathscr{C}^{1/2-2\kappa})$. The path-wise regularity of all the other stochastic objects follows from the next lemma, provided the function γ is chosen appropriately.

LEMMA 2.13. *There exists a function $\gamma_t \in C^1(\mathbb{R}_+,\mathbb{R})$ such that*

$$|\gamma_t| + \langle t\rangle |\dot{\gamma}_t| \lesssim \log \langle t\rangle, \qquad t \geqslant 0. \tag{2.20}$$

and such that the vector $\mathbb{W}$ is almost surely in $\mathfrak{S}$ where $\mathfrak{S}$ is the Banach space

$$\mathfrak{S} = C([0,\infty],\mathfrak{W}) \cap \{\mathbb{W}^{\langle 3\rangle} \in L^2(\mathbb{R}_+,\mathscr{C}^{-1/2-\kappa}), \mathbb{W}^{\langle 2\rangle \diamond \langle 2\rangle} \in L^1(\mathbb{R}_+,\mathscr{C}^{-\kappa})\}$$

with

$$\mathfrak{W} = \mathfrak{W}_\kappa := \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-1-\kappa} \times \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-\kappa} \times \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-\kappa},$$

and equipped with the norm

$$\|\mathbb{W}\|_{\mathfrak{S}} := \|\mathbb{W}\|_{C([0,\infty],\mathfrak{W})} + \|\mathbb{W}^{\langle 3\rangle}\|_{L^2(\mathbb{R}_+,\mathscr{C}^{-1/2-\kappa})} + \|\mathbb{W}^{\langle 2\rangle \diamond \langle 2\rangle}\|_{L^1(\mathbb{R}_+,\mathscr{C}^{-\kappa})}.$$

The norm $\|\mathbb{W}\|_{\mathfrak{S}}$ belongs to all $L^p(\mathbb{P})$ spaces. Moreover the averages of the Besov norms $B^\alpha_{q,r}$ of the components of $\mathbb{W}$ of regularity α are uniformly bounded in the volume $|\Lambda|$ if $r < \infty$.

Proof. The proof is based on the observation that one can choose γ in such a way that every component $\mathbb{W}^{(i)}$ of the vector $\mathbb{W}$ is such that $(\Delta_q \mathbb{W}^{(i)}_t(x))_{t \geqslant 0}$ for $q \geqslant -1$ and $x \in \Lambda$ is a martingale wrt. the Brownian filtration (possibly modulo a deterministic term we can control). This can be seen by writing these terms as iterated stochastic integrals. For example, introducing the notation $\mathrm{d}w_s(k) = \langle k\rangle^{-1}\sigma_s(k)\mathrm{d}B^k_s$ we can write

$$\mathbb{W}^2_T(x) \;=\; 24 \sum_{k_1,k_2} e^{i(k_1+k_2)\cdot x} \int_0^T \int_0^{s_2} \mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)$$

so, recalling the definition of Littlewood-Paley kernels ϱ_i from Appendix A, we have

$$\Delta_i \mathbb{W}^2_T(x) = 24 \sum_{k_1,k_2} e^{i(k_1+k_2)\cdot x} \varrho_i(k_1+k_2) \int_0^T \int_0^{s_2} \mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2).$$

By Burkholder's inequality and Fubini's theorem

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant T} \|\Delta_i \mathbb{W}^2_t\|^p_{L^p}\right] &\lesssim \left(\sum_{k_1,k_2} \varrho_i(k_1+k_2) \int_0^T \int_0^{s_2} \frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2} \frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2\right)^{p/2} \\ &\lesssim 2^{p(2+\kappa)i/2},\end{aligned}$$

uniformly in T and so

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant T} \|\mathbb{W}^2_T\|^p_{B^{-1-\kappa}_{p,p}}\right] &\leqslant \mathbb{E}\left[\left(\sum_i 2^{p(-1-\kappa)i} \sup_{t\leqslant T} \|\Delta_i \mathbb{W}^2_t\|^p_{L^p}\right)\right] \\ &\lesssim \sum_i 2^{p(-1-\kappa)i}\, \mathbb{E}\left[\sup_{t\leqslant T} \|\Delta_i \mathbb{W}^2_t\|^p_{L^p}\right] \\ &\lesssim \sum_i 2^{p(-1-\kappa)i} 2^{p(1+\kappa/2)i} \lesssim \sum_i 2^{-pi\kappa/2} < +\infty\end{aligned}$$

By Besov embedding this implies that $\mathbb{E}[\sup_{T<\infty} \|\mathbb{W}^2_T\|^p_{B^{-1-\kappa}_{p,q}}]$ is finite for any $p, q < \infty$ uniformly in the volume and $\mathbb{E}[\|\mathbb{W}^2_T\|^p_{C\mathscr{C}^{-1-}}]$ is finite. Since $\mathbb{W}^2_T$ is a continuous, L^2-bounded martingale, it converges and therefore it belongs to $C([0,\infty],\mathscr{C}^{-1-})$. The same reasoning can be carried out for the more complicated terms $\mathbb{W}^{\langle 3\rangle}$, $\mathbb{W}^{[3]\circ 1}$, $\mathbb{W}^{2\diamond[3]}$, $\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}$. The details can be found in Section 2.9. □

For convenience of the reader we summarize the probabilistic estimates in Table 3.1.

$\mathbb{W}^1$	$\mathbb{W}^2$	$\mathbb{W}^{\langle 3\rangle}$	$\mathbb{W}^{[3]}$	$\mathbb{W}^{[3]\circ 1}$	$\mathbb{W}^{2\diamond[3]}$	$\mathbb{W}^{\langle 2\rangle\diamond\langle 2\rangle}$
$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{-1-}$	$C\mathscr{C}^{-1/2-}\cap L^2\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{1/2-}$	$C\mathscr{C}^{0-}$	$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{0-}\cap L^1\mathscr{C}^{0-}$

Table 2.1. Regularities of the various stochastic objects, the domain of the time variable is understood to be $[0,\infty]$. Estimates in these norms hold a.s. and in $L^p(\mathbb{P})$ for all $p\geqslant 1$ (see Lemma 2.13).

Remark 2.14. The requirement that $\mathbb{W}^{\langle 3\rangle}\in L^2\mathscr{C}^{-1/2-}$ will be used in Section 2.6 to establish equicoercivity and to relax the variational problem to a suitable space of measures.

We are now ready to perform a change of variables which renormalizes the variational functional.

Lemma 2.15. *Define $l=l^T(u)\in\mathbb{H}_a$, $Z=Z(u)\in C([0,\infty],H^{1/2-\kappa})$, $K=K(u)\in C([0,\infty],H^{1-\kappa})$ such that*

$$\begin{array}{ll} Z_t(u):=I_t(u), & \\ l_t^T(u):=u_t+\lambda\mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle}+\lambda\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^2\succ Z_t^\flat(u)), & t\geqslant 0.\\ K_t(u):=I_t(w(u)),\ \text{with } w_t(u):=-\lambda\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^2\succ Z_t^\flat(u))+l_t^T(u), & \end{array}\tag{2.21}$$

Then the functional $F_T(u)$ defined in eq. (2.18) takes the form

$$F_T(u) = \mathbb{E}\left[\Phi_T(\mathbb{W},Z(u),K(u))+\lambda\fint(Z_T(u))^4+\frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right],$$

where

$$\Phi_T(\mathbb{W},Z,K):=f(W_T+Z_T)+\sum_{i=1}^6\Upsilon^{(i)},$$

$$\begin{aligned}
\Upsilon_T^{(1)} &:= -\frac{\lambda}{2}\mathfrak{K}_2(\mathbb{W}_T^2,K_T,K_T)+\frac{\lambda}{2}\fint(\mathbb{W}_T^2\prec K_T)K_T-\lambda^2\fint(\mathbb{W}_T^2\prec\mathbb{W}_T^{[3]})K_T,\\
\Upsilon_T^{(2)} &:= \lambda\fint(\mathbb{W}_T^2\succ(Z_T-Z_T^\flat))K_T,\\
\Upsilon_T^{(3)} &:= \lambda\int_0^T\fint(\mathbb{W}_t^2\succ\dot{Z}_t^\flat)K_t\mathrm{d}t,\\
\Upsilon_T^{(4)} &:= 4\lambda\fint W_TK_T^3-12\lambda^2\fint W_T\mathbb{W}_T^{[3]}K_T^2+12\lambda^3\fint W_T(\mathbb{W}_T^{[3]})^2K_T,\\
\Upsilon_T^{(5)} &:= -2\lambda^2\fint\gamma_TZ_T^\flat(Z_T-Z_T^\flat)-\lambda^2\fint\gamma_T(Z_T-Z_T^\flat)^2-2\lambda^2\int_0^T\fint\gamma_tZ_t^\flat\dot{Z}_t^\flat\mathrm{d}t,\\
\Upsilon_T^{(6)} &:= -\lambda^2\fint\mathbb{W}_T^{2\diamond[3]}K_T-\frac{\lambda^2}{2}\int_0^T\fint\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}(Z_t^\flat)^2\mathrm{d}t-\frac{\lambda^2}{2}\int_0^T\mathfrak{K}_{3,t}(\mathbb{W}_t^2,\mathbb{W}_t^2,Z_t^\flat,Z_t^\flat)\mathrm{d}t.
\end{aligned}$$

Here $\mathfrak{K}_2$ and $\mathfrak{K}_{3,t}$ are linear forms defined in Proposition A.14 and A.15 in Appendix A (and recalled in the proof below). Moreover we have chosen the renormalization constant δ_T appearing in equation (2.17) to be

$$\begin{aligned}\delta_T := & -\frac{\lambda^2}{2}\mathbb{E}\int_0^T\fint(\mathbb{W}_t^{\langle 3\rangle})^2\mathrm{d}t+\frac{\lambda^3}{2}\mathbb{E}\fint\mathbb{W}_T^2(\mathbb{W}_T^{[3]})^2\\ & +2\lambda^3\gamma_T\mathbb{E}\fint W_T\mathbb{W}_T^{[3]}-4\lambda^4\mathbb{E}\fint W_T(\mathbb{W}_T^{[3]})^3.\end{aligned}\tag{2.22}$$

Proof.

Step 1. We are going to absorb the mixed terms (2.16) via the quadratic cost function. To do so we develop them along the flow of the scale parameter via Itô formula. For the first we have

$$\lambda\fint\mathbb{W}_T^3Z_T=\lambda\int_0^T\fint\mathbb{W}_t^3\dot{Z}_t\mathrm{d}t+\text{martingale},$$

and we can cancel the first term on the r.h.s. by introducing

$$w_t := u_t + \lambda \mathbb{1}_{t \leqslant T} \mathbb{W}_t^{\langle 3 \rangle}, \qquad t \geqslant 0, \tag{2.23}$$

into the cost functional to get

$$\lambda \fint \mathbb{W}_T^3 Z_T + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s = -\frac{\lambda^2}{2}\int_0^T \fint (\mathbb{W}_t^{\langle 3 \rangle})^2 \mathrm{d}t + \frac{1}{2}\int_0^\infty \|w_s\|_{L^2}^2 \mathrm{d}s + \text{martingale},$$

where we used that J_t is self-adjoint. Taking into account (here and below) that the martingale term will average to zero, we have replaced the divergent term $\fint \mathbb{W}_T^3 Z_T$ with a divergent but purely stochastic term $\int_0^T \fint (\mathbb{W}_t^{\langle 3 \rangle})^2 \mathrm{d}t$ which does not affect anymore the variational problem and can be explicitly removed by adding its average to δ_T. As a consequence, we are no more able to control $(Z_t)_t$ in H^1 and we should rely on the relation (2.21) and on a control over the H^1 norm of $(K_t)_t$ coming from the residual quadratic term $\|w\|_{\mathcal{H}}^2$.

Step 2. From (2.23) we have the relation

$$Z_T = -\lambda \mathbb{W}_T^{[3]} + K_T,$$

which can be used to expand the second mixed divergent term in (2.16) as

$$\frac{\lambda}{2}\fint \mathbb{W}_T^2 Z_T^2 = \frac{\lambda^3}{2}\fint \mathbb{W}_T^2 (\mathbb{W}_T^{[3]})^2 - \lambda^2 \fint \mathbb{W}_T^2 \mathbb{W}_T^{[3]} K_T + \frac{\lambda}{2}\fint \mathbb{W}_T^2 K_T^2. \tag{2.24}$$

Again, the first term on the r.h.s. a purely stochastic object and will give a contribution independent of the drift u and absorbed in δ_T. We are still not done since this operation has left two new divergent terms on the r.h.s. of eq. (2.24): the H^1 regularity of K_T is not enough to control the products with $\mathbb{W}^2$ which has regularity $\mathscr{C}^{-1-\kappa}$, a bit below -1. In order to proceed further we will isolate the divergent parts of these products via a paraproduct decomposition (see Appendix A for details) and expand

$$\begin{aligned}
-\lambda^2 \fint \mathbb{W}_T^2 \mathbb{W}_T^{[3]} K_T + \frac{\lambda}{2}\fint \mathbb{W}_T^2 K_T^2 \; = \; & \lambda \fint (\mathbb{W}_T^2 \succ Z_T) K_T - \lambda^2 \fint (\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]}) K_T \\
& - \lambda^2 \fint (\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]}) K_T + \frac{\lambda}{2}\fint (\mathbb{W}_T^2 \prec K_T) K_T \\
& + \frac{\lambda}{2}\left(\fint (\mathbb{W}_T^2 \circ K_T) K_T - \fint (\mathbb{W}_T^2 \succ K_T) K_T \right).
\end{aligned}$$

The first two terms will require renormalizations which we put in place in Step 3 below. All the other terms will be well behaved and we collect them in $\Upsilon_T^{(1)}$. In particular we observe that the last one can be rewritten as

$$\frac{\lambda}{2}\left(\fint (\mathbb{W}_T^2 \circ K_T) K_T - \fint (\mathbb{W}_T^2 \succ K_T) K_T \right) = -\frac{\lambda}{2}\mathfrak{K}_2(\mathbb{W}_T^2, K_T, K_T)$$

introducing the trilinear form $\mathfrak{K}_2$ whose properties are detailed in Proposition A.14 below.

Step 3. As we anticipated, the resonant term $\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]}$ needs renormalization. In the expression of F_T in (2.18) we have the counterterm $-2\lambda^2 \gamma_T \fint W_T Z_T$ available, which we put now in use writing

$$-\lambda^2 \fint (\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]}) K_T - 2\lambda^2 \gamma_T \fint W_T Z_T = -\lambda^2 \fint \underbrace{(\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]} + 2\gamma_T W_T)}_{\mathbb{W}_T^{2\diamond[3]}} K_T + 2\lambda^3 \gamma_T \fint W_T \mathbb{W}_T^{[3]}.$$

The first contribution is collected in $\Upsilon_T^{(6)}$ and the expectation of the second will contribute to δ_T.

As far as the term $\lambda \fint (\mathbb{W}_T^2 \succ Z_T) K_T$ is concerned, we want to absorb it into $\int \|w_s\|^2 \mathrm{d}s$ like we did with the linear term in Step 2. Before we can do this we must be sure that, after applying Itô's formula, it will be still possible to use $\fint Z_T^4$ to control some of the growth of this term. Indeed the quadratic dependence in K_T (via Z_T) cannot be fully taken care of by the quadratic cost $\int \|w_s\|^2 \mathrm{d}s$.

We decompose

$$\lambda \fint (\mathbb{W}_T^2 \succ Z_T) K_T = \lambda \fint (\mathbb{W}_T^2 \succ Z_T^\flat) K_T + \lambda \fint (\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat)) K_T$$

and using the fact that the functions $Z_T - Z_T^\flat$ and $K_T - K_T^\flat$ are spectrally supported outside of a ball or radius cT we will be able to show that the second term is nice enough as $T \to \infty$ to not require further analysis and we collect it in $\Upsilon_T^{(2)}$. For the first we apply Itô's formula to decompose it along the flow of scales as

$$\lambda \fint (\mathbb{W}_T^2 \succ Z_T^\flat) K_T = \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t + \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ \dot{Z}_t^\flat) K_t \mathrm{d}t + \text{martingale}.$$

The second term will be fine and we collect it in $\Upsilon_T^{(3)}$.

Step 4. We are left with the singular term $\int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t$. Using eq. (2.21) and expanding w in the residual quadratic cost function obtained in Step 1, we compute

$$\begin{aligned} \lambda \int_0^T \fint (\mathbb{W}_t^2 \succ Z_t^\flat) \dot{K}_t \mathrm{d}t + \frac{1}{2} \int_0^\infty \|w_t\|_{L^2}^2 \mathrm{d}t &= -\frac{\lambda^2}{2} \int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))^2 \mathrm{d}t + \frac{1}{2} \int_0^\infty \|l_t\|_{L^2}^2 \mathrm{d}t \\ &= -\frac{\lambda^2}{2} \int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))(J_t(\mathbb{W}_t^2 \succ Z_t^\flat)) \mathrm{d}t + \frac{1}{2} \|l\|_{\mathcal{H}}^2 \end{aligned} \tag{2.25}$$

To renormalize the first term on the r.h.s. we observe that the remaining counterterm can be rewritten as

$$-\lambda^2 \gamma_T \fint Z_T^2 = -\lambda^2 \gamma_T \fint (Z_T^\flat)^2 - 2\lambda^2 \gamma_T \fint Z_T^\flat (Z_T - Z_T^\flat) - \lambda^2 \gamma_T \fint (Z_T - Z_T^\flat)^2. \tag{2.26}$$

Differentiating in T the first term in the r.h.s. of eq. (2.26) we get

$$-\lambda^2 \gamma_T \fint (Z_T^\flat)^2 = -\lambda^2 \int_0^T \fint \dot{\gamma}_t (Z_t^\flat)^2 \mathrm{d}t - 2\lambda^2 \int_0^T \fint \gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t. \tag{2.27}$$

The last term in eq. (2.27) and the last two contributions in (2.26) are collected in $\Upsilon_T^{(5)}$. The first contribution in eq. (2.27) has the right form to be used as a counterterm for the resonant product in (2.25). Using the commutator $\mathfrak{K}_{3,t}$ introduced in Proposition A.15 we have

$$\begin{aligned} &-\frac{\lambda^2}{2} \int_0^T \fint [(J_t(\mathbb{W}_t^2 \succ Z_t^\flat))(J_t(\mathbb{W}_t^2 \succ Z_t^\flat)) + 2\dot{\gamma}_t (Z_t^\flat)^2] \mathrm{d}t \\ &= -\frac{\lambda^2}{2} \int_0^T \fint \underbrace{[(J_t \mathbb{W}_t^2) \circ (J_t \mathbb{W}_t^2) + 2\dot{\gamma}_t]}_{\mathbb{W}^{\langle 2 \rangle \diamond \langle 2 \rangle}} (Z_t^\flat)^2 \mathrm{d}t - \frac{\lambda^2}{2} \int_0^T \mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^\flat, Z_t^\flat) \mathrm{d}t \end{aligned}$$

and we collect both terms in $\Upsilon_T^{(6)}$.

Step 5. Finally, we are left with the cubic term which we rewrite as

$$4\lambda \fint W_T Z_T^3 = -4\lambda^4 \fint W_T (\mathbb{W}_T^{[3]})^3 + 12\lambda^3 \fint W_T (\mathbb{W}_T^{[3]})^2 K_T - 12\lambda^2 \fint W_T \mathbb{W}_T^{[3]} K_T^2 + 4\lambda \fint W_T K_T^3.$$

The average of the first term is collected in δ_T while all the remaining terms in $\Upsilon_T^{(4)}$. At last we have established the claimed decomposition since the residual cost functional, from eq. (2.25) is indeed $\|l\|_{\mathcal{H}}^2/2$. □

2.5. Bounds

The aim of this section is to give upper and lower bounds on $\mathcal{W}_T(f)$ uniformly on T and $|\Lambda|$. In particular we will prove the bounds of Corollary 2.2 taking the explicit dependence on the coupling constant λ into account.

Lemma 2.16. *There exists a finite constant C, which does not depend on Λ, such that*

$$\sup_T |\mathcal{W}_T(f)| \leqslant C.$$

Proof. Observe that, from Lemma 2.15 and from the analysis in Section 2.8, we have that

$$|\Phi_T(\mathbb{W}, Z, K)| \leqslant Q_T + \varepsilon \left(\lambda \|Z_T\|_{L^4}^4 + \frac{1}{2} \int_0^\infty \|l_t^T(u)\|_{L^2}^2 \mathrm{d}t \right),$$

which immediately gives

$$-\mathbb{E}[Q_T] \leqslant -\mathbb{E}[Q_T] + (1-\varepsilon)\mathbb{E}\left(\lambda \|Z_T\|_{L^4}^4 + \frac{1}{2} \int_0^\infty \|l_t^T(u)\|_{L^2}^2 \mathrm{d}t \right) \leqslant \mathcal{W}_T(f). \tag{2.28}$$

On the other hand for any suitable drift $\check{u} \in \mathbb{H}_a$ we get the bound

$$\mathcal{W}_T(f) \leqslant \mathbb{E}[Q_T] + (1+\varepsilon)\mathbb{E}\left(\lambda \|I_T(\check{u})\|_{L^4}^4 + \frac{1}{2} \int_0^\infty \|l_t^T(\check{u})\|_{L^2}^2 \mathrm{d}t \right), \tag{2.29}$$

where

$$l_t^T(\check{u}) = \check{u}_t + \lambda \mathbb{1}_{t \leqslant T} J_t(\mathbb{W}_t^3 + \mathbb{W}_t^2 \succ (I_t(\check{u}))^\flat). \tag{2.30}$$

Therefore it remains to produce an appropriate drift $\check{u}$ for which the r.h.s. in eq. (2.29) is finite (and so uniformly in $|\Lambda|$ and of order $o(\lambda^3)$).

One possible strategy is to try and choose $\check{u}$ such that $l^T(\check{u}) = 0$, however this fails since estimates on this choice of drift via Gronwall's inequality would rely on the Besov-Hölder norm of $\mathbb{W}^2$ for which we do not have any uniform control in the volume. In order to overcome this problem we decompose $\mathbb{W}^2$ and use weighted estimates similarly as done in [72] in the SPDE context.

Consider the decomposition

$$\mathbb{W}_s^2 = \mathcal{U}_\geqslant \mathbb{W}_s^2 + \mathcal{U}_\leqslant \mathbb{W}_s^2,$$

where the random field $\mathcal{U}_\geqslant \mathbb{W}_s^2$ is constructed as follows. Let φ be smooth function, positive and supported on $[-2,2]^3$ and such that $\sum_{m \in \Lambda \cap \mathbb{Z}^d} \varphi^2(\bullet - m) = 1$. Denote $\varphi_m := \varphi(\bullet - m)$. Let $\tilde{\chi}$ be a smooth function supported in $B(0,1)$, denote by $\mathcal{X}_{>N}$ the Fourier multiplier operator $\tilde{\chi}(\mathrm{D}/N)$ and similarly $\mathcal{X}_{\leqslant N} := (1 - \tilde{\chi}(\mathrm{D}/N))$. Set $L_m(s) := (1 + \|\varphi_m \mathbb{W}_s^2\|)_{\mathscr{C}^{-1-\delta}}^{\frac{1}{2\delta}}$, let

$$\mathcal{U}_> \mathbb{W}_s^2 := \sum_{m \in \Lambda \cap \mathbb{Z}^d} \varphi_m \mathcal{X}_{>L_m(s)}(\varphi_m \mathbb{W}_s^2)$$

and

$$\mathcal{U}_\leqslant \mathbb{W}_s^2 := \sum_{m \in \Lambda \cap \mathbb{Z}^d} \varphi_m \mathcal{X}_{\leqslant L_m(s)}(\varphi_m \mathbb{W}_s^2).$$

(with slight abuse of notation we drop the dependence on time of the operators $\mathcal{U}_\leqslant, \mathcal{U}_>$).

Observe that the laws of both $\mathcal{U}_> \mathbb{W}_s^2$ and $\mathcal{U}_\leqslant \mathbb{W}_s^2$ are translation invariant w.r.t to translations by $m \in \Lambda \cap \mathbb{Z}^d$. By [114], Theorem 2.4.7 and Bernstein inequality

$$\begin{aligned} \|\mathcal{U}_> \mathbb{W}_s^2\|_{\mathscr{C}^{-1-3\delta}} &\lesssim \sup_m \|\mathcal{X}_{>L_m(s)}(\varphi_m \mathbb{W}_s^2)\|_{\mathscr{C}^{-1-3\delta}} \\ &\lesssim \sup_m \frac{1}{1 + \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}}} \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}} \lesssim 1. \end{aligned}$$

Furthermore for a polynomial weight ρ (see Appendix A for precisions on the weights and the weighted spaces $L^p(\rho)$, $\mathscr{C}^\alpha(\rho)$ and $B_{p,q}^\alpha(\rho)$ used below):

$$\begin{aligned} \|\mathcal{U}_\leqslant \mathbb{W}_s^2\|_{\mathscr{C}^{-1+\delta}(\rho^2)} &\lesssim \sup_m \|\varphi_m \mathcal{U}_\leqslant \mathbb{W}_s^2\|_{\mathscr{C}^{-1+\delta}(\rho^2)} \\ &\lesssim \sup_m (1 + \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}}) \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho^2)} \\ &\lesssim \sup_m \rho(m) \, (1 + \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}}) \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)} \\ &\lesssim \sup_m (1 + \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)}) \|\varphi_m \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)} \\ &\lesssim 1 + \|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}(\rho)}^2, \end{aligned} \tag{2.31}$$

where we used the possibility to compare weighted and unweighted norms once localized via φ_m. We now let $\check{u}$ be the solution to the linear integral equation

$$\check{u}_t = -\lambda \mathbb{1}_{t \leqslant T}[\mathbb{W}_t^{\langle 3 \rangle} + J_t \mathcal{U}_> \mathbb{W}_t^2 \succ \theta_t(I_t(\check{u}))], \qquad t \geqslant 0, \tag{2.32}$$

which can be solved globally. For $2\delta < 1/2$, $p \geqslant 1$ and $t \in [0, T]$, we have

$$\|I_t(\check{u})\|_{B_{p,p}^{1/2-2\delta}(\rho)} \lesssim \lambda \int_0^t \Big[\|J_s^2 \mathbb{W}_s^3\|_{B_{p,p}^{1/2-2\delta}(\rho)} + \lambda \|J_s^2 \mathcal{U}_> \mathbb{W}_s^2 \succ \theta_s(I_s(\check{u}))\|_{B_{p,p}^{1/2-2\delta}(\rho)} \Big] \mathrm{d}s$$
$$\lesssim \lambda \int_0^t \frac{\mathrm{d}s}{\langle s \rangle^{1/2+\delta}} \|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1/2-\delta}(\rho)} + \lambda \int_0^t \frac{\mathrm{d}s}{\langle s \rangle^{1+\delta}} \|\mathcal{U}_> \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}} \|I_s(\check{u})\|_{B_{p,p}^{1/2-2\delta}(\rho)}.$$

Gronwall's lemma implies that, for $t \in [0, T]$:

$$\begin{aligned}
\|I_t(\check{u})\|_{B_{p,p}^{1/2-\delta}(\rho)} &\lesssim \left(\lambda \int_0^T \frac{\mathrm{d}s}{\langle s \rangle^{1/2+\delta}} \|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1/2-\delta}(\rho)} \right) \exp\left(\lambda \int_0^T \frac{\|\mathcal{U}_> \mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta}} \mathrm{d}s}{\langle s \rangle^{1+\delta}} \right) \\
&\lesssim \left(\lambda \int_0^T \frac{\mathrm{d}s}{\langle s \rangle^{1/2+\delta}} \|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1/2-\delta}(\rho)} \right) \\
&\lesssim \lambda \|\mathbb{W}^{\langle 3 \rangle}\|_{L^2(\mathbb{R}_+, B_{p,p}^{-1/2-\delta}(\rho))}.
\end{aligned} \tag{2.33}$$

Note that eq. (2.33) is also valid replacing the weighted norm $B_{p,p}^{1/2-\delta}(\rho)$ with the standard (normalized) norm $B_{p,p}^{1/2-\delta}$, from which, using Besov embedding we deduce:

$$\sup_T \mathbb{E} \|I_T(\check{u})\|_{L^4}^4 \lesssim \lambda^4 \mathbb{E} \left(\int_0^\infty \frac{\mathrm{d}s}{\langle s \rangle^{1/2+\delta}} \|J_s \mathbb{W}_s^3\|_{B_{4,4}^{-1/2-\delta}} \right)^4 \lesssim \lambda^4.$$

Computing $l^T(\check{u})$ from eq. (2.30) and (2.32), we obtain

$$l_t^T(\check{u}) = \lambda \mathbb{1}_{t \leqslant T} J_t \mathcal{U}_\leqslant \mathbb{W}_t^2 \succ \theta_t(I_t(\check{u})), \qquad t \geqslant 0.$$

It remains to prove that $\mathbb{E}[\|l^T(\check{u})\|_{\mathcal{H}}^2] \lesssim O(\lambda^3)$ uniformly in $T > 0$. Note that, for $s \in [0, T]$,

$$\begin{aligned}
\|J_s \mathcal{U}_\leqslant \mathbb{W}_s^2 \succ \theta_s(I_s(\check{u}))\|_{L^2(\rho^3)} &\lesssim \frac{1}{\langle s \rangle^{1/2+\delta/2}} \|\mathcal{U}_\leqslant \mathbb{W}_s^2 \succ \theta_s(I_s(\check{u}))\|_{B_{2,2}^{-1+\delta/2}(\rho^3)} \\
&\lesssim \frac{1}{\langle s \rangle^{1/2+\delta/2}} \|\mathcal{U}_\leqslant \mathbb{W}_s^2\|_{\mathscr{C}^{-1+\delta/2}(\rho^2)} \|I_s(\check{u})\|_{B_{2,2}^{1/2-3\delta}(\rho)}.
\end{aligned} \tag{2.34}$$

We know that the distribution of $\check{u}$ is invariant under translation by $m \in \Lambda \cap \mathbb{Z}^d$. Recalling that $\sum_{m \in \Lambda \cap \mathbb{Z}^d} \varphi^2(\bullet - m) = 1$ and letting ρ be a polynomial weight with sufficient decay and such that $\rho^5 \geqslant \varphi^2$, we have

$$\begin{aligned}
\mathbb{E}[\|l^T(\check{u})\|_{\mathcal{H}}^2] &= \lambda^2 \mathbb{E}[\|s \mapsto \mathbb{1}_{s \leqslant T} J_s \mathcal{U}_\leqslant \mathbb{W}_s^2 \succ \theta_s(I_s(\check{u}))\|_{\mathcal{H}}^2] \\
&\leqslant \lambda^2 \sum_{m \in \Lambda \cap \mathbb{Z}^d} \mathbb{E}[\|s \mapsto \varphi(\bullet - m) J_s \mathcal{U}_\leqslant \mathbb{W}_s^2 \succ \theta_s(I_s(\check{u}))\|_{\mathcal{H}}^2] \\
\text{(by trans. inv.)} &\lesssim \lambda^2 |\Lambda| \mathbb{E}[\|s \mapsto \mathbb{1}_{s \leqslant T} \varphi J_s \mathcal{U}_\leqslant \mathbb{W}_s^2 \succ \theta_s(I_s(\check{u}))\|_{\mathcal{H}}^2] \\
\text{(using } \rho^5 \geqslant \varphi^2\text{)} &\lesssim \lambda^2 \int_0^T \mathrm{d}s \, \mathbb{E}[\|J_s \mathcal{U}_\leqslant \mathbb{W}_s^2 \succ \theta_s(I_s(\check{u}))\|_{L^2(\rho^5)}^2] \\
\text{(by eq. (2.34))} &\lesssim \lambda^2 \int_0^T \frac{\mathrm{d}s}{\langle s \rangle^{1+\delta}} \mathbb{E}\Big[\|\mathcal{U}_\leqslant \mathbb{W}_s^2\|_{\mathscr{C}^{-1+\delta/2}(\rho^2)}^2 \|I_s(\check{u})\|_{B_{2,2}^{1/2-3\delta}(\rho)}^2 \Big] \\
&\lesssim \lambda^2 \int_0^T \frac{\mathrm{d}s}{\langle s \rangle^{1+\delta}} \mathbb{E}\Big[\lambda^2 \|\mathcal{U}_\leqslant \mathbb{W}_s^2\|_{\mathscr{C}^{-1+\delta/2}(\rho^2)}^4 + \lambda^{-2} \|I_s(\check{u})\|_{B_{2,2}^{1/2-3\delta}(\rho)}^4 \Big] \\
\text{(by eqs. (2.33),(2.31))} &\lesssim \lambda^4 \int_0^\infty \frac{\mathrm{d}s}{\langle s \rangle^{1+\delta}} \Big[1 + \mathbb{E} \|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta/2}(\rho)}^8 + \lambda \mathbb{E} \|\mathbb{W}^{\langle 3 \rangle}\|_{L^2(\mathbb{R}_+, B_{p,p}^{-1/2-\delta}(\rho))}^4 \Big] \\
&\lesssim \lambda^4 \Big[1 + \sup_{s \geqslant 0} \mathbb{E} \|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta/2}(\rho)}^8 + \lambda \mathbb{E} \|\mathbb{W}^{\langle 3 \rangle}\|_{L^2(\mathbb{R}_+, B_{p,p}^{-1/2-\delta}(\rho))}^4 \Big] \\
&\lesssim O(\lambda^4).
\end{aligned}$$

The last inequality is the consequence of bounds on the two expectations on the r.h.s. obtained as follows. For p sufficiently large we have

$$\left[\mathbb{E}\|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta/2}(\rho)}^8\right]^{p/8} \leqslant \mathbb{E}\|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\delta/2}(\rho)}^p \leqslant \mathbb{E}\|\mathbb{W}_s^2\|_{B_{p,p}^{-1-\delta}(\rho^p)}^p$$

$$= \sum_{i\geqslant -1} 2^{i(-1-\delta/2)p} \int_\Lambda \mathrm{d}x |\rho(x)|^p \mathbb{E}|\Delta_i \mathbb{W}_s^2(x)|^p \lesssim \sum_{i\geqslant -1} 2^{i(-1-\delta/2)p} \mathbb{E}|\Delta_i \mathbb{W}_s^2(0)|^p \lesssim 1,$$

uniformly in $s \geqslant 0$. Similarly, we have

$$\left[\mathbb{E}\|\mathbb{W}_s^3\|_{B_{p,p}^{-\delta/2}(\rho))}^4\right]^{p/4} \leqslant \mathbb{E}\|\mathbb{W}_s^3\|_{B_{p,p}^{-\delta/2}(\rho)}^p \lesssim \mathbb{E}|\mathbb{W}_s^3(0)|^p.$$

By Lemma 2.53

$$\mathbb{E}|\mathbb{W}_s^3(0)|^p \lesssim (\mathbb{E}|\mathbb{W}_s^3(0)|^2)^{p/2} \lesssim \langle s\rangle^{3p/2},$$

and using the standard multiplier bounds for J_s we conclude

$$\begin{aligned}
\mathbb{E}\|\mathbb{W}^{\langle 3\rangle}\|_{L^2(\mathbb{R}_+, B_{p,p}^{-1/2-\delta}(\rho))}^4 &\lesssim \mathbb{E}\left(\int_0^\infty \|J_s \mathbb{W}_s^3\|_{B_{p,p}^{-1/2-\delta}(\rho)}^2 \mathrm{d}s\right)^2 \\
&\lesssim \mathbb{E}\left(\int_0^\infty \left\|\frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D}\rangle} \mathbb{W}_s^3\right\|_{B_{p,p}^{-1/2-\delta}(\rho)}^2 \mathrm{d}s\right)^2 \\
&\lesssim \mathbb{E}\left(\int_0^\infty \langle s\rangle^{-1-\delta}\left(\langle s\rangle^{-3/2}\|\mathbb{W}_s^3\|_{B_{p,p}^{-\delta/2}(\rho)}\right)^2 \mathrm{d}s\right)^2 \\
&\lesssim \int_0^\infty \langle s\rangle^{-1-\delta} \mathbb{E}\left(\langle s\rangle^{-3/2}\|\mathbb{W}_s^3\|_{B_{p,p}^{-\delta/2}(\rho)}\right)^4 \mathrm{d}s \\
&\lesssim 1.
\end{aligned}$$

□

Remark 2.17. The decomposition of the noise is similar to the one given in [72] but differs in the fact that we choose the frequency cutoff dependent on the size of the noise instead of the point, to preserve translation invariance. The price to pay is that the decomposition is nonlinear in the noise, however this does not present any inconvenience in our context.

2.6. Gamma convergence

In this section we establish the Γ-convergence of the variational functional obtained in Lemma 2.15 as $T \to \infty$. Γ-convergence is a notion of convergence introduced by De Giorgi which is well suited for the study of variational problems. The book [24] is a nice introduction to Γ-convergence in the context of the calculus of variations. For the convenience of the reader we recall here the basic definitions and results.

Definition 2.18. *Let $\mathcal{T}$ be a topological space and let $F, F_n: \mathcal{T} \to (-\infty, \infty]$. We say that the sequence of functionals $(F_n)_n$ Γ-converges to F iff*

i. For every sequence $x_n \to x$ in $\mathcal{T}$

$$F(x) \leqslant \liminf_{n\to\infty} F_n(x_n);$$

ii. For every point x there exists a sequence $x_n \to x$ (called a recovery sequence) such that

$$F(x) \geqslant \limsup_{n\to\infty} F_n(x_n).$$

Definition 2.19. *A sequence of functionals $F_n: \mathcal{T} \to (-\infty, \infty]$ is called equicoercive if there exists a compact set $\mathcal{K} \subseteq \mathcal{T}$ such that for all $n \in \mathbb{N}$*

$$\inf_{x\in\mathcal{K}} F_n(x) = \inf_{x\in\mathcal{T}} F_n(x).$$

A fundamental consequence of Γ-convergence is the convergence of minima.

Theorem 2.20. *If $(F_n)_n$ Γ-converges to F and $(F_n)_n$ is equicoercive, then F admits a minimum and*

$$\min_{\mathcal{T}} F = \lim_{n\to\infty} \inf_{\mathcal{T}} F_n.$$

For a proof see [44].

In this section we allow all constants to depend on the volume $|\Lambda|$: this is not critical since, at this point, the aim is to obtain explicit formulas at fixed Λ.

We denote

$$\mathcal{H}^{\alpha,p} := L^2([0,\infty); W^{\alpha,p}), \qquad \alpha\in\mathbb{R}, 1<p<\infty,$$

and by $\mathcal{H}_w^{\alpha,p}$ the reflexive Banach space $\mathcal{H}^{\alpha,p}$ endowed with the weak topology. With this definitions we have $\mathcal{H}^\alpha = \mathcal{H}^{\alpha,2}$ and $\mathcal{H}=\mathcal{H}^{0,2}$. Moreover for small enough $\kappa>0$ (fixed once and for all) we let $\mathcal{L} := \mathcal{H}^{-1/2-\kappa,3}$. This space will be useful as it gives sufficient control over Z:

Lemma 2.21. *For κ small enough, $u\mapsto Z(u)$ is a compact map $\mathcal{L}\to C([0,\infty],L^4)$.*

Proof. By definition of Z we have for any $0<\varepsilon<1/8-\kappa/2$,

$$\begin{aligned}
\|Z_{t_2}(u)-Z_{t_1}(u)\|_{W^{\varepsilon,4}} &= \left\|\int_{t_1}^{t_2} J_s u_s \mathrm{d}s\right\|_{W^{\varepsilon,4}} \leqslant \int_{t_1}^{t_2}\left\|\frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D}\rangle}u_s\right\|_{W^{\varepsilon,4}}\mathrm{d}s\\
&\lesssim \int_{t_1}^{t_2}\|\langle\mathrm{D}\rangle^{-1+\varepsilon}u_s\|_{W^{\varepsilon,4}}\frac{\mathrm{d}s}{\langle s\rangle^{1/2+\varepsilon}}\\
&\lesssim \int_{t_1}^{t_2}\|\langle\mathrm{D}\rangle^{-1+\varepsilon}u_s\|_{W^{1/4+\varepsilon,3}}\frac{\mathrm{d}s}{\langle s\rangle^{1/2+\varepsilon}}\\
&\lesssim \left(\int_0^\infty \|u_s\|^2_{W^{-1/2-\kappa,3}}\mathrm{d}s\right)^{1/2}\left(\int_{t_1}^{t_2}\frac{\mathrm{d}s}{\langle s\rangle^{1+2\varepsilon}}\right)^{1/2}\\
&\lesssim \left(\int_{t_1}^{t_2}\frac{\mathrm{d}s}{\langle s\rangle^{1+2\varepsilon}}\right)^{1/2}\|u\|_{\mathcal{L}}.
\end{aligned}$$

where we have used the Sobolev embedding $W^{1/4+\varepsilon,3}\hookrightarrow W^{\varepsilon,4}$. Since

$$\lim_{t_1\to t_2}\int_{t_1}^{t_2}\frac{\mathrm{d}s}{\langle s\rangle^{1+2\varepsilon}}=0, \qquad \int_0^\infty \frac{\mathrm{d}s}{\langle s\rangle^{1+2\varepsilon}}\mathrm{d}s<\infty,$$

for any $t_2\in[0,\infty]$, we can conclude by the Rellich–Kondrachov embedding theorem and the Ascoli–Arzelá theorem, that bounded sets in $\mathcal{L}$ are mapped to compact sets in $C([0,\infty],L^4)$, proving the claim. □

In the sequel, by an abuse of notation, we will denote both a generic element of $\mathfrak{S}$ and the canonical random variable on $\mathfrak{S}$ by

$$\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2, \mathbb{X}^{\langle 3\rangle}, \mathbb{X}^{[3]\circ 1}, \mathbb{X}^{\langle 2\rangle\diamond\langle 2\rangle}, \mathbb{X}^{2\diamond[3]}).$$

We will need the following lemma, which establishes point-wise convergence for the functional Φ_T defined in Lemma 2.15.

Lemma 2.22. *Define $l^\infty(u) = l^\infty(\mathbb{X},u)\in\mathbb{H}_a$ by*

$$l_t^\infty(u) := u_t + \lambda\mathbb{X}_t^{\langle 3\rangle} + \lambda J_t(\mathbb{X}_t^2 \succ Z_t^\flat(u)), \qquad t\geqslant 0. \tag{2.35}$$

For any sequence $(\mathbb{X}^T,u^T)_T$ such that $u^T\to u$ in $\mathcal{L}_w$, $l^T = l^T(\mathbb{X}^T,u^T)\to l = l^\infty(\mathbb{X},u)$ in $\mathcal{H}_w$ and

$$\begin{aligned}
\mathbb{X}^T &= (\mathbb{X}^{T,1}, \mathbb{X}^{T,2}, \mathbb{X}^{T,\langle 3\rangle}, \mathbb{X}^{T,[3]\circ 1}, \mathbb{X}^{T,\langle 2\rangle\diamond\langle 2\rangle}, \mathbb{X}^{T,2\diamond[3]})\\
&\downarrow\\
\mathbb{X} &= (\mathbb{X}^1, \mathbb{X}^2, \mathbb{X}^{\langle 3\rangle}, \mathbb{X}^{[3]\circ 1}, \mathbb{X}^{\langle 2\rangle\diamond\langle 2\rangle}, \mathbb{X}^{2\diamond[3]})
\end{aligned}$$

in $\mathfrak{S}$ we have

$$\lim_{T\to\infty}\Phi_T(\mathbb{X}^T, Z(u^T), K(u^T)) = \Phi_\infty(\mathbb{X}, Z(u), K(u)).$$

Here $Z_t(u) = I_t(u)$, we let $K_t(u) := Z_t(u) - \lambda\mathbb{X}_t^{[3]}$ and Φ_∞ is defined by

$$\Phi_\infty(\mathbb{X}, Z(u), K(u)) := f(\mathbb{X}_\infty^1 + Z_\infty(u)) + \sum_{i=1}^6 \Upsilon_\infty^{(i)}(\mathbb{X}, Z(u), K(u)),$$

with $\Upsilon_\infty^{(i)}(\mathbb{X}, Z, K) = \Upsilon_\infty^{(i)}$ given by

$$\begin{aligned}
\Upsilon_\infty^{(1)} &:= \frac{\lambda}{2}\mathfrak{K}_2(\mathbb{X}_\infty^2, K_\infty, K_\infty) + \frac{\lambda}{2}\fint(\mathbb{X}_\infty^2 \prec K_\infty)K_\infty - \lambda^2\fint(\mathbb{X}_\infty^2 \prec \mathbb{X}_\infty^{[3]})K_\infty,\\
\Upsilon_\infty^{(2)} &= 0,\\
\Upsilon_\infty^{(3)} &:= \lambda\int_0^\infty \fint(\mathbb{X}_t^2 \succ \dot{Z}_t^\flat)K_t \mathrm{d}t,\\
\Upsilon_\infty^{(4)} &:= 4\lambda\fint\mathbb{X}_\infty^1 K_\infty^3 - 12\lambda^2\fint(\mathbb{X}_\infty^1\mathbb{X}_\infty^{[3]})K_\infty^2 + 12\lambda^3\fint\mathbb{X}_\infty^1(\mathbb{X}_\infty^{[3]})^2K_\infty,\\
\Upsilon_\infty^{(5)} &:= -2\lambda^2\int_0^\infty\fint\gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t,\\
\Upsilon_\infty^{(6)} &:= -\frac{\lambda^2}{2}\fint\mathbb{X}_\infty^{2\diamond[3]}K_\infty - \lambda^2\int_0^\infty\fint\mathbb{X}_t^{\langle 2\rangle\diamond\langle 2\rangle}(Z_t^\flat)^2\mathrm{d}t - \frac{\lambda^2}{2}\int_0^\infty \mathfrak{K}_{3,t}(\mathbb{X}_t^2, \mathbb{X}_t^2, Z_t^\flat, Z_t^\flat)\mathrm{d}t,
\end{aligned}$$

where $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_{3,t}$ are the multilinear forms defined in Proposition A.13, Proposition A.14 and Proposition A.15 respectively and where, with abuse of notation, we let

$$\begin{aligned}
\mathbb{X}_\infty^1\mathbb{X}_\infty^{[3]} &:= \mathbb{X}_\infty^1 \succ \mathbb{X}_\infty^{[3]} + \mathbb{X}_\infty^1 \prec \mathbb{X}_\infty^{[3]} + \mathbb{X}_\infty^{[3]\circ 1},\\
\mathbb{X}_\infty^1(\mathbb{X}_\infty^{[3]})^2 &:= \mathbb{X}_\infty^1(\mathbb{X}_\infty^{[3]} \circ \mathbb{X}_\infty^{[3]}) + 2\mathbb{X}_\infty^{[3]\circ 1}\mathbb{X}_\infty^{[3]} + 2\mathfrak{K}_1(\mathbb{X}_\infty^{[3]}, \mathbb{X}_\infty^{[3]}, \mathbb{X}_\infty^1)\\
&\quad + 2\mathbb{X}_\infty^1 \succ (\mathbb{X}_\infty^{[3]} \succ \mathbb{X}_\infty^{[3]}) + 2\mathbb{X}_\infty^1 \prec (\mathbb{X}_\infty^{[3]} \succ \mathbb{X}_\infty^{[3]}).
\end{aligned} \tag{2.36}$$

Proof. Lemma 2.21 implies that for any $u^T \to u$ in $\mathcal{L}_w$ we have $Z(u^T) \to Z(u)$ in $C([0,\infty], L^4)$ and by the convergence of $l^T \to l$ in $\mathcal{H}_w$ we have also $K(u^T) \to K(u)$ in $C([0,\infty], H^{1-\kappa})$. The products $\mathbb{X}_T^{T,1}\mathbb{X}_T^{T,[3]}$ and $\mathbb{X}_T^{T,1}(\mathbb{X}_T^{T,[3]})^2$ can be decomposed using paraproducts and, after identifying the resonant products with the corresponding stochastic objects in $\mathbb{X}^T$, we obtain the finite T analogs of the expressions in eq. (2.36). After this preprocessing, it is easy to see by continuity that we have $\mathbb{X}_T^{T,1}\mathbb{X}_T^{T,[3]} \to \mathbb{X}_\infty^1\mathbb{X}_\infty^{[3]}$ and $\mathbb{X}_T^{T,1}(\mathbb{X}_T^{T,[3]})^2 \to \mathbb{X}_\infty^1(\mathbb{X}_\infty^{[3]})^2$ in $\mathscr{C}^{1/2-\kappa}$. For $\Upsilon^{(1)}$ and $\Upsilon^{(4)}$ and the first term of $\Upsilon^{(6)}$ the statement follows from the fact that they are bounded multilinear forms on $\mathfrak{S} \times C([0,\infty], H^{1/2-\kappa}) \times C([0,\infty], H^{1-\kappa})$. For $\Upsilon^{(2)}$ and the first two terms of $\Upsilon^{(5)}$ convergence to 0 follows from the bounds established in Lemma 2.46 and the proof Lemma 2.49 (in particular eq. (2.62) and eq. (2.63)). For $\Upsilon^{(3)}$, the last term of $\Upsilon^{(5)}$ and the last two terms of $\Upsilon^{(6)}$ we can establish point-wise convergence under the time integrals since the integrands are again bounded (uniformly in time) multilinear forms, and conclude by dominated convergence. □

Going back to our particular setting recall that from Lemma 2.15 we learned

$$\mathcal{W}_T(f) = \inf_{u\in\mathbb{H}_a} F_T(u),$$

with

$$F_T(u) = \mathbb{E}\left[\Phi_T(\mathbb{W}, Z(u), K(u)) + \lambda\|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right],$$

where $l^T(u), Z(u), K(u)$ are functions of u according to eq. (2.21). This form of the functional is appropriate to analyze the limit $T\to\infty$ and obtain the main result of the paper, stated precisely in the following theorem.

Theorem 2.23. *We have*

$$\lim_{T\to\infty}\mathcal{W}_T(f) = \mathcal{W}(f) := \inf_{u\in\mathbb{H}_a} F_\infty(u),$$

where

$$F_{\infty}(u)=\mathbb{E}\left[\Phi_{\infty}(\mathbb{W},Z(u),K(u))+\lambda\|Z_{\infty}(u)\|_{L^4}^4+\frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right],$$

with Φ_∞ *and* l^∞ *introduced in Lemma 2.22.*

Proof. The statement is a direct consequence of Theorem 2.27 below. □

In order to use Γ-convergence, we need to modify the variational setting to guarantee enough compactness and continuity uniformly as $T\to\infty$.

As long as T is finite, the original potential V_T is bounded below so in particular we have

$$-C_T+\mathbb{E}\left[\frac{1}{2}\|u\|_{\mathcal{H}}^2\right]\leqslant F_T(u). \tag{2.37}$$

which quantifies the coercivity of F_T. Unfortunately, this estimate does not survive the limit. However the analytic estimates contained in Section 2.8 below on the renormalized control problem allow to infer that there exists a small $\delta\in(0,1)$, and a finite constant $C>0$ independent of T, such that

$$-C+(1-\delta)\mathbb{E}\left[\lambda\|Z_T(u)\|_{L^4}^4+\frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right]\leqslant F_T(u), \tag{2.38}$$

and

$$F_T(u)\leqslant C+(1+\delta)\mathbb{E}\left[\lambda\|Z_T(u)\|_{L^4}^4+\frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right]. \tag{2.39}$$

Moreover the cost functional $\lambda\|Z_T(u)\|_{L^4}^4+\frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2$ control the $\mathcal{L}$ norm of u uniformly in T, modulo constants depending only on $\|\mathbb{W}\|_{\mathfrak{S}}$ and which are bounded in average uniformly in T. More precisely we have (in a more general setting, useful below)

Lemma 2.24. *Let μ be a probability measure on $\mathfrak{S}\times\mathcal{L}$ with first marginal* $\mathrm{Law}_{\mathbb{P}}(\mathbb{W})$ *and denote with $(\mathbb{X},u)$ the canonical variable on $\mathfrak{S}\times\mathcal{L}$. Then there exists a constant C, depending only on λ, such that*

$$\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2]\lesssim C+2\lambda\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4]+\mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2].$$

Proof. We use $\|l^T(u)\|_{\mathcal{L}}\lesssim\|l^T(u)\|_{\mathcal{H}}$ in the bound

$$\begin{aligned}
\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] &\lesssim \lambda\mathbb{E}_\mu[\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2]+\lambda\mathbb{E}_\mu[\|s\mapsto J_s(\mathbb{X}_s^2\succ\theta_sZ_T(u))\|_{\mathcal{L}}^2]+\mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2]\\
&\lesssim \lambda\mathbb{E}_\mu[\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2]+\lambda\mathbb{E}_\mu\left[\int_0^\infty\frac{\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\kappa}}^2}{\langle s\rangle^{1+\kappa}}\|Z_T(u)\|_{L^4}^2\mathrm{d}s\right]\\
&\quad+\mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2]\\
&\lesssim \lambda\mathbb{E}_\mu[\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}}^2]+\frac{\lambda}{2}\mathbb{E}_\mu\left[\int_0^\infty\frac{\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\kappa}}^4}{\langle s\rangle^{1+\kappa}}\mathrm{d}s\right]+2\lambda\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4]\\
&\quad+\mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2].
\end{aligned}$$

□

From this we conclude that we can relax the optimization problem and ask that $u\in\mathbb{L}_a$ where $\mathbb{L}_a$ is the space of predictable processes in $\mathcal{L}$:

$$\mathcal{W}_T(f)=\inf_{u\in\mathbb{L}_a}F_T(u).$$

For future reference note that eq. (2.38) implies also that for any sequence $(u^T)_T$ such that $F_T(u^T)$ remains bounded we must have that also

$$\sup_T\mathbb{E}[\|l^T(u^T)\|_{\mathcal{H}}^2]<\infty. \tag{2.40}$$

To prove Γ-convergence we need to set up the problem in a space with a topology which, on the one hand is strong enough to enable to prove the Γ-liminf inequality, and on the other hand allows to obtain enough compactness from F_T. Almost sure convergence on $\mathfrak{S} \times \mathcal{L}$ would allow for the former but is too strong for the latter. For this reason we need a setting based on convergence in law as made precise in the following definition.

Definition 2.25. *Denote by* $(\mathbb{X}, u)$ *be the canonical variables on* $\mathfrak{S} \times \mathcal{L}$ *and consider the space of probability measures*

$$\mathcal{Y} := \{\mu \in \mathcal{P}(\mathfrak{S} \times \mathcal{L}) \mid \mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] < \infty\}$$

equipped with the following topology: $\mu_n \to \mu$ *iff*

a) μ_n *converges to* μ *weakly on* $\mathfrak{S} \times \mathcal{L}_w$,

b) $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$.

Let

$$\mathcal{X} := \{\mu \in \mathcal{Y} \mid \mu = \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, u) \text{ for some } u \in \mathbb{L}_a\}$$

and denote by $\bar{\mathcal{X}} \subseteq \mathcal{Y}$ *the closure of* $\mathcal{X}$ *in* $\mathcal{Y}$.

Remark 2.26. Condition (b) allows to exclude pathological points in $\bar{\mathcal{X}}$ and makes possible Lemma 2.34 below.

With these new notations we have

$$\mathcal{W}_T(f) = \inf_{\mu \in \mathcal{X}} \breve{F}_T(\mu), \tag{2.41}$$

where

$$\breve{F}_T(\mu) := \mathbb{E}_\mu\left[\Phi_T(\mathbb{X}, Z(u), K(u)) + \lambda \|Z_T(u)\|_{L^4}^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right]$$

and where $\mathbb{E}_\mu$ denotes the expectation on $\mathfrak{S} \times \mathcal{L}$ wrt. the probability measure μ. We also define the corresponding limiting functional as

$$\breve{F}_\infty(\mu) := \mathbb{E}_\mu\left[\Phi_\infty(\mathbb{X}, Z(u), K(u)) + \lambda \|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|l^\infty(u)\|_{\mathcal{H}}^2\right]. \tag{2.42}$$

Finally we can state the key result of this section.

Theorem 2.27. *The family* $(\breve{F}_T)_T$ Γ*–converges to* $\breve{F}_\infty$ *on* $\bar{\mathcal{X}}$. *Moreover*

$$\lim_T \mathcal{W}_T(f) = \lim_T \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}_T(\mu) = \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}_\infty(\mu) = \mathcal{W}(f).$$

Proof.
Step 1. *(Relaxation)* We will prove below that:

a) the family $(\breve{F}_T)_T$ is indeed equicoercive on $\bar{\mathcal{X}}$ (Lemma 2.29);

b) the variational problems for $\breve{F}_T$ (with $T < \infty$ or $T = \infty$) on $\mathcal{X}$ and on $\bar{\mathcal{X}}$ are equivalent (Lemma 2.35 and Lemma 2.38).

Step 2. *(liminf inequality)* Consider a sequence $\mu^T \to \mu$ in $\bar{\mathcal{X}}$. We need to prove that

$$\liminf_{T \to \infty} \breve{F}_T(\mu^T) \geqslant \breve{F}_\infty(\mu).$$

It is enough to prove this statement for a subsequence, the full statement follows from the fact that every sequence has a subsequence satisfying the inequality. Take a subsequence (not relabeled) such that

$$\sup_T \breve{F}_T(\mu^T) < \infty. \tag{2.43}$$

If there is no such subsequence there is nothing to prove. Otherwise tightness for the subsequence follows like in the proof of equicoercivity in Lemma 2.29 below. Then invoking the Skorokhod representation theorem of [81] we can extract a subsequence (again, not relabeled) and find random variables $(\tilde{\mathbb{X}}^T, \tilde{u}^T)_T$ and $(\tilde{\mathbb{X}}, \tilde{u})$ on some probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ such that $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde{\mathbb{X}}^T, \tilde{u}^T) = \mu^T$, $\mathrm{Law}_{\tilde{\mathbb{P}}}(\tilde{\mathbb{X}}, \tilde{u}) = \mu$ and almost surely $\tilde{\mathbb{X}}^T \to \tilde{\mathbb{X}}$ in $\mathfrak{S}$, $\tilde{u}^T \to \tilde{u}$ in $\mathcal{L}_w$. Note that $\tilde{l}^T := l^T(\tilde{\mathbb{X}}^T, \tilde{u}^T) \to l := l^\infty(\tilde{\mathbb{X}}, u)$ in $\mathcal{L}_w$ and using (2.43) we deduce that the almost sure convergence $l^T \to l$ in $\mathcal{H}_w$, maybe modulo taking another subsequence, again not relabeled. Note that, by the analytic estimates of Section 2.8 (which hold point-wise on the probability space) we have

$$\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2}\|l^T(\tilde{u}^T)\|^2_{\mathcal{H}} + Q(\tilde{\mathbb{X}}^T) \geqslant 0,$$

for some $L^1(\tilde{\mathbb{P}})$ random variable $Q(\tilde{\mathbb{X}}^T)$ such that $\mathbb{E}_{\tilde{\mathbb{P}}}[Q(\tilde{\mathbb{X}}^T)] = \mathbb{E}[Q(\mathbb{W})]$ (for example we can take $Q(\tilde{\mathbb{X}}^T) = C(1 + \|\tilde{\mathbb{X}}^T\|^p_{\mathfrak{S}})$ for some large enough p). Fatou's lemma and Lemma 2.22 then give

$$\begin{aligned}
\liminf_{T\to\infty} \breve{F}_T(\mu^T) &= \liminf_{T\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}}\left[\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2}\|l^T\|^2_{\mathcal{H}}\right] \\
&= \liminf_{T\to\infty} \mathbb{E}_{\tilde{\mathbb{P}}}\left[\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2}\|l^T\|^2_{\mathcal{H}} + Q(\tilde{\mathbb{X}}^T)\right] - \mathbb{E}[Q(\mathbb{W})] \\
&\geqslant \mathbb{E}_{\tilde{\mathbb{P}}} \liminf_{T\to\infty}\left[\Phi_T(\tilde{\mathbb{X}}^T, Z(\tilde{u}^T), K(\tilde{u}^T)) + \lambda\|Z_T(\tilde{u}^T)\|^4_{L^4} + \frac{1}{2}\|l^T\|^2_{\mathcal{H}} + Q(\tilde{\mathbb{X}}^T)\right] - \mathbb{E}[Q(\mathbb{W})] \\
&\geqslant \mathbb{E}_{\tilde{\mathbb{P}}}\left[\Phi_\infty(\tilde{\mathbb{X}}, Z(\tilde{u}), K(\tilde{u})) + \lambda\|Z_\infty(\tilde{u})\|^4_{L^4} + \frac{1}{2}\|l^\infty(\tilde{u})\|^2_{\mathcal{H}}\right] = \breve{F}_\infty(\mu),
\end{aligned}$$

which is the Γ-liminf inequality.

Step 3. *(limsup inequality)* Now all that remains is constructing a recovery sequence, for this we can again assume w.l.o.g that $\breve{F}_\infty(\mu) < \infty$. From Lemma 2.37 there is μ_L such that $|\breve{F}_\infty(\mu) - \breve{F}_\infty(\mu_L)| < \frac{1}{L}$ and (2.50) is satisfied. Then choosing $\mu_L^T = \mathrm{Law}_{\mu_L}(\mathbb{X}, \mathbb{1}_{\{t\leqslant T\}} u_t)$ we obtain that $l^T(\mathbb{1}_{\{\cdot\leqslant T\}} u) = \mathbb{1}_{\{\cdot\leqslant T\}} l^\infty(u)$, so $\|l^T(\mathbb{1}_{\{\cdot\leqslant T\}} u)\|_{\mathcal{H}} \leqslant \|l^\infty(u)\|_{\mathcal{H}}$, and $\|Z_T(\mathbb{1}_{\{\cdot\leqslant T\}} u)\|^4_{L^4} = \|Z_T(u)\|^4 \leqslant \|u\|^4_{\mathcal{L}}$, which is integrable by (2.50). By dominated convergence and Lemma 2.22 we obtain $\lim_{T\to\infty} \breve{F}_T(\mu_L^T) = \breve{F}_\infty(\mu_L)$. Extracting a suitable diagonal sequence gives the required recovery sequence. □

The rest of this section contains the auxiliary lemmas required to complete the proof of the previous theorem.

Lemma 2.28. *Let $G \subseteq \bar{\mathcal{X}}$ such that $\sup_{\mu\in G} \mathbb{E}_\mu[\|u\|^2_{\mathcal{L}}] < \infty$. Then G is tight on $\mathfrak{S} \times \mathcal{L}_w$ and in particular compact in $\bar{\mathcal{X}}$.*

Proof. Observe that for all $\mu \in G$, $\mathrm{Law}_\mu(\mathbb{X}) = \mathrm{Law}_{\mathbb{P}}(\mathbb{W})$ and that $\mathrm{Law}_{\mathbb{P}}(\mathbb{W})$ on $\mathfrak{S}$ is tight since $\mathfrak{S}$ is a separable metric space, so for any $\varepsilon > 0$, we can find a compact set $\mathcal{K}^1_\varepsilon \subset \mathfrak{S}$ such that $\mu((\mathfrak{S} \setminus \mathcal{K}^1_\varepsilon) \times \mathcal{L}) < \varepsilon/2$. Now let $\mathcal{K}^2_\varepsilon := \mathcal{K}^1_\varepsilon \times B(0, C) \subset \mathfrak{S} \times \mathcal{L}$, for some large C to be chosen later. Then $\mathcal{K}^2_\varepsilon$ is a compact subset of $\mathfrak{S} \times \mathcal{L}_w$ and

$$\mathbb{P}_\mu[(\mathbb{X}, u) \notin \mathcal{K}^2_\varepsilon] \leqslant \frac{\varepsilon}{2} + \frac{1}{C}\mathbb{E}_\mu[\|u\|^2_{\mathcal{L}}].$$

Choosing $C > \sup_{\mu\in G} 2\,\mathbb{E}_\mu[\|u\|^2_{\mathcal{L}}]/\varepsilon$ gives tightness of the family G. □

Lemma 2.29. *The family $(\breve{F}_T)_T$ is equicoercive on $\bar{\mathcal{X}}$.*

Proof. Define for some $K > 0$ large enough

$$\mathcal{K} := \{\mu \in \bar{\mathcal{X}} : \mathbb{E}_\mu[\|u\|^2_{\mathcal{L}}] \leqslant K\}.$$

Note that $\mathcal{K}$ is compact from Lemma 2.28. From eq. (2.38) we have

$$\lambda \mathbb{E}_\mu[\|Z_T(u)\|^4_{L^4}] + \frac{1}{2}\mathbb{E}_\mu[\|l^T(u)\|^2_{\mathcal{H}}] \leqslant C + 2\breve{F}_T(\mu).$$

Indeed, note that the analytic estimates of Section 2.8 are path-wise and holds also wrt. $(\mathbb{X}, u)$ under the measure μ (the point is that here u is not necessarily adapted to $\mathbb{X}$), while for the probabilistic estimates on $Q_T(\mathbb{W})$ we have $\mathbb{E}[Q_T(\mathbb{W})] = \mathbb{E}_\mu[Q_T(\mathbb{X})]$ since $\mathrm{Law}_\mu(\mathbb{X}) = \mathrm{Law}_{\mathbb{P}}(\mathbb{W})$. From this we deduce that for some $C, c > 0$

$$\begin{aligned} & \breve{F}_T(\mu) \\ \geqslant\ & \frac{\lambda}{2}\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4] + \frac{1}{4}\mathbb{E}_\mu[\|l^T(u)\|_{\mathcal{H}}^2] - C \\ \geqslant\ & c\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] - C \end{aligned}$$

where in the last line we have used Lemma 2.24. Therefore $\inf_{\mu \in \mathcal{K}^c} \breve{F}_T(\mu) \geqslant cK - C$. On the other hand from eq. (2.39) it follows that $\sup_T \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}_T(\mu) < \infty$. So for K large enough

$$\inf_{\mu \in \bar{\mathcal{X}}} \breve{F}_T(\mu) = \inf_{\mu \in \mathcal{K}} \breve{F}_T(\mu).$$

□

To be able to use this equicoercivity we will need to show that we can extend the infimum in (2.41) to $\bar{\mathcal{X}}$. For this we will first need some properties of the space $\bar{\mathcal{X}}$. In particular we will need to show that measures with sufficiently high moments are dense in $\bar{\mathcal{X}}$ in a way which behaves well with respect to $\breve{F}_T$. With this aim we introduce some useful approximations.

Definition 2.30. *Let $u \in \mathcal{L}$, $N \in \mathbb{N}$, and $(\eta_\varepsilon)_{\varepsilon > 0}$ be a smooth Dirac sequence on Λ and $(\varphi_\varepsilon)_{\varepsilon > 0}$ be another smooth Dirac sequence compactly supported on $\mathbb{R}_+ \times \Lambda$. Denote by $*_\Lambda$ the convolution only wrt the space variable, and by $*$ the space-time convolution. Define the following approximations of the identity:*

$$\begin{aligned} (\mathrm{reg}_{x,\varepsilon}(u)) &:= u *_\Lambda \eta_\varepsilon, \\ (\mathrm{reg}_{t:x,\varepsilon}(u))(t) &:= e^{-\varepsilon t} u * \varphi_\varepsilon(t) = e^{-\varepsilon t} \int_0^t u(t-s) *_\Lambda \varphi_\varepsilon(s)\, \mathrm{d}s. \end{aligned}$$

Let

$$\tilde{T}^N(u) := \inf\left\{ t \geqslant 0 \,\middle|\, \int_0^t \|u(s)\|_{W^{-1/2-\kappa,3}}^2 \mathrm{d}s \geqslant N \right\},$$

and

$$(\mathrm{cut}_N(u))(t) := u(t) \mathbb{1}_{\{t \leqslant \tilde{T}^N(u)\}}.$$

Observe the following properties of these maps:

- $\mathrm{reg}_{x,\varepsilon}$ *is a continuous map* $\mathcal{L}_w \to \mathcal{H}_w$ *and* $\mathcal{L} \to \mathcal{H}$;
- $\mathrm{reg}_{t:x,\varepsilon}$ *is a continuous map* $\mathcal{L}_w \to \mathcal{H}$;
- cut_N *is continuous as a map* $\mathcal{L} \to B(0, N) \subset \mathcal{L}$;
- *if u is a predictable process then $\mathrm{reg}_{x,\varepsilon}(u)$, $\mathrm{reg}_{t:x,\varepsilon}(u)$, $\mathrm{cut}_N(u)$ will also be predictable.*

Furthermore we have the bounds

$$\|\mathrm{reg}_{x,\varepsilon}(u)\|_{\mathcal{L}}, \|\mathrm{reg}_{t:x,\varepsilon}(u)\|_{\mathcal{L}}, \|\mathrm{cut}_N(u)\|_{\mathcal{L}} \leqslant \|u\|_{\mathcal{L}}.$$

uniformly in ε, N, and for every $u \in \mathcal{L}$,

$$\lim_{\varepsilon \to 0} \|\mathrm{reg}_{x,\varepsilon}(u) - u\|_{\mathcal{L}} = \lim_{\varepsilon \to 0} \|\mathrm{reg}_{t:x,\varepsilon}(u) - u\|_{\mathcal{L}} = \lim_{N \to \infty} \|\mathrm{cut}_N(u) - u\|_{\mathcal{L}} = 0.$$

With abuse of notation, for $\mu \in \mathcal{P}(\mathfrak{S} \times \mathcal{L})$ and $f: \mathcal{L} \to \mathcal{L}$, we let

$$f_* \mu = (\mathrm{Id}, f)_* \mu = \mathrm{Law}_\mu(\mathbb{X}, f(u)).$$

Remark 2.31. Let us briefly comment on the rationale for these approximations. $\mathrm{reg}_{t:x,\varepsilon}$ will be used when one wants to obtain a sequence of weakly convergent measures on $\mathfrak{S} \times \mathcal{H}$ or $\mathfrak{S} \times \mathcal{L}$ from a sequence of measures weakly convergent on $\mathfrak{S} \times \mathcal{L}_w$. $\mathrm{reg}_{x,\varepsilon}$ will be used when one wants to obtain a measure on $\mathfrak{S} \times \mathcal{H}$ from one on $\mathfrak{S} \times \mathcal{L}$, while preserving the estimates on the moments of $Z(u)$ since $Z(u *_\Lambda \eta_\varepsilon) = Z(u) *_\Lambda \eta_\varepsilon$.

Lemma 2.32. *Let $\mu \in \bar{\mathcal{X}}$. There exist $(\mu_n)_n$ in $\mathcal{X}$ such that $\mu_n \to \mu$ on $\mathfrak{S} \times \mathcal{L}$ (now with the norm topology) and $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$.*

Proof. By definition of $\bar{\mathcal{X}}$ of there exists $\tilde{\mu}_n \to \mu$ weakly on $\mathfrak{S} \times \mathcal{L}_w$. Then $(\mathrm{reg}_{t:x,\varepsilon})_* \tilde{\mu}_n \to (\mathrm{reg}_{t:x,\varepsilon})_* \mu$ on $\mathfrak{S} \times \mathcal{L}$ as $n \to \infty$, and since $(\mathrm{reg}_{t:x,\varepsilon})_* \mu \to \mu$ weakly on $\mathfrak{S} \times \mathcal{L}$ as $\varepsilon \to 0$, we obtain the statement by taking a diagonal sequence. □

Lemma 2.33. *Let $\mu_n \to \mu$ on $\mathfrak{S} \times \mathcal{L}$, such that $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$. Then*

1. *for every Lipschitz function f on $\mathcal{L}$, $\mathbb{E}_{\mu_n}[f(u)] \to \mathbb{E}_\mu[f(u)]$;*
2. *for every Lipschitz function f on $C([0,\infty], L^4)$ we have $\mathbb{E}_{\mu_n}[f(Z(u))] \to \mathbb{E}_\mu[f(Z(u))]$.*

Proof. Let f be a Lipschitz function on $\mathcal{L}$ with Lipschitz constant L. Let $\eta \in C(\mathbb{R}, \mathbb{R})$ be supported on $B(0,2)$ with $\eta = 1$ on $B(0,1)$, and $\eta_N(x) = \eta(x/N)$. Then $u \mapsto f(u)\, \eta_N(\|u\|_{\mathcal{L}})$ is bounded,

$$\lim_{n\to\infty} \mathbb{E}_{\mu_n}[f(u)\, \eta_N(\|u\|_{\mathcal{L}})] = \mathbb{E}_\mu[f(u)\, \eta_N(\|u\|_{\mathcal{L}})],$$

and

$$\begin{aligned}
\mathbb{E}_{\mu_n}[f(u)\eta_N(\|u\|_{\mathcal{L}})] - \mathbb{E}_{\mu_n}[f(u)] &= \mathbb{E}_{\mu_n}[(f(u)\eta_N(\|u\|_{\mathcal{L}}) - f(u))\mathbb{1}_{\{\|u\|_{\mathcal{L}} \geqslant N\}}] \\
&\leqslant \mathbb{E}_{\mu_n}[2L\|u\|_{\mathcal{L}}\mathbb{1}_{\{\|u\|_{\mathcal{L}} \geqslant N\}}] \\
&\leqslant 2L\mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2]^{1/2} \mu_n(\|u\|_{\mathcal{L}} \geqslant N) \\
&\leqslant \frac{2L}{N}\mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2].
\end{aligned}$$

Using that $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$ we have

$$\begin{aligned}
\lim_{n\to\infty} |\mathbb{E}_{\mu_n}[f(u)] - \mathbb{E}_\mu[f(u)]| &\leqslant \left|\lim_{n\to\infty} \mathbb{E}_{\mu_n}[f(u)\, \eta_N(\|u\|_{\mathcal{L}}^2)] - \mathbb{E}_\mu[f(u)\, \eta_N(\|u\|_{\mathcal{L}}^2)]\right| \\
&\quad + \sup_n |\mathbb{E}_{\mu_n}[f(u)\, \eta_N(\|u\|_{\mathcal{L}}^2)] - \mathbb{E}_{\mu_n}[f(u)]| \\
&\quad + |\mathbb{E}_\mu[f(u)\, \eta_N(\|u\|_{\mathcal{L}}^2)] - \mathbb{E}_\mu[f(u)]| \\
&\leqslant \frac{4L}{N} \sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] \lesssim N^{-1},
\end{aligned}$$

and sending $N \to \infty$ gives the statement. The second statement follows from the first and Lemma 2.21. □

The next lemma proves that we can approximate measures in $\bar{\mathcal{X}}$ by measures with bounded support in the second marginal which are still in $\bar{\mathcal{X}}$.

Lemma 2.34. *Let $\mu \in \bar{\mathcal{X}}$ such that $E_\mu[\|Z_T(u)\|_{L^4}^4] + E_\mu[\|u\|_{\mathcal{L}}^2] < \infty$. For any $L > 0$ there exists $\mu_L \in \bar{\mathcal{X}}$ such that $\|u\|_{\mathcal{L}} \leqslant L$, μ_L-almost surely, $\mu_L \to \mu$ weakly on $\mathfrak{S} \times \mathcal{L}$ as $L \to \infty$,*

$$\mathbb{E}_{\mu_L}[\|Z_T(u)\|_{L^4}^4] \to \mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4], \quad \textit{and} \quad \mathbb{E}_{\mu_L}[\|u\|_{\mathcal{L}}^2] \to \mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2].$$

Furthermore for any μ_L there exists $(\mu_{L,n})_n \subset \mathcal{X}$ such that $\|u\|_{\mathcal{L}} \leqslant L$, $\mu_{L,n}$-almost surely and $\mu_{L,n} \to \mu_L$ weakly on $\mathfrak{S} \times \mathcal{L}_w$.

Proof.
Step 1 First let us show how to approximate μ with $\tilde{\mu}_L$ which are defined such that $\|Z_T(u)\|_{L^4} \leqslant L$, $\tilde{\mu}_L$ almost surely. As $\mu \in \bar{\mathcal{X}}$, there exists $(\mu_n)_n \subset \mathcal{X}$ such that $\mu_n \to \mu$ on $\mathfrak{S} \times \mathcal{L}$ and $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$. Since $\mu_n \in \mathcal{X}$ there exist $(u^n)_n$ adapted such that $\mu_n = \mathrm{Law}(\mathbb{W}, u^n)$. Define $\tilde{Z}_s^n := \mathbb{E}\left[\int_0^T J_t u_t^n \mathrm{d}t \middle| \mathcal{F}_s\right] = \int_0^T J_t \mathbb{E}[u_t^n | \mathcal{F}_s] \mathrm{d}t$. Then $\tilde{Z}$ is a martingale with continuous paths in $L^4(\Lambda)$. Define the stopping time $T_{L,n} = \inf\{t \in [0,T] | \|\tilde{Z}_t^n\|_{L^4} \geqslant L\}$ where the infimum is equal to T if the set is empty. Observe that $\tilde{Z}_{T_{L,n}} = \int_0^T J_t \mathbb{E}[u_t^n | \mathcal{F}_{T_{L,n}}] \mathrm{d}t = Z_T(u^{L,n})$ with $u_t^{L,n} := \mathbb{E}[u_t^n | \mathcal{F}_{T_{L,n}}]$ adapted, by optional sampling, and almost surely $\|\tilde{Z}_{T_L}\|_{L^4} \leqslant L$. Now set $\tilde{\mu}_{L,n} := \mathrm{Law}_{\mathbb{P}}(\mathbb{W}, u^{L,n})$.
Step 1.1 (Tightness) The next goal is to show that for fixed L, we can select a suitable convergent subsequence from $(\tilde{\mu}_{L,n})_n$. For this we first show that $(\tilde{\mu}_{L,n})_n$ is tight on $\mathfrak{S} \times \mathcal{L}_w$. From the definition of $\mathcal{X}$ we have that $\sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty$, and by construction

$$\sup_n \mathbb{E}_{\tilde{\mu}_{L,n}}[\|u\|_{\mathcal{L}}^2] \leqslant \sup_n \mathbb{E}_{\mathbb{P}}[\|\mathbb{E}[u_t^n | \mathcal{F}_{T_{L,n}}]\|_{\mathcal{L}}^2] \leqslant \sup_n \mathbb{E}_{\mathbb{P}}[\|u^n\|_{\mathcal{L}}^2] = \sup_n \mathbb{E}_{\mu_n}[\|u\|_{\mathcal{L}}^2] < \infty,$$

which gives tightness according to Lemma 2.28. We can then select a subsequence which converges on $\mathcal{L}_w$.

Step 1.2 (Bounds) Let $\tilde{\mu}_L$ be the limit of the sequence constructed in Step 1.1. In this step we prove bounds on the relevant moments of $\tilde{\mu}_L$. Let f_1^M, f_2^M be sequences of functions on $\mathbb{R}$ which are Lipschitz, convex and monotone for every M, while for every $x \in \mathbb{R}$

$$0 \leqslant f_1^M(x) \leqslant x^2, \qquad \lim_{M \to \infty} f_1^M(x) = x^2,$$
$$0 \leqslant f_2^M(x) \leqslant x^4, \qquad \lim_{M \to \infty} f_2^M(x) = x^4.$$

Then $f_1^M(\|u\|_{\mathcal{L}})$ is a lower-semi continuous positive function on $\mathcal{L}_w$ so by the Portmanteau lemma we have

$$\mathbb{E}_{\tilde{\mu}_L}[f_1^M(\|u\|_{\mathcal{L}})] \leqslant \liminf_{n \to \infty} \mathbb{E}_{\tilde{\mu}_{L,n}}[f_1^N(\|u\|_{\mathcal{L}})],$$

and since it is also Lipschitz continuous and convex we have

$$\begin{aligned} \liminf_{n \to \infty} \mathbb{E}_{\tilde{\mu}_{L,n}}[f_1^M(\|u\|_{\mathcal{L}})] &= \liminf_{n \to \infty} \mathbb{E}_{\mathbb{P}}[f_1^M(\|\mathbb{E}[u_n|\mathcal{F}_{T_{L,n}}]\|_{\mathcal{L}})] \\ &\leqslant \liminf_{n \to \infty} \mathbb{E}_{\mathbb{P}}[f_1^M(\|u_n\|_{\mathcal{L}})] = \mathbb{E}_\mu[f_1^M(\|u\|_{\mathcal{L}})]. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbb{E}_{\tilde{\mu}_L}[\|u\|_{\mathcal{L}}^2] &= \lim_{M \to \infty} \mathbb{E}_{\tilde{\mu}_L}[f_1^M(\|u\|_{\mathcal{L}})] \\ &\leqslant \lim_{M \to \infty} \mathbb{E}_\mu[f_1^M(\|u\|_{\mathcal{L}})] = \mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2]. \end{aligned}$$

Proceeding similarly for Z, we see that $f_2^M(\|Z_T\|_{L^4})$ is a continuous function on L^4 bounded below, Lipschitz continuous and convex on L^4 so we again can estimate

$$\mathbb{E}_{\tilde{\mu}_L}[f_2^M(\|Z_T\|_{L^4})] = \lim_{n \to \infty} \mathbb{E}_{\tilde{\mu}_{L,n}}[f_2^M(\|Z_T\|_{L^4})],$$

$$\begin{aligned} \mathbb{E}_{\tilde{\mu}_L}[f_2^N(\|Z_T\|_{L^4})] &= \lim_{n \to \infty} \mathbb{E}_{\tilde{\mu}_{L,n}}[f_2^M(\|Z_T\|_{L^4})] \\ &= \lim_{n \to \infty} \mathbb{E}_{\mathbb{P}}[f_2^M(\|\mathbb{E}[Z_T(u_n)|\mathcal{F}_{T_{L,n}}]\|_{L^4})] \\ &\leqslant \lim_{n \to \infty} \mathbb{E}_{\mathbb{P}}[f_2^M(\|Z_T(u_n)]\|_{L^4})] = \mathbb{E}_\mu[f_2^M(\|Z_T(u_n)]\|_{L^4})]. \end{aligned}$$

Taking $N \to \infty$, we obtain

$$\mathbb{E}_{\tilde{\mu}_L}[\|Z_T\|_{L^4}^4] \leqslant \mathbb{E}_\mu[\|Z_T\|_{L^4}^4].$$

Step 1.3 (Weak convergence) Now we prove weak convergence of $\tilde{\mu}_L$ to μ on $\mathfrak{S} \times \mathcal{L}$. Let $f: \mathfrak{S} \times \mathcal{L} \to \mathbb{R}$ be bounded and continuous. By dominated convergence and continuity of f, $\lim_\varepsilon \mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u))] = \mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, u)]$. Using furthermore that $(\mathbb{X}, u) \mapsto f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u))$ is continuous on $\mathfrak{S} \times \mathcal{L}_w$ and Lemma 2.21 in the 5th line below, we can estimate

$$\begin{aligned} & \lim_{L \to \infty} |\mathbb{E}_\mu[f(\mathbb{X}, u)] - \mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, u)]| \\ = & \lim_{L \to \infty} \lim_{\varepsilon \to 0} \left| \lim_{n \to \infty} \mathbb{E}_{\mu_n}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u^n))] - \mathbb{E}_{\tilde{\mu}_{L,n}}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u^n))] \right| \\ = & \lim_{L \to \infty} \lim_{\varepsilon \to 0} \left| \lim_{n \to \infty} \mathbb{E}_{\mathbb{P}}[f(\mathbb{W}, \mathrm{reg}_{t:x,\varepsilon}(u^n)) - f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}])] \right| \\ = & \lim_{L \to \infty} \lim_{\varepsilon \to 0} \left| \lim_{n \to \infty} \mathbb{E}_{\mathbb{P}}[f(\mathbb{W}, \mathrm{reg}_{t:x,\varepsilon}(u^n)) - f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}])\mathbb{1}_{\{T_L < \infty\}}] \right| \\ \leqslant & \lim_{L \to \infty} \lim_{\varepsilon \to 0} \left| \lim_{n \to \infty} \mathbb{E}_{\mathbb{P}}[f(\mathbb{W}, \mathrm{reg}_{t:x,\varepsilon}(u^n)) - f(\mathbb{W}, \mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}])\mathbb{1}_{\{\|u^n\|_{\mathcal{L}} > cL\}}] \right| \\ \leqslant & \frac{2}{c} \left(\sup_{\mathfrak{S} \times \mathcal{L}} |f| \right) \lim_{L \to \infty} \sup_n \frac{\mathbb{E}[\|u^n\|_{\mathcal{L}}^2]}{L^2} = 0. \end{aligned}$$

Step 2 In this step we improve the approximation to have bounded support. Let $\mu_n \to \mu$ be the subsequence selected in Step 1.1. Recall that $\mu_n = \mathrm{Law}(\mathbb{W}, u^n)$ with adapted u^n. Define $\tilde{Z}_t^{n,N} := \mathbb{E}[Z_T(\mathrm{cut}_N(u)) \mid \mathcal{F}_t]$, and similarly to Step 1, $T_{n,L,N} := \inf\{t \geqslant 0 | \|\tilde{Z}_t^{n,N}\|_{L^4} \geqslant L\}$. Set $u^{n,N,L} := \mathbb{E}[\mathrm{cut}_N(u) \mid \mathcal{F}_{T_{n,L,N}}]$, then $\|u^{n,N,L}\|_{\mathcal{L}} \leqslant N$ uniformly in n and $\mathbb{P}$-almost surely, so $\mu_{n,L,N} = \mathrm{Law}(\mathbb{W}, u^{n,N,L})$ is tight on $\mathfrak{S} \times \mathcal{L}_w$ and we can select a weakly convergent subsequence. Denote the limit by $\mu_{L,N}$. Now we follow the strategy from Step 1.

Step 2.1 (Bounds) We now prove bounds on $\mu_{L,N}$ uniformly in L, N similarly to step 1.2. Let f_1^M be defined like in Step 1.2. Then again we have

$$\begin{aligned}\liminf_{n\to\infty}\mathbb{E}_{\mu_{n,L,N}}[f_1^M(\|u\|_{\mathcal{L}})] &= \liminf_{n\to\infty}\mathbb{E}_{\mathbb{P}}[f_1^M\|\mathbb{E}[\mathrm{cut}_N(u^n)|\mathcal{F}_{T_{n,L,N}}]\|_{\mathcal{L}})]\\ &\leqslant \lim_{n\to\infty}\mathbb{E}_{\mathbb{P}}[f_1^M(\|\mathrm{cut}_N(u^n)\|_{\mathcal{L}})]\\ &= \lim_{n\to\infty}\mathbb{E}_{\mu_n}[f_1^M(\|\mathrm{cut}_N(u)\|_{\mathcal{L}})]\leqslant\mathbb{E}_{\mu}[f_1^M(\|u\|_{\mathcal{L}})].\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}_{\mu_{L,N}}[\|u\|_{\mathcal{L}}^2] &= \lim_{M\to\infty}\mathbb{E}_{\tilde{\mu}_{L,N}}[f_1^M(\|u\|_{\mathcal{L}})]\\ &\leqslant \lim_{M\to\infty}\liminf_{n\to\infty}\mathbb{E}_{\mu_{n,L,N}}[f_1^M(\|u\|_{\mathcal{L}})]\\ &\leqslant \lim_{M\to\infty}\mathbb{E}_{\mu}[f_1^M(\|u\|_{\mathcal{L}})]=\mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2].\end{aligned}$$

Step 2.1 (Weak convergence) Now we prove that $\mu_{L,N}\to\tilde{\mu}_L$ weakly on $\mathcal{L}$. Let $f\colon\mathfrak{S}\times\mathcal{L}\to\mathbb{R}$ be bounded and continuous. By dominated convergence and continuity of f, $\lim_{\varepsilon}\mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X}, \mathrm{reg}_{t:x,\varepsilon}(u))]=\mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X},u)]$, and furthermore since $f(\mathbb{X},\mathrm{reg}_{t:x,\varepsilon}(u))$ is continuous on $\mathfrak{S}\times\mathcal{L}_w$ we have (recall that $\tilde{T}^N(u^n)$ is introduced in Definition 2.30)

$$\begin{aligned}&\lim_{N\to\infty}|\mathbb{E}_{\tilde{\mu}_L}[f(\mathbb{X},u)]-\mathbb{E}_{\mu_{L,N}}[f(\mathbb{X},u)]|\\ =\ &\lim_{N\to\infty}\lim_{\varepsilon\to0}\left|\lim_{n\to\infty}\mathbb{E}_{\mu_{n,L}}[f(\mathbb{X},\mathrm{reg}_{t:x,\varepsilon}(u))]-\mathbb{E}_{\tilde{\mu}_{n,L,N}}[f(\mathbb{X},\mathrm{reg}_{t:x,\varepsilon}(u))]\right|\\ =\ &\lim_{N\to\infty}\lim_{\varepsilon\to0}\left|\lim_{n\to\infty}\mathbb{E}_{\mathbb{P}}[f(\mathbb{W},\mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}])-f(\mathbb{W},\mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(\bar{u}^{n,N})|\mathcal{F}_{T_{n,L,N}}])]\right|\\ =\ &\lim_{N\to\infty}\lim_{\varepsilon\to0}\left|\lim_{n\to\infty}\mathbb{E}_{\mathbb{P}}[(f(\mathbb{W},\mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}])-\right.\\ &\left.f(\mathbb{W},\mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(\bar{u}^{n,N})|\mathcal{F}_{T_{n,L,N}}]))\mathbb{1}_{\{\tilde{T}^N(u^n)<\infty\}}]\right|\\ \leqslant\ &\lim_{N\to\infty}\sup_{\varepsilon}\left|\sup_n\mathbb{E}_{\mathbb{P}}[(f(\mathbb{W},\mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(u^n)|\mathcal{F}_{T_L}])-\right.\\ &\left.f(\mathbb{W},\mathbb{E}[\mathrm{reg}_{t:x,\varepsilon}(\bar{u}^{n,N})|\mathcal{F}_{T_{n,L,N}}]))\mathbb{1}_{\{\|u^n\|_{\mathcal{L}}\}>N\}}]\right|\\ \leqslant\ &\left(\sup_{\mathfrak{S}\times\mathcal{L}}|f|\right)\lim_{N\to\infty}\sup_n\frac{\mathbb{E}[\|u^n\|_{\mathcal{L}}^2]}{N^2}\\ =\ &0\end{aligned}$$

Step 3. We now put everything together. Since all $\mu_{L,N}$ are supported on the set $\{u\colon\|Z_T(u)\|_{L^4}\leqslant L\}$, weak convergence and Lemma 2.21 imply

$$\lim_{N\to\infty}\mathbb{E}_{\mu_{N,L}}[\|Z_T(u)\|_{L^4}^4]=\mathbb{E}_{\tilde{\mu}_L}[\|Z_T(u)\|_{L^4}^4].$$

By the Portmanteau lemma,

$$\liminf_{N\to\infty}\mathbb{E}_{\mu_{N,L}}[\|u\|_{\mathcal{L}}^2]\geqslant\mathbb{E}_{\tilde{\mu}_L}[\|u\|_{\mathcal{L}}^2], \tag{2.44}$$

and

$$\liminf_{L\to\infty}\mathbb{E}_{\tilde{\mu}_L}[\|u\|_{\mathcal{L}}^2]\geqslant\mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]$$

which together with Step 1.2 imply $\lim_{L\to\infty}\mathbb{E}_{\tilde{\mu}_L}[\|u\|_{\mathcal{L}}^2]=\mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]$, and by the same argument $\lim_{L\to\infty}\mathbb{E}_{\tilde{\mu}_L}[\|Z_T(u)\|_{L^4}^4]=\mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4]$. For any $\delta>0$ we can choose a $\tilde{\mu}_L$ such that

$$|\mathbb{E}_{\tilde{\mu}_L}[\|Z_T(u)\|_{L^4}^4]-\mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4]|+|\mathbb{E}_{\tilde{\mu}_L}[\|u\|_{\mathcal{L}}^2]-\mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]|\leqslant\delta.$$

By (2.44)

$$\mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]\geqslant\liminf_{N\to\infty}\mathbb{E}_{\mu_{N,L}}[\|u\|_{\mathcal{L}}^2]\geqslant\mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]-\delta,$$

and we can choose N large enough so that

$$|\mathbb{E}_{\mu_{N,L}}[\|Z_T(u)\|_{L^4}^4]-\mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4]|+|\mathbb{E}_{\mu_{N,L}}[\|u\|_{\mathcal{L}}^2]-\mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2]|\leqslant\delta,$$

which implies the statement of the theorem. □

Lemma 2.35. *If $T<\infty$ we have*

$$\inf_{\mu\in\mathcal{X}}\breve{F}_T(\mu)=\inf_{\mu\in\bar{\mathcal{X}}}\breve{F}_T(\mu).$$

Proof. To prove the claim it is enough to show that for any $\mu\in\bar{\mathcal{X}}$, for any $\alpha>0$, there exists a sequence $\mu_n\in\mathcal{X}$ such that $\limsup_{n\to\infty}\breve{F}_T(\mu_n)\leqslant\breve{F}_T(\mu)+\alpha$. W.l.o.g we can assume that $\breve{F}_T(\mu)<\infty$. Observe that, as long as $T<\infty$ we can also express

$$\breve{F}_T(\mu)=\mathbb{E}_\mu\left[\frac{1}{|\Lambda|}V_T(\mathbb{X}_T^1+Z_T(u))+\frac{1}{2}\|u\|_{\mathcal{H}}^2\right],$$

and deduce that $\mathbb{E}_\mu[\|u\|_{\mathcal{H}}^2]<\infty$ since V_T is bounded below at fixed T. By Lemma 2.34 there exists a sequence $(\mu_L)_L\subset\bar{\mathcal{X}}$, such that $\mu_L(\|u\|_{\mathcal{L}}\leqslant L)=1$, $\mu_L\to\mu$ on $\mathfrak{S}\times\mathcal{L}$ and by weak convergence and domination,

$$\mathbb{E}_{\mu_L}[\|Z_T(u)\|_{L^4}^4]\to\mathbb{E}_\mu[\|Z_T(u)\|_{L^4}^4],\qquad\mathbb{E}_{\mu_L}[\|u\|_{\mathcal{L}}^2]\to\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2].$$

First we have to improve the regularity of μ_L to get convergence on $\mathfrak{S}\times\mathcal{H}_w$ but without affecting our control on the moments of Z_T, so let $\mu_L^\varepsilon:=(\mathrm{reg}_{x,\varepsilon})_*\mu_L$ and $\mu^\varepsilon:=(\mathrm{reg}_{x,\varepsilon})_*\mu$. Then

$$\mathbb{E}_{\mu_L^\varepsilon}[\|Z_T(u)\|_{L^4}^4]\to E_{\mu^\varepsilon}[\|Z_T(u)\|_{L^4}^4],\qquad\mathbb{E}_{\mu_L^\varepsilon}[\|u\|_{\mathcal{H}}^2]\to E_{\mu^\varepsilon}[\|u\|_{\mathcal{H}}^2],$$

and $\mu_L^\varepsilon\to\mu^\varepsilon$ on $\mathfrak{S}\times\mathcal{H}$. By continuity of $\breve{F}_T$ and the bound (2.39), $\breve{F}_T(\mu_L^\varepsilon)\to\breve{F}_T(\mu^\varepsilon)$ as $L\to\infty$ and $\breve{F}_T(\mu^\varepsilon)\to\breve{F}_T(\mu)$ as $\varepsilon\to0$. In particular we can find L and ε such that $|\breve{F}_T(\mu_L^\varepsilon)-\breve{F}_T(\mu)|<\alpha/2$. By Lemma 2.34 there exists a sequence $(\mu_{n,L})_{n,L}\subset\mathcal{X}$ such that each measure $\mu_{n,L}$ is supported on $\mathfrak{S}\times B(0,L)$ and $\mu_{n,L}\to\mu_L$ weakly on $\mathfrak{S}\times\mathcal{H}_w$. Setting $\mu_{n,L}^{\varepsilon,\delta}:=(\mathrm{reg}_{t;x,\delta})_*(\mathrm{reg}_{x,\varepsilon})_*\mu_{n,L}$ and $\mu_L^{\varepsilon,\delta}:=(\mathrm{reg}_{t;x,\delta})_*(\mathrm{reg}_{x,\varepsilon})_*\mu_L$ we have $\mu_{n,L}^{\varepsilon,\delta}\to\mu_L^{\varepsilon,\delta}$ on $\mathfrak{S}\times\mathcal{H}$ with norm topology. It is not hard too see that $V_T(\mathbb{X}_T^1+Z_T(u))\lesssim_T\|\mathbb{X}\|_{\mathfrak{S}}^4+\|u\|_{\mathcal{H}}^4$ and since on the support of $\mu_{n,L}^{\varepsilon,\delta}$, $\|u\|_{\mathcal{H}}\leqslant L$ and the first marginal of $\mu_{n,L}^{\varepsilon,\delta}$ is fixed we have again by domination and weak convergence

$$\lim_{n\to\infty}\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}\left[\frac{1}{|\Lambda|}V_T(\mathbb{X}_T^1+Z_T(u))+\frac{1}{2}\|u\|_{\mathcal{H}}^2\right]=\mathbb{E}_{\mu_L^{\varepsilon,\delta}}\left[\frac{1}{|\Lambda|}V_T(\mathbb{X}_T^1+Z_T(u))+\frac{1}{2}\|u\|_{\mathcal{H}}^2\right]$$

and by dominated convergence (since $\mu_L^{\varepsilon,\delta}$ is supported on $\mathfrak{S}\times B(0,L)$) we can find a δ such that $|\breve{F}_T(\mu_L^{\varepsilon,\delta})-\breve{F}_T(\mu_L^\varepsilon)|<\alpha/2$ which proves the statement. □

The proof of Lemma 2.35 does not apply when $T=\infty$. An additional difficulty derives from the fact that in approximating the drift u we might destroy the regularity of $l^\infty(u)$, since now $l^\infty(u)$ needs to be more regular than u, contrary to the finite T case. To resolve this problem we need to be able to smooth out the remainder without destroying the bound on $Z_T(u)$. To do so smoothing $l^\infty(u)$ directly, and constructing a corresponding new u will not work, since $l^\infty(u)$ by itself does not give enough control on u and $Z(u)$. However we are still able to prove the following lemma by regularizing an "augmented" version of $l^\infty(u)$.

Lemma 2.36. *There exists a family of continuous functions* rem_ε: $\mathfrak{S}\times\mathcal{L}\to\mathcal{L}$, *which are also continuous* $\mathfrak{S}\times\mathcal{L}_w\to\mathcal{L}_w$, *such that for any* $T\in[0,\infty]$,

$$\begin{aligned}\|\mathrm{rem}_\varepsilon(\mathbb{X},u)\|_{\mathcal{L}}&\lesssim\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}},\\ \|Z_T(\mathrm{rem}_\varepsilon(\mathbb{X},u))\|_{L^4}&\lesssim\|\mathbb{X}\|_{\mathfrak{S}}+\|Z_T(u)\|_{L^4},\\ \|l^\infty(\mathrm{rem}_\varepsilon(\mathbb{X},u))\|_{\mathcal{H}}^2&\lesssim_\varepsilon(1+\|\mathbb{X}\|_{\mathfrak{S}})^4+\|Z_\infty(u)\|_{L^4}^4+\|u\|_{\mathcal{L}}^2,\end{aligned}$$

and $\|l^\infty(\mathrm{rem}_\varepsilon(\mathbb{X},u))\|_{\mathcal{H}}$ *depends continuously on* $(\mathbb{X},u)\in\mathfrak{S}\times\mathcal{L}$. *Furthermore*

$$\mathrm{rem}_\varepsilon(\mathbb{X},u)\to u\ \text{ in }\mathcal{L},$$

and if $l^\infty(u)\in\mathcal{H}$

$$l^\infty(\mathrm{rem}_\varepsilon(\mathbb{X},u))\to l^\infty(u)\text{ in }\mathcal{H}\text{ as }\varepsilon\to0.$$

Proof. Let $\mathbb{X}^2 = \mathcal{U}_{\leqslant}\mathbb{X}^2 + \mathcal{U}_{>}\mathbb{X}^2$ be the decomposition introduced in Section 2.5, and observe that for any $c > 0$ we can easily modify it to ensure that $\|\mathcal{U}_{>}\mathbb{X}^2\|_{\mathscr{C}^{-1-\kappa}} < c$, almost surely for any $\mu \in \bar{\mathcal{X}}$ and for any $1 \leqslant p < \infty$, $\mathbb{E}_{\mu}[\|\mathcal{U}_{\leqslant}\mathbb{X}^2\|^p_{\mathscr{C}^{-1+\kappa}}] \leqslant C$ where C depends on $|\Lambda|$, κ, c, p. Now set $\tilde{l}_t(u) = -\lambda J_t(\mathcal{U}_{\leqslant}\mathbb{X}^2_t \succ Z^{\flat}_t(u)) + l^{\infty}_t(u)$. Then u satisfies

$$u_s = -\lambda\mathbb{X}^{\langle 3\rangle}_s - \lambda J_s(\mathcal{U}_{>}\mathbb{X}^2_s \succ Z^{\flat}_s) + \tilde{l}_s(u).$$

From this equation we can see that, like in Section 2.5,

$$\|u\|_{\mathcal{L}} \lesssim \lambda\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \lambda\int_0^{\infty}\frac{1}{\langle s\rangle^{1+\varepsilon}}\|\mathcal{U}_{>}\mathbb{X}^2_s\|_{\mathscr{C}^{-1-\kappa}}\mathrm{d}s\|u\|_{\mathcal{L}} + \|\tilde{l}_s(u)\|_{\mathcal{L}},$$

and choosing c small enough we get

$$\|u\|_{\mathcal{L}} \lesssim \lambda\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|\tilde{l}(u)\|_{\mathcal{L}}. \tag{2.45}$$

Similarly we observe that

$$Z_T(u) = -\lambda\mathbb{X}^{[3]}_T - \lambda\int_0^T J^2_s(\mathcal{U}_{>}\mathbb{X}^2_s \succ Z^{\flat}_s(u))\mathrm{d}s + Z_T(\tilde{l}(u)),$$

so again with c small enough and since $Z^{\flat}_s = \theta_s Z_T$ for $s \leqslant T$:

$$\|Z_T(u)\|_{L^4} \lesssim \lambda\left\|\mathbb{X}^{[3]}_T\right\|_{L^4} + \|Z_T(\tilde{l}(u))\|_{L^4}. \tag{2.46}$$

Conversely, it is not hard to see that we have the inequalities

$$\|Z_T(\tilde{l}(u))\|_{L^4} \lesssim \lambda\left\|\mathbb{X}^{[3]}_T\right\|_{L^4} + \|Z_T(u)\|_{L^4}, \tag{2.47}$$

and

$$\|\tilde{l}(u)\|_{\mathcal{L}} \lesssim \lambda\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}. \tag{2.48}$$

Clearly the map $(\mathbb{X}, u) \mapsto (\mathbb{X}, \tilde{l}(u))$ is continuous as a map $\mathfrak{S} \times \mathcal{L} \to \mathfrak{S} \times \mathcal{L}$ and using Lemma 2.21 also as a map $\mathfrak{S} \times \mathcal{L}_w \to \mathfrak{S} \times \mathcal{L}_w$, and the inverse is clearly continuous $\mathfrak{S} \times \mathcal{L} \to \mathfrak{S} \times \mathcal{L}$. We now show that it is also continuous as a map $\mathfrak{S} \times \mathcal{L}_w \to \mathfrak{S} \times \mathcal{L}_w$. Assume that $\tilde{l}(u^n) \to l(u)$ weakly, since then $\|l(u^n)\|_{\mathcal{L}}$ bounded, this implies by (2.45) that also $\|u^n\|_{\mathcal{L}}$ is bounded, and so we can select a weakly convergent subsequence, converging to $u^{\star}$. Then $u^{\star}$ solves the equation

$$u^{\star}_s = -\lambda\mathbb{X}^{\langle 3\rangle}_s - \lambda J_s(\mathcal{U}_{>}\mathbb{X}^2_s \succ Z^{\flat}_s(u^{\star})) + \tilde{l}_s(u),$$

(which can be seen for example by testing with some $h \in \mathcal{L}^*$) which implies that $u^{\star} = u$ (e.g. by Gronwall). Now define $\mathrm{rem}_{\varepsilon}(u)$ to be the solution to the equation

$$\mathrm{rem}_{\varepsilon}(u) = -\lambda\mathbb{X}^{\langle 3\rangle}_s - \lambda J_s(\mathcal{U}_{>}\mathbb{X}^2_s \succ Z^{\flat}_s(\mathrm{rem}_{\varepsilon}(u))) + \mathrm{reg}_{x,\varepsilon}(\tilde{l}_s(u)).$$

Then by the properties discussed above $(\mathbb{X}, u) \mapsto (\mathbb{X}, \mathrm{rem}_{\varepsilon}(u))$ is continuous in both the weak and the norm topology and we also have from (2.45) and (2.48) that

$$\|\mathrm{rem}_{\varepsilon}(u)\|_{\mathcal{L}} \lesssim \lambda\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}.$$

From (2.46) we have

$$\|Z_T(\mathrm{rem}_{\varepsilon}(u))\|_{L^4} \lesssim \lambda\left\|\mathbb{X}^{[3]}_T\right\|_{L^4} + \|Z_T(u)\|_{L^4},$$

and by definition of $\mathrm{rem}_{\varepsilon}(u)$

$$\begin{aligned}\|\tilde{l}(\mathrm{rem}_{\varepsilon}(u))\|_{\mathcal{H}} &= \|\mathrm{reg}_{x,\varepsilon}(\tilde{l}(u))\|_{\mathcal{H}}\\ &\lesssim_{\varepsilon} \lambda\|\mathbb{X}^{\langle 3\rangle}\|_{\mathcal{L}} + \|u\|_{\mathcal{L}}.\end{aligned} \tag{2.49}$$

Now observe that

$$\begin{aligned}\|l^{\infty}(\mathrm{rem}_{\varepsilon}(u))\|^2_{\mathcal{H}} &\lesssim \|s \mapsto \lambda J_s(\mathcal{U}_{\leqslant}\mathbb{X}^2_s \succ Z^{\flat}_s(\mathrm{rem}_{\varepsilon}(u)))\|^2_{\mathcal{H}} + \|\tilde{l}(\mathrm{rem}_{\varepsilon}(u))\|^2_{\mathcal{H}}\\ &\lesssim_{\varepsilon} \lambda\int_0^{\infty}\frac{1}{\langle s\rangle^{1+\kappa}}\|\mathcal{U}_{\leqslant}\mathbb{X}^2_s\|^2_{\mathscr{C}^{-1+\kappa}}\|Z^{\flat}_s(\mathrm{rem}_{\varepsilon}(u))\|^2_{L^4}\mathrm{d}s + \lambda\|\mathbb{X}^{\langle 3\rangle}\|^2_{\mathcal{L}} + \|u\|^2_{\mathcal{L}}\\ &\lesssim \lambda(1 + \|\mathbb{X}\|_{\mathfrak{S}})^4 + \|Z_{\infty}(\mathrm{rem}_{\varepsilon}(u))\|^4_{L^4} + \|u\|^2_{\mathcal{L}}\\ &\lesssim \lambda(1 + \|\mathbb{X}\|_{\mathfrak{S}})^4 + \|Z_{\infty}(u)\|^4_{L^4} + \|u\|^2_{\mathcal{L}}.\end{aligned}$$

Since also $\|\lambda J_t(\mathcal{U}_{\leqslant}\mathbb{X}_t^2 \succ Z_t^{\flat}(\text{rem}_{\varepsilon}(u)))\|_{\mathcal{H}}$ depends continuously on $(\mathbb{X}, u)$ (both in the weak and strong topology on $\mathcal{L}$) we obtain the statement. □

Lemma 2.37. *For any $\mu \in \bar{\mathcal{X}}$ such that $\breve{F}_{\infty}(\mu) < \infty$ there exists a sequence of measures $\mu_L \in \bar{\mathcal{X}}$ such that*

i. For any $p < \infty$,

$$\mathbb{E}_{\mu_L}[\|u\|_{\mathcal{L}}^p] + \mathbb{E}_{\mu_L}[\|l^{\infty}(u)\|_{\mathcal{H}}^p] < \infty, \tag{2.50}$$

ii. $\mu_L \to \mu$ weakly on $\mathfrak{S} \times \mathcal{L}$ and $\text{Law}_{\mu_L}(l^{\infty}(u)) \to \text{Law}_{\mu}(l^{\infty}(u))$ weakly on $\mathcal{H}$,

iii.

$$\lim_{L \to \infty} \breve{F}_{\infty}(\mu_L) = \breve{F}_{\infty}(\mu),$$

iv. For any μ_L there exists a sequence $\mu_{n,L} \in \mathcal{X}$ such that

$$\sup_n (\mathbb{E}_{\mu_{n,L}}[\|u\|_{\mathcal{L}}^p] + \mathbb{E}_{\mu_{n,L}}[\|l^{\infty}(u)\|_{\mathcal{H}}^p]) < \infty, \tag{2.51}$$

$\mu_{n,L} \to \mu_L$ weakly on $\mathfrak{S} \times \mathcal{L}_w$ and $\text{Law}_{\mu_{n,L}}(l^{\infty}(u)) \to \text{Law}_{\mu}(l^{\infty}(u))$ weakly on $\mathcal{H}_w$.

Proof. By Lemma 2.34 there exists a sequence $\mu_{\tilde{L}} \to \mu$ weakly on $\mathfrak{S} \times \mathcal{L}$ such that

$$\mathbb{E}_{\mu_{\tilde{L}}}[\|Z_T(u)\|_{L^4}^4] \to \mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4], \qquad \mathbb{E}_{\mu_{\tilde{L}}}[\|u\|_{\mathcal{L}}^2] \to \mathbb{E}_{\mu}[\|u\|_{\mathcal{L}}^2],$$

and $\mu_{\tilde{L}}$ is supported on $\mathfrak{S} \times B(0, \tilde{L}) \subset \mathfrak{S} \times \mathcal{L}$. Now set $\mu_{\tilde{L}}^{\varepsilon} := (\text{rem}_{\varepsilon})_* \mu_{\tilde{L}}$. Then $\mu_{\tilde{L}}^{\varepsilon} \to \mu^{\varepsilon} := (\text{rem}_{\varepsilon})_* \mu$ on $\mathfrak{S} \times \mathcal{L}$ and by the bounds from Lemma 2.36 also $\mathbb{E}_{\mu_{\tilde{L}}^{\varepsilon}}[\|Z_T(u)\|_{L^4}^4] \to \mathbb{E}_{\mu^{\varepsilon}}[\|Z_T(u)\|_{L^4}^4]$ and $\mathbb{E}_{\mu_{\tilde{L}}^{\varepsilon}}[\|l^{\infty}(u)\|_{\mathcal{H}}^2] \to \mathbb{E}_{\mu^{\varepsilon}}[\|l^{\infty}(u)\|_{\mathcal{H}}^2]$. The bounds from Lemma 2.36 imply also $\mathbb{E}_{\mu^{\varepsilon}}[\|Z_T(u)\|_{L^4}^4] \to \mathbb{E}_{\mu}[\|Z_T(u)\|_{L^4}^4]$, $\mathbb{E}_{\mu^{\varepsilon}}[\|l^{\infty}(u)\|_{\mathcal{H}}^2] \to \mathbb{E}_{\mu}[\|l^{\infty}(u)\|_{\mathcal{H}}^2]$, and furthermore

$$\mathbb{E}_{\mu_{\tilde{L}}^{\varepsilon}}[\|u\|_{\mathcal{L}}^p] \lesssim \mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|_{\mathfrak{S}}^p + \|u\|_{\mathcal{L}}^p) \lesssim \mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|_{\mathfrak{S}}^p) + \tilde{L}^p,$$

and similarly

$$\mathbb{E}_{\mu_{\tilde{L}}^{\varepsilon}}[\|l^{\infty}(u)\|_{\mathcal{L}}^p] \lesssim \mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|_{\mathfrak{S}}^p + \|u\|_{\mathcal{L}}^p) \lesssim \mathbb{E}_{\mu_{\tilde{L}}}(\|\mathbb{X}\|_{\mathfrak{S}}^p) + \tilde{L}^p,$$

and by continuity of $\breve{F}_{\infty}$ and domination using (2.39) we are also able to deduce that we can find ε small enough and $\tilde{L}$ large enough depending on ε such that $|\breve{F}_{\infty}(\mu^{\varepsilon}) - \breve{F}_{\infty}(\mu)| < 1/2L$ and $|\breve{F}_{\infty}(\mu_{\tilde{L}}^{\varepsilon}) - \breve{F}_{\infty}(\mu^{\varepsilon})| < 1/2L$. Choosing $\mu_L = \mu_{\tilde{L}}^{\varepsilon}$ we obtain the first three points of the Lemma. For the fourth point recall that from Lemma 2.34 we have sequences $\mu_{n,\tilde{L}} \to \mu_{\tilde{L}}$ weakly on $\mathfrak{S} \times \mathcal{L}_w$, and $\mu_{n,\tilde{L}} \in \mathcal{X}$, which have support in $\mathfrak{S} \times B(0, \tilde{L})$ and since rem_{ε} is continuous on $\mathfrak{S} \times \mathcal{L}_w$ setting $\mu_{n,\tilde{L}}^{\varepsilon} := (\text{reg}_{\varepsilon})_* \mu_{n,\tilde{L}}$ we obtain the desired sequence. □

Lemma 2.38. *If $T = \infty$ we have*

$$\inf_{\mu \in \mathcal{X}} \breve{F}_{\infty}(\mu) = \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}_{\infty}(\mu).$$

Proof. One can now proceed very similarly to the proof of Lemma 2.35. Let $\mu \in \bar{\mathcal{X}}$ such that $\breve{F}_{\infty}(\mu) < \infty$. By Lemma 2.37, for any $L, \mu \in \bar{\mathcal{X}}$, there exists a μ_L such that $|\breve{F}_{\infty}(\mu) - \breve{F}_{\infty}(\mu_L)| < 1/L$, and a sequence $(\mu_{n,L})_n$ such that $\mu_{n,L} \in \mathcal{X}$, $\mu_{n,L} \to \mu_L$ weakly on $\mathfrak{S} \times \mathcal{L}_w$, and such that (2.51) is satisfied. Define $\mu_{n,L}^{\varepsilon,\delta} := \text{Law}(\mathbb{X}, \text{rem}_{\varepsilon}(\text{reg}_{t:x,\varepsilon}(u)))$, and observe that now $\mu_{n,L}^{\varepsilon,\delta} \to \mu_L^{\varepsilon,\delta}$ on $\mathfrak{S} \times \mathcal{L}$, $\text{Law}_{\mu_{n,L}^{\varepsilon,\delta}}(\mathbb{X}, l^{\infty}(u)) \to \text{Law}_{\mu_L^{\varepsilon,\delta}}(\mathbb{X}, l^{\infty}(u))$ on $\mathfrak{S} \times \mathcal{H}$, and that we have $\sup_n (\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|u\|_{\mathcal{L}}^p] + \mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|l^{\infty}(u)\|_{\mathcal{H}}^p]) < \infty$. Then for some $\chi \in C(\mathbb{R}, \mathbb{R})$, $\chi = 1$ on $B(0,1)$ supported on $B(0,2)$, for any $N \in \mathbb{N}$, the function

$$\begin{aligned} &\chi\left(\frac{\|\mathbb{X}\|_{\mathfrak{S}} + \|u\|_{\mathcal{L}} + \|l^{\infty}(u)\|_{\mathcal{H}}}{N}\right)\left(\Phi_{\infty}(\mathbb{X}, Z(u), K(u)) + \lambda \|Z_{\infty}(u)\|_{L^4}^4 + \frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right) \\ =:\ &\tilde{\chi}_N(\mathbb{X}, u)\left(\Phi_{\infty}(\mathbb{X}, Z(u), K(u)) + \lambda \|Z_{\infty}(u)\|_{L^4}^4 + \frac{1}{2}\|l^{\infty}(u)\|_{\mathcal{H}}^2\right) \end{aligned}$$

is bounded and continuous on $\mathfrak{S}\times\mathcal{L}$, and so by weak convergence

$$\begin{aligned}
&\lim_{n\to\infty}|\breve{F}_\infty(\mu_{n,L}^{\varepsilon,\delta})-\breve{F}_\infty(\mu_L^{\varepsilon,\delta})|\\
\leqslant\ &\lim_{n\to\infty}\left|\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}\left[\tilde{\chi}_N(\mathbb{X},u)\left(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|_{L^4}^4+\frac{1}{2}\|l^\infty(u)\|_{\mathcal{H}}^2\right)\right]-\right.\\
&\qquad\left.-\mathbb{E}_{\mu_L^{\varepsilon,\delta}}\left[\tilde{\chi}_N(\mathbb{X},u)\left(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|_{L^4}^4+\frac{1}{2}\|l^\infty(u)\|_{\mathcal{H}}^2\right)\right]\right|\\
&+\sup_n\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}\left[\left|(1-\tilde{\chi}_N(\mathbb{X},u))\left(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|_{L^4}^4+\frac{1}{2}\|l^\infty(u)\|_{\mathcal{H}}^2\right)\right|\right]\\
&+\mathbb{E}_{\mu_L^{\varepsilon,\delta}}\left[\left|(1-\tilde{\chi}_N(\mathbb{X},u))\left(\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|_{L^4}^4+\frac{1}{2}\|l^\infty(u)\|_{\mathcal{H}}^2\right)\right|\right]\\
\leqslant\ &2\sup_n\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}\left[\mathbb{1}_{\{\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^\infty(u)\|_{\mathcal{H}}>N\}}\left|\Phi_\infty(\mathbb{X},Z(u),K(u))+\lambda\|Z_\infty(u)\|_{L^4}^4+\frac{1}{2}\|l^\infty(u)\|_{\mathcal{H}}^2\right|\right]\\
\lesssim\ &\sup_n\left(\mu_{n,L}^{\varepsilon,\delta}(\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^\infty(u)\|_{\mathcal{H}}>N)\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|\mathbb{X}\|_{\mathfrak{S}}^p+\|u\|_{\mathcal{L}}^8+\|l^\infty(u)\|_{\mathcal{H}}^4]\right)\\
\lesssim\ &\sup_n\left(\frac{1}{N}\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|\mathbb{X}\|_{\mathfrak{S}}+\|u\|_{\mathcal{L}}+\|l^\infty(u)\|_{\mathcal{H}}]\mathbb{E}_{\mu_{n,L}^{\varepsilon,\delta}}[\|\mathbb{X}\|_{\mathfrak{S}}^p+\|u\|_{\mathcal{L}}^8+\|l^\infty(u)\|_{\mathcal{H}}^4]\right)\\
\to\ &0\quad\text{as } N\to\infty
\end{aligned}$$

As we can find ε,δ such that $|\breve{F}_\infty(\mu_L^{\varepsilon,\delta})-\breve{F}_\infty(\mu_L)|<1/L$ we conclude. □

2.7. Large Deviations

In this section we want to discuss a Laplace principle for the Φ_3^4 measure in the "small noise limit". We introduce the family $\nu_T^\hbar$ of measures given by

$$\int g(\phi)\nu_T^\hbar(\mathrm{d}\phi)=\frac{\mathbb{E}\left[g(\hbar^{1/2}W_T)\,e^{-\frac{1}{\hbar}V_T^\hbar(\hbar^{1/2}W_T)}\right]}{\mathscr{Z}_T^\hbar},\tag{2.52}$$

where simililarly to above

$$V_T^\hbar(\varphi):=\lambda\int_\Lambda(|\varphi(\xi)|^4-a_T^\hbar|\varphi(\xi)|^2+b_T^\hbar)\mathrm{d}\xi\qquad\mathscr{Z}_T^\hbar:=\int e^{-V_T^\hbar(\phi_T)}\vartheta(\mathrm{d}\phi)$$

for any bounded measurable $g:\mathscr{S}'(\Lambda)\to\mathbb{R}$. In this section we will take all integrals to be non-normalized for simplicity since we wish to derive a large deviations principle for fixed volume.

This corresponds (modulo renormalization) to the measure heuristically defined by

$$e^{-\frac{1}{\hbar}\int\lambda\varphi(\xi)^4+\varphi(\xi)^2+|\nabla\varphi(\xi)|^2\mathrm{d}\xi}\mathrm{d}\varphi$$

Our goal is now to show that $\nu^\hbar$ given as the weak limit of $\nu_T^\hbar$ satisfies a Laplace principle according to the following definition.

Definition 2.39. *A sequence of measures $\rho_\hbar$ on $\mathscr{S}'(\Lambda)$ satisfies the Laplace principle with rate function I if for any continuous bounded function $f:\mathscr{S}'(\Lambda)\to\mathbb{R}$*

$$\lim_{\hbar\to0}-\hbar\log\int e^{-\frac{1}{\hbar}f(\psi)}\mathrm{d}\rho_\hbar(\psi)=\inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi)+I(\psi)\}.$$

Our goal for the rest of this section will be to prove the following theorem:

Theorem 2.40. *The sequence of measures $\nu_T^\hbar$ converges to a unique limit $\nu^\hbar$ as $T\to\infty$. Furthermore $\nu^\hbar$ satisfies a Laplace principle with rate function*

$$I(\psi)=\lambda\int\psi^4+\int\psi^2+\int|\nabla\psi|^2\tag{2.53}$$

as $\hbar\to0$.

We will have to analyze the quantity

$$\mathcal{W}_T^\hbar(f) = -\hbar \log \mathbb{E}\left[e^{-\frac{1}{\hbar}(f(\hbar^{1/2}W_T)+V_T^\hbar(\hbar^{1/2}W_T))}\right]$$

First we have to choose $V_T^\hbar$ appropriatly. Set $W_T^\hbar = \hbar^{1/2}W_T$. Again we choose $a_T^\hbar, b_T^\hbar$ so that

$$V_T(W_T^\hbar) = \int [\![(W_T^\hbar)^4]\!] - \gamma_T^\hbar \int [\![(W_T^\hbar)^2]\!] + \delta_T^\hbar,$$

where now the Wick product is taken with respect to law of $W_T^\hbar$ and not W_T, and $\gamma^\hbar$, $\delta^\hbar$ will be fixed below. Now most of the analysis from sections 2.4, 2.5, 2.6 carries over into this situation for fixed $\hbar$. By the BD formula we have

$$\mathcal{W}_T^\hbar(f) = \inf_{u \in \mathbb{H}_a} \tilde{F}_T^\hbar(u)$$

with

$$\tilde{F}_T^\hbar(u) = \mathbb{E}\left[f(\hbar^{1/2}W_T + \hbar^{1/2}I_T(u)) + V_T^\hbar(W_T^\hbar + \hbar^{1/2}I_T(u)) + \frac{\hbar}{2}\int_0^T \|u_s\|_{L^2}^2\right].$$

And by the change $\hbar^{1/2}u \to u$ of variables we get

$$\mathcal{W}_T^\hbar(f) = \inf_{u \in \mathbb{H}_a} F_T^\hbar(u),$$

with

$$F_T^\hbar(u) = \mathbb{E}\left[f(\hbar^{1/2}W_T + I_T(u)) + V_T^\hbar(W_T^\hbar + I_T(u)) + \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2\right].$$

Lemma 2.41. *Define $l = l^{T,\hbar}(u) \in \mathbb{H}_a$, $Z = Z(u) \in C([0,\infty], H^{1/2-\kappa})$, $K = K^\hbar(u) \in C([0,\infty], H^{1-\kappa})$ such that*

$$\begin{array}{ll} Z_t(u) := I_t(u), & \\ l_t^{T,\hbar}(u) := u_t + \lambda\hbar^{3/2}\mathbb{1}_{t\leqslant T}\mathbb{W}_t^{\langle 3\rangle} + \lambda\hbar\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^2 \succ Z_t^\flat(u)), & t \geqslant 0. \\ K_t^\hbar(u) := I_t(w^\hbar(u)), \text{ with } w_t^\hbar(u) := -\lambda\hbar\mathbb{1}_{t\leqslant T}J_t(\mathbb{W}_t^2 \succ Z_t^\flat(u)) + l_t^{T,\hbar}(u), & \end{array} \tag{2.54}$$

Then the functional $F_T(u)$ takes the form

$$F_T^\hbar(u) \;=\; \mathbb{E}\left[f(\hbar W_T + Z_T) + \Phi_T^\hbar(\mathbb{W}, Z(u), K(u)) + \lambda \fint (Z_T(u))^4 + \frac{1}{2}\|l^{T,\hbar}(u)\|_{\mathcal{H}}^2\right],$$

where

$$\Phi_T^\hbar(\mathbb{W}, Z, K^\hbar) := \sum_{i=1}^6 \Upsilon_T^{(i),\hbar},$$

$$\begin{aligned}
\Upsilon_T^{(1),\hbar} &:= -\frac{\lambda}{2}\hbar\mathfrak{K}_2(\mathbb{W}_T^2, K_T^\hbar, K_T^\hbar) + \frac{\lambda}{2}\hbar\fint(\mathbb{W}_T^2 \prec K_T^\hbar)K_T^\hbar - \lambda^2\hbar^{5/2}\fint(\mathbb{W}_T^2 \prec \mathbb{W}_T^{[3]})K_T^\hbar,\\
\Upsilon_T^{(2),\hbar} &:= \lambda\hbar\int(\mathbb{W}_T^2 \succ (Z_T - Z_T^\flat))K_T^\hbar,\\
\Upsilon_T^{(3),\hbar} &:= \lambda\hbar\int_0^T\int(\mathbb{W}_t^2 \succ \dot{Z}_t^\flat)K_t^\hbar \mathrm{d}t,\\
\Upsilon_T^{(4),\hbar} &:= 4\lambda\hbar^{1/2}\int W_T K_T^3 - 12\lambda^2\hbar^2\int W_T\mathbb{W}_T^{[3]}(K_T^\hbar)^2 + 12\lambda^3\hbar^{7/2}\int W_T(\mathbb{W}_T^{[3]})^2K_T^\hbar,\\
\Upsilon_T^{(5),\hbar} &:= -2\lambda^2\hbar^2\int\gamma_T Z_T^\flat(Z_T - Z_T^\flat) - \lambda^2\hbar^2\int\gamma_T(Z_T - Z_T^\flat)^2 - 2\lambda^2\hbar^2\int_0^T\int\gamma_t Z_t^\flat\dot{Z}_t^\flat \mathrm{d}t,\\
\Upsilon_T^{(6),\hbar} &:= -\lambda^2\hbar^{5/2}\int\mathbb{W}_T^{2\diamond[3]}K_T^\hbar - \frac{\lambda^2}{2}\hbar^2\int_0^T\int\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}(Z_t^\flat)^2\mathrm{d}t - \frac{\lambda^2}{2}\hbar^2\int_0^T\mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^\flat, Z_t^\flat)\mathrm{d}t.
\end{aligned}$$

Moreover we have chosen $\gamma_T^\hbar = \hbar^2\gamma_T$ and the renormalization constant $\delta_T^\hbar$ to be

$$\begin{aligned}
\delta_T^\hbar \;:=\; & -\frac{\lambda^2}{2}\hbar^3\mathbb{E}\int_0^T\int(\mathbb{W}_t^{\langle 3\rangle})^2\mathrm{d}t + \frac{\lambda^3}{2}\hbar^4\mathbb{E}\int\mathbb{W}_T^2(\mathbb{W}_T^{[3]})^2\\
& + 2\lambda^3\hbar^2\gamma_T^\hbar\mathbb{E}\int W_T\mathbb{W}_T^{[3]} - 4\lambda^4\hbar^5\mathbb{E}\int W_T(\mathbb{W}_T^{[3]})^3.
\end{aligned} \tag{2.55}$$

The proof of Lemma 2.41 is analogous to the proof of Lemma 2.15.

Now for fixed $\hbar$ reasoning analogous to Theorem 2.23 we can conclude that

$$\begin{aligned}
&\lim_{T\to\infty} \mathcal{W}_T^{\hbar}(f)\\
=\ & \mathcal{W}^{\hbar}(f)\\
=\ & \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[f(\hbar^{1/2}W_\infty + I_\infty(u)) + \Phi_\infty^{\hbar}(\mathbb{W}^{\varepsilon}, Z(u), K^{\hbar}(u)) + \lambda\|Z_\infty\|_{L^4(\Lambda)}^4 + \frac{1}{2}\|l^{\hbar}(u)\|_{L^2(\mathbb{R}_+\times\Lambda)}^2\right]
\end{aligned}$$

where

$$l_t^{\hbar}(u) = u_t + \lambda\hbar^{3/2}\mathbb{W}_t^{\langle 3\rangle} + \lambda\hbar J_t(\mathbb{W}_t^2 \succ Z_t^{\flat}(u)).$$

and $\Phi_\infty^{\hbar}(\mathbb{W}, Z, K^{\hbar}) := \sum_{i=1}^6 \Upsilon_\infty^{(i),\hbar}$

$$\begin{aligned}
\Upsilon_\infty^{(1),\hbar} &:= -\frac{\lambda}{2}\hbar\mathfrak{K}_2(\mathbb{W}_\infty^2, K_\infty^{\hbar}, K_\infty^{\hbar}) + \frac{\lambda}{2}\hbar\int(\mathbb{W}_T^2 \prec K_\infty^{\hbar})K_\infty^{\hbar} - \lambda^2\hbar^{5/2}\int(\mathbb{W}_\infty^2 \prec \mathbb{W}_\infty^{[3]})K_\infty^{\hbar},\\
\Upsilon_\infty^{(2),\hbar} &:= 0\\
\Upsilon_\infty^{(3),\hbar} &:= \lambda\hbar\int_0^\infty\int(\mathbb{W}_t^2 \succ \dot{Z}_t^{\flat})K_t^{\hbar}\mathrm{d}t,\\
\Upsilon_\infty^{(4),\hbar} &:= 4\lambda\hbar^{1/2}\int W_\infty(K_\infty^{\hbar})^3 - 12\lambda^2\hbar^2\int W_\infty\mathbb{W}_\infty^{[3]}(K_\infty^{\hbar})^2 + 12\lambda^3\hbar^{7/2}\int W_\infty(\mathbb{W}_\infty^{[3]})^2K_\infty^{\hbar},\\
\Upsilon_\infty^{(5),\hbar} &:= 2\lambda^2\hbar^2\int_0^\infty\int\gamma_t Z_t^{\flat}\dot{Z}_t^{\flat}\mathrm{d}t,\\
\Upsilon_\infty^{(6),\hbar} &:= -\lambda^2\hbar^{5/2}\int\mathbb{W}_\infty^{2\diamond[3]}K_\infty^{\hbar} - \frac{\lambda^2}{2}\hbar^2\int_0^\infty\int\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}(Z_t^{\flat})^2\mathrm{d}t\\
&\quad -\frac{\lambda^2}{2}\hbar^2\int_0^\infty\mathfrak{K}_{3,t}(\mathbb{W}_t^2, \mathbb{W}_t^2, Z_t^{\flat}, Z_t^{\flat})\mathrm{d}t.
\end{aligned}$$

Now it remains to prove

$$\lim_{\hbar\to 0}\mathcal{W}^{\hbar}(f) = \inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi) + I(\psi)\}$$

with I defined by (2.53). For this in analogy with Section 2.6 we introduce

$$\breve{F}^{\hbar}(\mu) := \mathbb{E}_\mu\left[f(\hbar^{1/2}W_\infty + I_\infty(u)) + \Phi_\infty^{\hbar}(\mathbb{X}, Z(u), K^{\hbar}(u)) + \lambda\|Z_\infty\|_{L^4}^4 + \frac{1}{2}\|l^{\hbar}(u)\|_{\mathcal{H}}^2\right]$$

where the functional is again defined on the space $\mathcal{X}$ and

$$\mathcal{W}^{\hbar}(f) = \inf_{\mu\in\mathcal{X}}\breve{F}^{\hbar}(\mu)$$

and and in the same way as for Lemma 2.38 we can show that

$$\mathcal{W}^{\hbar}(f) = \inf_{\mu\in\bar{\mathcal{X}}}\breve{F}^{\hbar}(\mu)$$

Next we claim that taking

$$\breve{F}^0(\mu) = \mathbb{E}_\mu\left[f(Z_\infty(u)) + \lambda\|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|u\|_{\mathcal{H}}^2\right]$$

we have the following statements

LEMMA 2.42. *The family $\breve{F}^{\hbar}(\mu)$ is equicoercive on $\bar{\mathcal{X}}$*

Proof. In analogy with Section 2.4 it is not hard to see that

$$\breve{F}^{\hbar}(\mu) \geqslant -C + \frac{1}{2}\mathbb{E}\left[\lambda\|Z_\infty\|_{L^4}^4 + \frac{1}{2}\|l^{\hbar}(u)\|_{\mathcal{H}}^2\right]$$

and also that $\mathbb{E}_\mu[\|u\|_{\mathcal{L}}^2] \lesssim C + 2\lambda\mathbb{E}_\mu[\|Z_\infty(u)\|_{L^4}^4] + \mathbb{E}_\mu[\|l^{\hbar}(u)\|_{\mathcal{H}}^2]$, Define the set

$$\mathcal{K} = \{\mu: \mathbb{E}_\mu[\|Z_\infty(u)\|_{L^4}^4] + \mathbb{E}_\mu[\|l^{\hbar}(u)\|_{\mathcal{H}}^2] \leqslant K\}$$

note that $\mathcal{K}$ is compact from Lemma 2.28 . Now we can prove in the same way as in Section 2.5 that

$$\sup_{\hbar} \inf_{\mu \in \bar{\mathcal{X}}} \breve{F}^{\hbar}(\mu) < \infty.$$

So choosing K large enough we have that

$$\inf_{\mu \notin \mathcal{K}} \breve{F}^{\hbar}(\mu) \geqslant cK - C > \sup_{\hbar} \inf_{u \in \bar{\mathcal{X}}} \breve{F}^{\hbar}(\mu)$$

which implies

$$\inf_{\mu \in \bar{\mathcal{X}}} \breve{F}^{\hbar}(\mu) = \inf_{\mu \in \mathcal{K}} \breve{F}^{\hbar}(\mu)$$

□

Lemma 2.43. *$\breve{F}^{\hbar}(\mu)$ Γ-converges to $\breve{F}^0(\mu)$ on $\bar{\mathcal{X}}$*

Proof.

Step 1. First we prove the liminf inequality. Consider a sequence $\mu_{\hbar} \rightarrow \mu$ in $\bar{\mathcal{X}}$, our aim is to prove that

$$\liminf_{\hbar \rightarrow 0} \breve{F}^{\hbar}(\mu^{\hbar}) \geqslant \breve{F}^0(\mu).$$

As before it is enough to prove the statement for a subsequence of $\mu^{\hbar}$. W.l.o.g we can assume that

$$\sup_{\hbar} (\mathbb{E}_{\mu^{\hbar}}[\|Z_{\infty}(u)\|_{L^4}^4] + \mathbb{E}_{\mu^{\hbar}}[\|l^{\hbar}(u)\|_{\mathcal{H}}^2]) < \infty. \tag{2.56}$$

Now from the estimates in Section 2.8 and the definition of $\Phi_{\infty}^{\hbar}$ we observe that for any $\mu_{\hbar} \in \bar{\mathcal{X}}$

$$\mathbb{E}_{\mu^{\hbar}}[|\Phi_{\infty}^{\hbar}|] \leqslant \hbar^{1/2} (\mathbb{E}_{\mu^{\hbar}}[\|Z_{\infty}(u)\|_{L^4}^4] + \mathbb{E}_{\mu^{\hbar}}[\|l^{\hbar}(u)\|_{\mathcal{H}}^2]),$$

so for for any sequence satisfying (2.56) $\mathbb{E}_{\mu^{\hbar}}[|\Phi_{\infty}^{\hbar}|] \rightarrow 0$. By the Portmanteau Lemma we have

$$\liminf_{\hbar \rightarrow 0} \mathbb{E}_{\mu^{\hbar}}[\|Z_{\infty}(u)\|_{L^4}^4] \geqslant \mathbb{E}_{\mu}[\|Z_{\infty}(u)\|_{L^4}^4].$$

We claim that also

$$\liminf_{\hbar \rightarrow 0} \mathbb{E}_{\mu^{\hbar}}[\|l^{\hbar}(u)\|_{\mathcal{H}}^2] \geqslant \mathbb{E}_{\mu}[\|u\|_{\mathcal{H}}^2].$$

For this we find a subsequence of $\mu_{\hbar}$ (not relabeled) and a sequence of random variables $(\mathbb{X}^{\hbar}, u^{\hbar})$ on a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ such that $\mathrm{Law}(\mathbb{X}^{\hbar}, u^{\hbar}) = \mu_{\hbar}$ and $\mathbb{X}^{\hbar} \rightarrow \mathbb{X}$ in $\mathfrak{S}$ and $u^{\hbar} \rightarrow u^0$ in $\mathcal{L}_w$, where $\mathrm{Law}(\mathbb{X}, u^0) = \mu$. Then from eq. (2.56) we can pick a further subsequence such that $l^{\hbar}(u^{\hbar}) \rightarrow l^{\star}$ in $\mathcal{H}_w$. The definition of $l^{\hbar}$ implies

$$l_t^{\hbar}(u) \;=\; u_t + \lambda \hbar^{3/2} \mathbb{X}_t^{\langle 3 \rangle, \hbar} + \lambda \hbar J_t(\mathbb{X}_t^{2, \hbar} \succ Z_t^{\flat}(u))$$

and testing and taking limits this implies

$$l^{\star} = u,$$

which implies our claim. Now from weak convergence it follows that provided that f is bounded and continuous on $H^{-\delta}(\Lambda)$.

$$\lim_{\hbar \rightarrow 0} \mathbb{E}_{\mu_{\hbar}}[f(\hbar^{1/2} X_{\infty} + Z_{\infty}(u))] = \mathbb{E}_{\mu}[f(Z_{\infty}(u))],$$

which completes Step 1.

Step 2. Now we construct the recovery sequence , more precisely we prove that for every μ there exists $\mu^{\hbar} \rightarrow \mu$ such that

$$\limsup_{\hbar \rightarrow 0} \breve{F}^{\hbar}(\mu^{\hbar}) \leqslant \breve{F}^0(\mu).$$

By Lemma 2.34 for any μ such that $\breve{F}^0(\mu) < \infty$ (otherwise there is nothing to prove) we have a sequence μ^n such that $\|u\|_{\mathcal{L}} \leqslant n$ μ^n-almost surely and $\lim_{n \rightarrow \infty} \breve{F}^0(\mu^n) = \breve{F}^0(\mu)$. By a diagonal argument it is enough to find sequences $\mu^{n, \hbar}$ such that

$$\limsup_{\hbar \rightarrow 0} \breve{F}^{\hbar}(\mu^{n, \hbar}) \leqslant \breve{F}^0(\mu^n).$$

For this consider $u^\hbar$ to be the solution to

$$u_t = u_t^\hbar + \lambda\hbar^{3/2}\mathbb{X}_t^{\langle 3\rangle} + \lambda\hbar J_t(\mathbb{X}_t^2 \succ Z_t^\flat(u^\hbar)).$$

This can be proven to exist by a standart fixpoint and Gronwall argument. Applying J and integrating we obtain

$$Z_t(u^\hbar) = Z_t(u) - \lambda\hbar^{3/2}\mathbb{X}_t^{[3]} - \lambda\hbar\int_0^t J_s^2(\mathbb{X}_s^2 \succ Z_s^\flat(u^\hbar))\mathrm{d}s \tag{2.57}$$

so by Gronwall's lemma

$$\begin{aligned}\|Z_t(u^\hbar)\|_{L^4(\Lambda)} &\leqslant \sup_{s\leqslant t}\left(\|Z_t(u)\|_{L^4} + \lambda\hbar^{3/2}\left\|\mathbb{X}_t^{[3]}\right\|_{L^4(\Lambda)}\right)e^{\lambda\hbar\int_0^t\langle s\rangle^{-(2-\delta)}\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\delta}(\Lambda)}\mathrm{d}s}\\ &\leqslant \sup_{s\leqslant t}\left(\|u\|_{\mathcal{L}}^2 + \lambda\hbar^{3/2}\left\|\mathbb{X}_t^{[3]}\right\|_{L^4(\Lambda)}^2\right) + e^{2\lambda\hbar\int_0^t\langle s\rangle^{-(2-\delta)}\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\delta}(\Lambda)}\mathrm{d}s} \end{aligned} \tag{2.58}$$

This together with the definition of $u^\hbar$ implies that

$$\begin{aligned}&\|Z_t(u^\hbar) - Z_t(u)\|_{L^4}^4\\ \leqslant\ &\hbar\left(\lambda\hbar^{3/2}\left\|\mathbb{X}_t^{[3]}\right\|_{L^4}^4 - \sup_{s\leqslant t}\left(\|u\|_{\mathcal{L}}^8 + \lambda\hbar^{3/2}\left\|\mathbb{X}_t^{[3]}\right\|_{L^4(\Lambda)}^8\right) + e^{8\lambda\hbar\int_0^t\langle s\rangle^{-(2-\delta)}\|\mathbb{X}_s^2\|_{\mathscr{C}^{-1-\delta}(\Lambda)}\mathrm{d}s}\right)\\ &+\lambda\hbar^{3/2}\left\|\mathbb{X}_t^{[3]}\right\|_{L^4}\end{aligned}$$

So by dominated convergence

$$\mathbb{E}_{\mu^n}\|Z_t(u^\hbar) - Z_t(u)\|_{L^4}^4 \to 0$$

since $\|u\|_{\mathcal{L}} \leqslant n$ μ^n almost surely. Now set $\mu^{n,\hbar} = \mathrm{Law}_{\mu^n}(\mathbb{X},\ u^\hbar)$. Clearly $l^\hbar(u^\hbar) = u$, so $\mathbb{E}_{\mu^{n,\hbar}}[\|l^\hbar(u)\|_{\mathcal{H}}^2] = \mathbb{E}_{\mu^n}[\|u\|_{\mathcal{H}}^2]$ and we have already established that $\mathbb{E}_{\mu^{n,\hbar}}[\|Z_\infty(u)\|_{L^4(\Lambda)}^4] \to \mathbb{E}_{\mu^n}[\|Z_\infty(u)\|_{L^4(\Lambda)}^4]$.

Now together with the fact that

$$\mathbb{E}_{\mu^{n,\hbar}}[|\Phi_\infty^\hbar(\mathbb{W}^\hbar, Z(u), K^\hbar(u))|] \leqslant \hbar^{1/2}(\mathbb{E}_{\mu^{n,\hbar}}[\|Z_\infty(u)\|_{L^4(\Lambda)}^4] + \mathbb{E}_{\mu^{n,\hbar}}[\|l^\hbar(u)\|_{\mathcal{H}}^2]) \to 0$$

we can conclude. □

LEMMA 2.44.

$$\inf_{\mu\in\breve{\mathcal{X}}}\breve{F}^0(\mu) = \inf_{u\in\mathbb{H}_a}F(u) = \inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi) + I(\psi)\}$$

where

$$F(u) = \mathbb{E}\left[f(Z_\infty(u)) + \lambda\|Z_\infty(u)\|_{L^4}^4 + \frac{1}{2}\|u\|_{\mathcal{H}}^2\right]$$

Proof. The first equality can easily be proven in the same fashion as Lemma 2.38. We only prove the second equality.

Step 1. First we prove

$$\inf_{u\in\mathbb{H}_a}F(u) \leqslant \inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi) + I(\psi)\}.$$

Restricting the infimum to processes of the form

$$u_s = J_s\langle\nabla\rangle^2\psi$$

with $\psi\in H^2(\Lambda)$, we see that

$$Z_\infty(u) = \int_0^\infty J_s u_s\mathrm{d}s = \int_0^\infty J_s^2\langle\nabla\rangle^2\psi\mathrm{d}s = \psi$$

we also compute

$$\|u\|_{\mathcal{H}}^2 = \int_0^\infty\int_\Lambda u_s^2\mathrm{d}s = \int_0^\infty\langle J_s^2\langle\nabla\rangle^2\psi, \langle\nabla\rangle^2\psi\rangle_{L^2(\Lambda)}\mathrm{d}s = \langle\psi, \langle\nabla\rangle^2\psi\rangle_{L^2(\Lambda)} = \|\psi\|_{H^1}^2$$

so

$$\inf_{u\in\mathbb{H}_a}F(u) \leqslant \inf_{u_s = J_s\langle\nabla\rangle^2\psi}F(u) = \inf_{\psi\in H^2}\{f(\psi) + I(\psi)\} = \inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi) + I(\psi)\}$$

where the last equality follows from approximation of $\psi \in H^1(\Lambda)$ with $H^2(\Lambda)$ functions, and $f(\psi)+I(\psi)$ is understood to be $=\infty$ if $\psi \notin H^1(\Lambda)$.

Step 2.We now prove the converse inequality

$$\inf_{u\in\mathbb{H}_a} F(u) \geqslant \inf_{\psi\in\mathscr{S}'(\Lambda)} \{f(\psi)+I(\psi)\}.$$

Recall that from the proof of Lemma 2.6 $\|u\|_{\mathcal{H}} \geqslant \|Z_\infty(u)\|_{H^1}$ so

$$\begin{aligned}\inf_{u\in\mathbb{H}_a} F(u) &\geqslant \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[f(Z_\infty(u))+\lambda\|Z_\infty(u)\|_{L^4}^4+\frac{1}{2}\|Z_\infty(u)\|_{H^1}^2\right]\\ &\geqslant \inf_{\psi\in\mathscr{S}'(\Lambda)}\{f(\psi)+I(\psi)\}.\end{aligned}$$

□

2.8. Analytic estimates

In this section we collect a series of analytic estimates which together allow to establish the point wise bounds (2.38) and (2.39) and the continuity required for Lemma 2.22. First of all note that

$$\begin{aligned}\|K_t\|_{H^{1-\kappa}}^2 &\lesssim \lambda^2\int_0^t \frac{1}{\langle t\rangle^{1+\delta}}\|\mathbb{W}_s^2\|_{B_{4,\infty}^s}^2 \mathrm{d}s\, \|Z_T\|_{L^4}^2+\int_0^t\|l_s\|_{L^2}^2\mathrm{d}s\\ &\lesssim \lambda^3\left(\int_0^t\frac{1}{\langle t\rangle^{1+\delta}}\|\mathbb{W}_s^2\|_{B_{4,\infty}^s}^2\mathrm{d}s\right)^2+\lambda\|Z_T\|_{L^4}^4+\int_0^t\|l_s\|_{L^2}^2\mathrm{d}s,\end{aligned}\tag{2.59}$$

which implies that quadratic functions of the norm $\|K_t\|_{H^{1-\kappa}}$ with small coefficients can always be controlled, uniformly in $[0,\infty]$, by the coercive term

$$\lambda\,⨍ Z_T^4+\frac{1}{2}\int_0^\infty\|l_s\|_{L^2}^2\mathrm{d}s.$$

Lemma 2.45. *For any small $\varepsilon>0$ there exists $\delta>0$ such that*

$$|\Upsilon_T^{(1)}|\leqslant C(\varepsilon,\delta)E(\lambda)Q_T+\varepsilon\|K_T\|_{H^{1-\delta}}^2+\varepsilon\lambda\|Z_T\|_{L^4}^4.$$

Proof. By Proposition A.14,

$$\begin{aligned}\lambda|\mathfrak{K}_2(\mathbb{W}_T^2,K_T,K_T)| &\lesssim \lambda\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}\|K_T\|_{B_{7/3,2}^{9/16}}^2\lesssim\lambda\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}\|K_T\|_{B_{7/3,7/3}^{5/8}}^2\\ &\lesssim \lambda\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}\|K_T\|_{H^{7/8}}^{10/7}\|K_T\|_{B_{4,4}^0}^{4/7}\\ &\lesssim \lambda^6\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}^7+\|K_T\|_{H^{7/8}}^2+\lambda\|K_T\|_{L^4}^4.\end{aligned}\tag{2.60}$$

By Proposition A.10,

$$\left|\lambda⨍(\mathbb{W}_T^2\prec K_T)K_T\right| \lesssim \lambda\|\mathbb{W}_T^2\|_{B_{7,\infty}^{-9/8}}\|K_T\|_{B_{7/3,2}^{9/16}}^2$$

which can be estimated in the same way, and finally

$$\begin{aligned}\left|\lambda^2⨍(\mathbb{W}_T^2\prec\mathbb{W}_T^{[3]})K_T\right| &\lesssim \lambda^2\|\mathbb{W}_T^2\|_{B_{4,4}^{-1-\delta/2}}\|\mathbb{W}_T^{[3]}\|_{B_{4,4}^{1/2-\delta/2}}\|K_T\|_{H^{1/2+\delta}}\\ &\leqslant C(\varepsilon)\,\lambda^4\left(\|\mathbb{W}_T^2\|_{B_{4,4}^{-1-\delta/2}}\|\mathbb{W}_T^{[3]}\|_{B_{4,4}^{1/2-\delta/2}}\right)^2+\varepsilon\|K_T\|_{H^{1/2+\delta}}^2.\end{aligned}$$

□

Lemma 2.46. *For any small $\varepsilon>0$ there exists $\delta>0$ such that*

$$\left|\Upsilon_T^{(2)}\right| \leqslant T^{-\delta}(C(\varepsilon,\delta)E(\lambda)Q_T+\varepsilon\|K\|_{H^{1-\delta}}+\varepsilon\lambda\|Z_T\|_{L^4})$$

Proof. Using the spectral support properties of the various terms we observe that

$$\|\mathbb{W}_T^2\|_{B_{p,q}^{-1+\delta}}\lesssim\|\mathbb{W}_T^2\|_{B_{p,q}^{-1+\delta}}T^{2\delta},$$

and

$$T^{2\delta}\|Z_T-Z_T^\flat\|_{L^2}\lesssim\|Z_T-Z_T^\flat\|_{H^{2\delta}}\lesssim\|Z_T-Z_T^\flat\|_{H^{1/2-\delta}}^{\frac{2\delta}{1/2-\delta}}\|Z_T-Z_T^\flat\|_{L^2}^{\frac{1/2-3\delta}{1/2-\delta}}$$
$$\lesssim\|Z_T\|_{H^{1/2-\delta}}^{\frac{2\delta}{1/2-\delta}}\|Z_T\|_{L^2}^{\frac{1/2-3\delta}{1/2-\delta}},$$

where we used also interpolation and the L^2 bound $\|Z_T^\flat\|_{L^2}\lesssim\|Z_T\|_{L^2}$. We recall also that

$$Z_T=K_T+\lambda\mathbb{W}_T^{[3]}. \tag{2.61}$$

Therefore we estimate as follows

$$\lambda⨍(\mathbb{W}_T^2\succ(Z_T-Z_T^\flat))K_T \;=\; \lambda⨍(\mathbb{W}_T^2\succ(K_T-K_T^\flat))K_T+\lambda^2⨍\big(\mathbb{W}_T^2\succ\big(\mathbb{W}_T^{[3]}-\mathbb{W}_T^{[3],\flat}\big)\big)K_T$$

For the second term we can estimate

$$\begin{aligned}\lambda^2⨍\big(\mathbb{W}_T^2\succ\big(\mathbb{W}_T^{[3]}-\mathbb{W}_T^{[3],\flat}\big)\big)K_T &\lesssim \lambda^2\|W_T^2\|_{B^{-1+\delta}_{4,\infty}}\big\|\mathbb{W}_T^{[3]}-\mathbb{W}_T^{[3],\flat}\big\|_{B^0_{4,2}}\|K_T\|_{H^{1-\delta}}\\ &\lesssim \lambda^2T^{-\delta}\|W_T^2\|_{B^{-1-\delta}_{4,\infty}}\big\|\mathbb{W}_T^{[3]}\big\|_{B^{3\delta}_{4,2}}\|K_T\|_{H^{1-\delta}},\end{aligned}$$

while for the first term we get

$$\begin{aligned}\lambda⨍(\mathbb{W}_T^2\succ(K_T-K_T^\flat))K_T &\lesssim \lambda\|W_T^2\|_{B^{-1/2-\delta}_{7,\infty}}\|K_T\|_{B^0_{7/3,2}}\|K_T\|_{B^{1/2+\delta}_{7/3,2}}\\ &\lesssim \lambda\|W_T^2\|_{B^{-1-\delta}_{7,\infty}}T^{1/2}T^{-1/2-\delta}\|K_T\|^2_{B^{1/2+\delta}_{7/3,2}}\\ &\lesssim \lambda T^{-\delta}\|W_T^2\|_{B^{-1-\delta}_{7,\infty}}\|K_T\|^2_{B^{1/2+\delta}_{7/3,2}},\end{aligned}$$

which we can again estimate like in Lemma 2.45. □

Lemma 2.47. *For any small $\hbar>0$ there exists $\delta>0$ such that*

$$\big|\Upsilon_T^{(3)}\big| \;\leqslant\; C(\varepsilon,\delta)E(\lambda)Q_T+\varepsilon\sup_{0\leqslant t\leqslant T}\|K_t\|^2_{H^{1-\delta}}+\varepsilon\lambda\|Z_T\|^4_{L^4}.$$

Proof. First note that for $t\geqslant 11$ we have $\dot{\theta}_t(\mathrm{D})=(\langle\mathrm{D}\rangle/t^2)\dot{\tilde{\theta}}(\langle\mathrm{D}\rangle/t)$. In particular $\dot{Z}_t^\flat$ is spectrally supported in an annulus with inner radius $t/4$ and outer radius $t/3$. Then for any $\beta\in[0,1]$

$$\|\dot{Z}_t^\flat\|_{B^{s+\beta}_{p,q}}=\left\|\dot{\tilde{\theta}}\left(\frac{\langle\mathrm{D}\rangle}{t}\right)\frac{\langle\mathrm{D}\rangle}{t^2}Z_T\right\|_{B^{s+\beta}_{p,q}}\lesssim\left\|\dot{\tilde{\theta}}\left(\frac{\langle\mathrm{D}\rangle}{t}\right)\frac{\langle\mathrm{D}\rangle^{1+\beta}}{t^{2+\beta}}Z_T\right\|_{B^{s+\beta}_{p,q}}\lesssim\frac{\|Z_T\|_{B^s_{p,q}}}{\langle t\rangle^{1+\beta}}.$$

The same estimate holds trivially for $t\leqslant 11$.

By Proposition A.10, for any $\varepsilon>0$ there exists $\delta>0$ such that

$$\begin{aligned}&\left|\lambda\int_0^T⨍(\mathbb{W}_t^2\succ\dot{Z}_t^\flat)K_t\,\mathrm{d}t\right|\lesssim\lambda\int_0^T\|\mathbb{W}_t^2\|_{B^{-1+\delta}_{6,\infty}}\|\dot{Z}_t^\flat\|_{B^0_{3,2}}\|K_t\|_{H^{1-\delta}}\mathrm{d}t\\ &\lesssim\lambda\int_0^T\|\mathbb{W}_t^2\|_{B^{-1+\delta}_{6,\infty}}\|Z_T\|_{B^{3\delta}_{3,2}}\|K_t\|_{H^{1-\delta}}\frac{\mathrm{d}t}{\langle t\rangle^{1+3\delta}}\\ &\lesssim\lambda\|Z_T\|_{B^{4\delta}_{3,3}}\sup_{0\leqslant t\leqslant T}\|K_t\|_{H^{1-\delta}}\int_0^T\|\mathbb{W}_t^2\|_{B^{-1+\delta}_{6,\infty}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}\\ &\lesssim\lambda\|Z_T\|^{1/2}_{H^{1/2-\delta}}\|Z_T\|^{1/2}_{B_{4,4}}\sup_{0\leqslant t\leqslant T}\|K_t\|_{H^{1-\delta}}\int_0^T\|\mathbb{W}_t^2\|_{B^{-1+\delta}_{6,\infty}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}\\ &\lesssim\lambda\|Z_T\|^{1/2}_{L^4}\sup_{0\leqslant t\leqslant T}\|K_t\|^{3/2}_{H^{1-\delta}}\int_0^T\|\mathbb{W}_t^2\|_{B^{-1+\delta}_{6,\infty}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}\\ &+\lambda^{3/2}\|Z_T\|^{1/2}_{L^4}\sup_{0\leqslant t\leqslant T}\|K_t\|_{H^{1-\delta}}\|\mathbb{W}_T^{[3]}\|^{1/2}_{H^{4\delta}}\int_0^T\|\mathbb{W}_t^2\|_{B^{-1+\delta}_{6,\infty}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}\end{aligned}$$

and again

$$\lambda\|Z_T\|_{L^4}^{1/2}\sup_{0\leqslant t\leqslant T}\|K_t\|_{H^{1-\delta}}^{3/2}\int_0^T\|\mathbb{W}_t^2\|_{B_{7,\infty}^{-1+\delta}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}$$
$$\leqslant C\lambda^7\int_0^T\|\mathbb{W}_t^2\|_{B_{7,\infty}^{-1+\delta}}^8\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}+\varepsilon\sup_{0\leqslant t\leqslant T}\|K_t\|_{H^{1-\delta}}^2+\varepsilon\lambda\|Z_T\|_{L^4}^4.$$

While

$$\lambda^{3/2}\|Z_T\|_{L^4}^{1/2}\sup_{0\leqslant t\leqslant T}\|K_t\|_{H^{1-\delta}}\|\mathbb{W}_T^{[3]}\|_{H^{4\delta}}^{1/2}\int_0^T\|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1+\delta}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}$$
$$\leqslant C\lambda^{11/3}\int_0^T\|\mathbb{W}_t^2\|_{B_{7,\infty}^{-1+\delta}}^{8/3}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}\|\mathbb{W}_T^{[3]}\|_{H^{4\delta}}^{8/6}+\sup_{0\leqslant t\leqslant T}\|K_t\|_{H^{1-\delta}}^2+\lambda\|Z_T\|_{L^4}.$$

□

Lemma 2.48. *For any small $\varepsilon>0$ there exists $\delta>0$ such that*

$$|\Upsilon_T^{(4)}|\leqslant C(\varepsilon,\delta)E(\lambda)Q_T+\varepsilon\|K_T\|_{H^{1-\delta}}^2+\varepsilon\lambda\|Z_T\|_{L^4}^4.$$

Proof. Using Lemma 2.9 we establish that

$$\left|\lambda\fint W_TK_T^3\right|\leqslant E(\lambda)\|W_T\|_{W^{-1/2-\varepsilon,p}}^K+\delta(\|K_T\|_{H^{1-\varepsilon}}^2+\lambda\|K_T\|_{L^4}^4).$$

Next, we can write,

$$\lambda^3\left|\fint W_T(\mathbb{W}_T^{[3]})^2K_T\right|\lesssim\lambda^3\left|\fint W_T(\mathbb{W}_T^{[3]}\succ\mathbb{W}_T^{[3]})K_T\right|+\lambda^3\|W_T\|_{B_{6,\infty}^{-1/2-\delta}}\|\mathbb{W}_T^{[3]}\|_{B_{6,4}^{-1/2-\delta}}^2\|K_T\|_{H^{1-\varepsilon}}.$$

which can be easily estimated by Young's inequality. Decomposing

$$W_T(\mathbb{W}_T^{[3]}\succ\mathbb{W}_T^{[3]})=W_T\succ(\mathbb{W}_T^{[3]}\succ\mathbb{W}_T^{[3]})+W_T\prec(\mathbb{W}_T^{[3]}\succ\mathbb{W}_T^{[3]})+W_T\circ(\mathbb{W}_T^{[3]}\succ\mathbb{W}_T^{[3]}).$$

We can estimate the first two terms by

$$\lambda^3\left|\fint W_T\succ(\mathbb{W}_T^{[3]}\succ\mathbb{W}_T^{[3]})K_T\right|\lesssim\lambda^3\|W_T\|_{B_{6,\infty}^{-1/2-\delta}}\|\mathbb{W}_T^{[3]}\|_{B_{6,2}^0}^2\|K_T\|_{H^{1-\varepsilon}},$$

and

$$\lambda^3\left|\fint W_T\prec(\mathbb{W}_T^{[3]}\succ\mathbb{W}_T^{[3]})K_T\right|\lesssim\lambda^3\|W_T\|_{B_{6,2}^{-1/2-\delta}}\|\mathbb{W}_T^{[3]}\|_{B_{6,\infty}^0}^2\|K_T\|_{H^{1-\varepsilon}}.$$

Young's inequality gives then the appropriate result. For the final term we use Proposition A.13 to get

$$\lambda^3\left|\fint W_T\circ(\mathbb{W}_T^{[3]}\succ\mathbb{W}_T^{[3]})K_T\right|$$
$$\lesssim\lambda^3\left|\fint\mathbb{W}_T^{[3]}\mathbb{W}_T^{1\circ[3]}K_T\right|+\lambda^3\|W_T\|_{B_{4,\infty}^{-1/2-\delta}}\|\mathbb{W}_T^{[3]}\|_{B_{4,2}^{-1/2-\delta}}^2\|K_T\|_{H^{1-\delta}}$$
$$\lesssim\lambda^3\|\mathbb{W}_T^{[3]}\|_{B_{4,\infty}^{1/2-\delta}}\|\mathbb{W}_T^{1\circ[3]}\|_{B_{4,2}^{-\delta}}\|K_T\|_{H^{1-\delta}}+\lambda^3\|W_T\|_{B_{4,\infty}^{-1/2-\delta}}\|\mathbb{W}_T^{[3]}\|_{B_{4,2}^{-1/2-\delta}}^2\|K_T\|_{H^{1-\delta}}$$
$$\lesssim\lambda^6C(\delta,\varepsilon)\Big[\|\mathbb{W}_T^{[3]}\|_{B_{4,\infty}^{1/2-\delta}}\|\mathbb{W}_T^{1\circ[3]}\|_{B_{4,2}^{-\delta}}+\|W_T\|_{B_{4,\infty}^{-1/2-\delta}}\|\mathbb{W}_T^{[3]}\|_{B_{4,2}^{-1/2-\delta}}^2\Big]^2+\varepsilon\|K_T\|_{H^{1-\delta}}^2.$$

For the last term we estimate

$$\left|\lambda^2\fint(W_T\mathbb{W}_T^{[3]})K_T^2\right|\lesssim\lambda^2\|W_T\mathbb{W}_T^{[3]}\|_{B_{7,\infty}^{-1/2-\delta}}\|K_T\|_{B_{7/3,2}^{1/2+\delta}}^2,$$

which can be estimated like in Lemma 2.45 after we observe that

$$\|W_T\mathbb{W}_T^{[3]}\|_{B_{7,\infty}^{-1/2-\delta}}\leqslant\|W_T\succ\mathbb{W}_T^{[3]}\|_{B_{7,\infty}^{-1/2-\delta}}+\|W_T\circ\mathbb{W}_T^{[3]}\|_{B_{7,\infty}^{-1/2-\delta}}+\|W_T\prec\mathbb{W}_T^{[3]}\|_{B_{7,\infty}^{-1/2-\delta}}$$
$$\lesssim\|W_T\|_{B_{14,\infty}^{-1/2-\delta}}\|\mathbb{W}_T^{[3]}\|_{B_{14,\infty}^0}+\|\mathbb{W}_T^{1\circ[3]}\|_{B_{7,\infty}^{-\delta}}$$

and use Lemma 2.52 to bound $\mathbb{W}_T^{1\circ[3]}$. □

LEMMA 2.49. *For any small $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|\Upsilon_T^{(5)}| \leqslant C_\varepsilon E(\lambda) \left[\frac{|\gamma_T|}{\langle T \rangle^{1/4}} + \int_0^T \frac{|\gamma_t| \, \mathrm{d}t}{\langle t \rangle^{5/4}} \right]^2 + \varepsilon \|Z_T\|_{H^{1/2-\delta}}^2 + \varepsilon \lambda \|Z_T\|_{L^4}^4.$$

Proof. We can estimate

$$\left| \lambda^2 \gamma_T \fint Z_T^\flat (Z_T - Z_T^\flat) \right| \leqslant \lambda^2 |\gamma_T| \|Z_T^\flat\|_{L^2} \|Z_T - Z_T^\flat\|_{L^2} \lesssim \lambda^2 \frac{|\gamma_T|}{\langle T \rangle^{1/4}} \|Z_T^\flat\|_{L^2} \|Z_T - Z_T^\flat\|_{H^{1/4}}, \tag{2.62}$$

and

$$\left| \lambda^2 \gamma_T \fint (Z_T - Z_T^\flat)^2 \right| \leqslant \lambda^2 |\gamma_T| \|Z_T - Z_T^\flat\|_{L^2}^2 \lesssim \lambda^2 \frac{|\gamma_T|}{\langle T \rangle^{1/4}} \|Z_T^\flat - Z_T\|_{L^2} \|Z_T - Z_T^\flat\|_{H^{1/4}}. \tag{2.63}$$

For the last term we can apply the estimate

$$\left| \lambda^2 \int_0^T \fint \gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t \right| \leqslant \lambda^2 \int_0^T |\gamma_t| \|Z_t^\flat\|_{L^2} \|\dot{Z}_t^\flat\|_{L^2} \mathrm{d}t \lesssim \lambda^2 \|Z_T\|_{L^2} \|Z_T\|_{H^{1/4}} \int_0^T \frac{|\gamma_t| \, \mathrm{d}t}{\langle t \rangle^{5/4}}.$$

Collecting these bounds we get

$$|\Upsilon_T^{(5)}| \lesssim C_\varepsilon \lambda^7 \left[\frac{|\gamma_T|}{\langle T \rangle^{1/4}} + \int_0^T \frac{|\gamma_t| \, \mathrm{d}t}{\langle t \rangle^{5/4}} \right]^2 + \lambda \varepsilon \|Z_T\|_{L^4}^4 + \varepsilon \|Z_T\|_{H^{1/2-\delta}}^2. \qquad \square$$

Remark 2.50. Note that

$$\sup_T \left[\frac{|\gamma_T|}{\langle T \rangle^{1/4}} + \int_0^T \frac{|\gamma_t| \, \mathrm{d}t}{\langle t \rangle^{5/4}} \right] < \infty,$$

provided γ_T does not grow too fast in T which is indeed guaranteed by the choice of renormalization made in Lemma 2.52.

LEMMA 2.51. *For any small $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|\Upsilon_T^{(6)}| \leqslant C(\varepsilon, \delta) E(\lambda) Q_T + \varepsilon \|K_T\|_{H^{1-\delta}}^2 + \varepsilon \lambda \|Z_T\|_{L^4}^4.$$

Proof. We start by observing that

$$\lambda^2 \left| \fint (\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]} + 2\gamma_T W_T) K_T \right| \lesssim \lambda^2 \|\mathbb{W}_T^{2\diamond[3]}\|_{W^{-1/2-\varepsilon,2}} \|K_T\|_{W^{1/2+\varepsilon,2}}.$$

and using Lemma 2.52 and eq. (2.59) we have this term under control. Next split

$$\left| \frac{\lambda^2}{2} \mathbb{E} \int_0^T \fint [(J_t(\mathbb{W}_t^2 \succ Z_t^\flat))^2 + 2\dot{\gamma}_t (Z_t^\flat)^2] \mathrm{d}t \right|$$

$$\lesssim \frac{\lambda^2}{2} \left| \int_0^T \fint ((J_t \mathbb{W}_t^2 \succ Z_t^\flat))^2 - (J_t \mathbb{W}_t^2 \circ J_t \mathbb{W}_t^2)(Z_t^\flat)^2 \mathrm{d}t \right| + \lambda^2 \left| \int_0^T \fint \mathbb{W}_t^{\langle 2 \rangle \diamond \langle 2 \rangle} (Z_t^\flat)^2 \mathrm{d}t \right|.$$

Recall that $t^{1/2} J_t$ is a Fourier multiplier with symbol

$$\langle k \rangle^{-1} (-2\rho'(\langle k \rangle / t) \rho(\langle k \rangle / t) (\langle k \rangle / t))^{1/2} = \langle k \rangle^{-1} \eta(\langle k \rangle / t),$$

where η is a smooth function supported in an annulus of radius 1. From this we prove that $t^{1/2} J_t$ satisfies the assumptions of Proposition A.12 with $m = -1$. Therefore

$$\|J_t(\mathbb{W}_t^2 \succ Z_t^\flat) - (J_t \mathbb{W}_t^2) \succ Z_t^\flat\|_{H^{1/4-2\delta}} \lesssim \langle t \rangle^{-1/2} \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}} \|Z_t^\flat\|_{B_{3,3}^{-1/4-\delta}},$$

and by Proposition A.7,

$$\|J_t(\mathbb{W}_t^2 \succ Z_t^\flat)\|_{H^{-2\delta}} + \|J_t(\mathbb{W}_t^2 \succ Z_t^\flat)\|_{H^{-2\delta}} \lesssim \langle t \rangle^{-1/2-\delta} \|\mathbb{W}_t^2\|_{B_{6,\infty}^{-1-\delta}} \|Z_t^\flat\|_{B_{3,3}^0}.$$

Therefore

$$\left|\frac{\lambda^2}{2}\int_0^T \fint (J_t(\mathbb{W}_t^2 \succ Z_t^\flat))^2 \mathrm{d}t - \frac{\lambda^2}{2}\int_0^T \fint ((J_t\mathbb{W}_t^2) \succ Z_t^\flat)^2 \mathrm{d}t\right|$$

$$\lesssim \lambda^2 \sup_{t\leqslant T}\left[\|Z_t^\flat\|_{B^0_{3,3}}\|Z_t^\flat\|_{B^{1/4-\delta}_{3,3}}\right]\int_0^T \|\mathbb{W}_t^2\|^2_{B^{-1-\delta}_{6,\infty}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}$$

$$\lesssim \lambda^2 \sup_{t\leqslant T}[\|Z_t^\flat\|_{L^4}\|Z_t^\flat\|_{H^{1/2-\delta}}]\int_0^T \|\mathbb{W}_t^2\|^2_{B^{-1-\delta}_{6,\infty}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}},$$

which can be easily estimated by Young's inequality. From Proposition A.14 and Proposition A.4

$$\left|\frac{\lambda^2}{2}\fint ((J_t\mathbb{W}_t^2 \succ Z_t^\flat))^2 - \frac{\lambda^2}{2}\fint (J_t\mathbb{W}_t^2 \succ Z_t^\flat)\circ J_t\mathbb{W}_t^2\, Z_t^\flat\right| \lesssim \lambda^2\|J_t\mathbb{W}_t^2\|^2_{B^{-1-\delta}_{6,\infty}}\|Z_t^\flat\|_{B^{-1/4-\delta}_{3,\infty}}\|Z_t^\flat\|_{B^0_{3,3}}$$

and by interpolation

$$\lesssim \lambda^2\|J_t\mathbb{W}_t^2\|^2_{B^{-1-\delta}_{6,\infty}}\|Z_t^\flat\|_{L^4}\|Z_t^\flat\|_{H^{1/2-\delta}}.$$

The integrability of this term in time follows from the inequality

$$\|J_t\mathbb{W}_t^2\|^2_{B^{-1-\delta}_{6,\infty}} \lesssim \langle t\rangle^{-1-2\delta}\|\mathbb{W}_t^2\|^2_{B^{-1-\delta}_{6,\infty}}.$$

Using again Proposition A.7 for $t^{1/2}J_t$ gives the estimate. Applying Proposition A.13 and Proposition A.4 we get

$$\lambda^2\|(J_t\mathbb{W}_t^2 \succ Z_t^\flat)\circ J_t\mathbb{W}_t^2 - (J_t\mathbb{W}_t^2\circ J_t\mathbb{W}_t^2)(Z_t^\flat)\|_{B^0_{3/2,\infty}} \lesssim \lambda^2\|J_t\mathbb{W}_t^2\|^2_{B^{-1-\delta}_{6,\infty}}\|Z_t^\flat\|_{B^{3\delta}_{3,\infty}}.$$

and after using duality and interpolation we obtain

$$\frac{\lambda^2}{2}\left|\int_0^T \fint ((J_t\mathbb{W}_t^2 \succ Z_t^\flat))^2 - (J_t\mathbb{W}_t^2\circ J_t\mathbb{W}_t^2)(Z_t^\flat)^2\mathrm{d}t\right|$$

$$\lesssim \lambda^2 \sup_{t\leqslant T}[\|Z_t^\flat\|_{L^4}\|Z_t^\flat\|_{H^{1/2-\delta}}]\int_0^T \|\mathbb{W}_t^2\|^2_{B^{-1-\delta}_{6,\infty}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}$$

$$\lesssim \varepsilon\left(\frac{1}{2}\sup_{t\leqslant T}\|Z_t^\flat\|^2_{H^{1/2-\delta}} + \lambda\|Z_T\|^4_{L^4}\right) + C(\varepsilon,\delta)\lambda^7\left(\int_0^T \|\mathbb{W}_t^2\|^2_{B^{-1-\delta}_{6,\infty}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}\right)^4$$

$$\lesssim \varepsilon\left(\frac{1}{2}\|Z_T\|^2_{H^{1/2-\delta}} + \lambda\|Z_T\|^4_{L^4}\right) + C(\varepsilon,\delta)\lambda^7\int_0^T \|\mathbb{W}_t^2\|^8_{B^{-1-\delta}_{6,\infty}}\frac{\mathrm{d}t}{\langle t\rangle^{1+\delta}}.$$

Finally we have

$$\lambda^2\left|\int_0^T \fint \mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}(Z_t^\flat)^2\mathrm{d}t\right| \lesssim \lambda^2\left[\int_0^T \left\|\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}\right\|_{L^4}\mathrm{d}t\right]\|Z_T\|_{H^\varepsilon}\|Z_T\|_{L^4}$$

$$\leqslant C(\varepsilon)\lambda^7\left[\int_0^T \|\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}\|_{L^4}\mathrm{d}t\right]^4 + \lambda\varepsilon\|Z_T\|^4_{L^4} + \varepsilon\|Z_T\|^2_{H^{1/2-\delta}}.$$

Using eq. (2.61) to control $\|Z_T\|_{H^{1/2-\delta}}$ in terms of K_T we obtain the claim. □

2.9. Stochastic estimates

In this section we close our argument proving the following lemmas which give uniform estimates as $T\to\infty$ of some of the stochastic terms appearing in our analytic estimates.

Lemma 2.52. *For any $\varepsilon>0$ and any $p>1, r<\infty, q\in[1,\infty]$, there exists a constant $C(\varepsilon,p,q)$ which does not depend on Λ such that*

$$\sup_T \mathbb{E}\left[\left\|W_T\circ W_T^{[3]}\right\|^p_{B^{-\varepsilon}_{r,q}}\right] \leqslant C(\varepsilon,p,q). \tag{2.64}$$

Moreover there exists a function $\gamma_t \in C^1(\mathbb{R}_+, \mathbb{R})$ *such that for any* $\varepsilon > 0$ *and any* $p > 1$*,*

$$\sup_T \mathbb{E}\Big[\big\|(\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]} - 2\gamma_T W_T)\big\|_{B_{r,q}^{-1/2-\varepsilon}}^p\Big] \leqslant C(\varepsilon, p, q), \tag{2.65}$$

$$\mathbb{E}\Bigg[\bigg(\int_0^\infty \|J_t \mathbb{W}_t^2 \circ J_t \mathbb{W}_t^2 - 2\dot{\gamma}_t\|_{B_{r,q}^{-\varepsilon}} \mathrm{d}t\bigg)^p\Bigg] \leqslant C(\varepsilon, p, q). \tag{2.66}$$

$$\sup_t \mathbb{E}[\|J_t \mathbb{W}_t^2 \circ J_t \mathbb{W}_t^2 - 2\dot{\gamma}_t\|_{B_{r,q}^{-\varepsilon}}] \leqslant C(\varepsilon, p, q)$$

and

$$|\gamma_t| + \langle t \rangle |\dot{\gamma}_t| \lesssim 1 + \log\langle t \rangle, \qquad t \geqslant 0. \tag{2.67}$$

Furthermore γ *is independent of* Λ*. By Besov embedding, the Besov-Hölder norms of these objects are also uniformly bounded in* T *(but not uniformly in* Λ*).*

Proof. We will concentrate in proving the bounds on the renormalized terms in eqs. (2.65) and (2.66) and leave to the reader to fill the details for the easier term in eq. (2.64). Recall the representation of $(W_t)_t$ in terms of the family of Brownian motions $(B_t^n)_{t,n}$ in eq. (2.4). Wick's products of the Gaussian field W_T can be represented as iterated stochastic integrals wrt. $(B_t^n)_{t,n}$. In particular, if we let $\mathrm{d}w_s(k) = \langle k \rangle^{-1} \sigma_s(k) \mathrm{d}B_s^k$, we have

$$\mathbb{W}_T^2(x) = 12[\![W_T^2]\!](x) = 24 \sum_{k_1, k_2} e^{i(k_1 + k_2) \cdot x} \int_0^T \int_0^{s_2} \mathrm{d}w_{s_1}(k_1) \mathrm{d}w_{s_2}(k_2),$$

$$\mathbb{W}_T^{[3]}(x) = 24 \sum_{k_1, k_2, k_3} e^{ik_{(123)} \cdot x} \int_0^T \int_0^{s_3} \int_0^{s_2} \bigg(\int_{s_3}^T \frac{\sigma_u^2(k_{(123)})}{\langle k_{(123)} \rangle^2} \mathrm{d}u\bigg) \mathrm{d}w_{s_1}(k_1) \mathrm{d}w_{s_2}(k_2) \mathrm{d}w_{s_3}(k_3),$$

where we denote $k_{(1\cdots n)} := k_1 + \cdots + k_n$ for any $n \geqslant 2$. Now products of iterated integrals can be decomposed in sums of iterated integrals and we get

$$\begin{aligned}
\Delta_q(\mathbb{W}_T^{2\diamond[3]})(x) &= \Delta_q\big(\mathbb{W}_T^2 \circ \mathbb{W}_T^{[3]} - 2\gamma_T W_T\big)(x) \\
&= \sum_{k_1, \ldots, k_5} \int_{A_T^5} G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots5}) \mathrm{d}w_{s_1}(k_1) \cdots \mathrm{d}w_{s_5}(k_5) \\
&\quad + \sum_{k_1, \ldots, k_3} \int_{A_T^3} G_{1,q}^{2\diamond[3]}((s,k)_{1\cdots3}) \mathrm{d}w_{s_1}(k_1) \cdots \mathrm{d}w_{s_3}(k_3) \\
&\quad + \sum_{k_1} \int_{A_T^1} G_{2,q}^{2\diamond[3]}((s,k)_1) \mathrm{d}w_{s_1}(k_1),
\end{aligned} \tag{2.68}$$

where $A_T^n := \{0 \leqslant s_1 < \cdots < s_n \leqslant T\} \subseteq [0,T]^n$ and where the deterministic kernels are given by

$$\begin{aligned}
G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots5}) &:= (24^2)\, \varrho_q(k_{(1\cdots5)}) e^{i(k_{(1\cdots5)}) \cdot x} \sum_{\sigma \in \mathrm{Sh}(2,3)} \sum_{i \sim j} \times \\
&\quad \times \varrho_i(k_{(\sigma_1\sigma_2)})\, \varrho_j(k_{(\sigma_3\sigma_4\sigma_5)}) \bigg(\int_{s_{\sigma_5}}^T \frac{\sigma_u(k_{(\sigma_3\sigma_4\sigma_5)})^2}{\langle k_{(\sigma_3\sigma_4\sigma_5)} \rangle^2} \mathrm{d}u\bigg), \\
G_{1,q}^{2\diamond[3]}((s,k)_{1\cdots3}) &:= (24^2)\, \varrho_q(k_{(1\cdots3)}) e^{i(k_{(1\cdots3)}) \cdot x} \sum_{\sigma \in \mathrm{Sh}(1,2)} \sum_{i \sim j} \sum_p \int_0^T \mathrm{d}r \frac{\sigma_r(p)^2}{\langle p \rangle^2} \times \\
&\quad \times \varrho_i(k_{\sigma_1} + p)\, \varrho_j(k_{(\sigma_2\sigma_3)} - p) \bigg(\int_{s_{\sigma_3} \vee r}^T \frac{\sigma_u(k_{(\sigma_2\sigma_3)} - p)^2}{\langle k_{(\sigma_2\sigma_3)} - p \rangle^2} \mathrm{d}u\bigg), \\
G_{2,q}^{2\circ[3]}((s,k)_1) &:= (24^2)\, \varrho_q(k_1) e^{ik_1 \cdot x} \sum_{i \sim j} \sum_{p_1, p_2} \int_0^T \mathrm{d}r_1 \int_0^T \mathrm{d}r_2 \frac{\sigma_{r_1}(p_1)^2}{\langle p_1 \rangle^2} \frac{\sigma_{r_2}(p_2)^2}{\langle p_2 \rangle^2} \times \\
&\quad \times \varrho_i(p_1 + p_2)\, \varrho_j(k_1 - p_1 - p_2) \bigg(\int_{r_1 \vee r_2 \vee s_1}^T \frac{\sigma_u(k_1 - p_1 - p_2)^2}{\langle k_1 - p_1 - p_2 \rangle^2} \mathrm{d}u\bigg), \\
G_{2,q}^{2\diamond[3]}((s,k)_1) &:= G_{2,q}^{2\circ[3]}((s,k)_1) - 2\gamma_T \varrho_q(k_1)\, e^{ik_1 \cdot x},
\end{aligned}$$

where $\mathrm{Sh}(k,l)$ is the set of permutations σ of $\{1,\ldots,k+l\}$ keeping the orders $\sigma(1)<\cdots<\sigma(k)$ and $\sigma(k+1)<\cdots<\sigma(k+l)$ and where, for any symbol z, we denote with expression of the form $z_{1\cdots n}$ the vector $(z_1,\ldots,z_n)$. Estimation of $\Delta_q(\mathbb{W}_T^2\circ\mathbb{W}_T^{[3]})(x)$ reduces then to estimate each of the three iterated integrals using BDG inequalities to get, for any $p\geqslant 2$,

$$I_{0,q}=\left\{\mathbb{E}\left[\left|\sum_{k_1,\ldots,k_5}\int_{A_T^5}G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_5}(k_5)\right|^{2p}\right]\right\}^{1/p}$$

$$\lesssim\sum_{k_1,\ldots,k_5}\int_{A_T^5}\left|G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5})\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5.$$

The kernel $G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5})$ being a symmetric function of its argument, we can simplify this expression into an integral over $[0,T]^5$:

$$I_{0,q}\lesssim\sum_{k_1,\ldots,k_5}\int_{[0,T]^5}\left|G_{0,q}^{2\diamond[3]}((s,k)_{1\cdots 5})\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5.$$

Under the measure $\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_5$, we have

$$\left|\int_{s_{\sigma_5}}^T\frac{\sigma_u(k_{(\sigma_3\sigma_4\sigma_5)})^2}{\langle k_{(\sigma_3\sigma_4\sigma_5)}\rangle^2}\mathrm{d}u\right|\lesssim\frac{1}{\langle k_{\sigma_5}\rangle^2}.$$

Therefore with some standard estimates we can reduce this to

$$I_{0,q}\lesssim\sum_{k_1,\ldots,k_5}\int_{[0,T]^5}\frac{\varrho_q(k_{(1\cdots 5)})^2}{\langle k_5\rangle^4}\mathbb{1}_{k_{(12)}\sim k_{(345)}}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5$$

$$\lesssim\sum_{k_1,\ldots,k_5}\int_{[0,T]^5}\frac{\varrho_q(k_{(1\cdots 5)})^2}{\langle k_5\rangle^4}\mathbb{1}_{k_{(12)}\sim k_{(345)}}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_5}(k_5)^2}{\langle k_5\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_5$$

$$\lesssim\sum_{k_1,\ldots,k_5}\frac{\varrho_q(k_{(1\cdots 5)})^2}{\langle k_5\rangle^4}\mathbb{1}_{k_{(12)}\sim k_{(345)}}\frac{1}{\langle k_1\rangle^2}\cdots\frac{1}{\langle k_5\rangle^2}$$

$$\lesssim\sum_{p_1,p_2}\mathbb{1}_{p_1\sim p_2}\varrho_q(p_1+p_2)^2\sum_{k_1,\ldots,k_5}\frac{1}{\langle k_5\rangle^4}\mathbb{1}_{k_{(12)}=p_1,k_{(345)}=p_2}\frac{1}{\langle k_1\rangle^2}\cdots\frac{1}{\langle k_5\rangle^2}$$

$$\lesssim\sum_{p_1,p_2}\mathbb{1}_{p_1\sim p_2}\varrho_q(p_1+p_2)^2\frac{1}{\langle p_1\rangle}\frac{1}{\langle p_2\rangle^4}\lesssim\sum_{p_1,r}\varrho_q(r)^2\frac{1}{\langle p_1\rangle}\frac{1}{\langle p_1+r\rangle^4}\lesssim\sum_r\varrho_q(r)^2\frac{1}{\langle r\rangle^2}\lesssim 2^q.$$

Now by similar reasoning we also have

$$\left|G_{1,q}^{2\diamond[3]}((s,k)_{1\cdots 3})\right|\lesssim\sum_{\sigma\in\mathrm{Sh}(1,2)}|\varrho_q(k_{(1\cdots 3)})|\sum_{i\sim j}\sum_p\int_0^T\mathrm{d}r\frac{\sigma_r(p)^2|\varrho_i(k_{\sigma_1}+p)\varrho_j(k_{(\sigma_2\sigma_3)}-p)|}{\langle p\rangle^2\langle k_{\sigma_1}+p\rangle^2}$$

$$\lesssim\sum_{\sigma\in\mathrm{Sh}(1,2)}\frac{|\varrho_q(k_{(1\cdots 3)})|}{\langle k_{\sigma_1}\rangle}$$

so

$$I_{1,q}=\left\{\mathbb{E}\left[\left|\sum_{k_1,\ldots,k_3}\int_{A_T^3}G_{1,q}^{2\diamond[3]}((s,k)_{1\cdots 5})\mathrm{d}y_{s_1}(k_1)\cdots\mathrm{d}y_{s_3}(k_3)\right|^{2p}\right]\right\}^{1/p}$$

$$\lesssim\sum_{k_1,\ldots,k_3}\int_{[0,T]^3}\left|\sum_{\sigma\in\mathrm{Sh}(1,2)}\frac{|\varrho_q(k_{(1\cdots 3)})|}{\langle k_{\sigma_1}\rangle}\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_3}(k_3)^2}{\langle k_3\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_3$$

$$\lesssim\sum_{k_1,\ldots,k_3}\frac{|\varrho_q(k_{(1\cdots 3)})|^2}{\langle k_1\rangle^4\langle k_2\rangle^2\langle k_3\rangle^2}\lesssim\sum_r\frac{\varrho_q(r)^2}{\langle r\rangle^2}\lesssim 2^q.$$

Finally, we note that the same strategy cannot be applied to the first chaos, since the kernel $G^{2\circ[3]}_{2,q}$ cannot be uniformly bounded. We let

$$\begin{aligned}A_T(s_1,k_1) &:= (24^2)\sum_{i\sim j}\sum_{q_1,q_2}\int_0^T \mathrm{d}r_1\int_0^T \mathrm{d}r_2\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\times\\ &\quad\times\varrho_i(q_1+q_2)\varrho_j(k_1-q_1-q_2)\left(\int_{r_1\vee r_2\vee s_1}^T\frac{\sigma_u^2(k_1-q_1-q_2)}{\langle k_1-q_1-q_2\rangle^2}\mathrm{d}u\right),\end{aligned}$$

so

$$G^{2\diamond[3]}_{2,q}((s,k)_1)=\varrho_q(k_1)e^{ik_1\cdot x}[A_T(s_1,k_1)-2\gamma_T].$$

Observe that

$$\begin{aligned}A_T(0,0) &= (12^2\cdot 2)\sum_{q_1,q_2}\int_0^T \mathrm{d}r_1\int_0^T \mathrm{d}r_2\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\times\\ &\quad\times\int_{r_1\vee r_2}^T\frac{\sigma_u^2(q_1+q_2)}{\langle q_1+q_2\rangle^2}\mathrm{d}u\sum_{i\sim j}\varrho_i(q_1+q_2)\varrho_j(-q_1-q_2).\end{aligned}$$

We choose γ_T as

$$\gamma_T=A_T(0,0)=(12^2\cdot 2)\sum_{q_1,q_2}\int_0^T\mathrm{d}u\int_0^u\mathrm{d}r_1\int_0^u\mathrm{d}r_2\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\frac{\sigma_u^2(q_1+q_2)}{\langle q_1+q_2\rangle^2}\tag{2.69}$$

where we used the fact that for all $q\in\mathbb{R}^d$ we have $\sum_{i\sim j}\varrho_i(q)\varrho_j(q)=1$, since $\int f\circ g=\int fg$. Note that, as claimed,

$$|\gamma_T|\lesssim\sum_{q_1,q_2}\frac{\mathbb{1}_{|q_1|,|q_2|,|q_1+q_2|\lesssim T}}{\langle q_1\rangle^2\langle q_2\rangle^2\langle q_1+q_2\rangle^2}\lesssim 1+\log\langle T\rangle.$$

Now

$$A_T(s_1,k_1)-2\gamma_T=(24^2\cdot 6)\sum_{q_1,q_2}\int_0^T\mathrm{d}r_1\mathrm{d}r_2\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\sum_{i\sim j}\varrho_i(q_1+q_2)\times$$

$$\times\left(\varrho_j(k_1-q_1-q_2)\int_{s_1\vee r_1\vee r_2}^T\frac{\sigma_u^2(k_1-q_1-q_2)}{\langle k_1-q_1-q_2\rangle^2}\mathrm{d}u-\varrho_j(q_1+q_2)\int_{r_1\vee r_2}^T\frac{\sigma_u^2(q_1+q_2)}{\langle q_1+q_2\rangle^2}\mathrm{d}u\right)$$

so there existst a constatn C such that, when $|q_1+q_2|\geqslant C|k_1|$, the quantity in round brackets can be estimated by $|k_1|\langle q_1+q_2\rangle^{-4}$ while when $|q_1+q_2|\leqslant C|k_1|$ it is estimated by $\langle q_1+q_2\rangle^{-2}$. We have

$$\begin{aligned}|A_T(s_1,k_1)-\gamma_T| &\lesssim\sum_{q_1,q_2}\frac{1}{\langle q_1\rangle^2}\frac{1}{\langle q_2\rangle^2}\frac{1}{\langle q_1+q_2\rangle^2}\left(\mathbb{1}_{|q_1+q_2|\leqslant C|k_1|}+\mathbb{1}_{|q_1+q_2|\geqslant C|k_1|}\frac{|k_1|}{\langle q_1+q_2\rangle^2}\right)\\ &\lesssim 1+\log\langle k_1\rangle.\end{aligned}$$

And then with this choice of γ_T the kernel $G^{2\diamond[3]}_{2,q}$ stays uniformly bounded as $T\to\infty$ and satisfies

$$\left|G^{2\diamond[3]}_{2,q}((s,k)_1)\right|\lesssim\varrho_q(k_1)\log\langle k_1\rangle.$$

From this we easily deduce that

$$I_{2,q}=\left\{\mathbb{E}\left[\left|\sum_{k_1}\int_{A_T}G^{2\diamond[3]}_{2,q}((s,k)_1)\mathrm{d}y_{s_1}(k_1)\right|^{2p}\right]\right\}^{1/p}\lesssim q2^q,\qquad q\geqslant -1.$$

All together these estimates imply that

$$\mathbb{E}\left\|\Delta_q\mathbb{W}^{2\diamond[3]}_T\right\|^{2p}_{L^{2p}}\lesssim(q2^{q/2})^{2p},\qquad q\geqslant -1.$$

Standard argument allows to deduce eq. (2.65). The analysis of the other renormalized product proceeds similarly. Let

$$V(t):=\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}=J_t\mathbb{W}_t^2\circ J_t\mathbb{W}_t^2-2\dot{\gamma}_t,\qquad t\geqslant 0.$$

First note that by definition of Besov spaces we have

$$\mathbb{E}\left[\left(\int_0^\infty \|V(t)\|_{B^{-\varepsilon-d/r}_{r,r}}\mathrm{d}t\right)^p\right] \lesssim \mathbb{E}\left[\left(\int_0^\infty \left(\sum_q 2^{-qr(\varepsilon+d/r)}\|\Delta_q V(t)\|_{L^r}\right)^{1/r}\mathrm{d}t\right)^p\right].$$

By Minkowski's integral inequality this is bounded by

$$\lesssim \left(\int_0^\infty \mathrm{d}t\left\{\mathbb{E}\left[\left(\sum_q 2^{-qr(\varepsilon+d/r)}\|\Delta_q V(t)\|_{L^r}^r\right)^{p/r}\right]\right\}^{1/p}\right)^p.$$

When $r \geqslant p$ Jensen's inequality and Fubini's theorem give

$$\lesssim \left(\int_0^\infty \mathrm{d}t\left\{\sum_q 2^{-qr(\varepsilon+d/r)}\int_\Lambda \frac{\mathrm{d}x}{|\Lambda|}\mathbb{E}[|\Delta_q V(t)(x)|^r]\right\}^{1/r}\right)^p.$$

Finally hypercontractivity and stationarity allow to reduce this to bound

$$\lesssim \left(\int_0^\infty \mathrm{d}t\left\{\sum_q 2^{-qr(\varepsilon+d/r)}\,(\mathbb{E}[|\Delta_q V(t)(0)|^2])^{r/2}\right\}^{1/r}\right)^p.$$

Letting $I_q(t) = \mathbb{E}[|\Delta_q V(t)(0)|^2]$ we have

$$\mathbb{E}\left[\left(\int_0^\infty \|\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle}\|_{B^{-\varepsilon-d/r}_{r,r}}\mathrm{d}t\right)^p\right] \lesssim \left(\int_0^\infty \mathrm{d}t\left\{\sum_q 2^{-qr(\varepsilon+d/r)}\,(I_q(t))^{r/2}\right\}^{1/r}\right)^p.$$

Now we decompose the random field $\Delta_q(\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle})(x)$ into homogeneous stochastic integral as above and obtain

$$\begin{aligned}\Delta_q(\mathbb{W}_t^{\langle 2\rangle\diamond\langle 2\rangle})(x) \;=\; & \sum_{k_1,\ldots,k_4}\int_{A_t^4} G_{0,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{1\cdots 4})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_4}(k_4)\\ & +\sum_{k_1,k_2}\int_{A_t^2} G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12})\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\\ & +G_{2,q}^{\langle 2\rangle\diamond\langle 2\rangle}\end{aligned} \tag{2.70}$$

with

$$\begin{aligned}G_{0,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{1\cdots 4}) \;=\; & (24^2)\,\varrho_q(k_{(1\cdots 4)})e^{i(k_{(1\cdots 4)})\cdot x}\times\\ & \times\sum_{\sigma\in\mathrm{Sh}(2,2)}\sum_{i\sim j}\varrho_i(k_{(\sigma_1\sigma_2)})\,\varrho_j(k_{(\sigma_3\sigma_4)})\frac{\sigma_t(k_{(\sigma_1\sigma_2)})}{\langle k_{(\sigma_1\sigma_2)}\rangle}\frac{\sigma_t(k_{(\sigma_3\sigma_4)})}{\langle k_{(\sigma_3\sigma_4)}\rangle}\\ G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12}) \;=\; & (24^2)\,\varrho_q(k_{(12)})e^{i(k_{(12)})\cdot x}\sum_{\sigma\in\mathrm{Sh}(1,1)}\sum_{i\sim j}\sum_q\times\\ & \times\int_0^t \mathrm{d}r\frac{\sigma_r^2(q)}{\langle q\rangle^2}\varrho_i(k_{\sigma_1}+q)\,\varrho_j(k_{\sigma_2}-q)\left(\frac{\sigma_t(k_{\sigma_1}+q)}{\langle k_{\sigma_1}+q\rangle}\frac{\sigma_t(k_{\sigma_2}-q)}{\langle k_{\sigma_2}-q\rangle}\right)\\ G_{2,q}^{\langle 2\rangle\diamond\langle 2\rangle} \;=\; & (24^2)\mathbb{1}_{q=-1}\sum_{i\sim j}\sum_{q_1,q_2}\int_0^t\mathrm{d}r_1\int_0^t\mathrm{d}r_2\times\\ & \times\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\varrho_i(q_1+q_2)\,\varrho_j(-q_1-q_2)\frac{\sigma_t(q_1+q_2)^2}{\langle q_1+q_2\rangle^2}\\ & -2\dot{\gamma}_t\mathbb{1}_{q=-1}.\end{aligned}$$

Using our choice of γ_T in eq. (2.69) we have that

$$\dot{\gamma}_t = (12^2\cdot 2)\sum_{q_1,q_2}\int_0^t\mathrm{d}r_1\int_0^t\mathrm{d}r_2\frac{\sigma_{r_1}(q_1)^2}{\langle q_1\rangle^2}\frac{\sigma_{r_2}(q_2)^2}{\langle q_2\rangle^2}\frac{\sigma_t^2(q_1+q_2)}{\langle q_1+q_2\rangle^2},$$

which implies also that

$$G_{2,q}^{\langle 2\rangle\diamond\langle 2\rangle}=0,\quad \text{and}\qquad |\dot{\gamma}_t|\lesssim\frac{1+\log\langle t\rangle}{\langle t\rangle}.$$

as claimed. We pass now to estimate the other two chaoses. The technique is the same we used above. Consider first

$$\begin{aligned}
I_{0,q}(t) &:= \mathbb{E}\left[\left|\sum_{k_1,\ldots,k_4}\int_{A_t^4} G_{0,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{1\cdots 4})\mathrm{d}w_{s_1}(k_1)\cdots\mathrm{d}w_{s_4}(k_4)\right|^2\right]\\
&\lesssim \sum_{k_1,\ldots,k_4}\int_{A_t^4}\left|G_{0,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{1\cdots 4})\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_4}(k_4)^2}{\langle k_4\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_4\\
&\lesssim \sum_{k_1,\ldots,k_4}\int_{[0,t]^4}\left|G_{0,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{1\cdots 4})\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_4}(k_4)^2}{\langle k_4\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_4\\
&\lesssim \sum_{k_1,\ldots,k_4}\varrho_q(k_{(1\cdots 4)})^2\int_{[0,t]^4}\frac{\sigma_t^2(k_{(12)})}{\langle k_{(12)}\rangle^2}\frac{\sigma_t^2(k_{(34)})}{\langle k_{(34)}\rangle^2}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\cdots\frac{\sigma_{s_4}(k_4)^2}{\langle k_4\rangle^2}\mathrm{d}s_1\cdots\mathrm{d}s_4\\
&\lesssim \sum_{k_1,\ldots,k_4}\varrho_q(k_{(1\cdots 4)})^2\frac{\sigma_t^2(k_{(12)})}{\langle k_{(12)}\rangle^2}\frac{\sigma_t^2(k_{(34)})}{\langle k_{(34)}\rangle^2}\frac{1}{\langle k_1\rangle^2}\cdots\frac{1}{\langle k_4\rangle^2}\\
&\lesssim \frac{\mathbb{1}_{2^q\lesssim t}}{\langle t\rangle^6}\sum_{k_1,\ldots,k_4}\frac{\varrho_q(k_{(1\cdots 4)})^2}{\langle k_1\rangle^2\langle k_2\rangle^2\langle k_3\rangle^2\langle k_4\rangle^2}\lesssim\frac{\mathbb{1}_{2^q\lesssim t}}{\langle t\rangle^6}2^{4q}
\end{aligned}$$

where we used that $|\sigma_t(x)|\lesssim t^{-1/2}\mathbb{1}_{x\sim t}$. Now taking $\varepsilon+d/r>0$ we have

$$\begin{aligned}
\int_0^\infty \mathrm{d}t\left\{\sum_q 2^{-qr(\varepsilon+d/r)}\,(I_{0,q}(t))^{r/2}\right\}^{1/r} &\lesssim \int_0^\infty \mathrm{d}t\left\{\sum_{q:2^q\lesssim t}\frac{2^{qr(2-\varepsilon-d/r)}}{\langle t\rangle^{3r}}\right\}^{1/r}\\
&\lesssim \int_0^\infty\frac{\mathrm{d}t}{\langle t\rangle^{1+\varepsilon+d/r}}\lesssim 1.
\end{aligned}$$

Taking into account that $|k_1|,|k_2|\lesssim t$ we can estimate

$$\left|G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12})\right|\lesssim|\varrho_q(k_{(12)})|\sum_p\frac{\mathbb{1}_{|p|\lesssim t}}{\langle p\rangle^2}\left(\frac{\sigma_t(k_1+p)}{\langle k_1+p\rangle}\frac{\sigma_t(k_2-p)}{\langle k_2-p\rangle}\right)\lesssim|\varrho_q(k_{(12)})|\langle t\rangle^{-2},$$

from which we deduce that

$$\begin{aligned}
I_{1,q}(t) &:= \mathbb{E}\left[\left|\sum_{k_1,k_2}\int_{A_t^2} G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12})\mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\right|^2\right]\\
&\lesssim \sum_{k_1,k_2}\int_{A_t^2}\left|G_{1,q}^{\langle 2\rangle\diamond\langle 2\rangle}((s,k)_{12})\right|^2\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\frac{\sigma_{s_2}(k_2)^2}{\langle k_2\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\\
&\lesssim \langle t\rangle^{-4}\sum_{k_1,k_2}|\varrho_q(k_{(12)})|^2\int_{[0,t]^2}\frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2}\frac{\sigma_{s_2}(k_2)^2}{\langle k_2\rangle^2}\mathrm{d}s_1\mathrm{d}s_2\\
&\lesssim \langle t\rangle^{-4}\sum_{k_1,k_2}|\varrho_q(k_{(12)})|^2\frac{\mathbb{1}_{k_1\lesssim t}}{\langle k_1\rangle^2}\frac{\mathbb{1}_{k_2\lesssim t}}{\langle k_2\rangle^2}\lesssim\langle t\rangle^{-4}2^{2q}\mathbb{1}_{2^q\lesssim t},
\end{aligned}$$

and then, as for $I_{0,q}$, we have

$$\int_0^\infty\mathrm{d}t\left(\sum_q 2^{-qr(\varepsilon+d/r)}\,(I_{1,q}(t))^{r/2}\right)^{1/r}\lesssim\int_0^\infty\frac{\mathrm{d}t}{\langle t\rangle^2}\left(\sum_q 2^{qr(1-\varepsilon-d/r)}\mathbb{1}_{2^q\lesssim t}\right)^{1/r}\lesssim 1,$$

as claimed. From these estimates standard arguments give eq. (2.66). □

Lemma 2.53. *We have*

$$\mathbb{E}[\|\mathbb{W}_T^3\|_{L^p}^p]^{1/p} \lesssim T^{3/2}.$$

This implies that $\mathbb{W}^{\langle 3\rangle} \in C\big([0,\infty], B_{p,p}^{-1/2-\kappa}\big) \cap L^2\big(\mathbb{R}_+, B_{p,p}^{-1/2-\kappa}\big)$ *for any* $p<\infty$ *uniformly in the volume and* $\mathbb{W}^{\langle 3\rangle} \in C([0,\infty], \mathscr{C}^{-1/2-\kappa}) \cap L^2(\mathbb{R}_+, \mathscr{C}^{-1/2-\kappa})$.

Proof. Observe that

$$\mathbb{W}_T^3(x) = 12[\![W_T^3]\!](x) = 24 \sum_{k_1,k_2,k_3} e^{i(k_1+k_2+k_3)\cdot x} \int_0^T \int_0^{s_1} \int_0^{s_2} \mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3).$$

By space homogeneity, we get for any p,

$$\begin{aligned}
\mathbb{E}[\|\mathbb{W}_T^3(x)\|_{L^p}^p] &= \mathbb{E}[\|\mathbb{W}_T^3(0)\|^p] \\
&= \mathbb{E}\left[\left|\sum_{k_1,k_2,k_3} \int_0^T \int_0^{s_2} \int_0^{s_1} \mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3)\right|^p\right] \\
&\lesssim \left(\mathbb{E}\left[\left|\sum_{k_1,k_2,k_3} \int_0^T \int_0^{s_2} \int_0^{s_1} \mathrm{d}w_{s_1}(k_1)\mathrm{d}w_{s_2}(k_2)\mathrm{d}w_{s_3}(k_3)\right|^2\right]\right)^{p/2} \\
&= \left(\sum_{k_1,k_2,k_3} \int_0^T \int_0^{s_2} \int_0^{s_1} \frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2} \cdots \frac{\sigma_{s_3}(k_3)^2}{\langle k_3\rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_3\right)^{p/2} \\
&\lesssim (T^{3/2})^p.
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{k_1,k_2,k_3} \int_0^T \int_0^{s_2} \int_0^{s_1} \frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2} \cdots \frac{\sigma_{s_3}(k_3)^2}{\langle k_3\rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_3 \\
\leqslant &\sum_{k_1,k_2,k_3} \int_0^T \int_0^T \int_0^T \frac{\sigma_{s_1}(k_1)^2}{\langle k_1\rangle^2} \cdots \frac{\sigma_{s_3}(k_3)^2}{\langle k_3\rangle^2} \mathrm{d}s_1 \cdots \mathrm{d}s_3 \\
= &\left(\sum_k \int_0^T \frac{\sigma_s(k)^2}{\langle k\rangle^2} \mathrm{d}s\right)^3 \lesssim T^3.
\end{aligned}$$

Now the remaining properties follow by the fact that σ_t is supported in an annulus of radius t, so

$$\|\mathbb{W}_t^{\langle 3\rangle}\|_{B_{p,p}^{-1/2-\kappa}} = \left\|\frac{\sigma_t(\mathrm{D})}{\langle \mathrm{D}\rangle} \mathbb{W}_t^3\right\|_{B_{p,p}^{-1/2-\kappa}} \lesssim \|\sigma_t(\mathrm{D})\mathbb{W}_t^3\|_{B_{p,p}^{-3/2-\kappa}} \lesssim \langle t\rangle^{-1/2-\kappa} \langle t\rangle^{-3/2} \|\mathbb{W}_t^3\|_{L^p}$$

and the Hölder estimates follow by Besov embedding (but with constants which depends on the volume). □

Chapter 3

Φ_3^4 via Girsanov Transform

3.1. Introduction

The Φ_3^4 measure on the three dimensional torus $\Lambda = \mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ is the probability measure ν on distributions $\mathscr{S}'(\Lambda)$ corresponding to the formal functional integral

$$\nu(\mathrm{d}\varphi) = \text{``}\left\{\frac{1}{Z}\exp\left[-\lambda\int_\Lambda(\varphi^4 - \infty\varphi^2)\mathrm{d}x\right]\mu(\mathrm{d}\varphi)\right\}\text{''} \tag{3.1}$$

where μ is the law of the Gaussian free field with covariance $(1-\Delta)^{-1}$ on Λ, Z a normalization constant and λ the coupling constant. The ∞ appearing in this expression reminds us that many things are wrong with this recipe. The key difficulty can be traced to the fact that the measure we are looking for is not absolutely continuous wrt. the reference measure μ. This fact seems part of the folklore even if we could not find a rigorous proof for it in the available literature apart from a work of Albeverio and Liang [3] which however refers to the Euclidean fields at time zero.

As already motioned in section 1.2.1 in recent years the rigorous study of the Φ_3^4 model has been pursued from the point of view of *stochastic quantization*. In the original formulation of Parisi–Wu [106], stochastic quantization is a way to introduce additional degrees of freedom (in particular a dependence on a fictious time) in order to obtain an *equation* whose solutions describe a measure of interest, in this case the Φ_3^4 measure on Λ as in (3.1) or its counterpart in the full space.

A conceptual advantage of stochastic quantization is that it is insensitive to questions of absolute continuity wrt. to a reference measure. This, on the other hand, is the main difficulty of the Gibbsian point of view as expressed in eq. (3.1). In order to explore further the tradeoffs of different approaches we have developed in Chapter 2 a variational method for the construction and *description* of Φ_3^4. We were able to provide an explicit formula for the Laplace transform of Φ_3^4 in terms of a stochastic control problem in which the controlled process represents the scale-by-scale evolution of the interacting random field.

This chapter is the occasion to explore further this point of view by constructing a novel measure via a random translation of the Gaussian free field and by proving that the Φ_3^4 measure can be obtained as an absolutely continuous perturbation thereof. Without entering into technical details now, let us give the broad outline of this construction. We consider a Brownian martingale $(W_t)_{t\geqslant 0}$ with values in $\mathscr{S}'(\Lambda)$ and such that W_t is a regularization of the Gaussian free field μ at (Fourier) scale t. Let us denote $\mathbb{P}$ its law and $\mathbb{E}$ the corresponding expectation. In particular, $W_t \to W_\infty$ in law as $t\to\infty$ and W_∞ has law μ. We can identify the Φ_3^4 measure ν as the weak limit $\nu^T \to \nu$ as $T\to\infty$ of the family of probability measures $(\nu^T)_{T\geqslant 0}$ on $\mathscr{S}'(\Lambda)$ defined as

$$\nu^T(\cdot) = \mathbb{P}^T(W_T \in \cdot),$$

where $\mathbb{P}^T$ is the measure on paths $(W_t)_{t\geqslant 0}$ with density

$$\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\mathbb{P}} = \frac{1}{Z_T}e^{-V_T(W_T)},$$

and

$$V_T(\varphi) := \lambda\int_\Lambda(\varphi(x)^4 - a_T\varphi(x)^2 + b_T)\mathrm{d}x,$$

is a quartic polynomial in the field φ with $(a_T, b_T)_T$ a family of (suitably diverging) renormalization constants. The presence of the scale parameter $t \in \mathbb{R}_+$ allows to introduce a filtration and a family of measures $\mathbb{Q}^v$ defined as the Girsanov transformation

$$\left.\frac{\mathrm{d}\mathbb{Q}^v}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_T} = \exp\left(L_T^v - \frac{1}{2}\langle L^v\rangle_T\right), \qquad L_t^v = \int_0^t \langle v_s, \mathrm{d}W_s\rangle_{L^2(\Lambda)} \tag{3.2}$$

where $(\langle L^v\rangle_t)_{t\geqslant 0}$ is the quadratic variation of the (scalar) local martingale $(L_t^v)_{t\geqslant 0}$ and $(v_t)_{t\geqslant 0}$ is an adapted process with values in $L^2(\Lambda)$. Let

$$D_T := \frac{1}{Z_T} e^{-V_T(W_T)} \left(\frac{\mathrm{d}\mathbb{Q}^v}{\mathrm{d}\mathbb{P}}\right)^{-1},$$

be the density of $\mathbb{P}^T$ wrt. $\mathbb{Q}^v$. We will show that it is possible to choose v in such a way that the family $(D_T)_{T\geqslant 0}$ is uniformly integrable under $\mathbb{Q}^v$ and that $D_T \to D_\infty$ weakly in $L^1(\mathbb{Q}^v)$. With particular choice of v we call $\mathbb{Q}^v$ the *drift* measure: it is the central object of this paper. By Girsanov's theorem the canonical process $(W_t)_{t\geqslant 0}$ satisfies the equation

$$\mathrm{d}W_t = v_t \mathrm{d}t + \mathrm{d}\tilde{W}_t, \qquad t \geqslant 0,$$

where $(\tilde{W}_t)_{t\geqslant 0}$ is a Gaussian martingale under $\mathbb{Q}^v$ (and has law equal to that of $(W_t)_{t\geqslant 0}$ under $\mathbb{P}$, that is is a regularized Gaussian free field). We will show also that the drift v_t can be written as a (polynomial) function of $(\tilde{W}_s)_{s\in[0,t]}$, that is $v_t = \tilde{V}_t((\tilde{W}_s)_{s\in[0,t]})$. Therefore we have an explicit description of the process $(W_t)_{t\geqslant 0}$ under the drift measure $\mathbb{Q}^v$ as the unique solution of the path-dependent SDE

$$\mathrm{d}W_t = \tilde{V}_t((\tilde{W}_s)_{s\in[0,t]})\mathrm{d}t + \mathrm{d}\tilde{W}_t, \qquad t \geqslant 0. \tag{3.3}$$

Let us note that this formula expresses the "interacting" random field $(W_t)_t$ as a function of the "free" field $(\tilde{W}_t)_t$. In this respect it shares very similar technical merits with the stochastic quantization approach.

Intuitively this new measure $\mathbb{Q}^v$ is half way between the variational description in 2 and the (formal) Gibbsian description of eq. (3.1). It constitutes a measure which is relatively explicit, easy to construct and analyze and which can be used as reference measure for Φ_3^4, very much like the Gaussian free field can be used as reference measure for Φ_2^4 [68].

As an application we provide a self-contained proof of the singularity of the Φ_3^4 measure ν wrt. the Gaussian free field μ. As we already remarked the singularity of Φ_3^4 seems to belongs to the folklore and we were not able to trace any written proof of that. However, M. Hairer, during a conference at Imperial College in 2019, showed us an unpublished proof of him of singularity using the stochastic quantization equation. Our proof and his are very similar and we do not claim any essential novelty in this respect. Albeit the proof is quite straightforward we wrote down all the details in order to provide a reference for this fact. The main contribution of the present paper remains that of describing the drift measure as a novel object in the context of Φ_3^4 and similar measures.

Our proof of singularity, in particular also shows that the drift measure $\mathbb{Q}^v$ is singular wrt. $\mathbb{P}$. The intuitive reason is that the drift $(V_t)_{t\geqslant 0}$ in the SDE (3.3) is not regular enough (as $t \to \infty$) to be along Cameron–Martin directions for the law $\mathbb{P}$ of the process $(W_t)_{t\geqslant 0}$ and therefore the Girsanov transform (3.2) gives a singular measure when extended all the way to $T = +\infty$.

Notations. Let us fix some notations and objects.

- For $a \in \mathbb{R}^d$ we let $\langle a\rangle := (1 + |a|^2)^{1/2}$. $B(x, r) \subseteq \mathbb{R}$ denotes the open ball of center $x \in \mathbb{R}$ and radius $r > 0$.
- The constant $\varepsilon > 0$ represents a small positive number which can be different from line to line.

- Denote with $\mathscr{S}(\Lambda)$ the space of Schwartz functions on Λ and with $\mathscr{S}'(\Lambda)$ the dual space of tempered distributions. The notation $\hat{f}$ or $\mathscr{F}f$ stands for the space Fourier transform of f and we will write $g(\mathrm{D})$ to denote the Fourier multiplier operator with symbol $g\colon \mathbb{R}^n \to \mathbb{R}$, i.e. $\mathscr{F}(g(\mathrm{D})f) = g\mathscr{F}f$.
- $B^{\alpha}_{p,q} = B^{\alpha}_{p,q}(\Lambda)$ denotes the Besov spaces of regularity α and integrability indices p, q as usual. $\mathscr{C}^{\alpha} = \mathscr{C}^{\alpha}(\Lambda)$ is the Hölder–Besov space $B^{\alpha}_{\infty,\infty}$, $W^{\alpha,p} = W^{\alpha,p}(\Lambda)$ denote the standard fractional Sobolev spaces defined by the norm $\|f\|_{W^{s,q}} := \|\langle \mathrm{D}\rangle^s f\|_{L^q}$ and $H^{\alpha} = W^{\alpha,2}$. The symbols $\prec, \succ, \circ$ denotes spatial paraproducts wrt. a standard Littlewood–Paley decomposition. The reader is referred to Appendix A for an overview of the functional spaces and paraproducts.

3.2. The setting

The setting of this chapter is the same of that of Chapter 2. In this section we will briefly recall it and also state some results from that chapter which will be needed below. They concern the Boué–Dupuis formula and certain estimates which are relevant to our analysis of absolute continuity.

Let $\Omega := C(\mathbb{R}_+; \mathscr{C}^{-3/2-\varepsilon}(\Lambda))$ and $\mathscr{F}$ be the Borel σ–algebra of Ω. On $(\Omega, \mathscr{F})$ consider the probability measure $\mathbb{P}$ which makes the canonical process $(X_t)_{t\geqslant 0}$ a cylindrical Brownian motion on $L^2(\Lambda)$ and let $(\mathscr{F}_t)_{t\geqslant 0}$ the associated filtration. In the following $\mathbb{E}$ without any qualifiers will denote expectations wrt. $\mathbb{P}$ and $\mathbb{E}_{\mathbb{Q}}$ will denote expectations wrt. some other measure $\mathbb{Q}$.

On the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ there exists a collection $(B^n_t)_{n\in(\mathbb{Z})^3}$ of complex (2-dimensional) Brownian motions, such that $\overline{B^n_t} = B^{-n}_t$, B^n_t, B^m_t independent for $m \neq \pm n$ and $X_t = \sum_{n\in\mathbb{Z}^3} e^{i\langle n,\cdot\rangle} B^n_t$, for example in $\mathscr{S}'(\Lambda)$.

Fix some decreasing $\rho \in C^{\infty}_c(\mathbb{R}_+, \mathbb{R}_+)$ such that $\rho = 1$ on $B(0, 9/10)$ and $\operatorname{supp}\rho \subset B(0,1)$. For $x \in \mathbb{R}^3$ let $\rho_t(x) := \rho(\langle x\rangle/t)$ and

$$\sigma_t(x) := \left(\frac{\mathrm{d}}{\mathrm{d}t}(\rho^2_t(x))\right)^{1/2} = (-2(\langle x\rangle/t)\rho(\langle x\rangle/t)\rho'(\langle x\rangle/t))^{1/2}/t^{1/2}.$$

Denote $J_s = \sigma_s(\mathrm{D})\langle \mathrm{D}\rangle^{-1}$ and consider the process $(W_t)_{t\geqslant 0}$ defined by

$$W_t := \int_0^t J_s \mathrm{d}X_s = \sum_{n\in\mathbb{Z}^3} e^{i\langle n,\cdot\rangle} \int_0^t \frac{\sigma_s(n)}{\langle n\rangle} dB^n_s, \qquad t \geqslant 0. \tag{3.4}$$

It is a centered Gaussian process with covariance

$$\mathbb{E}[\langle W_t, \varphi\rangle \langle W_s, \psi\rangle] = \sum_{n\in\mathbb{Z}^3} \frac{\rho^2_{\min(s,t)}(n)}{\langle n\rangle^2} \hat{\varphi}(n)\overline{\hat{\psi}(n)},$$

for any $\varphi, \psi \in \mathscr{S}(\Lambda)$ and $t, s \geqslant 0$, by Fubini theorem and Ito isometry. By dominated convergence $\lim_{t\to\infty}\mathbb{E}[\langle W_t, \varphi\rangle \langle W_t, \psi\rangle] = \sum_{n\in\mathbb{Z}^3} \langle n\rangle^{-2} \hat{\varphi}(n)\overline{\hat{\psi}(n)}$ for any $\varphi, \psi \in L^2(\Lambda)$. For any finite "time" T the random field W_T on Λ has a bounded spectral support and the stopped process $W^T_t = W_{t\wedge T}$ for any fixed $T > 0$, is in $C(\mathbb{R}_+, C^{\infty}(\Lambda))$. Furthermore $(W^T_t)_t$ only depends on a finite subset of the Brownian motions $(B^n)_{n\in\mathbb{Z}^3}$.

Observe that J_t satisfies the following bound

$$\|J_t f\|_{B^{s+1-\alpha}_{p,p}} \lesssim \langle t\rangle^{-\alpha-1/2} \|f\|_{B^s_{p,p}}$$

for any function $f \in B^s_{p,p}$ with $p \in [1, \infty]$ and $s \in \mathbb{R}$ and for any $\alpha \in \mathbb{R}$.

We will denote by $[\![W_t^n]\!]$, $n=1,2,3$, the n-th Wick-power of the Gaussian random variable W_t (under $\mathbb{P}$) and recall the convenient notations $\mathbb{W}_t^2:=12[\![W_t^2]\!]$, $\mathbb{W}_t^3:=4[\![W_t^3]\!]$. Furthermore we will write $[\![(\langle \mathrm{D}\rangle^{-1/2}W_t)^n]\!], n\in\mathbb{N}$ for the n-th Wick-power of $\langle \mathrm{D}\rangle^{-1/2}W_t$. It exists for any $0<t<\infty$ and any $n\geqslant 1$ since it is easy to see that $\langle \mathrm{D}\rangle^{-1/2}W_t$ has a covariance with a diagonal behavior which can be controlled by $\log\langle t\rangle$. These Wick powers converge as $T\to\infty$ in spaces of distributions with regularities given in the following table:

W	$\mathbb{W}^2$	$s\mapsto J_s\mathbb{W}_s^3$	$[\![(\langle \mathrm{D}\rangle^{-1/2}W)^n]\!]$
$C\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{-1-}$	$C\mathscr{C}^{-1/2-}\cap L^2\mathscr{C}^{-1/2-}$	$C\mathscr{C}^{0-}$

Table 3.1. Regularities of the various stochastic objects. The domain of the time variable is understood to be $[0,\infty]$, $C\mathscr{C}^\alpha=C([0,\infty];\mathscr{C}^\alpha)$ and $L^2\mathscr{C}^\alpha=L^2(\mathbb{R}_+;\mathscr{C}^\alpha)$. Estimates in these norms holds a.s. and in $L^p(\mathbb{P})$ for all $p\geqslant 1$ (see Chapter 2).

We denote by $\mathbb{H}_a$ the space of $(\mathscr{F}_t)_{t\geqslant 0}$-progressively measurable processes which are $\mathbb{P}$-almost surely in $\mathcal{H}:=L^2(\mathbb{R}_+\times\Lambda)$. We say that an element v of $\mathbb{H}_a$ is a *drift*. Below we will need also drifts belonging to $\mathcal{H}^\alpha:=L^2(\mathbb{R}_+;H^\alpha(\Lambda))$ for some $\alpha\in\mathbb{R}$ where $H^\alpha(\Lambda)$ is the Sobolev space of regularity $\alpha\in\mathbb{R}$ and we will denote the corresponding space with $\mathbb{H}_a^\alpha$. For any $v\in\mathbb{H}_a$ define the measure $\mathbb{Q}^v$ on Ω by

$$\frac{\mathrm{d}\mathbb{Q}^v}{\mathrm{d}\mathbb{P}}=\exp\left[\int_0^\infty v_s\mathrm{d}X_s-\frac{1}{2}\int_0^\infty\|v_s\|_{L^2}^2\mathrm{d}s\right].$$

Denote with $\mathbb{H}_c\subseteq\mathbb{H}_a$ the set of drifts $v\in\mathbb{H}_a$ for which $\mathbb{Q}^v(\Omega)=1$, and set $W^v:=W-I(v)$, where

$$I_t(v)=\int_0^t J_s v_s\mathrm{d}s.$$

We will need also the following objects. For all $t\geqslant 0$ let $\theta_t:\mathbb{R}^3\to[0,1]$ be a smooth function such that

$$\begin{array}{lcl}\theta_t(\xi)\sigma_s(\xi) & = & 0 \text{ for } s\geqslant t,\\ \theta_t(\xi) & = & 1 \text{ for } |\xi|\leqslant t/2 \quad \text{provided that } t\geqslant T_0\end{array}\tag{3.5}$$

for some $T_0>0$. For example one can fix smooth functions $\tilde{\theta}$, η: $\mathbb{R}^3\to\mathbb{R}_+$ such that $\tilde{\theta}(\xi)=1$ if $|\xi|\leqslant 1/2$ and $\tilde{\theta}(\xi)=0$ if $|\xi|\geqslant 2/3$, $\eta(\xi)=1$ if $|\xi|\leqslant 1$ and $\eta(\xi)=0$ if $|\xi|\geqslant 2$. Then let $\tilde{\theta}_t(\xi):=\tilde{\theta}(\xi/t)$ and define

$$\theta_t(\xi):=(1-\eta(\xi))\tilde{\theta}_t(\xi)+\zeta(t)\eta(\xi)\tilde{\theta}_t(\xi),$$

where $\zeta(t):\mathbb{R}_+\to\mathbb{R}$ is a smooth function such that $\zeta(t)=0$ for $t\leqslant 10$ and $\zeta(t)=1$ for $t\geqslant 3$. Then eq. (3.5) holds with $T_0=3$. Let

$$f^\flat:=\theta(\mathrm{D})f\tag{3.6}$$

for any $f\in\mathscr{S}'(\Lambda)$.

Our aim here to study the measures ν_T defined on $\mathscr{C}^{-1/2-\varepsilon}$ as

$$\frac{\mathrm{d}\nu_T}{\mathrm{d}\mathbb{P}}=e^{-V_T(W_T)}$$

with

$$V_T(\varphi):=\lambda\int_\Lambda(\varphi^4-a_T\varphi^2+b_T)\mathrm{d}x,\qquad \varphi\in C^\infty(\Lambda),\tag{3.7}$$

and suitable a_T, $b_T\to\infty$. For convenience the measure ν_T is not normalized and, wrt. to the notations in the introduction we have

$$\frac{\mathrm{d}\mathbb{P}^T}{\mathrm{d}\nu_T}=\frac{1}{\nu_T(\Omega)}.$$

Recall the following results of Chapter 2.

THEOREM 3.1. *For any $a_T,b_T\in\mathbb{R}$, and $f:\mathscr{C}^{-1/2-\varepsilon}(\Lambda)\to\mathbb{R}$ with linear growth let*

$$V_T^f(\varphi):=f(\varphi)+V_T(\varphi),$$

where V_T is given by (3.7). Then the variational formula

$$-\log \mathbb{E}\big[e^{-V_T^f(W_T)}\big] = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[V_T^f(W_T + I_T(u)) + \frac{1}{2}\int_0^T \|u_t\|_{L^2(\Lambda)}^2 \mathrm{d}t\right] \tag{3.8}$$

holds for any finite T.

This is a consequence of the more general Boué–Dupuis formula.

THEOREM 3.2. (BD FORMULA) *Assume $F: C([0,T], C^\infty(\Lambda)) \to \mathbb{R}$, be Borel measurable and such that there exist $p, q \in (1,\infty)$, with $1/p + 1/q = 1$, $\mathbb{E}[|F(W)|^p] < \infty$ and $\mathbb{E}[|e^{-F(W)}|^q] < \infty$ (where we can regard W as an element of $C([0,T], C^\infty(\Lambda))$ by restricting to $[0,T]$). Then*

$$-\log \mathbb{E}[e^{-F(W)}] = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[F(W + I(u)) + \frac{1}{2}\int_0^T \|u_s\|_{L^2(\Lambda)}^2\right]. \tag{3.9}$$

We will use several times below eq. (3.9) in order to control exponential integrability of various functionals. By a suitable choice of renormalization and a change of variables in the control problem (3.8) we were able in Chapter 2 to control the functional in Theorem 3.1 uniformly up to infinity.

THEOREM 3.3. *There exist a sequence $(a_T, b_T)_T$ with $a_T, b_T \to \infty$ as $T \to \infty$, such that*

$$\begin{aligned} &\mathbb{E}\left[V_T^f(W_T + I_T(u)) + \frac{1}{2}\int_0^T \|u_t\|_{L^2(\Lambda)} \mathrm{d}t\right] \\ = \; &\mathbb{E}\left[\Psi_T^f(W, I(u)) + \lambda \int_\Lambda (I_T(u))^4 + \frac{1}{2}\|l^T(u)\|_{\mathcal{H}}^2\right] \end{aligned}$$

where (recall that $I_t^\flat(u) = \theta(\mathrm{D}) I_t(u)$ by (3.6))

$$l_t^T(u) := u_t + \lambda \mathbb{1}_{t \leqslant T} J_t \mathbb{W}_t^3 + \lambda \mathbb{1}_{t \leqslant T} J_t(\mathbb{W}_t^2 \succ I_t^\flat(u)) \tag{3.10}$$

and the functionals $\Psi_T^f: C([0,T], C^\infty(\Lambda)) \times C([0,T], C^\infty(\Lambda)) \to \mathbb{R}$ satisfy the following bound

$$|\Psi_T^f(W, I(u))| \leqslant Q_T(W) + \frac{1}{4}(\|I_T(u)\|_{L^4}^4 + \|l^T(u)\|_{\mathcal{H}}^2)$$

where $Q_T(W)$ is a function of W independent of u and such that $\sup_T \mathbb{E}[|Q_T(W)|] < \infty$.

As a consequence we obtain the following corollary (cfr. Corollary 2.2 in Chapter 2)

COROLLARY 3.4. *For $f: \mathscr{C}^{-1/2-\varepsilon}(\Lambda) \to \mathbb{R}$ with linear growth the bound*

$$-C \leqslant \mathbb{E}_{\nu^T}[e^f] \leqslant C,$$

holds, with a constant C independent of T. In particular ν_T is tight on $\mathscr{C}^{-1/2-\varepsilon}$.

3.3. CONSTRUCTION OF THE DRIFT MEASURE

We start now to implement the strategy discussed in the introduction: identify a translated measure sufficiently similar to Φ_3^4. Intuitively the Φ_3^4 measure should give rise to a canonical process which is a shift of the Gaussian Free Field with a drift of the form given by eq. (3.10). Indeed this drift u should be the optimal drift in the variational formula. A small twist is given by the fact that the relevant Gaussian Free Field entering these considerations is not the process $W = W(X)$ but that obtained from the shifted canonical process $X_t^u = X_t - \int_0^t u_s \mathrm{d}s$ which we denote by

$$W^u := W(X^u) = W - I(u).$$

Moreover, to prevent explosion at finite time, we have to modify the drift in large scales and add a coercive term.This will also allow later to prove some useful estimates. As a consequence, we define the functional

$$\Xi_s(W, u) := -\lambda J_s \mathbb{W}_s^3 - \lambda \mathbb{1}_{\{s \geqslant \bar{T}\}} J_s(\mathbb{W}_s^2 \succ I_s^\flat(u)) + J_s \langle \mathrm{D} \rangle^{-1/2}(\llbracket(\langle \mathrm{D} \rangle^{-1/2} W_s)^n\rrbracket), \qquad s \geqslant 0, \tag{3.11}$$

where $\bar{T} > 0$, $n \in \mathbb{N}$ are constants which will be fixed later on and where we understand all the Wick renormalizations to be given functions of W, i.e. polynomials in W where the constants are determined according to the law of W under $\mathbb{P}$. We look now for the solution u of the equation

$$u = \Xi(W^u, u) = \Xi(W - I(u), u). \tag{3.12}$$

Expanding the Wick polynomials appearing in $\Xi(W - I(u), u)$ we obtain the equation

$$\begin{aligned} u_s &= \Xi(W - I(u), u) \\ &= -\lambda J_s[\mathbb{W}_s^3 - \mathbb{W}_s^2 I_s(u) + 12 W_s (I_s(u))^2 - 4(I_s(u))^3] \\ &\quad -\lambda \mathbb{1}_{\{s \geqslant \bar{T}\}} J_s[((\mathbb{W}_s^2 - 24 W_s I_s(u) + 12(I_s(u))^2)) \succ I_s^\flat(u)] \\ &\quad + \sum_{i=0}^n \binom{n}{i} J_s \langle \mathrm{D} \rangle^{-1/2} [\llbracket (\langle \mathrm{D} \rangle^{-1/2} W_s)^i \rrbracket (-\langle \mathrm{D} \rangle^{-1/2} I_s(u))^{n-i}] \end{aligned} \tag{3.13}$$

for all $s \geqslant 0$. This is an integral equation for $t \mapsto u_t$ with smooth coefficients depending smoothly on W and can be solved via standard methods. Since the coefficients are of polynomial growth the solution could explode in finite time. Note that for any finite time the process $(u_s)_{s>0}$ has bounded spectral support. As a consequence we can solve the equation in L^2 and as long as $\int_0^t \|u\|_{L^2}^2 \mathrm{d}s$ is finite we can see from the equation that $\sup_{s \leqslant t} \|u_s\|_{L^2}^2$ is finite. By the existence of local solutions we have that, for all $N \geqslant 0$, the stopping time

$$\tau_N := \inf \left\{ t \geqslant 0 \middle| \int_0^t \|u_s\|_{L^2}^2 \mathrm{d}s \geqslant N \right\},$$

is strictly positive $\mathbb{P}$-almost surely and u exists up to the (explosion) time $T_{\exp} := \sup_{N \in \mathbb{N}} \tau_N$. The following lemma will help show that $\mathbb{P}$-almost surely $T_{\exp} = +\infty$ and will also be very useful below.

LEMMA 3.5. *Let*

$$\mathrm{Aux}_s(W, w) := \sum_{i=0}^n \binom{n}{i} J_s \langle \mathrm{D} \rangle^{-1/2} (\llbracket (\langle \mathrm{D} \rangle^{-1/2} W_s)^i \rrbracket (\langle \mathrm{D} \rangle^{-1/2} I_s(w))^{n-i}).$$

then we have

$$\mathbb{E} \int_0^t \|w_s\|_{L^2}^2 \mathrm{d}s + \sup_{s \leqslant t} \mathbb{E} \|I_s(w)\|_{W^{-1/2, n+1}}^{n+1} \lesssim 1 + \int_0^t (2\mathbb{E} \|w_s + g_s\|_{L^2}^2 + 4\mathbb{E} \|g_s - \mathrm{Aux}_s(W, w)\|_{L^2}^2) \mathrm{d}s,$$

uniformly in $t \geqslant 0$, for any pair of adapted processes $w, g \in L^2(\mathbb{P}, \mathcal{H})$ such that

$$\mathbb{E} \int_0^t \|g_s - \mathrm{Aux}_s(W, w)\|_{L^2}^2 \, \mathrm{d}s < \infty.$$

Proof. Take $\iota_n = \inf \{ t > 0 : \int_0^t \|w_s\|_{L^2}^2 \mathrm{d}s > N \}$. By Ito's formula we have

$$\int_0^{t \wedge \iota_N} \int_\Lambda \mathrm{Aux}_s(W, w) w_s \mathrm{d}s = \overline{\mathrm{Aux}}_{t \wedge \iota_N}(W, w) + \text{martingale}$$

where

$$\overline{\mathrm{Aux}}_t(W, w) := \sum_{i=0}^n \frac{1}{n+1-i} \binom{n}{i} \int_\Lambda (\llbracket (\langle \mathrm{D} \rangle^{-1/2} W_t)^i \rrbracket (\langle \mathrm{D} \rangle^{-1/2} I_t(w))^{n+1-i}).$$

Integrating over the probability space and using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E} \left(\int_0^{t \wedge \iota_N} \|w_s\|_{L^2}^2 \mathrm{d}s + 4\, \overline{\mathrm{Aux}}_{t \wedge \iota_N}(W, w) \right) \\ = \;& \mathbb{E} \left[\int_\Lambda \mathbb{1}_{\{t \leqslant \iota_N\}} (w_t^2 + 4\, \mathrm{Aux}_t(W, w) w_t) \right] \\ \leqslant \;& \mathbb{E} \left[\mathbb{1}_{\{t \leqslant \iota_N\}} \left(2\|w_t + g_t\|_{L^2}^2 + 4 \int_\Lambda (\mathrm{Aux}_t(W, w) - g_t) w_t - \|w_t\|_{L^2}^2 \right) \right] \\ \leqslant \;& 2\mathbb{E} \mathbb{1}_{\{t \leqslant \iota_N\}} \|w_t + g_t\|_{L^2}^2 + 4\mathbb{E} \mathbb{1}_{\{t \leqslant \iota_N\}} \|g_t - \mathrm{Aux}_t(W, w)\|_{L^2}^2. \end{aligned}$$

where g_t is an arbitrary function. By Lemma 3.11 below, we have constants c, C and a random variable $Q_T(W)$ such that

$$\sup_{t\in\mathbb{R}}\sup_{N\in\mathbb{N}}\mathbb{E}[|Q_{t\wedge\iota_N}(W)|]<\infty,$$

and for any stopping time τ

$$\begin{aligned} c\int_0^\tau \|w_s\|_{L^2}^2\mathrm{d}s + c\|I_\tau(w)\|_{W^{-1/2,n+1}}^{n+1} - Q_\tau(W) &\leqslant \int_0^\tau \|w_s\|_{L^2}^2\mathrm{d}s + \overline{\mathrm{Aux}}_\tau(W,w)\\ &\leqslant C\|I_\tau(w)\|_{W^{-1/2,n+1}}^{n+1} + C\int_0^\tau \|w_s\|_{L^2}^2\mathrm{d}s + Q_\tau(W). \end{aligned}$$

As a consequence, we deduce

$$\begin{aligned} &\mathbb{E}\int_0^t \mathbb{1}_{\{s\leqslant\iota_N\}}\|w_s\|_{L^2}^2\mathrm{d}s + \mathbb{E}\|I_{\iota_N}(w)\|_{W^{-1/2,n+1}}^{n+1}\\ \lesssim\ & 1+\int_0^t (2\mathbb{E}\mathbb{1}_{\{s\leqslant\iota_N\}}\|w_s+g_s\|_{L^2}^2 + 4\mathbb{1}_{\{s\leqslant\iota_N\}}\mathbb{E}\|g_s-\mathrm{Aux}_s(W,w)\|_{L^2}^2)\mathrm{d}s.\\ \leqslant\ & 1+\int_0^t (2\mathbb{E}\|w_s+g_s\|_{L^2}^2 + 4\mathbb{E}\|g_s-\mathrm{Aux}_s(W,w)\|_{L^2}^2)\mathrm{d}s. \end{aligned}$$

And we can conclude by sending $N\to\infty$ and using Fatou's Lemma. □

In particular, taking $w=-\mathbb{1}_{t\leqslant\tau_N}u$ and $g=-w$, we have

$$\mathbb{E}\int_0^t \|\mathbb{1}_{s\leqslant\tau_N}u_s\|_{L^2}^2\mathrm{d}s + \sup_{s\leqslant t}\mathbb{E}\|I_s(\mathbb{1}_{\cdot\leqslant\tau_N}u)\|_{W^{-1/2,n+1}}^{n+1} \lesssim 1+\int_0^t \mathbb{E}(\mathbb{1}_{s\leqslant\tau_N}\|u_s-\mathrm{Aux}_s(W,-u)\|_{L^2}^2)\mathrm{d}s,$$

for all $t\leqslant T$, where, using (3.13),

$$\begin{aligned} u_s-\mathrm{Aux}_s(W,-u) \ =\ & -\lambda J_s[\mathbb{W}_s^3-\mathbb{W}_s^2 I_s(u)+12W_s(I_s(u))^2-4(I_s(u))^3]\\ & -\lambda\mathbb{1}_{\{s\geqslant\bar{T}\}}J_s[((\mathbb{W}_s^2-24W_sI_s(u)+12(I_s(u))^2))\succ I_s^\flat(u)]. \end{aligned} \tag{3.14}$$

Then, for any $s\leqslant T$ we have

$$\mathbb{E}(\mathbb{1}_{s\leqslant\tau_N}\|u_s-\mathrm{Aux}_s(W,-u)\|_{L^2}^2)\leqslant C_T+\mathbb{E}\|I_s(\mathbb{1}_{s\leqslant\tau_N}u)\|_{W^{-1/2,n+1}}^{n+1},$$

provided n is chosen sufficiently large. Using Gronwall's inequality this gives $\mathbb{E}\int_0^T\|\mathbb{1}_{s\leqslant\tau_N}u_s\|_{L^2}^2\mathrm{d}s\lesssim C_T$, and we can let $N\to\infty$ to obtain

$$\mathbb{E}\int_0^T\|u_s\|_{L^2}^2\mathrm{d}s\lesssim C_T$$

which implies $T_{\exp}=+\infty$. In addition and by construction, the process $u_t^N:=\mathbb{1}_{\{t\leqslant\tau_N\}}u_t$ satisfies Novikov's condition, so it is in $\mathbb{H}_c$ and Girsanov's transformation allows us to define the probability measure $\mathbb{Q}^{u^N}$ on $C(\mathbb{R}_+,\mathscr{C}^{-1/2-\varepsilon}(\Lambda))$ given by

$$\mathrm{d}\mathbb{Q}^{u^N}:=e^{\int_0^\infty u_s^N\mathrm{d}X_s-\frac{1}{2}\int_0^\infty\|u_s^N\|_{L^2(\Lambda)}^2\mathrm{d}s}\mathrm{d}\mathbb{P},$$

under which $X_t^{u^N}=X_t-\int_0^t u_s^N\mathrm{d}s$ is a cylindrical Brownian motion. Moreover, under $\mathbb{Q}^{u^N}$ the process $(W_t^{u^N}:=\int_0^t J_s\mathrm{d}X_s^{u^N})_{t\geqslant0}$ has the same law as $(W_t)_{t\geqslant0}$ under $\mathbb{P}$. We observe also that $W_s^{u^N}=W_s^u$ for $0\leqslant s\leqslant\tau_N$ and that u satisfies the equation

$$u_s=-\lambda J_s\mathbb{W}_s^{u,3}-\lambda\mathbb{1}_{\{s\geqslant\bar{T}\}}J_s(\mathbb{W}_s^{u,2}\succ I_t^\flat(u))+J_s\langle\mathrm{D}\rangle^{-1/2}([\![(\langle\mathrm{D}\rangle^{-1/2}W_s^u)^n]\!]),\qquad s\in[0,\tau_N], \tag{3.15}$$

where we introduced the notations $\mathbb{W}_s^{u,3}:=4[\![(W_s^u)^3]\!]$ and $\mathbb{W}_s^{u,2}:=12[\![(W_s^u)^2]\!]$. Note that here the Wick powers are still taken to be given functions of W, i.e we are still taking the Wick ordering with respect to the law of W under $\mathbb{P}$ (or, equivalently, the law of W^{u^N} under $\mathbb{Q}^{u^N}$).

If we think of the terms containing W^u as given (that is, we ignore their dependence on u), eq. (3.15) is a linear integral equation in u which can be estimated via Gronwall-type arguments. In order to do so, let us denote by $U: H \mapsto \hat{u}$ the solution map of the equation

$$\hat{u} = \Xi(H, \hat{u}). \tag{3.16}$$

This last equation is linear and therefore has nice global solutions (let's say in $C(\mathbb{R}_+, L^2)$) and by uniqueness and eq. (3.15) we have $u_t = U_t(W^u)$ for $t \in [0, T_{\exp})$. From this perspective the residual dependence on u will not play any role since under the shifted measure the law of the process W^u does not depend on u. By standard paraproduct estimates (see Appendix A) we have

$$\begin{aligned}\|I_t(u)\|_{L^\infty} &\lesssim \tilde{H}_t + \int_0^t \mathbb{1}_{\{s \geqslant \bar{T}\}} \|J_s^2(\mathbb{W}_s^{u,2} \succ I_s^\flat(u))\|_{L^\infty} \mathrm{d}s \\ &\lesssim \tilde{H}_t + \bar{T}^{-\varepsilon} \int_0^t \langle s \rangle^{-3/2} \|\mathbb{W}_s^{u,2}\|_{\mathscr{C}^{-1-\varepsilon}} \|I_s^\flat(u)\|_{L^\infty} \mathrm{d}s,\end{aligned}$$

where we have crucially exploited the presence of the cutoff $\mathbb{1}_{\{s \geqslant \bar{T}\}}$ to introduce the small factor $\bar{T}^{-\varepsilon}$ and we have employed the notation

$$\tilde{H}_t := \int_0^t [\|J_s^2 \mathbb{W}_s^{u,3}\|_{L^\infty} + \|J_s \langle \mathrm{D} \rangle^{-1/2} (⟦(\langle \mathrm{D} \rangle^{-1/2} W_s^u)^n⟧)\|_{L^\infty}] \mathrm{d}s$$

$$\lesssim \int_0^t \frac{1}{\langle s \rangle^{1/2-\varepsilon}} \|J_s \mathbb{W}_s^{u,3}\|_{\mathscr{C}^{-1/2-\varepsilon}} \mathrm{d}s + \int_0^t \frac{1}{\langle s \rangle^{3/2}} \|⟦(\langle \mathrm{D} \rangle^{-1/2} W_s^u)^n⟧\|_{H^{-1/2}} \mathrm{d}s.$$

By Gronwall's lemma,

$$\sup_{t \leqslant \tau_N} \|I_t(u)\|_{L^\infty} \lesssim \tilde{H}_{\tau_N} \exp\left(C \bar{T}^{-\varepsilon} \int_0^{\tau_N} \|\mathbb{W}_s^{u,2}\|_{\mathscr{C}^{-1-\varepsilon}} \frac{\mathrm{d}s}{\langle s \rangle^{1+\varepsilon}} \right). \tag{3.17}$$

Under $\mathbb{Q}^{u^N}$, the terms in $\tilde{H}_{\tau_N}$ are in all the L^p spaces by hypercontractivity and moreover for any $p \geqslant 1$ one can choose $\bar{T}$ large enough so that also the exponential term is in L^p. Using eq. (3.15) it is then not difficult to show that $\mathbb{E}_{\mathbb{Q}^{u^{N_1}}}[\|u^{N_2}\|_{\mathcal{H}^{-1/2-\varepsilon}}^p] < \infty$ for any $p > 1$ (again provided we take $\bar{T}$ large enough depending on p) as long as $N_1 > N_2$. By the spectral properties of J and the equation for u, the process $t \mapsto \mathbb{1}_{\{t \leqslant T\}} u_t$ is spectrally supported in a ball of radius T, so we get in particular that

$$\mathbb{E}_{\mathbb{Q}^{u^{N_1}}}\left[\int_0^{\tau_{N_2} \wedge T} \|u_s\|_{L^2}^2 \mathrm{d}s \right] \lesssim T^{1+\varepsilon},$$

uniformly for any choice of $N_1 \geqslant N_2 \geqslant 0$.

Lemma 3.6. *The family $(\mathbb{Q}^{u^N})_N$ weakly converges to a limit $\mathbb{Q}^u$ on $C(\mathbb{R}_+, \mathscr{C}^{-3/2-\varepsilon})$. Under $\mathbb{Q}^u$ it holds $T_{\exp} = \infty$ almost surely and $\mathrm{Law}_{\mathbb{Q}^u}(X^u) = \mathrm{Law}_{\mathbb{P}}(X)$. Moreover for any finite T*

$$\frac{\mathrm{d}\mathbb{Q}^u|_{\mathscr{F}_T}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}_T}} = \exp\left(\int_0^T u_s \mathrm{d}X_s - \frac{1}{2} \int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s \right).$$

Proof. Consider the filtration $(\mathscr{G}_N = \mathscr{F}_{\tau_N})_N$ and observe that $(\mathbb{Q}^{u^N}|_{\mathscr{G}_N})_N$ is a consistent family of inner regular probability distributions and therefore there exists a unique extension $\mathbb{Q}^u$ to $\mathscr{G}_\infty = \vee_N \mathscr{G}_N$. Next observe that $\{T_{\exp} < \infty\} = \bigcup_{T \in \mathbb{N}} \{T_{\exp} < T\} \subset \bigcup_{T \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} \{\tau_N < T\}$ and that for any $N, T < \infty$, we have

$$\mathbb{E}_{\mathbb{Q}^u}\left[\int_0^{\tau_N \wedge T} \|u_s\|_{L^2}^2 \mathrm{d}s \right] = \mathbb{E}_{\mathbb{Q}^{u^N}}\left[\int_0^{\tau_N \wedge T} \|u_s\|_{L^2}^2 \mathrm{d}s \right] \lesssim T^{1+\varepsilon}.$$

On the event $\{\tau_N \leqslant T\}$ we have

$$\int_0^{\tau_N \wedge T} \|u_s\|_{L^2}^2 \mathrm{d}s = N,$$

and therefore we also have $\mathbb{Q}^u(\{\tau_N \leqslant T\}) \leqslant CT^{1+\varepsilon}N^{-1}$ which in turn implies $\mathbb{Q}^u(T_{\exp} < T) = 0$. This proves that $T_{\exp} = +\infty$ under $\mathbb{Q}^u$, almost surely. As a consequence we can extend $\mathbb{Q}^u$ to all of $\mathscr{F} = \vee_T \mathscr{F}_T$ since for any $A \in \mathscr{F}_T$ we can set

$$\mathbb{Q}^u(A) = \mathbb{Q}^u(A \cap \{T_{\exp} = +\infty\}) = \lim_N \mathbb{Q}^u(A \cap \{T_{\exp} = +\infty, \tau_N \geqslant T\}) = \lim_N \mathbb{Q}^{u^N}(A \cap \{\tau_N \geqslant T\}).$$

If $A \in \mathscr{F}_T$ then by monotone convergence

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_A(X^u)] &= \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}(X^u)] = \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^{u^N}}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}(X^{u^N})] \\ &= \lim_{N\to\infty} \mathbb{E}_{\mathbb{P}}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}(X)] = \mathbb{E}_{\mathbb{P}}[\mathbb{1}_A(X)] \end{aligned}$$

This establishes that $\mathrm{Law}_{\mathbb{Q}^u}(X^u) = \mathrm{Law}_{\mathbb{P}}(X)$. On the other hand if $A \in \mathscr{F}_T$ we have, using the martingale property of the Girsanov density,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_A] &= \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}] = \lim_{N\to\infty} \mathbb{E}_{\mathbb{Q}^{u^N}}[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}}] \\ &= \lim_{N\to\infty} \mathbb{E}\Big[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}} e^{\int_0^{\tau_N} u_s \mathrm{d}X_s - \frac{1}{2}\int_0^{\tau_N} \|u_s\|_{L^2}^2 \mathrm{d}s}\Big] \\ &= \lim_{N\to\infty} \mathbb{E}\Big[\mathbb{1}_{A\cap\{T\leqslant\tau_N\}} e^{\int_0^T u_s \mathrm{d}X_s - \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s}\Big]. \\ &= \mathbb{E}\Big[\mathbb{1}_A e^{\int_0^T u_s \mathrm{d}X_s - \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s}\Big] \end{aligned}$$

by monotone convergence and the fact that $T_{\exp} = \infty$ $\mathbb{P}$-almost surely. Therefore

$$\frac{\mathrm{d}\mathbb{Q}^u|_{\mathscr{F}_T}}{\mathrm{d}\mathbb{P}|_{\mathscr{F}_T}} = e^{\int_0^T u_s \mathrm{d}X_s - \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s},$$

as claimed. □

The following lemma will also be useful in the sequel and it is a consequence of the above discussion:

Lemma 3.7. *For any $p > 1$ there exists a suitable choice of $\bar{T}$ such that*

$$\mathbb{E}_{\mathbb{Q}^u}\Big[\sup_{t\geqslant 0} \|I_t(u)\|_{L^\infty}^p\Big] < \infty.$$

Proof. This follows from the bound (3.17), after choosing $\bar{T}$ large enough. □

3.3.1. Proof of absolute continuity

In this section we prove that the measure μ_T is absolutely continuous with respect to the measure $\mathbb{Q}^u$ we constructed in Lemma 3.6. First recall that the measures ν_T defined on Ω as

$$\frac{\mathrm{d}\nu_T}{\mathrm{d}\mathbb{P}} = e^{-V_T(W_T)}$$

can be described, using Lemma 3.6, as a perturbation of $\mathbb{Q}^u$ with density D_T given by

$$D_T := \frac{\mathrm{d}\nu_T}{\mathrm{d}\mathbb{Q}^u}\bigg|_{\mathscr{F}_T} = \frac{\mathrm{d}\nu_T}{\mathrm{d}\mathbb{P}}\bigg|_{\mathscr{F}_T} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^u}\bigg|_{\mathscr{F}_T} = e^{-V_T(W_T) - \int_0^T u \mathrm{d}X + \frac{1}{2}\int_0^T \|u_t\|_{L^2}^2 \mathrm{d}t},$$

at least on $\mathscr{F}_T$.

Lemma 3.8. *There exists a $p > 1$, such that for any $K > 0$,*

$$\sup_T \mathbb{E}_{\mathbb{Q}^u}\big[|D_T|^p \mathbb{1}_{\{\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}} \leqslant K\}}\big] < \infty.$$

in particular, the family $(D_T)_T$ is uniformly integrable under $\mathbb{Q}^u$.

Proof. The proof of the first claim is given in Section 3.3.2 below. For the second claim fix $\varepsilon > 0$. Our aim is to show that there there exists $\delta > 0$ such that $\mathbb{Q}^u(A) < \delta$ implies $\int_A D_T \mathrm{d}\mathbb{Q}^u < \varepsilon$. From corollary 3.4 for any $\varepsilon > 0$ there exists a $K > 0$ such that

$$\varepsilon / 2 > \nu_T(\{\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}} \geqslant K\}) = \int_{\{\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}} \geqslant K\}} D_T \mathrm{d}\mathbb{Q}^u.$$

Then for any $A \in \mathscr{F}$ such that $\mathbb{Q}^u(A)^{(p-1)/p} < \varepsilon / \left(2 \sup_T \mathbb{E}_{\mathbb{Q}^u}\left[|D_T|^p \mathbb{1}_{\{\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}} \leqslant K\}}\right]\right)$

$$\begin{aligned}
& \int_A D_T \mathrm{d}\mathbb{Q}^u \\
= & \int_{A \cap \{\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}} \geqslant K\}} D_T \mathrm{d}\mathbb{Q}^u + \int_{A \cap \{\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}} \leqslant K\}} D_T \mathrm{d}\mathbb{Q}^u \\
\leqslant & \ \varepsilon / 2 + \sup_T \mathbb{E}_{\mathbb{Q}^u}\left[|D_T|^p \mathbb{1}_{\{\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}} \leqslant K\}}\right] \mathbb{Q}^u(A)^{(p-1)/p} \\
\leqslant & \ \varepsilon
\end{aligned}$$

□

COROLLARY 3.9. *The family of measures $(\nu_T)_{T \geqslant 0}$ is sequentially compact w.r.t. strong convergence on $(\Omega, \mathscr{F})$. Furthermore any accumulation point is absolutely continuous with respect to $\mathbb{Q}^u$.*

Proof. We choose a sub-sequence (not relabeled) such that $D_T \to D_\infty$ weakly in $L^1(\mathbb{Q}^u)$, for some $D_\infty \in L^1(\mathbb{Q}^u)$. It always exists by uniform integrability. We now claim that for any $A \in \mathscr{F}$

$$\lim_{T \to \infty} \nu_T(A) = \int_A D_\infty \mathrm{d}\mathbb{Q}^u.$$

It is enough to check this for $A \in \mathscr{F}_S$ for any $S \in \mathbb{R}_+$ since these generate $\mathscr{F}$. But there we have for $T \geqslant S$,

$$\nu_T(A) = \int_A D_T \mathrm{d}\mathbb{Q}^u \to \int_A D_\infty \mathrm{d}\mathbb{Q}^u$$

by weak L^1 convergence. □

Recall that the Φ^4_3 measure can be defined as a weak limit of the measures ν^T on $\mathscr{C}^{-1/2-\varepsilon}$ given by

$$\int f(\varphi) \nu^T(\mathrm{d}\varphi) = \int f(\varphi) e^{-V_T(\varphi)} \vartheta_T(\mathrm{d}\varphi) = \mathbb{E}_{\mathbb{P}}[f(W_T) e^{-V_T(W_T)}]$$

where ϑ_T is the gaussian measure with covariance $\rho_T^2(\mathrm{D}) \langle \mathrm{D} \rangle^{-2}$. From this together with the above considerations we see that any accumulation point ν^∞ of ν^T satisfies

$$\tilde{\nu}_\infty(A) = \mathbb{E}_{\mathbb{Q}^u}[\mathbb{1}_A(W_\infty) D_\infty] \tag{3.18}$$

for some $D_\infty \in L^1(\mathbb{Q}^u)$.

3.3.2. L^p bounds

Now we will prove local L^p-bounds on the density D_T. In the sequel we will denote $\tilde{W} = W^u$, with u satisfying (3.13), namely $u = U(\tilde{W})$. Before we proceed let us study how the functional $U(\tilde{W})$ behaves under shifts of $\tilde{W}$, since later we will want to apply the Boué–Dupuis formula and this kind of behavior will be crucial. Let $w \in L^2([0, \infty) \times \Lambda)$ and denote

$$u^w := U(\tilde{W} + I(w)) \quad \text{and} \quad h^w := U(\tilde{W} + I(w)) + w = u^w + w.$$

The process h^w satisfies

$$h^w - w = u^w = \Xi(\tilde{W} + I(w), u^w).$$

More explicitly, for all $s \geqslant 0$ we have

$$\begin{aligned} h_s^w - w_s &= -4\lambda J_s[\![\tilde{W}_s^3]\!] - 12\lambda J_s[\![\tilde{W}_s^2]\!] I_s(w) - 12\lambda J_s \tilde{W}_s (I_s(w))^2 - 4\lambda J_s (I_s(w))^3 \\ & \quad -12\lambda \mathbb{1}_{\{s \geqslant \bar{T}\}} J_s([\![\tilde{W}_s^2]\!] \succ I_s^\flat(u^w)) - 24\lambda \mathbb{1}_{\{s \geqslant \bar{T}\}} (J_s(\tilde{W}_s I_s(w) \succ I_s^\flat(u^w))) \\ & \quad -12\lambda \mathbb{1}_{\{s \geqslant \bar{T}\}} J_s((I_s(w))^2 \succ I_s^\flat(u^w)) \\ & \quad + \sum_{i=0}^n \binom{n}{i} J_s[\![(\langle \mathrm{D} \rangle^{-1/2} \tilde{W}_s)^i]\!] (\langle \mathrm{D} \rangle^{-1/2} I_s(w))^{n-i}. \end{aligned}$$

Decomposing

$$[\![\tilde{W}_s^2]\!] I_s(w) = [\![\tilde{W}_s^2]\!] \succ \theta_s I_s(w) + [\![\tilde{W}_s^2]\!] \succ (1-\theta_s) I_s(w) + [\![\tilde{W}_s^2]\!] \circ I_s(w) + [\![\tilde{W}_s^2]\!] \prec I_s(w),$$

we can write

$$u^w = U(\tilde{W} + I(w)) = -4\lambda J_s[\![\tilde{W}_s^3]\!] - 12\lambda J_s([\![\tilde{W}_s^2]\!] \succ I_s^\flat(h^w)) + r_s^w, \tag{3.19}$$

with

$$\begin{aligned} r_s^w &= -12\lambda J_s[\![\tilde{W}_s^2]\!] \succ (1-\theta_s) I_s(w) - 12\lambda J_s([\![\tilde{W}_s^2]\!] \circ I_s(w)) - 12\lambda J_s[\![\tilde{W}_s^2]\!] \prec I_s(w) \\ & \quad -12\lambda J_s \tilde{W}_s (I_s(w))^2 - 4\lambda J_s (I_s(w))^3 - 24\lambda \mathbb{1}_{\{s \geqslant \bar{T}\}} (J_s(\tilde{W}_s I_s(w) \succ \theta_s I_s^\flat(u^w))) \\ & \quad -12\lambda \mathbb{1}_{\{s \geqslant \bar{T}\}} J_s((I_s(w))^2 \succ I_s^\flat(u^w)) + 12\lambda \mathbb{1}_{\{s < \bar{T}\}} J_s([\![\tilde{W}_s^2]\!] \succ I_s^\flat(u^w)) \\ & \quad + \sum_{i=0}^n \binom{n}{i} J_s \langle \mathrm{D} \rangle^{-1/2} [[\![(\langle \mathrm{D} \rangle^{-1/2} W_s)^i]\!] (\langle \mathrm{D} \rangle^{-1/2} I_s(w))^{n-i}]. \end{aligned} \tag{3.20}$$

The first two terms in (3.19) will be used for renormalization while the remainder r^w contains terms of higher regularity which will have to be estimated in the sequel.

Proof of Lemma 3.8. Observe that

$$\mathbb{1}_{\{\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}} \leqslant K\}} \gtrsim_{K,n} \exp(-\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}}^n) = \exp(-\|\tilde{W}_\infty + I_\infty(U(\tilde{W}))\|_{\mathscr{C}^{-1/2-\varepsilon}}^n)$$

and

$$|D_T|^p = e^{-p\left[V_T(\tilde{W}_T + I(U(\tilde{W}))) + \int_0^T U(\tilde{W}) \mathrm{d}\tilde{X} + \frac{1}{2}\int_0^T \|U_t(\tilde{W})\|_{L^2}^2 \mathrm{d}t\right]}.$$

Combining these two facts we have

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}^u}\big[|D_T|^p \mathbb{1}_{\{\|W\|_{\mathscr{C}^{-1/2-\varepsilon}} \leqslant K\}}\big] \\ & \gtrsim_{K,n} \mathbb{E}_{\mathbb{Q}^u}\bigg[\exp\bigg(-p\bigg(V_T(\tilde{W}_T + I_T(U(\tilde{W}))) + \int_0^T U_t(\tilde{W}) \mathrm{d}\tilde{X}_t + \frac{1}{2}\int_0^T \|U_t(\tilde{W})\|_{L^2}^2 \mathrm{d}t\bigg) \\ & \qquad - \|\tilde{W}_\infty + I_\infty(U(\tilde{W}))\|_{\mathscr{C}^{-1/2-\varepsilon}}^n\bigg)\bigg] \\ & = \mathbb{E}\bigg[\exp\bigg(-p\bigg(V_T(W_T + I_T(U(W))) + \int_0^T U_t(W) \mathrm{d}X_t + \frac{1}{2}\int_0^T \|U_t(W)\|_{L^2}^2 \mathrm{d}t\bigg) \\ & \qquad - \|W_\infty + I_\infty(U(W))\|_{\mathscr{C}^{-1/2-\varepsilon}}^n\bigg)\bigg]. \end{aligned}$$

The Boué–Dupuis formula (3.9) provides the variational bound

$$\begin{aligned} & -\log \mathbb{E}_{\mathbb{Q}^u}\big[|D_T|^p \mathbb{1}_{\{\|W\|_{\mathscr{C}^{-1/2-\varepsilon}} \leqslant K\}}\big] \\ & \quad \gtrsim \inf_{w \in \mathbb{H}_a} \mathbb{E}\bigg[p\bigg(V_T(W_T + I_T(h^w)) + \frac{1}{2}\int_0^T \|h^w\|_{L^2}^2 \mathrm{d}t\bigg) \\ & \qquad + \frac{1-p}{2}\int_0^T \|w_t\|_{L^2}^2 \mathrm{d}t + \|W_\infty + I_\infty(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^n + \frac{1}{2}\int_T^\infty \|w_t\|_{L^2}^2 \mathrm{d}t\bigg] \end{aligned}$$

where we have set $h^w = w + U(W + I(w))$ as above. Recall now that from Theorem 3.3 there exists a constant C, independent of T , such that for each h^w,

$$\mathbb{E}\bigg[p\bigg(V_T(W_T + I_T(h^w)) + \frac{1}{2}\int_0^T \|h^w\|_{L^2}^2 \mathrm{d}t\bigg)\bigg] \geqslant -C + \frac{1}{4}\mathbb{E}_{\mathbb{P}}\bigg[\lambda\|I_T(h^w)\|_{L^4}^4 + \int_0^T \|l^T(h^w)\|_{L^2}^2\bigg]$$

where

$$l_t^T(h^w) = h_t^w + \lambda \mathbb{1}_{t\leqslant T} J_t \mathbb{W}_t^3 + \lambda \mathbb{1}_{t\leqslant T} J_t(\mathbb{W}_t^2 \succ I_t^\flat(h^w)).$$

Using eq. (3.19) we compute

$$\begin{aligned}
\mathbb{1}_{t\leqslant T} l_t^T(h^w) &= \mathbb{1}_{t\leqslant T} h_t^w + \lambda \mathbb{1}_{t\leqslant T} J_t \mathbb{W}_t^3 + \lambda \mathbb{1}_{t\leqslant T} J_t(\mathbb{W}_t^2 \succ I_t^\flat(h^w)) \\
&= \mathbb{1}_{t\leqslant T}(u_t^w + w_t) + \lambda \mathbb{1}_{t\leqslant T} J_t \mathbb{W}_t^3 + \lambda \mathbb{1}_{t\leqslant T} J_t(\mathbb{W}_t^2 \succ I_t^\flat(h^w)) \\
&= \mathbb{1}_{t\leqslant T}(r_t^w + w_t).
\end{aligned}$$

At this point we need a lower bound for

$$\begin{aligned}
\mathbb{E}\bigg[\frac{1}{4}\bigg(\lambda\|I_T(h^w)\|_{L^4}^4 + \int_0^T \|r_t^w + w_t\|_{L^2}^2 \mathrm{d}t\bigg) + \frac{1-p}{2}\int_0^T \|w_t\|_{L^2}^2 \mathrm{d}t \\
+ \|W_\infty + I_\infty(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^n + \frac{1}{2}\int_T^\infty \|w_t\|_{L^2}^2 \mathrm{d}t\bigg] - C.
\end{aligned}$$

Given that we need to take $p > 1$, estimating this expression presents a difficulty in the fact that the term $\int_0^T \|w_t\|_{L^2}^2 \mathrm{d}t$ appears with a negative coefficient. Note that this term cannot easily be controlled via $\int_0^T \|r_t^w + w_t\|_{L^2}^2 \mathrm{d}t$ since the contribution r^w, see eq. (3.20), contains factors which are homogeneous in w of order up to 3. This is the reason we had to localize the estimate, introduce the "good" term $\|W_\infty + I_\infty(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^n$, and introduce the term $J_s\langle \mathrm{D}\rangle^{-1/2}(\llbracket(\langle \mathrm{D}\rangle^{-1/2}W_s)^n\rrbracket)$ in (3.11) which will help us to control the growth of r^w. Indeed in Lemma 3.10 below, a Gronwall argument will allow us to show that $\int_0^T \|w_t\|_{L^2}^2 \mathrm{d}t$ can be bounded by a combination of the other "good" terms as

$$\mathbb{E}\bigg[\int_0^T \|w\|_{L^2}^2 \mathrm{d}t\bigg] \lesssim \mathbb{E}\bigg[\|I_T^\flat(h)\|_{L^4}^4 + \|I_T^\flat(h)\|_{\mathscr{C}^{-1/2-\varepsilon}}^n + \int_0^T \|w_t + r_t^w\|_{L^2}^2 \mathrm{d}t + 1\bigg].$$

This implies that for $1 < p \ll 2$,

$$\begin{aligned}
&-\log \mathbb{E}_{\mathbb{Q}^u}\big[|D_T|^p \mathbb{1}_{\{\|W\|_{\mathscr{C}^{-1/2-\varepsilon}} \leqslant K\}}\big] \\
&\geqslant \inf_{w\in\mathbb{H}_a} \mathbb{E}\bigg\{\frac{1}{4}\bigg[\lambda\|I_T(h^w)\|_{L^4}^4 + \int_0^T \|l_t^T(h^w)\|_{L^2}^2 \mathrm{d}t\bigg] \\
&\qquad + (1-p)C\bigg[\|I_T^\flat(h^w)\|_{L^4}^4 + \|I_T^\flat(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^n + \int_0^T \|l_t^T(h^w)\|_{L^2}^2 \mathrm{d}t\bigg] \\
&\qquad + \|W_\infty + I_\infty(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^n\bigg\} - C \\
&\geqslant -C
\end{aligned}$$

which gives the claim. Note that here we used the bound

$$\begin{aligned}
\mathbb{E}\|I_\infty(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^n &\lesssim \mathbb{E}\|W_\infty\|_{\mathscr{C}^{-1/2-\varepsilon}}^n + \mathbb{E}\|W_\infty + I_\infty(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^n \\
&\lesssim C + \mathbb{E}\|W_\infty + I_\infty(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^n
\end{aligned}$$

as well as the fact that $\|I_t^\flat(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}} \lesssim \|I_\infty(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}$ to conclude. □

The following lemmas complete the proof.

Lemma 3.10. *For $n \in \mathbb{N}$ odd and large enough*

$$\mathbb{E}\int_0^T \|w_s\|_{L^2}^2 \mathrm{d}s \lesssim \mathbb{E}\int_0^T \|w_s + r_s^w\|^2 \mathrm{d}s + \mathbb{E}\|I_T^\flat(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^{n+1} + \|I_T^\flat(h^w)\|_{L^4}^4 + 1.$$

Proof. Let us recall the notation

$$\mathrm{Aux}_s(W,w):=\sum_{i=0}^{n}\binom{n}{i}J_s\langle \mathrm{D}\rangle^{-1/2}([\![(\langle \mathrm{D}\rangle^{-1/2}W_s)^i]\!](\langle \mathrm{D}\rangle^{-1/2}I_s(w))^{n-i}).$$

Write $r_s^w=\tilde{r}_s^w+\mathrm{Aux}_s(W,w)$ and observe that by Lemma 3.5 we with $g=r^w$ we have

$$\mathbb{E}\int_0^t\|w_s\|_{L^2}^2\mathrm{d}s+\sup_{s\leqslant t}\mathbb{E}\|I_s(w)\|_{W^{-1/2,n+1}}^{n+1}\lesssim 1+\int_0^t(2\mathbb{E}\|w_s+r_s\|_{L^2}^2+4\mathbb{E}\|\tilde{r}_s^w\|_{L^2}^2)\mathrm{d}s,$$

Now by Lemma 3.12 below

$$\langle t\rangle^{1+\varepsilon}\|\tilde{r}_t^w\|_{L^2}^2 \lesssim \int_0^t\|w_s\|_{L^2}^2\mathrm{d}s+\|I_t(w)\|_{W^{-1/2,n+1}}^{n+1}+\|I_t^\flat(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^{n+1}+\|I_t^\flat(h^w)\|_{L^4}^4+Q_t(W)$$

for a random variable $Q_t(W)$ such that $\sup_{t\in\mathbb{R}}\mathbb{E}[Q_t(W)]<\infty$. Now Gronwalls inequality allows us to conclude. □

Lemma 3.11. *There exists constants c,C and a random variable $Q_T(W)$ such that for any stopping time τ*

$$\sup_T\mathbb{E}[|Q_{\tau\wedge T}(W)|]<\infty,$$

and

$$\begin{aligned}&-Q_\tau(W)+c\int_0^\tau\|w_s\|_{L^2}^2\mathrm{d}s+c\|I_\tau(w)\|_{W^{-1/2,n+1}}^{n+1}\\ \leqslant{}& \int_0^\tau\|w_s\|_{L^2}^2\mathrm{d}s+\overline{\mathrm{Aux}}_\tau(W,w)\\ \leqslant{}& C\|I_\tau(w)\|_{W^{-1/2,n+1}}^{n+1}+C\int_0^\tau\|w_s\|_{L^2}^2\mathrm{d}s+Q_\tau(W)\end{aligned}$$

Proof. We recall that

$$\begin{aligned}\overline{\mathrm{Aux}}_\tau(W,w) &= \sum_{i=0}^{n}\frac{1}{n+1-i}\binom{n}{i}\int_\Lambda([\![(\langle \mathrm{D}\rangle^{-1/2}W_\tau)^i]\!](\langle \mathrm{D}\rangle^{-1/2}I_\tau(w))^{n+1-i})\\ &= \sum_{i=1}^{n}\frac{1}{n+1-i}\binom{n}{i}\int_\Lambda([\![(\langle \mathrm{D}\rangle^{-1/2}W_\tau)^i]\!](\langle \mathrm{D}\rangle^{-1/2}I_\tau(w))^{n+1-i})\\ &\quad+\frac{1}{n+1}\|I_\tau(w)\|_{W^{-1/2,n+1}}^{n+1}\end{aligned}$$

and since $\mathbb{E}[\sup_{T<\infty}\|[\![(\langle \mathrm{D}\rangle^{-1/2}W_T)^i]\!]\|_{\mathscr{C}^{-\varepsilon}}^p]<\infty$ for any $p<\infty$ and any $\varepsilon>0$ it is enough to bound $\|(\langle \mathrm{D}\rangle^{-1/2}I_\tau(w))^{n+1-i}\|_{B_{1,1}^\varepsilon}^q$ for some $q>1$ by the terms $\|I_\tau(w)\|_{W^{-1/2,n+1}}^{n+1}$ and $\|I_\tau(w)\|_{H^1}^2\lesssim\int_0^\tau\|w_s\|_{L^2}^2\mathrm{d}s$. By interpolation we can estimate, for $i\geqslant 1$,

$$\begin{aligned}\|(\langle \mathrm{D}\rangle^{-1/2}I_\tau(w))^{n+1-i}\|_{B_{1,1}^\varepsilon} &\lesssim \|\langle \mathrm{D}\rangle^{-1/2}I_\tau(w)\|_{B_{n,1}^\varepsilon}^n+C\\ &\lesssim \|I_\tau(w)\|_{W^{-1/2,n+1}}^{n-\frac{1}{(n-1)}}\|I_\tau(w)\|_{H^1}^{\frac{1}{n-1}}+C \qquad \left(\text{let }\varepsilon=\frac{1}{n(n-1)}\right)\end{aligned}$$

Choosing $q=n/\left(n-\frac{1}{(n-1)}\right)>1$, we have

$$\left(\|I_\tau(w)\|_{W^{-1/2,n+1}}^{n-\frac{1}{(n-1)}}\|I_\tau(w)\|_{H^1}^{\frac{1}{n-1}}\right)^q=\|I_\tau(w)\|_{W^{-1/2,n+1}}^n\|I_\tau(w)\|_{H^1}^{\frac{n}{(n-1)n-1}}.$$

Now for n large enough $\frac{n}{(n-1)n-1}\leqslant\frac{2}{n+1}$ and using Young's inequality we can estimate

$$\begin{aligned}\|I_\tau(w)\|_{W^{-1/2,n+1}}^n\|I_\tau(w)\|_{H^1}^{\frac{n}{(n-1)n-1}} &\lesssim \|I_\tau(w)\|_{W^{-1/2,n+1}}^n\left(\|I_\tau(w)\|_{H^1}^{\frac{2}{n+1}}+1\right)\\ &\lesssim \|I_\tau(w)\|_{W^{-1/2,n+1}}^{n+1}+\|I_\tau(w)\|_{H^1}^2+1\end{aligned}$$

□

Lemma 3.12. *Let*

$$\begin{aligned}\tilde{r}_s^w &= -12\lambda J_s[\![W_s^2]\!]\succ(1-\theta_s)I_s(w)+12\lambda J_s([\![W_s^2]\!]\circ I_s(w))+12\lambda J_s[\![W_s^2]\!]\prec I_s(w)\\ &\quad -12\lambda J_sW_s(I_s(w))^2-4\lambda J_s(I_s(w))^3-24\lambda(J_s(W_sI_s(w)\succ I_s^\flat(u^w)))\\ &\quad -12\lambda J_s((I_s(w))^2\succ I_s^\flat(u^w))+\lambda\mathbb{1}_{\{s<\bar{T}\}}J_s(\mathbb{W}_s^2\succ I_s^\flat(u^w)).\end{aligned}$$

Setting $h^w=u+w$, there exists a random variable $Q_t(W)$ such that $\sup_t\mathbb{E}[|Q_t(W)|]<\infty$ *and*

$$\langle t\rangle^{1+\varepsilon}\|\tilde{r}_t^w\|^2\lesssim\int_0^t\|w_s\|_{L^2}^2\mathrm{d}s+\|I_t(w)\|_{W^{-1/2,n+1}}^{n+1}+\|I_t^\flat(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^{n+1}+\|I_t^\flat(h^w)\|_{L^4}^4+Q_t(W).$$

Proof. Note that

$$\begin{aligned}\|\mathbb{1}_{\{s<\bar{T}\}}J_s(\mathbb{W}_s^2\succ I_s^\flat(u^w))\|_{L^2}^2 &\lesssim_{\bar{T}} \frac{1}{\langle s\rangle^2}\|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\varepsilon}}^2\|I_s^\flat(u^w)\|_{L^4}^2\\ &\lesssim \frac{1}{\langle s\rangle^2}(\|\mathbb{W}_s^2\|_{\mathscr{C}^{-1-\varepsilon}}^4+\|I_s^\flat(u^w)\|_{L^4}^4).\end{aligned}$$

Moreover $h^w=u^w+w$ implies

$$\|I_t^\flat(u^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^{n+1}\lesssim\|I_t^\flat(w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^{n+1}+\|I_t^\flat(h^w)\|_{\mathscr{C}^{-1/2-\varepsilon}}^{n+1},$$

and $\|I_t^\flat(u^w)\|_{L^4}^4\lesssim\|I_t^\flat(h^w)\|_{L^4}^4+\|I_t^\flat(w)\|_{L^4}^4$. From Lemma 3.19 we get

$$\|I_t^\flat(w)\|_{L^4}^4\lesssim C+\int_0^t\|w_s\|_{L^2}^2\mathrm{d}s+\|I_t(w)\|_{W^{-1/2,n+1}}^{n+1}.$$

The estimation for the other terms is easy but technical and postponed until Section 3.5. □

3.4. Singularity of Φ_3^4 w.r.t. the free field

The goal of this section is to prove that the Φ_3^4 measure is singular with respect to the Gaussian free field. For this we have to find a set $S\subseteq\mathscr{C}^{-1/2-\varepsilon}(\Lambda)$ such that $\mathbb{P}(W_\infty\in S)=1$ and $\mathbb{Q}^u(W_\infty\in S)=0$. Together with (3.18), this will imply singularity. We claim that setting

$$S:=\left\{f\in\mathscr{C}^{-1/2-\varepsilon}(\Lambda):\frac{1}{T_n^{1/2+\delta}}\int_\Lambda[\![(\theta_{T_n}f)^4]\!]\to 0\right\}$$

for some suitable sub-sequence T_n and a small $\delta>0$, does the job. Here

$$[\![(\theta_Tf)^4]\!]=(\theta_Tf)^4-6\mathbb{E}[(\theta_TW_\infty(0))^2](\theta_Tf)^2+3\mathbb{E}[(\theta_TW_\infty(0))^2]^2$$

denotes the Wick ordering with respect to the Gaussian free field. Let us prove first that indeed $\mathbb{P}(W_\infty\in S)=1$ for some T_n. For later use we define

$$\mathbb{W}_t^{\theta_T,3}=4(\theta_TW_t)^3-12\mathbb{E}[(\theta_TW_t(0))^2](\theta_TW_t)$$

and

$$\mathbb{W}_t^{\theta_T,2}=12((\theta_TW_t)^2-\mathbb{E}[(\theta_TW_t(0))^2]).$$

Lemma 3.13. *For any $\delta>0$*

$$\lim_{T\to\infty}\mathbb{E}\left[\left(\frac{1}{T^{(1+\delta)/2}}\int_\Lambda[\![(\theta_TW_\infty)^4]\!]\right)^2\right]=0.$$

Proof. Wick products correspond to iterated Ito integrals. Introducing the notation

$$\mathrm{d}w_t^{\theta_T}=\theta_TJ_t\mathrm{d}X_t,$$

we can verify by Ito formula that

$$\int_\Lambda [\![\theta_T W_\infty^4]\!] = \int_0^\infty \int_\Lambda \mathbb{W}_t^{\theta_T,3} \mathrm{d}w_t^{\theta_T} = \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t.$$

Since $\theta_T J_t = 0$ for $t \geqslant T$, Ito isometry gives

$$\mathbb{E}\left|\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t\right|^2 = \mathbb{E}\int_0^T \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3})^2 \mathrm{d}t.$$

Then, again by Ito formula the expectation on the r.h.s. can be estimated as

$$\begin{aligned}
\mathbb{E}\left[\int_\Lambda (\mathbb{W}_t^{\theta_T,3})^2\right] &= 4\mathbb{E}\left[\left|\sum_{k_1,k_2,k_3} \int_0^t \int_0^{s_1} \int_0^{s_2} \mathrm{d}w_{s_1}^{\theta_T}(k_1) \mathrm{d}w_{s_2}^{\theta_T}(k_2) \mathrm{d}w_{s_3}^{\theta_T}(k_3)\right|^2\right] \\
&= 24\mathbb{E}\left[\sum_{k_1,k_2,k_3} \int_0^t \int_0^{s_1} \int_0^{s_2} \frac{\theta_T^2(k_1)\sigma_{s_1}^2(k_1)}{\langle k_1\rangle^2} \frac{\theta_T^2(k_2)\sigma_{s_2}^2(k_2)}{\langle k_2\rangle^2} \frac{\theta_T^2(k_3)\sigma_{s_3}^2(k_3)}{\langle k_3\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3\right] \\
&\leqslant 24\mathbb{E}\left[\sum_{k_1,k_2,k_3} \int_0^t \int_0^t \int_0^t \frac{\sigma_{s_1}^2(k_1)}{\langle k_1\rangle^2} \frac{\sigma_{s_2}^2(k_2)}{\langle k_2\rangle^2} \frac{\sigma_{s_3}^2(k_3)}{\langle k_3\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3\right] \\
&\lesssim t^3
\end{aligned}$$

Now recall that $\|J_t f\|_{L^2(\Lambda)} \lesssim \langle t\rangle^{-3/2} \|f\|_{L^2(\Lambda)}$ to conclude:

$$\mathbb{E}\left[\frac{1}{T^{1+\delta}} \int_0^T \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3})^2 \mathrm{d}t\right] \leqslant \frac{1}{T^{1+\delta}} \int_0^T \frac{1}{t^3} \mathbb{E}[\|(\theta_T \mathbb{W}_t^{\theta_T,3})\|_{L^2(\Lambda)}^2] \mathrm{d}t \to 0 \qquad \square$$

The lemma implies that $\frac{1}{T^{(1+\delta)/2}} \int_\Lambda [\![(\theta_T W_\infty)^4]\!] \to 0$ in $L^2(\mathbb{P})$. So there exists a sub-sequence T_n such that $\frac{1}{T_n^{(1+\delta)/2}} \int_\Lambda [\![(\theta_{T_n} W_\infty)^4]\!] \to 0$ almost surely.

The next step of the proof is to check that $\mathbb{Q}^u(W_\infty \in S) = 0$. More concretely we will show that for a sub-sequence of T_n (not relabeled)

$$\frac{1}{T_n^{1-\delta}} \int_\Lambda [\![(\theta_{T_n} W_\infty)^4]\!] \to -\infty,$$

$\mathbb{Q}^u$ almost surely. Observe that

$$\begin{aligned}
\int_\Lambda [\![(\theta_T W_\infty)^4]\!] &= \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t \\
&= \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t^u + \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} u_t \mathrm{d}t \\
&= \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t^u - \lambda \int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t \mathbb{W}_t^{u,3} \mathrm{d}t \\
&\quad -\lambda \int_{\bar{T}}^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t (\mathbb{W}_t^{u,2} \succ I_t^\flat(u)) \mathrm{d}t \\
&\quad + \int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t \langle D\rangle^{-1/2} [\![(\langle D\rangle^{-1/2} W_t^u)^n]\!] \mathrm{d}t.
\end{aligned}$$

We expect the term

$$\int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t \mathbb{W}_t^{u,3} \mathrm{d}t$$

to go to infinity faster than $T^{1-\delta}$, $\mathbb{Q}^u$-almost surely. To actually prove it, we start by a computation in average.

LEMMA 3.14. *It holds*

$$\lim_{T\to\infty} \frac{1}{T^{1-\delta}} \mathbb{E}\left[\int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t \mathbb{W}_t^3 \mathrm{d}t\right] = \infty.$$

Proof. Recall that $\mathrm{d}w^{\theta_T}_t = \theta_T J_t \mathrm{d}X_t$. With a slight abuse of notation we can write

$$
\begin{aligned}
&\int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}^{\theta_T,3}_t) J_t \mathbb{W}^3_t \mathrm{d}t\\
= &\; 16 \int_0^\infty \sum_k \frac{\theta_T(k)\sigma^2_t(k)}{\langle k\rangle^2} \left(\sum_{k_1+k_2+k_3=k} \int_0^t \int_0^{s_1} \int_0^{s_2} \mathrm{d}w^{\theta_T}_{s_1}(k_1) \mathrm{d}w^{\theta_T}_{s_2}(k_2) \mathrm{d}w^{\theta_T}_{s_3}(k_3) \right.\\
&\times \left. \sum_{k_1+k_2+k_3=k} \int_0^t \int_0^{s_1} \int_0^{s_2} \mathrm{d}w_{s_1}(k_1) \mathrm{d}w_{s_2}(k_2) \mathrm{d}w_{s_3}(k_3) \right) \mathrm{d}t
\end{aligned}
$$

and by Ito isometry

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k_1+k_2+k_3=k} \int_0^t \int_0^{s_1} \int_0^{s_2} \mathrm{d}w^{\theta_T}_{s_1}(k_1) \mathrm{d}w^{\theta_T}_{s_2}(k_2) \mathrm{d}w^{\theta_T}_{s_3}(k_3) \right.\\
&\times \left. \sum_{k_1+k_2+k_3=k} \int_0^t \int_0^{s_1} \int_0^{s_2} \mathrm{d}w_{s_1}(k_1) \mathrm{d}w_{s_2}(k_2) \mathrm{d}w_{s_3}(k_3) \right]\\
= &\; 6 \sum_{k_1+k_2+k_3=k} \int_0^t \int_0^{s_1} \int_0^{s_2} \frac{\theta_T(k_1)\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2} \frac{\theta_T(k_2)\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2} \frac{\theta_T(k_3)\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3
\end{aligned}
$$

For T large enough and since σ^2 and θ are positive, we have

$$
\begin{aligned}
&\int_0^\infty \sum_k \frac{\theta_T(k)\sigma^2_t(k)}{\langle k\rangle^2} \sum_{k_1+k_2+k_3=k} \int_0^t \int_0^{s_1} \int_0^{s_2} \frac{\theta_T(k_1)\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2} \frac{\theta_T(k_2)\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2} \frac{\theta_T(k_3)\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}t\\
\geqslant &\int_{T/8}^{T/2} \sum_k \frac{\sigma^2_t(k)}{\langle k\rangle^2} \sum_{k_1+k_2+k_3=k} \int_0^{T/8} \int_0^{s_1} \int_0^{s_2} \frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2} \frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2} \frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}t
\end{aligned}
$$

Introduce the notation $\mathbb{Z}^3_+ = \{n \in \mathbb{Z}^3 : n = (n_1, n_2, n_3) \,\mathrm{with}\, n_i \geqslant 0\}$. After restricting the sum to $(\mathbb{Z}^3_+)^3$ we get the bound

$$
\begin{aligned}
\geqslant &\int_{T/8}^{T/2} \sum_k \frac{\sigma^2_t(k)}{\langle k\rangle^2} \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^3_+\\ k_1+k_2+k_3=k}} \int_{3T/32}^{T/8} \int_{3T/32}^{s_1} \int_{3T/32}^{s_2} \frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2} \frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2} \frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \mathrm{d}t\\
\gtrsim &\; \frac{1}{T^2} \sum_{k\in\mathbb{Z}^3_+} (\rho_{T/2}(k) - \rho_{T/8}(k)) \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^3_+\\ k_1+k_2+k_3=k}} \int_{3T/32}^{T/8} \int_{3T/32}^{s_1} \int_{3T/32}^{s_2} \frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2} \frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2} \frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3
\end{aligned}
$$

Now, for large enough T if $k_1+k_2+k_3=k$ and $\langle k_i\rangle \leqslant T/8$ then $\langle k\rangle \leqslant T/2 \times 0.9$. Furthermore if T large enough and $k_1, k_2, k_3 \in \mathbb{Z}^3_+$ and $k_1+k_2+k_3=k$, while $\langle k_i\rangle \geqslant (3T/32) \times 0.9$ (recall that if $\langle k_i\rangle < (3T/32) \times 0.9$ and $s > 3T/32$ then $\sigma_s(k_1) = 0$) we have $\langle k\rangle \geqslant T/8$. So for any k for which the integral is nonzero we have $\rho_{T/2}(k) - \rho_{T/8}(k) = 1$ (recall that $\rho = 1$ on $B(0, 9/10)$ and $\rho = 0$ outside of $B(0,1)$). This implies

$$
\begin{aligned}
&\frac{1}{T^2} \sum_{k\in\mathbb{Z}^3_+} (\rho_{T/2}(k) - \rho_{T/8}(k)) \sum_{\substack{k_1,k_2,k_3\in\mathbb{Z}^3_+\\ k_1+k_2+k_3=k}} \int_{3T/32}^{T/8} \int_{3T/32}^{s_1} \int_{3T/32}^{s_2} \frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2} \frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2} \frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3\\
= &\; \frac{1}{T^2} \sum_{k_1,k_2,k_3\in\mathbb{Z}^3_+} \int_{3T/32}^{T/8} \int_{3T/32}^{s_1} \int_{3T/32}^{s_2} \frac{\sigma^2_{s_1}(k_1)}{\langle k_1\rangle^2} \frac{\sigma^2_{s_2}(k_2)}{\langle k_2\rangle^2} \frac{\sigma^2_{s_3}(k_3)}{\langle k_3\rangle^2} \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3\\
\gtrsim &\; T
\end{aligned}
$$

□

Next we upgrade this bound to almost sure divergence.

LEMMA 3.15. *There exists a $\delta_0 > 0$ such that for any $\delta_0 \geqslant \delta > 0$, ,there exists a sequence $(T_n)_n$ such that* $\mathbb{P} -$ almost surely

$$\frac{1}{T_n^{1-\delta}} \int_0^\infty \int_\Lambda (\theta_{T_n} J_t \mathbb{W}_t^{\theta_{T_n},3}) J_t \mathbb{W}_t^3 \mathrm{d}t \to \infty.$$

Proof. Define

$$G_T := \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t \mathbb{W}_t^3 \mathrm{d}t + \sup_{t<\infty} \|W_t\|_{\mathscr{C}^{-1/2-\varepsilon}}^K.$$

We will show that $e^{-G_T} \to 0$ in $L^1(\mathbb{P})$, which implies that there exists a sub-sequence T_n such that $e^{-G_{T_n}} \to 0$ almost surely. From this our statement follows. By the Boué–Dupuis formula

$$\begin{aligned}
-\log\mathbb{E}[e^{-G_T}] &= \inf_{v\in\mathbb{H}_a} \mathbb{E}\Big[\frac{1}{T^{1-\delta}} 16 \int_0^\infty \int_\Lambda (\theta_T J_t [\![\theta_T((W_t + I_t(v))^3)]\!]) J_t [\![(W_t + I_t(v))^3]\!] \mathrm{d}t + \\
&\quad + \sup_{t<\infty} \|W_t + I_t(v)\|_{\mathscr{C}^{-1/2-\varepsilon}}^K + \frac{1}{2}\int_0^\infty \|v_t\|_{L^2}^2 \mathrm{d}t\Big] \\
&= \inf_{v\in\mathbb{H}_a} \mathbb{E}\Big[\frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t \mathbb{W}_t^3 \mathrm{d}t + \\
&\quad + \frac{1}{T^{1-\delta}} \sum_{(i,j)\in\{0,1,2,3\}^2\setminus(0,0)} \int_0^T \int_\Lambda A_t^i B_t^j \mathrm{d}t \\
&\quad + \sup_{t<\infty} \|W_t + I_t(v)\|_{\mathscr{C}^{-1/2-\varepsilon}}^K + \frac{1}{2}\int_0^\infty \|v_t\|_{L^2}^2 \mathrm{d}t\Big] \\
&\geqslant \inf_{v\in\mathbb{H}_a} \mathbb{E}\Big[\frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda (\theta_T J_t \mathbb{W}_t^{\theta_T,3}) J_t \mathbb{W}_t^3 \mathrm{d}t \\
&\quad + \frac{1}{T^{1-\delta}} \sum_{(i,j)\in\{0,1,2,3\}^2\setminus(0,0)} \int_0^T \int_\Lambda A_t^i B_t^j \mathrm{d}t \\
&\quad + \frac{1}{2}\sup_{t<\infty} \|I_t(v)\|_{\mathscr{C}^{-1/2-\varepsilon}}^K - C \sup_{t<\infty} \|W_t\|_{\mathscr{C}^{-1/2-\varepsilon}}^K + \frac{1}{2}\int_0^\infty \|v_t\|_{L^2}^2 \mathrm{d}t\Big]
\end{aligned}$$

where where have used that $\theta_T J_t = 0$ for $t \geqslant T$ and introduced the notations, for $0 \leqslant i \leqslant 3$,

$$A_t^i := 4\binom{3}{i} J_t \theta_T([\![(\theta_T W_t)^{3-i}]\!](\theta_T I_t(v))^i),$$

and

$$B_t^i := 4\binom{3}{i} J_t([\![W_t^{3-i}]\!](I_t(v))^i).$$

Our aim now to prove that the last three terms are bounded below uniformly as $T \to \infty$ (while we already know that the first one diverges). For $i \in \{1, 2, 3\}$

$$\|A_t^i\|_{L^2}^2 + \|B_t^i\|_{L^2}^2 \lesssim \langle t \rangle^{-1+\delta} (\|I_t(u)\|_{\mathscr{C}^{-1/2-\varepsilon}}^K + \|I_t(u)\|_{H^1}^2 + Q_t(W))$$

by Lemmas 3.21 and 3.23. Here $Q_t(W)$ is a random variable only depending on W such that $\sup_t \mathbb{E}[|Q_t(W)|^p] < \infty$ for any $p < \infty$. Then

$$\begin{aligned}
&\frac{1}{T^{1-\delta}} \sum_{(i,j)\in\{0,1,2,3\}^2\setminus(0,0)} \int_0^T \int_\Lambda |A_t^i B_t^j| \mathrm{d}t \\
\leqslant\; &\frac{1}{T^{1-\delta}} \sum_{(i,j)\in\{1,2,3\}^2} \int_0^T \|A_t^i\|_{L^2}^2 + \|B_t^j\|_{L^2}^2 \mathrm{d}t \\
&+ \frac{1}{T^{1-\delta}} \sum_{i\in\{1,2,3\}} \int_0^T \|A_t^0\|_{L^2} \|B_t^i\|_{L^2} \mathrm{d}t + \frac{1}{T^{1-\delta}} \sum_{i\in\{1,2,3\}} \int_0^T \|A_t^i\|_{L^2} \|B_t^0\|_{L^2} \mathrm{d}t.
\end{aligned}$$

Now for the first term we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\frac{1}{T^{1-\delta}}\sum_{(i,j)\in\{1,2,3\}^2}\int_0^T \|A_t^i\|_{L^2}^2+\|B_t^j\|_{L^2}^2\mathrm{d}t\right]\\
=\;&\mathbb{E}\left[\frac{1}{T^{1-\delta}}\sum_{(i,j)\in\{1,2,3\}^2}\int_0^T \langle t\rangle^{-1+\delta}(\|I_t(v)\|_{\mathscr{C}^{-1/2-\varepsilon}}^K+\|I_t(v)\|_{H^1}^2+Q_t(W))\mathrm{d}t\right]\\
=\;&\frac{C}{T^{1-2\delta}}\mathbb{E}\left[\sup_t(\|I_t(v)\|_{\mathscr{C}^{-1/2-\varepsilon}}^K+\|I_t(v)\|_{H^1}^2)\right]+\frac{C}{T^{1-2\delta}}.
\end{aligned}
$$

For the second term we use that $\|A_t^0\|_{L^2}\leqslant Q_t(W)$ so

$$
\begin{aligned}
&\frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T\|A_t^0\|_{L^2}\|B_t^i\|_{L^2}\mathrm{d}t\right]\\
\leqslant\;&\frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T\langle t\rangle^{-1/2}\|A_t^0\|_{L^2}^2\mathrm{d}t+\int_0^T\langle t\rangle^{1/2}\|B_t^i\|_{L^2}^2\mathrm{d}t\right]\\
\lesssim\;&\frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T\langle t\rangle^{-1/2}\|A_t^0\|_{L^2}^2\mathrm{d}t\right]\\
&+\frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T\langle t\rangle^{-1/2+\delta}(\|I_t(v)\|_{\mathscr{C}^{-1/2-\varepsilon}}^K+\|I_t(v)\|_{H^1}^2+Q_t(W))\mathrm{d}t\right]\\
\lesssim\;&\frac{C}{T^{1/2-2\delta}}\mathbb{E}\left[\sup_t(\|I_t(v)\|_{\mathscr{C}^{-1/2-\varepsilon}}^K+\|I_t(v)\|_{H^1}^2)\right]+\frac{C}{T^{1/2-2\delta}}
\end{aligned}
$$

Since $\sup_t\ \|I_t(v)\|_{H^1}^2\lesssim\int_0^\infty\|v_t\|_{L^2}^2\mathrm{d}t$ in total we obtain for T large enough. The third term is estimated analogously.

$$
\begin{aligned}
&-\log\mathbb{E}[e^{-G_T}]\\
\geqslant\;&\inf_{v\in\mathbb{H}_a}\mathbb{E}\left[\frac{1}{T^{1-\delta}}\int_0^\infty\int_\Lambda(\theta_TJ_t\mathbb{W}_t^{\theta_T,3})J_t\mathbb{W}_t^3\mathrm{d}t+\left(\frac{1}{2}-\frac{C}{T^{1/2-2\delta}}\right)\sup_{t<\infty}\|I_t(v)\|_{\mathscr{C}^{-1/2-\varepsilon}}^K\right.\\
&\left.-C\sup_{t<\infty}\|W_t\|_{\mathscr{C}^{-1/2-\varepsilon}}^K+\left(\frac{1}{2}-\frac{C}{T^{1/2-2\delta}}\right)\int_0^\infty\|v_t\|_{L^2}^2\mathrm{d}t-\frac{C}{T^{1/2-2\delta}}\right]\\
\geqslant\;&\mathbb{E}\left[\frac{1}{T^{1-\delta}}\int_0^\infty\int_\Lambda(\theta_TJ_t\mathbb{W}_t^{\theta_T,3})J_t\mathbb{W}_t^3\mathrm{d}t\right]-C\to\infty
\end{aligned}
$$

□

Next we prove an estimate which will help with the proof of the main theorem.

Lemma 3.16. *We have*

$$\sup_T\mathbb{E}_{\mathbb{Q}^u}\left[\int_0^\infty\int_\Lambda\frac{1}{t^{1+\delta}}(\theta_TJ_t\mathbb{W}_t^{\theta_T,3})^2\mathrm{d}t\right]<\infty.$$

Furthermore, there exists a (deterministic) sub-sequence $(T_n)_n$ such that

$$\frac{1}{T_n^{1/2+\delta}}\left|\int_0^\infty\int_\Lambda\theta_{T_n}J_t\mathbb{W}_t^{\theta_{T_n},3}\mathrm{d}X_t^u\right|\to 0$$

$\mathbb{Q}^u$ almost surely.

Proof. Recall that under $\mathbb{Q}^u$ we have $W_t=W_t^u+I_t(u)$ where u is defined above by (3.13) and $\mathrm{Law}_{\mathbb{Q}^u}(W^u)=\mathrm{Law}_{\mathbb{P}}(W)$. With this in mind we compute

$$\int_0^T\int_\Lambda\frac{1}{t^{1+\delta}}(\theta_TJ_t\mathbb{W}_t^{\theta_T,3})^2\mathrm{d}t=\sum_{i,j\leqslant 3}\int_0^T\int_\Lambda\frac{1}{t^{1+\delta}}A_t^iA_t^j\mathrm{d}t,$$

where, as above,

$$A_t^i=4\binom{3}{i}J_t\theta_T(\llbracket(\theta_TW_t^u)^{3-i}\rrbracket(\theta_TI_t(u))^i).$$

By Lemmas 3.21 and 3.23 we have that $\mathbb{E}_{\mathbb{Q}^u}[\|A^i_t\|^2_{L^2}] \leqslant C$ so the Cauchy–Schwartz inequality gives the result. □

Theorem 3.17. *There exists a sequence $(T_n)_n$ such that, $\mathbb{Q}^u$ almost surely,*

$$\frac{1}{T_n^{1-\delta}} \int_\Lambda [\![(\theta_{T_n} W_\infty)^4]\!] \to -\infty.$$

Proof. We have

$$\int_\Lambda [\![(\theta_T W_\infty)^4]\!] = \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t.$$

Now since $\mathrm{d}X_t = \mathrm{d}X^u_t + u_t \mathrm{d}t$ we have

$$\begin{aligned}
& \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X_t \\
= & \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X^u_t + \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} u_t \mathrm{d}t \\
= & \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} \mathrm{d}X^u_t - \frac{\lambda}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \mathbb{W}_t^{u,3} \mathrm{d}t \\
& - \frac{\lambda}{T^{1-\delta}} \int_{\bar{T}}^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t (\mathbb{W}_t^{u,2} \succ I^\flat_t(u)) \mathrm{d}t \\
& + \frac{1}{T^{1-\delta}} \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \langle D \rangle^{-1/2} [\![(\langle D \rangle^{-1/2} W^u_t)^n]\!] \mathrm{d}t.
\end{aligned}$$

The first term goes to 0 $\mathbb{Q}^u$-almost surely by Lemma 3.16. To analyze the third term we estimate

$$\begin{aligned}
& \frac{1}{T^{1-\delta}} \int_{\bar{T}}^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t (\mathbb{W}_t^{u,2} \succ I^\flat_t(u)) \mathrm{d}t \\
= & \frac{1}{T^{1-\delta}} \int_{\bar{T}}^T \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t (\mathbb{W}_t^{u,2} \succ I^\flat_t(u)) \mathrm{d}t \\
\leqslant & \frac{1}{T^{1-\delta}} \int_{\bar{T}}^T \|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2} \|J_t (\mathbb{W}_t^{u,2} \succ I^\flat_t(u))\|_{L^2} \mathrm{d}t \\
\lesssim & \frac{1}{T^{1-\delta}} \int_{\bar{T}}^T t^{-1/2+\delta/2} \|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2} \|\mathbb{W}_t^{u,2}\|_{\mathscr{C}^{-1-\delta/2}} \|I_t(u)\|_{L^2} \mathrm{d}t \\
\leqslant & T^{-1/2-2\delta} \left(\int_{\bar{T}}^T \|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|^2_{L^2} \mathrm{d}t \right)^{1/2} \\
& \times T^{-1/2+2\delta} \left(\int_{\bar{T}}^T t^{-1+\delta} (\|\mathbb{W}_t^{u,2}\|_{\mathscr{C}^{-1-\delta/2}} \|I_t(u)\|_{L^2})^2 \right)^{1/2}
\end{aligned} \tag{3.21}$$

By the computation from Lemma 3.16 we have

$$\mathbb{E}_{\mathbb{Q}^u}\left[T^{-1/2-2\delta} \left(\int_{\bar{T}}^T \|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|^2_{L^2} \mathrm{d}t \right)^{1/2} \right] \to 0,$$

and $\sup_t \mathbb{E}_{\mathbb{Q}^u}[(\|\mathbb{W}_t^{u,2}\|_{\mathscr{C}^{-1-\delta/2}} \|I_t(u)\|_{L^2})^2] < \infty$, so (3.21) converges to 0 in $L^1(\mathbb{Q}^u)$. For the fourth term we proceed in the same way:

$$\begin{aligned}
& \left| \int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \langle D \rangle^{-1/2} [\![(\langle D \rangle^{-1/2} W^u_t)^n]\!] \mathrm{d}t \right| \\
= & \left| \int_0^T \int_\Lambda \theta_T J_t \mathbb{W}_t^{\theta_T,3} J_t \langle D \rangle^{-1/2} [\![(\langle D \rangle^{-1/2} W^u_t)^n]\!] \mathrm{d}t \right| \\
\leqslant & \int_0^T \|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2} \|J_t \langle D \rangle^{-1/2} [\![(\langle D \rangle^{-1/2} W^u_t)^n]\!]\|_{L^2} \mathrm{d}t \\
\lesssim & \int_0^T (\|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2}) t^{-2+\delta} \|[\![(\langle D \rangle^{-1/2} W^u_t)^n]\!]\|_{H^{-\delta}} \mathrm{d}t \\
\leqslant & \left(\int_0^T t^{-2(1-\delta)} (\|\theta_T J_t \mathbb{W}_t^{\theta_T,3}\|_{L^2})^2 \right)^{1/2} \left(\int_0^T t^{-2(1-\delta)} \|[\![(\langle D \rangle^{-1/2} W^u_t)^n]\!]\|^2_{H^{-\delta}} \right)^{1/2}
\end{aligned}$$

which is bounded in expectation uniformly in T, so the fourth term goes to 0 in $L^1(\mathbb{Q}^u)$ as well. It remains to analyze the second term. Again introducing the notation

$$A^i_t = 4\binom{3}{i} J_t \theta_T(\llbracket(\theta_T W^u_t)^{3-i}\rrbracket(\theta_T I_t(u))^i),$$

$$\mathbb{W}^{\theta_T,u,3}_t = 4\llbracket(\theta_T W^u_t)^3\rrbracket,$$

we have

$$\begin{aligned} & \frac{1}{T^{1-\delta}}\int_0^\infty \int_\Lambda \theta_T J_t \mathbb{W}^{\theta_T,3}_t J_t \mathbb{W}^{u,3}_t \mathrm{d}t \\ = & \frac{1}{T^{1-\delta}}\int_0^T \int_\Lambda \theta_T J_t \mathbb{W}^{\theta_T,u,3}_t J_t \mathbb{W}^{u,3}_t \mathrm{d}t + \sum_{1\leqslant i\leqslant 3} \frac{1}{T^{1-\delta}}\int_0^T \int_\Lambda A^i_t J_t \mathbb{W}^{u,3}_t \mathrm{d}t. \end{aligned}$$

Now observe that

$$\frac{1}{T^{1-\delta}}\int_0^T \int_\Lambda \theta_T J_t \mathbb{W}^{\theta_T,u,3}_t J_t \mathbb{W}^{u,3}_t \mathrm{d}t \;{}_{\mathbb{Q}^u}\!\!\sim_{\mathbb{P}}\; \frac{1}{T^{1-\delta}}\int_0^T \int_\Lambda \theta_T J_t \mathbb{W}^{\theta_T,3}_t J_t \mathbb{W}^3_t \mathrm{d}t,$$

so the limsup of this is ∞ almost surely. To estimate the sum we again observe that for $i \geqslant 3$ $\mathbb{E}_{\mathbb{Q}^u}[\|A^i_t\|^2_{L^2}] \lesssim \langle t\rangle^{-1+\delta}$ and by Young's inequality

$$\begin{aligned} \int_0^T \int_\Lambda A^i_t J_t \mathbb{W}^{u,3}_t \mathrm{d}t & \leqslant \int_0^T \int_\Lambda \|A^i_t\|_{L^2} \|J_t \mathbb{W}^{u,3}_t\|_{L^2} \mathrm{d}t \\ & \leqslant \int_0^T \int_\Lambda \langle t\rangle^{1/3} \|A^i_t\|_{L^2} \langle t\rangle^{-1/3} \|J_t \mathbb{W}^{u,3}_t\|_{L^2} \mathrm{d}t \\ & \leqslant \int_0^T \int_\Lambda \langle t\rangle^{2/3} \|A^i_t\|^2_{L^2} + \int_0^T \int_\Lambda \langle t\rangle^{-2/3} \|J_t \mathbb{W}^{u,3}_t\|^2_{L^2} \mathrm{d}t. \end{aligned}$$

Taking expectation we obtain

$$\begin{aligned} & \frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T \int_\Lambda A^i_t J_t \mathbb{W}^{u,3}_t \mathrm{d}t\right] \\ \leqslant & \frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T \int_\Lambda \langle t\rangle^{2/3} \|A^i_t\|^2_{L^2}\right] + \frac{1}{T^{1-\delta}}\mathbb{E}\left[\int_0^T \int_\Lambda \langle t\rangle^{-2/3} \|J_t \mathbb{W}^{u,3}_t\|^2_{L^2} \mathrm{d}t\right] \\ \lesssim & \frac{1}{T^{1-\delta}}\int_0^T \int_\Lambda \langle t\rangle^{-1/3+\delta} + \frac{1}{T^{1-\delta}}\int_0^T \int_\Lambda \langle t\rangle^{-2/3} \mathrm{d}t \to 0 \end{aligned}$$

We have deduced that

$$\frac{1}{T^{1-\delta}}\int_\Lambda \llbracket(\theta_T W_\infty)^4\rrbracket = -\frac{1}{T^{1-\delta}}\int_0^T \int_\Lambda \theta_T J_t \mathbb{W}^{\theta_T,u,3}_t J_t \mathbb{W}^{u,3}_t \mathrm{d}t + R_T,$$

where $R_T \to 0$ in $L^1(\mathbb{Q}^u)$. We can conclude by selecting a sub-sequence $(T_n)_n$ such that

$$\frac{1}{T_n^{1-\delta}}\int_0^{T_n} \int_\Lambda \theta_T J_t \mathbb{W}^{\theta_{T_n},u,3}_t J_t \mathbb{W}^{u,3}_t \mathrm{d}t \to \infty$$

$\mathbb{Q}^u$-almost surely and $R_{T_n} \to 0$, $\mathbb{Q}^u$-almost surely. □

3.5. Some analytic estimates

We collect in this final section various technical estimates needed to complete the proof of Lemma 3.12.

Proposition 3.18. *Let $1 < p < \infty$ and $p_1, p_2, p_1', p_2' > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1'} + \frac{1}{p_2'} = \frac{1}{p}$. Then for every $s, \alpha \geqslant 0$*

$$\|\langle \mathrm{D}\rangle^s (fg)\|_{L^p} \lesssim \|\langle \mathrm{D}\rangle^{s+\alpha} f\|_{L^{p_2}} \|\langle \mathrm{D}\rangle^{-\alpha} g\|_{L^{p_1}} + \|\langle \mathrm{D}\rangle^{s+\alpha} g\|_{L^{p_1'}} \|\langle \mathrm{D}\rangle^{-\alpha} f\|_{L^{p_2'}}.$$

Proof. See [75]. □

Lemma 3.19. *There exists* $\varepsilon > 0, n \in \mathbb{N}$ *such that for any* $\delta > 0$ *there exists* $C_\delta < \infty$ *for which the following inequality holds for any* $\phi \in H^1(\Lambda)$

$$\|\phi\|_{L^4}^{4+\varepsilon} \leqslant C\|\phi\|_{W^{-1/2,n+1}}^{n+1} + \delta\|\phi\|_{H^1}^2 + C_\delta.$$

Proof.

$$\begin{aligned}\int \phi^4 \mathrm{d}x &\leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}\|\langle \mathrm{D}\rangle^{1/2}\phi^3\|_{L^{8/7}} \\ &\leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}\|\langle \mathrm{D}\rangle^{1/2}\phi\|_{L^{8/3}}\|\phi\|_{L^4}^2 \\ &\leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}\|\phi\|_{H^1}^{1/2}\|\phi\|_{L^4}^{5/2}\end{aligned}$$

So

$$(\|\phi\|_{L^4}^4)^{21/20} \leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}^{21/20}\|\phi\|_{H^1}^{21/40}\|\phi\|_{L^4}^{104/40}$$

and applying Young's inequality with the exponents $(32, 32/9, 32/22)$, we obtain

$$\begin{aligned}\|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}^{21/20}\|\phi\|_{H^1}^{21/40}\|\phi\|_{L^4}^{104/40} &\leqslant C_\delta\|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}^{168/5} + \delta\|\phi\|_{H^1}^{16/9} + \delta\|\phi\|_{L^4}^{208/55} \\ &\leqslant \|\langle \mathrm{D}\rangle^{-1/2}\phi\|_{L^8}^{34} + \delta\|\phi\|_{H^1}^2 + \delta(\|\phi\|_{L^4}^4)^{21/20} + C_\delta\end{aligned}$$

and subtracting $\delta(\|\phi\|_{L^4}^4)^{21/20}$ on both sides of the inequality gives the result. □

Lemma 3.20. *The following estimates hold with* $\varepsilon > 0$ *small enough*

$$\|J_t(\llbracket W_t^2\rrbracket \succ (1-\theta_t)I_t(w))\|_{L^2}^2 \lesssim \frac{1}{\langle t\rangle^{1+\varepsilon}}\left(\int_0^t \|w_s\|^2\mathrm{d}s + \|I_t(w)\|_{W^{-1/2,n+1}}^n + \|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}^n\right)$$

$$\|J_t(\llbracket W_t^2\rrbracket \circ I_t(w))\|_{L^2}^2 \lesssim \frac{1}{\langle t\rangle^{1+\varepsilon}}\left(\int_0^t \|w_s\|^2\mathrm{d}s + \|I_t(w)\|_{W^{-1/2,n+1}}^n + \|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}^n\right)$$

$$\|J_t\llbracket W_t^2\rrbracket \prec I_t(w)\|_{L^2}^2 \lesssim \frac{1}{\langle t\rangle^{1+\varepsilon}}\left(\int_0^t \|w_s\|^2\mathrm{d}s + \|I_t(w)\|_{W^{-1/2,n+1}}^n + \|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}^n\right)$$

Proof. We observe that since $\llbracket W_t^2\rrbracket$ is spectrally supported in a ball or radius $\sim t$

$$\|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1+\varepsilon}} \lesssim \langle t\rangle^{2\varepsilon}\|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}.$$

For the first estimate we know that $(1-\theta_t)I_t(w)$ is supported in an annulus of radius $\sim t$, so $\|(1-\theta_t)I_t(w)\|_{L^2} \lesssim \langle t\rangle^{-1+\varepsilon}\|I_t(w)\|_{H^{1-\varepsilon}}$ and furthermore by interpolation $\|I_t(w)\|_{H^{1-\varepsilon}} \lesssim \|I_t(w)\|_{H^1}^{1-\varepsilon}\|I_t(w)\|_{L^2}^{\varepsilon} \lesssim \|I_t(w)\|_{H^1}^{1-\varepsilon}\|I_t(w)\|_{L^4}^{\varepsilon}$. By definition $\langle t\rangle^{1/2}J_t$ is a uniformly bounded Fourier multiplier regularizing by 1, and putting everything together, by paraproduct estimates

$$\begin{aligned}\|J_t(\llbracket W_t^2\rrbracket \succ (1-\theta_t)I_t(w))\|_{L^2}^2 &\lesssim \langle t\rangle^{-1}\langle t\rangle^{2\varepsilon}\langle t\rangle^{-2+2\varepsilon}\|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}^2\|I_t(w)\|_{H^{1-\varepsilon}}^2 \\ &\lesssim \langle t\rangle^{-1}\langle t\rangle^{2\varepsilon}\langle t\rangle^{-2+2\varepsilon}\|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}^2\|I_t(w)\|_{H^1}^{2-2\varepsilon}\|I_t(w)\|_{L^4}^{2\varepsilon} \\ (\varepsilon = 2/7) \quad &\lesssim \langle t\rangle^{-3/2}(\|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}^{14} + \|I_t(w)\|_{H^1}^2 + \|I_t(w)\|_{L^4}^4) \\ &\lesssim \langle t\rangle^{-3/2}\left(\int_0^t \|w\|^2\mathrm{d}s + \|I_t(w)\|_{W^{-1/2,n+1}}^n + \|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}^{14}\right)\end{aligned}$$

For the second term in addition observe that the function $\langle t\rangle^{1/2}J_t$ is spectrally supported in an annulus of radius $\sim t$, and regularizes by 1 so again by estimates for the resonant product

$$\begin{aligned}\|J_t(\llbracket W_t^2\rrbracket \circ I_t(w))\|_{L^2}^2 &\lesssim \langle t\rangle^{-3}\|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1+2\varepsilon}}^2\|I_t(w)\|_{H^{1-\varepsilon}}^2 \\ &\lesssim \langle t\rangle^{-3}\langle t\rangle^{6\varepsilon}\|\llbracket W_t^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}^2\|I_t(w)\|_{H^{1-\varepsilon}}^2\end{aligned}$$

For the third estimate again applying paraproduct estimates and the properties of J,

$$\|J_t(\llbracket W_s^2\rrbracket \prec I_t(w))\|_{L^2}^2 \lesssim \langle t\rangle^{-3+4\varepsilon}\|\llbracket W_s^2\rrbracket\|_{\mathscr{C}^{-1-\varepsilon}}^2\|I_t(w)\|_{H^{1-\varepsilon}}^2.$$

Now, the claim follows from interpolation and Young's inequality

$$\begin{aligned} & \|[\![W_t^2]\!]\|^2_{\mathscr{C}^{-1-\varepsilon}}\|I_t(w)\|^2_{H^{1-\varepsilon}} \\ \lesssim\; & \|[\![W_t^2]\!]\|^2_{\mathscr{C}^{-1-\varepsilon}}\|I_t(w)\|^{2-2\varepsilon}_{H^1}\|I_t(w)\|^{2\varepsilon}_{L^4} \\ (\varepsilon=2/7) \qquad \lesssim\; & \|[\![W_t^2]\!]\|^{14}_{\mathscr{C}^{-1-\varepsilon}}+\|I_t(w)\|^2_{H^1}+\|I_t(w)\|^4_{L^4} \\ \lesssim\; & \left(\int_0^t \|w_s\|^2_{L^2}\mathrm{d}s+\|I_t(w)\|^n_{W^{-1/2,n+1}}+\|[\![W_t^2]\!]\|^{14}_{\mathscr{C}^{-1-\varepsilon}}\right). \end{aligned}$$

□

LEMMA 3.21. *Let $f\in C([0,\infty],\mathscr{C}^{-1/2-\varepsilon})$ and $g\in C([0,\infty],H^1)$ such that f_t,g_t have spectral support in a ball of radius proportional to t. There exists $n\in\mathbb{N}$ such that the following estimates hold:*

$$\|J_t(f_t g_t^2)\|^2_{L^2}\lesssim\langle t\rangle^{-3/2}\|f_t\|^2_{\mathscr{C}^{-1/2-\delta}}\|g_t\|^4_{L^4},$$

$$\|J_t(f_t g_t^2)\|^2_{L^2}\lesssim\langle t\rangle^{-3/2}(\|f_t\|^n_{\mathscr{C}^{-1/2-\delta}}+\|g_t\|^2_{H^1}+\|g_t\|^n_{W^{-1/2,n}}),$$

and

$$\|J_t(g_t^3)\|^2_{L^2}\lesssim\langle t\rangle^{-3/2}(\|g_t\|^2_{H^1}+\|g_t\|^n_{W^{-1/2,n}}).$$

Proof. By the spectral properties of J_t,

$$\|J_t(f_t g_t^2)\|^2_{L^2}\lesssim\langle t\rangle^{-3}\|f_t\|^2_{L^\infty}\|g_t\|^4_{L^4}\lesssim\langle t\rangle^{-3/2}\|f_t\|^2_{\mathscr{C}^{-1/2-\delta}}\|g_t\|^4_{L^4}.$$

Applying Young's inequality with exponents $\left(\frac{n}{2},\frac{n/2}{(n/2-1)}\right)$ with n such that $\frac{2n}{(n/2-1)}\leqslant 4+\varepsilon$ where ε is chosen as in Lemma 3.19 we have

$$\begin{aligned}\langle t\rangle^{-3/2}\|f_t\|^2_{\mathscr{C}^{-1/2-\delta}}\|g_t\|^4_{L^4} &\leqslant \langle t\rangle^{-3/2}(\|f_t\|^n_{\mathscr{C}^{-1/2-\delta}}+\|g_t\|^{4+\varepsilon}_{L^4}) \\ &\leqslant \langle t\rangle^{-3/2}(\|f_t\|^n_{\mathscr{C}^{-1/2-\delta}}+\|g_t\|^n_{W^{-1/2,n}}+\|g_t\|^2_{H^1})\end{aligned}$$

Now the second estimate follows from chosing n large enough (depending on δ) and using Besov embedding after taking $f=g$. □

LEMMA 3.22. *The following estimates hold*

$$\langle t\rangle^{1+\varepsilon}\|J_s(W_s I_t(w)\succ I_t^\flat(u))\|^2_{L^2}\lesssim\|I_t(w)\|^{4+\varepsilon}_{L^4}+\|I_t^\flat(u)\|^4_{L^4}+\|W_t\|^n_{\mathscr{C}^{-1/2-\varepsilon}},$$

$$\langle t\rangle^{1+\varepsilon}\|J_s((I_s(w))^2\succ I_s^\flat(u))\|^2_{L^2}\lesssim\|I_t(w)\|^{4+\varepsilon}_{L^4}+\|I_t^\flat(u)\|^n_{\mathscr{C}^{-1/2-\varepsilon}}.$$

Proof. For the first estimate we again use the spectral properties of $W,I,$ and J and obtain by paraproduct estimate

$$\begin{aligned}\|J_s(W_t I_t(w)\succ I_t^\flat(u))\|^2_{L^2} &\lesssim \langle t\rangle^{-3}\|W_t\|^2_{L^\infty}\|I_t(w)\|^2_{L^4}\|I_t^\flat(u)\|^2_{L^4} \\ &\lesssim \langle t\rangle^{-3}\langle t\rangle^{1+4\varepsilon}\|W_t\|^2_{\mathscr{C}^{-1/2-\varepsilon}}\|I_t(w)\|^2_{L^4}\|I_t^\flat(u)\|^2_{L^4}\end{aligned}$$

and the claim follows by Young's inequality. For the second

$$\|J_s((I_s(w))^2\succ I_s^\flat(u))\|^2_{L^2}\lesssim\langle t\rangle^{2-2\varepsilon}\|(I_s(w))\|^4_{L^4}\|I_t^\flat(u)\|^2_{\mathscr{C}^{-1/2-\varepsilon}},$$

and the claim follows again by Young's inequality. □

LEMMA 3.23. *Let $f_t\in C([0,\infty],\mathscr{C}^{-1/2-\delta})$ and $g_t\in C([0,\infty],H^1)$ such that f_t, g_t have spectral support in a ball of radius proportional to t. Then the following estimates hold*

$$\|(J_t(f_t g_t))\|^2_{L^2}\lesssim\langle t\rangle^{-1+2\delta}\|f_t\|^2_{\mathscr{C}^{-1-\delta}}\|g_t\|^2_{L^2}$$

$$\|(J_t(f_t g_t))\|^2_{L^2}\lesssim\langle t\rangle^{-1+2\delta}(\|f_t\|^8_{\mathscr{C}^{-1-\delta}}+\|g_t\|^4_{H^{-1}}+\|g_t\|^2_{H^1})$$

Proof.

$$\|(J_t(f_t g_t))\|^2_{L^2}\lesssim\langle t\rangle^{-3}\|f_t\|^2_{L^\infty}\|g_t\|^2_{L^2}\lesssim\langle t\rangle^{-1+2\delta}\|f_t\|^2_{\mathscr{C}^{-1-\delta}}\|g_t\|^2_{L^2}.$$

This proves the first estimate. For the second we continue

$$\begin{aligned}\langle t\rangle^{-1+2\delta}\|f_t\|^2_{\mathscr{C}^{-1-\delta}}\|g_t\|^2_{L^2} &\lesssim \langle t\rangle^{-1+2\delta}\|f_t\|^2_{\mathscr{C}^{-1-\delta}}\|g_t\|_{H^1}\|g_t\|_{H^{-1}}\\ &\lesssim \langle t\rangle^{-1+2\delta}(\|f_t\|^8_{\mathscr{C}^{-1-\delta}}+\|g_t\|^4_{H^{-1}}+\|g_t\|^2_{H^{-1}}).\end{aligned}$$

□

Lemma 3.24. *It holds*

$$\int_0^T\int_\Lambda (J_t(\mathbb{W}_t^2 \succ I_t^\flat(w)))^2 \lesssim T^{3\delta}\left(\sup_t \|\mathbb{W}_t^2\|^2_{\mathscr{C}^{-1-\delta}}\right)\left(\sup_t \|I_t(w)\|^2_{L^2}\right),$$

and

$$\int_0^T\int_\Lambda (J_t(\mathbb{W}_t^2 \succ I_t^\flat(w)))^2 \lesssim T^{3\delta}\left(\sup_t \|I_t(w)\|^4_{H^{-1}}+\int_0^T \|w_t\|^2_{L^2}\mathrm{d}t+\sup_t \|\mathbb{W}_t^2\|^8_{\mathscr{C}^{-1-\delta}}\right).$$

Proof. This follows in the same fashion as Lemma 3.23 . □

Chapter 4

A Stochastic Control approach to Sine-Gordon

4.1. Introduction

In this chapter we will consider the Sine-Gordon measure formally given by

$$\nu_{\mathrm{SG}}(\mathrm{d}\phi) = \frac{1}{Z} e^{-\lambda \int \cos(\beta\varphi)} \mu(\mathrm{d}\varphi) \tag{4.1}$$

where μ is the Gaussian free field on $\mathbb{R}^2$ with mass m , that is the Gaussian measure with covariance $(m^2 - \Delta)^{-1}$ and Z is a normalization factor. Again we run into small scale divergencies as in Chapter 2 and Chapter 3, however this time we also have to deal with large scale divergencies (since we work on $\mathbb{R}^2$ instead of a bounded domain). These are essentially due to the fact that realizations of the GFF do not exhibit decay in space. As a first step we approximate with (4.1) with

$$\nu_{\mathrm{SG}}^{T,\rho}(\mathrm{d}\phi) = \frac{1}{Z_{T,\rho}} e^{-\lambda\alpha(T) \int \rho \cos(\beta\varphi)} \mu_T(\mathrm{d}\varphi) \tag{4.2}$$

where as in the previous chapters μ_T will be a family of Gaussian measures on spaces of smooth functions such that as $T \to \infty$ $\mu_T \to \mu$ and $\rho \in C_c^\infty(\mathbb{R}^2, [0, 1])$ is a spatial cutoff while $Z_{T,\rho}$ is a normalization factor. Note also that we have introduced the constant $\alpha(T) \to \infty$ as $T \to \infty$. This is necessary to prevent the measure from becoming trivial in the $T \to \infty$ limit, and plays the same role as Wick ordering for Φ^4_2. We then want to achieve the following goals

1. Prove that a weak limit of (4.2) exists as $T \to \infty, \rho \to 1$.
2. Characterize it via a variational formula and as a random shift of the GFF.
3. Prove that it satisfies the Osterwalder Schrader axioms

The main difficulty in implementing this program compared to Chapter 2 is the aforementioned infrared divergence. Let us briefly sketch our strategy for dealing with this: We will study the Laplace transform

$$\int e^{-f(\phi)} \nu_{\mathrm{SG}}^{\rho,T}(\mathrm{d}\phi)$$

for $f(\phi)$ depending only on the value of ϕ in a bounded region. Then the Boué-Dupuis (Theorem 2.4 above or Corollary 4.14 below) formula will give

$$\begin{aligned} &-\log \int e^{-f(\phi)} \nu_{\mathrm{SG}}^{\rho,T}(\mathrm{d}\phi) \\ = &\inf_{u \in \mathbb{H}_a} \mathbb{E}\left[f(W_T + I_T(u)) + \lambda\alpha(T) \int \rho \cos(\beta W_T + \beta I_T(u)) + \frac{1}{2} \int_0^T \|u\|^2_{L^2(\mathbb{R}^2)} \right] \\ &- \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[\lambda\alpha(T) \int \rho \cos(\beta W_T + \beta I_T(u)) + \frac{1}{2} \int_0^T \|u\|^2_{L^2(\mathbb{R}^2)} \right] \end{aligned}$$

where $I_T(u)$ is a linear map behaving similarly to $(m^2 - \Delta)^{-1/2}$ as in Chapter 2 and Chapter 3.

Denoting by u^f a minimizer of

$$\mathbb{E}\left[f(W_T + I_T(u)) + \lambda\alpha(T) \int \rho \cos(\beta W_T + \beta I_T(u)) + \frac{1}{2} \int_0^T \|u\|^2_{L^2(\mathbb{R}^2)} \right],$$

provided it exists (which we will show) we end up with

$$
\begin{aligned}
& -\log \int e^{-f(\phi)} \nu_{\mathrm{SG}}^{\rho,T}(\mathrm{d}\phi) \\
= \;& \mathbb{E}\bigg[f(W_T + I_T(u^f)) + \lambda\alpha(T) \int \rho(\cos(\beta W_T + \beta I_T(u^f)) - \cos(\beta W_T + \beta I_T(u^0))) \\
& + \frac{1}{2}\int_0^T \|u^f\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{2}\int_0^T \|u^0\|_{L^2(\mathbb{R}^2)}^2 \bigg]
\end{aligned}
$$

and we want to control this quantity in the $T \to \infty, \rho \to 1$ limit. Since after we send $\rho \to 1$ we will have an integral over $\mathbb{R}^2$, we will need to have some decay of $u^f - u^0$ to be able to control the limit. In fact we will show that $u^f - u$ decay's exponentially fast away from the support of f or more precisely

$$
\int_0^T \int \exp(\gamma x) |u_t^f(x) - u_t^0(x)|^2 \mathrm{d}x \mathrm{d}t < \infty \tag{4.3}
$$

for some $\gamma > 0$. So one could say that the main contribution of this chapter is controlling the dependence of the minimizer on perturbations in bounded regions. A related issue is the problem of controlling the dependence of u^0 on ρ as $\rho \to 1$ and showing that it has a unique limit, which is needed for proving rotation invariance of the measure.

We will carry our our analysis of the Sine-Gordon model in the case $\beta^2 < 4\pi$. However it is known that the measure can be (and has been) constructed for the range $\beta^2 < 8\pi$, [48, 89, 49].For $\beta^2 > 4\pi$ the Sine Gordon measure becomes singular with respect to the free field, even in finite volume, similarly to Φ_3^4, as shown in Chapter 3. At the thresholds $\beta^2 \geqslant 8\pi(1 - 1/2n)$ $n \in \mathbb{N}$ the partition function acquires more and more divergent contributions which require renormalization, however contrary to Φ_3^4 it is sufficient to subtract constant terms from the potential to make the measure convergent, in particular no "mass renormalization" is necessary. It would be very interesting to extend our analysis to the range $\beta^2 < 8\pi$ (or even any other threshold beyond 4π). Before we move on to the main results of this chapter let us make some conventions.

Convention 4.1.

- For $a \in \mathbb{R}^d$ we let $\langle a \rangle := (1 + |a|^2)^{1/2}$. $B(x, r) \subseteq \mathbb{R}$ denotes the open ball of center $x \in \mathbb{R}$ and radius $r > 0$.
- Denote with $\mathscr{S}(\Lambda)$ the space of Schwartz functions on Λ and with $\mathscr{S}'(\Lambda)$ the dual space of tempered distributions. The notation $\hat{f}$ or $\mathscr{F} f$ stands for the space Fourier transform of f and we will write $g(\mathrm{D})$ to denote the Fourier multiplier operator with symbol $g: \mathbb{R}^n \to \mathbb{R}$, i.e. $\mathscr{F}(g(\mathrm{D}) f) = g \mathscr{F} f$.
- $B_{p,q}^\alpha = B_{p,q}^\alpha(\mathbb{R}^2)$ denotes the Besov spaces of regularity α and integrability indices p, q as usual. By $B_{p,q}^\alpha(\langle x \rangle^{-n})$ we denote the weighted Besov space with weight $\langle \cdot \rangle^{-n}$ see Appendix A for details . $\mathscr{C}^\alpha = \mathscr{C}^\alpha(\mathbb{R}^2)$ is the Hölder–Besov space $B_{\infty,\infty}^\alpha$, $W^{\alpha,p} = W^{\alpha,p}(\mathbb{R}^2)$ denote the standard fractional Sobolev spaces defined by the norm $\|f\|_{W^{s,q}} := \|\langle \mathrm{D} \rangle^s f\|_{L^q}$ and $H^\alpha = W^{\alpha,2}$. The reader is referred to Appendix A for an overview of the functional spaces and paraproducts.

Convention 4.2. In the sequel C will be a large constant which changes from line to line and can depend on β^2, but non on λ, ρ, T. δ will be a small constant which changes from line to line. Furthermore for $\beta^2 \in [0, \tilde{\beta}]$, $\tilde{\beta} < 4\pi$, C will always be uniformly bounded from above, δ will be uniformly bounded above and away from 0.

Convention 4.3. Throughout the chapter we will frequently compute Gradients and Hessian of functionals on $f: L^2(\mathbb{R}^2) \to \mathbb{R}$. We will always interpret $\nabla f(\varphi)$, to be an element $L^2(\mathbb{R}^2)$ by the Riesz representation theorem. Similarly we will always interpret Hess $f(\varphi)$ to be an operator on $L^2(\mathbb{R})$.

Convention 4.4. We will say that family of spacial cutoffs $\rho^N \in C_c^\infty(\mathbb{R}^2, [0,1])$ converges to 1 or $\rho^N \to 1$ if for any $K > 0$ there exists $N_0 \in \mathbb{N}$ such that $\rho^N(x) = 1$ for any $x \in B(0,K)$ and $N \geqslant N_0$. Often we will drop the index N and simply write $\rho \to 1$ in this case.

Let us now state the results we will prove in this chapter.

4.1.1. Overview of the results

4.1.1.1. Descriptions of the measure

In Chapter 2 we obtained a variational description of the Laplace transform of Φ_3^4 in finite volume. The importance of this result lies in fact that Φ_3^4 is not absolutely continuous with respect to the free field and so it is hard to describe explicitly. Similarly to Φ_2^4 the Sine Gordon measure for $\beta^2 < 4\pi$ on the other hand is absolutely continuous with respect to the free field in finite volume. However this property is lost once one removes the infrared cutoff. For this reason it is still interesting to obtain a description of the Laplace transform of the Sine Gordon measure in infinite volume. Our first result will indeed be such a description (we adapt the notation for weighted spaces introduced in the appendix):

Theorem 4.5. *$\nu_{\mathrm{SG}}^{T,\rho}$ weakly converges as $T \to \infty, \rho \to 1$ to a measure ν_{SG} on $\mathscr{S}'(\mathbb{R}^2)$. Furthermore ν_{SG} satisfies*

$$-\log \int e^{-f(\varphi)} \nu_{\mathrm{SG}}(\mathrm{d}\varphi) = \inf_{u \in \mathbb{D}^f} \mathbb{E}\left[f(W_{0,\infty} + I_{0,\infty}(u) + I_{0,\infty}(u^\infty)) + \Psi(u) + \frac{1}{2}\int_0^\infty \|u_t\|_{L^2}^2 \mathrm{d}t \right]$$

where similarly to Chapter 2:

- *W is a gaussian process and a martingale, and $\mathrm{Law}(W_T) = \mu_T$, $\mathbb{H}_a \subseteq L^2(\mathbb{P}, L^2(\mathbb{R}_+ \times \mathbb{R}^2))$ is the space of square integrable processes adapted to the filtration generated by W.*
- *$u^\infty \in L^\infty(\mathbb{P}, L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)) \cap L^2(\mathbb{P}, L^2(\mathbb{R}_+, L^2(\langle x \rangle^{-n})))$ where by $L^2(\langle x \rangle^{-n})$ we denote the space equipped with the norm $\|f\|_{L^2(\langle x \rangle^{-n})}^2 = \int_{\mathbb{R}^2} \langle x \rangle^{-n} f(x) \mathrm{d}x$. u^∞ does not depend on f.*
- *I is a linear map improving regularity by 1, see eq. (4.1.2.1) below for details.*
- *$\Psi(u)$ is a functional of u which also depends on u^∞ and W, but not on f, it will be specified below.*
- *$\mathbb{D}^f$ is a subspace of $\mathbb{H}_a$ containing drifts with exponential decay in space, it will also be specified below.*

We will also obtain a description of the Sine Gordon measure as a random shift of a Gaussian measure, similarly to how we described the drift measure in Chapter 3.

Theorem 4.6. *There exists a random variable $I \in L^\infty(\mathbb{P}, W^{1,\infty}(\mathbb{R}^2))$ such that*

$$\nu_{\mathrm{SG}} = \mathrm{Law}_{\mathbb{P}}(W_\infty + I).$$

Furthermore the Law of (W_∞, I) is invariant under the action of the Euclidean group.

From this we immediately obtain that ν_{SG} has gaussian tails:

Corollary 4.7. *For any ε there exists an $\alpha > 0$ such that*

$$\int e^{+\alpha \|\varphi\|_{H^{-\varepsilon}(\langle x \rangle^{-n})}^2} \mathrm{d}\nu_{\mathrm{SG}} < \infty.$$

The estimates we will prove for this will be strong enough to partially recover the results of [62].

Theorem 4.8. *ν_{SG} satisfies the Osterwalder-Schrader axioms. Furthermore the clustering is exponential and ν_{SG} is non-Gaussian.*

Proof. Euclidean invariance follows from Theorem 4.6. Corollary 4.7 implies that the measure is exponentially integrable. Reflection Positivity is proved in Section 4.8.1 and exponential clustering is proved in section 4.8.2 while non Gaussianity is shown in Section 4.8.3. □

4.1.1.2. Large deviations

As in Section 2.7 we will discuss large deviations for ν_{SG} in a semi-classical limit. For this we have to introduce Planck's constant into the measure. Indeed we want to look at the measure formally given by

$$\mathrm{d}\nu_{\mathrm{SG},\hbar} = \frac{1}{Z_\hbar} e^{-\frac{1}{\hbar}\int \lambda\cos(\beta\varphi(x)) + \frac{1}{2}m^2\varphi(x)^2 + \frac{1}{2}|\nabla\varphi(x)|^2\mathrm{d}x}\mathrm{d}\varphi.$$

This can be rewritten as

$$\nu_{\mathrm{SG},\hbar}(\mathrm{d}\varphi) = \frac{1}{Z_\hbar} e^{-\frac{\lambda}{\hbar}\int\cos(\hbar^{1/2}\beta\varphi)}\mu(\mathrm{d}\varphi).$$

where $Z_\hbar$ is normalization constant and we are interested in the limit $\hbar\to 0$. These measure can be made sense of in the same way as ν_{SG}. We will prove

Theorem 4.9. *$\nu_{\mathrm{SG},\hbar}$ satisfies a large deviations principle with rate function*

$$I(\varphi) = \lambda\int(\cos(\varphi(x)) - 1)\mathrm{d}x + \frac{1}{2}m^2\int\varphi^2(x)\mathrm{d}x + \frac{1}{2}\int|\nabla\varphi(x)|^2\mathrm{d}x.$$

or equivalently

$$\lim_{\hbar\to 0} - \hbar\log\int e^{-\frac{1}{\hbar}f(\varphi)}\mathrm{d}\nu_{\mathrm{SG},\hbar} = \inf_{\varphi\in H^{-1}(\langle x\rangle^{-n})}\{f(\varphi) + I(\varphi)\}.$$

Similar results were obtained by Lacoin Rhodes and Vargas for the Liouville measure on a compact surface [88, 90], however to our knowledge this is the first such result in infinite volume.

4.1.2. Strategy

In order to achieve (4.3) we would like to use the convexity of

$$\lambda\alpha(T)\int\rho\cos(\beta W_T + \beta I_T(u)) + \frac{1}{2}\int_0^T\|u_t\|_{L^2}^2\mathrm{d}t$$

in u. However it is quite obvious that even if λ is small this functional is not convex for large T since $\alpha(T)\to\infty$ as $T\to\infty$. To remedy this we will make use of the stochastic control structure of the problem.

4.1.2.1. Polchinski equation and stochastic control

Let X_t be a cylindrical Brownian motion on $L^2(\mathbb{R}^2)$ and let

$$W_{s,t} = \int_s^t Q_l\mathrm{d}X_l$$

with $Q_t = \left(\frac{1}{t^2}e^{-(m^2-\Delta)/t}\right)^{1/2}$. We use here the heat kernel instead of a decomposition with compact support in Fourier space, because it has exponential decay in space. This will be useful in proving exponential clustering. One can check that Law $(W_{0,T})\to\mu$ by computing the covariance. Let us introduce the effective potential for $\varphi\in L^2(\mathbb{R}^2)$

$$V_{t,T}^f(\varphi) = -\log\mathbb{E}\left[\exp\left(-f(W_{t,T}+\varphi) - \lambda\alpha(T)\int\rho\cos(\beta W_{t,T}+\beta\varphi)\right)\right]$$

with f being a sufficiently nice functional (to be made more precise below).

We will show below following [15] that $V_{t,T}^f(\varphi)$ satisfies the Polchinski equation [108]

$$\frac{\partial}{\partial t}v_{t,T}(\varphi)+\frac{1}{2}\mathrm{Tr}(\dot{C}_t\,\mathrm{Hess}\,v_{t,T}(\varphi))-\frac{1}{2}\|Q_t\nabla v_{t,T}(\varphi)\|_{L^2(\mathbb{R}^2)}^2=0.$$

By the BD formula we will also have writing $V_T^f(\varphi)=f(\varphi)-\lambda\alpha(T)\int\rho\cos(\beta\varphi)$

$$V_{t,T}^f(\varphi)=\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[V_T^f(W_{t,T}+I_{t,T}(u)+\varphi)+\frac{1}{2}\int_t^T\|u_s\|_{L^2}^2\mathrm{d}s\right],$$

where $\mathbb{H}_a$ is as in the previous chapters the space of square integrable adapted processes, $W_{t,T}$ will be a Gaussian process described more precisely below and

$$I_{t,T}(u)=\int_t^T Q_s u_s\mathrm{d}s.$$

Now by dynamic programming (see Proposition 4.12 below) we have

$$V_{s,T}^f(\varphi)=\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[V_{t,T}^f(W_{s,t}+I_{s,t}(u)+\varphi)+\frac{1}{2}\int_s^t\|u_w\|_{L^2}^2\mathrm{d}w\right].$$

What allows use convexity to control the minimizer is the following:

- For $t\geqslant T/2$ and λ small enough $\lambda\alpha(T)\int\rho\cos(\beta W_{t,T}+I_{t,T}(u)+\beta\varphi)+\frac{1}{2}\int_s^t\|u_w\|_{L^2}^2\mathrm{d}w$ is convex in u.
- $V_{t,T}^0=\lambda\alpha(t)\int\rho\cos(\beta\varphi)+R_{t,T}(\varphi)$ where $\sup_{\varphi\in L^2(\mathbb{R}^2)}\|\mathrm{Hess}\,R_{t,T}(\varphi)\|$ is bounded uniformly in t,T.
- $\nabla(V_{t,T}^f(\varphi)-V_{t,T}^0(\varphi))$ is small away from the support of f, where we interpret $\nabla V_{t,T}^f$ according to Convention 4.3.

The first fact is quite obvious. The second and the third fact are nontrivial and will be discussed in Section 4.4 and Section 4.3 respectivly.

4.1.2.2. Outline of the chapter

In Section 4.2 we will recall the derivation of the Polchinski equation for $v_{t,T}$, and recall some notions from stochastic optimal control and some properties of the renormalized cosine, establishing that it converges to a well defined random distribution. In Section 4.3 we will derive (4.3) provided that the value function satisfies some properties. In Section 4.4 we will establish the necessary control on the Hessian to apply the result of Section 4.3 and obtain (4.3). In Section 4.5 we will refine our estimates to understand the dependence of minimizer u^0 on the spacial cutoff ρ. Section 4.6 will be used to derive Theorem 4.5 from the preceeding analysis. In Section 4.7 we will show how to express expectations under the Sine Gordon measure in terms of the minimizer in the variational problem and prove Theorem 4.6. In the final two sections of the chapter we will establish the Osterwalder-Schrader Axioms and the Large Deviations Principle (Theorems 4.8 and 4.9) respectively.

4.2. Setup

4.2.1. Stochastic optimal control

We consider the decomposition (with $L=(m^2-\Delta)$)

$$L^{-1}=\int_0^\infty Q_t^2\mathrm{d}t$$

where

$$Q_t=\left(\frac{1}{t^2}e^{-L/t}\right)^{1/2}.$$

We denote by

$$\mathcal{C}_t = \int_0^t Q_s^2 \mathrm{d}s = L^{-1} e^{-L/t}, \tag{4.4}$$

and by $K_t(x, y)$ the kernel of $\mathcal{C}_t$. From the definitions one can see that

$$K_t(x,y) = \int_0^t e^{-m^2/s}\left(\frac{1}{s^2}\frac{s}{4\pi}e^{-4s|x-y|^2}\right)\mathrm{d}s = \int_0^t e^{-m^2/s^2}\left(\frac{1}{4\pi s}e^{-4s|x-y|^2}\right)\mathrm{d}s$$

so

$$K_t(x,x) = \int_0^t e^{-m^2/s^2}\left(\frac{1}{4\pi s}\right)\mathrm{d}s = \mathbb{1}_{t\geqslant 1}\frac{1}{4\pi}\log t + C(t)$$

where $\sup_{t\in\mathbb{R}_+} C(t) < \infty$. Let $0 \leqslant s < t$ and $u \in L^2([s,t], L^2(\mathbb{R}^2))$. For later use we introduce the notation

$$I_{s,t}(u) = \int_s^t Q_l u_l \mathrm{d}l.$$

We are interested in studying the quantities

$$v_{t,T}(\varphi) = -\log \mathbb{E}[\exp(-V_T(\varphi + W_{t,T}))]$$

where $W_{t,T} = \int_t^T Q_s \mathrm{d}X_s$,with X being a cylindrical Brownian motion on $L^2(\mathbb{R}^2)$, and $Z_{t,T} = \exp(-v_{t,T})$, for $\varphi \in L^2(\mathbb{R}^2)$.

For the rest of this chapter we will denote by $C^n(L^2(\mathbb{R}^2))$ functions $L^2(\mathbb{R}^2) \to \mathbb{R}$ which are n times continuously Fréchet differntiable with bounded derivatives. Next we can derive a Hamilton-Jacobi-Bellmann equation for $v_{t,T}$, known in the physics literature as the Polchinski equation.

Proposition 4.10. *Assume that $V_T \in C^2(L^2(\mathbb{R}^2))$. Then $v_{t,T}$ satisfies*

$$\frac{\partial}{\partial t} v_{t,T}(\varphi) + \frac{1}{2}\mathrm{Tr}(\dot{\mathcal{C}}_t \,\mathrm{Hess}\, v_{t,T}(\varphi)) - \frac{1}{2}\|Q_t \nabla v_{t,T}(\varphi)\|^2_{L^2(\mathbb{R}^2)} = 0$$

$$v_{T,T}(\varphi) = V_T(\varphi).$$

Furthermore if $V_T \in C^2(L^2(\mathbb{R}^2))$ then $v_{t,T} \in C([0,T], C^2(L^2(\mathbb{R}^2))) \cap C^1([0,T], C(L^2(\mathbb{R}^2)))$.

Proof. Write $Z_{t,T} = \exp(-v_{t,T}) = \mathbb{E}[\exp(-V_T(\varphi + W_{t,T}))]$. Noting that $W_{t,T}$ has covariance $C_T - C_t$ it is not hard to see that

$$\begin{aligned}\frac{\partial}{\partial t} Z_{t,T} &= \frac{\partial}{\partial t}\mathbb{E}[\exp(-V_T(\varphi + W_{t,T}))] \\ &= -\mathbb{E}[\langle W_{t,T}, (\mathcal{C}_T - \mathcal{C}_t)^{-2}\dot{\mathcal{C}}_t W_{t,T}\rangle_{L^2(\mathbb{R}^2)} \exp(-V_T(\varphi + W_{t,T}))].\end{aligned}$$

Now using Gaussian integration by parts (see [17] Exercise 2.1.3)

$$\begin{aligned}& -\mathbb{E}[\langle W_{t,T}, (\mathcal{C}_T - \mathcal{C}_t)^{-2}\dot{\mathcal{C}}_t W_{t,T}\rangle_{L^2(\mathbb{R}^2)} \exp(-V_T(\varphi + W_{t,T}))] \\ =\; & -\mathrm{Tr}(\dot{\mathcal{C}}_t \,\mathrm{Hess}\, Z_{t,T}(\varphi)).\end{aligned}$$

Applying chain rule we get

$$\begin{aligned}\frac{\partial}{\partial t} v_{t,T} &= -\frac{\partial}{\partial t}\log Z_{t,T} \\ &= -\frac{\frac{\partial}{\partial t} Z_{t,T}}{Z_{t,T}} \\ &= \frac{\mathrm{Tr}(\dot{C}_t \,\mathrm{Hess}\, Z_{t,T}(\varphi))}{Z_{t,T}} \\ &= e^{v_{t,T}} \mathrm{Tr}(\dot{\mathcal{C}}_t \,\mathrm{Hess}\, e^{-v_{t,T}}) \\ &= -\mathrm{Tr}(\dot{\mathcal{C}}_t \,\mathrm{Hess}\, v_{t,T}) + \langle \nabla v_{t,T}, \dot{\mathcal{C}}_t \nabla v_{t,T}\rangle_{L^2(\mathbb{R}^2)}\end{aligned}$$

For the second statement differentiating under the expectation we obtain $Z_{t,T}(\varphi)\in C^2(L^2(\mathbb{R}^2))$, so using our first computation we can deduce from this that also $Z_{t,T}\in C^1([0,T],C(L^2(\mathbb{R}^2)))$. Now observing that if $V_T\in C^2(L^2(\mathbb{R}^2))$ then $\inf_{t,\varphi} Z_{t,T}(\varphi)>0$, and using chain rule we can conclude. □

DEFINITION 4.11. *Let $T>0$, H be a Hilbert space and $V_T\colon H\to\mathbb{R}$ measurable,bounded below. Let X_t be a cylindrical process on some Hilbert space Ξ. Let Λ be a Polish space and $u\colon[0,T]\to\Lambda$ be a process adapted to X_t. Let $Y_{s,t}(\varphi,u)$ be a solution to the equation*

$$\mathrm{d}Y_{s,t}(u,\varphi)=\beta(t,Y_{s,t}(u,\varphi),u_t)\mathrm{d}t+\sigma(t,Y_{s,t}(u,\varphi),u_t)\mathrm{d}X_t \tag{4.5}$$

$$Y_s(u,\varphi)=\varphi.$$

Where $\beta\colon[0,T]\times H\times\Lambda\to H$ and $\sigma\colon[0,T]\times H\times\Lambda\to\mathcal{L}(\Xi,H)$ are measurable. Then we say that $V_{t,T}$ is the value function on the stochastic control problem if

$$V_{t,T}(\varphi)=\inf_{u\in A([s,T])}\mathbb{E}\left[V_T(Y_{s,T}(u,\varphi))+\int_s^T l_t(Y_{s,t},u_t)\mathrm{d}t\right],$$

with $l\colon[0,T]\times H\times\Lambda\to\mathbb{R}$ measurable, bounded below and we denote by $A([s,t])$ the space of all processes $u\colon[s,t]\to\Lambda$ which are adapted to X_t.

PROPOSITION 4.12. (DYNAMIC PROGRAMMING) *$V_{t,T}$ as defined above satisfies for any $S<T$*

$$V_{t,T}(\varphi)=\inf_{u\in A([t,S])}\mathbb{E}\left[V_{S,T}(Y_{t,S}(u,\varphi))+\int_t^S l_s(Y_{t,s},u_t)\mathrm{d}t\right].$$

For a proof see [52] Theorem 2.24.

Now assume that $\sigma(t,Y_t,u_t)$ is self adjoint.We can associate a HJB equation to the control problem from Definition 4.11 . It is:

$$\frac{\partial}{\partial t}v(t,\varphi)+\frac{1}{2}\inf_{a\in\Lambda}[\mathrm{Tr}(\sigma^2(t,\varphi,a)\mathrm{Hess}\,v(t,\varphi))+\langle\nabla v,\beta(t,\varphi,a)\rangle_H+l(t,\varphi,a)]=0. \tag{4.6}$$

$$v(T,\varphi)=V_T(\varphi)$$

We have the following theorem relating (4.6) to the solution of the control problem:

PROPOSITION 4.13. (VERIFICATION) *Assume that $v\in C([0,T],C^{2,\mathrm{loc}}(H))\cap C^{1,\mathrm{loc}}([0,T],C(H))$ and v solves (4.6) with $v(T,\varphi)=V_T(\varphi)$. Furthermore assume that there exists $u\in A([t,T])$ and Y such that u,Y satisfy (4.5) and*

$$u_t\in\mathrm{argmin}_{a\in\Lambda}[\mathrm{Tr}(\sigma^2(t,Y_t,u_t)\,\mathrm{Hess}\,v(t,Y_t))+\langle\nabla v(t,Y_t),\beta(t,Y_t,a)\rangle_H+l(t,Y_t,a)]. \tag{4.7}$$

Then $v(t,\varphi)=V_{t,T}(\varphi)$ and the pair u,Y is optimal.

For a proof see [52] Theorem 2.36. Now consider the case $H=\Lambda=L^2(\mathbb{R}^2)$ and

$$\begin{aligned}\beta(t,\varphi,a)&=Q_t a\\ \sigma(t,\varphi,a)&=Q_t\\ l(t,Y_t,a)&=\frac{1}{2}\|a\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Then (4.7) becomes a minimization problem for a quadratic functional and reduces to

$$u_t=-Q_t\nabla v(t,Y_{s,t}).$$

This means if we can solve the equation

$$\mathrm{d}Y_{s,t}=-Q_t\nabla v(t,Y_{s,t})\mathrm{d}t+Q_t\mathrm{d}X_t, \tag{4.8}$$

we can apply the verification theorem. Furthermore in this case (4.6) takes the form

$$\frac{\partial}{\partial t}v(t,\varphi)+\frac{1}{2}\mathrm{Tr}(\dot{\mathcal{C}}_t\mathrm{Hess}\,v(t,\varphi))-\frac{1}{2}\|Q_t\nabla v(t,\varphi)\|_{L^2(\mathbb{R}^2)}^2=0. \tag{4.9}$$

since

$$\begin{aligned}
&\inf_{a\in\Lambda}\,[\,\mathrm{Tr}(\sigma(t,\varphi,a)\mathrm{Hess}\,v(t,\varphi))+\langle\nabla v,\beta(t,\varphi,a)\rangle_H+l(t,\varphi,a)]\\
=&\inf_{a\in\Lambda}\left[\,\mathrm{Tr}(Q_t^2\mathrm{Hess}\,v(t,\varphi))+\langle\nabla v,Q_ta\rangle_{L^2(\mathbb{R}^2)}+\frac{1}{2}\|a\|_{L^2(\mathbb{R}^2)}^2\right]\\
=&\frac{1}{2}\mathrm{Tr}(\dot{\mathcal{C}}_t\mathrm{Hess}\,v(t,\varphi))-\frac{1}{2}\|Q_t\nabla v(t,\varphi)\|_{L^2(\mathbb{R}^2)}^2
\end{aligned}$$

Corollary 4.14.

$$-\log\mathbb{E}[e^{-V_T(\varphi+W_{t,T})}]=\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[V_T(Y_{s,T}(u,\varphi))+\frac{1}{2}\int_s^T\|u_t\|_{L^2}^2\mathrm{d}t\right]$$

where $\mathbb{H}_a$ is the space of processes adapted to X_t such that $\mathbb{E}[\int_0^\infty\|u_t\|_{L^2}^2\mathrm{d}t]$ and $Y_t(u,\varphi)$ satisfies

$$\mathrm{d}Y_{s,t}(u,\varphi)=-Q_tu_t\mathrm{d}t+Q_t\mathrm{d}W_t$$

$$Y_{s,s}(u,\varphi)=\varphi.$$

Note that $Y_{s,T}(u,\varphi)=\varphi+W_{t,T}+I_{t,T}(u)$. Furthermore the infimum on the r.h.s is attained

Proof. As already noted $v_{t,T}=-\log\mathbb{E}[e^{-V_T(\varphi+W_{t,T})}]$ satisfies the HJB equation (4.9) and is in $C([0,T],C^2(L^2(\mathbb{R}^2)))$, so $\nabla v_{t,T}$ is Lipschitz continuous uniformly in T and bounded. By a standard fix-point argument we can then solve (4.8), and so applying the verification theorem we obtain

$$-\log\mathbb{E}[e^{-V_T(\varphi+W_{t,T})}]=\inf_{u\in A([s,T])}\mathbb{E}\left[V_T(Y_{s,T}(u,\varphi))+\frac{1}{2}\int_s^T\|u_t\|_{L^2}^2\mathrm{d}t\right].$$

Since V_T is bounded below we can clearly restrict the infimum on the right hand side to $u\in\mathbb{H}_a$. □

This proof of the Boue-Dupuis formula is very similar to the one that can be found in [37] Chapters 8.1.3 and 8.1.4.

4.2.2. Martingale cutoff and renormalized cosine

We recall the definition of the regularized GFF as

$$W_t=W_{0,t}=\int_0^tQ_s\mathrm{d}X_s$$

where X_s is a cylindrical Brownian motion on L^2. We can calculate:

$$\mathbb{E}[W_t(x)W_t(y)]=K_t(x,y).$$

Now it is not hard to see from Ito's formula that the quantity

$$e^{\frac{\beta^2}{2}K_t(x,x)}\cos(\beta W_t(x))=:\alpha(t)\cos(\beta W_t(x)) \tag{4.10}$$

is a martingale. We will write

$$\begin{aligned}
[\![\cos(\beta W_t)]\!](x)&=\alpha(t)\cos(\beta W_t(x))\\
[\![\sin(\beta W_t)]\!](x)&=\alpha(t)\sin(\beta W_t(x))\\
[\![e^{i\beta W_t}]\!](x)&=\alpha(t)e^{iW_t(x)}
\end{aligned}$$

We claim that is $[\![\cos(\beta W_t)]\!]$ bounded in $L^2(\mathbb{P},H^{-1+\delta}(\langle x\rangle^{-n}))$ uniformly in t,g. Since it is also a martingale it converges almost surely. To prove this the following lemma will be helpful:

Lemma 4.15.

$$|\mathbb{E}[\llbracket\cos(\beta W_t)\rrbracket(x)\llbracket\cos(\beta W_t)\rrbracket(y)]-1|\leqslant\frac{1}{|x-y|^{\beta^2/2\pi}}$$

Proof. Recall that $e^{\frac{\beta^2}{2}K_t(x,x)}:=\alpha(t)\leqslant C\langle t\rangle^{\frac{\beta^2}{8\pi}}$. By Ito's formula

$$\mathrm{d}\llbracket\cos(\beta W_t)\rrbracket(x)=-\beta\alpha(t)\sin(\beta W_t(x))\mathrm{d}W_t(x),$$

so by Ito's isometry:

$$\begin{aligned}
& |\mathbb{E}[\llbracket\cos(\beta W_t)\rrbracket(x)\llbracket\cos(\beta W_t)\rrbracket(y)]-1| \\
= \; & \beta^2\left|\mathbb{E}\int_0^t\alpha^2(t)\sin(\beta W_s(x))\sin(\beta W_s(y))\mathrm{d}\langle W_s(x)W_s(y)\rangle\right| \\
\leqslant \; & \beta^2\int_0^t\alpha^2(t)|\mathbb{E}[\sin(\beta W_s(x))\sin(\beta W_s(y))]|Q_s(x,y)\mathrm{d}s \\
\leqslant \; & C\beta^2\int_0^t\alpha^2(t)Q_s(x,y)\mathrm{d}t \\
\leqslant \; & C\beta^2\int_0^t\langle s\rangle^{\frac{\beta^2}{4\pi}}e^{-m^2/s^2}\left(\frac{1}{4\pi s}e^{-4s|x-y|^2}\right) \\
\leqslant \; & C\beta^2\int_0^\infty\langle s\rangle^{\frac{\beta^2}{4\pi}}e^{-m^2/s^2}\left(\frac{1}{4\pi s}e^{-4s|x-y|^2}\right) \\
\leqslant \; & C\beta^2\frac{1}{|x-y|^{\beta^2/2\pi}}
\end{aligned}$$

where in the last line we have used the change of variables $s'=s|x-y|^2$. □

Lemma 4.16. *Let N_t be a family of random functions in L^∞ such that*

$$|\mathbb{E}[N_t(x)N_t(y)]|\leqslant C|x-y|^{-\gamma}$$

with $\gamma<2$. Then for any $\delta>0$ small enough

$$\sup_t\mathbb{E}\left[\|N_t\|^2_{H^{-\gamma/2-\delta}(\langle x\rangle^{-n})}\right]\leqslant C.$$

Proof. Recall the devinition of the Littlewood-Palye blocks $\Delta_iN_t=\varphi_i*N_t$. Using the Littlewood-Paley characterization we can estimate

$$\begin{aligned}
& \mathbb{E}\left[\|N_t\|^2_{H^{-\gamma/2-\delta}(\langle x\rangle^{-n})}\right] \\
= \; & \sum_i 2^{-(\gamma-2\delta)}\mathbb{E}[\|\Delta_iN_t\|^2_{L^2(\langle x\rangle^{-n})}] \\
\leqslant \; & C\sup_i 2^{-\gamma-\delta}\mathbb{E}[\|\Delta_iN_t\|^2_{L^2(\langle x\rangle^{-n})}].
\end{aligned}$$

Now φ_i satisfies,by interpolation

$$\left\|\int\langle x-y\rangle^n\varphi_i(x-y)\langle x-y\rangle^n\varphi_i(x-z)\mathrm{d}x\right\|_{L^k(\mathrm{d}y\mathrm{d}z)}\lesssim\|\varphi_i\|^{2(1-1/k)}_{L^2(\langle x\rangle^n)}\|\varphi_i\|^{2/k}_{L^1(\langle x\rangle^n)}.$$

since

$$\begin{aligned}
\left\|\int\langle x-y\rangle^{-n}|\varphi_i(x-y)|\langle x-y\rangle^{-n}|\varphi_i(x-z)|\mathrm{d}x\right\|_{L^\infty} & \leqslant \|\varphi_i\|^2_{L^2(\langle x\rangle^n)} \\
\left\|\int\langle x-y\rangle^{-n}|\varphi_i(x-y)|\langle x-y\rangle^{-n}|\varphi_i(x-z)|\mathrm{d}x\right\|_{L^2} & \leqslant \|\varphi_i\|^2_{L^1(\langle x\rangle^n)}
\end{aligned}$$

Recall that

$$\sup_i\|\varphi_i\|_{L^1(\langle x\rangle^n)}\leqslant C \qquad \text{and} \qquad \|\varphi_i\|_{L^2(\langle x\rangle^n)}\leqslant C2^i$$

Using thsi we get:

$$
\begin{aligned}
& \left[\int \rho \mathbb{E}(\Delta_i N_t)^2 \mathrm{d}x\right] \\
= & \mathbb{E}\left[\int \langle x\rangle^{-2n} \int \varphi_i(x-y) N_t(y) \mathrm{d}y \int \varphi_i(x-z) N_t(z) \mathrm{d}z \mathrm{d}x\right] \\
\leqslant & C\left[\iint \langle x-y\rangle^n |\varphi_i(x-y)| \langle x-y\rangle^n |\varphi_i(x-z)| \mathrm{d}x \frac{\langle y\rangle^{-n}\langle z\rangle^{-n}}{|y-z|^\gamma} \mathrm{d}y \mathrm{d}z\right] \\
\leqslant & C\left\|\int \langle x-y\rangle^n |\varphi_i(x-y)| \langle x-y\rangle^n |\varphi_i(x-z)| \mathrm{d}x\right\|_{L^k(\mathrm{d}y\mathrm{d}z)} \left\|\frac{\langle y\rangle^{-n}\langle z\rangle^{-n}}{|y-z|^\gamma}\right\|_{L^l(\mathrm{d}y\mathrm{d}z)} \\
\leqslant & C(2^{2i})^{1-1/k}
\end{aligned}
$$

where we have chosen l such that $\gamma/(2+\delta) < l < \gamma/2$, $1/k+1/l=1$, so

$$
1-\frac{1}{k}=\frac{1}{q}<\gamma/(2+\delta)
$$

and

$$
(2^{2i})^{1-1/k} < (2^{2i})^{\gamma/(2+\delta)},
$$

so all together, since $\left\|\frac{\langle x\rangle^{-n}\langle y\rangle^{-n}}{|y-z|^\gamma}\right\|_{L^l(\mathrm{d}y\mathrm{d}z)} < \infty$ we deduce

$$
\mathbb{E}[\|\Delta_i N_t\|^2_{L^2(\langle x\rangle^{-n})}]^2 < (2^i)^{2\gamma/(2+\delta)}
$$

and choosing δ small enough such that $2\gamma/(2+\delta) < \gamma+\delta$ we can conclude. □

Lemma 4.17.

$$
\sup_{t<\infty} \mathbb{E}\left[\|[\![\cos(\beta W_t)]\!]\|^p_{B^{-\beta^2/4\pi-\delta}_{p,p}}\right] < \infty
$$

Proof.

$$
\begin{aligned}
& \mathbb{E}\left[\|[\![\cos(\beta W_t)]\!]\|^2_{B^{-\beta^2/4\pi-2\delta}_{p,p}}\right]^p \\
= & \sum_i 2^{-p(2\varepsilon-\beta^2/2\pi)} \mathbb{E}[\|\Delta_i [\![\cos(\beta W_t)]\!]\|^p_{L^p}] \\
\leqslant & C \sup_i 2^{-p\beta^2/2\pi(1+\delta)} \mathbb{E}[\|\Delta_i [\![\cos(\beta W_t)]\!]\|^p_{L^p}]
\end{aligned}
$$

Using this we can estimate using Fubini's theorem and BDG inequality

$$
\begin{aligned}
& C \sup_i 2^{-p\beta^2/2\pi(1+\delta)} \mathbb{E}[\|\Delta_i [\![\cos(\beta W_t)]\!]\|^p_{L^p}] \\
= & C \sup_i 2^{-p\beta^2/2\pi(1+\delta)} \mathbb{E}\left[\int \rho |(\Delta_i [\![\cos(\beta W_t)]\!])|^p\right] \\
= & C \sup_i 2^{-p\beta^2/2\pi(1+\delta)} \left[\int \rho \mathbb{E}|(\Delta_i [\![\cos(\beta W_t)]\!])|^p\right] \\
\leqslant & C \sup_i 2^{-p\beta^2/2\pi(1+\delta)} \left[\int \rho \mathbb{E}|(\Delta_i [\![\cos(\beta W_t)]\!])|^2\right]^{p/2} \\
\leqslant & C \mathbb{E}\left[\|[\![\cos(\beta W_t)]\!]\|^2_{H^{-\beta^2/4\pi-\delta}(\rho)}\right]^{p/2}
\end{aligned}
$$

and now the statement follows from Lemma 4.15 and Lemma 4.16. □

Definition 4.18. *Since* $[\![\cos(\beta W_t)]\!]$ *is a martingale and*

$$
\sup_t \mathbb{E}\left[\|[\![\cos(\beta W_t)]\!]\|^p_{B^{-\beta^2/4\pi-2\delta}_{p,p}(\langle x\rangle^{-n})}\right] < \infty
$$

it converges in $L^p\big(\mathbb{P}, B_{p,p}^{-\beta^2/4\pi-2\delta}(\langle x\rangle^{-n})\big)$ *to a limit. We will denote this limit by* $[\![\cos(\beta W_\infty)]\!]$ *(and analogously for* $\alpha(t)\sin(W_t)$ *and* $\alpha(t)e^{iW_t}$*).*

Remark 4.19. From Lemma 4.15 we see that as $\beta\to 0$ $\mathbb{E}[\|\Delta_i([\![\cos(\beta W_t)]\!]-1)\|_{L^2(\langle x\rangle^{-n})}^2]\to 0$. Together with Lemma 4.17 we can easily deduce from this that

$$\mathbb{E}\Big[\|([\![\cos(\beta W_t)]\!]-1)\|^2_{B_{p,p}^{-\beta^2/4\pi-3\varepsilon}(\langle x\rangle^{-n+1})}\Big]\to 0, \mathbb{E}\Big[\|([\![\sin(\beta W_t)]\!])\|^2_{B_{p,p}^{-\beta^2/4\pi-3\varepsilon}(\langle x\rangle^{-n+1})}\Big]\to 0.$$

At this point we are ready to define the approximate measures $\nu_{\mathrm{SG}}^{\rho,T}$ in a precise way.

Definition 4.20. *A Let* μ_T *be a Gaussian measure with covariance* $\mathcal{C}_T(m^2-\Delta)^{-1}$ *and* $\rho\in C_c^\infty(\mathbb{R}^2,[0,1])$*. Then we define*

$$\nu_{\mathrm{SG}}^{\rho,T}(\mathrm{d}\phi)=\frac{1}{Z_{T,\rho}}\exp\bigg(-\alpha(T)\int_{\mathbb{R}^2}\rho(x)\cos(\beta\phi(x))\mathrm{d}s\bigg)\mu_T(\mathrm{d}\phi)$$

and $Z_{T,\rho}$ *is the normalization constant*

$$Z_{T,\rho}=\int\exp\bigg(-\alpha(T)\int_{\mathbb{R}^2}\rho(x)\cos(\beta\phi(x))\mathrm{d}s\bigg)\mu_T(\mathrm{d}\phi).$$

4.2.3. Weighted estimates

In this section we collect some estimates on Q,I which will be important in the sequel. We invite the reader to read this section only superficially and to return to it once the estimates discussed here become important.

Lemma 4.21.

$$\|(m^2-\Delta)^{1/2}I_{0,\infty}(u)\|_{L^2}^2\leqslant\int_0^\infty\|u_s\|_{L^2}^2\mathrm{d}s$$

Proof.

$$\begin{aligned}
&\int_{\mathbb{R}^2}((m^2-\Delta)^{1/2}I_{0,\infty}(u))^2\mathrm{d}x\\
=&\int_{\mathbb{R}^2}(m^2+|k|^2)(\mathscr{F}I_{0,\infty}(u)(k))^2\mathrm{d}k\\
=&\int_{\mathbb{R}^2}(m^2+|k|^2)\bigg(\int_0^\infty\frac{1}{t}e^{-(m^2+|k|^2)/2t}\mathscr{F}u_t(k)\mathrm{d}t\bigg)^2\mathrm{d}k\\
\leqslant&\int_{\mathbb{R}^2}(m^2+|k|^2)\bigg(\int_0^\infty\frac{1}{t^2}e^{-(m^2+|k|^2)/t}\mathrm{d}t\bigg)\int_0^\infty(\mathscr{F}u_s(k))^2\mathrm{d}s\mathrm{d}k\\
=&\int_{\mathbb{R}^2}\int_0^\infty(\mathscr{F}u_s(k))^2\mathrm{d}s\mathrm{d}k\\
=&\int_0^\infty\|u_s\|_{L^2}^2\mathrm{d}s
\end{aligned}$$

□

Definition 4.22. *Let* $A\subset\mathbb{R}^2$*,* $r\in\mathbb{R}$*, We define the weight*

$$\omega^{A,r}(x)=\exp(rd(x,A))$$

where $d(x,A)=\inf_{y\in A}|x-y|$*.*

Definition 4.23. *For a set* $A\subseteq\mathbb{R}^2$*,*$r\in\mathbb{R}$ *we define the weighted* L^p *spaces*

$$\|f\|_{L^{p,r}(A)}=\bigg(\int(w^{A,r}(x))^pf^p(x)\mathrm{d}x\bigg)^{1/p}$$

And

$$\|f\|_{W^{1,p,r}(A)} = \|f\|_{L^{p,r}(A)} + \left(\int (w^{A,r}(x))^p (\nabla f(x))^p \mathrm{d}x\right)^{1/p}$$

We will also set $H^{1,r}(A) = W^{1,2,r}(A)$. *Furthermore we will set*

$$\|f\|_{L^{p,r}} = \|f\|_{L^{p,r}(B(0,1))}, \qquad \|f\|_{W^{1,p,r}} = \|f\|_{W^{1,p,r}(B(0,1))}.$$

It is not hard to see that denoting $A_i = \{y: i-1 \leqslant d(y,A) \leqslant i\}$ there exist $c, C > 0$ such that

$$c\left(\sum_{i=1}^{\infty} \exp(2ri)\|\mathbb{1}_{A^i} f\|_{L^2}^2\right)^{1/2} \leqslant \|f\|_{L^{2,r}(A)} \leqslant C\left(\sum_{i=1}^{\infty} \exp(2ri)\|\mathbb{1}_{A^i} f\|_{L^2(A)}^2\right)^{1/2}.$$

LEMMA 4.24. *Let* $r > 0$. *Then for* $f \in L^{2,r_1}(A), g \in L^{2,r_2}(B)$

$$\int fg\mathrm{d}x \leqslant \exp(-(r_1 \wedge r_2) d(A,B)) \|f\|_{L^{2,r_1}(A)} \|g\|_{L^{2,r_2}(B)}$$

where $d(A,B) = \inf_{x \in A, y \in B} |x-y|$.

Proof.

$$\begin{aligned} & \int fg\mathrm{d}x \\ \leqslant & \int \exp(r_1\, d(x,A)) \exp(r_2\, d(x,B)) \exp(-r_1 \wedge r_2\, d(A,B)) f(x)\, g(x) \mathrm{d}x \\ = & \exp(-r_1 \wedge r_2\, d(A,B)) \int \exp(r_1\, d(x,A)) f \exp(r_2\, d(x,B)) g \mathrm{d}x \\ \leqslant & \exp(-(r_1 \wedge r_2) d(A,B)) \|f\|_{L^{2,r_1}(A)} \|g\|_{L^{2,r_2}(B)} \end{aligned}$$

where we have used that by triangle inequality

$$r_1\, d(x,A) + r_2\, d(x,B) - r_1 \wedge r_2\, d(A,B) \geqslant 0.$$

□

LEMMA 4.25. *For any* $\gamma > 0, n \leqslant 0$

$$\|f\|_{L^2(\langle x\rangle^{-n})} \leqslant C \langle d(0,A)\rangle^{-n/2} \|f\|_{L^{2,\gamma}(A)}$$

Proof.

$$\begin{aligned} & \int f^2(x) \langle x\rangle^{-n} \mathrm{d}x \\ = & \int f^2(x) e^{2d(x,A)} e^{-2d(x,A)} \langle x\rangle^{-n} \mathrm{d}x \\ \leqslant & \int f^2(x) e^{2d(x,A)} \langle d(x,A)\rangle^{-n} \langle x\rangle^{-n} \mathrm{d}x \\ \leqslant & C \langle d(0,A)\rangle^{-n} \int f^2(x) e^{2d(x,A)} \mathrm{d}x \end{aligned}$$

□

LEMMA 4.26. *Let* $s \in \{0,1\}$ $r > 0$ *and* $f \in W_p^{s,r}$ *is supported on* $B(0,N)^c$, $N \geqslant 1$. *Then*

$$\|f\|_{W_p^{s,r-\kappa}} \leqslant N^{-\kappa} \|f\|_{W_p^{s,r}}$$

Proof.

$$
\begin{aligned}
& \left(\int f^p \exp((r-\kappa)p|x|)\mathrm{d}x\right)^{1/p}\\
=\ & \left(\int_{|x|\geqslant N} f^p \exp((r-\kappa)\,p|x|)\mathrm{d}x\right)^{1/p}\\
\leqslant\ & N^{-\kappa}\left(\int f^p \exp(rp|x|)\mathrm{d}x\right)^{1/p}\\
=\ & N^{-\kappa}\|f\|_{L^{p,r}}
\end{aligned}
$$

This proves the claim with $s=0$. Applying this inequality also to ∇f we obtain the full statment. □

Lemma 4.27.

$$\|Q_t f\|_{L^\infty} \leqslant t^{-1}\|f\|_{L^\infty}$$

Proof. This follows directly from Young's inequality. □

Lemma 4.28. *Assume that $t/2 \leqslant s \leqslant t$, or $0 \leqslant t \leqslant 1$ then*

$$\|I_{s,t}(u)\|_{L^\infty} \leqslant C\|u\|_{L^\infty([s,t]\times\mathbb{R}^2)}.$$

Proof.

$$
\begin{aligned}
& \sup_x \left|\int_s^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi} l^{1/2}} e^{-2l|x-y|^2} u_l(y)\mathrm{d}l\mathrm{d}y\right|\\
\leqslant\ & \sup_x \int_s^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi} l^{1/2}} e^{-2l|x-y|^2} \mathrm{d}l\mathrm{d}y \|u\|_{L^\infty([s,t]\times\mathbb{R}^2)}\\
\leqslant\ & \int_s^t e^{-\frac{1}{2}m^2/l} l^{-1}\mathrm{d}l\|u\|_{L^\infty([s,t]\times\mathbb{R}^2)}
\end{aligned}
$$

Now in the case $t/2 \leqslant s \leqslant t$

$$\int_s^t e^{-\frac{1}{2}m^2/l} l^{-1}\mathrm{d}l\|u\|_{L^\infty([s,t]\times\mathbb{R}^2)} \leqslant \int_{t/2}^t l^{-1}\mathrm{d}l\|u\|_{L^\infty([s,t]\times\mathbb{R}^2)} \leqslant \log 2\,\|u\|_{L^\infty([s,t]\times\mathbb{R}^2)}$$

and in the case $0 \leqslant t \leqslant 1$

$$\int_s^t e^{-\frac{1}{2}m^2/l} l^{-1}\mathrm{d}l\|u\|_{L^\infty([s,t]\times\mathbb{R}^2)} \leqslant \int_0^1 e^{-\frac{1}{2}m^2/l}\mathrm{d}l\|u\|_{L^\infty([s,t]\times\mathbb{R}^2)} \leqslant C\|u\|_{L^\infty([s,t]\times\mathbb{R}^2)}.$$

□

Lemma 4.29.

$$\|I_{s,t}(u)\|_{W^{1,\infty}} \leqslant C\|\langle l\rangle^{1/2+\delta} u_l\|_{L_l^\infty([s,t]\times\mathbb{R}^2)}$$

Proof.

$$
\begin{aligned}
& \sup_x \left|\int_s^t \int_{\mathbb{R}^2} \nabla_x e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi} l^{1/2}} e^{-2l|x-y|^2} u_l(y)\mathrm{d}l\mathrm{d}y\right|\\
=\ & \sup_x \left|\int_s^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{2(x-y)l^{1/2}}{\sqrt{\pi}} e^{-2l|x-y|^2} u_l(y)\mathrm{d}l\mathrm{d}y\right|\\
\leqslant\ & \sup_x \left|\int_s^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{2(x-y)\langle l\rangle^{-\delta}}{\sqrt{\pi}} e^{-2l|x-y|^2} \langle l\rangle^{1/2+\delta} u_l(y)\mathrm{d}l\mathrm{d}y\right|\\
\leqslant\ & \int_s^t \langle l\rangle^{-\delta} \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{2(x-y)}{\sqrt{\pi}} e^{-2l|x-y|^2} \mathrm{d}y\mathrm{d}l \|\langle l\rangle^{1/2+\delta} u_l\|_{L_l^\infty([s,t]\times\mathbb{R}^2)}
\end{aligned}
$$

$$
\begin{aligned}
\leqslant & \left|\int_s^1 \langle l\rangle^{-\delta} \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{2(x-y)}{\sqrt{\pi}} e^{-2l|x-y|^2} \mathrm{d}y\mathrm{d}l\right| \|\langle l\rangle^{1/2+\delta} u_l\|_{L_l^\infty([s,t]\times\mathbb{R}^2)} \\
& + \left|\int_1^t \langle l\rangle^{-\delta} \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{2(x-y)}{\sqrt{\pi}} e^{-2l|x-y|^2} \mathrm{d}y\mathrm{d}l \|\langle l\rangle^{1/2+\delta} u_l\|_{L_l^\infty([s,t]\times\mathbb{R}^2)}\right| \\
\leqslant & \left|\int_s^1 \langle l\rangle^{-\delta} \int_{\mathbb{R}^2} e^{-m|x-y|} |x-y| \mathrm{d}y\mathrm{d}l\right| \|\langle l\rangle^{1/2+\delta} u_l\|_{L_l^\infty([s,t]\times\mathbb{R}^2)} \\
& \left|\int_1^t \langle l\rangle^{-\delta} \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{2(x-y)}{\sqrt{\pi}} e^{-2l|x-y|^2} \mathrm{d}y\mathrm{d}l\right| \|\langle l\rangle^{1/2+\delta} u_l\|_{L_l^\infty([s,t]\times\mathbb{R}^2)} \\
\leqslant & C \|\langle l\rangle^{1/2+\delta} u_l\|_{L_l^\infty([s,t]\times\mathbb{R}^2)} \\
& + \left|\int_1^t \langle l\rangle^{-\delta} \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} e^{-l|x-y|^2} \mathrm{d}y\mathrm{d}l\right| \|\langle l\rangle^{1/2+\delta} u_l\|_{L_l^\infty([s,t]\times\mathbb{R}^2)} \\
\leqslant & C \|\langle l\rangle^{1/2+\delta} u_l\|_{L_l^\infty([s,t]\times\mathbb{R}^2)}
\end{aligned}
$$

□

Definition 4.30. *For $A \subseteq \mathbb{R}^2$ we say that $u \in L^2([0,\infty)\times\mathbb{R}^2)$, is in $D^r(A)$ if*

$$
\|u\|_{D^r(A)} := \left(\int_0^\infty \|u_t\|_{L^{2,r}}^2 \mathrm{d}t\right)^{1/2} < \infty.
$$

Lemma 4.31. *Let $A \subset \mathbb{R}^2$, and assume that $-m+\kappa \leqslant r \leqslant m-\kappa$, $s \leqslant t$,*

$$
\|I_{s,t}(u)\|_{L^{2,r}(A)} \leqslant C \langle s\rangle^{-1/2} \|u\|_{D^r(A)}
$$

where the constant depends on κ.

Proof. It is enough to prove the inequality for $s,t \leqslant 1$ and $s,t \geqslant 1$, then the general case will follow from $I_{s,t}(u) = I_{s,1}(u) + I_{1,t}(u)$. In the proof we will use several times that

$$
e^{rd(x,A)} e^{-|r||x-y|} \leqslant e^{rd(y,A)}
$$

For $s,t \geqslant 1$

$$
\begin{aligned}
& \int_{\mathbb{R}^2} \left|\int_s^t \int e^{rd(x,A)} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi} l^{1/2}} e^{-2l|x-y|^2} u_l(y) \mathrm{d}l\mathrm{d}y\right|^2 \mathrm{d}x \\
\leqslant & \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} e^{rd(x,A)} \left(\int_s^t e^{-m^2/l} \frac{1}{4\pi l} e^{-4l|x-y|^2} \mathrm{d}l\right)^{1/2} \left(\int_s^t u_l^2(y) \mathrm{d}l\right)^{1/2} \mathrm{d}y\right)^2 \mathrm{d}x \\
\leqslant & C s^{-1} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} e^{rd(x,A)} \left(\frac{1}{|x-y|^2} e^{-4s|x-y|^2}\right)^{1/2} \left(\int_s^t u_l^2(y) \mathrm{d}l\right)^{1/2} \mathrm{d}y\right)^2 \mathrm{d}x \\
\leqslant & C s^{-1} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} e^{rd(x,A)} \frac{1}{|x-y|} e^{-2s|x-y|^2} \left(\int_s^t u_l^2(y) \mathrm{d}l\right)^{1/2} \mathrm{d}y\right)^2 \mathrm{d}x \\
\leqslant & C s^{-1} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{1}{|x-y|} e^{-s|x-y|^2} \left(\int_s^t e^{2rd(y,A)} u_l^2(y) \mathrm{d}l\right)^{1/2} \mathrm{d}y\right)^2 \mathrm{d}x \\
\leqslant & C s^{-1} \|u\|_{D^r(A)}^2,
\end{aligned}
$$

where in the last line we have used Young's inequality. We now treat the $s,t \leqslant 1$ case.

$$
\begin{aligned}
& \|I_{s,t}(u)\|_{L^{2,r}(A)}^2 \\
\leqslant & C \int \exp(2rd(x,A)) \int_s^t \left|\int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi} l^{1/2}} e^{-2l|x-y|^2} u_l(y) \mathrm{d}y\right|^2 \mathrm{d}x\mathrm{d}l
\end{aligned}
$$

Note that $e^{-\frac{1}{2}m^2/l}\frac{1}{\sqrt{4\pi}l^{1/2}}e^{-2l|x-y|^2} \leqslant C_\kappa e^{-(m-\kappa)|x-y|}$ so using Jensen's inequality

$$\begin{aligned}
&\|I_{s,t}(u)\|_{L^{2,r}(A)}^2 \\
\leqslant\ & \int_s^t \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{rd(x,A)} e^{-\frac{1}{2}m^2/l} \frac{1}{\sqrt{4\pi}l^{1/2}} e^{-2l|x-y|^2} u_l(y)\mathrm{d}y \right|^2 \mathrm{d}x\mathrm{d}l \\
\leqslant\ & C\int_s^t \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{-(m-\kappa/2)|x-y|} e^{rd(x,A)} u_l(y)\mathrm{d}y \right|^2 \mathrm{d}x\mathrm{d}l \\
\leqslant\ & C_\kappa \int_s^t \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{-(m-\kappa/2-r)|x-y|} e^{rd(y,A)} u_l(y)\mathrm{d}y \right|^2 \mathrm{d}x\mathrm{d}l \\
\leqslant\ & C_\kappa \int_s^t \|e^{rd(y,A)} u_l(y)\|_{L^2}^2 \mathrm{d}y\mathrm{d}l \\
\leqslant\ & C_\kappa \|u\|_{D^r(A)}^2,
\end{aligned}$$

as long as $m-r-\kappa \geqslant 0$ and we have used Young's inequality. □

Lemma 4.32. *Let $A\subset\mathbb{R}^2$, and assume that $-m/2 \leqslant r \leqslant m/2$, $s \leqslant t$,Then for any $\delta>0$*

$$\|I_{s,t}(u)\|_{H^{1,r}(A)} \leqslant C\|u\|_{D^r(A)}$$

Proof. We first discuss the case $s,t \geqslant m$. We calculate

$$\begin{aligned}
&\|\nabla I_{s,t}(u)\|_{L^{2,r}(A)}^2 \\
=\ & \int \exp(2rd(x,A)) \left| \int_s^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \nabla_x \frac{1}{\sqrt{4\pi}l^{1/2}} e^{-2l|x-y|^2} u_l(y)\mathrm{d}l\mathrm{d}y \right|^2 \mathrm{d}x \\
=\ & \int \exp(2rd(x,A)) \left| \int_s^t \int_{\mathbb{R}^2} e^{-\frac{1}{2}m^2/l} \frac{2l^{1/2}(x-y)}{\sqrt{\pi}} e^{-2l|x-y|^2} u_l(y)\mathrm{d}l\mathrm{d}y \right|^2 \mathrm{d}x \\
\leqslant\ & \int \exp(2rd(x,A)) \left(\int_s^t \int_{\mathbb{R}^2} \frac{2l^{1/2}|x-y|}{\sqrt{\pi}} e^{-2l|x-y|^2} |u_l(y)|\mathrm{d}l\mathrm{d}y \right)^2 \mathrm{d}x \\
\leqslant\ & \int \left(\int_s^t \int_{\mathbb{R}^2} \frac{2l^{1/2}|x-y|}{\sqrt{\pi}} e^{-l|x-y|^2} (e^{rd(y,A)}|u_l(y)|)\mathrm{d}l\mathrm{d}y \right)^2 \mathrm{d}x \\
\leqslant\ & \int \left(\int_{\mathbb{R}^2} |x-y| \left(\int_s^t \frac{2l}{\pi} e^{-2l|x-y|^2}\mathrm{d}l \right)^{1/2} \left(\int_s^t (e^{2rd(y,A)}|u_l(y)|)^2 \mathrm{d}l \right)^{1/2} \mathrm{d}y \right)^2 \mathrm{d}x
\end{aligned}$$

Now integrating by parts

$$\begin{aligned}
&\left(\int_s^t \frac{2l}{\pi} e^{-2l|x-y|^2}\mathrm{d}l \right) \\
=\ & \frac{1}{2\pi|x-y|^2}(se^{-2l|x-y|^2} - te^{-2t|x-y|^2}) + \frac{1}{2\pi|x-y|^2} \int_s^t e^{-2l|x-y|^2}\mathrm{d}l \\
=\ & \frac{1}{2\pi|x-y|^2}(se^{-2l|x-y|^2} - te^{-2t|x-y|^2}) + \frac{1}{2\pi|x-y|^4} e^{-2s|x-y|^2} - e^{-2t|x-y|^2}
\end{aligned}$$

So taking square root

$$\begin{aligned}
&\left(\int_s^t \frac{2l}{\pi} e^{-2l|x-y|^2}\mathrm{d}l \right)^{1/2} \\
\leqslant\ & \frac{1}{\sqrt{2}\pi|x-y|}(s^{1/2}e^{-s|x-y|^2} + t^{1/2}e^{-t|x-y|^2}) + \frac{1}{\sqrt{2}\pi|x-y|^2}(e^{-s|x-y|^2} + e^{-t|x-y|^2})
\end{aligned}$$

Now plugging the first term back in our original computation we get

$$\begin{aligned}
&\int\left(\int_{\mathbb{R}^2}(s^{1/2}\,e^{-s|x-y|^2}-t^{1/2}\,e^{-t|x-y|^2})\left(\int_s^t(e^{rd(x,A)}|u_l(y)|)^2\mathrm{d}l\right)^{1/2}\mathrm{d}y\right)^2\mathrm{d}x\\
\leqslant\ &2\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}s\,e^{-2s|x-y|^2}\int_s^t(e^{rd(x,A)}|u_l(y)|)^2\mathrm{d}l\mathrm{d}y\right)^2\mathrm{d}x\\
\leqslant\ &2\int\int_s^t(e^{rd(x,A)}|u_l(y)|)^2\mathrm{d}l\mathrm{d}y=\|u\|^2_{D^r(A)}.
\end{aligned}$$

by Young's inequality, since $\|se^{-2s|y|^2}\|_{L^1(\mathbb{R}^2)}\leqslant C$ uniformly in s. Pluggin the second term we get

$$\begin{aligned}
&\int\left(\int_{\mathbb{R}^2}\left(\left|\frac{1}{2|x-y|}e^{-2s|x-y|^2}-e^{-2t|x-y|^2}\right|\right)\left(\int_s^t(e^{rd(x,A)}|u_l(y)|)^2\mathrm{d}l\right)^{1/2}\mathrm{d}y\right)^2\mathrm{d}x\\
\leqslant\ &\|u\|^2_{D^r(A)}
\end{aligned}$$

by Young's inequality since

$$\sup_{s,t\geqslant m}\left\|\left(\frac{1}{2|y|}(e^{-2s|y|^2}-e^{-2t|y|^2})\right)\right\|_{L^1(\mathbb{R}^2)}\leqslant C$$

for $s,t\geqslant m$. For $s,t\leqslant m$ we compute using $e^{-\frac{1}{2}m^2/l}e^{-2l|x-y|^2}\leqslant e^{-m|x-y|}$

$$\begin{aligned}
&\|\nabla I_{s,t}(u)\|^2_{L^{2,r}(A)}\\
=\ &\int\exp(2rd(x,A))\left|\int_s^t\int_{\mathbb{R}^2}e^{-\frac{1}{2}m^2/l}\frac{2l^{1/2}(x-y)}{\sqrt{\pi}}e^{-2l|x-y|^2}u_l(y)\mathrm{d}l\mathrm{d}y\right|^2\mathrm{d}x\\
\leqslant\ &\int\exp(2rd(x,A))\left|\int_s^t\int_{\mathbb{R}^2}e^{-(m-\kappa)|x-y|}u_l(y)\mathrm{d}l\mathrm{d}y\right|^2\mathrm{d}x\\
\leqslant\ &\int\left(\exp(rd(x,A))\left(\int_{\mathbb{R}^2}e^{-2(m-\kappa)|x-y|}\int_s^t u_l^2(y)\mathrm{d}l\right)^{1/2}\mathrm{d}y\right)^2\mathrm{d}x\\
\leqslant\ &C\int\left(\int_{\mathbb{R}^2}e^{-(m-\kappa)|x-y|}\int_s^t e^{rd(y,A)}u_l^2(y)\mathrm{d}l\mathrm{d}y\right)^2\mathrm{d}x\\
\leqslant\ &C\|u\|^2_{D^r}
\end{aligned}$$

again by Young's inequality. In the case $s\leqslant m, t>m$ we write $I_{s,t}(u)=I_{s,m}(u)+I_{m,t}(u)$ and we can reduce the problem to the previous two cases. □

4.3. LOCALITY

The main goal of this section is to prove that the value function satisfies certain locality properties: If the terminal data is perturbed by a functional whose gradient is supported in a bounded set, the effect of this perturbation on the value function will be small away from that bounded set. To encode this we will need the following definition.

DEFINITION 4.33. *For a functional $G\colon L^2(\mathbb{R}^2)\to\mathbb{R}$ and $A\subseteq\mathbb{R}^2$ we define the quantities*

$$|G|_{1,\infty}=\sup_{\varphi\in L^2(\mathbb{R}^2)}\|\nabla G(\varphi)\|_{L^\infty(\mathbb{R}^2)}$$

$$|G|^A_{1,2,r}=\sup_{\varphi\in L^2(\mathbb{R}^2)}\|\nabla G(\varphi)\|_{L^{2,r}(A)}$$

$$|G|_2=\sup_{\varphi\in L^2(\mathbb{R}^2)}\|\mathrm{Hess}\,G(\varphi)\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}$$

$$|G|_{2,\iota}=\sup_{\varphi\in L^2(\mathbb{R}^2)}\|\mathrm{Hess}\,G(\varphi)\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}+\sup_{h>\iota}\sup_{A\subseteq\mathbb{R}^2}\sup_{\varphi\in L^2(\mathbb{R}^2)}\|\mathrm{Hess}\,G(\varphi)\|_{L^{2,h}(A)\to L^{2,h-\iota}(A)}$$

where

$$\|\text{Hess}\, G(\varphi)\|_{H \to L} = \sup_{\psi \in H} \frac{\|\text{Hess}\, G(\varphi)\psi\|_L}{\|\psi\|_L}.$$

We will also need the following notation.

NOTATION 4.34. *In the sequel $c > 0$ will always be a small constant such that*

$$m - 4(\lambda + c) \sum_{n \in \mathbb{N}} 2^{n\delta} > 0$$

and we will denote

$$c_t = (\lambda + c) \sum_{n : 2^n \geqslant t} 2^{n\delta}.$$

We will be interested in value functions of the form with $V_T^f \in C^2(L^2(\mathbb{R}^2), \mathbb{R})$

$$\begin{aligned} V_{s,T}^f(\varphi) &= \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[V_T^f(W_{s,T} + I_{s,T}(u) + \varphi) + \frac{1}{2}\int_s^T \|u_w\|_{L^2}^2 \mathrm{d}w\right] \\ &= \inf_{u \in \mathbb{H}_a} F^{\varphi,f}(u) \end{aligned} \tag{4.11}$$

where $V_T^f = f + V_T$ with $|f|_{1,2,r} < \infty$ and $|V_T|_{2,\iota} + |V_T|_{1,\infty} \leqslant \lambda$ with λ small enough. We will denote the minimizer of the r.h.s of (4.11) by u^f. In this section we will consider an "abstract" V_T such that the corresponding value function satisfies the following hypotheses, in the subsequent sections we will then further specify $V_T(\varphi) = V_T^\rho = \alpha(T) \int \rho \cos(\beta\varphi)$ and show that this example satisfies our hypotheses.

HYPOTHESIS A. *Assume that $V_T = V_T^0$ satisfies*

$$|V_T|_{2,c\langle T\rangle^{-\delta}} \leqslant \lambda \langle T\rangle^{1/2-\delta}.$$

This hypothesis is only a restriction on the terminal condition and will be easy to verify. However we will also require the following hypothesis which is more tricky:

HYPOTHESIS B. *Assume that $V_T = V_T^0$ is such that $V_{t,T}$ given by*

$$V_{t,T}(\varphi) = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[V_T^f(W_{t,T} + I_{t,T}(u) + \varphi) + \frac{1}{2}\int_t^T \|u_w\|_{L^2}^2 \mathrm{d}w\right],$$

satisfies

$$|V_{t,T}|_{2,c\langle t\rangle^{-\delta}} \leqslant \lambda \langle t\rangle^{1/2-\delta}.$$

Our goal will be to show the following propositions:

PROPOSITION 4.35. *Assume that V_T satisfies Hypothesis B. Then for any $2(\lambda + c)\langle t\rangle^{-\delta} < r < m$ there exists a $C > 0$ such that*

$$|V_{t,T}^f - V_{t,T}^0|_{1,2,r-c_t} \leqslant C|f|_{1,2,r}.$$

and

PROPOSITION 4.36. *Assume that V_T satisfies Hypothesis B. Then for any $B \subseteq \mathbb{R}^2$.*

$$\mathbb{E}\left[\int_0^T \|u^f - u^0\|_{L^{2,r-2c_t(B)}}^2 \mathrm{d}t\right] \leqslant C\, |f|_{1,2,r}^B.$$

Our strategy in proving these bounds will be the following: We will prove them for $t \geqslant T/2$ where we can use that the convexity provided by the term

$$\frac{1}{2}\int_s^t \|u_w\|_{L^2}^2 \mathrm{d}w$$

beats the semiconvexity of V_T. Under these conditions we will be able to show that

$$V_{t,T}^f = V_{t,T}^0 + \tilde{f}$$

with $|\tilde{f}|_{1,2,r-c\langle t\rangle^{-\delta}} \leqslant C|f|_{1,2,r}$. Then using the assumption of $V_{t,T}^0$ we will iterate the argument to obtain the full statement. For technical reasons we will prove very similar statements under weaker hypotheses:

Hypothesis C. *$V_T = V_T^0$ satisfies*

$$|V_T|_2 \leqslant \lambda \langle T\rangle^{1/2-\delta}. \tag{4.12}$$

Hypothesis D. *$V_T = V_T^0$ is such that satisfies $V_{t,T}$ given by*

$$V_{t,T}(\varphi) = \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[V_T^f(W_{t,T} + I_{t,T}(u) + \varphi) + \frac{1}{2}\int_t^T \|u_w\|_{L^2}^2 \mathrm{d}w\right]$$

satisfies

$$|V_{t,T}|_2 \leqslant \lambda \langle t\rangle^{1/2-\delta}.$$

We then have

Proposition 4.37. *Assume that V_T satisfies Hypothesis D. Then*

$$|V_{t,T}^f - V_{t,T}^0|_{1,2} \leqslant C|f|_{1,2}.$$

The proof is again analogous to the proof of Proposition 4.35.

Proposition 4.38. *Assume that V_T satisfies Hypothesis D. Then*

$$\mathbb{E}[\|u^f - u^0\|_{D^0}^2] \leqslant C|f|_{1,2,0}.$$

4.3.1. Interlude: A formula for the gradient of the value function

Before we proceed with the proof let us discuss a formula to represent the gradient of the value function which will be useful. It can be considered to be a version of the *Envelope Theorem* [93], see also [5]. Take $V \in C^2(L^2(\mathbb{R}^2), \mathbb{R})$ and consider

$$\begin{aligned} V_{s,t}(\varphi) &= \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[V_t(W_{s,t} + I_{s,t}(u) + \varphi) + \frac{1}{2}\int_s^t \|u_w\|_{L^2}^2 \mathrm{d}w\right] \\ &= \inf_{u\in\mathbb{H}_a} F(u). \end{aligned}$$

Now denote by

$$\mathcal{X} = \{\mu\colon \mu = \mathrm{Law}(W, u) \text{ with } u \in \mathbb{H}_a\}$$

and

$$\bar{\mathcal{X}} = \left\{\mu\colon \exists \mu_n \to \mu \text{ weakly on } C([s,t], \mathscr{C}^{-\varepsilon}(\langle x\rangle^{-n})) \times L_w^2([s,t]\times\mathbb{R}^2) \text{ s.th. } \mu_n \in \mathcal{X}, \sup_n \mathbb{E}_{\mu_n}[\|u\|_{D^0}^2] < \infty\right\}$$

where as usual L_w^2 denotes L^2 equipped with the weak topology, and it is possible to show that indeed $W \in C([s,t], \mathscr{C}^{-\varepsilon}(\langle x\rangle^{-n}))$ almost surely, see for instance [72], Theorem 3.1.

One can prove analogously Lemma 2.32 in Chapter 2 that

$$\bar{\mathcal{X}} = \left\{\mu\colon \exists \mu_n \to \mu \text{ weakly on } C([s,t], \mathscr{C}^{-\varepsilon}(\langle x\rangle^{-n})) \times L^2([s,t]\times\mathbb{R}^2) \text{ s.th. } \mu_n \in \mathcal{X}, \sup_n \mathbb{E}_{\mu_n}[\|u\|_{D^0}^2] < \infty\right\},$$

where by abuse of nation we have denoted by $(W_{s,t},\ u)$ the canonical variables on $C([s,\ t], \mathscr{C}^{-\varepsilon}(\langle x\rangle^{-n})) \times L^2([s,t]\times\mathbb{R}^2)$. We can define

$$\tilde{F}(\mu) = \mathbb{E}_\mu\left[V_t(W_{s,t} + I_{s,t}(u) + \varphi) + \frac{1}{2}\int_s^t \|u_w\|_{L^2}^2 \mathrm{d}w\right].$$

From now on we will consider $\tilde{F}$ as a functional on $\bar{\mathcal{X}}$. Note that using continuity and boundedness of V_t we can easily show that

$$\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[V_t(W_{s,t}+I_{s,t}(u)+\varphi)+\frac{1}{2}\int_s^t\|u_w\|_{L^2}^2\mathrm{d}w\right]=\inf_{\mu\in\mathcal{X}}\tilde{F}=\inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}.$$

With this setup we can prove the following Lemma:

Lemma 4.39.

$$\langle\nabla V_{s,t}(\varphi),\psi\rangle_{L^2}=\inf_{\mu\in\operatorname{argmin}\tilde{F}}\mathbb{E}_\mu[\langle\nabla V_t(W_{s,t}+I_{s,t}(u)+\varphi),\psi\rangle_{L^2}]$$

where argmin$\tilde{F}$ *denotes the set of minimizers of* $\tilde{F}$*.*

Proof. Define the functional

$$F^\varepsilon(\mu)=\frac{\mathbb{E}_\mu\left[V_t(W_{s,t}+I_{s,t}(u)+\varphi+\varepsilon\psi)+\frac{1}{2}\int_s^t\|u_w\|_{L^2}^2\mathrm{d}w\right]-\inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}}{\varepsilon}$$

and observe that

$$\lim_{\varepsilon\to0}\inf_{\mu\in\bar{\mathcal{X}}}F^\varepsilon=\langle\nabla V_{s,t}(\varphi),\psi\rangle_{L^2}.$$

So it is enough to show that

$$\lim_{\varepsilon\to0}\inf_{\mu\in\bar{\mathcal{X}}}F^\varepsilon=\inf_{\mu\in\operatorname{argmin}\tilde{F}}\mathbb{E}_\mu[\langle\nabla V_t(W_{s,t}+I_{s,t}(u)+\varphi),\psi\rangle_{L^2}].$$

We will show F^ε Γ-converges (see Section 2.6) to $(F)'$ where

$$(F)'(\mu)=\begin{cases}\mathbb{E}_\mu[\langle\nabla V_t(W_{s,t}+I_{s,t}(u)+\varphi),\psi\rangle_{L^2}] \text{ if } \mu\in\operatorname{argmin}\tilde{F}\\ \infty \text{ otherwise}\end{cases}$$

and furthermore F^ε is equicoercive on $\bar{\mathcal{X}}$. To prove equicoercivity set $\mathcal{K}=\{\mu:\mathbb{E}_\mu[\int_s^t\|u_w\|_{L^2}^2\mathrm{d}w]\leqslant K\}$ for $K>0$ to be chosen later. It is not hard to see that $\mathcal{K}$ is compact in $\bar{\mathcal{X}}$ (see Lemma 2.28 above). Now

$$\begin{aligned}F^\varepsilon(\mu) &= \frac{\mathbb{E}_\mu\left[V_t(W_{s,t}+I_{s,t}(u)+\varphi+\varepsilon\psi)+\frac{1}{2}\int_s^t\|u_w\|_{L^2}^2\mathrm{d}w\right]-\inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}}{\varepsilon}\\ &= \frac{1}{\varepsilon}\left(\mathbb{E}_\mu\left[V_t(W_{s,t}+I_{s,t}(u)+\varphi)+\frac{1}{2}\int_s^t\|u_w\|_{L^2}^2\mathrm{d}w\right]-\inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}\right)\\ &\quad+\frac{1}{\varepsilon}\mathbb{E}_\mu[V_t(W_{s,t}+I_{s,t}(u)+\varphi+\varepsilon\psi)-V_t(W_{s,t}+I_{s,t}(u)+\varphi)]\\ &= \frac{1}{\varepsilon}\left(\tilde{F}-\inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}\right)+\mathcal{O}(1)\end{aligned}$$

where $\mathcal{O}(1)$ denotes functionals which are bounded uniformly in μ,ε. This implies $\sup_\varepsilon\inf_\mu F^\varepsilon\leqslant C$.

On the other hand since V is bounded we have that

$$F^\varepsilon(\mu)\geqslant\frac{1}{\varepsilon}\left(\frac{1}{2}\mathbb{E}_\mu\left[\int_s^t\|u_w\|_{L^2}^2\mathrm{d}w\right]-C\right)$$

so $\inf_{\mu\in\mathcal{K}^c}F^\varepsilon(\mu)\geqslant\frac{1}{\varepsilon}(K-C)$ so choosing K large enough we obtain

$$\inf_{\mu\in\bar{\mathcal{X}}}F^\varepsilon(\mu)=\inf_{\mu\in\mathcal{K}}F^\varepsilon(\mu),$$

which proves equicoercivity. To prove the liminf inequality of Γ-convergence we consider a sequence $\mu^\varepsilon\to\mu$ in $\bar{\mathcal{X}}$ and distinguish two cases $\mu\in\operatorname{argmin}\tilde{F}$ and $\mu\notin\operatorname{argmin}\tilde{F}$. For the first case recall from above that

$$\begin{aligned}F^\varepsilon(\mu^\varepsilon) &= \frac{1}{\varepsilon}\left(\tilde{F}(\mu)-\inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}\right)\\ &\quad+\frac{1}{\varepsilon}\mathbb{E}_{\mu^\varepsilon}[V_t(W_{s,t}+I_{s,t}(u)+\varphi+\varepsilon\psi)-V_t(W_{s,t}+I_{s,t}(u)+\varphi)]\\ &\geqslant \frac{1}{\varepsilon}\mathbb{E}_{\mu^\varepsilon}[V_t(W_{s,t}+I_{s,t}(u)+\varphi+\varepsilon\psi)-V_t(W_{s,t}+I_{s,t}(u)+\varphi)]\\ &= \mathbb{E}_{\mu^\varepsilon}[\langle\nabla V_t(W_{s,t}+I_{s,t}(u)+\varphi),\psi\rangle_{L^2}]+\mathcal{O}(\varepsilon)\end{aligned}$$

where in the last line we have used Taylor expansion. Now

$$\begin{aligned}&\mathbb{E}_{\mu^\varepsilon}[\langle \nabla V_t(W_{s,t}+I_{s,t}(u)+\varphi),\psi\rangle_{L^2}]\\ \rightarrow\ &\mathbb{E}_{\mu}[\langle \nabla V_t(W_{s,t}+I_{s,t}(u)+\varphi),\psi\rangle_{L^2}]\end{aligned}$$

by continuity and boundedness of ∇V. For the second case ($\mu \notin \operatorname{argmin} \tilde{F}$) we consider

$$\begin{aligned}\liminf_{\varepsilon\to 0} F^\varepsilon(\mu^\varepsilon) &= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\tilde{F}(\mu^\varepsilon)-\inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}\right)\\ &\quad+\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}_{\mu^\varepsilon}[V_t(W_{s,t}+I_{s,t}(u)+\varphi+\varepsilon\psi)-V_t(W_{s,t}+I_{s,t}(u)+\varphi)]\\ &= \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\tilde{F}(\mu^\varepsilon)-\inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}\right)+\mathcal{O}(1)\\ &= \infty\end{aligned}$$

where in the last line we have used that since $\tilde{F}$ is lower semincontinuous so

$$\liminf_{\varepsilon\to 0}\tilde{F}(\mu^\varepsilon)\geqslant \tilde{F}(\mu)$$

and since $\mu \notin \operatorname{argmin}\tilde{F}$ there exits $c>0$ and ε_0 such that for any $\varepsilon<\varepsilon_0$, $\tilde{F}(\mu^\varepsilon)-\inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}>c$. Now we can conclude that Γ-convergence holds by observing that we can take the recovery sequence constant. □

Lemma 4.40. *Assume that for small $\lambda>0,\alpha<1$: and that $V_T\in C^2(L^2(\mathbb{R}^2))$.*

$$\|\operatorname{Hess} V_T\|_{L^2\to L^2}\leqslant \lambda\langle t\rangle^\alpha. \tag{4.13}$$

Then recalling that $F^{\varphi,0}(u)$ defined by

$$F^{\varphi,0}(u)=\mathbb{E}\left[V_T(W_{t,T}+I_{t,T}(u)+\varphi)+\frac{1}{2}\int_t^T\|u_w\|_{L^2}^2\mathrm{d}w\right]$$

is strongly convex in u on D^0, with constant $1/4$. A minimizer of $F^{\varphi,0}$ exists by Proposition 4.13 and is unique. We denote it by u^φ. Furthermore u^φ satisfies for any $u\in\mathbb{H}_a$,

$$\mathbb{E}\left[\int\nabla V_T(W_{t,T}+I_{t,T}(u^\varphi)+\varphi)I_{t,T}(u)+\int_t^T\int_{\mathbb{R}^2}u_s^\varphi u_s\mathrm{d}s\right]=0 \tag{4.14}$$

Proof. Strong convexity follows from strong convexity of $\int_t^T\|u_w\|_{L^2}^2\mathrm{d}w$, the assumption on V_T and Lemma 4.31. Existence of the minimizer follows from Proposition 4.14 and uniqueness is implied by convexity. To prove (4.14) we proceed in the standard way. Since

$$\frac{F^{\varphi,0}(u^\varphi+\varepsilon u)-F^{\varphi,0}(u^\varphi)}{\varepsilon}\geqslant 0 \text{ for any } u\in\mathbb{H}_a$$

we can take the limit and obtain

$$\begin{aligned}&0\\ \leqslant\ &\lim_{\varepsilon\to 0}\frac{F^{\varphi,0}(u^\varphi+\varepsilon u)-F^{\varphi,0}(u^\varphi)}{\varepsilon}\\ =\ &\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\mathbb{E}[V_T(W_{t,T}+I_{t,T}(u^\varphi+\varepsilon u)+\varphi)-V_T(W_{t,T}+I_{t,T}(u^\varphi)+\varphi)])\\ &+\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^T\|u_s^\varphi+\varepsilon u_s\|_{L^2}^2\mathrm{d}s-\int_t^T\|u_s^\varphi\|_{L^2}^2\mathrm{d}s\right]\\ =\ &\mathbb{E}\left[\int\nabla V_T(W_{t,T}+I_{t,T}(u^\varphi)+\varphi)I_{t,T}(u)+\int_t^T\int u_s^\varphi u_s\mathrm{d}s\right]\end{aligned}$$

This gives the statement since the converse inequality can be obtained by replacing u with $-u$. □

Remark 4.41. Strong convexity of $F^{\varphi,0}$ on D^0, with constant $1/4$ is equivalent to

$$\begin{aligned}&F^{\varphi,0}(u^1)-F^{\varphi,0}(u^2)\\ \geqslant\;&\mathbb{E}\left[\int_{\mathbb{R}^2}\nabla V_T(W_{t,T}+I_{t,T}(u^2)+\varphi)I_{t,T}(u^1-u^2)+\int_t^T\int_{\mathbb{R}^2}(u_s^1-u_s^2)u_s^2\mathrm{d}t+\frac{1}{4}\int_t^T\|u_s^1-u_s^2\|_{L^2}^2\mathrm{d}s\right]\end{aligned}$$

for any $u^1,u^2\in\mathbb{H}_a$ which together with (4.14) implies

$$\begin{aligned}&F^{\varphi,0}(u)-F^{\varphi,0}(u^\varphi)\\ \geqslant\;&\mathbb{E}\left[\frac{1}{4}\int_t^T\|u_s-u_s^\varphi\|_{L^2}^2\mathrm{d}s\right]\end{aligned}$$

for any $u\in\mathbb{H}_a$ and u^φ the minimizer from Lemma 4.40.

LEMMA 4.42. *With the assumptions and notation from Lemma 4.40 and we have*

$$\nabla V_{t,T}(\varphi)=\mathbb{E}[\nabla V(W_{t,T}+I_{t,T}(u^\varphi)+\varphi)].$$

Proof. The proof is similar to the proof of Lemma 4.39. Consider the functional

$$F^\varepsilon(u)=\frac{F^{\varphi+\varepsilon\psi}(u)-\inf_{u\in\mathbb{H}_a}F^\varphi(u)}{\varepsilon}=\frac{F^{\varphi+\varepsilon\psi}(u)-F^\varphi(u^\varphi)}{\varepsilon}$$

And observe that $\langle\nabla V_{t,T}(\varphi),\psi\rangle=\lim_{\varepsilon\to0}\inf_{u\in\mathbb{H}_a}F^\varepsilon(u)$. Now

$$F^{\varphi+\varepsilon\psi}(u)=\mathbb{E}[V_T(W_{t,T}+I_{t,T}(u)+\varphi+\varepsilon\psi)-V_T(W_{t,T}+I_{t,T}(u)+\varphi)]+F^\varphi(u)$$

so

$$F^\varepsilon(u)=\frac{1}{\varepsilon}(F^\varphi(u)-F^\varphi(u^\varphi))+\mathcal{O}(1)$$

where again $\mathcal{O}(1)$ is a term uniformly bounded in u and ε. This implies that

$$\liminf_{\varepsilon\to0}F^\varepsilon(u^\varepsilon)\geqslant\liminf_{\varepsilon\to0}\frac{1}{\varepsilon}(F^\varphi(u^\varepsilon)-F^\varphi(u^\varphi))-C\geqslant\frac{1}{2\varepsilon}\mathbb{E}\left[\int_t^T\|u_s^\varepsilon-u_s^\varphi\|^2\mathrm{d}s\right]-C$$

So $\liminf_{\varepsilon\to0}F^\varepsilon(u^\varepsilon)<\infty$ implies that for a subsequence (not relabeled)

$$\lim_{\varepsilon\to0}\mathbb{E}\left[\int_t^T\|u_s^\varepsilon-u_s^\varphi\|^2\mathrm{d}s\right]=0. \tag{4.15}$$

Now (4.15) implies that, provided that $V_T\in C^2(L^2(\mathbb{R}^2))$

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}[V_T(W_{t,T}+I_{t,T}(u^\varepsilon)+\varphi+\varepsilon\psi)-V_T(W_{t,T}+I_{t,T}(u^\varepsilon)+\varphi)]\\ =\;&\lim_{\varepsilon\to0}\mathbb{E}\left[\int_0^1\langle\nabla V_T(W_{t,T}+I_{t,T}(u^\varepsilon)+\theta\varepsilon\psi+\varphi),\psi\rangle\mathrm{d}\theta\right]\\ =\;&\mathbb{E}[\langle\nabla V_T(W_{t,T}+I_{t,T}(u^\varphi)+\varphi),\psi\rangle]\end{aligned}$$

so

$$\begin{aligned}&\liminf_{\varepsilon\to0}F^\varepsilon(u^\varepsilon)\\ \geqslant\;&\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\mathbb{E}[V_T(W_{t,T}+I_{t,T}(u^\varepsilon)+\varphi+\varepsilon\psi)-V_T(W_{t,T}+I_{t,T}(u^\varepsilon)+\varphi)]+\liminf_{\varepsilon\to0}\frac{1}{\varepsilon}(F^\varphi(u^\varepsilon)-F^\varphi(u^\varphi))\\ \geqslant\;&\mathbb{E}[\langle\nabla V_T(W_{t,T}+I_{t,T}(u^\varphi)+\varphi),\psi\rangle]\end{aligned}$$

which implies

$$\liminf_{\varepsilon\to0}\inf_{u\in\mathbb{H}_a}F^\varepsilon(u)\geqslant\mathbb{E}[\langle\nabla V_T(W_{t,T}+I_{t,T}(u^\varphi)+\varphi),\psi\rangle].$$

For the converse inequality it is enough to observe that

$$\liminf_{\varepsilon\to 0} \inf_{u\in\mathbb{H}_a} F^{\varepsilon}(u) \leqslant \lim_{\varepsilon\to 0} F^{\varepsilon}(u^{\varphi}) = \mathbb{E}[\langle\nabla V_T(W_{t,T}+I_{t,T}(u^{\varphi})+\varphi),\psi\rangle]. \qquad\square$$

4.3.2. Locality at high frequencies

Lemma 4.43. *Assume that V_T satisfies Hypothesis C. Then $F^{\varphi,f}(u)$ is defined by*

$$F^{\varphi,f}(u) = \mathbb{E}\left[f(W_{t,T}+I_{t,T}(u)+\varphi)+V_T(W_{t,T}+I_{t,T}(u)+\varphi)+\frac{1}{2}\int_t^T\|u_w\|_{L^2}^2\mathrm{d}w\right]$$

Assume that $u^{\varphi,f}$ is such that $F^{\varphi,f}(u^{\varphi,f})\leqslant\inf_{u\in\mathbb{H}_a}F^{\varphi,f}(u)+\varepsilon$. Let u^{φ} be the unique minimizer of $F^{\varphi,0}(u)$. Then

$$\mathbb{E}[\|u^{\varphi,f}-u^{\varphi}\|_{D^0}^2]\leqslant 4\langle t\rangle^{-1/2}|f|_{1,2,0}+\varepsilon. \tag{4.16}$$

Proof. We prove that if eq (4.16) is violated $u^{\varphi,f}$ cannot be a minimizer because

$$F^{\varphi,f}(u^{\varphi}) < F^{\varphi,f}(u^{\varphi,f}).$$

Indeed by assumptions on $u^{f,\varphi}$

$$\begin{aligned}
& \varepsilon\\
\geqslant\ & F^{\varphi,f}(u^{\varphi,f})-F^{\varphi,f}(u^{\varphi})\\
=\ & \mathbb{E}[f(W_{t,T}+I_{t,T}(u^{\varphi,f})+\varphi)-f(W_{t,T}+I_{t,T}(u^{\varphi})+\varphi)]\\
& +F^{\varphi,0}(u^{\varphi,f})-F^{\varphi,0}(u^{\varphi})\\
\geqslant\ & -|f|_{1,2,0}\mathbb{E}[\|I_{t,T}(u^{\varphi,f})-I_{t,T}(u^{\varphi})\|_{L^2(\mathbb{R}^2)}]+\frac{1}{4}\mathbb{E}[\|u^{\varphi,f}-u^{\varphi}\|_{D^0}^2]\\
\geqslant\ & -\langle t\rangle^{-1/2}|f|_{1,2,0}\mathbb{E}[\|u^{\varphi,f}-u^{\varphi}\|_{D^0}^2]^{1/2}+\frac{1}{4}\mathbb{E}[\|u^{\varphi,f}-u^{\varphi}\|_{D^0}^2]
\end{aligned}$$

which implies the statement. $\square$

Our next aim is to show that if (4.13) holds perturbing V_T by a functional f such that $|f|_{1,2,r}<\infty$ amounts to perturbing the value function at time t by an $\tilde{f}$ with similar properties, provided $\langle t\rangle\geqslant T/2$. For this we will need the following notations

Notation 4.44. *Let $u\in L^2([0,\infty)\times\mathbb{R}^2)$ and $B\subseteq\mathbb{R}^2$. Define $B_N=\{x\in\mathbb{R}^2: N-1\leqslant d(x,B)\leqslant N\}$, $B^{+n}=\bigcup_{N\geqslant n}B^N$ and denote by $u^N=\mathbb{1}_{B^N}(u-u^{\varphi})$*

$$\bar{u}^{-n}=u^{\varphi}+\sum_{N\leqslant n}u^N,\ u^{-n}=\sum_{N\leqslant n}u^N$$

and

$$u^{+n}=u-\bar{u}^{-n}.$$

Lemma 4.45. *We consider a random functional $f: L^2(\mathbb{R}^2)\to\mathbb{R}$ satisfying*

$$\mathbb{E}[(|f|_{1,2,r}^B)^2]<\infty$$

for some $B\subseteq\mathbb{R}^2, r>0$. Assume also that V_T satisfies Hypothesis A. Now for $\langle t\rangle\geqslant T/2$ consider the variational problem

$$\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[f(Y_{t,T}(u,\varphi))+V_T(Y_{t,T}(u,\varphi))+\frac{1}{2}\int_t^T\|u_s\|_{L^2}^2\mathrm{d}s\right]=\inf_{u\in\mathbb{H}_a}F^{\varphi,f}(u)$$

Then for any u such that $F^{\varphi,f}(u)\leqslant\inf_{u\in\mathbb{H}_a}F^{\varphi,f}(u)+\varepsilon$, $\varepsilon\geqslant 0$ we have for any $r<m-c\langle T\rangle^{-\delta}$

$$\mathbb{E}\left[\int_t^T\mathbb{1}_{B^N}\|u_s-u_s^{\varphi}\|_{L^2}^2\mathrm{d}s\right]^{1/2}\leqslant(\langle t\rangle^{-1/2}\mathbb{E}[|f|_{1,2,r}^2]^{1/2}+\exp(rn)\varepsilon)\exp(-(r-2\lambda\,\langle t\rangle^{-1+\alpha})\,n),$$

where u^{φ} is the unique minimizer of $F^{\varphi,0}$.

Proof. $F^{\varphi,f}(u)\leqslant\inf_{u\in\mathbb{H}_a}F^{\varphi,f}(u)+\varepsilon$ implies that

$$\begin{aligned}&F^{\varphi,f}(u)-F^{\varphi,f}(u^{-n})\\=&F^{\varphi,0}(u)-F^{\varphi,0}(u^{-n})+\mathbb{E}[f(W_{t,T}+I_{t,T}(u)+\varphi)-f(W_{t,T}+I_{t,T}(u^{-n})+\varphi)]\\\leqslant&\varepsilon\end{aligned}$$

Then from Remark 4.41 we have

$$\begin{aligned}&F^{\varphi,0}(u)-F^{\varphi,0}(u^{-n})\\\geqslant&\mathbb{E}\left[\int_{\mathbb{R}^2}\nabla V_T(W_{t,T}+I_{t,T}(\bar{u}^{-n})+\varphi)I_{t,T}(u^{+n})+\int_t^T\int_{\mathbb{R}^2}\bar{u}_s^{-n}u_s^{+n}\mathrm{d}s+\frac{1}{4}\|u^{+n}\|_{D^0}^2\right]\\=&\mathbb{E}\left[\int_{\mathbb{R}^2}\nabla V_T(W_{t,T}+I_{t,T}(\bar{u}^{-n})+\varphi)I_{t,T}(u^{+n})+\int_t^T\int_{\mathbb{R}^2}u_s^{\varphi}u_s^{+n}\mathrm{d}s+\frac{1}{4}\|u^{+n}\|_{D^0}^2\right]\\=&\mathbb{E}\left[\int_{\mathbb{R}^2}\nabla V_T(W_{t,T}+I_{t,T}(u^{\varphi})+\varphi)I_{t,T}(u^{+n})+\int_t^T\int_{\mathbb{R}^2}u_s^{\varphi}u_s^{+n}\mathrm{d}s+\frac{1}{4}\|u^{+n}\|_{D^0}^2\right]\\&+\mathbb{E}\left[\int_{\mathbb{R}^2}(\nabla V_T(W_{t,T}+I_{t,T}(\bar{u}^{-n})+\varphi)-\nabla V_T(W_{t,T}+I_{t,T}(u^{\varphi})+\varphi))I_{t,T}(u^{+n})\right]\\=&\mathbb{E}\left[\int_0^1\int_{\mathbb{R}^2}I_{t,T}(u^{-n})(\operatorname{Hess}V_T(W_{t,T}+I_{t,T}(u^{\varphi})+\theta I_{t,T}(u^{-n})+\varphi)\,I_{t,T}(u^{+n}))\mathrm{d}\theta+\frac{1}{4}\|u^{+n}\|_{D^0}^2\right]\end{aligned}$$

where in the last line we have used (4.14) and the fundamental theorem of calculus. Now using Lemma 4.24 ,with $\gamma=m-c\langle T\rangle^{-\delta}$, denoting

$$\mathcal{V}_\theta=\operatorname{Hess}V_T(W_{t,T}+I_{t,T}(u^{\varphi})+\theta I_{t,T}(u^{-n})+\varphi)$$

we get:

$$\begin{aligned}&\mathbb{E}\left[\int_0^1\int_{\mathbb{R}^2}I_{t,T}(u^{-n})(\mathcal{V}_\theta\,I_{t,T}(u^{+n}))\mathrm{d}\theta+\frac{1}{4}\|u^{+n}\|_{D^0}^2\right]\\\geqslant&-\mathbb{E}\left[\sum_{N\leqslant n}\int_0^1\int_{\mathbb{R}^2}|I_{t,T}(u^N)(\mathcal{V}_\theta I_{t,T}(u^{+n}))|\mathrm{d}\theta\right]+\frac{1}{4}\mathbb{E}[\|u^{+n}\|_{D^0}^2]\\\geqslant&-\mathbb{E}\left[\sum_{N\leqslant n}\exp(-\gamma(n-N))\|I_{t,T}(u^N)\|_{L^{2,\gamma}(B^N)}\sup_\theta\|\mathcal{V}_\theta I_{t,T}(u^{+n})\|_{L^{2,\gamma}(B^{+n})}\right]+\frac{1}{4}\mathbb{E}[\|u^{+n}\|_{D^0}^2]\\\geqslant&-\lambda T^\alpha\langle t\rangle^{-1}\sum_{N\leqslant n}\exp(-\gamma(n-N))\mathbb{E}[\|u^N\|_{D^m(B_N)}^2]^{1/2}\mathbb{E}[\|u^{+n}\|_{D^m(B^{+N})}^2]^{1/2}+\frac{1}{4}\mathbb{E}[\|u^{+n}\|_{D^0}^2]\\\geqslant&-\lambda T^\alpha\langle t\rangle^{-1}\sum_{N\leqslant n}\exp(-\gamma(n-N))\mathbb{E}[\|u^N\|_{D^0}^2]^{1/2}\mathbb{E}[\|u^{+n}\|_{D^0}^2]^{1/2}+\frac{1}{4}\mathbb{E}[\|u^{+n}\|_{D^0}^2].\end{aligned}$$

Now

$$\begin{aligned}&|\mathbb{E}[f(W_{t,T}+I_{t,T}(u)+\varphi)-f(W_{t,T}+I_{t,T}(u^{-n})+\varphi)]|\\\leqslant&\langle t\rangle^{-1/2}\exp(-rn)\mathbb{E}[(|f|_{1,2,r}^B)^2]^{1/2}\,\mathbb{E}[\|u^{+n}\|_{D^m(B^{+n})}]^{1/2}\end{aligned}$$

Now recall that from our assumption on u we must have $F^{\varphi,0}(u)-F^{\varphi,0}(u^{-n})<\varepsilon$, so

$$\begin{aligned}0\geqslant&-\lambda T^\alpha\langle t\rangle^{-1}\sum_{n\leqslant N}\exp(-\gamma(n-N))\mathbb{E}[\|u^N\|_{D^0}^2]^{1/2}\mathbb{E}[\|u^{+n}\|_{D^0}^2]^{1/2}\\&-\langle t\rangle^{-1/2}\exp(-rn)\mathbb{E}[(|f|_{1,2,r}^B)^2]\,\mathbb{E}[\|u^{+n}\|_{D^0}^2]^{1/2}+\frac{1}{4}\mathbb{E}[\|u^{+n}\|_{D^0}^2]-\varepsilon,\end{aligned}$$

which implies that

$$\begin{aligned}&\mathbb{E}[\|u^{n+1}\|_{D^0}^2]^{1/2}\\\leqslant&\mathbb{E}[\|u^{+n}\|_{D^0}^2]^{1/2}\\\leqslant&\lambda T^\alpha\langle t\rangle^{-1}\sum_{N\leqslant n}\exp(-\gamma(n-N))\mathbb{E}[\|u_s^N\|_{D^0}^2]^{1/2}+\langle t\rangle^{-1/2}\exp(-rn)\mathbb{E}[(|f|_{1,2,r}^B)^2]^{1/2}+\varepsilon.\end{aligned}$$

Setting $a_n = \mathbb{E}[\|u^{n+1}\|_{D^0}^2]^{1/2}$ we have the inequality

$$\begin{aligned} a_n &\leqslant \lambda T^\alpha \langle t\rangle^{-1} \sum_{N\leqslant n-1} \exp(-\gamma(n-N)) a_N + \langle t\rangle^{-1/2}\mathbb{E}[(|f|_{1,2,r}^B)^2]^{1/2}\exp(-rn)+\varepsilon \\ &\leqslant \lambda T^\alpha \langle t\rangle^{-1} \sum_{N\leqslant n-1} \exp(-r(n-N)) a_N + \langle t\rangle^{-1/2}\mathbb{E}[(|f|_{1,2,r}^B)^2]^{1/2}\exp(-rn)+\varepsilon, \end{aligned}$$

and introducing $\tilde{a}_n = \exp(rn)a_n$ this is equivalent to

$$\begin{aligned} \tilde{a}_n &\leqslant \lambda T^{1/2-\delta}\langle t\rangle^{-1}\sum_{N\leqslant n} \tilde{a}_N + \mathbb{E}[(|f|_{1,2,r}^B)^2]^{1/2}\langle t\rangle^{-1/2} + \exp(rn)\varepsilon \\ &\leqslant 2\lambda\langle t\rangle^{-1/2-\delta}\sum_{N\leqslant n} \tilde{a}_N + \mathbb{E}[(|f|_{1,2,r}^B)^2]^{1/2}\langle t\rangle^{-1/2} + \exp(rn)\varepsilon. \end{aligned}$$

With this in mind discrete Gronwall lemma [42] gives

$$\tilde{c}_n \leqslant (\mathbb{E}[(|f|_{1,2,r}^B)^2]^{1/2}\langle t\rangle^{-1/2} + \varepsilon\exp(rn))\exp(2\lambda\langle t\rangle^{-1/2+\delta}n),$$

which in turn implies

$$c_n \leqslant (\mathbb{E}[(|f|_{1,2,r}^B)^2]^{1/2}\langle t\rangle^{-1/2} + \varepsilon\exp(rn))\exp(2\lambda\langle t\rangle^{-1/2+\delta}n - rn). \qquad \square$$

Definition 4.46. *We write* $\mathcal{Y} = \{\mu \colon \mu = \mathrm{Law}(W, u, u^\varphi)$ with u^φ minimizer of $F^{\varphi,0}$, $u\in\mathbb{H}_a\}$ *and take* $\bar{\mathcal{Y}}$ *to the closure of* $\mathcal{Y}$ *under weak convergence on* $\mathscr{C}^{-\varepsilon}(\langle x\rangle^{-n})\times L^2(\mathbb{R}_+\times\mathbb{R}^2)\times L^2(\mathbb{R}_+\times\mathbb{R}^2)$. *Observe that* $\mathcal{X} = P^*\mathcal{Y}, \bar{\mathcal{X}} = P^*\bar{\mathcal{Y}}$ *where* P *is the projection on the first two components.*

Lemma 4.47. *Assume* V_T *satisfies Hypothesis A. We consider* f *(deterministic) satisfying*

$$|f|_{1,2,r}^B < \infty,$$

for some $B\subseteq\mathbb{R}^2, 0\leqslant r < m - c\langle t\rangle^{-\delta}$ *and* $\langle t\rangle \geqslant T/2$. *Define*

$$V_{t,T}^f(\varphi) = \inf_{u\in\mathbb{H}_a}\mathbb{E}\left[f(Y_{t,T}(u,\varphi)) + V_T(Y_{t,T}(u,\varphi)) + \frac{1}{2}\int_t^T \|u_s\|_{L^2}^2 \mathrm{d}s\right] =: \inf_{u\in\mathbb{H}_a} F^{\varphi,f}(u) = \inf_{\mu\in\bar{\mathcal{X}}}\tilde{F}^{\varphi,f}(\mu)$$

and let $u^\varphi\in\mathbb{H}_a$ *be the unique minimizer of* $F^{\varphi,0}(u)$. *Then for any* $\mu\in\bar{\mathcal{Y}}$ *such that* $P^*\mu\in \mathrm{argmin}\tilde{F}^{\varphi,f}(\mu)$ *we have*

$$\mathbb{E}_\mu\left[\|u-u^\varphi\|_{D^{r-(2\lambda+c)\langle t\rangle^{-\delta/2}}(B)}^2\right]^{1/2} \leqslant C\langle t\rangle^{-1/2+\delta}|f|_{1,2,r}^B.$$

In particular if $u^{f,\varphi}\in\mathbb{H}_a$ *is a minimizer of* $F^{\varphi,f}$ *then*

$$\mathbb{E}\left[\|u^{f,\varphi}-u^\varphi\|_{D^{r-(2\lambda+c)\langle t\rangle^{-\delta/2}}(B)}^2\right]^{1/2} \leqslant C\langle t\rangle^{-1/2+\delta}|f|_{1,2,r}^B.$$

Proof. It is not hard to see that for μ such that $P^*\mu\in\mathrm{argmin}\tilde{F}^{\varphi,f}(\mu)$ we have u^ε such that $\mathrm{Law}(W,u^\varepsilon,u^\varphi)\to\mu$ and $F^{\varphi,f}(u^\varepsilon)\leqslant \inf_{u\in\mathbb{H}_a}F^{\varphi,f}(u)+\varepsilon$. Now from Lemma 4.45 we know

$$\mathbb{E}\left[\int_t^T \|(u_s^\varepsilon)^n\|_{L^2}^2\mathrm{d}s\right]^{1/2} \leqslant (C\langle t\rangle^{-1/2} + \varepsilon\exp(rn))\exp(2\lambda\langle t\rangle^{-1+\delta}n - rn),$$

recall Notation 4.44. So

$$\begin{aligned} &\mathbb{E}_\mu\left[\frac{1}{2}\int_t^T \|(u_s)^n\|_{L^2}^2\mathrm{d}s\right]^{1/2} \\ \leqslant\;& \liminf_{\varepsilon\to 0}\mathbb{E}\left[\frac{1}{2}\int_t^T \|(u_s^\varepsilon)^n\|_{L^2}^2\mathrm{d}s\right]^{1/2} \\ \leqslant\;& C\langle t\rangle^{-1/2}|f|_{1,2,r}^B\exp(2\lambda\langle t\rangle^{-1+\alpha}n - rn). \end{aligned}$$

Now for any $\kappa > 0$

$$
\begin{aligned}
& \mathbb{E}_\mu\left[\int_t^T \|u_s - u_s^\varphi\|^2_{L^{2,r-2\lambda\langle t\rangle^{-1/2-\delta}-\kappa}(B)} \mathrm{d}s\right] \\
\leqslant \ & C\mathbb{E}_\mu\left[\int_t^T \sum_{N=0}^{\infty} \exp(2(r - 2\lambda\langle t\rangle^{-1/2-\delta} - \kappa)N)\mathbb{1}_{B_N}\|u_s - u^\varphi\|^2_{L^2}\mathrm{d}s\right] \\
= \ & C\mathbb{E}_{\mu^\varepsilon}\left[\int_t^T \sum_{N=0}^{\infty} \exp(2(r - 2\lambda\langle t\rangle^{-1/2-\delta})N)\exp(-2\kappa N)\mathbb{1}_{B_N}\|u_s - u^\varphi\|^2_{L^2}\mathrm{d}s\right] \qquad (4.17) \\
\leqslant \ & \frac{C}{2\kappa}\sup_N \exp(2(r - 2\lambda\langle t\rangle^{-1/2-\delta})N)\mathbb{E}\left[\int_s^T \mathbb{1}_{B_N}\|u_s - u^\varphi\|^2_{L^2}\right] \\
\leqslant \ & \frac{C}{2\kappa}\langle t\rangle^{-1/2-\delta}|f|^B_{1,2,r}
\end{aligned}
$$

if we choose $\kappa = c\langle t\rangle^{-\delta}$ we obtain the statement. □

Lemma 4.48. *Assume V_T satisfies Hypothesis A. We consider f (deterministic) satisfying*

$$|f|^B_{1,2,r} < \infty$$

for some $B \subseteq \mathbb{R}^2, 0 \leqslant r < m - \iota$ and $\langle t\rangle \geqslant T/2$ Then

$$V^f_{t,T}(\varphi) = f_{t,T} + V^0_{t,T}(\varphi)$$

where $f_{t,T}$ satisfies for any $c > 0$

$$|f_{t,T}|^B_{1,2,r-2(\lambda+c)\langle t\rangle^{-\delta}} \leqslant (1 + C\langle t\rangle^{-1/2+\delta})|f|^B_{1,2,r}.$$

Proof. By Lemma 4.39 we have

$$
\begin{aligned}
& \langle\nabla V^f_{t,T}(\varphi), \psi\rangle_{L^2} \\
= \ & \inf_{\mu\in\operatorname{argmin}\tilde{F}^{\varphi,f}} \mathbb{E}_\mu[\langle\nabla V^f_T(W_{t,T} + I_{t,T}(u) + \varphi), \psi\rangle_{L^2}] \\
= \ & \inf_{\mu\in\operatorname{argmin}\tilde{F}^{\varphi,f}} \mathbb{E}_\mu[\langle\nabla f(W_{t,T} + I_{t,T}(u) + \varphi), \psi\rangle_{L^2} + \langle\nabla V_T(W_{t,T} + I_{t,T}(u) + \varphi), \psi\rangle_{L^2}]
\end{aligned}
$$

and so by Lemmas 4.39 and 4.42 with u^φ being the minimzer of

$$\mathbb{E}\left[V_T(W_{t,T} + I_{t,T}(u) + \varphi) + \frac{1}{2}\int_t^T \|u_w\|^2_{L^2}\mathrm{d}w\right],$$

we can compute

$$
\begin{aligned}
& |\langle\nabla V^f_{t,T}(\varphi), \psi\rangle_{L^2} - \langle\nabla V^0_{t,T}(\varphi), \psi\rangle_{L^2}| \\
= \ & \Bigg|\left(\inf_{\mu\in\operatorname{argmin}\tilde{F}^{\varphi,f}} \mathbb{E}_\mu[\langle\nabla f(W_{t,T} + I_{t,T}(u) + \varphi), \psi\rangle_{L^2} + \langle\nabla V_T(W_{t,T} + I_{t,T}(u) + \varphi), \psi\rangle_{L^2}]\right) \\
& - \mathbb{E}_\mu[\langle\nabla V_T(W_{t,T} + I_{t,T}(u^\varphi) + \varphi), \psi\rangle_{L^2}]\Bigg| \\
\leqslant \ & \sup_{\mu\in\operatorname{argmin}\tilde{F}^f} \mathbb{E}_\mu[|\langle\nabla f(W_{t,T} + I_{t,T}(u) + \varphi), \psi\rangle_{L^2}|] \\
& + \sup_{\mu\in\operatorname{argmin}\tilde{F}^f} |\mathbb{E}_\mu[\langle\nabla V_T(W_{t,T} + I_{t,T}(u) + \varphi), \psi\rangle_{L^2} - \langle\nabla V_T(W_{t,T} + I_{t,T}(u^\varphi) + \varphi), \psi\rangle_{L^2}]| \\
\leqslant \ & |f|^B_{1,2,r}\|\psi\|_{L^{2,-r}(B)} + |V_T|_{2,c\langle T\rangle^{-\delta}} \\
& \times \sup_{\mu\in\operatorname{argmin}\tilde{F}^f} \mathbb{E}_\mu\left[\|I_{s,t}(u - u^\varphi)\|_{L^{2,r-(2\lambda+c)\langle t\rangle^{-\delta}-c\langle T\rangle^{-\delta}}(B)}\right]\|\psi\|_{L^{2,-r+(2\lambda+c)\langle t\rangle^{-\delta}+c\langle T\rangle^{-\delta}}(B)} \\
\leqslant \ & |f|^B_{1,2,r}\|\psi\|_{L^{2,-r}(B)} + \lambda\langle t\rangle^{-1/2}|V_T|_{2,\iota}\mathbb{E}_\mu\left[\|(u - u^\varphi)\|_{D^{r-(2\lambda+c)\langle t\rangle^{-\delta}}(B)}\right]\|\psi\|_{L^{2,-r+2(\lambda+c)\langle t\rangle^{-\delta}}(B)} \\
\leqslant \ & |f|^B_{1,2,r}\|\psi\|_{L^{2,-r}(B)} + C\lambda\langle t\rangle^{-\delta}|f|^B_{1,2,r}\|\psi\|_{L^{2,-r+2(\lambda+c)\langle t\rangle^{-\delta}}(B)} \\
\leqslant \ & |f|^B_{1,2,r}(1 + C\lambda\langle t\rangle^{-\delta})\|\psi\|_{L^{2,-r+2(\lambda+c)\langle t\rangle^{-\delta}}(B)},
\end{aligned}
$$

and we can conclude by duality. □

Analogously we also have:

LEMMA 4.49. *Assume V_T satisfies Hypothesis C. We consider f (deterministic) satisfying*

$$|f|_{1,2,0}<\infty$$

Then for $\langle t\rangle \geqslant T/2$

$$V^f_{t,T}(\varphi)=f_{t,T}+V^0_{t,T}(\varphi),$$

where $f_{t,T}$ satisfies

$$|f_{t,T}|_{1,2,0}\leqslant (1+C\langle t\rangle^{-1/2+\delta})|f|_{1,2,0}.$$

The proof is analogous to the proof of Lemma 4.48.

4.3.3. Dependence on the initial condition

LEMMA 4.50. *With the assumptions and notations from Lemma 4.45 we have*

$$\mathbb{E}\left[\|u^{\varphi+\psi}-u^{\varphi}\|^2_{D^{r-2(\lambda+c)\langle t\rangle^{-\delta}}(A)}\right]^{1/2}\leqslant C\lambda T^{-\delta}\mathbb{E}[\|\psi\|^2_{D^r(A)}]^{1/2}.$$

Proof. We can set

$$f^{\psi}(\varphi)=V_T(\varphi+\psi)-V_T(\varphi)=\int_0^1\langle\nabla V_T(\varphi+\lambda\psi),\psi\rangle_{L^2}\mathrm{d}\lambda.$$

Then

$$\nabla f^{\psi}(\varphi)=\int_0^1\mathrm{Hess}\,V_T(\varphi+\lambda\psi)\psi\mathrm{d}\lambda$$

so

$$\mathbb{E}[\|\nabla f^{\psi}(\varphi)\|_{L^{2,r-\iota}(A)}]\leqslant |V_T|_{2,\iota}\mathbb{E}[\|\psi\|_{L^{2,r}(A)}]\leqslant\lambda T^{1/2-\delta}\mathbb{E}[\|\psi\|_{L^{2,r}(A)}].$$

Now applying Lemma 4.47 we with $f=f^{\psi}$ and $\varepsilon=0$, and estimating like in (4.47) we obtain:

$$\begin{aligned}
&\mathbb{E}_{\mu}\left[\|u^{\varphi+\psi}-u^{\varphi}\|^2_{D^{r-\iota-2(\lambda+c)\langle t\rangle^{-\delta}}(B)}\mathrm{d}s\right]\\
\leqslant\;&\mathbb{E}_{\mu}\left[\int_t^T\sum_{N=0}^{\infty}\exp(2(r-2(\lambda+c)\langle t\rangle^{-\delta}-c\langle t\rangle^{-\delta})N)\mathbb{1}_{B_N}\|u^{\varphi+\psi}-u^{\varphi}\|^2_{L^2}\mathrm{d}s\right]\\
=\;&\mathbb{E}_{\mu^{\varepsilon}}\left[\int_t^T\sum_{N=0}^{\infty}\exp(2(r-2\lambda\langle t\rangle^{-\delta}-c\langle t\rangle^{-\delta})N)\exp(-2c\langle t\rangle^{-\delta})\mathbb{1}_{B_N}\|u^{\varphi+\psi}-u^{\varphi}\|^2_{L^2}\mathrm{d}s\right] \qquad (4.18)\\
\leqslant\;&C\sup_N\exp(2(r-2\lambda\langle t\rangle^{-\delta}-c\langle t\rangle^{-\delta})N)\mathbb{E}\left[\int_t^T\mathbb{1}_{B_N}\|u^{\varphi+\psi}-u^{\varphi}\|^2_{L^2}\mathrm{d}t\right]\\
\leqslant\;&C\lambda T^{-\delta}\mathbb{E}[\|\psi\|^2_{L^{2,r}(B)}]^{1/2}.
\end{aligned}$$

□

4.3.4. Proofs of Propositions 4.35 and 4.36

For the remainder of this section we will denote by u^0 the minimizer of

$$\mathbb{E}\left[V_T(Y_{0,T}(u,\varphi))+\frac{1}{2}\int_0^T\|u_s\|^2_{L^2}\mathrm{d}s\right]$$

and by u^f the minimizer of

$$\mathbb{E}\left[V_T(Y_{0,T}(u,\varphi))+f(Y_{0,T}(u,\varphi))+\frac{1}{2}\int_0^T\|u_s\|^2_{L^2}\mathrm{d}s\right].$$

Proof of Proposition 4.35. We prove by induction that for any t such that $\langle t\rangle\geqslant 2^{-n}T$.

$$|V^f_{t,T}(\varphi)-V^0_{t,T}(\varphi)|_{1,2,r-c_t-\iota}\leqslant C_t|f|_{1,2,r},$$

where $c_t = 4(\lambda + c)\sum_{n:2^n \geqslant t} 2^{-n\delta}$ and $C_t = \prod_{n:2^n \geqslant t} (1 + C2^{-n\delta})$. Note that

$$\sup_{t>0} C_t \leqslant \exp\left(\sum_{n\in\mathbb{N}} \log((1 + C2^{-n\delta}))\right) \leqslant \exp\left(C\sum_{n\in\mathbb{N}} 2^{-n\delta}\right) < \infty.$$

Assume the statement has been proven for $t \geqslant \tilde{t} = 2^{-(n-1)}T$. By Proposition 4.12 we have

$$V_{t,T}^f(\varphi) = \mathbb{E}\left[V_{\tilde{t},T}^0(Y_{t,\tilde{t}}(u,\varphi)) + \tilde{f}(Y_{t,\tilde{t}}(u,\varphi)) + \frac{1}{2}\int_t^{\tilde{t}} \|u_w\|_{L^2}^2 \mathrm{d}w\right]$$

with $\tilde{f} = V_{\tilde{t},T}^f(\varphi) - V_{\tilde{t},T}^0(\varphi)$. Now by Lemma 4.48 we know that

$$\begin{aligned}
& |V_{t,T}^f(\varphi) - V_{t,T}^0(\varphi)|_{1,2,r-c_t} \\
\leqslant\ & |V_{t,T}^f(\varphi) - V_{t,T}^0(\varphi)|_{1,2,r-c_{\tilde{t}}+c\langle t\rangle^{-\delta}+2(2\lambda+c)\langle t\rangle^{-\delta}} \\
\leqslant\ & (1 + C\langle t\rangle^{-\delta})|\tilde{f}|_{1,2,r-c_{\tilde{t}}} \\
\leqslant\ & (1 + C\langle t\rangle^{-\delta})C_{\tilde{t}}|f|_{1,2,r} \\
\leqslant\ & C_t|f|_{1,2,r}.
\end{aligned}$$

□

We now prove Proposition 4.36.

Proof of Proposition 4.36. For the purposes of this argument we fix C (it can depend on the constants from the previous statment but cannot change from line to line).We may assume $C \leqslant \lambda^{-1}$ choosing λ small enough. We show that for $n \geqslant 0, r_n = r - c_0 - c_n$

$$\mathbb{E}\left[\int_{2^N}^{2^{N+1}} \|u^{\varphi,f} - u^{\varphi,0}\|_{L^{2,r_n}(B)}^2\right]^{1/2} \leqslant C2^{-N\delta}|f|_{1,2,r}^B$$

and

$$\mathbb{E}\left[\int_0^1 \|u_t^{\varphi,f} - u_t^{\varphi,0}\|_{L^{2,r_n}(B)}^2\right]^{1/2} \leqslant C\,|f|_{1,2,r}^B.$$

From this the statement will follow. To prove the second inequality we observe that by Proposition 4.12, u^f, u^0 are the minimizers for

$$\mathbb{E}\left[V_{\tilde{t},T}^0(Y_{t,\tilde{t}}(u,Y_{0.t}(u^f))) + f_{\tilde{t}}(Y_{t,\tilde{t}}(u,Y_{0,t}(u^f))) + \frac{1}{2}\int_t^{\tilde{t}} \|u_w\|_{L^2}^2 \mathrm{d}w\right]$$

and

$$\mathbb{E}\left[V_{\tilde{t},T}^0(Y_{t,\tilde{t}}(u,Y_{0,t}(u^0))) + \frac{1}{2}\int_t^{\tilde{t}} \|u_w\|_{L^2}^2 \mathrm{d}w\right]$$

respectively. Denote $f_{\tilde{t}} = V_{\tilde{t},T}^f - V_{\tilde{t},T}^0$. Now by our assumptions on V^0 we have by Lemma 4.47

$$\int_0^1 \|u_t^f - u_t^0\|_{D^{r-c_0-(2\lambda+c)2^{-\delta}}(B)}^2 \leqslant C|f_1|_{1,2,r-c_0}^B$$

Now we proceed by induction. Assume we have proved the inequality $N \leqslant n$, now want to prove it for $n+1$.Note that by Lemma 4.50 we get for any γ with $m - c_0 > \gamma > 0$:

$$\mathbb{E}\left[\int_{2^n}^{2^{n+1}} \|u_t^0 - \tilde{u}_t\|_{L^{2,\gamma-(2\lambda+c)2^{-n\delta}}(B)}^2 \mathrm{d}t\right]^{1/2} \leqslant C\lambda 2^{-\delta}\mathbb{E}[\|Y_{0,2^n}(u^0) - Y_{0,2^n}(u^f)\|_{L^{2,\gamma}(B)}^2]^{1/2}.$$

Here $\tilde{u}$ minimizes

$$\mathbb{E}\left[V_{2^{n+1},T}^0(Y_{t,\tilde{t}}(u,Y_{0,t}(u^f))) + \frac{1}{2}\int_{2^n}^{2^{n+1}} \|u_w\|_{L^2}^2 \mathrm{d}w\right].$$

Since by the induction assumption we have

$$
\begin{aligned}
& \mathbb{E}[\|Y_{0,2^n}(u^0)-Y_{0,2^n}(u^f)\|^2_{L^{2,\tilde{r}_n}}]^{1/2} \\
\leqslant & \sum_{N\leqslant n} \mathbb{E}[\|I_{2^{N-1},2^N}(u^0-u^f)\|^2_{L^{\tilde{r}_n}}]^{1/2} \\
\leqslant & \ 2C|f|^A_{1,2,r}\left(\sum_{N\leqslant n} 2^{-N\delta}\right)
\end{aligned}
$$

And by Lemma 4.47 we have

$$
\mathbb{E}\left[\int_{2^n}^{2^{n+1}} \|u^f_t-\tilde{u}_t\|^2_{L^{2,r_n-(2\lambda+c)2^{-n\delta}}(B)}\mathrm{d}t\right]^{1/2} \leqslant C2^{-n\delta}|f|_{1,2,r}
$$

and so by triangle inequality

$$
\begin{aligned}
& \mathbb{E}\left[\int_{2^n}^{2^{n+1}} \|u^{\varphi,f}_t-u^{\varphi}_t\|^2_{L^{2,r_{n+1}}(B)}\mathrm{d}t\right]^{1/2} \\
\leqslant & \ \mathbb{E}\left[\int_{2^n}^{2^{n+1}} \|u^{\varphi,f}_t-\tilde{u}^{\varphi}_t\|^2_{L^{2,r_{n+1}}(B)}\mathrm{d}t\right]^{1/2}+\mathbb{E}\left[\int_{2^n}^{2^{n+1}} \|u^{\varphi}_t-\tilde{u}^{\varphi}_t\|^2_{L^{2,r_{n+1}}(B)}\mathrm{d}t\right]^{1/2} \\
\leqslant & \ C2^{-n\delta}|f|^B_{1,2,r}+C^2\lambda 2^{-n\delta}\,|f_1|^B_{1,2,r}\sum_{N\leqslant n} 2^{-N\delta} \\
\leqslant & \ C\,|f|^B_{1,2,r}\sum_{N\leqslant n+1} 2^{-N\delta}
\end{aligned}
$$

where in the last line we have used $\lambda\leqslant C^{-1}$.This proves the statement. □

4.4. BOUNDS ON THE HESSIAN

In this section we will consider the case

$$
V^{\rho,R}_T(\varphi)=\lambda\alpha(T)\int_{\mathbb{R}^2}\rho(x)\cos(\beta\varphi(x))\mathrm{d}x+R(\varphi),
$$

where $\beta^2<4\pi,\alpha$ is defined by (4.10) and $R\in C^2(L^2(\mathbb{R}^2))$ satisfies

$$
|R(\varphi)|_{1,\infty}\leqslant\lambda, \qquad |R(\varphi)|_{2,c\langle T\rangle^{-\delta}}\leqslant C\lambda^2. \tag{4.19}
$$

The reason we denote the perturbation by R and not f as in the previous section is to emphasize the different properties. By f we usually denote a functional satisfying $|f|_{1,2,r}<\infty$ for some $r\geqslant 2$, while R usually satisfies (4.19). Again we are interested in the value function

$$
V^{\rho,R}_{t,T}(\varphi)=\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[V^{\rho,R}_T(Y_{t,T}(u,\varphi))+\frac{1}{2}\int_t^T\|u_s\|^2_{L^2}\mathrm{d}s\right] \tag{4.20}
$$

Our goal is to show that $V^{\rho,R}_{t,T}$ satisfies the assumptions of Proposition 4.35, or more precisely Hypothesis B.

THEOREM 4.51. *$V^{\rho,R}_{t,T}$ defined by (4.20) can be written as*

$$
V^{\rho,R}_{t,T}(\varphi)=\lambda\alpha(t)\int\rho\cos(\beta\varphi)+R_{t,T}(\varphi),
$$

where $R_{t,T}$ satisfies

$$
|R_{t,T}|_{1,\infty}+|R_{t,T}|_{2,c\langle t\rangle^{-\delta}}\leqslant C\lambda^2.
$$

C is independent of R,t,T,λ,ρ. In particular $V^{\rho,R}_{t,T}$ satisfied Hypothesis B for λ small enough.

Theorem 4.51 will be established in Section 4.4.1 below. This might seem hopeless at first since for this $V_{t,T}^{\rho,R}$ has to be smaller then the terminal data $V_T^{\rho,R}$. However what saves us is the martingale property of the renormalized cosine and the fact that we can expect $I(u)$ to be small. Indeed we have the following lemma.

Lemma 4.52. *Assume that $\langle t\rangle \geqslant T/2$. There exists a unique $u^{\varphi}\in\mathbb{H}_a$ such that*

$$\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[V_T^{\rho,R}(Y_{t,T}(u,\varphi))+\frac{1}{2}\int_t^T\|u_s\|_{L^2}^2\mathrm{d}s\right]=\mathbb{E}\left[V_T^{\rho,R}(Y_{t,T}(u^{\varphi},\varphi))+\frac{1}{2}\int_t^T\|u_s^{\varphi}\|_{L^2}^2\mathrm{d}s\right]$$

and

$$\|u^{\varphi}\|_{L^\infty([t,T]\times\mathbb{R}^2)}\leqslant 4\lambda\beta\langle t\rangle^{-1/2-\delta}.$$

Proof. Note that the assumptions of Lemma 4.42 are satisfied and so u^{φ} exists and is unique. Furthermore

$$\nabla V_{t,T}^{\rho,R}(\varphi)=\mathbb{E}[\nabla V_T^{\rho,R}(W_{t,T}+I_{t,T}(u^{\varphi})+\varphi)].$$

In particular $|V_{t,T}|_{1,\infty}\leqslant|V_T|_{1,\infty}\leqslant\lambda\beta\alpha(T)$. Now by the Verification Principle

$$u_s^{\varphi}=-Q_s\nabla V_{s,T}^{\rho,R}(Y_{t,s}),$$

where Y is the solution to the equation

$$\mathrm{d}Y_{t,s}=-Q_s\nabla V_{s,T}^{\rho,R}(Y_{t,s})\mathrm{d}s+Q_s\mathrm{d}X_s,\qquad Y_t=\varphi.$$

So by Lemma 4.27, provided $\langle t\rangle\geqslant T/2$:

$$\|u_t^{\varphi}\|_{L^\infty(\mathbb{R}^2)}\leqslant\langle t\rangle^{-1}\sup_{\varphi\in L^2(\mathbb{R}^2)}\|\nabla V_{t,T}^{\rho,R}(\varphi)\|_{L^\infty(\mathbb{R}^2)}=|V_{t,T}|_{1,\infty}\leqslant 2\lambda\langle t\rangle^{-1}\alpha(T)\leqslant 4\lambda\beta\langle t\rangle^{-1/2-\delta}.\qquad\square$$

We also introduce the map

$$\mathcal{R}_{t,T}^{\rho}(R)(\varphi)=V_{t,T}^{\rho,R}(\varphi)-\lambda\alpha(t)\int\rho(x)\cos(\beta\varphi(x))\mathrm{d}x\tag{4.21}$$

Let us discuss some properties of $\mathcal{R}_{t,T}^{\rho}$.

Lemma 4.53. *If $R\in C^2(L^2(\mathbb{R}^2))$ then so is $\mathcal{R}_{s,t}^{\rho}(R)$ for any $s,t>0$. Furthermore $\mathcal{R}^{\rho}$ has the following semi-group property:*

$$\mathcal{R}_{s,t}^{\rho}(\mathcal{R}_{t,T}^{\rho}(R))=\mathcal{R}_{s,T}^{\rho}(R).$$

Proof. For the first statement observe that by Proposition 4.10 and Corollary 4.14 we have that

$$V_{t,T}^{\rho,R}(\varphi)=\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[\lambda\alpha(T)\int\rho\cos(\beta Y_{t,T}(u,\varphi))+R(Y_{t,T}(u,\varphi))+\frac{1}{2}\int_t^T\|u_s\|_{L^2}^2\mathrm{d}s\right]$$

is in $C^2(L^2(\mathbb{R}^2))$. Since $\lambda\alpha(t)\int\rho\cos(\beta\varphi)$ is clearly in $C^2(L^2(\mathbb{R}^2))$ the definition of $\mathcal{R}_{s,t}^{\rho}$ implies the statement. For the second statement we see that by Proposition 4.12

$$\begin{aligned}\mathcal{R}_{s,T}^{\rho}(R)(\varphi) &= \inf_{u\in\mathbb{H}_a}\mathbb{E}\left[\lambda\alpha(T)\int\rho\cos(\beta Y_{s,T}(u,\varphi))+R(Y_{s,T}(u,\varphi))+\frac{1}{2}\int_s^T\|u_s\|_{L^2}^2\mathrm{d}s\right]\\ &\quad -\lambda\alpha(s)\int\rho\cos(\beta\varphi)\\ &= \inf_{u\in\mathbb{H}_a}\mathbb{E}\left[V_{t,T}^{\rho,R}(Y_{s,t}(u,\varphi))+\frac{1}{2}\int_s^t\|u_s\|_{L^2}^2\mathrm{d}s\right]-\lambda\alpha(s)\int\rho\cos(\beta\varphi)\\ &= \inf_{u\in\mathbb{H}_a}\mathbb{E}\left[\lambda\alpha(t)\int\rho\cos(\beta Y_{s,t}(u,\varphi))+\mathcal{R}_{t,T}^{\rho}(R)(Y_{s,t}(u,\varphi))+\frac{1}{2}\int_s^t\|u_s\|_{L^2}^2\mathrm{d}s\right]\\ &\quad -\lambda\alpha(s)\int\rho\cos(\beta\varphi)\mathrm{d}x\\ &= \mathcal{R}_{s,t}^{\rho}(\mathcal{R}_{t,T}^{\rho}(R))(\varphi).\end{aligned}$$

$\square$

4.4.1. Bounds on the remainder

LEMMA 4.54. *Assume that $\langle t\rangle \geqslant T/2$. Then for any R with $|R(\varphi)|_{1,\infty} \leqslant \lambda$ and $\iota \geqslant 0$:*

$$\begin{array}{lll} |\mathcal{R}^{\rho}_{t,T}(R)|_{1,\infty} & \leqslant & |R|_{1,\infty} + C\lambda^2\langle t\rangle^{-\delta} \\ |\mathcal{R}^{\rho}_{t,T}(R)|_{2,\iota+(\lambda+c)\langle t\rangle^{-\delta}} & \leqslant & |R|_{2,\iota} + C\lambda^2\langle t\rangle^{-\delta} \\ |\mathcal{R}^{\rho}_{t,T}(R)|_{2} & \leqslant & |R|_{2} + C\lambda^2\langle t\rangle^{-\delta} \end{array}$$

and for any $A \subseteq \mathbb{R}^2, m-\iota-2(\lambda+c)\langle t\rangle^{-\delta} \geqslant r \geqslant 0$ there exists a constant C_ρ dependent on ρ such that

$$|\mathcal{R}^{\rho}_{t,T}(R)|^{A}_{1,2,r} \leqslant |R|^{A}_{1,2,r} + C_\rho\langle t\rangle^{-\delta}$$

$$|\mathcal{R}^{\rho}_{t,T}(R) - \mathcal{R}^{\rho}_{t,T}(G)|^{A}_{1,2,r-\iota-2(\lambda+c)\langle t\rangle^{-\delta}} \leqslant (1 + C_\rho\langle t\rangle^{-\delta} + C_\rho\langle t\rangle^{-\delta}|R|_{2,\iota})(|R-G|^{A}_{1,2,r}). \tag{4.22}$$

Proof. Denote by $u^{\varphi,R}$ the minimizer of

$$\mathbb{E}\left[\lambda\alpha(T)\int\rho(x)\cos(\beta Y_{t,T}(u,\varphi))\mathrm{d}x + R(Y_{t,T}(u,\varphi)) + \frac{1}{2}\int_t^T \|u_s\|^2_{L^2}\mathrm{d}s\right].$$

By Lemma 4.42 we have

$$\begin{aligned} & \nabla V^{\rho,R}_{t,T}(\varphi) \\ = \; & \mathbb{E}[\nabla V^{\rho,R}_{T}(W_{t,T} + I_{t,T}(u^{\varphi,R}) + \varphi)] \\ = \; & \mathbb{E}[-\lambda\beta\alpha(T)\rho(x)\sin(\beta(W_{t,T} + I_{t,T}(u^{\varphi,R}) + \varphi)) + \nabla R(W_{t,T} + I_{t,T}(u^{\varphi,R}) + \varphi)] \\ = \; & -\mathbb{E}[\lambda\beta\alpha(T)\rho\sin(\beta(W_{t,T} + \varphi))] \\ & -\mathbb{E}\left[\lambda\beta\alpha(T)\int_0^1 \rho\cos(\beta(W_{t,T} + \varphi + \theta I_{t,T}(u^{\varphi,R})))I_{t,T}(u^{\varphi,R})\mathrm{d}\theta\right] \\ & +\mathbb{E}[\nabla R(W_{t,T} + I_{t,T}(u^{\varphi,R}) + \varphi)] \end{aligned} \tag{4.23}$$

and recall that $\mathbb{E}[\lambda\beta\alpha(T)\rho(x)\sin(\beta(W_{t,T} + \varphi))] = \mathbb{E}[\lambda\beta\alpha(t)\rho(x)\sin(\beta\varphi)]$. We now write

$$\tilde{R}(\varphi) = -\mathbb{E}\left[\lambda\beta\alpha(T)\int_0^1 \rho\cos(\beta(W_{t,T} + \varphi + \theta I_{t,T}(u^{\varphi,R})))I_{t,T}(u^{\varphi,R})\mathrm{d}\theta\right].$$

and the previous computation gives $\nabla\mathcal{R}^{\rho}_{t,T}(R)(\varphi) = \mathbb{E}[\nabla R(W_{t,T} + I_{t,T}(u^{\varphi,R}) + \varphi)] + \tilde{R}(\varphi)$. Since by Lemma 4.52 $\|I_{t,T}(u^{\varphi})\|_{L^\infty} \leqslant \|u^{\varphi}\|_{L^\infty([t,T]\times\mathbb{R}^2)} \leqslant 4\lambda\beta\langle t\rangle^{-1/2-\delta}$ we have for $\beta^2/8\pi \leqslant 1/2-\delta$ and $\langle t\rangle \geqslant T/2$

$$\sup_{\varphi\in L^2(\mathbb{R}^2)} \|\tilde{R}(\varphi)\|_{L^\infty} \leqslant 4\lambda^2\beta^2 t^{-\delta}.$$

Furthermore

$$\begin{aligned} & \|\tilde{R}(\varphi) - \tilde{R}(\psi)\|_{L^{2,r-2(\lambda+c)\langle t\rangle^{-\delta}}} \\ = \; & \left\|\mathbb{E}\left[\lambda\beta\alpha(T)\int_0^1 \rho\sin(\beta(W_{t,T} + \varphi + \theta I_{t,T}(u^{\varphi,R})))I_{t,T}(u^{\varphi,R})\mathrm{d}\theta\right]\right. \\ & \left. -\mathbb{E}\left[\lambda\beta\alpha(T)\int_0^1 \rho\sin(\beta(W_{t,T} + \psi + \theta I_{t,T}(u^{\psi,R})))I_{t,T}(u^{\psi,R})\mathrm{d}\theta\right]\right\|_{L^{2,r-2(\lambda+c)\langle t\rangle^{-\delta}}} \\ \leqslant \; & \lambda\beta\alpha(T)\left\|\mathbb{E}\left[\int_0^1 \rho\sin(\beta(W_{t,T} + \varphi + \theta I_{t,T}(u^{\varphi,R})))(I_{t,T}(u^{\varphi,R}) - I_{t,T}(u^{\psi,R}))\mathrm{d}\theta\right\|_{L^{2,r-2(\lambda+c)\langle t\rangle^{-\delta}}}\right. \\ & +\left\|\int_0^1 \rho(\sin(\beta(W_{t,T} + \varphi + \theta I_{t,T}(u^{\varphi,R})))\right. \\ & \left.\left. -\sin(\beta(W_{t,T} + \psi + \theta I_{t,T}(u^{\psi,R}))))I_{t,T}(u^{\psi,R})\mathrm{d}\theta\right]\right\|_{L^{2,r-2(\lambda+c)\langle t\rangle^{-\delta}}} \\ \leqslant \; & \lambda\beta\alpha(T)\mathbb{E}[\|I_{t,T}(u^{\varphi,R}) - I_{t,T}(u^{\psi,R})\|_{L^{2,r-2(\lambda+c)\langle t\rangle^{-\delta}}}] + \lambda\beta\alpha(T)\mathbb{E}[\|I_{t,T}(u^{\psi,R})\|_{L^\infty} \\ & \times(\|\varphi-\psi\|_{L^{2,r-2(\lambda+c)\langle t\rangle^{-\delta}}} + \|I_{t,T}(u^{\varphi,R}) - I_{t,T}(u^{\psi,R})\|_{L^{2,r-2(\lambda+c)\langle t\rangle^{-\delta}}})] \\ \leqslant \; & C\lambda^2\langle t\rangle^{-\delta}\|\varphi-\psi\|_{L^{2,r}} \end{aligned}$$

where in the last line we have used Lemmas 4.50 and 4.31. Analogously we obtain

$$\|\tilde{R}(\varphi)-\tilde{R}(\psi)\|_{L^2}\leqslant C\lambda^2\langle t\rangle^{-\delta}\|\varphi-\psi\|_{L^2}.$$

Clearly

$$\|\mathbb{E}[\nabla R(W_{t,T}+I_{t,T}(u^{\varphi,R})+\varphi)]\|_{L^\infty}\leqslant\sup_{\xi\in L^2}\|\nabla R(\xi)\|_{L^\infty},$$

while

$$\begin{aligned}&\|\mathbb{E}[\nabla R(W_{t,T}+I_{t,T}(u^{\varphi,R})+\varphi)]-\mathbb{E}[\nabla R(W_{t,T}+I_{t,T}(u^{\psi,R})+\psi)]\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}-\iota}}\\ \leqslant\;&|R(\varphi)|_{2,\iota}\mathbb{E}[\|(\varphi-\psi)\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}}}+\|I_{t,T}(u^{\varphi,R}-u^{\psi,R})\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}}}]\\ \leqslant\;&|R(\varphi)|_{2,\iota}(1+C\lambda\langle t\rangle^{-\delta})\|(\varphi-\psi)\|_{L^{2,r}},\end{aligned}$$

combining these bounds gives (4.54). To prove the second statement clearly

$$\begin{aligned}&\sup_\varphi\|\mathbb{E}[\nabla R(W_{t,T}+I_{t,T}(u^{\varphi,R})+\varphi)]\|_{L^{2,r}(A)}\\ \leqslant\;&\sup_\varphi\|R(\varphi)\|_{L^{2,r}(A)},\end{aligned}$$

while

$$\begin{aligned}&\lambda\beta\alpha(T)\left\|\int_0^1\rho(x)\sin(\beta(W_{t,T}+\varphi+\lambda I_{t,T}(u^{\varphi,R})))I_{t,T}(u^{\varphi,R})\mathrm{d}\lambda\right\|_{L^{2,r}(A)}\\ \leqslant\;&\lambda\beta\alpha(T)\|\rho I_{t,T}(u^{\varphi,R})\|_{L^{2,r}(A)}\\ \leqslant\;&C_\rho\lambda\alpha(T)\|I_{t,T}(u^{\varphi,R})\|_{L^\infty}\\ \leqslant\;&C_\rho\langle t\rangle^{-\delta}.\end{aligned}$$

On the other hand we can write $G=R-R+G$ and so applying Lemma 4.47 we obtain

$$\begin{aligned}&\mathbb{E}[\|I_{t,T}(u^{\varphi,R}-u^{\varphi,G})\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}}}]\\ \leqslant\;&C\langle t\rangle^{-1/2}\mathbb{E}\big[\|u^{\varphi,R}-u^{\varphi,G}\|_{D^{r-2\lambda\langle t\rangle^{-\delta}}}\big]\\ \leqslant\;&C\langle t\rangle^{-1/2-\delta}|R-G|_{1,2,r},\end{aligned}$$

where we recall that D^r was introduced in definition 4.30. With this in mind we estimate

$$\begin{aligned}&\|\nabla\mathcal{R}_{t,T}(R)(\varphi)-\nabla\mathcal{R}_{t,T}(G)(\varphi)\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}-\iota}(A)}\\ \leqslant\;&\mathbb{E}\bigg[\bigg\|\lambda\beta\alpha(T)\int_0^1\rho\sin(\beta(W_{t,T}+\varphi+\theta I_{t,T}(u^{\varphi,R})))I_{t,T}(u^{\varphi,R})\mathrm{d}\theta\\ &-\lambda\beta\alpha(T)\int_0^1\rho\sin(\beta(W_{t,T}+\varphi+\theta I_{t,T}(u^{\varphi,G})))I_{t,T}(u^{\varphi,G})\mathrm{d}\theta\bigg\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}-\iota}(A)}\bigg]\\ &+\mathbb{E}\Big[\|\nabla R(W_{t,T}+I_{t,T}(u^{\varphi,R})+\varphi)-\nabla G(W_{t,T}+I_{t,T}(u^{\varphi,G})+\varphi)\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}-\iota}(A)}\Big]\\ \leqslant\;&\lambda\alpha(T)\mathbb{E}\Big[\|I_{t,T}(u^{\varphi,R}-u^{\varphi,G})\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}-\iota}(A)}\Big]\\ &+\lambda\alpha(T)\mathbb{E}\Big[\|I_{t,T}(u^{\varphi,G})\|_{L^\infty}\|I_{t,T}(u^{\varphi,R}-u^{\varphi,G})\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}-\iota}(A)}\Big]\\ &+\sup_\varphi\|\nabla(R-G)(\varphi)\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}}(A)}+|R|_{2,\iota}\mathbb{E}\Big[\|I_{t,T}(u^{\varphi,R}-u^{\varphi,G})\|_{L^{2,r-2\lambda\langle t\rangle^{-\delta}}(A)}\Big]\\ \leqslant\;&(1+C\langle t\rangle^{-\delta})|(R-G)(\varphi)|^A_{1,2,r-2\lambda\langle t\rangle^{-\delta}}+C\langle t\rangle^{-\delta}|R|_{2,\iota}\,|(R-G)(\varphi)|^A_{1,2,r}.\end{aligned}$$

□

We can now iterativly apply this lemma to obtain:

Lemma 4.55. *Assume that $|R|_{1,\infty}<\lambda/2$. Let $t\geqslant 0$. Then for λ small enough*

$$V^{\rho,R}_{t,T}(\varphi)=\lambda\alpha(t)\int\rho\cos(\beta\varphi)+R_t(\varphi)$$

where R_t satisfies, with C_t defined by $C_t = C\sum_{N:2^N \geqslant t} 2^{-\delta N}$,

$$\begin{aligned} |R_t|_{2,\iota+(\lambda+c)c_t} &\leqslant \lambda^2 C_t + |R|_{2,\iota} \\ |R_t|_2 &\leqslant \lambda^2 C_t + |R_t|_2 \\ |R|_{1,\infty} &\leqslant \lambda^2 C_t + |R|_{1,\infty}. \end{aligned}$$

Proof. First we prove the statement for $\langle t\rangle \geqslant 2^{-N}T$. Assume the statement holds for $\tilde{t}$: $\langle\tilde{t}\rangle \geqslant 2^{-N+1}T$. Then by dynamic programming

$$\begin{aligned} & V_{t,T}^{\rho,R}(\varphi) \\ = & \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[\lambda\alpha(\tilde{t})\int\rho\cos(\beta Y_{t,\tilde{t}}(u,\varphi)) + R_{\tilde{t}}(Y_{t,\tilde{t}}(u,\varphi)) + \frac{1}{2}\int_t^{\tilde{t}}\|u_s\|_{L^2}^2 \mathrm{d}s\right] \end{aligned}$$

where $|R_{\tilde{t}}(\varphi)|_{2,\iota+c_{\tilde{t}}} \leqslant \lambda^2 C_{\tilde{t}} + |R|_{2,\iota}$, $|R_{\tilde{t}}(\varphi)|_{1,\infty} \leqslant \lambda^2 C_{\tilde{t}} + |R|_{1,\infty}$. Choosing λ small enough we get $|R_{\tilde{t}}|_{1,\infty} \leqslant \lambda$. Then we can apply Lemma 4.54 and deduce that

$$|R_t|_{2,\iota+(\lambda+c)c_t} \leqslant |R_{\tilde{t}}|_{2,\iota+(\lambda+c)c_{\tilde{t}}} + C\lambda^2\langle t\rangle^{-\delta} \leqslant \lambda^2 C_{\tilde{t}} + \lambda^2\langle t\rangle^{-\delta} + |R|_{2,\iota} = \lambda^2 C_t + |R|_{2,\iota},$$

and analogously for the bound on $|R_t|_2, |R|_{1,\infty}$. ☐

Finally we prove another lemma about $\mathcal{R}$ which will be useful when removing the UV cutoff.

Lemma 4.56. *Assume that $|R|_{2,(\lambda+c)\langle T\rangle^{-\delta}} \leqslant \lambda$. Then for any $t \geqslant 0$:*

$$|\mathcal{R}_{t,T}^{\rho}(R) - \mathcal{R}_{t,T}^{\rho}(G)|_{1,2,r-3c_t}^{A} \leqslant C_\rho(|R-G|_{1,2,r}^{A}).$$

Proof. In Lemma 4.54 the statement has been proven for $\langle t\rangle \geqslant T/2$, furthermore Lemma 4.55 gives, for λ small enough $|\mathcal{R}_{t,T}^{\rho}(R)|_{2,(\lambda+c)\langle T\rangle^{-\delta}+(\lambda+c)\langle t\rangle^{-\delta}} \leqslant |R|_{2,2(\lambda+c)\langle t\rangle^{-\delta}} + C\lambda^2\langle t\rangle^{-\delta} \leqslant \lambda$. Define $C_{\rho,t} = \prod_{n:2^n \geqslant t}(1+(C_\rho+\lambda)2^{-n\delta})$ with C_ρ being the constant from (4.22) Now assume we have proven the statement for $\langle t\rangle \geqslant \tilde{t} = 2^{-(n-1)}T$. By Lemma 4.53 and the induction assumptions

$$\begin{aligned} & |\mathcal{R}_{t,T}^{\rho}(R) - \mathcal{R}_{t,T}^{\rho}(G)|_{1,2,r-c_t}^{A} \\ = & |\mathcal{R}_{t,2t}^{\rho}(\mathcal{R}_{2t,T}^{\rho}(R)) - \mathcal{R}_{t,2t}^{\rho}(\mathcal{R}_{2t,T}^{\rho}(G))|_{1,2,r-2(\lambda+c)\langle t\rangle^{-\delta}-(\lambda+c)\langle t\rangle^{-\delta}-c_{2t}}^{A} \\ \leqslant & (1+(C_\rho+\lambda)\langle t\rangle^{-\delta})|(\mathcal{R}_{2t,T}^{\rho}(R)) - (\mathcal{R}_{2t,T}^{\rho}(G))|_{1,2,r-2c_{2t}}^{A} \\ \leqslant & (1+C_\rho\langle t\rangle^{-\delta})C_{\rho,2t}|R-G|_{1,2,r}^{A} \\ = & C_{\rho,t}|R-G|_{1,2,r}^{A}. \end{aligned}$$

☐

4.5. Dependence on the spatial cutoff

Lemma 4.57. *Let $u^{\varphi,\rho}$ be the minimizer for*

$$\mathbb{E}\left[\lambda\alpha(T)\int\rho\cos(\beta Y_{0,T}(u,\varphi)) + \frac{1}{2}\int_0^T\|u_s\|_{L^2}^2 \mathrm{d}s\right]$$

in $\mathbb{H}_a$. Assume that $\rho^1,\rho^2:\mathbb{R}^2\to\mathbb{R}$, $-1\leqslant\rho^i\leqslant 1$. Then there exists a $\gamma>0$ such that

$$\mathbb{E}\left[\int_0^T\left\|u_t^{\varphi,\rho^1} - u_t^{\varphi,\rho^2}\right\|_{L^{2,\gamma}(\mathrm{supp}(\rho^1-\rho^2))}^2 \mathrm{d}t\right] \leqslant C(1+|\mathrm{supp}(\rho^1-\rho^2)|),$$

where $|\mathrm{supp}(\rho^1-\rho^2)|$ denotes the measure of the support of $\rho^1-\rho^2$ and C does not depend on T.

Proof. To prove this observe that form eq. (4.22) we have

$$V_{t,T}^{\rho^1}(\varphi) = V_{t,T}^{\rho^2}(\varphi) + f_t(\varphi) + \lambda\alpha(t)\int(\rho_1-\rho_2)\cos(\beta\varphi) \tag{4.24}$$

with $|f_t|_{1,2,r}^{\mathrm{supp}(\rho^1-\rho^2)} \leqslant C(1+|\mathrm{supp}(\rho^1-\rho^2)|)$ with $r = m - c_0$ and C does not depend on t, T. Note also that

$$|\alpha(t)(\rho^1-\rho^2)\cos(\beta\varphi)|_{1,2,r} \leqslant \alpha(t)|\mathrm{supp}(\rho^1-\rho^2)|^{1/2}.$$

By Proposition 4.12 with $i \in \{1,2\}$ u^{φ,ρ^i} restricted to $[t_1,t_2]$ is a minimizer in $\mathbb{H}_a$ for

$$\mathbb{E}\left[V_{t_2,T}^{\rho^i}(Y_{t_1,t_2}(u,Y_{0,t_1}(u^{\varphi,\rho^i},\varphi)))+\frac{1}{2}\int_{t_1}^{t_2}\|u_s\|_{L^2}^2\mathrm{d}s\right].$$

Now we again prove by induction that

$$\mathbb{E}\left[\int_{2^n}^{2^{n+1}}\|u_t^{\varphi,\rho^1}-u_t^{\varphi,\rho^2}\|_{L^{2,r_n}(\mathrm{supp}(\rho^1-\rho^2))}^2\mathrm{d}t\right]\leqslant C2^{-n\delta}$$

for some C independent of N,T and $r_n=r-c_0-c\sum_{N\leqslant n}2^{-N\delta}$ (recall Notation 4.34). The proof is analogous to the proof of Proposition 4.36: Note that u^{φ,ρ^i} minimizes

$$\mathbb{E}\left[V_{1,T}^{\rho^i}(Y_{0,1}(u,\varphi))+\frac{1}{2}\int_0^1\|u_s\|_{L^2}^2\mathrm{d}s\right].$$

Since

$$V_{1,T}^{\rho_1}(\varphi)-V_{1,T}^{\rho_2}(\varphi)=f_1(\varphi)+\lambda\alpha(1)\int_{\mathbb{R}^2}(\rho_1-\rho_2)\cos(\beta\varphi)$$

with $|f|_{1,2,r}\leqslant C|\mathrm{supp}(\rho^1-\rho^2)|^{1/2}$ we obtain by applying Lemma 4.47 and using eq. (4.24)

$$\mathbb{E}\left[\int_0^1\|u_t^{\varphi,\rho^1}-u_t^{\varphi,\rho^2}\|_{L^{2,r-c_1}(\mathrm{supp}(\rho^1-\rho^2))}^2\mathrm{d}t\right]\leqslant C(1+|\mathrm{supp}(\rho^1-\rho^2)|).$$

Now by dynamic programming u^{φ,ρ^i} restricted to $[2^n,2^{n+1}]$ minimizes the functional

$$\mathbb{E}\left[\lambda\alpha(2^{n+1})\int\rho^i\cos(\beta Y_{2^n,2^{n+1}}(u,\ Y_{0,2^n}(u^{\varphi,\rho^i})))\ +\ \mathcal{R}_{2^{n+1},T}^{\rho_i}(0)(Y_{2^n,2^{n+1}}(u,\ Y_{0,2^n}(u^{\varphi,\rho^i})))\ +\ \frac{1}{2}\int_{2^n}^{2^{n+1}}\|u_s\|_{L^2}^2\mathrm{d}s\right].$$

Now setting $\tilde{u}\in\mathbb{H}_a$ to be the minimizer of the functional

$$\mathbb{E}\left[\lambda\alpha(2^{n+1})\int\rho^1(\cos(\beta Y_{2^n,2^{n+1}}(u,\ Y_{0,2^n}(u^{\varphi,\rho^2})))\ +\ \mathcal{R}_{2^{n+1},T}^{\rho^1}(0)(Y_{2^n,2^{n+1}}(u,\ Y_{0,2^n}(u^{\varphi,\rho^2})))\ +\ \frac{1}{2}\int_{2^n}^{2^{n+1}}\|u_s\|_{L^2}^2\mathrm{d}s\right]$$

we have that

$$\begin{aligned}&\mathbb{E}\left[\int_{2^n}^{2^{n+1}}\|u_t^{\varphi,\rho^1}-\tilde{u}_t\|_{L^{2,r_n-c2^{-n+1}}(\mathrm{supp}(\rho^1-\rho^2))}^2\mathrm{d}t\right]^{1/2}\\ \leqslant\ & 2^{-n\delta}\mathbb{E}[\|Y_{0,2^n}(u^{\varphi,\rho^1})-Y_{0,2^n}(u^{\varphi,\rho^2})\|_{L^{2,r_n}(\mathrm{supp}(\rho^1-\rho^2))}^2]^{1/2}\end{aligned}$$

from Lemma 4.50, and analogolously to the proof of Proposition 4.36 we can show that from the induction assumption it follows that for λ small enough

$$\mathbb{E}[\|Y_{0,2^n}(u^{\varphi,\rho^1})-Y_{0,2^n}(u^{\varphi,\rho^2})\|_{L^{2,r_n}(\mathrm{supp}(\rho^1-\rho^2))}^2]^{1/2}\leqslant 2^{-n\delta}C(1+|\mathrm{supp}(\rho^1-\rho^2)|)\sum_{N\leqslant n}2^{-N\delta}.$$

Applying Lemma 4.47 and using eq. (4.24) we know that

$$\mathbb{E}\left[\int_{2^n}^{2^{n+1}}\|u_t^{\varphi,\rho^2}-\tilde{u}_t\|_{L^{2,r-c2^{-n\delta}}(\mathrm{supp}(\rho^1-\rho^2))}^2\mathrm{d}t\right]^{1/2}\leqslant 2C(1+|\mathrm{supp}(\rho^1-\rho^2)|^{1/2})2^{-n\delta}.$$

Adding things up we deduce the claim by induction. □

Lemma 4.58. *Let $u^{\varphi,\rho}$ be a minimizer for*

$$\mathbb{E}\left[\lambda\alpha(T)\int\rho\cos(\beta Y_{0,T}(u,\varphi)+\frac{1}{2}\int_0^T\|u_s\|_{L^2}^2\mathrm{d}s\right].$$

Assume that $\rho^1,\rho^2:\mathbb{R}^2\to\mathbb{R}$, $-1\leqslant\rho^i\leqslant 1$ and for any $|x|\leqslant N$ $\rho^1(x)=\rho^2(x)=1$.Let $a>3$ Then

$$\mathbb{E}\left[\int_0^T\|u_t^{\varphi,\rho^1}-u_t^{\varphi,\rho^2}\|_{L^2(\langle x\rangle^{-a})}^2\mathrm{d}t\right]^{1/2}\leqslant C\langle N\rangle^{-(a-3)}.$$

from which it trivially follows that for any $\gamma>0$

$$\mathbb{E}\left[\int_0^T\|u_t^{\varphi,\rho^1}-u_t^{\varphi,\rho^2}\|_{L^{2,-\gamma}}^2\mathrm{d}t\right]^{1/2}\leqslant C\langle N\rangle^{-(a-3)},$$

and C does not depend on N,T or the ρ^i(provided they satisfy the assumptions).

Proof. Write $\tilde{\rho}^n=\rho^1+\mathbb{1}_{A_k}(\rho^2-\rho^1)$ with $A_k=\{x:k-1\leqslant|x|<k\}$. Then $\sum_{k\geqslant N}\tilde{\rho}^n=\rho^2$. Now we estimate using Lemma 4.57 and Lemma 4.25:

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\|u_t^{\varphi,\rho^1}-u_t^{\varphi,\rho^2}\|_{L^2(\langle x\rangle^{-a})}^2\mathrm{d}t\right]^{1/2}\\
\leqslant&\sum_{k\geqslant N}\mathbb{E}\left[\int_0^T\|u_t^{\varphi,\rho^1}-u_t^{\varphi,\tilde{\rho}^n}\|_{L^2(\langle x\rangle^{-a})}^2\mathrm{d}t\right]^{1/2}\\
\leqslant&\sum_{k\geqslant N}\langle n\rangle^{-a}\mathbb{E}\left[\int_0^T\|u_t^{\varphi,\rho^1}-u_t^{\varphi,\tilde{\rho}^n}\|_{L^{2,\gamma}(A_n)}^2\mathrm{d}t\right]^{1/2}\\
\leqslant&C\sum_{k\geqslant N}\langle n\rangle^{-a}|A_n|\\
\leqslant&C\langle N\rangle^{-(a-3)}.
\end{aligned}$$

□

Lemma 4.59. *Assume that R satisfies $|R|_{1,\infty}\leqslant\lambda,|R|_{2,\iota}\leqslant\lambda$ with λ sufficiently small and let ι such that $\log(t-T)\iota<m-c_t$.*

$$|\mathcal{R}_{t,T}^{\rho_1}(R)-\mathcal{R}_{t,T}^{\rho_2}(R)|_{1,2,m-\log_2(t-T)\iota-c_t}^{\mathrm{supp}(\rho^1-\rho^2)}\leqslant C|\mathrm{supp}(\rho^1-\rho^2)|^{1/2}\langle t\rangle^{-\delta}$$

In particular if $\iota\leqslant\langle T\rangle^{-\delta}$ then for T large enough

$$|\mathcal{R}_{t,T}^{\rho_1}(R)-\mathcal{R}_{t,T}^{\rho_2}(R)|_{1,2,m-\langle T\rangle^{-\delta/2}-c_t}^{\mathrm{supp}(\rho^1-\rho^2)}\leqslant C|\mathrm{supp}(\rho^1-\rho^2)|^{1/2}\langle t\rangle^{-\delta}$$

and

$$|\mathcal{R}_{t,T}^{\rho_1}(0)-\mathcal{R}_{t,T}^{\rho_2}(0)|_{1,2,m-c_t}^{\mathrm{supp}(\rho^1-\rho^2)}\leqslant C|\mathrm{supp}(\rho^1-\rho^2)|^{1/2}\langle t\rangle^{-\delta}.$$

Proof. Recall from eq. (4.23) that

$$\begin{aligned}
&\nabla(\mathcal{R}_{t,T}^{\rho_1}(R)(\varphi)-\mathcal{R}_{t,T}^{\rho_2}(R)(\varphi))\\
=&\mathbb{E}[\nabla R(W_{t,T}+I_{t,T}(u^{\varphi,\rho^1})+\varphi)-\nabla R(W_{t,T}+I_{t,T}(u^{\varphi,\rho^2})+\varphi)]\\
&-\lambda\beta\alpha(T)\mathbb{E}\left[\int_0^1\rho^1\sin(\beta(W_{t,T}+\varphi+\theta I_{t,T}(u^{\varphi,\rho^1})))I_{t,T}(u^{\varphi,\rho^1})\mathrm{d}\theta\right.\\
&\left.-\int_0^1\rho^2\sin(\beta(W_{t,T}+\varphi+\theta I_{t,T}(u^{\varphi,\rho^2})))I_{t,T}(u^{\varphi,\rho^2})\mathrm{d}\theta\right].
\end{aligned}$$

We first prove the statement for $\langle t\rangle\geqslant T/2$. Denote by $A=\mathrm{supp}(\rho^1-\rho^2)$ now we can estimate the first term by applying Lemma 4.47, we have

$$\mathbb{E}\left[\int_t^T\|u_s^{\varphi,\rho^1}-u_s^{\varphi,\rho^2}\|_{L^{2,m-c\langle t\rangle^{-\delta}}(A)}^2\mathrm{d}s\right]^{1/2}\leqslant C\lambda\langle t\rangle^{-\delta}|A|^{1/2}.$$

So

$$
\begin{aligned}
& \|\mathbb{E}[\nabla R(W_{t,T}+I_{t,T}(u^{\varphi,\rho^1})+\varphi)-\nabla R(W_{t,T}+I_{t,T}(u^{\varphi,\rho^2})+\varphi)]\|_{L^{2,m-\iota-c\langle t\rangle^{-\delta}}(A)}\\
\leqslant\ & |R|_{2,\iota}\mathbb{E}\Big[\|u^{\varphi,\rho^1}-u^{\varphi,\rho^2}\|^2_{D^{m-c\langle t\rangle^{-\delta}}(A)}\Big]^{1/2}\\
\leqslant\ & |R|_{2,\iota}\langle t\rangle^{-\delta}|A|^{1/2}.
\end{aligned}
$$

and furthermore we can decompose

$$
\begin{aligned}
& \rho^1\sin(\beta(W_{t,T}+\theta I_{t,T}(u^{\varphi,\rho^1})+\varphi))I_{t,T}(u^{\varphi,\rho^1})\\
& -\rho^2\sin(\beta(W_{t,T}+\varphi+\theta I_{t,T}(u^{\varphi,\rho^2})))I_{t,T}(u^{\varphi,\rho^2})\\
=\ & \rho^1\sin(\beta(W_{t,T}+\varphi+\theta I_{t,T}(u^{\varphi,\rho^1})))I_{t,T}(u^{\varphi,\rho^1}-u^{\varphi,\rho^2})\\
& \rho^1(\sin(\beta(W_{t,T}+\theta I_{t,T}(u^{\varphi,\rho^1})+\varphi))-\sin(\beta(W_{t,T}+\theta I_{t,T}(u^{\varphi,\rho^2})+\varphi)))I_{t,T}(u^{\varphi,\rho^2})\\
& +(\rho^2-\rho^1)\sin(\beta(W_{t,T}+\varphi+\lambda I_{t,T}(u^{\varphi,\rho^2})))I_{t,T}(u^{\varphi,\rho^2})
\end{aligned}
$$

and we can estimate

$$
\begin{aligned}
& \lambda\beta\alpha(T)\mathbb{E}\Big[\|\rho^1\sin(\beta(W_{t,T}+\varphi+\theta I_{t,T}(u^{\varphi,\rho^1})))I_{t,T}(u^{\varphi,\rho^1}-u^{\varphi,\rho^2})\|_{L^{2,m-c\langle t\rangle^{-\delta}}(A)}\Big]\\
\leqslant\ & \lambda\beta\alpha(T)\mathbb{E}\Big[\int_t^T\|u^{\varphi,\rho^1}-u^{\varphi,\rho^2}\|^2_{L^{2,m-c\langle t\rangle^{-\delta}}(A)}\Big]^{1/2}\\
\leqslant\ & C\lambda^2\langle t\rangle^{-\delta}|A|^{1/2}
\end{aligned}
$$

similarly

$$
\begin{aligned}
& \lambda\beta\alpha(T)\mathbb{E}\Big[\|\rho^1(x)(\sin(\beta(W_{t,T}+\theta I_{t,T}(u^{\varphi,\rho^1})+\varphi))-\sin(\beta(W_{t,T}+\theta I_{t,T}(u^{\varphi,\rho^2})+\\
& \varphi)))I_{t,T}(u^{\varphi,\rho^2})\|_{L^{2,m-c\langle t\rangle^{-\delta}}(A)}\Big]\\
\leqslant\ & \lambda\beta\alpha(T)\mathbb{E}\Big[\|\rho^1(x)I_{t,T}(u^{\varphi,\rho^1}-u^{\varphi,\rho^2})I_{t,T}(u^{\varphi,\rho^2})\|_{L^{2,m-c\langle t\rangle^{-\delta}}(A)}\Big]\\
\leqslant\ & \lambda\beta\mathbb{E}\Big[\int_t^T\|u_s^{\varphi,\rho^1}-u_s^{\varphi,\rho^2}\|^2_{L^{2,m-c\langle t\rangle^{-\delta}}(A)}\mathrm{d}s\|I_{t,T}(u^{\varphi,\rho^2})\|_{L^\infty}\Big]\\
\leqslant\ & C\lambda^2\langle t\rangle^{-1/2-\delta}|A|^{1/2}
\end{aligned}
$$

and for the last term

$$
\begin{aligned}
& \lambda\beta\alpha(T)\mathbb{E}[\|(\rho^2(x)-\rho^1(x))\sin(\beta(W_{t,T}+\varphi+\lambda I_{t,T}(u^{\varphi,\rho^2})))I_{t,T}(u^{\varphi,\rho^2})\|_{L^{2,m}(A)}]\\
\leqslant\ & \lambda\beta\alpha(T)\mathbb{E}[\|I_{t,T}(u^{\varphi,\rho^2})\|_{L^\infty}\|\rho^2(x)-\rho^1(x)\|_{L^{2,m}(A)}]\\
\leqslant\ & C\lambda^2\langle t\rangle^{-\delta}\|\rho^2(x)-\rho^1(x)\|_{L^{2,m}(A)}\\
\leqslant\ & C\lambda^2\langle t\rangle^{-\delta}|\mathrm{supp}(\rho^1-\rho^2)|^{1/2}
\end{aligned}
$$

Putting things together implies the statement for $\langle t\rangle\geqslant T/2$. Define $a_n=n\iota-c_{2^n}$. Now for the general statement we proceed by induction: We claim that for $\langle t\rangle\geqslant T/2^n$

$$
|\mathcal{R}^{\rho_1}_{t,T}(R)-\mathcal{R}^{\rho_2}_{t,T}(R)|^{\mathrm{supp}(\rho^1-\rho^2)}_{1,2,m-a_n}\leqslant C_t|A|^{1/2}\sum_{m\geqslant n}2^{-m\delta}
$$

where $C_t=C\prod_{i:2^i\geqslant t}(1+2^{(i-1)\delta})$

Assume the statement is proven for $\langle t\rangle\geqslant\tilde{T}=T/2^{n-1}$. By choosing λ small enough we can assume that $|\mathcal{R}^{\rho_1}_{t,T}(R)|_{1,\infty}+|\mathcal{R}^{\rho_1}_{t,T}(R)|_{2,\iota}\leqslant\lambda$ from Lemma 4.55. Then triangle inequality we can write

$$
\begin{aligned}
& |\mathcal{R}^{\rho_1}_{t,T}(R)-\mathcal{R}^{\rho_2}_{t,T}(R)|^A_{1,2,m-a_n}\\
=\ & |\mathcal{R}^{\rho_1}_{t,2t}(\mathcal{R}^{\rho_1}_{2t,T}(R))-\mathcal{R}^{\rho_2}_{t,2t}(\mathcal{R}^{\rho_2}_{2t,T}(R))|^A_{1,2,m-a_{n-1}-c\langle t\rangle^{-\delta}-\iota}\\
\leqslant\ & |\mathcal{R}^{\rho_1}_{t,2t}(\mathcal{R}^{\rho_1}_{2t,T}(R))-\mathcal{R}^{\rho_2}_{t,2t}(\mathcal{R}^{\rho_1}_{2t,T}(R))|^A_{1,2,m-a_{n-1}-c\langle t\rangle^{-\delta}-\iota}\\
& +|\mathcal{R}^{\rho_2}_{t,2t}(\mathcal{R}^{\rho_2}_{2t,T}(R))-\mathcal{R}^{\rho_2}_{t,2t}(\mathcal{R}^{\rho_1}_{2t,T}(R))|^A_{1,2,m-a_{n-1}-c\langle t\rangle^{-\delta}-\iota}
\end{aligned}
$$

now by our previous considerations

$$
\begin{aligned}
& |\mathcal{R}^{\rho_1}_{t,2t}(\mathcal{R}^{\rho_1}_{2t,T}(R)) - \mathcal{R}^{\rho_2}_{t,2t}(\mathcal{R}^{\rho_1}_{2t,T}(R))|^A_{1,2,m-a_{n-1}-c\langle t\rangle^{-\delta}-\iota}\\
\leqslant\ & C|A|^{1/2}\langle t\rangle^{-\delta}
\end{aligned}
$$

and by Lemma 4.54

$$
\begin{aligned}
& |\mathcal{R}^{\rho_2}_{t,2t}(\mathcal{R}^{\rho_2}_{2t,T}(R)) - \mathcal{R}^{\rho_2}_{t,2t}(\mathcal{R}^{\rho_1}_{2t,T}(R))|^A_{1,2,m-a_{n-1}-c\langle t\rangle^{-\delta}-\iota}\\
\leqslant\ & (1+\langle t\rangle^{-\delta} + |\mathcal{R}^{\rho_1}_{2t,T}(R)|_{2,\iota+c_{2t}})|\mathcal{R}^{\rho_2}_{2t,T}(R) - \mathcal{R}^{\rho_1}_{2t,T}(R)|^A_{1,2,m-a_{n-1}}\\
\leqslant\ & (1+\langle t\rangle^{-\delta} + \langle t\rangle^{-\delta}|R|_{2,\iota})|\mathcal{R}^{\rho_2}_{2t,T}(R) - \mathcal{R}^{\rho_1}_{2t,T}(R)|^A_{1,2,m-a_{n-1}}
\end{aligned}
$$

so putting thing together

$$
\begin{aligned}
& |\mathcal{R}^{\rho_1}_{t,T}(R) - \mathcal{R}^{\rho_2}_{t,T}(R)|^A_{1,2,m-a_n}\\
\leqslant\ & C|A|^{1/2}\langle t\rangle^{-\delta} + (1+\langle t\rangle^{-\delta} + \langle t\rangle^{-\delta}|R|_{2,\iota})|\mathcal{R}^{\rho_2}_{2t,T}(R) - \mathcal{R}^{\rho_1}_{2t,T}(R)|^A_{1,2,m-a_{n-1}}\\
\leqslant\ & C|A|^{1/2}\langle t\rangle^{-\delta} + C_{2t}(1+2\langle t\rangle^{-\delta})|A|^{1/2}\sum_{m\geqslant n} 2^{-m\delta}\\
\leqslant\ & C_t|A|^{1/2}\sum_{m\geqslant n+1} 2^{-m\delta} + C|A|^{1/2}\langle t\rangle^{-\delta}\\
\leqslant\ & C_t|A|^{1/2}\sum_{m\geqslant n} 2^{-m\delta}.
\end{aligned}
$$

□

4.6. Variational description

The purpose of this section is to establish Theorem 4.5. We restate it here in a more precise form. First we need the following definition.

Definition 4.60. *Take $r = m - 2c_0$ (recall that m is the "bare mass of the theory",more precisely our base Gaussian measure has covariance $m^2 - \Delta$) with c_0 defined in Notation 4.34. Take a C sufficiently large (to be fixed below) but independent of f. We define the set*

$$\mathbb{D}^f = \left\{u \in \mathbb{H}_a : \mathbb{E}\left[\int_0^\infty \|u_t\|^2_{D^r(A)}\mathrm{d}t\right] \leqslant C|f|_{1,2,m}\right\}.$$

Theorem 4.61.

$$
\begin{aligned}
& -\log\int e^{-f(\phi)}\nu^{\rho,T}_{\mathrm{SG}}(\mathrm{d}\phi)\\
=\ & -\lim_{T\to\infty,\rho\to 1}\left(\log\mathbb{E}\left[\exp\left(-f(W_{t,T}) - \alpha(T)\int \rho\cos(\beta W_{t,T})\right)\right]\right.\\
& \left. -\log\mathbb{E}\left[\exp\left(-\alpha(T)\int\rho\cos(\beta W_{t,T})\right)\right]\right)\\
=\ & \inf_{u\in\mathbb{D}^f} G^{\rho,f}(u)
\end{aligned}
$$

where

$$
\begin{aligned}
& G^{\rho,f}(u)\\
=\ & \mathbb{E}\Big[f(W_{0,\infty} + I_{0,\infty}(u) + I_{0,\infty}(u^\infty))\\
& + \lambda\int [\![\cos(\beta W_{0,\infty})]\!](\cos((\beta I_{0,\infty}(u) + \beta I_{0,\infty}(u^\infty)) - \cos(\beta I_{0,\infty}(u^\infty))))\\
& + \lambda\int [\![\sin(\beta W_{0,\infty})]\!](\sin(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^\infty))) - \sin(\beta I_{0,\infty}(u^\infty)))\\
& + \frac{1}{2}\int_0^\infty \|u_t\|^2_{L^2}\mathrm{d}t + \int_0^\infty\int u_t u^\infty_t\mathrm{d}t\Big]
\end{aligned}
$$

and $\|\langle t\rangle^{1/2+\delta}u_t^{\infty}\|_{L^{\infty}(\mathbb{P},L_t^{\infty}(\mathbb{R}_+,L^{\infty}(\mathbb{R}^2)))}<\infty$.

Proof. By Corollary 4.14

$$-\log\mathbb{E}\left[\exp\left(-f(W_{t,T})-\alpha(T)\int\rho\cos(\beta W_{t,T})\right)\right]=\inf_{u\in\mathbb{H}_a}F^{f,\rho}(u)$$

with

$$F_T^{\rho,f}(u)=\mathbb{E}\left[f(Y)\lambda\alpha(T)\int\rho\cos(\beta W_{0,T}+\beta I_{0,T}(u))+\frac{1}{2}\int_0^T\|u_s\|_{L^2}^2\mathrm{d}s\right].$$

By Theorem 4.62 below

$$\lim_{T\to\infty}\inf_{u\in\mathbb{H}_a}F_T^{\rho}(u)=\inf_{u\in\mathbb{H}_a}F_{\infty}^{\rho}(u)$$

with

$$\begin{aligned}&F_{\infty}^{\rho}(u)\\=\;&\mathbb{E}\left[\lambda\int\rho[\![\cos(\beta W_{0,\infty})]\!]\cos(\beta(I_{0,\infty}(u)))+\lambda\int\rho[\![\sin(\beta W_{0,\infty})]\!]\sin(\beta(I_{0,\infty}(u)))+\frac{1}{2}\int_0^{\infty}\|u_s\|_{L^2}^2\mathrm{d}s\right].\end{aligned}$$

By Corollary 4.63 below if C in Definition 4.60 is chosen suffiently large

$$\lim_{T\to\infty}\inf_{u\in\mathbb{H}_a}F_T^{f,\rho}(u)-F_T^{0,\rho}(u)=\inf_{u\in\mathbb{D}^f(u)}G^{f,\rho}(u).$$

Here

$$\begin{aligned}&G^{f,\rho}(u)\\=\;&\mathbb{E}\Big[f(W_{0,\infty}+I_{\infty}(u)+I_{\infty}(u^{\infty,\rho}))\\&+\lambda\int\rho(x)[\![\cos(\beta W_{0,\infty})]\!](\cos(\beta(I_{0,\infty}(u)+I_{0,\infty}(u^{\infty,\rho})))-\cos(\beta I_{0,\infty}(u^{\infty,\rho})))\\&+\lambda\int\rho(x)[\![\sin(\beta W_{0,\infty})]\!](\sin(\beta(I_{0,\infty}(u)+I_{0,\infty}(u^{\infty,\rho}))-\sin(\beta I_{0,\infty}(u^{\infty,\rho}))))\\&+\frac{1}{2}\int_0^{\infty}\|u_t\|_{L^2}^2\mathrm{d}t+\int_0^{\infty}\int u_t u_t^{\infty,\rho}\mathrm{d}t\Big]\end{aligned}$$

and $u^{\infty,\rho}$ is the minimizer of F_{∞}^{ρ}, it satisfies $\|\langle t\rangle^{1/2+\delta}u_t^{\infty,\rho}\|_{L^{\infty}(\mathbb{P},L_t^{\infty}(\mathbb{R}_+,L^{\infty}(\mathbb{R}^2)))}<\infty$.

Now by Lemma 4.65 below as $\rho\to 1$, $u^{\rho,\infty}$converges in $L^2(\mathbb{P},L^2(\mathbb{R}_+,L^2(\langle x\rangle^{-k})))$, for k large enough, to a $u^{\infty}\in L^{\infty}(\mathbb{P},L^{\infty}(\mathbb{R}_+\times\mathbb{R}^2))$ which satisfies $\|\langle t\rangle^{1/2+\delta}u_t\|_{L^{\infty}(\mathbb{P},L_t^{\infty}(\mathbb{R}_+,L^{\infty}(\mathbb{R}^2)))}\leqslant C$. Furthermore by Proposition 4.66

$$G^{f,\rho}(u)\to G^f(u)$$

uniformly on $\mathbb{D}^f$ which proves the statement. □

4.6.1. Removing the UV cutoff

In this section we fix $\rho\in C_c^{\infty}(\mathbb{R}^2)$, $\varphi\in L^2(\mathbb{R}^2)$. We denote by $u^{T,\rho}\in\mathbb{H}_a$ a minimizer of

$$F_T^{\rho}(u)=\mathbb{E}\left[\lambda\alpha(T)\int\rho\cos(\beta W_{0,T}+\beta I_{0,T}(u))+\frac{1}{2}\int_0^T\|u_s\|_{L^2}^2\mathrm{d}s\right].$$

Theorem 4.62. *$u^{T,\rho}$ converges in $\mathbb{H}_a$ to some $u^{\infty,\rho}\in\mathbb{H}_a$ and $u^{\infty,\rho}$ minimizes the functional*

$$\begin{aligned}&F_{\infty}^{\rho}(u)\\=\;&\mathbb{E}\left[\lambda\int\rho[\![\cos(\beta W_{0,\infty})]\!]\cos(\beta(I_{0,\infty}(u)))+\lambda\int\rho[\![\sin(\beta W_{0,\infty})]\!]\sin(\beta(I_{0,\infty}(u)))+\frac{1}{2}\int_0^{\infty}\|u_s\|_{L^2}^2\mathrm{d}s\right]\end{aligned}$$

in $\mathbb{H}_a$. Furthermore $\|\langle t\rangle^{1/2+\delta}u^{\infty,\rho}\|_{L^{\infty}(\mathbb{P}\times[0,T]\times\mathbb{R}^2)}\leqslant C\lambda$.

Proof. By Lemma 4.55 we know that $V_{t,T}^{\rho}$ given by

$$V_{t,T}^{\rho}(\varphi)=\inf_{u\in\mathbb{H}_a}\mathbb{E}\left[\lambda\alpha(T)\int\rho\cos(\beta Y_{t,T}(u,\varphi))\mathrm{d}x+\frac{1}{2}\int_t^T\|u_s\|_{L^2}^2\mathrm{d}s\right]$$

satisfies

$$V_{t,T}^{\rho}(\varphi)=\alpha(t)\int\rho\cos(\beta\varphi)+R_{t,T}(\varphi)$$

with $|R_{t,T}|_{1,\infty}+|R_{t,T}|_{2,\iota}\leqslant C\lambda^2$ and for some $\iota\leqslant C\lambda^2$. Furthermore $|R_{t,T}|_{1,2,0}\leqslant C_\rho$. This implies that the equation

$$\mathrm{d}Y_t=-Q_t\nabla V_{t,T}^{\rho}(Y_t)+Q_t\mathrm{d}X_t$$

has a unique solution in $C([0,T],L^2(\mathbb{R}^2))$ by a standard fix-point argument and

$$u_t^{T,\rho}=-Q_t\nabla V_{t,T}^{\rho}(Y_t)$$

by Proposition 4.13 . By definition of $\mathcal{R}$ (see eq. (4.21)) we have for $T_1<T_2$:

$$V_{t,T_1}^{\rho}(\varphi)=\alpha(t)\int\rho\cos(\beta\varphi)+\mathcal{R}_{t,T^1}^{\rho}(0)(\varphi)$$

and

$$V_{t,T_2}^{\rho}(\varphi)=\alpha(t)\int\rho\cos(\beta\varphi)+\mathcal{R}_{t,T^2}^{\rho}(\mathcal{R}_{T^1,T^2}(0))(\varphi).$$

By Lemma 4.54

$$|\mathcal{R}_{t,T_1}(0)-\mathcal{R}_{t,T_2}(\mathcal{R}_{T^1,T^2}(0))|_{1,2,0}\leqslant C|\mathcal{R}_{T_1,T_2}(0)|_{1,2,0}\leqslant C_\rho\langle T_1\rangle^{-\delta} \tag{4.25}$$

so by Proposition 4.36 we have

$$\mathbb{E}\left[\int_0^{T_1}\|u_t^{T_1,\rho}-u_t^{T_2,\rho}\|_{L^2}^2\mathrm{d}t\right]\leqslant C_\rho\langle T_1\rangle^{-\delta}$$

Furthermore from we have

$$|V_{t,T}^{\rho}|_{1,2}\leqslant|\mathcal{R}_{t,T}^{\rho}(0)|_{1,2}+\alpha(t)|\rho\sin(\beta\,\cdot\,)|_{1,2}\leqslant C_\rho\langle t\rangle^{1/2-\delta}$$

so , for any $\varphi\in L^2(\mathbb{R}^2)$

$$\|Q_t\nabla V_{t,T}^{\rho}(\varphi)\|_{L^2([T_1,\infty]\times\mathbb{R}^2)}\leqslant C_\rho\left(\int_{T_1}^{\infty}(t^{-1/2}\langle t\rangle^{1/2-\delta})^2\mathrm{d}t\right)^{1/2}\leqslant C_\rho\langle T_1\rangle^{-\delta}.$$

This implies that $u^{T,\rho}$ is a Cauchy sequence in $L^2(\mathbb{P}\times[0,\infty]\times\mathbb{R}^2)$ so it converges to some $u^{\infty,\rho}$, which in turn implies $I_{t,\bar{T}}(u^{T,\rho})\to I_{t,\bar{T}}(u^{\infty,\rho})$ in $H^1(\mathbb{R}^2)$ for any $\bar{T}\in[0,\infty]$ by Lemma 4.32. We are now going to prove that that indeed $u^{\infty,\rho}\in\mathbb{H}_a$, for which we have to prove that it is adapted.

From (4.25) we have also for any $\tilde{T}<\infty$

$$\sup_{t\leqslant\tilde{T}}|V_{t,T_1}^{\rho}-V_{t,T_2}^{\rho}|_{1,2,0}\to 0.$$

So $\nabla V_{t,T}^{\rho}\to\nabla V_{t,\infty}^{\rho}$ locally uniformly on $\mathbb{R}_+\times L^2(\mathbb{R}^2)$ and from the fact that $|R_{t,T}(\varphi)|_2\leqslant C$ we deduce that

$$\sup_{t\leqslant\tilde{T}}|V_{t,\infty}^{\rho}|_2\leqslant C\langle\tilde{T}\rangle^{\beta^2/8\pi}<\infty.$$

So $\nabla V_{t,\infty}$ is Lipschitz in $L^2(\mathbb{R}^2)$. Putting things together we obtain that as $T\to\infty$, $\mathbb{P}$-almost surely

$$u_t^{T,\rho}=Q_t\nabla V_{t,T}^{\rho}(W_{0,t}+I_{0,t}(u^{T,\rho}))\to Q_t\nabla V_{t,\infty}^{\rho}(W_{0,t}+I_{0,t}(u^{\infty,\rho}))\text{ in }L^2(\mathbb{R}^2),$$

which implies that $u^{\infty,\rho}$ is adapted since $u^{T,\rho}$ is adapted. Now to prove that $u^{\infty,\rho}$ minimizes $F_\infty^{\rho}(u)$ we observe that

$$\begin{aligned}F_T^{\rho}(u) &= \mathbb{E}\left[\lambda\alpha(T)\int\rho\cos(\beta Y_{0,T}(u,\varphi))+\frac{1}{2}\int_0^T\|u_s\|_{L^2}^2\mathrm{d}s\right]\\ &= \mathbb{E}\bigg[\lambda\alpha(T)\int\rho\cos(\beta W_{0,T})\cos(\beta(I_{0,T}(u)+\varphi))\\ &\quad+\lambda\alpha(T)\int\rho\sin(\beta W_{0,T})\sin(\beta(I_{0,T}(u)+\varphi))+\frac{1}{2}\int_0^T\|u_s\|_{L^2}^2\mathrm{d}s\bigg].\end{aligned}$$

Now from Definition 4.18 $\alpha(T)\cos(\beta W_{0,T}) \to [\![\cos(\beta W_{0,\infty})]\!]$ and $\alpha(T)\sin(\beta W_{0,T}) \to [\![\sin(\beta W_{0,\infty})]\!]$ in $L^2(\mathbb{P}, H^{-1+\delta})$. Note also that $I_{0,T}(u^T) = I_{0,\infty}(u^T) \to I_{0,\infty}(u^\infty)$ in $H^{1-\delta}$ which implies

$$\begin{aligned} & \mathbb{E}\left[\int \rho\alpha(T)\cos(\beta W_{0,T})\cos(\beta(I_{0,T}(u)+\varphi))\right] \\ \longrightarrow\ & \mathbb{E}\left[\int \rho[\![\cos(\beta W_{0,\infty})]\!]\cos(\beta(I_{0,\infty}(u)+\varphi))\right] \end{aligned}$$

and the same holds for the term with sinus. By Fatou's lemma we also know that

$$\liminf_{T\to\infty}\mathbb{E}\left[\int_0^T \|u_s^T\|_{L^2}^2 \mathrm{d}s\right] \geqslant \mathbb{E}\left[\int_0^\infty \|u_s^\infty\|_{L^2}^2 \mathrm{d}s\right]$$

so we can deduce $\liminf_{T\to\infty} F_T^\rho(u^{T,\rho}) \geqslant F_\infty^\rho(u^{\infty,\rho})$, which also implies $\liminf_{T\to\infty} \inf F_T^\rho \geqslant F_\infty^\rho(u^{\infty,\rho})$. Now observe that analogously we can show that $\lim_{T\to\infty} F_T^\rho(u) = F_\infty^\rho(u)$ for any $u \in \mathbb{H}_a$ which implies that $\liminf_{T\to\infty} \inf_{u\in\mathbb{H}_a} F_T^\rho(u) \leqslant \inf_{u\in\mathbb{H}_a} F_\infty^\rho(u)$ so

$$F_\infty^\rho(u^{\infty,\rho}) = \liminf_{T\to\infty} \inf_{u\in\mathbb{H}_a} F_\infty^\rho(u) = \inf_{u\in\mathbb{H}_a} F_\infty^\rho()u.$$

For the L^∞ bound

$$\begin{aligned} \|u_t^{T,\rho}\|_{L^\infty} &\leqslant \sup_\varphi \|Q_t \nabla V_{t,T}^\rho(\varphi)\|_{L^\infty} \\ &\leqslant \langle t\rangle^{-1}(\lambda\|\alpha(t)\rho\sin(\beta\varphi)\|_{L^\infty} + |R|_{1,\infty}) \leqslant C\lambda\,\langle t\rangle^{-1}(\langle t\rangle^{1/2-\delta}) \\ &\leqslant C\lambda\langle t\rangle^{-1/2-\delta}, \end{aligned}$$

which implies the final statement. □

We now discuss what happens in the $f \neq 0$ case.

Corollary 4.63. *Let $f \in C^2(\mathbb{R}^2)$ satisfy $|f|_{1,2,m}^A < \infty$. Then one can rewrite*

$$\begin{aligned} & \inf_{u\in\mathbb{H}_a} F_T^{f,\rho}(u) \\ = & \inf_{u\in\mathbb{H}_a} \mathbb{E}\left[f(W_{0,T}+I_T(u)) + \alpha(T)\int \rho\cos(\beta(W_{0,T}+I_{0,T}(u))) + \frac{1}{2}\int_0^\infty \|u_t\|_{L^2}^2\right] \\ = & \inf_{u\in\mathbb{D}^r(A)} \mathbb{E}\Bigg[f(W_{0,T}+I_{0,T}(u)+I_{0,T}(u^{T,\rho})) + \alpha(T)\int \rho\cos(\beta(W_{0,T}+I_{0,T}(u)+I_{0,T}(u^{T,\rho}))) \\ & + \frac{1}{2}\int_0^\infty \|u_t\|_{L^2}^2 \mathrm{d}t + \int_0^\infty \int u_t u_t^{\rho,T} \mathrm{d}t + \frac{1}{2}\int_0^\infty \|u_t^{\rho,T}\|_{L^2}^2 \mathrm{d}t\Bigg] \\ =: & \inf_{u\in\mathbb{D}^f} \bar{F}_T^{f,\rho}(u) \end{aligned}$$

Note that here we have made a change of variables and introduced the functional $\bar{F}_T^{f,\rho}(u)$ defined by the second to last line.

Proof. From Theorem 4.51 we obtain that the assumptions of Proposition 4.36 (Hypothesis B) are satisfied. Then applying Proposition 4.36 we can deduce the statement. □

Proposition 4.64. *With $\bar{F}_T^{f,\rho}(u)$ defined as in 4.63 $\bar{F}_T^{f,\rho}(u) \to \bar{F}_\infty^{,f,\rho}(u)$ uniformly on $\mathbb{D}^f$, where*

$$\begin{aligned} & \bar{F}_\infty^{f,\rho}(u) \\ = & \mathbb{E}\Bigg[f(W_{0,\infty}+I_{0,\infty}(u)+I_{0,\infty}(u^{\infty,\rho})) + \lambda\int \rho[\![\cos(\beta W_{0,\infty})]\!]\cos(\beta(I_{0,\infty}(u)+I_{0,\infty}(u^{\infty,\rho}))) \\ & + \lambda\int \rho(x)[\![\sin(\beta W_{0,\infty})]\!]\sin(\beta(I_{0,\infty}(u)+I_{0,\infty}(u^{\infty,\rho}))) + \frac{1}{2}\int_0^\infty \|u_t\|_{L^2}^2 \mathrm{d}t + \int_0^\infty \int u_t u_t^{\rho,\infty} \mathrm{d}t \\ & + \frac{1}{2}\int_0^\infty \|u_t^{\rho,\infty}\|_{L^2}^2 \mathrm{d}t\Bigg] \end{aligned}$$

Proof. Note that clearly for $r>0$ $\|I_{0,T}(u)\|_{H^1} \leqslant \|u\|_{D^0} \leqslant \|u\|_{D^r}$, so for any $u \in \mathbb{D}^f$,

$$\mathbb{E}[\|I_{0,T}(u)\|_{H^1}^2] \leqslant C.$$

Now we decompose:

$$\begin{aligned}
& \bar{F}_{\infty}^{f,\rho}(u) - \bar{F}_T^{f,\rho}(u) \\
= & \mathbb{E}\bigg[\lambda \int \rho(x)(\llbracket\cos(\beta W_{0,\infty})\rrbracket - \llbracket\cos(\beta W_{0,T})\rrbracket)\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) \\
& + \lambda \int \rho(x)(\llbracket\cos(\beta W_{0,T})\rrbracket)(\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) - \cos(\beta(I_{0,T}(u) + I_{0,T}(u^{T,\rho})))) \\
& + \lambda \int \rho(x)(\llbracket\sin(\beta W_{0,\infty})\rrbracket - \llbracket\sin(\beta W_{0,T})\rrbracket)\sin(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) \\
& + \lambda \int \rho(x)(\llbracket\sin(\beta W_{0,T})\rrbracket)(\sin(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) - \sin(\beta(I_{0,T}(u) + I_{0,T}(u^{T,\rho})))) \\
& f(W_{0,T} + I_{0,T}(u) + I_{0,T}(u^{T,\rho})) - f(W_{0,\infty} + I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho})) \\
& + \int_0^{\infty} \int u_t(u_t^{\rho,\infty} - u^{\rho,T})\mathrm{d}t + \frac{1}{2}\int_0^{\infty} \|u_t^{\rho,\infty}\|_{L^2}^2 \mathrm{d}t - \frac{1}{2}\int_0^{\infty} \|u_t^{\rho,T}\|_{L^2}^2 \mathrm{d}t\bigg].
\end{aligned}$$

By Couchy-Schwarz gives:

$$\mathbb{E}\bigg[\int_0^{\infty} \int u_t(u_t^{\rho,\infty} - u^{\rho,T})\mathrm{d}t\bigg] \leqslant \mathbb{E}\bigg[\int_0^{\infty} \|u_t\|_{L^2}^2 \mathrm{d}t\bigg]^{1/2} \mathbb{E}\bigg[\int_0^{\infty} \|(u_t^{\rho,\infty} - u_t^{\rho,T})\|_{L^2}^2 \mathrm{d}t\bigg]^{1/2}$$

which goes to 0 uniformly on $\mathbb{D}^f$. Furthermore

$$\begin{aligned}
& \mathbb{E}\bigg[\int \rho(\llbracket\cos(\beta W_{0,\infty})\rrbracket - \llbracket\cos(\beta W_{0,T})\rrbracket)\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho})))\bigg]^2 \\
\leqslant & \mathbb{E}[\|\rho(\llbracket\cos(\beta W_{0,\infty})\rrbracket - \llbracket\cos(\beta W_{0,T})\rrbracket)\|_{H^{-1+\delta}}^2] \mathbb{E}[\|\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho})))\|_{H^{1-\delta}}^2] \\
\leqslant & C\mathbb{E}[\|\rho(\llbracket\cos(\beta W_{0,\infty})\rrbracket - \llbracket\cos(\beta W_{0,T})\rrbracket)\|_{H^{-1+\delta}}^2] \mathbb{E}[\|I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho})\|_{H^1}^2]
\end{aligned}$$

again this goes to 0 uniformly on $\mathbb{D}^f$. And we can proceed analogously for the sinus term. Furthermore

$$\begin{aligned}
& \mathbb{E}\bigg[\int \rho(\llbracket\cos(\beta W_{0,T})\rrbracket)(\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) - \cos(\beta(I_{0,T}(u) + I_{0,T}(u^{T,\rho}))))\bigg]^2 \\
\leqslant & \mathbb{E}[\|\rho(\llbracket\cos(\beta W_{0,T})\rrbracket)\|_{H^{-1+\delta}}^2] \mathbb{E}[\|(\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) - \cos(\beta(I_{0,T}(u) + I_{0,T}(u^{T,\rho}))))\|_{H^{1-\delta}}^2] \\
\leqslant & \mathbb{E}[\|\rho(\llbracket\cos(\beta W_{0,T})\rrbracket)\|_{H^{-1+\delta}}^2] \mathbb{E}[\|(\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) - \cos(\beta(I_{0,T}(u) + I_{0,T}(u^{T,\rho}))))\|_{L^2}^{2\delta} \|(\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) - \cos(\beta(I_{0,T}(u) + I_{0,T}(u^{T,\rho}))))\|_{H^1}^{2-2\delta}] \\
\leqslant & \mathbb{E}[\|\rho(\llbracket\cos(\beta W_{0,T})\rrbracket)\|_{H^{-1+\delta}}^2] \mathbb{E}[\|I_{T,\infty}(u) + I_{0,\infty}(u^{\infty,\rho} - u^{T,\rho})\|_{L^2}^{2\delta} \, \|I_{T,\infty}(u) + I_{0,\infty}(u^{\infty,\rho} - u^{T,\rho})\|_{H^1}^{2-2\delta}] \\
\leqslant & 2\mathbb{E}[\|\rho(\llbracket\cos(\beta W_{0,T})\rrbracket)\|_{H^{-1+\delta}}^2] \mathbb{E}\bigg[\langle T\rangle^{-\delta} \int_0^{\infty} \|u\|_{L^2}^2 \mathrm{d}t + \int_0^{\infty} \|u^{\infty,\rho} - u^{T,\rho}\|_{L^2}^2 \mathrm{d}t\bigg]
\end{aligned}$$

which again goes to 0 uniformly on $\mathbb{D}^f$. Again we can proceed analogously for the sinus. Finally

$$\begin{aligned}
& \mathbb{E}[f(W_{0,T} + I_{0,T}(u) + I_{0,T}(u^{T,\rho})) - f(W_{0,\infty} + I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))] \\
\leqslant & C\mathbb{E}[\|W_{T,\infty} + I_{T,\infty}(u) + I_{0,T}(u^{T,\rho} - u^{\infty,\rho}) + I_{T,\infty}(u^{\infty,\rho})\|_{H^{-1}}].
\end{aligned}$$

which also goes to 0 uniformly on $\mathbb{D}^f$. □

4.6.2. Removing the IR cutoff

Now we consider the functional

$$
\begin{aligned}
G^{\rho,f}(u) &= \bar{F}^{f,\rho}_{\infty}(u) - \inf_{u\in\mathbb{H}_a} F^{\rho}_{\infty}(u)\\
&= \bar{F}^{f,\rho}_{\infty}(u) - F^{\rho}_{\infty}(u^{\rho,\infty})\\
&= \mathbb{E}\bigg[f(W_{0,\infty} + I_{\infty}(u) + I_{\infty}(u^{\rho,\infty}))\\
&\quad + \lambda\int \rho(x)[\![\cos(\beta W_{0,\infty})]\!](\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) - \cos(\beta I_{0,\infty}(u^{\infty,\rho})))\\
&\quad + \lambda\int \rho(x)[\![\sin(\beta W_{0,\infty})]\!](\sin(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho})) - \sin(\beta I_{0,\infty}(u^{\infty,\rho}))))\\
&\quad + \frac{1}{2}\int_0^{\infty} \|u_t\|^2_{L^2}\mathrm{d}t + \int_0^{\infty}\int u_t u_t^{\rho,\infty}\mathrm{d}t\bigg]
\end{aligned}
$$

The goal of this section is to establish that $G^{\rho,f}(u)$ has a limit as $\rho\to 1$. We will always assume that $f\colon L^2(\mathbb{R}^2)\to\mathbb{R}$ is such that $|f|_{1,2,m}<\infty$. In particular f is Lipschitz on $L^{2,-m}(\mathbb{R}^2)$.

Lemma 4.65. *As $\rho\to 1$, $u^{\rho,\infty}$converges in $L^2(\mathbb{P},\, L^2(\mathbb{R}_+,\, L^2(\langle x\rangle^{-k})))$,for k large enough, to a $u^{\infty}\in L^{\infty}(\mathbb{P}, L^{\infty}(\mathbb{R}_+\times\mathbb{R}^2))$ which satisfies $\|\langle t\rangle^{1/2+\delta}u_t\|_{L^{\infty}(\mathbb{P},L^{\infty}_t(\mathbb{R}_+\times\mathbb{R}^2))}\leqslant C$. Furthermore the law of (W,u^{∞}) is invariant under the action of the Euclidean group, where an element of the Euclidean group $G=(R,a)$ $R\in O(2), a\in\mathbb{R}^2$ acts on functions by*

$$(Gf)(x) = f(Rx-a).$$

Proof. Take ρ^1,ρ^2 such that $\rho^1(x)=\rho^2(x)=1$ on $B(0,N)$. By Theorem 4.62 there exists T^N such that $\sum_{i=1}^2 \mathbb{E}[\|u^{T^N,\rho^i} - u^{\infty,\rho^i}\|^2_{L^2(\mathbb{R}_+,L^2(\langle x\rangle^{-k}))}]\leqslant\sum_{i=1}^2\mathbb{E}[\|u^{T^N,\rho^i}-u^{\infty,\rho^i}\|^2_{D^0}]\leqslant\langle N\rangle^{-a}$ for $a>0$. From Lemma 4.58 we know that $\mathbb{E}[\|u^{T,\rho^1}-u^{T,\rho^2}\|^2_{D^{-\gamma}}]\leqslant C\,\langle N\rangle^{-a}$ uniformly in T. So by triangle inequality for some k large enough

$$
\begin{aligned}
&\mathbb{E}[\|u^{\infty,\rho^1} - u^{\infty,\rho^2}\|^2_{L^2(\mathbb{R}_+,L^2(\langle x\rangle^{-k}))}]\\
\leqslant\ & \sum_{i=1}^2 \mathbb{E}[\|u^{T^N,\rho^i} - u^{\infty,\rho^i}\|^2_{L^2(\mathbb{R}_+,L^2(\langle x\rangle^{-k}))}] + \mathbb{E}[\|u^{T^N,\rho^1} - u^{T^N,\rho^2}\|^2_{L^2(\mathbb{R}_+,L^2(\langle x\rangle^{-k}))}]\\
\leqslant\ & \langle N\rangle^{-a}
\end{aligned}
$$

which implies our statement. To prove the second statement $u^{\infty}\in L^{\infty}(\mathbb{P}, L^{\infty}(\mathbb{R}_+\times\mathbb{R}^2))$ follows from the fact $\sup_{T<\infty}\sup_{\rho}\|\langle t\rangle^{1/2+\delta}u^{T,\rho}\|_{L^{\infty}(\mathbb{P},L^{\infty}(\mathbb{R}_+\times\mathbb{R}^2))}<\infty$, which was proven in Lemma 4.52. Now to prove Euclidean invariance we can recall from from Corollary 4.14 we can write

$$
\begin{aligned}
V^{\rho}_{t,T}(G\varphi) &= -\log\mathbb{E}\left[\exp\left(-\alpha(T)\int\rho\sin(\beta W_{t,T}+\beta G\varphi)\right)\right]\\
&= -\log\mathbb{E}\left[\exp\left(-\alpha(T)\int G^{-1}\rho\sin(\beta G^{-1}W_{t,T}+\beta\varphi)\right)\right]\\
&= -\log\mathbb{E}\left[\exp\left(-\alpha(T)\int G^{-1}\rho\sin(\beta W_{t,T}+\beta\varphi)\right)\right]\\
&= V^{G^{-1}\rho}_{t,T}(\varphi)
\end{aligned}
$$

Now it is not hard to see that $Gu^{\rho,T}$ is the minimizer of

$$GF_T(u) := F_T(Gu) = \mathbb{E}\left[\lambda\alpha(T)\int (G^{-1}\rho)\cos(\beta\tilde{W}_{0,T} + \beta I_{0,T}(u)) + \frac{1}{2}\int_0^T \|u_s\|^2_{L^2}\mathrm{d}s\right]$$

where $\tilde{W}_{0,T}=G^{-1}W_{0,T}$ and by the Verification Principle $Gu^{\rho,T}$ satisfies

$$Gu^{\rho,T}_t = -Q_t\nabla V^{G^{-1}\rho}_{t,T}(\tilde{Y}_t).$$

Here $\tilde{X} = G^{-1}X$ and $\tilde{Y}$ solves the equation

$$\mathrm{d}\tilde{Y}_t = -Q_t \nabla V_{t,T}^{G^{-1}\rho}(\tilde{Y}_t) + Q_t \mathrm{d}\tilde{X}_t, \qquad \tilde{Y}_0 = 0.$$

Now $\mathrm{Law}(\tilde{X}) = \mathrm{Law}(X)$ implies $\mathrm{Law}(\tilde{Y}) = \mathrm{Law}(\bar{Y})$ where $\bar{Y}$ is the solution to the equation

$$\mathrm{d}\bar{Y}_t = -Q_t \nabla V_{t,T}^{G^{-1}\rho}(\bar{Y}_t) + Q_t \mathrm{d}X_t, \qquad \bar{Y}_0 = 0$$

and $\mathrm{Law}((GX_t, Gu^{\rho,T})) = \mathrm{Law}\big(\big(GX_t, -Q_t \nabla V_{t,T}^{G^{-1}\rho}(\bar{Y}_t)\big)\big)$. Now observe that by verification

$$u_t^{G^{-1}\rho,T} = -Q_t \nabla V_{t,T}^{G^{-1}\rho}(\bar{Y}_t).$$

So in total $\mathrm{Law}((GX, Gu^{\rho,T})) = \mathrm{Law}\big(\big(X, u_t^{G^{-1}\rho,T}\big)\big)$. Taking $T \to \infty$ we obtain

$$\mathrm{Law}(GX, Gu^{\rho,\infty}) = \mathrm{Law}(X, u^{G^{-1}\rho,\infty}).$$

Now sending $\rho \to 1$ we get $\mathrm{Law}(G(X, u^{\infty})) = \lim_{\rho\to 1}\mathrm{Law}((GX, Gu^{\rho,\infty})) = \lim_{\rho\to 1}\mathrm{Law}((X, u^{G^{-1}\rho,\infty})) = \mathrm{Law}((X, u^{\infty}))$ by uniqueness. Since $W_{0,\infty} = \int_0^{\infty} Q_t X_t \mathrm{d}t$ this implies the statement. □

PROPOSITION 4.66. *As $\rho \to 1$ $G^{\rho,f}(u) \to G^f(u)$ uniformly on $\mathbb{D}^f$, with*

$$\begin{aligned} G^f(u) \;=\; \mathbb{E}\bigg[& f(W_{0,\infty} + I_{0,\infty}(u) + I_{0,\infty}(u^{\infty})) \\ & + \lambda \int [\![\cos(\beta W_{0,\infty})]\!](\cos((\beta I_{0,\infty}(u) + \beta I_{0,\infty}(u^{\infty})) - \cos(\beta I_{0,\infty}(u^{\infty})))) \\ & + \lambda \int [\![\sin(\beta W_{0,\infty})]\!](\sin(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))) - \sin(\beta I_{0,\infty}(u^{\infty}))) \\ & + \frac{1}{2}\int_0^{\infty} \|u_t\|_{L^2}^2 \mathrm{d}t + \int_0^{\infty}\int u_t u_t^{\infty} \mathrm{d}t \bigg] \end{aligned}$$

Proof.

$$\begin{aligned} & G^f(u) - G^{\rho,f}(u) \\ = \;& \mathbb{E}[f(W_{0,\infty} + I_{\infty}(u) + I_{\infty}(u^{\infty})) - f(W_{0,\infty} + I_{\infty}(u) + I_{\infty}(u^{\rho,\infty})) \\ & + \lambda \int \rho [\![\cos(\beta W_{0,\infty})]\!]((\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))) - \cos(\beta I_{0,\infty}(u^{\infty})) - (\cos(\beta(I_{0,\infty}(u) + \\ & I_{0,\infty}(u^{\infty,\rho}))) - \cos(\beta I_{0,\infty}(u^{\infty,\rho}))) \\ & + \lambda \int \rho [\![\sin(\beta W_{0,\infty})]\!]((\sin(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))) - \sin(\beta I_{0,\infty}(u^{\infty}))) - (\sin(\beta(I_{0,\infty}(u) + \\ & I_{0,\infty}(u^{\infty,\rho}))) - \sin(\beta I_{0,\infty}(u^{\infty,\rho}))) \\ & + \lambda \int (1-\rho)[\![\cos(\beta W_{0,\infty})]\!](\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))) - \cos(\beta I_{0,\infty}(u^{\infty}))) \\ & + \lambda \int (1-\rho)[\![\sin(\beta W_{0,\infty})]\!](\sin(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))) - \sin(\beta I_{0,\infty}(u^{\infty}))) \\ & + \int_0^{\infty}\int u_t(u_t^{\infty} - u_t^{\rho,\infty})\mathrm{d}t \bigg] \end{aligned}$$

So by Interpolation with L^{∞}, for q close enough to 1:

$$\begin{aligned} & \|((\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))) - \cos(\beta I_{0,\infty}(u^{\infty}))) - \cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) - \\ & \cos(\beta I_{0,\infty}(u^{\infty,\rho})))\|_{B_{q,q}^{1-\delta_1}(\langle x\rangle^k)} \\ \leqslant\; & 4\beta \int_0^1 \|(\sin(\theta\beta I_{0,\infty}(u) + \beta I_{0,\infty}(u^{\infty})) - \sin(\theta\beta I_{0,\infty}(u) + \beta I_{0,\infty}(u^{\infty,\rho}))I_{0,\infty}(u)\|_{W^{1,1,\gamma}}^{1-\delta_1} \mathrm{d}\theta \\ \leqslant\; & C\|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{H^{1,-\gamma}}^{1-\delta_1} \|I_{0,\infty}(u)\|_{H^{1,2\gamma}}^{1-\delta_1} \\ & + C\|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{L^{2,-\gamma}}^{\delta_2(1-\delta_1)} \|I_{0,\infty}(u)\|_{H^{1,2\gamma}}^{2(1-\delta_1)} \end{aligned}$$

where we have applied Lemma 4.67, using that $\|I(u^{\infty,\rho})\|_{W^{1,\infty}} \leqslant C$ from Lemma 4.65 and Lemma 4.29. It is clear that

$$\|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{L^{2,-\gamma}} \leqslant \|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{L^{\infty}}^{1-\delta} \|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{L^{2,-\gamma}}^{\delta}.$$

We can then use this estimate to obtain for p large enough such that $1/p+1/q=1$

$$\lambda\mathbb{E}\bigg[\int \rho[\![\cos(\beta W_{0,\infty})]\!]((\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))) - \cos(\beta I_{0,\infty}(u^{\infty}))) - \cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty,\rho}))) - \cos(\beta I_{0,\infty}(u^{\infty,\rho})))\bigg]$$

$$\begin{aligned}
&\leqslant C\mathbb{E}\Big[\|[\![\cos(\beta W_{0,\infty})]\!]\|_{B_{p,p}^{-1+\delta}(\langle x\rangle^{-k})}^{p}\Big]^{1/p} \mathbb{E}\big[\big(\|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{H^{1,-\gamma}}^{q(1-\delta_1)} \|I_{0,\infty}(u)\|_{H^{1,2\gamma}}^{q(1-\delta_1)} + \\
&\quad \|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{L^{2,-\gamma}}^{q\delta_2(1-\delta_1)} \|I_{0,\infty}(u)\|_{H^{1,2\gamma}}^{q2(1-\delta_1)}\big)\big]^{1/q} \\
&\leqslant \mathbb{E}\Big[\|[\![\cos(\beta W_{0,\infty})]\!]\|_{B_{p,p}^{-1+\delta}(\langle x\rangle^{-k})}^{p}\Big]^{1/p} \\
&\quad \times \mathbb{E}\big[\big(\|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{H^{1,-\gamma}}^{2q(1-\delta_1)}\big)\big]^{1/2q} \mathbb{E}\big[\|I_{0,\infty}(u)\|_{H^{1,2\gamma}}^{2q(1-\delta_1)}\big]^{1/2q} \\
&\quad + \mathbb{E}[\|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{H^{1,-\gamma}}^{2}]^{1/q-(1-\delta_1)} \mathbb{E}[\|I_{0,\infty}(u)\|_{H^{1,2\gamma}}^{2}]^{(1-\delta_1)},
\end{aligned}$$

provided that we choose $q<1/(1-\delta_1)$ and $\delta_2 = 2(1-q(1-\delta_1))/q(1-\delta_1)$. Now for $u\in\mathbb{D}^f$ the last line is bounded by

$$C\big(\mathbb{E}\big[\big(\|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{H^{1,-\gamma}}^{2q(1-\delta_1)}\big)\big]^{1/2q} + \mathbb{E}[\|I_{0,\infty}(u^{\infty,\rho}) - I_{0,\infty}(u^{\infty})\|_{H^{1,-\gamma}}^{2}]^{1/q-(1-\delta_1)}\big),$$

which goes to 0. We can proceed analogously for the sinus term. To estimate

$$\begin{aligned}
&\left|\int (1-\rho)[\![\cos(\beta W_{0,\infty})]\!](\cos(\beta(I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))) - \cos(\beta I_{0,\infty}(u^{\infty})))\right| \\
= &\left|\beta\int_0^1 \int (1-\rho)[\![\cos(\beta W_{0,\infty})]\!](\sin(\theta I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))I_{0,\infty}(u)\mathrm{d}\theta\right|
\end{aligned}$$

it is not hard to see that that $\|(1-\rho)f\|_{W^{1,1}(\langle x\rangle^{k})} \leqslant N^{-k/2}\|f\|_{W^{1,1}(\langle x\rangle^{k/2})}$, so interpolating between $W^{1,1,\gamma/2}$ and L^{∞} we have

$$\begin{aligned}
&\leqslant \mathbb{E}\Big[\|[\![\cos(\beta W_{0,\infty})]\!]\|_{B_{p,p}^{-1+\delta}(\langle x\rangle^{-k})}^{p}\Big]^{1/p} \\
&\quad \times \mathbb{E}\big[\|(1-\rho)((\sin(\beta(\theta I_{0,\infty}(u) + I_{0,\infty}(u^{\infty})))I_{0,\infty}(u))\|_{W^{1,1,\gamma/2}}^{(1-\delta)q}\big]^{1/q} \\
&\leqslant N^{-\gamma/2}\mathbb{E}\Big[\|[\![\cos(W_{0,\infty})]\!]\|_{B_{p,p}^{-1+\delta}(\langle x\rangle^{-k})}^{p}\Big]^{1/p} \\
&\quad \times \mathbb{E}\big[\|((\sin(\beta(\theta I_{0,\infty}(u) + I_{0,\infty}(u^{\infty})))I_{0,\infty}(u))\|_{W^{1,1,\gamma}}^{(1-\delta)q}\big]^{1/q}.
\end{aligned}$$

Now

$$\mathbb{E}[\|((\sin(\theta I_{0,\infty}(u) + I_{0,\infty}(u^{\infty}))I_{0,\infty}(u))\|_{W^{1,1}}^{1-\delta}]$$

can be estimated analogously to the above computations. Finally

$$\mathbb{E}\bigg[\int_0^{\infty}\int u_t(u_t^{\infty} - u_t^{\rho,\infty})\mathrm{d}t\bigg]^2 \leqslant \mathbb{E}[\|u\|_{D^{\gamma}}^2]\mathbb{E}[\|u^{\infty} - u^{\rho,\infty}\|_{D^{-\gamma}}^2] \to 0$$

and by definition of f

$$\mathbb{E}[|f(W_{0,\infty} + I_{\infty}(u) + I_{\infty}(u^{\infty})) - f(W_{0,\infty} + I_{\infty}(u) + I_{\infty}(u^{\rho,\infty}))|] \leqslant \mathbb{E}[\|u^{\infty} - u^{\rho,\infty}\|_{D^{-\gamma}}^2]^{1/2}$$

which allows us to conclude. □

Lemma 4.67. *Assume that* $\|f^1\|_{W^{1,\infty}} + \|f^2\|_{W^{1,\infty}} \leqslant C$. *Then*

$$\begin{aligned}
&\|((\cos(f^1+g) - \cos(f^2+g)g)\|_{W^{1,1,\gamma}} \\
\leqslant\ &C(\|f^1 - f^2\|_{H^{1,-\gamma}}\|g\|_{H^{1,2\gamma}} + \|f^1 - f^2\|_{L^{2,-\gamma}}^{\delta}\|g\|_{H^{1,2\gamma}}^2)
\end{aligned}$$

Proof. Set $w(x) = \exp(\gamma x)$. Then with $1/p+1/q+1/2=1$ and q close enough to 2 we have

$$
\begin{aligned}
& \|\nabla((\cos(f^1+g)-\cos(f^2+g)g)\|_{L^{1,1,\gamma}} \\
\leqslant\ & |\textstyle\int_{\mathbb{R}^2} w(x)(\cos(f^1+g)-\cos(f^2+g))\nabla g \mathrm{d}x| + |\textstyle\int_{\mathbb{R}^2} w(x)(\cos(f^1+g)-\cos(f^2+g)) g\nabla g \mathrm{d}x| \\
& + |\textstyle\int_{\mathbb{R}^2} w(x)(\cos(f^1+g)-\cos(f^2+g))\nabla f^1 g \mathrm{d}x| + |\textstyle\int_{\mathbb{R}^2} w(x)(\cos(f^1+g))(\nabla f^1-\nabla f^2) g \mathrm{d}x| \\
\leqslant\ & \textstyle\int_{\mathbb{R}^2} w(x)|f^1-f^2||\nabla g|\mathrm{d}x + \textstyle\int_{\mathbb{R}^2} w(x)|f^1-f^2||g||\nabla g|\mathrm{d}x + \textstyle\int_{\mathbb{R}^2} w(x)|f^1-f^2||\nabla f^1||g|\mathrm{d}x \\
& + \textstyle\int_{\mathbb{R}^2} w(x)|(\nabla f^1-\nabla f^2)||g|\mathrm{d}x \\
\leqslant\ & \|\nabla g\|_{L^{2,2\gamma}}\|\nabla f^1-\nabla f^2\|_{L^{2,-\gamma}} + \|f^1-f^2\|_{L^{p,p,-\gamma}}\|g\|_{L^q}\|\nabla g\|_{L^{2,2\gamma}} + \\
& \|\nabla f^1\|_{L^\infty}\|f^1-f^2\|_{L^{2,-\gamma}}\|g\|_{L^{2,2\gamma}} + \|\nabla f^1-\nabla f^2\|_{L^{2,-\gamma}}\|g\|_{L^{2,2\gamma}}
\end{aligned}
$$

Now using the Sobolev embedding

$$\|g\|_{L^q} \leqslant \|g\|_{H^1} \leqslant \|g\|_{H^{1,2\gamma}}$$

we have

$$
\begin{aligned}
& \|\nabla g\|_{L^{2,2\gamma}}\|\nabla f^1-\nabla f^2\|_{L^{2,-\gamma}} + \|f^1-f^2\|_{L^{p,p,-\gamma}}\|g\|_{L^q}\|\nabla g\|_{L^{2,2\gamma}} + \\
& \|\nabla f^1\|_{L^\infty}\|f^1-f^2\|_{L^{2,-\gamma}}\|g\|_{L^{2,2\gamma}} + \|\nabla f^1-\nabla f^2\|_{L^{2,-\gamma}}\|g\|_{L^{2,2\gamma}} \\
\leqslant\ & \|\nabla f^1-\nabla f^2\|_{L^{2,-\gamma}}(\|g\|_{L^{2,2\gamma}} + \|\nabla g\|_{L^{2,2\gamma}}) + \|\nabla f^1\|_{L^\infty}\|f^1-f^2\|_{L^{-\gamma}}\|g\|_{L^{2\gamma}} \\
& + \|f^1-f^2\|_{L^\infty}^{1-\delta}\|f^1-f^2\|_{L^{2,-\gamma}}^{\delta}\|g\|_{H^{1,2\gamma}}^2 \\
\leqslant\ & C(\|f^1-f^2\|_{H^{1,-\gamma}}\|g\|_{H^{1,2\gamma}} + \|f^1-f^2\|_{L^{2,-\gamma}}^{\delta}\|g\|_{H^{1,2\gamma}}^2)
\end{aligned}
$$

where in the last line we have applied the assumption $\|f^1\|_{W^{1,\infty}} + \|f^2\|_{W^{1,\infty}} \leqslant C$.

Now using that

$$\|((\cos(f^1+g)-\cos(f^2+g)g)\|_{L^{1,\gamma}} \leqslant \|f^1-f^2\|_{L^{2,-\gamma}}\|g\|_{L^{2,2\gamma}}$$

we can conclude. □

4.7. Characterization as a shifted Gaussian measure

This section is dedicated to proving Theorem 4.6. The following Lemma will be very useful in this endeavor

Lemma 4.68. *Let $f \in C^2(L^2(\mathbb{R}^2))$ satisfy*

$$|f|_{1,2,0}^A < \infty,$$

and $g \in C^2(L^2(\mathbb{R}^2))$ be such that

$$|g|_{1,\infty} + |g|_2 < \infty.$$

Then there exists an $s' > 0$ such that for all $0 \leqslant s < s'$

$$
\begin{aligned}
& \left(\int f e^{-sg} \mathrm{d}\nu_{\mathrm{SG}}^{\rho,T}\right)\left(\int e^{-sg} \mathrm{d}\nu_{\mathrm{SG}}^{\rho,T}\right) \\
=\ & \frac{\mathrm{d}}{\mathrm{d}t}\left(-\log \int e^{-tf-sg} \mathrm{d}\nu_{\mathrm{SG}}^{\rho,T}\right) \\
=\ & \mathbb{E}[f(W_{0,\infty} + I_{0,\infty}(u^{sg,T,\rho}))].
\end{aligned}
$$

Here $u^{sg,T,\rho}$ denotes the minimizer of

$$\mathbb{E}\left[\lambda\alpha(T)\int \rho\cos(\beta Y_{0,T}(u,0)) + sg(Y_{0,T}(u,0)) + \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s\right].$$

Proof. We set for $t \geqslant 0$

$$F^t(u) = \mathbb{E}\left[\lambda\alpha(T)\int \rho\cos(\beta Y_{0,T}(u,0)) + sg(Y_{0,T}(u,0)) + tf(Y_{0,T}(u,0)) + \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s\right].$$

By Verification this has a minimizer in $\mathbb{H}_a$ which we denote $u^{tf+sg,T,\rho}$. By Lemma 4.55 for s small enough the initial condition $V_T(\varphi) = \lambda\alpha(T)\int \rho\cos(\beta\varphi) + sg(\varphi)$ satisfies Hypothesis D in Section 4.3. So we can apply Proposition 4.38 to obtain

$$\mathbb{E}[\|u^{tf+sg,T,\rho} - u^{sg,T,\rho}\|_{D^0}^2] \leqslant Ct|f|_{1,2,0},$$

from which we can deduce that

$$\mathbb{E}[f(W_{0,T}+I_{0,T}(u^{tf+sg,T,\rho}))] \xrightarrow{t\to 0} \mathbb{E}[f(W_{0,T}+I_{0,T}(u^{sg,T,\rho}))].$$

Now observe that by Corollary 4.14 we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(-\log\int e^{-tf-sg}\mathrm{d}\mu_{\mathrm{SG}}^{\rho,T}\right)\\ =&\lim_{t\to 0}\frac{F^t(u^{tf+sg,T,\rho})-F^0(u^{sg,T,\rho})}{t}.\end{aligned}$$

Furthermore

$$\begin{aligned}&\liminf_{t\to 0}\frac{F^t(u^{tf+sg,T,\rho})-F^0(u^{sg,T,\rho})}{0}\\ =&\ \mathbb{E}[f(W_{0,T}+I_{0,T}(u^{tf+sg,T,\rho}))]+\liminf_{t\to 0}\frac{(F^0(u^{tf+sg,T,\rho})-F^0(u^{sg,\rho}))}{t}\\ \geqslant&\ \mathbb{E}[f(W_{0,T}+I_{0,T}(u^{sg,T,\rho}))].\end{aligned}$$

On the other hand

$$\begin{aligned}&\limsup_{t\to 0}\frac{F^t(u^{tf+sg,T,\rho})-F^0(u^{sg,T,\rho})}{t}\\ \leqslant&\ \limsup_{t\to 0}\frac{F^t(u^{sg,T,\rho})-F^0(u^{sg,T,\rho})}{t}\\ =&\ \mathbb{E}[f(W_{0,T}+I_{0,T}(u^{sg,T,\rho}))].\end{aligned}$$

□

We can now prove Theorem 4.6:

Proof of Theorem 4.6. Setting $g=0$ in Lemma 4.68 we obtain for any $f\in C^2(L^2(\mathbb{R}^2))$

$$\int f\mathrm{d}\nu_{\mathrm{SG}}^{\rho,T}=\mathbb{E}[f(W_{0,T}+I_{0,T}(u^{T,\rho}))]$$

where $u^{T,\rho}=u^{0,T,\rho}$ and we recognize that in Lemma 4.65 it was established that $u^{T,\rho}\to u^{\infty}$ in $L^2(\mathbb{P},L^2(\mathbb{R}_+,L^2(\langle x\rangle^{-k})))$ as $T\to\infty$, $\rho\to 1$. This implies that

$$I_{0,T}(u^{T,\rho})\to I_{0,\infty}(u^{\infty})\text{ in }L^2(\mathbb{P},L^2(\langle x\rangle^{-n}))$$

so if f is bounded and continuous on $H^{-1}(\langle x\rangle^{-n})$ we have

$$\lim_{\rho\to 1}\lim_{T\to\infty}\int f\mathrm{d}\nu_{\mathrm{SG}}^{\rho,T}=\mathbb{E}[f(W_{0,\infty}+I_{0,\infty}(u^{\infty}))]. \tag{4.26}$$

Recall also that $I_{0,\infty}(u^{\infty})\in L^{\infty}(\mathbb{P},L^{\infty}(\mathbb{R}^2))$. So we will have proven our theorem once we have extended (4.26) to any f which is continuous on $H^{-1}(\langle x\rangle^{-k})$. To do this we claim that for any $f\in C(H^{-1}(\langle x\rangle^{-k}))$ we can find a sequence $f_n\in C^2(L^2(\mathbb{R}^2))\cap C(H^{-1}(\langle x\rangle^{-k}))$ such that $\sup_{\varphi\in H^{-1}(\rho)}|f_n|\leqslant\sup_{\varphi\in H^{-1}(\langle x\rangle^{-k})}|f_n|$ and for any $\varphi\in H^{-1}(\langle x\rangle^{-k})$ $f_n(\varphi)\to f(\varphi)$.Here by $f_n\in C^2(L^2(\mathbb{R}^2))\cap C(H^{-1}(\langle x\rangle^{-k}))$ we mean that $f_n\in C^2(L^2(\mathbb{R}^2))$ and extends continuous to a functional in $C(H^{-1}(\langle x\rangle^{-k}))$. To do this let P_n be a sequence of projections in $H^{-1}(\langle x\rangle^{-k})$ on finite dimensional subspace on $H^{-1}(\langle x\rangle^{-k})$ such that $P_n\varphi\to\varphi$ in $H^{-1}(\rho)$ as $n\to\infty$. Defining $\tilde{f}_n=f\circ P_n$ we can find for any $\tilde{f}_n$ an f_n such that

$$\sup_{\varphi\in H^{-1}(\langle x\rangle^{-k})}|\tilde{f}_n(\varphi)-f_n(\varphi)|\leqslant 1/n$$

and $f_n\in C^2(L^2(\mathbb{R}^2))\cap C(H^{-1}(\langle x\rangle^{-k}))$. Taking a diagonal sequence we can conclude. □

4.8. Osterwalder Schrader Axioms

In this section we complete the proof of Theorem 4.8.

4.8.1. Reflection Positivity

To prove Reflection Positivity we prove that the measure ν_{SG} is a limit of reflection positive measures which is sufficient by Remark 1.20. We denote by $\nu_{\mathrm{SG}}^{\rho}:=\lim_{T\to\infty}\nu_{\mathrm{SG}}^{\rho,T}$. Since $\nu_{\mathrm{SG}}^{\rho}\to\nu_{\mathrm{SG}}$ as $\rho\to 1$ it is enough to construct a sequence $\nu_{\mathrm{SG}}^{\varepsilon,\rho}\to\nu_{\mathrm{SG}}^{\rho}$ such that $\nu_{\mathrm{SG}}^{\varepsilon,\rho}$ is reflection positive. We can take ρ being invariant under the time reflection $\Theta f(x_1,x_2)=:f(-x_1,x_2)$. To construct $\nu_{\mathrm{SG}}^{\varepsilon,\rho}$ we cannot smooth in the "physical time" direction since this would destroy reflection positivity. Instead define $\theta=\delta_0\otimes\eta$, $\theta\in\mathscr{S}'(\mathbb{R}^2)$ where $\eta\in C_c^{\infty}(\mathbb{R}^2)$. Also set $\theta^{\varepsilon}=\varepsilon^{-2}\theta(\cdot/\varepsilon)=\delta_0\otimes\eta_{\varepsilon}$ where $\eta_{\varepsilon}=\varepsilon^{-1}\eta(\cdot/\varepsilon)$. Finally we set $W_T^{\varepsilon}=\theta_{\varepsilon}*W_{0,T}$, $T\in[0,\infty]$. We define

$$\nu_{\mathrm{SG}}^{\varepsilon,\rho}=e^{-\lambda\int\rho\alpha^{\varepsilon}\cos(\beta W_{\infty}^{\varepsilon})}\mathrm{d}\mathbb{P}.$$

We will now proceed in three steps: In Step 1 we show that for the correct choice of α^{ε}

$$\alpha^{\varepsilon}\cos(\beta W_{\infty}^{\varepsilon})\to[\![\cos(\beta W_{\infty})]\!].$$

In Step 2 we show that for any $p>1$

$$\sup_{\varepsilon}\mathbb{E}[e^{-\lambda p\int\rho\alpha^{\varepsilon}\cos(\beta W_{\infty}^{\varepsilon})}]<\infty.$$

Steps 1 and 2 together imply that $\nu_{\mathrm{SG}}^{\varepsilon,\rho}\to\nu_{\mathrm{SG}}^{\rho}$. In Step 3 we prove that $\nu_{\mathrm{SG}}^{\varepsilon,\rho}$ is indeed reflection positive.

Step 1. Observe that

$$\mathbb{E}[\theta_{\varepsilon}*W_{0,T_1}(x)\,\theta_{\varepsilon}*W_{0,T_2}(y)]=(\theta_{\varepsilon}\otimes\theta_{\varepsilon}*K_{T_1\wedge T_2})(x,y).$$

Now observe that for $T\in[0,\infty]$, $K_T(x,y)=\bar{K}_T(x-y)$ with $\bar{K}_T(x)\leqslant-\frac{1}{4\pi}\log(T\wedge|x|)+g(x)$ with g a bounded function. Furthermore

$$(\theta_{\varepsilon}\otimes\theta_{\varepsilon}*K_T)(x,y)=(\theta_{\varepsilon}*\theta_{\varepsilon}*\bar{K}_T)(x-y).$$

Then it not hard to see that

$$\bar{K}^{\varepsilon}(x)=\theta_{\varepsilon}*\theta_{\varepsilon}*K_{\infty}=\frac{1}{4\pi}\log\left(\frac{1}{|x|\vee\varepsilon}\right)+g^{\varepsilon}(x).$$

with $\sup_{\varepsilon}\|g^{\varepsilon}\|_{L^{\infty}}<\infty$. From this we can deduce that $\theta_{\varepsilon}*W_{\infty}(x)$ is in $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ almost surely since for any bounded $U\subseteq\mathbb{R}^2$

$$\mathbb{E}\left[\int_U((\theta_{\varepsilon}*W_{0,\infty})(x))^2\mathrm{d}x\right]=|U|\bar{K}^{\varepsilon}(0).$$

We claim that for any $f\in C_c^{\infty}(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2}f\tau^{\varepsilon}:=\int fe^{\frac{\beta^2}{2}\bar{K}^{\varepsilon}(0)}e^{i\beta W_{\infty}^{\varepsilon}}\to\int f[\![e^{i\beta W_{\infty}}]\!]$$

where the convergence is in $L^2(\mathbb{P})$. To prove this we calculate

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_{\mathbb{R}^2}e^{\frac{1}{2}\beta^2\bar{K}^{\varepsilon}(0)}e^{i\beta W_{\infty}^{\varepsilon}(x)}f(x)-e^{\frac{1}{2}\beta^2K_T(0)}e^{i\beta W_T(x)}f(x)\mathrm{d}x\right|^2\right]\\
=\;&\mathbb{E}\left[\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{\beta^2\bar{K}^{\varepsilon}(0)}e^{i\beta(W_{\infty}^{\varepsilon}(x)-W_{\infty}^{\varepsilon}(y))}-e^{\frac{\beta^2}{2}(\bar{K}_T(0)+\bar{K}^{\varepsilon}(0))}e^{i\beta(W_T(x)-W_{\infty}^{\varepsilon}(y))}\right.\\
&\left.-e^{\frac{\beta^2}{2}(\bar{K}_T(0)+\bar{K}^{\varepsilon}(0))}e^{i\beta(W_{\infty}^{\varepsilon}(x)-W_T(y))}+e^{\beta^2\bar{K}_T(0)}e^{i\beta(W_T(y)-W_T(y))}f(x)f(y)\mathrm{d}x\mathrm{d}y\right]\\
=\;&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{\beta^2\bar{K}^{\varepsilon}(x-y)}+e^{\beta^2\bar{K}_T(x-y)}-2e^{\beta^2\mathbb{E}[W_T(x)W_{\infty}^{\varepsilon}(y)]}f(x)f(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

W.l.o.g we can take $f\geqslant 0$. Now since $\bar{K}^{\varepsilon}(x-y)\leqslant-\frac{1}{4\pi}\log|x-y|+C$,$K_T(x-y)\leqslant-\frac{1}{4\pi}\log|x-y|+C$ we have by dominated convergence and Fatou's lemma

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\lim_{T\to\infty}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{\beta^2\bar{K}^{\varepsilon}(x-y)}+e^{\beta^2\bar{K}_T(x-y)}-2e^{\beta^2\mathbb{E}[W_T(x)W_{\infty}^{\varepsilon}(y)]}f(x)f(y)\mathrm{d}x\mathrm{d}y.\\
=\;&0
\end{aligned}$$

which proves the claim. This clearly implies

$$\int \rho e^{\frac{1}{2}\beta^2 \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon}) \to \int \rho [\![\cos(\beta W_{\infty})]\!]$$

in $L^2(\mathbb{P})$. In particular we can select a subsequence (not relabeled) such that this implies that $\mathbb{P}-a.s$

$$e^{-\lambda \int \rho e^{\frac{1}{2}\beta^2 \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon})} \to e^{-\lambda \int \rho [\![\cos(\beta W_{\infty})]\!]}.$$

Step 2. Step 1 will imply that $\nu_{\mathrm{SG}}^{\varepsilon,\rho} \to \nu_{\mathrm{SG}}^{\rho}$ as soon as we have established that

$$\sup_{\varepsilon} \mathbb{E}\left[e^{-\lambda p \int \rho e^{\beta^2 \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon})}\right] < \infty.$$

From Corollary 4.14 we know

$$-\log \mathbb{E}\left[e^{-\lambda p \int \rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(W_{\infty}^{\varepsilon})}\right] = \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[\lambda p \int \rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta(W_{\infty}^{\varepsilon} + I^{\varepsilon}(u))) + \frac{1}{2}\int_0^{\infty} \|u_t\|_{L^2}^2 \mathrm{d}t\right]$$

with $I^{\varepsilon}(u) = \theta^{\varepsilon} * I_{0,\infty}(u)$. Expanding the cosine we get

$$\begin{aligned}
& \left|\int \rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta(W_{\infty}^{\varepsilon} + I^{\varepsilon}(u)))\right|^2 \\
= & \left|\int \rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon}) \cos(\beta I^{\varepsilon}(u))\right|^2 + \left|\int \rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \sin(\beta W_{\infty}^{\varepsilon}) \sin(\beta I^{\varepsilon}(u))\right|^2 \\
\leqslant & \mathbb{E}\left[\left\|\rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon})\right\|_{H^{-1}}^2\right] \mathbb{E}[\|\cos(\beta I^{\varepsilon}(u))\|_{H^1}^2] \\
& + \mathbb{E}\left[\left\|\rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \sin(\beta W_{\infty}^{\varepsilon})\right\|_{H^{-1}}^2\right] \mathbb{E}[\|\sin(\beta I^{\varepsilon}(u))\|_{H^1}^2] \\
\leqslant & C\left(\mathbb{E}\left[\left\|\rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon})\right\|_{H^{-1}}^2\right] + \mathbb{E}\left[\left\|\rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \sin(\beta W_{\infty}^{\varepsilon})\right\|_{H^{-1}}^2\right]\right) \mathbb{E}[\|I^{\varepsilon}(u)\|_{H^1}^2] \\
\leqslant & C\left(\mathbb{E}\left[\left\|\rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon})\right\|_{H^{-1}}^2\right] + \mathbb{E}\left[\left\|\rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \sin(\beta W_{\infty}^{\varepsilon})\right\|_{H^{-1}}^2\right]\right) \mathbb{E}\left[\int_0^{\infty} \|u_t\|_{L^2}^2 \mathrm{d}t\right]
\end{aligned}$$

where in the last line we have used Lemma 4.32. This implies by Young's inequality

$$\begin{aligned}
& \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[\lambda p \int \rho e^{\beta^2 \bar{K}^{\varepsilon}(0)} \cos(\beta(W_{\infty}^{\varepsilon} + I^{\varepsilon}(u))) + \frac{1}{2}\int_0^{\infty} \|u_t\|_{L^2}^2 \mathrm{d}t\right] \\
\geqslant & -C\left(\mathbb{E}\left[\left\|\rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon})\right\|_{H^{-1}}^2\right] + \mathbb{E}\left[\left\|\rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \sin(\beta W_{\infty}^{\varepsilon})\right\|_{H^{-1}}^2\right]\right) + \frac{1}{4}\mathbb{E}\left[\int_0^{\infty} \|u_t\|_{L^2}^2 \mathrm{d}t\right].
\end{aligned}$$

Now note that from a simple calculation we get

$$\mathbb{E}\left[\left|e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon}(x)) e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon}(y))\right|\right] \leqslant C \frac{1}{|x-y|^{\beta^2/2\pi}},$$

from which we can conclude by Lemma 4.16 that $\sup_{\varepsilon} \mathbb{E}\left[\left\|\rho e^{\frac{\beta^2}{2} \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon})\right\|_{H^{-1}}^2\right] < \infty$, so we can deduce that $\sup_{\varepsilon} \mathbb{E}\left[e^{-\lambda p \int \rho e^{\beta \bar{K}^{\varepsilon}(0)} \cos(\beta W_{\infty}^{\varepsilon})}\right] < \infty$.

Step 3. We now show that $\nu_{\mathrm{SG}}^{\varepsilon,\rho}$ are reflection positive. We can write

$$\nu_{\mathrm{SG}}^{\varepsilon,\rho} = e^{-\lambda S_{\varepsilon}^{\rho}(\phi)} \mu_F^{\varepsilon}(\mathrm{d}\phi), \quad \text{with} \quad S_{\varepsilon}^{\rho}(\phi) = e^{\frac{1}{2}\beta^2 \bar{K}^{\varepsilon}(0)} \int \rho \cos(\beta \phi)$$

where $\mu_F^{\varepsilon} = \mathrm{Law}(W_{\infty}^{\varepsilon})$ is the gaussian measure with covariance operator

$$C^{\varepsilon}(f) = \theta^{\varepsilon} * (m^2 - \Delta)^{-1} * \theta^{\varepsilon} f.$$

We claim that μ_F^{ε} is reflection positive. Since it is Gaussian by Theorem 6.2.2 in [67] it is enough to show that

$$\langle f, \Pi_+ \Theta C^{\varepsilon} \Pi_+ f \rangle_{L^2} \geqslant 0.$$

Where Π_+ is the projection on $L^2(\mathbb{R}_+\times\mathbb{R})$. Since the convolution with θ^ε commutes with Π_+ we have

$$\begin{aligned} & \langle f, \Pi_+ \Theta C^\varepsilon \Pi_+ f\rangle \\ = & \langle \Pi_+(\theta^\varepsilon * f), \Theta(m^2-\Delta)^{-1}\Pi_+(\theta^\varepsilon * f)\rangle \\ \geqslant & 0, \end{aligned}$$

where in the last line we have used reflection positivity of $(m^2-\Delta)^{-1}$. Now finally we prove that $\nu_{\mathrm{SG}}^{\varepsilon,\rho}$ is indeed reflection positive. Write

$$S_\varepsilon^{\rho,+}(\phi) = e^{\frac{\beta^2}{2}\bar{K}^\varepsilon(0)}\int_{\mathbb{R}_+\times\mathbb{R}} \rho\cos(\beta\phi).$$

Observe that provided ρ is symmetric

$$S_\varepsilon^\rho(\phi) = S_\varepsilon^{\rho,+}(\phi) + S_\varepsilon^{\rho,+}(\Theta\phi).$$

Then

$$\int F(\phi)\,\Theta F(\phi)\mathrm{d}\nu_{\mathrm{SG}}^{\varepsilon,\rho} = \int F(\phi)\,e^{-\lambda S_\varepsilon^{\rho,+}(\phi)}\Theta\big(F(\phi)\,e^{-\lambda S_\varepsilon^{\rho,+}(\phi)}\big)\mathrm{d}\mu_F^\varepsilon \geqslant 0$$

by reflection positivity of μ_F^ε.

4.8.2. Exponential clustering

In this section we want to study expectations under the Sine Gordon measure of the form

$$\int_{\mathscr{S}'(\mathbb{R}^2)} \prod_{i=1}^k \langle \psi_i, \phi\rangle_{L^2(\mathbb{R}^2)} \nu_{\mathrm{SG}}(\mathrm{d}\phi).$$

Our goal is to show that there exist constants $C = C(\{\psi_i\}_{i=1}^k)$ and an $m_p > 0$ independent of ψ, such that for any $a \in \mathbb{R}^2$

$$\begin{aligned} & \left|\int_{\mathscr{S}'(\mathbb{R}^2)} \prod_{i=1}^l \langle \psi_i, \phi\rangle_{L^2(\mathbb{R}^2)} \prod_{i=l+1}^k \langle \psi_i(\cdot+a), \phi\rangle \nu_{\mathrm{SG}}(\mathrm{d}\phi)\right. \\ & \left. -\int_{\mathscr{S}'(\mathbb{R}^2)} \prod_{i=1}^l \langle \psi_i, \phi\rangle_{L^2(\mathbb{R}^2)} \nu_{\mathrm{SG}}(\mathrm{d}\phi) \int_{\mathscr{S}'(\mathbb{R}^2)} \prod_{i=l+1}^k \langle \psi_i, \phi\rangle_{L^2(\mathbb{R}^2)} \nu_{\mathrm{SG}}(\mathrm{d}\phi)\right| \\ \leqslant & \; C\exp(-m_p|a|). \end{aligned}$$

In this subsection all constants will be allowed to depend on ψ_i. First note that a simple computation gives, for $f, g\colon H^{-1}(\langle x\rangle^{-n}) \to \mathbb{R}$ continuous,bounded

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\left(-\log\int_{\mathscr{S}'(\mathbb{R}^2)} e^{-tf-sg}\mathrm{d}\nu_{\mathrm{SG}}\right) = \int_{\mathscr{S}'(\mathbb{R}^2)} fg\mathrm{d}\nu_{\mathrm{SG}} - \int_{\mathscr{S}'(\mathbb{R}^2)} f\mathrm{d}\nu_{\mathrm{SG}}\int_{\mathscr{S}'(\mathbb{R}^2)} g\mathrm{d}\nu_{\mathrm{SG}}.$$

Lemma 4.69. *There exists a $\gamma > 0$ such that for any $f, g\colon H^{-1}(\langle x\rangle^{-n}) \to \mathbb{R}^2$ such that with $A, B \subseteq \mathbb{R}^2$*

$$|f|_{1,2,m}^A < \infty, |g|_{1,2,m}^B < \infty$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\left(-\log\int_{\mathscr{S}'(\mathbb{R}^2)} e^{-tf-sg}\mathrm{d}\nu_{\mathrm{SG}}\right) \leqslant C|f|_{1,2,m}^A |g|_{1,2,m}^B \exp(-\gamma d(A,B)).$$

Proof. By weak convergence it is enough to prove the statement for $\nu_{\mathrm{SG}}^{\rho,T}$ with C, γ uniform in ρ, T. By Lemma 4.68 we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\left(-\log\int_{\mathscr{S}'(\mathbb{R}^2)} e^{-tf-sg}\mathrm{d}\nu_{\mathrm{SG}}^{\rho,T}\right) = \lim_{s\to 0}\frac{1}{s}(\mathbb{E}[f(W_{0,\infty}+I_{0,\infty}(u^{sg,\rho}))] - \mathbb{E}[f(W_{0,\infty}+I_{0,\infty}(u^{0,\rho}))]).$$

Now from Theorem 4.51 and Proposition 4.36 we get

$$\|I_{0,T}(u^{sg,\rho}) - I_{0,T}(u^{0,\rho})\|_{L^{2,\gamma}(B)} \leqslant s|g|_{1,2,m}^B,$$

so we have by Lemma 4.24

$$\begin{aligned} & |\mathbb{E}[f(W_{0,T}+I_{0,T}(u^{sg,\rho}))]-\mathbb{E}[f(W_{0,T}+I_{0,T}(u^{0,\rho}))]| \\ \leqslant\ & |f|_{1,2,m}^{A}\|I_{0,T}(u^{sg,\rho})-I_{0,T}(u^{0,\rho})\|_{L^{2,\gamma}(B)}\exp(-\gamma d(A,B)) \\ \leqslant\ & s|f|_{1,2,m}^{A}|g|_{1,2,m}^{B}\exp(-\gamma d(A,B)) \end{aligned}$$

which implies the statement. □

Finally we are able to prove the exponential clustering: Take $\chi^N\in C_c^\infty(\mathbb{R},\mathbb{R})$ with $\chi^N(x)=1$ if $|x|\leqslant N$ and $\chi^N(x)=0$ if $|x|\geqslant N+1$, $\sup_{N\in\mathbb{N}}\|(\chi^N)'\|_{L^\infty}\leqslant C$. Now define

$$f^N(\phi)=\prod_{i=1}^{l}\langle\psi_i,\phi\rangle_{L^2(\mathbb{R}^2)}\chi^N(\|\phi\|_{H^{-1,-\gamma}(A)}),\qquad g^N(\phi)=\prod_{i=l+1}^{k}\langle\psi_i,\phi\rangle_{L^2(\mathbb{R}^2)}\chi^N(\|\phi\|_{H^{-1,-\gamma}}).$$

where we have introduced the norm

$$\|\phi\|_{H^{-1,-\gamma}(A)}=\|(m^2-\Delta)^{1/2}\exp(-\gamma(d(x,A))\phi\|_{L^2}.$$

Note that $\|\phi\|_{H^{-1,-\gamma}(A)}\leqslant C_{k,\gamma}\|\phi\|_{H^{-1}(\langle x\rangle^{-k})}$ for any $k\in\mathbb{N}$. Furthermore introduce

$$g^{N,a}(\phi)=\prod_{i=l+1}^{k}\langle\psi_i(\cdot+a),\phi\rangle_{L^2(\mathbb{R}^2)}\chi(\|\phi\|_{H^{-1,-\gamma}(A+a)}).$$

Observe that $f^N,g^N\in C^2(L^2(\mathbb{R}^2))$. Note that with $w(x)=\exp(-\gamma(d(x,A))$ by product rule

$$\begin{aligned} & \nabla f^N(\phi) \\ =\ & \chi^N(\|\phi\|_{H^{-1,-\gamma}})\sum_{j=1}^{l}\prod_{\substack{i=0\\ i\neq j}}^{l}\langle\psi_i,\phi\rangle_{L^2(\mathbb{R}^2)}\psi_j \\ & +\frac{(\chi^N)'(\|\phi\|_{H^{-1,-\gamma}})}{\|\phi\|_{H^{-1,-\gamma}}}\prod_{i=0}^{l}\langle\psi_i,\phi\rangle_{L^2(\mathbb{R}^2)}(w(1-\Delta)^{-1}w\phi) \end{aligned}$$

so since

$$\|w(1-\Delta)^{-1}w\phi\|_{L^{2,\gamma}}\leqslant\|(1-\Delta)^{-1}w\phi\|_{L^2}\leqslant C\|\phi\|_{H^{-1,-\gamma}}$$

$$|\nabla f^N(\phi)|_{1,2,\gamma}^{A}\leqslant CN^l\left(\prod_{j=1}^{l}|\psi_j|_{1,2,\gamma}\right)$$

and now by exponential integrability and translation invariance of ν_{SG}

$$\begin{aligned} & \int_{\mathscr{S}'(\mathbb{R}^2)}\left|\prod_{i=1}^{l}\langle\psi_i,\phi\rangle_{L^2(\mathbb{R}^2)}\prod_{i=l+1}^{k}\langle\psi_i(\cdot+a),\phi\rangle-f^N(\phi)g^{N,a}(\phi)\right|\nu_{\mathrm{SG}}(\mathrm{d}\phi) \\ \leqslant\ & C\int_{\{\|\phi\|_{H^{-1,-\gamma}(A+a)}\geqslant N\,\mathrm{or}\,\|\phi\|_{H^{-1,-\gamma}(A)}\geqslant N\}}\|\phi\|_{H^{-1,-\gamma}(A)}^{l}\|\phi\|_{H^{-1,-\gamma}(A+a)}^{k-l}\nu_{\mathrm{SG}}(\mathrm{d}\phi) \\ \leqslant\ & 2\nu_{\mathrm{SG}}(\|\phi\|_{H^{-1,-\gamma}(A)}\geqslant N)^{1/2}\int_{\mathscr{S}'(\mathbb{R}^2)}\|\phi\|_{H^{-1,-\gamma}(A)}^{4l}\nu_{\mathrm{SG}}(\mathrm{d}\phi)\int\|\phi\|_{H^{-1,-\gamma}(A+a)}^{4k-4l}\nu_{\mathrm{SG}}(\mathrm{d}\phi) \\ \leqslant\ & C2\nu_{\mathrm{SG}}(\|\phi\|_{H^{-1,-\gamma}(A)}\geqslant N)^{1/2}\int_{\mathscr{S}'(\mathbb{R}^2)}\|\phi\|_{H^{-1,-\gamma}(A)}^{4k}\nu_{\mathrm{SG}}(\mathrm{d}\phi) \\ \leqslant\ & Ce^{-N}. \end{aligned}$$

And analogous statements hold for

$$\int_{\mathscr{S}'(\mathbb{R}^2)}\left|\prod_{i=1}^{l}\langle\psi_i,\phi\rangle_{L^2(\mathbb{R}^2)}-f^N(\phi)\right|\nu_{\mathrm{SG}}(\mathrm{d}\phi),\ \int_{\mathscr{S}'(\mathbb{R}^2)}\left|\prod_{i=l+1}^{k}\langle\psi_i,\phi\rangle-g^N(\phi)\right|\nu_{\mathrm{SG}}(\mathrm{d}\phi).$$

Now by Lemma 4.69

$$
\begin{aligned}
& \left|\int_{\mathscr{S}'(\mathbb{R}^2)} f^N(\phi) g^{N,a}(\phi)\nu_{\mathrm{SG}}(\mathrm{d}\phi) - \int_{\mathscr{S}'(\mathbb{R}^2)} f^N(\phi)\nu_{\mathrm{SG}}(\mathrm{d}\phi)\int_{\mathscr{S}'(\mathbb{R}^2)} g^N(\phi)\nu_{\mathrm{SG}}(\mathrm{d}\phi)\right| \\
= & \left|\int_{\mathscr{S}'(\mathbb{R}^2)} f^N(\phi) g^{N,a}(\phi)\nu_{\mathrm{SG}}(\mathrm{d}\phi) - \int_{\mathscr{S}'(\mathbb{R}^2)} f^N(\phi)\nu_{\mathrm{SG}}(\mathrm{d}\phi)\int_{\mathscr{S}'(\mathbb{R}^2)} g^{N,a}(\phi)\nu_{\mathrm{SG}}(\mathrm{d}\phi)\right| \\
\leqslant & |\nabla f^N(\phi)|_{1,2,\gamma}^{A} |\nabla g^{N,a}(\phi)|_{1,2,\gamma}^{A+a} \exp(-\gamma a) \\
= & |\nabla f^N(\phi)|_{1,2,\gamma}^{A} |\nabla g^{N}(\phi)|_{1,2,\gamma}^{A} \exp(-\gamma a) \\
\leqslant & C N^k \exp(-\gamma a)
\end{aligned}
$$

Putting things together we have

$$
\begin{aligned}
& \int_{\mathscr{S}'(\mathbb{R}^2)} \prod_{i=1}^{l} \langle \psi_i, \phi\rangle_{L^2(\mathbb{R}^2)} \prod_{i=l+1}^{k} \langle \psi_i(\cdot + a), \phi\rangle \nu_{\mathrm{SG}}(\mathrm{d}\phi) \\
& - \int_{\mathscr{S}'(\mathbb{R}^2)} \prod_{i=1}^{l} \langle \psi_i, \phi\rangle_{L^2(\mathbb{R}^2)} \nu_{\mathrm{SG}}(\mathrm{d}\phi) \int_{\mathscr{S}'(\mathbb{R}^2)} \prod_{i=l+1}^{k} \langle \psi_i, \phi\rangle_{L^2(\mathbb{R}^2)} \nu_{\mathrm{SG}}(\mathrm{d}\phi) \\
\leqslant & \; C\,(N^k \exp(-\gamma a) + \exp(-N)) \\
N = \gamma|a| \quad = & \; C\,((\gamma a)^k \exp(-\gamma\,|a|) + \exp(-\gamma |a|)) \\
\leqslant & \; C \exp(-(1-\delta)\gamma |a|).
\end{aligned}
$$

4.8.3. Non Gaussianity

In this section we prove that ν_{SG} is indeed not a Gaussian measure. Assume ν_{SG} would be Gaussian, we can regard it as a gaussian measure on the Hilbert space $H^{-1}(\langle x\rangle^{-n})$ with $n \in \mathbb{N}$ sufficiently large. Then there exists a Banach space $\mathcal{H} \subseteq \mathscr{S}'(\mathbb{R}^2)$ and $M \in H^{-1}(\langle x\rangle^{-n})$ such that for any $\psi \in \mathcal{H}$

$$
\log \int e^{-\langle \psi, \phi\rangle} \mathrm{d}\nu_{\mathrm{SG}}(\mathrm{d}\phi) = \|\psi\|_{\mathcal{H}}^2 + (M, \psi)_{H^{-1}(\langle x\rangle^{-n})}
$$

(This follows easily from Lemma 5.1 in [83]). On the other hand we know that with $V_T^\rho(\phi) = \alpha(T)\int \rho(x)\cos(\phi(x))\mathrm{d}x$ by the Cameron-Martin theorem for the Gaussian Free Field

$$
\begin{aligned}
& \log \int e^{-\langle \psi, \phi\rangle} \mathrm{d}\nu_{\mathrm{SG}}(\mathrm{d}\phi) \\
= & \lim_{\rho \to 1, T \to \infty} \log \frac{1}{Z_{\rho,T}} \int e^{-\langle \psi, \phi\rangle} \mathrm{d}\nu_{\mathrm{SG}}^{\rho,T}(\mathrm{d}\phi) \\
= & \lim_{\rho \to 1, T \to \infty} \log \frac{1}{Z_{\rho,T}} \int e^{-\langle \psi, \phi\rangle} e^{-\lambda V_T^\rho(\phi)} \mathrm{d}\mu_T \\
= & \lim_{\rho \to 1, T \to \infty} \log \frac{1}{Z_{\rho,T}} \int e^{-\langle \psi, \mathcal{C}_T\phi\rangle} e^{-\lambda V_T^\rho(\mathcal{C}_T\phi)} \mathrm{d}\mu \\
= & \lim_{\rho \to 1, T \to \infty} \log \left(e^{\langle \mathcal{C}_T\psi, (m^2-\Delta)^{-1}\mathcal{C}_T\psi\rangle} \frac{1}{Z_{\rho,T}} \int e^{-\lambda V_T^\rho(\phi + (m^2-\Delta)^{-1}\psi)} \mathrm{d}\mu_T \right) \\
= & \lim_{\rho \to 1} \lim_{T \to \infty} (\langle \mathcal{C}_T\psi, (m^2-\Delta)^{-1}\mathcal{C}_T\psi\rangle + V_{0,T}^\rho((m^2-\Delta)^{-1}\psi) - V_{0,T}^\rho(0)).
\end{aligned}
$$

Recall that since $\sup_T |V_{0,T}^\rho|_{1,\infty} \leqslant C\lambda$ by Theorem 4.51 we have that for $\psi \in C_c^\infty$

$$
\begin{aligned}
& \|\psi\|_{\mathcal{H}}^2 - \langle \mathcal{C}_T\psi, (m^2-\Delta)^{-1}\mathcal{C}_T\psi\rangle \\
= & \log \int e^{-\langle \psi, \phi\rangle} \mathrm{d}\nu_{\mathrm{SG}}(\mathrm{d}\phi) - (M, \psi)_{H^{-1}(\langle x\rangle^{-n})} - \langle \mathcal{C}_T\psi, (m^2-\Delta)^{-1}\mathcal{C}_T\psi\rangle \\
\leqslant & \liminf_{\rho \to 1, T \to \infty} \log \int e^{-\langle \psi, \phi\rangle} \mathrm{d}\nu_{\mathrm{SG}}^{\rho,T}(\mathrm{d}\phi) - (M, \psi)_{H^{-1}(\langle x\rangle^{-n})} - \langle \mathcal{C}_T\psi, (m^2-\Delta)^{-1}\mathcal{C}_T\psi\rangle \\
\leqslant & \sup_{T < \infty, \rho \in C_c^\infty(\mathbb{R}^2, [0,1])} |V_{0,T}^\rho|_{1,\infty} \|(m^2-\Delta)^{-1}\psi\|_{L^1} - \|M\|_{H^1(\langle x\rangle^{-n})} \|\psi\|_{H^1(\langle x\rangle^{n})} \\
< & \infty.
\end{aligned}
$$

So in particular $\mathcal{H}$ contains C_c^∞ functions. We now show that $\lim_{\rho\to 1}\lim_{T\to\infty} V_{t,T}^\rho(\psi)$ is not a quadratic functional which will imply that

$$\lim_{\rho\to 1}\lim_{T\to\infty} \langle \mathcal{C}_T\psi, (m^2-\Delta)^{-1}\mathcal{C}_T\psi\rangle + V_{t,T}^\rho(\psi) - V_{0,T}^\rho(0) \neq \|\psi\|_{\mathcal{H}}^2 - (M,\psi)_{H^{-1}(\langle x\rangle^{-n})}.$$

giving a contradiction. Observe that

$$\nabla V_{0,T}^\rho(\psi) = \lambda\alpha(0)\sin(\psi) + \nabla R_{0,T}(\psi)$$

with $\sup_{\psi\in L^2(\mathbb{R}^2)}\|\nabla R_{0,T}(\psi)\|_{L^\infty} \leqslant C\lambda^2$,by Theorem 4.51. Now for a quadratic functional we would have $\nabla V(\psi)$ linear in ψ so

$$\lim_{T\to\infty,\rho\to 1} \nabla V_{0,T}^\rho(\psi+\varphi) + \nabla V_{0,T}^\rho(\psi-\varphi) - 2\nabla V_{0,T}^\rho(\psi) = 0. \tag{4.27}$$

However choosing ψ,φ such that on $\varphi,\psi\in C_c^\infty$ and for $x\in B(0,1)$ $\psi=\pi/2$ and $\varphi=\pi/4$. Then for any $x\in B(0,1)$

$$\lambda\alpha(0)\sin(\varphi(x)+\psi(x)) + \lambda\alpha(0)\sin(\psi(x)-\varphi(x)) - 2\lambda\alpha(0)\sin(\psi(x)) = \lambda(2\sqrt{2}/2-2) = \lambda(\sqrt{2}-2)$$

and since $\|\nabla R_{t,T}(\psi)\|_{L^\infty} \leqslant C\lambda^2$ this implies that for λ sufficiently small and $x\in B(0,1)$

$$\lim_{\rho\to 1}\lim_{T\to\infty} \nabla V_{t,T}^\rho(\psi+\varphi)(x) + \nabla V_{t,T}^\rho(\psi-\varphi)(x) - 2\nabla V_{t,T}^\rho(\psi)(x) > \lambda(\sqrt{2}-2)/2.$$

This is clearly a contradiction to (4.27).

4.9. LARGE DEVIATIONS

In this section we want to discuss a Laplace principle for the Sine-Gordon measure in the "semiclassical limit" as described in the introduction. We introduce the family $\nu_{\mathrm{SG},\hbar}^{T,\rho}$ of measures given by

$$\int_{\mathscr{S}'(\mathbb{R}^2)} g(\phi)\nu_{\mathrm{SG},\hbar}^{T,\rho}(\mathrm{d}\phi) = \frac{\mathbb{E}\Big[g(\hbar^{1/2}W_{0,T})\, e^{-\frac{\lambda}{\varepsilon}V_\hbar^{T,\rho}(\hbar^{1/2}W_{0,T})}\Big]}{Z_\hbar^{T,\rho}}, \tag{4.28}$$

where similarly as above

$$V_\hbar^{\rho,T}(\varphi) := \lambda\alpha^\hbar(T)\int_{\mathbb{R}^2}\cos(\beta\varphi(x))\mathrm{d}x \qquad Z_\hbar^{T,\rho} := \mathbb{E}\big[e^{-V_\hbar^{\rho,T}(W_{0,T})}\big]$$

for any bounded measurable $g\colon H^{-1}(\langle x\rangle^{-n})\to\mathbb{R}$. Here $\alpha^\hbar(T) = e^{\frac{\beta^2}{2}\hbar\bar{K}_T(0)}$ and $\alpha^\hbar(T)\cos(\hbar^{1/2}\beta W_{0,T})$ enjoys the same properties as $\alpha(T)\cos(\beta W_{0,T})$. It will also be convenient to introduce the unnormalized measures $\tilde{\nu}_{\mathrm{SG},\hbar}^{T,\rho} = Z_\hbar^{T,\rho}\nu_{\mathrm{SG},\hbar}^{T,\rho}$.

Note that this corresponds (modulo a normalization constant) to the measure heuristically defined by

$$e^{-\frac{1}{\hbar}\int_{\mathbb{R}^2}\lambda\alpha^\hbar(T)\cos(\beta\varphi(x)) + \frac{1}{2}m^2\varphi(x)^2 + \frac{1}{2}|\nabla\varphi(x)|^2\mathrm{d}x}\mathrm{d}\varphi.$$

Our goal is now to show that ν given as the weak limit of $\nu_{\mathrm{SG},\hbar}^{T,\rho}$ as $T\to\infty,\rho\to 1$ satisfies a Laplace principle as $\hbar\to 0$. We recall the definition of the Laplace principle.

DEFINITION 4.70. *A sequence of Borel measures ν_ε on a metric space S satisfies the Laplace principle with rate function I if for any continuous bounded function $f\colon S\to\mathbb{R}$*

$$-\lim_{\varepsilon\to 0}\varepsilon\log\int e^{-\frac{1}{\varepsilon}f(x)}\nu_\varepsilon(\mathrm{d}x) = \inf_{x\in S}\{f(x)+I(x)\}.$$

DEFINITION 4.71. *For a metric space S and let $I\colon S\to\mathbb{R}$ be a rate function. A set $D\subseteq C(S)$ is called rate function determining if any exponentially tight sequence ν_ε of measures on S such that*

$$-\lim_{\varepsilon\to 0}\varepsilon\log\int e^{-\frac{1}{\varepsilon}f}\mathrm{d}\nu_\varepsilon = \inf_{x\in S}\{f(x)+I(x)\},$$

for all $f \in D$ satisfies a large deviations principle with rate function I.

LEMMA 4.72. *Assume that $D \subseteq C(S)$ is bounded below, i.e $f \geqslant -C$ for any $f \in D$ with C independent of f. Furthermore assume that D isolates points i.e for each compact set $K \subseteq S$, $x \in S$ and $\varepsilon > 0$ there exists $f \in D$ such that*

- $|f(x)| < \varepsilon$
- $\inf_{y \in K} f(y) \geqslant 0$
- $\inf_{y \in K \cap B^c(x,\varepsilon)} f(y) \geqslant \varepsilon^{-1}$

Then D is rate function determining.

For a proof see [55] proposition 3.20.

LEMMA 4.73. *Let $S = H^{-1}(\langle x \rangle^{-n})$ for any $\gamma > 0$ Then*

$$D = C^2(L^2(\mathbb{R}^2), \mathbb{R}_+) \cap C(H^{-1}(\langle x \rangle^{-n})) \cap \{|f|_{1,2,m} < \infty\} \cap \{f \geqslant 0\}$$

is rate function determining.

Proof. We want to verify the assumptions of Lemma 4.72: By translating it is enough to verify the assumptions for $x = 0 \in H^{-1}(\langle x \rangle^{-n})$. Furthermore we can assume that $K \subseteq B(0, N)$ for some $N > 0$. Now choose $\chi \in C_c^{\infty}(\mathbb{R}, \mathbb{R}_+)$ such that $\chi(0) = 0$ and $\chi(y) \geqslant \varepsilon^{-1}$ if $N^2 \geqslant |y|^2 > \varepsilon$. $f(\varphi) = \chi(\|\varphi\|^2_{H^{-1,-m}})$ satisfies the requirement of Lemma 4.72. Clearly $f \in C^2(L^2(\mathbb{R}^2), \mathbb{R}_+) \cap C(H^{-1}(\langle x \rangle^{-n}))$, furthermore

$$\nabla f(\varphi) = 2\chi'(\|\varphi\|^2_{H^{-1,-m}})(w(1-\Delta)^{-1} w\varphi)$$

where $w(y) = \exp(-my)$. This implies that $|f|_{1,2,m} < \infty$ since

$$\|w(1-\Delta)^{-1} w\varphi\|_{L^{2,m}} \leqslant \|(1-\Delta)^{-1} w\varphi\|_{L^2} \leqslant \|\varphi\|_{H^{-1,-m}}. \qquad \square$$

4.9.1. Finite volume

In this section we will investigate Large Deviations of the the measures $\nu^{\rho}_{\mathrm{SG},\hbar} := \lim_{T\to\infty} \nu^{\rho,T}_{\mathrm{SG},\hbar}$. The fact that this limit exists can easily be seen as in Section 4.6.1. Let us also denote by $\tilde{\nu}^{\rho}_{\mathrm{SG},\hbar} = \lim_{T\to\infty} \tilde{\nu}^{\rho,T}_{\mathrm{SG},\hbar}$.

PROPOSITION 4.74. *The measures $\tilde{\nu}^{\rho}_{\mathrm{SG},\hbar}$ satisfy a large deviations principle with rate function*

$$\tilde{I}^{\rho}(\varphi) = \lambda \int \rho(x) \cos(\beta\varphi(x)) \mathrm{d}x + \frac{1}{2} m^2 \int \varphi^2(x) \mathrm{d}x + \frac{1}{2} \int |\nabla\varphi(x)|^2 \mathrm{d}x.$$

as $\hbar \to 0$.

Before we proceed with the proof let us observe that the discussion in Section 4.6.1 can be easily modified to obtain the following lemma.

LEMMA 4.75. *Assume that $f \in C^2(L^2(\mathbb{R}^2))$ then*

$$-\hbar \log \int e^{-\frac{1}{\varepsilon} f(\varphi)} \tilde{\nu}^{\rho}_{\mathrm{SG},\hbar}(\mathrm{d}\varphi) = \inf_{u \in \mathbb{H}_a} F^{\rho,f}_{\hbar}(u) = \inf_{u \in \mathbb{D}^f} F^{\rho,f}_{\hbar}(u)$$

with

$$\begin{aligned} F^{\rho,f}_{\hbar}(u) &= \mathbb{E}\bigg[f(\hbar^{1/2} W_{0,\infty} + I_{0,\infty}(u)) + \lambda \int \rho [\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!] \cos(\beta I_{0,\infty}(u)) \\ &\quad + \lambda \int \rho [\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!] \sin(\beta I_{0,\infty}(u)) + \frac{1}{2}\int_0^{\infty} \|u_s\|^2_{L^2} \mathrm{d}t \bigg] \end{aligned}$$

and $\mathbb{D}^f$ was introduced in Definition 4.60.

Proof. From Theorem 2.4 we have

$$
\begin{aligned}
&-\hbar\log\int e^{-\frac{1}{\varepsilon}f(\varphi)}\tilde{\nu}^{\rho}_{\mathrm{SG},\hbar}(\mathrm{d}\varphi)\\
=&\inf_{u\in\mathbb{H}_a}\mathbb{E}\bigg[f(\hbar^{1/2}W_{0,\infty}+\hbar I_{0,\infty}(u))+\lambda\int\rho[\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!]\cos(\beta\hbar I_{0,\infty}(u))\\
&+\lambda\int\rho[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!]\sin(\beta\hbar I_{0,\infty}(u))+\frac{\hbar}{2}\int_0^\infty\|u_s\|^2_{L^2}\mathrm{d}t\bigg]\\
=&\inf_{u\in\mathbb{H}_a}\mathbb{E}\bigg[f(\hbar^{1/2}W_{0,\infty}+I_{0,\infty}(u))+\lambda\int\rho[\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!]\cos(\beta I_{0,\infty}(u))\\
&+\lambda\int\rho[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!]\sin(\beta I_{0,\infty}(u))+\frac{1}{2}\int_0^\infty\|u_s\|^2_{L^2}\mathrm{d}t\bigg].
\end{aligned}
$$

where in the last line we have employed the change of variables $u\to\hbar^{-1/2}u$. Now the statement follows analogously to Section 4.6.1. □

Proof. (OF PROPOSITION 4.74) One can easily modify the bound from corollary 4.7 to be uniform in $\hbar$ and conclude that $\nu^{\rho}_{\mathrm{SG},\hbar}$ is exponentially tight on $H^{-1}(\langle x\rangle^{-n})$. So it is enough to show the statement for $f\in D$, D being defined in Lemma 4.73. We have that as $\hbar\to0$ $[\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!]\to1$ in $H^{-1}(\langle x\rangle^{-n})$ so

$$\sup_{u\in\mathbb{D}^f}\left|\int\rho[\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!]\cos(\beta I_{0,\infty}(u))-\int\rho\cos(\beta I_{0,\infty}(u))\right|\to0$$

and analogously

$$\sup_{u\in\mathbb{D}^f}\left|\int\rho[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!]\sin(\beta I_{0,\infty}(u))\right|\to0$$

since $[\![\sin(\hbar^{1/2}W_{0,\infty})]\!]\to0$. Since also $|f(\hbar^{1/2}W_{0,\infty}+I_{0,\infty}(u))-f(I_{0,\infty}(u))|\leqslant C\hbar^{1/2}\|W_{0,\infty}\|_{H^1(\langle x\rangle^{-n})}$ we have that

$$F^{\rho,f}_{\hbar}\to F^{\rho,f}_0$$

uniformly on $\mathbb{D}^f$, where

$$F^{\rho,f}_0(u)=\mathbb{E}\bigg[f(I_{0,\infty}(u))+\lambda\int\rho(x)\cos(\beta I_{0,\infty}(u))+\frac{1}{2}\int_0^\infty\|u_s\|^2_{L^2}\mathrm{d}t\bigg].$$

This implies that

$$\inf_{u\in\mathbb{D}^f}F^{\rho,f}_{\hbar}(u)\to\inf_{u\in\mathbb{D}^f}F^{\rho,f}_0(u).$$

Now from Lemma 4.84 below $\inf_{u\in\mathbb{D}^f}F^{\rho}_0(u)=\inf_{u\in\mathbb{H}_a}F^{\rho}_0(u)$. Finally Lemma 4.76 below shows that

$$\inf_{u\in\mathbb{H}_a}F^{\rho,f}_0(u)=\inf_{\psi\in\mathscr{S}'(\mathbb{R}^2)}\{f(\psi)+\tilde{I}^{\rho}(\psi)\}.$$ □

LEMMA 4.76. *Assume $\rho\in C^\infty_c(\mathbb{R}^2,[0,1])$. Then*

$$\inf_{u:L^2(\mathbb{P},L^2(\mathbb{R}_+\times\mathbb{R}^2))}F^{\rho}_0(u)=\inf_{u\in\mathbb{H}_a}F^{\rho}_0(u)=\inf_{\psi\in\mathscr{S}'(\mathbb{R}^2)}\{f(\psi)+\tilde{I}^{\rho}(\psi)\}.$$

Proof. Note that

$$\inf_{\psi\in\mathscr{S}'(\mathbb{R}^2)}\{f(\psi)+\tilde{I}^{\rho}(\psi)\}=\inf_{\psi\in H^1(\mathbb{R}^2)}\{f(\psi)+\tilde{I}^{\rho}(\psi)\}.$$

Step 1. First we prove

$$\inf_{u\in\mathbb{H}_a}F(u)\leqslant\inf_{\psi\in\mathscr{S}'(\mathbb{R}^2)}\{f(\psi)+\tilde{I}^{\rho}(\psi)\}.$$

Restricting the infimum to processes of the form

$$u_s=Q_s(m^2-\Delta)\psi$$

with $\psi \in H^2(\Lambda)$, we see that

$$I_{0,\infty}(u) = \int_0^\infty Q_s u_s \mathrm{d}s = \int_0^\infty Q_s^2 (m^2 - \Delta)\psi \mathrm{d}s = \psi.$$

We also compute

$$\|u\|_{D^0}^2 = \int_0^\infty \int_{\mathbb{R}^2} u_s^2 \mathrm{d}s = \int_0^\infty \langle Q_s^2 (m^2 - \Delta)\psi, (m^2 - \Delta)\psi \rangle_{L^2(\mathbb{R}^2)} = \langle \psi, (m^2 - \Delta)\psi \rangle_{L^2(\mathbb{R}^2)}$$

so

$$\inf_{u \in \mathbb{H}_a} F_0^{\rho,f}(u) \leqslant \inf_{u = Q_s (m^2 - \Delta)^{-1}\psi} F_0^{\rho,f}(u) = \inf_{\psi \in H^2(\mathbb{R}^2)} \{ f(\psi) + \tilde{I}^\rho(\psi) \} = \inf_{\psi \in H^1(\mathbb{R}^2)} \{ f(\psi) + \tilde{I}^\rho(\psi) \}$$

where the last equality follows from the density of the H^2 in H^1 and continuity of the functional in H^1.

Step 2.We now prove the converse inequality

$$\inf_{u \in L^2(\mathbb{P}, L^2(\mathbb{R}_+ \times \mathbb{R}^2))} F_0^{\rho,f}(u) \geqslant \inf_{\psi \in H^1(\mathbb{R}^2)} \{ f(\psi) + \tilde{I}^\rho(\psi) \}.$$

First note that from Lemma 4.21 $\|u\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^2)} \geqslant \|(m^2 - \Delta)^{1/2} I_{0,\infty}(u)\|_{L^2}$, so

$$\begin{aligned} \inf_{u \in \mathbb{H}_a} F_0^{\rho,f}(u) &\geqslant \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[f(I_{0,\infty}(u)) + \lambda \int \rho \cos(\beta I_{0,\infty}(u)) + \frac{1}{2} \int_{\mathbb{R}^2} ((m^2 - \Delta) I_{0,\infty}(u)) I_{0,\infty}(u) \right] \\ &\geqslant \inf_{\psi \in H^1(\mathbb{R}^2)} \{ f(\psi) + I^\rho(\psi) \}. \end{aligned}$$

Now

$$\inf_{u \in L^2(\mathbb{P}, L^2(\mathbb{R}_+ \times \mathbb{R}^2))} F_0^{\rho,f}(u) \geqslant \inf_{\psi \in H^1(\mathbb{R}^2)} \{ f(\psi) + \tilde{I}^\rho(\psi) \} \geqslant \inf_{u \in \mathbb{H}_a} F_0^{\rho,f}(u) \geqslant \inf_{u \in L^2(\mathbb{P}, L^2(\mathbb{R}_+ \times \mathbb{R}^2))} F_0^{\rho,f}(u)$$

which proves the statement. □

From this we can easily deduce the following

Corollary 4.77. *The measures $\nu_{\mathrm{SG},\hbar}^\rho$ satisfy a Large Deviations Principle with rate function*

$$I^\rho(\varphi) = \lambda \int \rho(x)(\cos(\beta\varphi(x)) - 1)\mathrm{d}s + \frac{1}{2} m^2 \int \varphi^2(x) \mathrm{d}x + \frac{1}{2} \int |\nabla \varphi(x)|^2 \mathrm{d}x$$

as $\hbar \to 0$.

Lemma 4.78. *For λ sufficiently small $u = 0$ is the unique minimizer of $F_0^{\rho,0}$.*

Proof. From Lemma 4.76

$$\inf_{\psi \in \mathscr{S}'(\mathbb{R}^2)} \tilde{I}^\rho(\psi) = \inf_{u \in \mathbb{H}_a} F_0^{\rho,0}.$$

and

$$\tilde{I}^\rho(\varphi) = \int \lambda \rho(x) \cos(\beta\varphi(x)) + \frac{1}{2} m^2 (\varphi(x))^2 \mathrm{d}x + \frac{1}{2} \int |\nabla \varphi|^2 \mathrm{d}x.$$

Now for λ small enough and $\rho \leqslant 1$

$$\lambda \rho(\cos(\varphi(x)) - 1) + \frac{m^2}{2} (\varphi(x))^2 \geqslant \frac{m^2}{4} (\varphi(x))^2$$

so

$$\tilde{I}^\rho(\varphi) - \tilde{I}^\rho(0) \geqslant \frac{m^2}{4} \int (\varphi(x))^2 \mathrm{d}x + \int |\nabla \varphi|^2 \mathrm{d}x.$$

so $\varphi = 0$ is the unique minimizer of $\tilde{I}^\rho(\varphi)$ and

$$\inf_{\varphi \in H^1(\mathbb{R}^2)} \tilde{I}^\rho(\varphi) = \lambda \int \rho(x) \mathrm{d}x.$$

On the other hand

$$F_0^{\rho,0}(0) = \lambda \int \rho(x) \mathrm{d}x = \inf_{\varphi \in H^1(\mathbb{R}^2)} \tilde{I}^\rho(\varphi) = \inf_{u \in \mathbb{H}_a} F_0^{\rho,0}.$$

So $u = 0$ is a minimizer of $F_0^{\rho,0}(0)$. Uniqueness follows since for λ small enough $F_0^{\rho,0}(u)$ is strongly convex in u. □

Lemma 4.79. *Let $u^{\hbar,\rho}$ be the minimizer of $F_{\hbar}^{\rho,0}(u)$. Then for λ sufficiently small*

$$\lim_{\hbar\to 0}\mathbb{E}[\|u^{\hbar,\rho}\|^2_{L^2(\mathbb{R}_+\times\mathbb{R}^2)}]=0.$$

Proof. We have already established in the proof of Proposition 4.74 above that

$$\lim_{\hbar\to 0}\sup_{u\in\mathbb{D}^f}|F_{\hbar}^{\rho,0}(u)-F_0^{\rho,0}(u)|=0.$$

On the other hand for $u^{\hbar,\rho}$ to be a minimizer we must have

$$F_{\hbar}^{\rho,0}(u^{\hbar,\rho})-F_{\hbar}^{\rho,0}(0)\leqslant 0.$$

Now

$$\begin{aligned}0 &\geqslant F_{\hbar}^{\rho,0}(u^{\hbar,\rho})-F_{\hbar}^{\rho,0}(0)\\ &\geqslant F_0^{\rho,0}(u^{\hbar,\rho})-F_0^{\rho,0}(0)-\sup_{u\in\mathbb{D}^f}|F_{\hbar}^{\rho,0}(u)-F_0^{\rho,0}(u)|\\ &\geqslant \frac{1}{4}\mathbb{E}[\|u^{\hbar,\rho}\|^2_{D^0}]-o(\hbar)\end{aligned}$$

Where in the last line we have used that $F_0^{\rho,0}$ is strongly convex and 0 is a minimizer of $F_0^{\rho,0}$. □

4.9.2. Infinite volume

We now want to discuss large deviations in infinite volume, i.e large deviations for $\nu_{\mathrm{SG},\hbar}$ where $\nu_{\mathrm{SG},\hbar}=\lim_{\rho\to 1}\nu^{\rho}_{\mathrm{SG},\hbar}$ and the limit is understood in a weak sense on $H^{-1}(\langle x\rangle^{-n})$ for n large enough. Recall that the variational description of ν_{SG} involves the process u^{∞} obtained as a limit of u^{ρ}. We can modify that construction and obtain the following:

Lemma 4.80. *Let $\rho^N\in C_c^{\infty}$satisfying $\rho^N(x)=1$ for $x\in B(0,N)$. Let $u^{\hbar,\rho}$ be the minimizer of $F_{\hbar}^{\rho,0}(u)$. There exist processes $u^{\hbar,\infty}$ such that*

$$\lim_{\rho^N\to 1}\sup_{\hbar}\mathbb{E}[\|u^{\hbar,\rho}-u^{\hbar,\infty}\|^2_{D^{-\gamma}}]\leqslant N^{-1}$$

and

$$\|\langle t\rangle^{1/2+\delta}u^{\hbar,\infty}\|_{L_t^{\infty}(L^{\infty})}\leqslant C.$$

Furthermore

$$\lim_{\hbar\to 0}\mathbb{E}[\|u^{\hbar,\infty}\|^2_{D^{-\gamma}}]=0$$

almost surely, where C is a deterministic constant (not depending on $\hbar$).

Proof. The first two statement's are an easy modification of Lemma 4.65. The second follows from the first and Lemma 4.79. □

One can easily modify Proposition 4.66 to obtain

Lemma 4.81.

$$-\hbar\log\int e^{-\frac{1}{\hbar}f}\mathrm{d}\nu_{\mathrm{SG},\hbar}=\inf_{u\in\mathbb{D}^f}G_{\hbar}^f(u)$$

where

$$\begin{aligned}G_{\hbar}^f(u) &= \mathbb{E}\Big[f(\hbar^{1/2}W_{0,\infty}+I_{0,\infty}(u)+I_{0,\infty}(u^{\hbar,\infty}))\\ &+\lambda\int [\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!](\cos(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\cos(\beta I_{0,\infty}(u^{\hbar,\infty})))\\ &+\lambda\int [\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!](\sin(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\sin(\beta I_{0,\infty}(u^{\hbar,\infty})))\\ &+\frac{1}{2}\int_0^{\infty}\|u_t\|^2_{L^2}\mathrm{d}t+\int_0^{\infty}\int u_t u_t^{\hbar,\infty}\mathrm{d}t\Big]\end{aligned}$$

Proposition 4.82. *Assume that $|f|_{1,2,m}<\infty$ and $f\colon H^{-1}(\langle x\rangle^{-n})\to\mathbb{R}$ be Lipschitz continuous.*

$$\lim_{\hbar\to 0}\sup_{u\in\mathbb{D}^f}|G_{\hbar}^f(u)-G_0^f(u)|=0$$

where

$$
\begin{aligned}
& G_0^f(u) \\
= & \mathbb{E}\left[f(I_{0,\infty}(u))+\lambda\int(\cos(\beta I_{0,\infty}(u))-1)+\frac{1}{2}\int_0^\infty\|u_t\|_{L^2}^2\mathrm{d}t\right].
\end{aligned}
$$

Proof. By Lipschitz continuity of f

$$
\begin{aligned}
& |f(\hbar^{1/2}W_{0,\infty}+I_{0,\infty}(u)+I_{0,\infty}(u^{\hbar,\infty}))-f(I_{0,\infty}(u))| \\
\leqslant & \hbar^{1/2}\mathbb{E}\|W_{0,\infty}\|_{H^{-1}(\langle x\rangle^{-n})}+\mathbb{E}[\|I_{0,\infty}(u^{\hbar,\infty})\|_{H^{-1}(\langle x\rangle^{-n})}] \\
\to & 0.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
& \left|\int[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!](\sin(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\sin(\beta I_{0,\infty}(u^{\hbar,\infty})))\right| \\
\leqslant & \|[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!]\|_{B_{p,p}^{-1+\delta}(\langle x\rangle^{-n})} \\
& \times\|(\sin(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\sin(\beta I_{0,\infty}(u^{\hbar,\infty})))\|_{B_{q,q}^{1-\delta}(\langle x\rangle^{-n})}
\end{aligned}
$$

Now for q close enough to 1 we have for any $\gamma>0$

$$
\begin{aligned}
& \|(\sin(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\sin(\beta I_{0,\infty}(u^{\hbar,\infty})))\|_{B_{q,q}^{1-\delta}(\langle x\rangle^{-n})} \\
\leqslant & \|(\sin(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\sin(\beta I_{0,\infty}(u^{\hbar,\infty})))\|_{W^{1,1,\gamma}}^{1-\delta} \\
\leqslant & \|(\cos(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\cos(\beta I_{0,\infty}(u^{\hbar,\infty})))\nabla I_{0,\infty}(u^{\hbar,\infty})\|_{L^{1,\gamma}}^{1-\delta} \\
& +\beta\|(\cos(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))\nabla I_{0,\infty}(u)\|_{L^{1,\gamma}}^{1-\delta} \\
& +\beta\|(\sin(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\sin(\beta I_{0,\infty}(u^{\hbar,\infty})))\|_{L^{1,\gamma}}^{1-\delta} \\
\leqslant & C(\|I_{0,\infty}(u)\|_{L^{2,2\gamma}}\|\nabla I_{0,\infty}(u^{\hbar,\infty})\|_{L^{2,-\gamma}}+\|\nabla I_{0,\infty}(u)\|_{L^{2,2\gamma}}+\|I_{0,\infty}(u)\|_{L^{2,2\gamma}})^{1-\delta}
\end{aligned}
$$

so

$$
\begin{aligned}
& \mathbb{E}\left|\int[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!](\sin(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\sin(\beta I_{0,\infty}(u^{\hbar,\infty})))\right| \\
\leqslant & C\mathbb{E}\left[\|[\![\sin(\hbar^{1/2}W_{0,\infty})]\!]\|_{B_{p,p}^{-1+\delta,-\gamma}}^{1/\delta}\right] \\
& \times\mathbb{E}[\|I_{0,\infty}(u)\|_{L^{2,2\gamma}}\|\nabla I_{0,\infty}(u^{\hbar,\infty})\|_{L^{2,-\gamma}}+\|\nabla I_{0,\infty}(u)\|_{L^{2,2\gamma}}+\|I_{0,\infty}(u)\|_{L^{2,2\gamma}}] \\
\leqslant & C\mathbb{E}\left[\|[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!]\|_{B_{p,p}^{-1+\delta,-\gamma}}^{1/\delta}\right]\mathbb{E}[\|I_{0,\infty}(u)\|_{L^{2,2\gamma}}^2]+\mathbb{E}[\|\nabla I_{0,\infty}(u^{\hbar,\infty})\|_{L^{2,-\gamma}}^2] \\
& +\mathbb{E}\left[\|[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!]\|_{B_{p,p}^{-1+\delta,-\gamma}}^{1/\delta}\right]\mathbb{E}[\|\nabla I_{0,\infty}(u)\|_{L^{2,2\gamma}}+\|I_{0,\infty}(u)\|_{L^{2,2\gamma}}]
\end{aligned}
$$

and since by Remark 4.19

$$
\mathbb{E}\left[\|[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!]\|_{B_{p,p}^{-1+\delta}(\langle x\rangle^{-n})}^{1/\delta}\right]\to 0,
$$

as $\hbar\to 0$, we have uniform convergence of this term to 0. We now rewrite

$$
\begin{aligned}
& \int[\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!](\cos(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\cos(\beta I_{0,\infty}(u^{\hbar,\infty}))) \\
& -\int(\cos(\beta I_{0,\infty}(u))-1) \\
= & \int([\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!]-1)(\cos(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\cos(\beta I_{0,\infty}(u^{\hbar,\infty}))) \\
& +\int(\cos(\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\cos(\beta I_{0,\infty}(u^{\hbar,\infty})))-\int(\cos(\beta I_{0,\infty}(u))-1).
\end{aligned}
$$

The first term can be estimated in the same way as the sinus term, provided we replace $[\![\sin(\hbar^{1/2}\beta W_{0,\infty})]\!]$ with $[\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!]-1$ which also satisfies

$$
\mathbb{E}\left[\|[\![\cos(\hbar^{1/2}\beta W_{0,\infty})]\!]-1\|_{B_{p,p}^{-1+\delta}(\langle x\rangle^{-n})}^{1/\delta}\right]\to 0.
$$

by Remark 4.19. For the second term by fundamental theorem of calculus we can write

$$\begin{aligned}&(\cos(\beta I_{0,\infty}(u)+I_{0,\infty}(u^{\hbar,\infty}))-\cos(I_{0,\infty}(u^{\hbar,\infty})))-(\cos(I_{0,\infty}(u))-1)\\ =\ & -\beta\int_0^1((\cos(\theta\beta I_{0,\infty}(u)+\beta I_{0,\infty}(u^{\hbar,\infty}))-\cos(\theta\beta I_{0,\infty}(u))I_{0,\infty}(u))\mathrm{d}\theta\\ =\ & -\beta\int_0^1\int_0^1((\cos(\theta\beta I_{0,\infty}(u)+\xi\beta I_{0,\infty}(u^{\hbar,\infty}))I_{0,\infty}(u^{\hbar,\infty})I_{0,\infty}(u))\mathrm{d}\theta\mathrm{d}\xi\end{aligned}$$

and so

$$\begin{aligned}&\mathbb{E}\left|\int_0^1\int_0^1((\cos(\theta\beta I_{0,\infty}(u)+\xi\beta I_{0,\infty}(u^{\hbar,\infty}))I_{0,\infty}(u^{\hbar,\infty})I_{0,\infty}(u))\mathrm{d}\theta\mathrm{d}\xi\right|\\ \leqslant\ & \mathbb{E}[\|I_{0,\infty}(u^{\hbar,\infty})\|_{L^{2,-\gamma}}\|I_{0,\infty}(u)\|_{L^{2,\gamma}}]\\ \leqslant\ & \mathbb{E}[\|I_{0,\infty}(u^{\hbar,\infty})\|_{L^{2,-\gamma}}^2]^{1/2}\mathbb{E}[\|I_{0,\infty}(u)\|_{L^{2,\gamma}}^2]^{1/2}\end{aligned}$$

which implies also that term converges to 0. Finally

$$\mathbb{E}\left[\int_0^\infty\int u_t u_t^{\hbar,\infty}\mathrm{d}t\right]\leqslant\mathbb{E}[\|u\|_{D^\gamma}\|u^{\hbar,\infty}\|_{D^{-\gamma}}]\leqslant\mathbb{E}[\|u\|_{D^\gamma}^2]^{1/2}\mathbb{E}[\|u^{\hbar,\infty}\|_{D^{-\gamma}}^2]^{1/2}$$

and we can conclude. □

We now relate G^f to the rate function.

Lemma 4.83.

$$\inf_{u\in\mathbb{D}^f}G_0^f(u)=\inf_{\psi\in H^1(\mathbb{R}^2)}\{f(\psi)+I(\psi)\}$$

Proof. By Lemma 4.84 below it is enough to show that

$$\inf_{u\in\mathbb{D}^f}G_0^f(u)=\inf_{\|\psi\|_{H^{1,\gamma}}\leqslant C|f|_{1,2,\gamma}}\{f(\psi)+I(\psi)\}$$

for some $\gamma>0$.

Step 1. First we prove

$$\inf_{u\in\mathbb{H}_a}F(u)\leqslant\inf_{\|\psi\|_{H^{1,\gamma}}\leqslant C|f|_{1,2,\gamma}}\{f(\psi)+I(\psi)\}.$$

Restricting the infimum to processes of the form

$$u_s=Q_s(m^2-\Delta)\psi$$

with $\psi\in H^2(\mathbb{R}^2)\cap H^{1,2\gamma}$,we see that

$$I_{0,\infty}(u)=\int_0^\infty Q_s u_s\mathrm{d}s=\int_0^\infty Q_s^2(m^2-\Delta)\psi\mathrm{d}s=\psi.$$

We also compute

$$\|u\|_{D^0}^2=\int_0^\infty\int_{\mathbb{R}^2}u_s^2\mathrm{d}s=\int_0^\infty\langle Q_s^2(m^2-\Delta)\psi,(m^2-\Delta)\psi\rangle_{L^2(\mathbb{R}62)}=\langle\psi,(m^2-\Delta)\psi\rangle_{L^2(\mathbb{R}^2)}$$

and with $w(x)=\exp(\gamma x)$

$$\|u\|_{D^\gamma}^2=\int_0^\infty\int_{\mathbb{R}^2}wu_s^2\mathrm{d}s==\int_0^\infty\langle wQ_s^2(m^2-\Delta)\psi,(m^2-\Delta)\psi\rangle_{L^2(\mathbb{R}^2)}=\langle w\psi,(m^2-\Delta)\psi\rangle_{L^2(\mathbb{R}^2)}$$

from which we can deduce that $\|u\|_{D^\gamma}^2\leqslant C\|\psi\|_{H^{1,\gamma}}$ and u is in $\mathbb{D}^f$. So

$$\begin{aligned}&\inf_{u\in\mathbb{D}^f}F(u)\\ \leqslant\ & \inf_{\substack{u_s=Q_s(m^2-\Delta)\psi\\ \psi\in H^2\\ \|\psi\|_{H^{1,2\gamma}}\leqslant C|f|_{1,2,m}}}F(u)\\ \leqslant\ & \inf_{\substack{\psi\in H^2\\ \|\psi\|_{H^{1,2\gamma}}\leqslant C|f|_{1,2,m}}}\{f(\psi)+I(\psi)\}\\ \leqslant\ & \inf_{\|\psi\|_{H^{1,2\gamma}}\leqslant C|f|_{1,2,m}}\{f(\psi)+I(\psi)\}\end{aligned}$$

where the last equality follows from the density of the H^2 in $H^{1,2\gamma}$ and continuity of the functional in H^1.

Step 2.We now prove the converse inequality

$$\inf_{u\in\mathbb{D}^f} F_0^\rho(u) \geqslant \inf_{\psi\in H^1(\mathbb{R}^2)} \{f(\psi)+I(\psi)\}.$$

Recall that from the proof of Lemma 4.21 $\|u\|_{D^0}\geqslant\|(m^2-\Delta)^{1/2}I_{0,\infty}(u)\|_{L^2}$ so

$$\begin{aligned}\inf_{u\in\mathbb{D}^f} F(u) &\geqslant \inf_{u\in\mathbb{D}^f}\mathbb{E}\left[f(I_{0,\infty}(u))+\lambda\int\rho\cos(\beta I_{0,\infty}(u))+\frac{1}{2}\int_{\mathbb{R}^2}((m^2-\Delta)I_{0,\infty}(u))I_{0,\infty}(u)\right]\\ &\geqslant \inf_{\psi\in H^1(\mathbb{R}^2)}\{f(\psi)+I(\psi)\}\end{aligned}$$

which proves the statement.

□

Lemma 4.84. *Assume that $2\gamma^2+\lambda<m^2$. Then for $\rho\in C^\infty(\mathbb{R}^2)$ and $\rho, |\nabla\rho|\leqslant 1$(note that this includes the $\rho=1$ case.)*

$$\inf_{\psi\in H^1(\mathbb{R}^2)} f(\varphi)+I^\rho(\varphi) = \inf_{\|\psi\|_{H^{1,\gamma}}\leqslant C|f|_{1,2,2\gamma}} f(\varphi)+I^\rho(\varphi)$$

Proof. By a standard argument we obtain that any minimizer of $f(\varphi)+I(\varphi)$ satisfies the Euler Lagrange equation

$$\nabla f(\varphi)+\lambda\rho\sin(\beta\varphi)+m^2\varphi-\Delta\varphi=0. \tag{4.29}$$

Now multiplying (4.29) with $w\varphi$ where $w(x)=\exp(2\gamma|x|)$ and integrating we obtain

$$\begin{aligned}0 &= \int w\nabla f(\varphi)\varphi+\lambda\int w\rho\sin(\beta\varphi)\varphi+m^2\int w\varphi^2-\int w\varphi\Delta\varphi\\ &= \int\rho\nabla f(\varphi)\varphi+\lambda\int w\rho\sin(\beta\varphi)\varphi+m^2\int w\varphi^2+\int w|\nabla\varphi|^2+\int\varphi\nabla w\cdot\nabla\varphi\end{aligned}$$

now observe that $\nabla w=2\gamma\frac{x}{|x|}\exp(2\gamma|x|)$ so $|\nabla w|\leqslant 2\gamma w$

$$\int|\varphi\nabla w\cdot\nabla\varphi|\leqslant 2\gamma^2\int w\varphi^2+\frac{1}{2}\int w|\nabla\varphi|^2$$

note also that

$$\lambda\int|\rho w\sin(\beta\varphi)\varphi|\leqslant\lambda\int w\varphi^2.$$

Since $m^2\int w\varphi^2+\int w|\nabla\varphi|^2>0$ we have

$$\begin{aligned}0 &= \int w\nabla f(\varphi)\varphi+\lambda\int w\rho\sin(\beta\varphi)\varphi+m^2\int w\varphi^2+\int w|\nabla\varphi|^2+\int\varphi\nabla w\cdot\nabla\varphi\\ &\geqslant (m^2-2\gamma^2+\lambda-\delta)\int w\varphi^2+\frac{1}{2}\int w|\nabla\varphi|^2-C|f|_{1,2,m}^2\end{aligned}$$

which implies

$$\int w\varphi^2+\frac{1}{2}\int w|\nabla\varphi|^2\leqslant C_\gamma|f|_{1,2,m}^2.$$

□

Bibliography

[1] Michael Aizenman. Geometric analysis of φ^4 fields and Ising models. I, II. *Comm. Math. Phys.*, 86(1):1–48, 1982.

[2] Michael Aizenman and Hugo Duminil-Copin. Marginal triviality of the scaling limits of critical 4d ising and φ_4^4 models. *arXiv preprint arXiv:1912.07973*, 2019.

[3] S. Albeverio and S. Liang. A remark on the nonequivalence of the time-zero ϕ_3^4-measure with the free field measure. *Markov Processes and Related Fields*, 14(1):159–164, 2008.

[4] Sergio Albeverio, Francesco C. De Vecchi, and Massimiliano Gubinelli. Elliptic stochastic quantization. *Ann. Probab.*, 48(4):1693–1741, 2020.

[5] Sergio Albeverio, Francesco C De Vecchi, Andrea Romano, and Stefania Ugolini. Mean-field limit for a class of stochastic ergodic control problems. *arXiv preprint arXiv:2003.06469*, 2020.

[6] Sergio Albeverio and Raphael Hø egh Krohn. The Wightman axioms and the mass gap for strong interactions of exponential type in two-dimensional space-time. *J. Functional Analysis*, 16:39–82, 1974.

[7] Sergio Albeverio, Raphael Hø egh Krohn, and Bogusł aw Zegarliński. Uniqueness and global Markov property for Euclidean fields: the case of general polynomial interactions. *Comm. Math. Phys.*, 123(3):377–424, 1989.

[8] Sergio Albeverio and Raphael Høegh-Krohn. Uniqueness and the global markov property for euclidean fields. the case of trigonometric interactions. *Comm. Math. Phys.*, 68(2):95–128, 1979.

[9] Sergio Albeverio and Seiichiro Kusuoka. The invariant measure and the flow associated to the ϕ_3^4-quantum field model. *Annali della Scuola Normale di Pisa - Classe di Scienze*, 2018. 10.2422/2036-2145.201809_008.

[10] Sergio Albeverio, Song Liang, and Boguslav Zegarlinski. Remark on the integration by parts formula for the ϕ_3^4-quantum field model. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 9(1):149–154, 2006. 10.1142/S0219025706002275.

[11] Sergio Albeverio, Francesco C. De Vecchi, and Massimiliano Gubinelli. The elliptic stochastic quantization of some two dimensional euclidean qfts, 2019.

[12] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer, January 2011.

[13] N. Barashkov and M. Gubinelli. A variational method for Φ_3^4. *Duke Math. J.*, 169(17):3339–3415, 2020.

[14] Nikolay Barashkov and Massimiliano Gubinelli. The Φ_3^4 measure via girsanov's theorem, 2020.

[15] Roland Bauerschmidt and Thierry Bodineau. Log-sobolev inequality for the continuum sine-gordon model. 2019.

[16] Roland Bauerschmidt, David C. Brydges, and Gordon Slade. A Renormalisation Group Method. III. Perturbative Analysis. *Journal of Statistical Physics*, 159(3):492–529, May 2015. 10.1007/s10955-014-1165-x.

[17] Roland Bauerschmidt, David C. Brydges, and Gordon Slade. Introduction to a renormalisation group method. *arXiv preprint arXiv:1907.05474*, 2019.

[18] G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicoló, E. Olivieri, E. Presutti, and E. Scacciatelli. Ultraviolet stability in Euclidean scalar field theories. *Communications in Mathematical Physics*, 71(2):95–130, June 1980. 10.1007/BF01197916.

[19] C. Borell. Diffusion equations and geometric inequalities. *Potential Analysis*, 12(1):49–71, 2000.

[20] M. Boué and P. Dupuis. A variational representation for certain functionals of Brownian motion. *The Annals of Probability*, 26(4):1641–1659, October 1998. 10.1214/aop/1022855876.

[21] J. Bourgain. Invariant measures for the Gross-Piatevskii equation. *J. Math. Pures Appl. (9)*, 76(8):649–702, 1997.

[22] J. Bourgain. Invariant measures for NLS in infinite volume. *Comm. Math. Phys.*, 210(3):605–620, 2000.

[23] Jean Bourgain. Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Comm. Math. Phys.*, 176(2):421–445, 1996.

[24] A. Braides. *Gamma-convergence for beginners*. Oxford Lecture Series in Mathematics and Its Applications, 22. Clarendon Press, 1 edition, 2002.

[25] Haïm Brezis and Petru Mironescu. Gagliardo-Nirenberg inequalities and non-inequalities: the full story. *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, 35(5):1355–1376, 2018. hal-01626613.

[26] Bjoern Bringmann. Invariant gibbs measures for the three-dimensional wave equation with a hartree nonlinearity i: Measures, 2020.

[27] Bjoern Bringmann. Invariant gibbs measures for the three-dimensional wave equation with a hartree nonlinearity ii: Dynamics, 2020.

[28] D. Brydges, J. Dimock, and T. R. Hurd. The short distance behavior of ϕ_3^4. *Communications in Mathematical Physics*, 172(1):143–186, 1995. MR1346375.

[29] D. C. Brydges. Lectures on the renormalisation group. In *Statistical mechanics*, volume 16 of *IAS/Park City Math. Ser.*, pages 7–93. Amer. Math. Soc., Providence, RI, 2009.

[30] D. C. Brydges, R. Fernández, and C. Ecublens. *Functional Integrals and their Applications*. 1993.

[31] D. C. Brydges, J. Fröhlich, and A. D. Sokal. A new proof of the existence and nontriviality of the continuum ϕ_2^4 and ϕ_3^4 quantum field theories. *Communications in Mathematical Physics*, 91(2):141–186, 1983.

[32] D. C. Brydges and T. Kennedy. Mayer expansions and the Hamilton-Jacobi equation. *Journal of Statistical Physics*, 48(1-2):19–49, July 1987. 10.1007/BF01010398.

[33] David C. Brydges and Gordon Slade. A Renormalisation Group Method. I. Gaussian Integration and Normed Algebras. *Journal of Statistical Physics*, 159(3):421–460, May 2015. 10.1007/s10955-014-1163-z.

[34] David C. Brydges and Gordon Slade. A Renormalisation Group Method. II. Approximation by Local Polynomials. *Journal of Statistical Physics*, 159(3):461–491, May 2015. 10.1007/s10955-014-1164-y.

[35] David C. Brydges and Gordon Slade. A Renormalisation Group Method. IV. Stability Analysis. *Journal of Statistical Physics*, 159(3):530–588, May 2015. 10.1007/s10955-014-1166-9.

[36] David C. Brydges and Gordon Slade. A Renormalisation Group Method. V. A Single Renormalisation Group Step. *Journal of Statistical Physics*, 159(3):589–667, May 2015. 10.1007/s10955-014-1167-8.

[37] Amarjit Budhiraja and Paul Dupuis. *Analysis and approximation of rare events*, volume 94 of *Probability Theory and Stochastic Modelling*. Springer, New York, 2019. Representations and weak convergence methods.

[38] Nicolas Burq and Nikolay Tzvetkov. Random data Cauchy theory for supercritical wave equations. I. Local theory. *Invent. Math.*, 173(3):449–475, 2008.

[39] Nicolas Burq and Nikolay Tzvetkov. Random data Cauchy theory for supercritical wave equations. II. A global existence result. *Invent. Math.*, 173(3):477–496, 2008.

[40] Rémi Catellier and Khalil Chouk. Paracontrolled distributions and the 3-dimensional stochastic quantization equation. *The Annals of Probability*, 46(5):2621–2679, 2018. 10.1214/17-AOP1235.

[41] Ajay Chandra, Trishen S. Gunaratnam, and Hendrik Weber. Phase transitions for φ_3^4, 2020.

[42] Dean S Clark. Short proof of a discrete gronwall inequality. *Discrete Applied Mathematics*, 16(3):279 – 281, 1987.

[43] Giuseppe Da Prato and Arnaud Debussche. Strong solutions to the stochastic quantization equations. *Ann. Probab.*, 31(4):1900–1916, 2003.

[44] A. Dal Maso. *An introduction to* Γ*-Convergence*. Birkhäuser Boston, 1993. 10.1007/978-1-4612-0327-8.

[45] François David, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Liouville quantum gravity on the Riemann sphere. *Comm. Math. Phys.*, 342(3):869–907, 2016.

[46] Pierre Deligne, Pavel Etingof, Daniel S. Freed, Lisa C. Jeffrey, David Kazhdan, John W. Morgan, David R. Morrison, and Edward Witten, editors. *Quantum fields and strings: a course for mathematicians. Vol. 1, 2*. American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997.

[47] Amir Dembo and Ofer Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[48] J. Dimock and T. R. Hurd. Construction of the two-dimensional sine-gordon model for $\beta\ lt; 8\pi$. *Comm. Math. Phys.*, 156(3):547–580, 1993.

[49] J. Dimock and T.R. Hurd. Sine-gordon revisited. *Annales Henri Poincaré*, 1(3):499–541, July 2000.

[50] Paul Dupuis and Richard S. Ellis. *A weak convergence approach to the theory of large deviations*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, 1997. A Wiley-Interscience Publication.

[51] J.-P. Eckmann and H. Epstein. Time-ordered products and schwinger functions. *Comm. Math. Phys.*, 64(2):95–130, 1978.

[52] Giorgio Fabbri, Fausto Gozzi, and Andrzej Święch. *Stochastic Optimal Control in Infinite Dimension*. Springer International Publishing, 2017.

[53] J. Feldman. The $\lambda\varphi_3^4$ field theory in a finite volume. *Communications in Mathematical Physics*, 37:93–120, 1974.

[54] Joel S. Feldman and Konrad Osterwalder. The Wightman axioms and the mass gap for weakly coupled $(\Phi^4)_3$ quantum field theories. *Ann. Physics*, 97(1):80–135, 1976.

[55] Jin Feng and Thomas G. Kurtz. *Large Deviations for Stochastic processes*. American Mathematical Society, 2015.

[56] W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer, 2nd edition, November 2005.

[57] H. Föllmer. An entropy approach to the time reversal of diffusion processes. In Prof M. Metivier and Prof E. Pardoux, editors, *Stochastic Differential Systems Filtering and Control*, number 69 in Lecture Notes in Control and Information Sciences, pages 156–163. Springer Berlin Heidelberg, 1985. 10.1007/BFb0005070.

[58] J. Fröhlich. On the triviality of λφd4 theories and the approach to the critical point in d(−) $>$ 4 dimensions. *Nuclear Physics*, 200:281–296, 1982.

[59] J. Fröhlich, B. Simon, and Thomas Spencer. Infrared bounds, phase transitions and continuous symmetry breaking. *Comm. Math. Phys.*, 50(1):79–95, 1976.

[60] Jürg Fröhlich. Schwinger functions and their generating functionals I. *Helv. Phys. Acta*, 47(3):265–306, 1974.

[61] Jürg Fröhlich and Erhard Seiler. The massive Thirring-Schwinger model (qed2): convergence of perturbation theory and particle structure. *Helv. Phys. Acta*, 49(6):889–924, 1976.

[62] Jurg Frohlich and Erhard Seiler. The Massive Thirring-Schwinger Model (QED in Two-Dimensions): Convergence of Perturbation Theory and Particle Structure. *Helv. Phys. Acta*, 49:889, 1976.

[63] M. Furlan and M. Gubinelli. Weak universality for a class of 3d stochastic reaction–diffusion models. *Probability Theory and Related Fields*, May 2018. 10.1007/s00440-018-0849-6.

[64] James Glimm and Arthur Jaffe . Positivity of the ϕ_3^4 Hamiltonian. *Fortschritte der Physik. Progress of Physics*, 21:327–376, 1973. MR0408581.

[65] James Glimm. Boson fields with the ϕ^4 interaction in three dimensions. *Communications in Mathematical Physics*, 10:1–47, 1968. MR0231601.

[66] James Glimm and Arthur Jaffe. Expansions in statistical physics. *Comm. Pure Appl. Math.*, 38(5):613–630, 1985.

[67] James Glimm and Arthur Jaffe. *Quantum physics*. Springer-Verlag, New York, second edition, 1987. A functional integral point of view.

[68] James Glimm and Arthur Jaffe. *Quantum Physics: A Functional Integral Point of View*. Springer-Verlag, New York, 2 edition, 1987.

[69] James Glimm, Arthur Jaffe, and Thomas Spencer. Phase transitions for ϕ_2^4 quantum fields. *Comm. Math. Phys.*, 45(3):203–216, 1975.

[70] M. Gubinelli, B. Ugurcan, and I. Zachhuber. Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions. *Stochastics and Partial Differential Equations: Analysis and Computations*, 2019. 10.1007/s40072-019-00143-9.

[71] Massimiliano Gubinelli and Martina Hofmanová. A PDE construction of the Euclidean ϕ_3^4 quantum field theory. *arXiv:1810.01700*, 2018.

[72] Massimiliano Gubinelli and Martina Hofmanová. Global Solutions to Elliptic and Parabolic ϕ^4 Models in Euclidean Space. *Communications in Mathematical Physics*, 368(3):1201–1266, 2019. 10.1007/s00220-019-03398-4.

[73] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski. Paracontrolled distributions and singular PDEs. *Forum of Mathematics. Pi*, 3:e6, 75, 2015. 10.1017/fmp.2015.2.

[74] Colin Guillarmou, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Conformal bootstrap in liouville theory. *arXiv preprint arXiv:2005.11530*, 2020.

[75] A. Gulisashvili and M. A. Kon. Exact Smoothing Properties of Schrödinger Semigroups. *American Journal of Mathematics*, 118(6):1215–1248, 1996. JSTOR 25098514.

[76] Trishen S. Gunaratnam, Tadahiro Oh, Nikolay Tzvetkov, and Hendrik Weber. Quasi-invariant gaussian measures for the nonlinear wave equation in three dimensions, 2018.

[77] M. Hairer. A theory of regularity structures. *Inventiones mathematicae*, 198(2):269–504, 2014. 10.1007/s00222-014-0505-4.

[78] Martin Hairer and Hao Shen. The dynamical sine-Gordon model. *Comm. Math. Phys.*, 341(3):933–989, 2016.

[79] Martin Hairer and Weijun Xu. Large-scale behavior of three-dimensional continuous phase coexistence models. *Communications on Pure and Applied Mathematics*, 71(4):688–746, 2018. 10.1002/cpa.21738.

[80] Arthur Jaffe. Introduction to quantum field theory. 2005.

[81] A. Jakubowski. The Almost Sure Skorokhod Representation for Subsequences in Nonmetric Spaces. *Theory Probab. Appl.*, (42(1)):167—-174, 1998.

[82] S. Janson. *Gaussian Hilbert Spaces*. Cambridge University Press, June 1997.

[83] Alexander Kukush. *Gaussian Measures in Hilbert Space*. Wiley, December 2019.

[84] A. Kupiainen. Renormalization Group and Stochastic PDEs. *Annales Henri Poincaré*, 17(3):497–535, 2016. 10.1007/s00023-015-0408-y.

[85] Antti Kupiainen. Constructive liouville conformal field theory. *arXiv preprint arXiv:1611.05243*, 2016.

[86] Antti Kupiainen and Joona Oikarinen. Stress-energy in Liouville conformal field theory. *J. Stat. Phys.*, 180(1-6):1128–1166, 2020.

[87] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Integrability of liouville theory: proof of the dozz formula. *arXiv preprint arXiv:1707.08785*, 2017.

[88] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. Semiclassical limit of liouville field theory. *Journal of Functional Analysis*, 273(3):875–916, aug 2017.

[89] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. A probabilistic approach of ultraviolet renormalisation in the boundary sine-gordon model. 2019.

[90] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. The semiclassical limit of liouville conformal field theory. 2019.

[91] J. Lehec. Representation formula for the entropy and functional inequalities. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, 49(3):885–899, 2013. MR3112438.

[92] J. Magnen and R. Sénéor. The infinite volume limit of the ϕ_3^4 model. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 24(2):95–159, 1976. MR0406217.

[93] Paul Milgrom and Ilya Segal. Envelope theorems for arbitrary choice sets. *Econometrica*, 70(2):583–601, 2002.

[94] A. Moinat and H. Weber. Space-time localisation for the dynamic ϕ_3^4 model. *arXiv:1811.05764*, November 2018. arXiv: 1811.05764.

[95] J.-C. Mourrat and H. Weber. The dynamic Φ_3^4 model comes down from infinity. *Comm. Math. Phys.*, 356(3):673–753, 2017.

[96] J.-C. Mourrat and H. Weber. Global well-posedness of the dynamic Φ^4 model in the plane. *The Annals of Probability*, 45(4):2398–2476, 2017. 10.1214/16-AOP1116.

[97] Jean-Christophe Mourrat and Hendrik Weber. The dynamic ϕ_3^4 model comes down from infinity. *Communications in Mathematical Physics*, 356(3):673–753, 2017. 10.1007/s00220-017-2997-4.

[98] Jean-Christophe Mourrat, Hendrik Weber, and Weijun Xu. Construction of Φ^4_3 diagrams for pedestrians. In *From particle systems to partial differential equations*, volume 209 of *Springer Proc. Math. Stat.*, pages 1–46. Springer, Cham, 2017. MR3746744.

[99] Edward Nelson. A quartic interaction in two dimensions. *Proceedings of the Conference on theMathematical Theory of Elementary Particles*, pages 69–73, 1966.

[100] Edward Nelson. Quantum fields and markoff fields. *Partial Differential Equations(Proc.Sympos. Pure Math, XXIII, held at the University of California, Berkeley, August 9–27,1971*, pages 413–420, 1973.

[101] Tadahiro Oh, Tristan Robert, Philippe Sosoe, and Yuzhao Wang. Invariant gibbs dynamics for the dynamical sine-gordon model. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, August 2020.

[102] Tadahiro Oh and Laurent Thomann. Invariant Gibbs measures for the 2-d defocusing nonlinear wave equations. *Ann. Fac. Sci. Toulouse Math. (6)*, 29(1):1–26, 2020.

[103] Tadahiro Oh and Nikolay Tzvetkov. Quasi-invariant Gaussian measures for the two-dimensional defocusing cubic nonlinear wave equation. *J. Eur. Math. Soc. (JEMS)*, 22(6):1785–1826, 2020.

[104] Konrad Osterwalder and Robert Schrader. Axioms for Euclidean Green's functions. *Comm. Math. Phys.*, 31:83–112, 1973.

[105] Felix Otto and Hendrik Weber. Quasilinear SPDEs via rough paths. *Archive for Rational Mechanics and Analysis*, 232(2):873–950, 2019. 10.1007/s00205-018-01335-8.

[106] G. Parisi and Y. S. Wu. Perturbation theory without gauge fixing. *Scientia Sinica. Zhongguo Kexue*, 24(4):483–496, 1981.

[107] Y. M. Park. The $\lambda\varphi_3^4$ Euclidean quantum field theory in a periodic box. *Journal of Mathematical Physics*, 16(11):2183–2188, 1975. 10.1063/1.522464.

[108] Joseph Polchinski. Renormalization and effective lagrangians. *Nuclear Physics B*, 231(2):269–295, 1984. 10.1016/0550-3213(84)90287-6.

[109] Daniel Revuz and Marc Yor. *Continuous Martingales and Brownian Motion*. Springer, 3rd edition, 2004. MR1725357.

[110] Rémi Rhodes and Vincent vargas. Lecture notes on gaussian multiplicative chaos and liouville quantum gravity. *arXiv preprint arXiv:1602.07323*, 2016.

[111] Thomas Spencer. The mass gap for the $P(\phi)_2$ quantum field model with a strong external field. *Comm. Math. Phys.*, 39:63–76, 1974.

[112] R. F. Streater and A. S. Wightman. *PCT, spin and statistics, and all that*. Princeton Landmarks in Physics. Princeton University Press, Princeton, NJ, 2000. Corrected third printing of the 1978 edition.

[113] Franco Strocchi. *An introduction to non-perturbative foundations of quantum field theory*, volume 158 of *International Series of Monographs on Physics*. Oxford University Press, Oxford, 2013.

[114] Hans Triebel. *Theory of Function Spaces II*. Springer, 1992.

[115] Pavlos Tsatsoulis and Hendrik Weber. Spectral gap for the stochastic quantization equation on the 2-dimensional torus. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(3):1204–1249, 2018.

[116] Nikolay Tzvetkov. Random data wave equations. In *Singular random dynamics*, volume 2253 of *Lecture Notes in Math.*, pages 221–313. Springer, Cham, [2019] ©2019.

[117] A. S. Üstünel. Variational calculation of Laplace transforms via entropy on Wiener space and applications. *Journal of Functional Analysis*, 267(8):3058–3083, 2014. 10.1016/j.jfa.2014.07.006.

[118] Kenneth G. Wilson. The renormalization group and critical phenomena. *Reviews of Modern Physics*, 55(3):583–600, 1983. 10.1103/RevModPhys.55.583.

[119] Kôsaku Yosida. *Functional analysis*, volume 123 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, sixth edition, 1980.

[120] X. Zhang. A variational representation for random functionals on abstract Wiener spaces. *Journal of Mathematics of Kyoto University*, 49(3):475–490, 2009. 10.1215/kjm/1260975036.

Appendix A

Besov spaces and paraproducts

In this section we will recall some well known results about Besov spaces, embeddings, Fourier multipliers and paraproducts. The reader can find full details and proofs in [12, 73] and for weighted spaces in [72, 96]. First recall the definition of Littlewood–Paley blocks. Let χ, ϱ be smooth radial functions $\mathbb{R}^d \to \mathbb{R}$ such that

- $\operatorname{supp} \chi \subseteq B(0, R)$, $\operatorname{supp} \varrho \subseteq B(0, 2R) \setminus B(0, R)$;
- $0 \leqslant \chi, \varrho \leqslant 1$, $\chi(\xi) + \sum_{j \geqslant 0} \varrho(2^{-j}\xi) = 1$ for any $\xi \in \mathbb{R}^d$;
- $\operatorname{supp} \varrho(2^{-j} \cdot) \cap \operatorname{supp} \varrho(2^{-i} \cdot) = \varnothing$ if $|i - j| > 1$.

Introduce the notations $\varrho_{-1} = \chi$, $\varrho_j = \varrho(2^{-j} \cdot)$ for $j \geqslant 0$. For any $f \in \mathscr{S}'(\Lambda)$ we define the operators $\Delta_j f := \varrho_j(\mathrm{D}) f$, $j \geqslant -1$.

Definition A.1. *Let $s \in \mathbb{R}, p, q \in [1, \infty]$. For a Schwarz distribution $f \in \mathscr{S}'(\Lambda)$ define the norm*

$$\|f\|_{B^s_{p,q}} = \|(2^{js} \|\Delta_j f\|_{L^p})_{j \geqslant -1}\|_{\ell^q}$$

where $\| \|_{L^p}$ denotes the normalized $L^p(\Lambda)$ norm. The space $B^s_{p,q}$ is the set of functions $f \in \mathscr{S}'(\Lambda)$ such that $\|f\|_{B^s_{p,q}} < \infty$ moreover $H^s = B^s_{2,2}$ are the usual Sobolev spaces, and we denote by $\mathscr{C}^s$ the closure of smooth functions in the $B^s_{\infty,\infty}$ norm.

Definition A.2. *A polynomial weight ρ is a function $\rho: \mathbb{R}^d \to \mathbb{R}_+$ of the form $\rho(x) = c\langle x \rangle^{-\sigma}$ for $\sigma, c \geqslant 0$. For a polynomial weight ρ let*

$$\|f\|_{L^p(\rho)} = \left(\int_{\mathbb{R}^d} |f(x)|^p \rho(x) \, \mathrm{d}x \right)^{1/p}$$

and by $L^p(\rho)$ the space of functions for which this norm is finite. For function defined on a torus in $\mathbb{R}^d$ we consider their periodic extensions on $\mathbb{R}^d$.

Definition A.3. *For a polynomial weight ρ let*

$$\|f\|_{L^p(\rho)} = \left(\int_{\mathbb{R}^d} |f(x)|^p \rho(x) \, \mathrm{d}x \right)^{1/p}$$

and by $L^p(\rho)$ the space of functions for which this norm is finite. For functions defined on the torus Λ we consider their periodic extensions on $\mathbb{R}^d$. Similarly we define the weighted Besov spaces $B^s_{p,q}(\rho)$ as the set of elements of $\mathscr{S}'(\mathbb{R}^d)$ for which the norm

$$\|f\|_{B^s_{p,q}(\rho)} = \|(2^{js} \|\Delta_j f\|_{L^p(\rho)})_{j \geqslant -1}\|_{\ell^q}$$

is finite and by $\mathscr{C}^s(\rho)$ those such that the norm

$$\|f\|_{\mathscr{C}^s(\rho)} = \|(2^{js} \|\rho \Delta_j f\|_{L^\infty})_{j \geqslant -1}\|_{\ell^\infty}$$

is finite.

Proposition A.4. *Let $\delta > 0$. We have for any $q_1, q_2 \in [1, \infty]$, $q_1 < q_2$*

$$\|f\|_{B^s_{p,q_2}} \leq \|f\|_{B^s_{p,q_1}} \leq \|f\|_{B^{s+\delta}_{p,\infty}}.$$

Furthermore, if we denote by $W^{s,p}$ the normalized fractional Sobolev spaces then for any $q \in [1, \infty]$

$$\|f\|_{B^s_{p,q}} \leq \|f\|_{W^{s+\delta,p}} \leq \|f\|_{B^{s+2\delta}_{p,\infty}}.$$

PROPOSITION A.5. *For any $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$, any $p, q \in [1, \infty]$ the Besov space $B^{s_1}_{p,q}$ is compactly embedded into $B^{s_2}_{p,q}$.*

DEFINITION A.6. *A smooth function η is said to be an S^m multiplier if for every multi-index α there exists a constant C_α such that*

$$\left| \frac{\partial^\alpha}{\partial \xi^\alpha} \eta(\xi) \right| \lesssim_\alpha (1 + |\xi|)^{m - |\alpha|}, \qquad \xi \in \mathbb{R}^d. \tag{A.1}$$

We say that a family η_t is a uniformly S^m multiplier if (A.1) is satisfied for every t with C_α independent of t.

PROPOSITION A.7. *Let η be an S^m multiplier, $s \in \mathbb{R}$, $p, q \in [1, \infty]$, and $f \in B^s_{p,q}$, then*

$$\| \eta(\mathrm{D}) f \|_{B^{s-m}_{p,q}} \lesssim \| f \|_{B^s_{p,q}}.$$

Furthermore the constant depends only on s, p, q, d and the constants C_α in eq. (A.1).

For a proof see [12] Lemma 2.78.

PROPOSITION A.8. *Let θ p, p_1, p_2 and s, s_1, s_2 be such that $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $s = \theta s_1 + (1-\theta) s_2$ and assume that $f \in W^{s_1, p_1} \cap W^{s_2, p_2}$. Then*

$$\| f \|_{W^{s,p}} \leqslant \| f \|^\theta_{W^{s_1, p_1}} \| f \|^{1-\theta}_{W^{s_2, p_2}}.$$

For a proof see [25].

DEFINITION A.9. *Let $f, g \in \mathscr{S}(\Lambda)$. We define the paraproducts and resonant product*

$$f \succ g = g \prec f := \sum_{j < i-1} \Delta_i f \Delta_j g, \qquad \text{and} \qquad f \circ g := \sum_{|i-j| \leqslant 1} \Delta_i f \Delta_j g.$$

Then

$$f g = f \prec g + f \circ g + f \succ g.$$

PROPOSITION A.10. *For any polynomial weight ρ, $\beta \leqslant 0, \alpha \in \mathbb{R}$ and $p_1, p_2 \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ we have the estimate*

$$\| f \succ g \|_{B^{\alpha+\beta}_{p,q}(\rho)} \;\lesssim\; \| f \|_{B^\alpha_{p_1,\infty}(\rho)} \| g \|_{B^\beta_{p_2,q}(\rho)}.$$

For any $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$ the estimate

$$\| f \circ g \|_{B^{\alpha+\beta}_{p,q}(\rho)} \;\lesssim\; \| f \|_{B^\alpha_{p_1,\infty}(\rho)} \| g \|_{B^\beta_{p_2,q}(\rho)}.$$

For a proof see Theorem 3.17 and Remark 3.18 in [96].

PROPOSITION A.11. *For any polynomial weights ν, ρ and $\beta \leqslant 0, \alpha \in \mathbb{R}$ we have*

$$\| f \succ g \|_{B^{\alpha+\beta}_{p,q}(\rho^p \nu)} \;\lesssim\; \| f \|_{\mathscr{C}^\alpha(\rho)} \| g \|_{B^\beta_{p,q}(\nu)}.$$

The proof is an easy modification of the proof of Theorem 3.17 in [96].

PROPOSITION A.12. *Assume $m \leqslant 0$, $\alpha \in (0,1), \beta \in \mathbb{R}$. Let η be an S^m multiplier and $q, p_1, p_2 \in [1, \infty]$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $f \in B^\beta_{p_1,\infty}$, $g \in B^\alpha_{p_1,\infty}$. Then for any $\delta > 0$.*

$$\| \eta(\mathrm{D})(f \succ g) - (\eta(\mathrm{D}) f \succ g) \|_{B^{\alpha+\beta-m-\delta}_{p,q}} \lesssim \| f \|_{B^\beta_{p_1,\infty}} \| g \|_{B^\alpha_{p_1,\infty}}.$$

The constant depends only on α, β, δ and the constants in (A.1).

For a proof see [12] Lemma 2.99.

PROPOSITION A.13. *Let $\alpha\in(0,1)$ $\beta,\gamma\in\mathbb{R}$ such that $\beta+\gamma<0$, $\alpha+\beta+\gamma>0$ and $p_1,p_2,p_3,p\in[1,\infty]$ such that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{p}$. Then there exists a trilinear form $\mathfrak{K}_1(f,g,h)$ such that ,*

$$\|\mathfrak{K}_1(f,g,h)\|_{B^{\alpha+\beta+\gamma}_{p,\infty}}\lesssim\|g\|_{B^{\alpha}_{p_1,\infty}}\|f\|_{B^{\beta}_{p_2,\infty}}\|h\|_{B^{\gamma}_{p_3,\infty}},$$

and when $f,g,h\in\mathscr{S}$ it has the form

$$\mathfrak{K}_1(f,g,h)=(f\succ g)\circ h-g(f\circ h).$$

Proof. The proof is a slight modification of the one given in [73]. Lemma 2.97 from [12] and an interpolation imply that $\|\Delta_j fg-\Delta_j(fg)\|_{L^p}\leq 2^{-j\alpha}\|f\|_{W^{\alpha,p_1}}\|g\|_{L^{p_2}}$. This in turn gives after some algebraic computation (see [73] for details) that

$$\Delta_j(f\succ g)=(\Delta_j f)\succ g+R_j(f,g),$$

with $\|R_j(f,g)\|_{L^p}\lesssim 2^{-j(\alpha+\beta)}\|f\|_{B^{\alpha}_{p_1,\infty}}\|g\|_{B^{\beta}_{p_2,\infty}}$. Now to prove the statement of the proposition observe that for smooth f,g,h we have

$$\mathfrak{K}_1(f,g,h)=\sum_{j,k\geqslant-1}\sum_{|i-j|\leqslant1}\Delta_j(f\succ\Delta_k g)\Delta_i h-\Delta_k g\Delta_j f\Delta_i h.$$

Now observe that the term $f\succ\Delta_k g$ has Fourier transform outside of $2^k B$ for some ball B independent of k, so choosing N large enough we can rewrite the sum as

$$\mathfrak{K}_1(f,g,h)=\sum_{j,k\geqslant-1}\sum_{|i-j|\leqslant1}\mathbb{1}_{k\leqslant i+N}(\Delta_j f\Delta_k g\Delta_i h+R_j(f,\Delta_k g))-\Delta_k g\Delta_j f\Delta_i h$$

$$\sum_{j,k\geqslant-1}\sum_{|i-j|\leqslant1}\mathbb{1}_{k\leqslant i+N}R_j(f,\Delta_k g)\Delta_i h-\mathbb{1}_{k\geqslant i+N}\Delta_k g\Delta_j f\Delta_i h.$$

Now we estimate the norm of the two terms separately. First note that for fixed j

$$\sum_{k\geqslant-1}\sum_{|i-j|\leqslant1}\mathbb{1}_{k\leqslant i+N}R_j(f,\Delta_k g)$$

has a Fourier transform supported in $2^j B$. By Lemma 2.69 from [12] it is enough to get an estimate on

$$\sup_k\left\|2^{(\alpha+\beta+\gamma)j}\sum_{j\geqslant-1}\sum_{|i-j|\leqslant1}\mathbb{1}_{k\leqslant i+N}R_j(f,\Delta_k g)\Delta_i h\right\|_{L^p}$$

to bound it in $B^{\alpha+\beta+\gamma}_{p,\infty}$, so by Hölder inequality,

$$\begin{aligned}\left\|\sum_{|i-j|\leqslant1}R_j\!\left(f,\sum_{k\geqslant-1}^{i+N}\Delta_k g\right)\!\Delta_i h\right\|_{L^p} &\lesssim \sum_{|i-j|\leqslant1}2^{-j(\alpha+\beta)}2^{-i\gamma}\|g\|_{B^{\alpha}_{p_1,\infty}}\|f\|_{B^{\beta}_{p_2,q_1}}\|h\|_{B^{\gamma}_{p_3,q_2}}\\ &\lesssim 2^{-j(\alpha+\beta+\gamma)}\|g\|_{B^{\alpha}_{p_1,\infty}}\|f\|_{B^{\beta}_{p_2,q_1}}\|h\|_{B^{\gamma}_{p_3,q_2}}.\end{aligned}$$

For the second term observe that for fixed k the Fourier transform of

$$\sum_{j\geqslant-1}\sum_{|i-j|\leqslant1}\mathbb{1}_{k\geqslant i+N}\Delta_k g\Delta_j f\Delta_i h$$

is supported in $2^k B$. Now we can estimate again by Hölder inequality

$$\begin{aligned}&\lesssim\left\|\sum_{j\geqslant-1}\sum_{|i-j|\leqslant1}\mathbb{1}_{k\geqslant i+N}\Delta_k g\Delta_j f\Delta_i h\right\|_{L^p}\\ &\lesssim 2^{-\alpha k}\sum_{j\geqslant-1}^{k+N}2^{-(\beta+\gamma)k}\mathbb{1}_{k\geqslant i+N}\|g\|_{B^{\alpha}_{p_1,\infty}}\|f\|_{B^{\beta}_{p_2,\infty}}\|h\|_{B^{\gamma}_{p_1,\infty}}\\ &\lesssim 2^{-j(\alpha+\beta+\gamma)}\|g\|_{B^{\alpha}_{p_1,\infty}}\|f\|_{B^{\beta}_{p_2,q_1}}\|h\|_{B^{\gamma}_{p_3,q_2}}.\end{aligned}$$

□

PROPOSITION A.14. *Assume $\beta \in (0, 1)$, $\alpha, \gamma \in \mathbb{R}$ such that $\alpha + \gamma < 0$, and $\alpha + \beta + \gamma = 0$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Then there exists a trilinear form $\mathfrak{K}_2(f, g, h)$ for which*

$$|\mathfrak{K}_2(f, g, h)| \lesssim \|f\|_{B^{\alpha}_{p_1,\infty}} \|g\|_{B^{\beta}_{p_2,q_1}} \|h\|_{B^{\gamma}_{p_3,q_2}},$$

and on smooth functions

$$\mathfrak{K}_2(f, g, h) = \fint [(f \succ g) h - g(f \circ h)].$$

Proof. This is modification of the proof of Lemma A.6 in [70]. Repeating an algebraic computation given in the proof of that lemma, we get that for smooth f, g, h we have

$$\mathfrak{K}_2(f, g, h) = \left(\sum_{j \leqslant i-1, |i-k| \leqslant L} - \sum_{i \sim k, j < i+L} \right) \fint (\Delta_i f \Delta_j g \Delta_k h),$$

for some $L \geqslant 1$. Then we estimate

$$\begin{aligned} |\mathfrak{K}_2(f, g, h)| &\lesssim \sum_{i \sim j \sim k} \|\Delta_i f \Delta_j g \Delta_k h\|_{L^1} \\ &\lesssim \sum_{i \sim j \sim k} \|\Delta_i f\|_{L^{p_1}} \|\Delta_j g\|_{L^{p_2}} \|\Delta_k h\|_{L^{p_3}} \\ &\lesssim \sup_i (2^{\alpha i} \|\Delta_i f\|_{L^{p_1}}) \sum_{j \sim k} 2^{(\beta+\gamma)k} \|\Delta_j g\|_{L^{p_2}} \|\Delta_k h\|_{L^{p_3}} \\ &\lesssim \|f\|_{B^{\alpha}_{p_1,\infty}} \|g\|_{B^{\beta}_{p_2,q_1}} \|h\|_{B^{\gamma}_{p_3,q_2}}. \end{aligned}$$

□

PROPOSITION A.15. *There exists a family $(\mathfrak{K}_{3,t})_{t \geqslant 0}$ of bounded multilinear forms on $\mathscr{C}^{-1-\kappa} \times \mathscr{C}^{-1-\kappa} \times H^{1/2-\kappa} \times H^{1/2-\kappa}$ such that for smooth $\varphi, \psi, g^{(1)}, g^{(2)}$ it holds*

$$\mathfrak{K}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)}) = \fint [J_t(\varphi \succ g^{(1)}) J_t(\psi \succ g^{(2)}) - (J_t \varphi \circ J_t \psi) g^{(1)} g^{(2)}],$$

and

$$|\mathfrak{K}_{3,t}(\varphi, \psi, g^{(1)}, g^{(2)})| \lesssim \frac{1}{\langle t \rangle^{1+\delta}} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}}.$$

Proof. Note that $\langle t \rangle^{1/2} J_t$ satisfies the assumptions of Proposition A.12 and with $m = -1$, therefore using also Proposition A.4

$$\|J_t(\varphi \succ g^{(1)}) - J_t \varphi \succ g^{(1)}\|_{H^{1/2-3\kappa}} \lesssim \langle t \rangle^{-1/2} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}}.$$

Therefore

$$\begin{aligned} &\left| \fint [J_t(\varphi \succ g^{(1)}) - (J_t \varphi \succ g^{(1)})] J_t(\psi \succ g^{(2)}) \right| \\ \lesssim\ & \|J_t(\varphi \succ g^{(1)}) - J_t \varphi \succ g^{(1)}\|_{H^{1/2-3\kappa}} \|J_t(\psi \succ g^{(2)})\|_{H^{-1/2+3\kappa}} \\ \lesssim\ & \langle t \rangle^{-1/2} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \langle t \rangle^{-1/2-\delta} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \end{aligned}$$

and by symmetry also

$$\begin{aligned} &\left| \fint [J_t(\varphi \succ g^{(1)}) J_t(\psi \succ g^{(2)}) - (J_t \varphi \succ g^{(1)})(J_t \psi \succ g^{(2)})] \right| \\ \lesssim\ & \langle t \rangle^{-1-\kappa} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \end{aligned}$$

Furthermore from Proposition A.14 and for sufficiently small $\kappa > 0$,

$$\begin{aligned} &\left| \fint (J_t \varphi \succ g^{(1)})(J_t \psi \succ g^{(2)}) - \fint ((J_t \varphi \succ g^{(1)}) \circ J_t \psi) g^{(2)}_t \right| \\ \lesssim\ & \|J_t \varphi\|_{\mathscr{C}^{-2\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|J_t \psi\|_{\mathscr{C}^{-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}} \\ \lesssim\ & \langle t \rangle^{-1-\kappa} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(2)}\|_{H^{1/2-\kappa}}. \end{aligned}$$

Applying Proposition-A.13 we get

$$\begin{aligned}
&\|(J_t\varphi^{(1)} \succ g^{(1)}) \circ J_t\psi_t - (J_t\varphi_t \circ J_t\psi_t)(g^{(1)})\|_{H^{-1/2+\kappa}} \\
\lesssim\ & \|J_t\varphi_t\|_{\mathscr{C}^{-2\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|J_t\psi_t\|_{\mathscr{C}^{-\kappa}} \\
\lesssim\ & \langle t\rangle^{-1-\delta} \|\varphi\|_{\mathscr{C}^{-1-\kappa}} \|g^{(1)}\|_{H^{1/2-\kappa}} \|\psi\|_{\mathscr{C}^{-1-\kappa}}
\end{aligned}$$

and putting things together gives the required estimate. □